Properties of dyons in $\mathcal{N} = 4$ theories at small charges

Aradhita Chattopadhyaya and Justin R. David

Centre for High Energy Physics, Indian Institute of Science,
C.V. Raman Avenue, Bangalore 560012, India

E-mail: aradhitac@iisc.ac.in, justin@iisc.ac.in

ABSTRACT: We study three properties of 1/4 BPS dyons at small charges in string compactifications which preserve $\mathcal{N} = 4$ supersymmetry. We evaluate the non-trivial constant present in the one loop statistical entropy for $\mathcal{N} = 4$ compactifications of type IIB theory on $K3 \times T^2$ orbifolded by an order $\mathbb{Z}_N$ freely acting orbifold g' including all CHL compactifications. This constant is trivial for the un-orbifolded model but we show that it contributes crucially to the entropy of low charge dyons in all the orbifold models. We then show that the meromorphic Jacobi form which captures the degeneracy of 1/4 BPS states for the first two non-trivial magnetic charges can be decomposed into an Appell-Lerch sum and a mock Jacobi form transforming under $\Gamma_0(N)$. This generalizes the earlier observation of Dabholkar-Murthy-Zagier to the orbifold models. Finally we study the sign of the Fourier coefficients of the inverse Siegel modular form which counts the index of 1/4 BPS dyons in $\mathcal{N} = 4$ models obtained by freely acting $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds of type II theory compactified on T^6. We show that sign of the index for sufficiently low charges and ensuring that it counts single centered black holes, violates the positivity conjecture of Sen which indicates that these states posses non-trivial hair.

KEYWORDS: Black Holes in String Theory, Conformal Field Models in String Theory, Superstring Vacua

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Contents

1 Introduction

One of the successes of string theory as a theory of quantum gravity lies in its microscopic understanding of the Bekenstein-Hawking entropy. In [1] it was shown that the statistical entropy of a system of branes carrying the same quantum numbers of the black hole agrees precisely with that of a class of extremal black holes in 5 dimensions. In 4 dimensions, starting with the original work of [2], the degeneracies of 1/4 BPS dyons in heterotic string theory compactified on T^6 and its generalizations to CHL compactifications [3] provide examples where a precise formula for the microscopic degeneracies of extremal black holes in known. The degeneracy of dyons or more precisely an index can be obtained from the Fourier coefficients of the inverse of an appropriate $\mathrm{Sp}(2, \mathbb{Z})$ Siegel modular form. On taking the large charge limit of the index, the logarithm of the index agrees precisely not only with the Bekenstein-Hawking entropy of the corresponding black hole with the same charges, but also with the sub-leading correction given by the Wald's generalization of the Bekenstein-Hawking formula [4, 5].[1]

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[1]See [6, 7] for reviews.

Given the success of the microscopic formula for the degeneracies of dyons in the large charge limit, it is natural to study its properties for dyons with small charges. This will hopefully provide more intuition on how a geometric description for the dyons in terms of metric can arise. In this paper we study three properties 1/4 BPS dyons in a class of $\mathcal{N} = 4$ compactifications of string theory. These compactifications arise from considering type IIB theory on $K3 \times T^2$ orbifolded by g' which acts as an $\mathbb{Z}_N$ automorphism on $K3$ together with a $1/N$ shift on one of the circles of T^2. This class includes all the CHL compactifications as well as more general ones where g' corresponds to conjugacy classes of the Mathieu group M_{23}. The orbifolds we study are listed table 1. The partition function of dyons in terms of inverse of a Siegel modular forms for these compactifications were constructed in [8, 9]. We also study certain $\mathbb{Z}_2, \mathbb{Z}_3$ compactifications of type II theory on T^6.

The degeneracy in the large charge limit is obtained by evaluating the leading saddle point of an integral and its one loop correction. The integral extracts out the Fourier coefficient of the inverse Siegel modular form. Let us call the logarithm of this saddle point approximation for the degeneracy $S_{\text{stat}}^{(1)}(Q, P)$ for definiteness where Q and P are the electric and magnetic charges of the dyon. It is $S_{\text{stat}}^{(1)}$ that agrees with the Bekenstein-Hawking formula and its generalization by Wald evaluated on a dyonic extremal black hole with the same set of charges. In [6] and [10], $S_{\text{sat}}^{(1)}$ was compared with the exact degeneracy of dyons at low charges.[2] For the $\mathcal{N} = 4$ heterotic string theory on T^6 or equivalently type IIB theory on $K3 \times T^2$ it was seen that $S_{\text{sat}}^{(1)}$ remarkably agrees with the exact degeneracy to within 2% even for the lowest admissible charge. We re-evaluate $S_{\text{sat}}^{(1)}$ in this paper for all $\mathcal{N} = 4$ compactifications arising as g' orbifolds of $K3 \times T^2$ with g' given in table 1 keeping track of a constant C_1 in $S_{\text{sat}}^{(1)}$. This constant is trivial that is, $\ln C_1 = 0$, for the un-orbifolded compactification on $K3 \times T^2$ and was not determined earlier for the orbifolds. We then compare the statistical entropy at one loop $S_{\text{sat}}^{(1)}$ to the exact degeneracy and show that the constant plays a crucial rule. In fact without C_1, the agreement with the exact degeneracy at low charges is off by more that 50% in most cases. The reader can directly go to tables 5 to 15 to appreciate the presence of C_1 in $S_{\text{sat}}^{(1)}$. The constant C_1 also contributes to all the subsequent saddle points. To demonstrate this, we evaluate the correction to $S_{\text{sat}}^{(1)}$ from the second saddle point for the 2A orbifold. We then briefly discuss the implication of this constant for the geometric description of the dyons.

The second property we study of low charge dyons is the Fourier-Jacobi coefficients of the inverse Siegel modular form at fixed magnetic charge of the dyon. The Fourier-Jacobi coefficients enumerate the degeneracy of dyons for arbitrary electric charge and angular momentum but at fixed magnetic charge. We study the case of the first two non-trivial magnetic charges. We show that these Fourier-Jacobi coefficients are meromorphic Jacobi-forms which can be decomposed into an Appell-Lerch sum, the polar part and a finite term which is a mock modular form. The finite part captures the degeneracy of single centered dyons. We perform this decomposition for all the orbifolds listed in table 1. This decomposition was done in [11] for the un orbifolded theory to reasonably high magnetic charges. For the CHL orbifolds corresponding to pA with $p = 2, 3, 5, 7$ at zero magnetic

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[2]These charges were such that the dyons are single centered.

charge $P^2 = 0$ was done in the appendix A.5 of [12]. In this paper we have generalized these observations to all the other CHL orbifolds as well as the others in table 1 to the first non-trivial order $P^2 = 2$. We observe that at this order there is a crucial identity among meromorphic Jacobi forms[3] that allows us to use the decompositions found by [11]. In fact for all the orbifolds we show that the mock modular form that occurs at the level of $P^2 = 2$ is the generating function of Hurwitz-Kronecker class numbers found by [11] for the un orbifolded theory. Thus our analysis generalizes these observations to all the orbifolds.

Finally we examine the 1/4 BPS dyons in orbifolds of type II compactifications on T^6 which preserve $\mathcal{N} = 4$ supersymmetry. These theories were originally constructed in [13] and the partition function of dyons in these theories was obtained in [14]. Our objective to study these models is to examine the positivity conjecture of [15] on the Fourier coefficients of the inverse Siegel modular forms which correspond to dyon partition functions. The crucial assumption that goes in to the positivity conjecture of [15] is that if the charges of the dyon is such that it is single centered, then the only sign to the index evaluating of the 1/4 BPS state arises from the fermionic zero modes that arise due to the breaking of $\mathcal{N} = 4$ supersymmetry by the dyon. This is because the single centered dyon is assumed to be spherically symmetric and therefore carries zero angular momentum. We study the signs of the Fourier coefficients that arise in the partition function of dyons of the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds of type II compactifications on T^6. We show that when the charges satisfy the condition that the dyon is single centered the sign of the index violates the positivity conjecture of [15]. We also use the criteria of subtracting out the polar part in the Fourier-Jacobi decomposition to identify the single dyons given a fixed magnetic charge and show that there are violations of the positivity conjecture. In fact we show that there is an infinite class of dyons which are single centered as well with magnetic charge $P^2 = 2$ which violate the positivity conjecture in the $\mathbb{Z}_2$ orbifold. These observations indicate the these dyons might posses hair modes which contribute to the sign of the 1/4 BPS index. This was one of the suggested ways, the positivity conjecture of [15] might be violated.

The common thread which runs through these results is that the all these properties are found at low charges of dyons and in orbifold models of $\mathcal{N} = 4$ compactifications. We see that there is more to be learned about 1/4 BPS dyons and their geometric description as black holes.

The organization of the paper is as follows. In section 2, we compare the statistical entropy of low charge dyons and the exact degeneracies for all orbifolds models given in table 1 and show that the constant C_1 contributes crucially to the entropy to $S^{(1)}_{\text{stat}}$. In section 3 we decompose the Fourier-Jacobi coefficient that occur in the expansion of the inverse Siegel modular form for the first two non-trivial magnetic charges but arbitrary electric and magnetic charge in terms of an Appell-Lerch sum and a mock modular form. This is performed in detail for the $2A$ orbifold. In section 4 we study the violation of the positivity conjecture by the Fourier coefficients of the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ toroidal orbifolds. Section 5 contains our conclusions. Appendix A contains the details regarding evaluating the constant C_1.

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[3]See equation (3.20) for the identity.

2 Degeneracy and statistical entropy at small charges

Consider $N = 4$ string compactifications obtained by considering type II B theories on $K3 \times T^2/\mathbb{Z}_N$ where the Z_N acts as a automorphism g' on $K3$ together with a $1/N$ shift on one of the circles of S^1. The action g' corresponds to the 26 classes of the Mathieu group M_{23}. For eg. The classes pA with $p = 2, 3, 4, 5, 6, 7, 8$ are known as Nikulin's automorphism. These compactifications are also known as CHL compactifications and they were introduced first as generalizations of models which dual to heterotic string compactifications with $\mathcal{N} = 4$ supersymmetry [16, 17]. Essentially these compactifications reduce the rank of the gauge group but preserve $\mathcal{N} = 4$ supersymmetry.

These CHL compactifications admit quarter BPS dyons. Let the charge vector for these dyons be $(\vec{Q}, \vec{P})$ in the heterotic frame. Let $d(\vec{Q}, \vec{P})$ denote the difference between the number of bosonic and fermions quarter BPS multiplets. In the region of moduli space where the type IIB theory is weakly coupled, $d(\vec{Q}, \vec{P})$ is given by

$$d(Q,P) = \frac{1}{N}(-1)^{Q \cdot P + 1} \int_{\mathcal{C}} d\tilde{\rho} d\tilde{\sigma} d\tilde{v} \; e^{-\pi i (N\tilde{\rho} Q^2 + \tilde{\sigma} P^2/N + 2\tilde{v} Q \cdot P)} \frac{1}{\tilde{\Phi}_k(\tilde{\rho}, \tilde{\sigma}, \tilde{v})}. \qquad (2.1)$$

The contour $\mathcal{C}$ is defined over a 3 dimensional subspace of the 3 complex dimensional space $(\tilde{\rho} = \tilde{\rho}_1 + i\tilde{\rho}_2, \tilde{\sigma} = \tilde{\sigma}_1 + i\tilde{\sigma}_2, \tilde{v} = \tilde{v}_1 + i\tilde{v}_2)$.

$$\tilde{\rho}_2 = M_1, \qquad \tilde{\sigma}_2 = M_2, \qquad \tilde{v}_2 = -M_3, \qquad (2.2)$$
$$0 \le \tilde{\rho}_1 \le 1, \qquad 0 \le \tilde{\sigma}_1 \le N, \qquad 0 \le \tilde{v}_1 \le 1.$$

Here M_1, M_2, M_3 are positive numbers, which are fixed and large and $M_3 \ll M_1, M_2$. The contour essentially implies that we perform the expansions first in $e^{2\pi i \tilde{\rho}}, e^{2\pi i \tilde{\sigma}}$ and then perform the expansion in $e^{-2\pi i \tilde{v}}$. The function $\tilde{\Phi}$, occurring in integrand is a Siegel modular form transforming under a subgroup of $\mathrm{Sp}(2, \mathbb{Z})$ with weight k. Explicitly, it is given by

$$\tilde{\Phi}(\rho, \sigma, v) = e^{2\pi i (\tilde{\rho} + \tilde{\sigma}/N + \tilde{v})} \qquad (2.3)$$

$$\prod_{b=0,1} \prod_{r=0}^{N-1} \prod_{\substack{k' \in \mathbb{Z} + \frac{r}{N}, l \in \mathcal{Z}, \\ j \in 2\mathbb{Z} + b \\ k', l \ge 0, \, j < 0 \; k' = l = 0}} (1 - e^{2\pi i (k'\sigma + l\rho + jv)})^{\sum_{s=0}^{N-1} e^{2\pi i s l / N} c_b^{r,s}(4k'l - j^2)}.$$

Here N is the order of the orbifold g'. The coefficients $c_b^{(r,s)}$ are read out from the expansion of the elliptic genus of $K3$ twisted by the action of g'. The twisted elliptic genus of $K3$ and its expansion is defined by

$$F^{(r,s)}(\tau, z) = \frac{1}{N} \mathrm{Tr}_{RR \, g'^r} \left[(-1)^{F_{K3} + \bar{F}_{K3}} g'^s e^{2\pi i z F_{K3}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right], \qquad (2.4)$$

$$= \sum_{b=0}^{1} \sum_{j \in 2\mathbb{Z} + b, \, n \in \mathbb{Z}/N} c_b^{(r,s)}(4n - j^2) e^{2\pi i n \tau + 2\pi i j z}.$$

$$0 \le r, s \le N - 1.$$

The trace in the above equation is taken over the Ramond-Ramond sector of the $\mathcal{N} = (4,4)$ super conformal field theory of $K3$ with central charge $(6,6)$ and F refers to the Fermion number. The twisted elliptic genera for the g' belonging to all the conjugacy classes of $M_{23} \subset M_{24}$ has been evaluated in [9, 18]. They take the form

$$F^{(0,0)}(\tau, z) = \alpha_{g'}^{(0,0)} A(\tau, z), \tag{2.5}$$
$$F^{(r,s)}(\tau, z) = \alpha_{g'}^{(r,s)} A(\tau, z) + \beta_{g'}^{(r,s)}(\tau) B(\tau, z),$$
$$r, s \in \{0, 1, \cdots N - 1\} \text{ with}(r, s) \neq (0, 0),$$

with

$$A(\tau, z) = \frac{\theta_2^2(\tau, z)}{\theta_2^2(\tau, 0)} + \frac{\theta_3^2(\tau, z)}{\theta_3^2(\tau, 0)} + \frac{\theta_4^2(\tau, z)}{\theta_4^2(\tau, 0)}, \tag{2.6}$$
$$B(\tau, z) = \frac{\theta_1^2(\tau, z)}{\eta^6(\tau)}. \tag{2.7}$$

$A(\tau, z)$ and $B(\tau, z)$ are Jacobi forms that transform under $\mathrm{SL}(2, \mathbb{Z})$ with index 1 and weight 0 and -2 respectively. The $\alpha_{g'}^{(r,s)}$ in (2.5) are numerical constants and $\beta_{g'}^{(r,s)}$ are weight 2 modular forms which transform under $\Gamma_0(N)$. For all g' corresponding to the conjugacy classes listed in table 1, the list of the twisted elliptic genera can be found in appendix E of [9]. The weight k, of the Siegel modular form is given by

$$k = \frac{1}{2} \sum_0^{N-1} c_0^{(0,s)}(0). \tag{2.8}$$

The weights of the Siegel modular forms corresponding to the twisted elliptic genera constructed in this paper is listed in table 2.

Now using the twisted elliptic genera and product form for the Siegel modular form in (2.3) we can obtain $d(\vec{Q}, \vec{P})$ as defined by its Fourier expansion in (2.1) for low values of the charges. For sufficiently large values of charges, the integral in (2.1) can be performed by evaluating the leading saddle point. The behaviour of the Siegel modular form $\tilde{\Phi}$ at the saddle point is determined by another modular form $\hat{\Phi}$ which is related to by a $\mathrm{Sp}(2, \mathbb{Z})$ transformation. Consider the transformation

$$\tilde{\rho} = \frac{1}{N} \frac{1}{2v - \rho - \sigma}, \quad \tilde{\sigma} = N \frac{v^2 - \rho\sigma}{2v - \rho - \sigma}, \quad \tilde{v} = \frac{v - \rho}{2v - \rho - \sigma}, \tag{2.9}$$

with inverse

$$\rho = \frac{\tilde{\rho}\tilde{\sigma} - \tilde{v}^2}{N\tilde{\rho}}, \quad \sigma = \frac{\tilde{\rho}\tilde{\sigma} - (\tilde{v} - 1)^2}{N\tilde{\rho}}, \quad v = \frac{\tilde{\rho}\tilde{\sigma} - \tilde{v}^2 + \tilde{v}}{N\tilde{\rho}}. \tag{2.10}$$

The saddle point is located at $v = 0$. This transformation results in the change of measure which is given by

$$d\tilde{\rho}d\tilde{\sigma}d\tilde{v} = -(2v - \rho - \sigma)^{-3}d\rho d\sigma dv. \tag{2.11}$$

Conjugacy Class	Order
1A	1
2A	2
3A	3
5A	5
7A	7
11A	11
23A/B	23
4B	4
6A	6
8A	8
14A/B	14
15A/B	15

Table 1. Conjugacy classes of M_{24} studied in the paper. We refer to the classes 2A, 3A, 5A, 7A, 4B, 6A, 8A as the CHL orbifolds.

Type 1	pA	4B	6A	8A	14A	15A
Weight	$\frac{24}{p+1} - 2$	3	2	1	0	0

Table 2. Weight of Siegel modular forms corresponding to classes in M_{23}.

Substituting this transformation we obtain

$$d(\vec{Q}, \vec{P}) = -\frac{1}{N}(-1)^{Q \cdot P + 1} \int_{\hat{\mathcal{C}}} d\rho d\sigma dv (2v - \rho - \sigma)^{-3} \tag{2.12}$$
$$\times \exp\left[-i\pi \left\{ \frac{v^2 - \rho\sigma}{2v - \rho - \sigma} P^2 + \frac{1}{2v - \rho - \sigma} Q^2 + 2\frac{v - \rho}{2v - \rho - \sigma} Q \cdot P \right\}\right]$$
$$\times \frac{1}{\tilde{\Phi}_k(\tilde{\rho}, \tilde{\sigma}, \tilde{v})}.$$

Here the variables $(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$ are now thought of functions of the variables (ρ, σ, v). The contour is also correspondingly mapped. Now the Siegel modular forms constructed from the twisted elliptic genus of $K3$ corresponding satisfy the relation

$$\tilde{\Phi}_k(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \tilde{\Phi}_k(\tilde{\sigma}/N, \tilde{\rho}N, \tilde{v}). \tag{2.13}$$

As shown in [5], this property results from the following equation satisfied by the twisted elliptic genera

$$\sum_{s=0}^{N-1} e^{-2\pi i l s/N} F^{(r,s)}(\tau, z) = \sum_{s=0}^{N-1} e^{-2\pi i r s/N} F^{(l,s)}(\tau, z). \tag{2.14}$$

In [5] it was verified that this property holds for all the orbifolds belonging to the class pA with $p = 1, 2, 3, 5, 7$. We have verified that this property remains to be true for all the orbifolds g' listed in table 1. This includes all the CHL orbifolds in addition to the

Conjugacy Class	$g^{(k+2)}(\rho)$
pA	$\eta^{k+2}(\rho)\eta^{k+2}(p\rho)$
4B	$\eta^4(4\rho)\eta^2(2\rho)\eta^4(\rho)$
6A	$\eta^2(\rho)\eta^2(2\rho)\eta^2(3\rho)\eta^2(6\rho)$
8A	$\eta^2(\rho)\eta(2\rho)\eta(4\rho)\eta^2(8\rho)$
14A	$\eta(\rho)\eta(2\rho)\eta(7\rho)\eta(14\rho)$
15A	$\eta(\rho)\eta(3\rho)\eta(5\rho)\eta(15\rho)$

Table 3. Factorization of $\hat{\Phi}_k(\rho,\sigma,v)$ as $\lim v \to 0$, $p \in \{1,2,3,5,7,11\}$.

new ones in the conjugacy class of M_{23}. Now using (2.13) with the transformation (2.9) we obtain

$$\tilde{\Phi}_k\left(\frac{v^2-\rho\sigma}{2v-\rho-\sigma}, \frac{1}{2v-\rho-\sigma}, \frac{v-\rho}{2v-\rho-\sigma}\right), \tag{2.15}$$
$$= -(i)^k C_1 (2v-\rho-\sigma)^k \hat{\Phi}(\rho,\sigma,v).$$

The last line defines the modular function $\hat{\Phi}_k$ related to $\tilde{\Phi}_k$ by the $\mathrm{Sp}(2,\mathbb{Z})$ transformation. For the case of the unorbifolded $K3$, $k = 10$ and since the Igusa cusp form is unique $\hat{\Phi}_{10}$ coincides with $\tilde{\Phi}_{10}$ and the constant $C_1 = 1$. Thus the leading saddle point is determined by the behaviour of the new modular form $\hat{\Phi}_k$ at $v \to 0$. Here C_1 is constant which is non-trivial for all the orbifolds in table 1 and we will show it plays an important role.[4] It can be shown using the product representation of $\hat{\Phi}_k$ that at $v \to 0$

$$\hat{\Phi}_k(\rho,\sigma,v)|_{v\to 0} = -4\pi^2 v^2 g(\rho)g(\sigma) \tag{2.16}$$

where $g(\tau)$ is a specific $\Gamma_0(N)$ form for each of the orbifold g' of weight $k+2$ For example for the $2A$ orbifold

$$g(\tau) = \eta^8(\tau)\eta^8(2\tau) \tag{2.17}$$

The list of the function $g(\tau)$ for each of the orbifolds is given in table 3. This was obtained for $pA, p = 2, 3, 5, 7$ orbifolds in [19] and for the remaining orbifolds of table 3 in [9]

The end result of evaluating the saddle at $v = 0$ and the one loop determinant is the following. Let us define the statistical entropy by

$$S_{\mathrm{stat}} \equiv \ln d(\vec{Q}, \vec{P}) \tag{2.18}$$

To obtain the result of S_{stat} we need to consider the statistical entropy function

$$S(\tau) = \frac{\pi}{2\tau_2}|Q - \tau P|^2 - \ln g(\tau) - \ln g(-\bar{\tau}) - (k+2)\ln(2\tau_2) - \ln(NC_1) + O(Q^{-2}, P^{-2}) \tag{2.19}$$

where $\tau = \tau_1 + i\tau_2$. Then the statistical entropy is obtained by evaluating $S(\tau)$ at its extremum τ_{extremum}

$$S_{\mathrm{stat}}^{(1)} = S(\tau)|_{\tau_{\mathrm{extremum}}}. \tag{2.20}$$

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[4]As we will subsequently demonstrate, we have chosen the phases so that C_1 is real.

The superscript (1) refers to the fact that the statistical entropy is obtained at one loop from the leading saddle. Note the presence of the constant C_1 in the statistical entropy function (2.19), this is the constant that relates the generating function for the degeneracies $\tilde{\Phi}_k$ and its $\mathrm{Sp}(2,\mathbb{Z})$ transform $\hat{\Phi}_k$ as given in (2.15). As mentioned earlier, $C_1 = 1$ for the unorbifolded $K3 \times T^2$ compactification, and therefore there is no contribution from this term to the statistical entropy function.

Our goal in the rest of this section is to evaluate this constant and compare its contribution in $S_{\text{stat}}^{(1)}$ to the exact entropy $S_{\text{stat}} = \ln d(\vec{Q}, \vec{P})$. We will perform this comparison for low values of charges as it is clear that for very large values of charges this constant will not play a relevant role. Such a comparison for low values of charges of the one loop statistical entropy function with the exact entropy was made for the un-orbifolded $K3 \times T^2$ compactification in [6] and later in [10]. It was seen that the statistical entropy function at one loop agrees with the exact entropy to 2% even for the lowest admissible charge. We will extend this comparison for all the orbifolds g' listed in table 1. We will see that the constant C_1 is non-trivial and depends on the orbifold and contributes crucially towards $S_{\text{stat}}^{(1)}$

2.1 $\hat{\Phi}_k$ and the constant C_1

In this section we determine the constant C_1 which occurs in the modular transformation relating $\tilde{\Phi}_k$ and $\hat{\Phi}_k$ (2.15). We will first follow the first principle method which defines $\hat{\Phi}_k$ in terms of the a 'threshold integral' relating it to the form $\tilde{\Phi}_k$ and obtain C_1. We then perform a simple cross check, by using the factorization property of $\hat{\Phi}_k$ given in (2.16). Before we proceed we simplify the modular transform relating these Siegel modular forms. We can also write (2.15) as

$$\tilde{\Phi}_k(\tilde{\rho}', \tilde{\sigma}', \tilde{v}') = -(i)^k C_1 (\tilde{\sigma}')^{-k} \hat{\Phi}_k \left(\tilde{\rho}' - \frac{(\tilde{v}')^2}{\tilde{\sigma}'}, \tilde{\rho}' - \frac{(\tilde{v}'-1)^2}{\tilde{\sigma}'}, \tilde{\rho}' - \frac{\tilde{v}'^2}{\tilde{\sigma}'} + \frac{\tilde{v}'}{\tilde{\sigma}'} \right) \qquad (2.21)$$

Here we have defined

$$\tilde{\rho}' = \frac{v^2 - \rho\sigma}{2v - \rho - \sigma}, \quad \tilde{\sigma}' = \frac{1}{2v - \rho - \sigma}, \quad \tilde{v}' = \frac{v - \rho}{2v - \rho - \sigma} \qquad (2.22)$$

and its inverse

$$\rho = \frac{\tilde{\rho}'\tilde{\sigma}' - \tilde{v}'^2}{\tilde{\sigma}'}, \quad \sigma = \frac{\tilde{\rho}'\tilde{\sigma}' - (\tilde{v}'-1)^2}{\tilde{\sigma}'}, \quad \tilde{v} = \frac{\rho'\tilde{\sigma}' - \tilde{v}'^2 + \tilde{v}}{\tilde{\sigma}'}. \qquad (2.23)$$

$\hat{\Phi}_k$ is invariant under the transformation [3].[5]

$$\hat{\Phi}_k(\rho, \sigma, v) = \hat{\Phi}_k(\rho, \sigma + \rho - 2v, v - \rho). \qquad (2.24)$$

Using this invariance we can re write the modular transformation (2.21) as

$$\tilde{\Phi}_k(\tilde{\rho}', \tilde{\sigma}', \tilde{v}') = -(i)^k C_1 (\tilde{\sigma}')^{-k} \hat{\Phi}_k \left(\tilde{\rho}' - \frac{\tilde{v}'^2}{\tilde{\sigma}'}, -\frac{1}{\tilde{\sigma}'}, \frac{\tilde{v}'}{\tilde{\sigma}} \right). \qquad (2.25)$$

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[5]This fact will also subsequently be evident from the final result of $\hat{\Phi}_k$ in terms of a 'threshold integral'.

To avoid cluttering, we will now refer to $(\tilde{\rho}', \tilde{\sigma}', \tilde{v}')$ as $(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$. The constant C_1 can be found by examining the construction of Siegel modular forms $\tilde{\Phi}_k$ and $\hat{\Phi}_k$ using 'threshold integrals'. For $\tilde{\Phi}_k$ we consider the integral

$$\tilde{\mathcal{I}}(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \sum_{r,s=0}^{N-1} \sum_{b=0}^{1} \tilde{\mathcal{I}}_{r,s,b} \tag{2.26}$$

$$\tilde{\mathcal{I}}_{r,s,b}(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{m_1,m_2,n_2 \in \mathbb{Z}, n_1 \in \mathbb{Z}+\frac{r}{N}, j \in 2\mathbb{Z}+b} \exp\left[2\pi i \tau \left(m_1 n_1 + m_2 n_2 + \frac{j^2}{4}\right)\right]$$

$$\times \exp\left(-\frac{\pi\tau_2}{\tilde{Y}}|n_2(\tilde{\rho}\tilde{\sigma} - \tilde{v}^2) + j\tilde{v} + n_1\tilde{\sigma} - m_1\tilde{\rho} + m_2|^2\right) e^{2\pi i m_1 s/N} h_b^{(r,s)}(\tau)$$

$$0 \le r,s \le (N-1) \tag{2.27}$$

where

$$\tilde{Y} = \det \mathrm{Im}(\tilde{\Omega}), \qquad \tilde{\Omega} = \begin{pmatrix} \tilde{\rho} & \tilde{v} \\ \tilde{v} & \tilde{\sigma} \end{pmatrix} \tag{2.28}$$

and $h_b^{(r,s)}$ are found by expanding the twisted elliptic genus as

$$F^{(r,s)}(\tau, z) = h_1^{r,s}(\tau)\theta_2(2\tau, 2z) + h_0^{r,s}\theta_3(\tau)(2\tau, 2z). \tag{2.29}$$

We can use the method of orbits to evaluate this integral as done in [19] and we obtain

$$\tilde{\mathcal{I}}(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = -2\ln[\det\mathrm{Im}\tilde{\Omega})^k] - 2\ln\tilde{\Phi}_k(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) - 2\ln\bar{\tilde{\Phi}}_k(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) - 2k\ln\kappa,$$

$$\kappa = \frac{8\pi}{3\sqrt{3}}e^{1-\gamma_E}, \tag{2.30}$$

where γ_E is the Euler-Mascheroni constant.

Now lets go over to the modular form $\hat{\Phi}_k$. From the $\mathrm{Sp}(2,\mathbb{Z})$ transformation given in (2.25) we see that this is equivalent to

$$\tilde{\Phi}\left(\rho - \frac{v^2}{\sigma}, -\frac{1}{\sigma}, -\frac{v}{\sigma}\right) = -(-i)^k C_1(-\sigma)^k \hat{\Phi}_k(\rho, \sigma, v). \tag{2.31}$$

From this transformation and from the threshold integral in (2.26) we see that we can obtain $\hat{\Phi}_k$ by the following replacements

$$m_2 \to n_1, \quad , n_1 \to -m_2, \quad m_1 \to -n_2, \quad n_2 \to m_1. \tag{2.32}$$

Therefore to construct $\hat{\Phi}$ we can consider the integral

$$\hat{\mathcal{I}}(\rho, \sigma, v) = \sum_{r,s=0}^{N-1} \hat{\mathcal{I}}(\rho, \sigma, v), \tag{2.33}$$

$$\hat{\mathcal{I}}(\rho, \sigma, v) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{m_1,n_1,n_2 \in \mathbb{Z}, m_1 \in \mathbb{Z}-\frac{r}{N}, j \in 2\mathbb{Z}+b} \exp\left[2\pi i \tau \left(m_1 n_1 + m_2 n_2 + \frac{j^2}{4}\right)\right]$$

$$\times \exp\left(-\frac{\pi\tau_2}{Y}|n_2(\rho\sigma - v^2) + j\,v + n_1\sigma - m_1\rho + m_2|^2\right) e^{-2\pi i n_2 s/N} h_b^{(r,s)}(\tau)$$

$$0 \le r,s \le (N-1). $$

Examining the integrals and using the relation between the coordinates we can see that

$$\hat{\mathcal{I}}(\rho,\sigma,v) = \tilde{\mathcal{I}}(\tilde{\rho},\tilde{\sigma},\tilde{v}). \tag{2.34}$$

In fact using a similar analysis one can also verify the invariance of $\hat{\Phi}$ under the transformation given in (2.24).

Evaluating the integral through the method of orbits we obtain

$$\hat{\mathcal{I}}(\rho,\sigma,v) = -2\ln[\det\text{Im}\Omega)^k] - 2\ln\hat{\Phi}_k(\rho,\sigma,\tilde{v}) - 2\ln\bar{\hat{\Phi}}_k(\rho,\sigma,v) - 2k\ln\kappa \tag{2.35}$$
$$-2(k+2)\ln(N),$$

where

$$\hat{\Phi}(\rho,\sigma,v) = -e^{2\pi i(\rho+\sigma+v)} \prod_{r,s=0}^{N-1} \prod_{\substack{k',l,b\in\mathbb{Z} \\ k',l,j>0}} \left(1 - e^{2\pi i r/N}\exp(2\pi i(k'\sigma+l\rho+jv))\right)^{\frac{1}{2}c^{r,s}(4k'l-j^2)}$$
$$\prod_{r,s=0}^{N-1} \prod_{\substack{k',l,b\in\mathbb{Z} \\ k',l,j>0}} \left(1 - e^{-2\pi i r/N}\exp(2\pi i(k'\sigma+l\rho+jv))\right)^{\frac{1}{2}c^{r,s}(4k'l-j^2)}.$$
$$\tag{2.36}$$

The details of the evaluation of the threshold integral again follow the methods of [19]. However since we are interested in keeping track of the constant in the last line of (2.35) we give some of the details of this in the appendix A.

Using the equality (2.34) and substituting the modular transformation (2.31) in equations (2.30) and (2.35) we find that

$$C_1 = N^{\frac{k+2}{2}} \tag{2.37}$$

It is clear that this approach does not fix the phase we have chosen in the modular transformation (2.15). To fix this phase and also perform a cross check on C_1, let us examine the relation given (2.15) in the $\tilde{v} \to 0$ limit. From the product representation of $\tilde{\Phi}_k$ given in (2.3) we can show that $\tilde{\Phi}_k$ factorizes as

$$\lim_{\tilde{v}\to 0} \tilde{\Phi}_k(\tilde{\rho},\tilde{\sigma},\tilde{v}) = -4\pi^2 g(\tilde{\rho})h(\tilde{\sigma}) \tag{2.38}$$

where $g(\tilde{\rho})$ and $h(\tilde{\sigma})$ are modular forms of weight $k+2$ transforming under subgroups of $\Gamma_0(N)$. The list of these forms for each the $\tilde{\Phi}_k$ corresponding to the orbifolds considered in this paper is given in table 4.

Taking the limit $\tilde{v} \to 0$ in (2.15), we obtain

$$4\pi^2 g(\tilde{\rho})\, h(\tilde{\sigma}) = -(i)^k\, C_1 \tilde{\sigma}^{-k} \left(\frac{\tilde{v}}{\tilde{\sigma}}\right)^2 g(\tilde{\rho})\, g\left(-\frac{1}{\tilde{\sigma}}\right) \tag{2.39}$$

From the list of the weight $k+2$ forms g given in table 3 we see that all g obey the property

$$g\left(-\frac{1}{\tilde{\sigma}}\right) = -(-i)^k\, N^{-\left(\frac{k+2}{2}\right)} h(\tilde{\sigma}) \tag{2.40}$$

Now from substituting (2.40) into (2.39) we confirm that C_1 is given by (2.37).

Conjugacy Class	k	$g(\rho)$	$h(\sigma)$
pA	$\frac{24}{p+1} - 2$	$\eta^{k+2}(\rho)\eta^{k+2}(p\rho)$	$\eta^{k+2}(\sigma)\eta^{k+2}(\sigma/p)$
4B	3	$\eta^4(4\rho)\eta^2(2\rho)\eta^4(\rho)$	$\eta^4(\frac{\sigma}{4})\eta^2(\frac{\sigma}{2})\eta^4(\sigma)$
6A	2	$\eta^2(\rho)\eta^2(2\rho)\eta^2(3\rho)\eta^2(6\rho)$	$\eta^2(\sigma)\eta^2(\frac{\sigma}{2})\eta^2(\frac{\sigma}{3})\eta^2(\frac{\sigma}{6})$
8A	1	$\eta^2(\rho)\eta(2\rho)\eta(4\rho)\eta^2(8\rho)$	$\eta^2(\sigma)\eta(\frac{\sigma}{2})\eta(\frac{\sigma}{4})\eta^2(\frac{\sigma}{8})$
14A	0	$\eta(\rho)\eta(2\rho)\eta(7\rho)\eta(14\rho)$	$\eta(\sigma)\eta(\frac{\sigma}{2})\eta(\frac{\sigma}{7})\eta(\frac{\sigma}{14})$
15A	0	$\eta(\rho)\eta(3\rho)\eta(5\rho)\eta(15\rho)$	$\eta(\sigma)\eta(\frac{\sigma}{3})\eta(\frac{\sigma}{5})\eta(\frac{\sigma}{15})$

Table 4. Factorization of $\tilde{\Phi}_k(\rho, \sigma, v)$ as $\lim v \to 0$ as shown in, $p \in \{1, 2, 3, 5, 7, 11\}$.

2.2 Comparison with statistical entropy at one loop

We compare the logarithm of the degeneracy obtained from the Fourier expansion given in (2.1)

$$S_{\text{stat}} = \ln d(Q, P) \tag{2.41}$$

and the statistical entropy at one loop which is given by

$$S_{\text{stat}}^{(1)} = \frac{\pi}{2\tau_2}|Q - \tau P|^2 - \ln g(\tau) - \ln g(-\bar{\tau}) - (k+2)\ln(2\tau_2) - \ln(NC_1), \tag{2.42}$$

$$\tau_1 = \frac{Q \cdot P}{P^2}, \qquad \tau_2 = \frac{1}{P^2}\sqrt{Q^2 P^2 - (Q \cdot P)^2}, \qquad C_1 = N^{\frac{k+2}{2}}.$$

We perform this comparison for all the orbifolds g' listed in table 1 for low value of charges. It is easy to evaluate the logarithm of the degeneracy by performing the Fourier expansion in Mathematica for low value of charges. Then the comparison with the statistical entropy is done both with and without constant $-\ln(NC_1)$. We see that for low values of charges, the constant is crucial in bringing the agreement of the statistical entropy to within a few percent of the actual degeneracy.

In the tables below δ and δ' are defined as follows

$$\delta = \frac{S_{\text{stat}} - S_{\text{stat}}^{(1)}}{S_{\text{stat}}} \times 100, \tag{2.43}$$

$$\delta' = \frac{\{S_{\text{stat}}^{(1)} + \ln(NC_1)\} - S_{\text{stat}}}{S_{\text{stat}}} \times 100$$

Thus δ' measures the percentage difference from S_{stat} without the constant $-\ln(NC_1)$ in the statistical entropy at one loop. We list the comparisons for orbifolds of the order N where N is prime first and then move to the non-prime cases.

2.3 Exponentially suppressed corrections: the 2A orbifold

In section 2.2 we have compared the contribution of the leading saddle point to the exact degeneracy We have shown that the constant C_1 present in the modular transformation (2.15) contributes substantially to the statistical entropy at low values of charges for the orbifold theories. In this section we show that this constant is also important in the sub-leading

$(Q^2, P^2 Q.P)$	$d(Q,P)$	S_{stat}	$S_{\text{stat}}^{(1)}$	δ	δ'
$(1, 2, 0)$	2164	7.67971	7.28409	5.15	50.28
$(1, 2, 1)$	360	5.8861	5.34077	9.26	68.14
$(1, 4, 1)$	4352	8.37839	8.39542	-0.2	41.16
$(2, 4, 0)$	198144	12.1967	11.727	3.85	32.27
$(1, 6, 1)$	36024	10.4919	11.1568	-6.33	26.7
$(3, 6, 0)$	15219528	16.5381	16.1699	2.22	23.18
$(3, 6, 3)$	149226	11.9132	11.624	2.43	31.52
$(3, 6, 4)$	2164	7.67971	7.28409	5.15	50.28

Table 5. Comparison of the statistical entropy and statistical entropy at one loop for 2A orbifold.

$(Q^2, P^2 Q.P)$	$d(Q,P)$	S_{stat}	$S_{\text{stat}}^{(1)}$	δ	δ'
$(2/3, 2, 0)$	540	6.29157	5.95751	5.31	75.16
$(2/3, 4, 0)$	3294	8.09986	8.12528	-0.31	53.94
$(2/3, 4, 1)$	378	5.93489	6.0378	-1.73	72.31
$(2/3, 6, 0)$	16200	9.69277	10.3187	-6.46	38.88
$(2/3, 6, 1)$	2646	7.8808	8.58224	-8.90	46.86
$(4/3, 6, 0)$	128706	11.7653	11.4413	2.75	40.10
$(4/3, 6, 1)$	37422	10.53	10.546	-0.15	41.58
$(2, 6, 0)$	820404	13.6176	13.2127	2.97	35.24
$(2, 6, 1)$	318267	12.6706	12.546	0.98	35.66
$(2, 6, 2)$	37818	10.5405	10.4379	0.97	42.66

Table 6. Comparison of the statistical entropy and statistical entropy at one loop for 3A orbifold.

$(Q^2, P^2 Q.P)$	$d(Q,P)$	S_{stat}	$S_{\text{stat}}^{(1)}$	δ	δ'
$(2/5, 2, 0)$	100	4.60517	4.57546	0.64	105
$(2/5, 4, 0)$	460	6.13123	6.30791	-2.88	75.86
$(2/5, 4, 1)$	20	2.99573	2.87779	3.94	165
$(2/5, 6, 0)$	1720	7.45008	8.08384	-8.51	56.30
$(2/5, 6, 1)$	125	4.82831	5.46281	-13.14	86.85
$(4/5, 6, 0)$	9180	9.12478	8.84163	3.10	56.02
$(4/5, 6, 1)$	1460	7.28619	7.49625	-2.88	63.38
$(6/5, 6, 0)$	39960	10.5956	10.1953	3.78	49.35
$(6/5, 6, 1)$	9345	9.1426	9.21235	-0.76	52.05
$(6/5, 6, 2)$	390	5.96615	5.88441	1.37	82.3

Table 7. Comparison of the statistical entropy and statistical entropy at one loop for 5A orbifold.

$(Q^2,\ P^2\, Q.P)$	$d(Q,P)$	S_{stat}	$S_{\text{stat}}^{(1)}$	δ	δ'
$(2/7,\ 2,\ 0)$	36	3.58352	3.74248	-4.43	131
$(2/7,\ 4,\ 0)$	138	4.92725	5.23041	-6.15	92.58
$(2/7,\ 6,\ 0)$	444	6.09582	6.76722	-11.01	68.79
$(2/7,\ 6,\ 1)$	18	2.89037	3.22552	-11.59	156.7
$(4/7,\ 6,\ 0)$	1916	7.55799	7.35616	2.67	67.04
$(4/7,\ 6,\ 1)$	210	5.34711	5.60863	-4.89	86.09
$(6/7,\ 6,\ 0)$	6892	8.83812	8.49212	3.91	58.96
$(6/7,\ 6,\ 1)$	1152	7.04925	7.23465	-2.63	66.38
$(6/7,\ 6,\ 2)$	18	2.89037	2.58629	10.52	178.83

Table 8. Comparison of the statistical entropy and statistical entropy at one loop for 7A orbifold.

$(Q^2,\ P^2\, Q.P)$	$d(Q,P)$	S_{stat}	$S_{\text{stat}}^{(1)}$	δ	δ'
$(6/11,\ 10,\ 0)$	4962	8.50956	8.32923	2.12	54.2
$(6/11,\ 10,\ 1)$	937	6.84268	7.13426	-4.26	74.35
$(6/11,\ 12,\ 0)$	11132	9.31758	9.22308	1.01	50.4
$(6/11,\ 12,\ 1)$	2558	7.72223	8.16335	-5.7	67.8
$(6/11,\ 12,\ 2)$	72	4.27667	4.36939	-2.16	114
$(6/11,\ 22,\ 0)$	366378	12.8114	13.1652	-2.76	40
$(6/11,\ 22,\ 1)$	139955	11.8491	12.4283	4.88	45.6
$(6/11,\ 22,\ 2)$	12760	9.45407	10.0209	5.99	56.7
$(6/11,\ 22,\ 3)$	114	4.7362	4.86058	2.6	103

Table 9. Comparison of the statistical entropy and statistical entropy at one loop for 11A orbifold.

$(Q^2,\ P^2\, Q.P)$	$d(Q,P)$	S_{stat}	$S_{\text{stat}}^{(1)}$	δ	δ'
$(4/23,\ 8,\ 0)$	91	4.51086	3.63606	19.3	84.87
$(6/23,\ 6,\ 0)$	103	4.63473	3.5562	23.27	78.2
$(6/23,\ 8,\ 0)$	190	5.24702	4.21628	19.6	70
$(6/23,\ 10,\ 0)$	312	5.743	4.86463	15.2	66.6
$(6/23,\ 10,\ 1)$	74	4.30407	2.90311	32.5	76.7

Table 10. Comparison of the statistical entropy and statistical entropy at one loop for 23A orbifold.

$(Q^2, P^2 Q.P)$	$d(Q,P)$	S_{stat}	$S^{(1)}$	δ	δ'
$(1/2, 2, 0)$	176	5.17048	4.9493	4.28	98.12
$(1/2, 4, 0)$	896	6.79794	6.84008	-0.62	70.75
$(1/2, 4, 1)$	80	4.38203	4.25615	2.87	113
$(1/2, 6, 0)$	3616	8.19312	8.7606	-6.92	52.29
$(1/2, 6, 1)$	480	6.17379	6.65185	-7.74	70.85
$(1, 4, 0)$	5024	8.52198	8.09089	5.06	61.99
$(1, 4, 1)$	832	6.72383	6.68116	0.63	72.79
$(3/2, 8, 0)$	491920	13.1061	12.7909	2.40	39.43
$(3/2, 8, 1)$	196960	12.1908	12.1281	0.51	40.31
$(3/2, 8, 2)$	23616	10.0697	10.0251	0.44	48.64

Table 11. Comparison of the statistical entropy and statistical entropy at one loop for 4B orbifold.

$(Q^2, P^2 Q.P)$	$d(Q,P)$	S_{stat}	$S_{\text{stat}}^{(1)}$	δ	δ'
$(1/3, 2, 0)$	40	3.68888	3.50247	5.05	150
$(1/3, 4, 0)$	148	4.99721	5.04572	-0.97	106
$(1/3, 6, 0)$	478	6.16961	6.61552	-7.23	79.9
$(2/3, 6, 0)$	2128	7.66294	7.38342	3.65	73.79
$(2/3, 6, 1)$	436	6.07764	6.03979	0.62	89.07
$(1, 12, 0)$	240612	12.3909	12.3009	0.726	44.11
$(1, 12, 1)$	106096	11.5721	11.6461	-0.639	45.81
$(1, 12, 2)$	13856	9.53647	9.55433	-0.187	56.18

Table 12. Comparison of the statistical entropy and statistical entropy at one loop for 6A orbifold.

$(Q^2, P^2 Q.P)$	$d(Q,P)$	S_{stat}	$S_{\text{stat}}^{(1)}$	δ	δ'
$(1/4, 2, 0)$	20	2.99573	3.0419	-1.54	172
$(1/4, 4, 0)$	68	4.21951	4.41444	-4.62	118
$(1/4, 6, 0)$	196	5.27811	5.82516	-10.36	88.13
$(1/4, 6, 1)$	10	2.30259	2.16053	6.17	231
$(1/4, 8, 0)$	504	6.22258	7.13849	-14.71	68.82
$(1/4, 8, 1)$	40	3.68888	4.06787	-10.27	130
$(3/4, 6, 0)$	2280	7.73193	7.48478	3.196	70.43
$(3/4, 6, 1)$	450	6.10925	6.19909	-1.47	83.62
$(3/4, 8, 0)$	6704	8.81046	8.59292	2.47	61.47
$(3/4, 8, 1)$	1728	7.45472	7.54122	-1.16	68.57

Table 13. Comparison of the statistical entropy and statistical entropy at one loop for 8A orbifold.

$(Q^2,\ P^2\,Q.P)$	$d(Q,P)$	S_{stat}	$S_{\text{stat}}^{(1)}$	δ	δ'
$(1/7,\ 2,\ 0)$	4	1.38629	1.5854	-14.3	395
$(1/7,\ 4,\ 0)$	10	2.30259	2.63796	-14.56	243.8
$(1/7,\ 6,\ 0)$	24	3.17805	3.72617	-17.2	183
$(2/7,\ 6,\ 0)$	70	4.2485	4.14076	2.5	121
$(2/7,\ 8,\ 0)$	156	5.04986	5.01278	0.73	103.7
$(3/7,\ 8,\ 0)$	406	6.00635	5.7842	3.7	84
$(5/7,\ 12,\ 0)$	11512	9.35115	9.12825	2.38	54
$(5/7,\ 12,\ 1)$	4156	8.33231	8.34407	0.14	63.4
$(5/7,\ 12,\ 2)$	292	5.67675	5.6847	-0.14	93

Table 14. Comparison of the statistical entropy and statistical entropy at one loop for 14A/B orbifold.

$(Q^2,\ P^2\,Q.P)$	$d(Q,P)$	S^{stat}	$S_{\text{stat}}^{(1)}$	δ	δ'
$(2/15,\ 2,\ 0)$	4	1.38629	1.32743	4.24	386
$(2/15,\ 4,\ 0)$	8	2.07944	2.33276	-12.18	272
$(2/15,\ 6,\ 0)$	20	2.99573	3.36918	-12.4	193
$(4/15,\ 6,\ 0)$	50	3.91202	3.79203	3	135
$(4/15,\ 8,\ 0)$	102	4.62497	4.62705	-0.044	117
$(8/15,\ 12,\ 0)$	2844	7.95297	7.76586	2.35	65.7
$(8/15,\ 12,\ 1)$	898	6.80017	6.83215	-0.47	80
$(8/15,\ 12,\ 2)$	40	3.68888	3.54063	4	142.8

Table 15. Comparison of the statistical entropy and statistical entropy at one loop for 15A/B orbifold.

saddles which are exponentially suppressed compared to the leading saddle. For this purpose we will study 2A orbifold, a similar analysis also applies for all the other orbifolds g' listed in table 1.

In [10] the first sub-leading saddle was analysed for the un-orbifolded theory of type II $K3 \times T^2$. We generalize this analysis to the case of an orbifold of order N. Starting from the degeneracy formula (2.1) and using the symmetry (2.13) and a change of variables we obtain the following

$$d(Q,P) = \frac{1}{N}(-1)^{Q\cdot P+1} \int d\tilde{\rho}\tilde{\sigma}d\tilde{v}\, e^{-\pi i(\tilde{\rho}P^2+\tilde{\sigma}Q^2+2\tilde{v}Q.P)}\frac{1}{\tilde{\Phi}_k(\tilde{\rho},\tilde{\sigma},\tilde{v})} \qquad (2.44)$$

The contour is same as that given in (2.2). Note, however the change in the arguments of the exponent. The saddle points of the integral occur at zeros of $\tilde{\Phi}_k$. These occur at the

following hyper surfaces

$$n_2(\tilde{\sigma}\tilde{\rho} - \tilde{v}^2) + b\tilde{v} + n_1\tilde{\sigma} - \tilde{\rho}m_1 + m_2 = 0, \tag{2.45}$$

$$m_1 \in N\mathbb{Z}, \qquad n_1 \in \mathbb{Z}, \qquad b \in 2\mathbb{Z}+1, \qquad m_2, n_2 \in \mathbb{Z},$$

$$m_1 n_1 + m_2 n_2 + \frac{b^2}{4} = \frac{1}{4}.$$

From the product representation of the modular form $\tilde{\Phi}_k$ it can be seen that it is invariant under the transformations $\tilde{\rho} \to \tilde{\rho}+1, \tilde{\sigma}+N, \tilde{v} \to \tilde{v}+1$. Applying these transformations, we can bring the locations of the points which characterize the hyper surface in (2.45) to

$$n_1 = 0, 1 \cdots (n_2 - 1), \tag{2.46}$$

$$m_1 = 0, N, 2N, \cdots (n_2-1)N,$$

$$b = 1, 3, 5, \cdots 2n_2 - 1.$$

Then m_2 is obtained by solving the last equation in (2.45).

Let us now specialize to the case of $N = 2$ and the second saddle $n_2 = 2$. Then using the conditions in (2.46), the points that characterize the hyper surface on which the zeros of $\tilde{\Phi}_k$ lie are given by

$$(m_1, n_1, m_2, n_2, j)_i = \{(0,0,0,2,1),\ (2,0,0,2,1), \tag{2.47}$$

$$(0,1,0,2,1),\ (2,1,-1,2,1),$$

$$(0,0,-1,2,3),\ (2,0,-1,2,3),$$

$$(0,1,-1,2,3),\ (2,1,-2,2,3).\}$$

Here the subscript $i = 1, \cdots 8$ labels the solution in the order written in the above equation. For illustration, the location of the first zero of the second saddle $n_2 = 2$ is given by the equation

$$2(\tilde{\rho}\tilde{\sigma} - \tilde{v}^2) + \tilde{v} = 0 \tag{2.48}$$

We now need to obtain the $\mathrm{Sp}(2,\mathbb{Z})$ transformation that allow us to determine how $\tilde{\Phi}_k$ behaves close to these zeros. Let write the $\mathrm{Sp}(2,\mathbb{Z})$ transformation more formally. We define the symplectic matrix

$$U_0^{-1} = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.49}$$

and construct

$$\hat{\Omega}_1 = (A_0\tilde{\Omega} + \tilde{B}_0)(C_0\tilde{\Omega} + D_0)^{-1}. \tag{2.50}$$

From examining this transformation we can see that (2.21) is written as

$$-(i)^k C_1 \hat{\Phi}_k(\hat{\Omega}_1) = \{\det(C_0\tilde{\Omega} + D_0)\}^k \tilde{\Phi}_k(\tilde{\Omega}). \tag{2.51}$$

Note that the expression for $\hat{v}'$ in terms of $\tilde{v}$ does not coincide with any of the zeros we have in the list (2.47). We need the transformation such that $\hat{\Phi}_k$ is evaluated at these zeros. For this, we perform a further $\text{Sp}(2, \mathbb{Z})$ transformation that keeps $\hat{\Phi}_k$ invariant. Then the transformation is restricted to the following sub-group G of $\text{Sp}(2, \mathbb{Z})$ defined by [10]

$$U_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad \rightarrow C_1 = \mathbf{O} \bmod N, \quad \det A_1 = 1 \bmod N, \quad \det D_1 = 1 \bmod N. \quad (2.52)$$

Let

$$\hat{\Omega} = (A_1 \hat{\Omega}_1 + \hat{B}_1)(C_1 \hat{\Omega}_1 + D_1)^{-1}, \quad (2.53)$$

then we define

$$\Phi_k(\hat{\Omega}) = \{\det(C_1 \hat{\Omega}_1 + D_1)\}^k \Phi_k(\hat{\Omega}_1). \quad (2.54)$$

Now combining (2.51) and (2.54) using the group property we obtain

$$-(i)^k C_1 \Phi_k(\hat{\Omega}) = \{\det(C\tilde{\Omega} + D)\}^k \tilde{\Phi}_k(\tilde{\Omega}) \quad (2.55)$$

where

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = U_1 U_0^{-1}. \quad (2.56)$$

We can use the additional degree of freedom of performing a transformation within the sub-group G defined in (2.52) so that $\hat{v}$ as a function of $\tilde{v}$ coincides with the locations of the zeros given in (2.47).

We construct U as follows:

1. First take U_1 to be any 4×4 matrix and evaluate $U_1 U_0^{-1}$.

2. Evaluate the action of $U = U_1 U_0^{-1}$ on $\tilde{\Omega}$, we get

$$\hat{\Omega} = (A\tilde{\Omega} + B)(C\tilde{\Omega} + D)^{-1}. \quad (2.57)$$

   Demand that the equation for $\hat{v} = 0$ in terms of $\tilde{v}$ coincides with the zeros given in (2.47).

3. Impose the conditions that result from $\text{Sp}(2, \mathbb{Z})$ on U_1.

4. Examine the resulting equations and finally impose the conditions of the sub-group G given in (2.52) on U_1.

These steps can be implemented in Mathematica and the final expression for $\hat{\Omega}$ written in terms of $\tilde{\Omega}$ for each the locations of the 8 zeros in (2.47) are given by

$$\hat{\Omega}_1 = \begin{pmatrix} \tilde{\rho} - \dfrac{\tilde{v}^2}{\tilde{\sigma}} & \dfrac{2\tilde{\rho}\tilde{\sigma} - 2\tilde{v}^2 + \tilde{v}}{\tilde{\sigma}} \\ \dfrac{2\tilde{\rho}\tilde{\sigma} - 2\tilde{v}^2 + \tilde{v}}{\tilde{\sigma}} & \dfrac{4\tilde{\rho}\tilde{\sigma} - 4(\tilde{v} - 1)\tilde{v} - 1}{\tilde{\sigma}} \end{pmatrix} \quad (2.58)$$

$$\hat{\Omega}_2 = \begin{pmatrix} \dfrac{\tilde{\rho}\tilde{\sigma} - \tilde{v}^2}{8\tilde{v}^2 - 4\tilde{v} + \tilde{\sigma} + \tilde{\rho}(4 - 8\tilde{\sigma})} & \dfrac{-2\tilde{v}^2 + \tilde{v} + 2\tilde{\rho}(\tilde{\sigma} - 1)}{8\tilde{v}^2 - 4\tilde{v} + \tilde{\sigma} + \tilde{\rho}(4 - 8\tilde{\sigma})} \\ \dfrac{-2\tilde{v}^2 + \tilde{v} + 2\tilde{\rho}(\tilde{\sigma} - 1)}{8\tilde{v}^2 - 4\tilde{v} + \tilde{\sigma} + \tilde{\rho}(4 - 8\tilde{\sigma})} & \dfrac{-4(\tilde{v} - 1)v + 4\tilde{\rho}\tilde{\sigma} - 1}{8\tilde{v}^2 - 4\tilde{v} + \tilde{\sigma} + \tilde{\rho}(4 - 8\tilde{\sigma})} \end{pmatrix}$$

$$\hat{\Omega}_3 = \begin{pmatrix} \tilde{\rho} - \dfrac{\tilde{v}^2}{\tilde{\sigma}} & \dfrac{-2\tilde{v}^2 + \tilde{v} + 2\tilde{\rho}\tilde{\sigma} + \tilde{\sigma}}{\tilde{\sigma}} \\[2mm] \dfrac{-2\tilde{v}^2 + \tilde{v} + 2\tilde{\rho}\tilde{\sigma} + \tilde{\sigma}}{\tilde{\sigma}} & \dfrac{-4(\tilde{v}-1)\tilde{v} + 4\tilde{\rho}\tilde{\sigma} + \tilde{\sigma} - 1}{\tilde{\sigma}} \end{pmatrix}$$

$$\hat{\Omega}_4 = \begin{pmatrix} \dfrac{\tilde{v}(\tilde{v}+2) - \tilde{\rho}(\tilde{\sigma}-4) + 1}{2\tilde{v}^2 - (2\tilde{\rho}+1)(\tilde{\sigma}-2)} & \dfrac{\tilde{v} + 2\tilde{\rho} + 1}{(2\tilde{\rho}+1)(\tilde{\sigma}-2) - 2\tilde{v}^2} + 1 \\[2mm] \dfrac{\tilde{v} + 2\tilde{\rho} + 1}{(2\tilde{\rho}+1)(\tilde{\sigma}-2) - 2\tilde{v}^2} + 1 & \dfrac{2\tilde{\rho}+1}{2\tilde{v}^2 - (2\tilde{\rho}+1)(\tilde{\sigma}-2)} - 1 \end{pmatrix}$$

$$\hat{\Omega}_5 = \begin{pmatrix} \dfrac{-4(\tilde{v}-1)\tilde{v} + 4\tilde{\rho}\tilde{\sigma} - 1}{\tilde{\sigma}} & \dfrac{\tilde{v}(3 - 2\tilde{v}) + 2\tilde{\rho}\tilde{\sigma} - 1}{\tilde{\sigma}} \\[2mm] \dfrac{\tilde{v}(3 - 2\tilde{v}) + 2\tilde{\rho}\tilde{\sigma} - 1}{\tilde{\sigma}} & \tilde{\rho} - \dfrac{(\tilde{v}-1)^2}{\tilde{\sigma}} \end{pmatrix}$$

$$\hat{\Omega}_6 = \begin{pmatrix} \dfrac{-4(\tilde{v}-1)\tilde{v} + \tilde{\rho}(4\tilde{\sigma}-2) - 1}{6\tilde{v}^2 - 8\tilde{v} + \tilde{\sigma} + \tilde{\rho}(4 - 6\tilde{\sigma}) + 2} & \dfrac{\tilde{v}(3 - 2\tilde{v}) + 2\tilde{\rho}(\tilde{\sigma}-1) - 1}{6\tilde{v}^2 - 8\tilde{v} + \tilde{\sigma} + \tilde{\rho}(4 - 6\tilde{\sigma}) + 2} \\[2mm] \dfrac{\tilde{v}(3 - 2\tilde{v}) + 2\tilde{\rho}(\tilde{\sigma}-1) - 1}{6\tilde{v}^2 - 8\tilde{v} + \tilde{\sigma} + \tilde{\rho}(4 - 6\tilde{\sigma}) + 2} & \dfrac{\tilde{\rho}\tilde{\sigma} - (\tilde{v}-1)^2}{6\tilde{v}^2 - 8\tilde{v} + \tilde{\sigma} + \tilde{\rho}(4 - 6\tilde{\sigma}) + 2} \end{pmatrix}$$

$$\hat{\Omega}_7 = \begin{pmatrix} \dfrac{-4(\tilde{v}-1)\tilde{v} + 4\tilde{\rho}\tilde{\sigma} + \tilde{\sigma} - 1}{\tilde{\sigma}} & \dfrac{\tilde{v}(3 - 2\tilde{v}) + 2\tilde{\rho}\tilde{\sigma} + \tilde{\sigma} - 1}{\tilde{\sigma}} \\[2mm] \dfrac{\tilde{v}(3 - 2\tilde{v}) + 2\tilde{\rho}\tilde{\sigma} + \tilde{\sigma} - 1}{\tilde{\sigma}} & -\dfrac{(\tilde{v}-1)^2}{\tilde{\sigma}} + \tilde{\rho} + 1 \end{pmatrix}$$

$$\hat{\Omega}_8 = \begin{pmatrix} \dfrac{2\tilde{\rho}+1}{2(\tilde{v}-2)\tilde{v} - 2\tilde{\rho}(\tilde{\sigma}-2) - \tilde{\sigma} + 4} & \dfrac{\tilde{v} - 2(\tilde{\rho}+1)}{2(\tilde{v}-2)\tilde{v} - 2\tilde{\rho}(\tilde{\sigma}-2) - \tilde{\sigma} + 4} + 1 \\[2mm] \dfrac{\tilde{v} - 2(\tilde{\rho}+1)}{2(\tilde{v}-2)\tilde{v} - 2\rho(\tilde{\sigma}-2) - \tilde{\sigma} + 4} + 1 & \dfrac{-\tilde{v}^2 + \tilde{\rho}\tilde{\sigma} + \tilde{\sigma}}{2(\tilde{v}-2)\tilde{v} - 2\tilde{\rho}(\tilde{\sigma}-2) - \tilde{\sigma} + 4} \end{pmatrix}.$$

We can now find the behaviour of the $\tilde{\Phi}_k$ at these zeros. Using the transformation in (2.55) and the factorization property of $\hat{\Phi}_k$ in (2.16), we find that

$$\tilde{\Phi}_k(\tilde{\rho}, \tilde{\sigma}, \tilde{v})|_{\text{zeros}} \rightarrow (i)^k [\det(C\tilde{\Omega} + D)]^{-k} 4\pi^2 \hat{v} g(\hat{\rho}) g(\hat{\sigma}). \tag{2.59}$$

Since we are restricting our attention to the 2A case, $k = 6$ and $g(\hat{\rho}) = \eta^8(\rho)\eta^8(2\rho)$. In the above equation for each of the zeros we have to substitute the corresponding value of $\hat{\rho}, \hat{\sigma}, \hat{v}$ in terms of $\tilde{\rho}, \tilde{\sigma}, \tilde{v}$ and the value of $C\tilde{\Omega} + D$ can also be read from (2.58). The next step is to perform the contour integral over $\tilde{v}$ in (2.44), which will pick up the residue at the double pole at $\hat{v} = 0$. Then perform the saddle point integration over $\hat{\rho}$ and $\hat{v}$. To the leading order the saddle point is obtained by minimizing the exponent in (2.44) given by

$$E = -\pi i (\tilde{\rho} P^2 + \tilde{\sigma} Q^2 + 2\tilde{v} Q \cdot P) \tag{2.60}$$

subject to the constraint of the location of the zero given in (2.45). The result for the saddle point is given by

$$(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = i \left[2n_2 \sqrt{Q^2 P^2 - (Q \cdot P)^2} \right]^{-1} (Q^2, P^2, Q \cdot P) - \frac{1}{n_2} \left(n_1, -m_1, \frac{j}{2} \right). \tag{2.61}$$

Repeating the analysis of [10] for the and keeping track of the differences for the case of 2A we see that the value at the sub-leading saddle including the one loop corrections is

Q^2	1	2	3	3	3	6	6
P^2	2	4	6	6	6	6	6
$Q \cdot P$	0	0	0	1	3	0	1
$d^{(2)}$	19.751	91.058	1679.22	895.668	0	83807.5	63637.7

Table 16. Contribution of the second saddle to the degeneracy: $\Delta d(Q, P)$ for 2A orbifold.

given by

$$
\begin{aligned}
\Delta d(Q, P)|_i = {} & \frac{1}{NC_1} \frac{1}{n_2} \exp(\pi \sqrt{Q^2 P^2 - (Q \cdot P)^2}) \\
& \times \left\{ [\det(C\tilde{\Omega} + D)]^{k+2} (g(\hat{\rho}) g(\hat{\sigma}))^{-1} (1 + O(Q^{-2}, P^{-2})) \right\} \\
& \times (-1)^{Q \cdot P} \exp \left[i \frac{\pi}{n_2} (n_1 P^2 - m_1 Q^2 - j Q \cdot P) \right] \bigg|_{\text{saddle}_i}.
\end{aligned}
$$

(2.62)

Here $n_2 = 2, k = 6$, and $g(\hat{\rho}) = \eta^8(\hat{\rho}) \eta^8(2\hat{\rho})$ and $N = 2$. The constant C_1 is given by $C_1 = 2^4$; $\hat{\rho}, \hat{\sigma}$ as well as $\det(C\tilde{\Omega} + D)$ are evaluated by examining the matrices given in (2.58) for each of the 8 saddles. This dependence on the zeros of $\tilde{\Phi}_k$ is indicated by the subscript 'i'.[6] The complete contribution of all saddles at $n_2 = 2$ is given by

$$
\Delta d(Q, P) = \sum_{i=1}^{8} \Delta d(Q, P)|_i.
$$

(2.63)

We have listed the contributions to the 2nd saddle to the degeneracies for a few low lying charges of dyons in the 2A orbifold in table 16.

We mention two consistency checks on evaluating the correction to the degeneracies at the second saddle. The contribution to the degeneracy from the second saddle $\Delta d(Q, P)$ has to be real, however the contribution from each of $\Delta d(Q, P)|_i$ need not be real. One of the checks is that on summing the contribution of the 8 saddles we obtain a real number. The second check is that the solutions of the matrices given in (2.58) are not unique. For example for 5th solution it is also possible to choose the matrix consistent with all the requirements discussed earlier.

$$
\hat{\Omega}'_5 = \begin{pmatrix} \frac{(1 - 2\tilde{v})^2 - 4\tilde{\rho}\tilde{\sigma}}{8(\tilde{v} - 1)\tilde{v} - (8\tilde{\rho} + 1)\tilde{\sigma} + 2} & \frac{\tilde{v}(3 - 2\tilde{v}) + 2\tilde{\rho}\tilde{\sigma} - 1}{-8(\tilde{v} - 1)\tilde{v} + 8\tilde{\rho}\tilde{\sigma} + \tilde{\sigma} - 2} \\ \frac{\tilde{v}(3 - 2\tilde{v}) + 2\tilde{\rho}\tilde{\sigma} - 1}{-8(\tilde{v} - 1)\tilde{v} + 8\tilde{\rho}\tilde{\sigma} + \tilde{\sigma} - 2} & \frac{(\tilde{v} - 2)\tilde{v} - \tilde{\rho}(\tilde{\sigma} - 2) + 1}{8(\tilde{v} - 1)\tilde{v} - (8\tilde{\rho} + 1)\tilde{\sigma} + 2} \end{pmatrix}.
$$

(2.64)

However we have explicitly verified that the correction $\Delta d(Q, P)_5$ is independent of the choice of the matrix $\hat{\Omega}_5$ or $\hat{\Omega}'_5$ for the charges listed in table 16. Finally we mention that for the charge $Q^2 = 3, P^2 = 6, Q \cdot P = 3$ the contribution to the second saddle vanishes. Such a property was also observed for the un-orbifolded theory in [6].

---

[6]We have kept track of all signs for $k = 6$. Note that we obtain that sign of $j Q \cdot P$ in the second line opposite to that obtained in [10].

In conclusion the observation that the constant C_1 is an important factor of all the saddle points, not just the leading one at $n_1 = 1$. It also occurs in the modular transformation relating $\tilde{\Phi}_k$ to all the other saddle points. We have demonstrated this for the $2A$ orbifold and for the saddle at $n_2 = 2$, but it is clear from the derivation that it will persist for all the other sub-leading saddles as well the other orbifolds. Finally we would also like to mention that the contribution of all the sub-leading saddles for the case of the un-orbifolded theory was evaluated in [20]. This was done by a characterization of all the solutions of the zeros of Φ_{10} for all the sub-leading saddles by determining the $\mathrm{Sp}(2,\mathbb{Z})$ transformation which maps the zeros to the zero at $v = 0$. It will be interesting to repeat this analysis for the $\mathbb{Z}_N$ orbifolds considered in this paper. Performing this would lead to a Farey tale like expansion of the dyon partition function for these theories. This would be important to understand the sub-leading contributions from semi-classical geometry.

2.4   Implications of the constant C_1

We briefly discuss the implication of the presence of the non-trivial constant C_1 in the one loop saddle point approximation to the entropy $S_{\mathrm{stat}}^{(1)}$. In the large charge limit, the $1/4$ BPS dyon can be described by an extremal black hole with the same charge (Q, P). We can evaluate the Wald's generalization of the Bekenstein-Hawking entropy to this black hole. This is done by considering the four derivative correction to the effective action of the $\mathcal{N} = 4$ string theory given by the Gauss-Bonnet term[7]

$$\mathcal{L} = \phi(a, S) \left(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right), \qquad (2.65)$$

where a, S is the axion and dilaton respectively in the heterotic description of the theory. Using the analysis of [23, 24] we can compute the function ϕ. It is given by

$$\phi(a, S) = -\frac{1}{64\pi^2} \left[(k+2)\ln S + \ln g(a+iS) + \ln g(-a+iS) \right] + \text{constant} \qquad (2.66)$$

Here g is the same function that occurs in the statistical entropy function given in (2.19) and listed in table 3 for all the orbifolds. However the constant present in (2.66) cannot be fixed. This is because the a constant coefficient in the Gauss-Bonnet term is a total derivative and therefore does not affect the equations of motion. The evaluation of the Gauss-Bonnet term is done through a string amplitude calculation which can determine the effective action only up to on shell terms. A constant in Gauss-Bonnet cannot be determined using this procedure. Now going evaluating the Wald entropy of the extremal dyonic black hole, one obtains the same minimizing problem as encountered in (2.19). That is

$$S(a, S) = \frac{\pi}{2S} |Q + (a+iS)P|^2 - \ln g(a+iS) - \ln g(-a+iS) - (k+2)\ln(2S)$$
$$+ \text{constant}. \qquad (2.67)$$

The Wald entropy is given by

$$S_{\mathrm{Wald}} = S(a, S)|_{\mathrm{minimum}}. \qquad (2.68)$$

---

[7]See [6] for a review.

Therefore the undetermined constant in the Gauss-Bonnet term turns out to be the $O(Q^0, P^0)$ term in the Wald entropy. Thus it is cleat that a string amplitude calculation will not be able to fix this constant coefficient in the Gauss-Bonnet term.

Recently beginning with the works of [25, 26] the method of localization in $AdS_2 \times S^2$ has proposed to evaluate the entropy of these black holes exactly from a geometric description. All of the works so far address only the un-orbifolded theory of compactification on $K3 \times T^2$. From the analysis in this paper it is clear that the constant C_1 substantially contributes to the entropy at low charges for all the orbifolds. We see that it is a $O(Q^0, P^0)$ term. The method of localization is exact and is hoped that partition functions agree with the microscopic description at finite order in charges. Therefore an important test for the method of localization is to reproduce this constant C_1.

3 Fourier-Jacobi coefficients of $1/\tilde{\Phi}_k$

Consider the following Fourier expansion if the partition function of dyons of the un-orbifolded theory, ie. type II compactified on $K3 \times T^2$,[8]

$$\frac{1}{\Phi_{10}(q,p,y)} = \sum_{m=-1}^{\infty} \psi_m(\tau, z) p^m, \qquad q = e^{2\pi i \tau}, p = e^{2\pi i \sigma}, y = e^{2\pi i z}. \tag{3.1}$$

The Fourier-Jacobi coefficients ψ_m after multiplying by $\eta^{24}(\tau)$ are meromorphic Jacobi forms of weight 2 and index m. Note that the Fourier-Jacobi coefficients count degeneracies for a given magnetic charge of the dyon. In [11] it was shown that the Fourier-Jacobi coefficients ψ_m for $m > 0$ admits the following unique decomposition

$$\psi_m(\rho, z) = \psi_m^{\mathrm{P}}(\tau, z) + \psi_m^{\mathrm{F}}(\tau, z), \tag{3.2}$$

in which the polar part $\psi_m^P(\tau, z)$ has the same pole structure in the z-plane as $\psi_m(\tau, z)$ and $\psi_m^{\mathrm{F}}(\tau, z)$ has no poles. The polar part is given by the following Appell-Lerch sum

$$\psi_m^{\mathrm{P}}(\tau, z) = \frac{p_{24}(m+1)}{\eta^{24}(\tau)} \mathcal{A}_{2,m}(\tau, z), \tag{3.3}$$

$$\mathcal{A}_{2,m}(\tau, z) = \sum_{s \in \mathbb{Z}} \frac{q^{ms^2 + s} y^{2ms+1}}{(1 - q^s y)^2}.$$

The Appell-Lerch sum exhibits wall-crossing and therefore ψ_m^{F} capture the degeneracies of multi-centered black holes. The Fourier coefficients of $\psi_m^{\mathrm{F}}(\tau, z)$ are independent of the choice of contour in the (z, τ) space and counts the degeneracies of immortal black holes. Further more $\psi_m^{\mathrm{F}}(\tau, z)$ is a mock Jacobi form.

In this section we generalize these observations for the partition function of dyons in type II compactifications on $K3 \times T^2/Z_{\mathbb{N}}$, where the orbifold is performed by the action of g' given in table 1. This observation is made for the 2 lowest values of magnetic charges. In this case ψ_m after multiplying by an appropriate $\Gamma_0(N)$ form are meromorphic Jacobi forms under $\Gamma_0(N)$. We will begin with the case of $2A$ and then move to all the other orbifolds.

---

[8]We have changed variables $\rho \to \tau$ and $v \to z$ to confirm with the standard notation for Jacobi forms.

3.1 The 2A orbifold

The relevant Siegel modular form for the $2A$ orbifold is $\tilde{\Phi}_6$. In the product form given in (2.3), the input required for its construction is given by the twisted elliptic genus of the $2A$ orbifold which is given by

$$
\begin{aligned}
F^{(0,0)} &= 4A\left(\tau, z\right), &(3.4)\\
F^{(0,1)} &= \frac{4}{3}A\left(\tau, z\right) - \frac{2}{3}B\left(\tau, z\right)\mathcal{E}_2\left(\tau\right), \\
F^{(1,0)} &= \frac{4}{3}A\left(\tau, z\right) + \frac{1}{3}B\left(\tau, z\right)\mathcal{E}_2\left(\frac{\tau}{2}\right), \\
F^{(1,1)} &= \frac{4}{3}A\left(\tau, z\right) + \frac{1}{3}B\left(\tau, z\right)\mathcal{E}_2\left(\frac{\tau+1}{2}\right).
\end{aligned}
$$

Here A and B are defined in (2.6) and $\mathcal{E}_N$ is defined as

$$
\mathcal{E}_N(\tau) = NE_2(N\tau) - E_2(\tau). \tag{3.5}
$$

Using the expansions of the twisted elliptic genus $c^{(r,s)}$ as defined in (2.4) we can construct $\tilde{\Phi}_6$. It is easy to see that for the $2A$ orbifold, the expansions for the inverse of $\tilde{\Phi}_6$ is given by

$$
\frac{1}{\tilde{\Phi}_6(q,p,y)} = \sum_{m=-1}^{\infty} \psi_m(\tau, z) p^{\frac{m}{2}}. \tag{3.6}
$$

Note that the expansion in terms of the magnetic variable y can now take half integral values. Furthermore from (2.1) we see that m is related to the magnetic charge by $P^2 = 2m$. From the explicit construction of Φ_6 and the expansion in terms of y, we can read out the following

$$
\begin{aligned}
[\eta(\tau)\eta(2\tau)]^8\psi_{-1} &= -\frac{1}{B(2\tau, z)}, &(3.7)\\
[\eta(\tau)\eta(2\tau)]^8\psi_0 &= -2\frac{F^{(1,0)}(2\tau, z)}{B(2\tau, z)}, \\
[\eta(\tau)\eta(2\tau)]^8\psi_1 &= -\frac{1}{B(2\tau, z)} \\
&\quad \times \left[F^{(0,0)}(\tau, z) + F^{(0,1)}(\tau + 1/2, z) + 2[F^{(1,0)}(2\tau, z)]^2 + F^{(1,0)}(4\tau, z^2)\right].
\end{aligned}
$$

We define

$$
g^{(8)}(\tau) = [\eta(\tau)\eta(2\tau)]^8(\tau). \tag{3.8}
$$

The next step is to follow the procedure of [11] and write down an Appell-Lerch sum whose poles and residues coincide with the weight 2 Jacobi forms transforming under $\Gamma_0(2)$ on the left hand side of (3.7). We do not consider $m = -1$, since the Appell-Lerch sum diverges. This is easy to see, because the meromorphic Jacobi form $1/B(2\tau, z)$ is identical to the function for the un-orbifolded theory with the replacement of $\tau \to 2\tau$.

$P^2 = 0, m = 0$. Let us examine the case $m = 0$. Using the expression for the twisted elliptic genus in (3.4) we see that it reduces to

$$-g^{(8)}(\tau)\psi_0 = \frac{8}{3}\frac{A(2\tau,z)}{B(2\tau,z)} + \frac{2}{3}\mathcal{E}_2(\tau). \tag{3.9}$$

Now we can re-write

$$4\frac{A(2\tau,z)}{B(2\tau,z)} = -12\sum_{n\in\mathbb{Z}}\frac{q^{2n}y}{(1-q^{2n}y)^2} - E_2(2\tau), \tag{3.10}$$

where E_2 is the non-holomorphic Eisenstein series of weight 2. This identity was used in [11].[9] Combining (3.9) and (3.10) we see that the polar and the finite part is given by

$$g^{(8)}(\tau)\psi_0^{\mathrm{P}} = 8\sum_{n\in\mathbb{Z}}\frac{q^{2n}y}{(1-q^{2s}y)^2}, \tag{3.11}$$

$$g^{(8)}(\tau)\psi_0^{\mathrm{F}} = \frac{2}{3}\left(E_2(2\tau) - \mathcal{E}_2(\tau)\right).$$

Therefore we have decomposed the meromorphic Jacobi form that occurs in the expansion (3.6) at $m = 0$ to a polar part and a finite term. The finite term contains the mock modular form $E_2(2\tau)$ as well as $\mathcal{E}_2(\tau)$ both of which transforms under $\Gamma_0(2)$ with weight 2. In appendix B we show that $E_2(2\tau)$ is a Mock modular form of weight 2 in $\Gamma_0(2)$.

$P^2 = 2, m = 1.\mathbf{y}$ The first step in the analysis of the meromorphic Jacobi from that occurs at $m = 1$ is to use the equations in the expression of the twisted elliptic genus (3.4) to obtain

$$-g^{(8)}(\tau)\psi_1 = \frac{44}{9}\frac{A^2(2\tau,z)}{B(2\tau,z)} + \frac{8}{3}\frac{A(\tau,z)}{B(2\tau,z)} + \frac{16}{9}A(2\tau,z)\mathcal{E}_2(\tau) + \frac{2}{9}B(2\tau,z)\mathcal{E}_2^2(\tau)$$

$$+ \frac{1}{4}E_4(2\tau)B(2\tau,z) + \frac{4}{3}\frac{\theta_2^2(2\tau,z)}{\theta_2^2(2\tau)}\mathcal{E}_2(2\tau) - \frac{2}{3}\frac{\theta_3^2(2\tau,z)}{\theta_3^2(2\tau)}\mathcal{E}_2(\tau+1/2). \tag{3.12}$$

Here we have also used the identities

$$\frac{3}{4}E_4(\tau)B(\tau,z)^2 = 4\left(A(2\tau,2z) + A\left(\frac{\tau}{2},z\right) + A\left(\frac{\tau+1}{2},z\right) - A^2(\tau,z)\right) \tag{3.13}$$

$$\frac{B(2\tau,z^2)}{B(\tau,z)} = 4\frac{\theta_2^2(\tau,z)}{\theta_2^2(\tau)}, \qquad \frac{B\left(\frac{\tau}{2},2z\right)}{B(\tau,z)} = \frac{\theta_4^2(\tau,z)}{\theta_4^2(\tau)}, \qquad \frac{B\left(\frac{\tau+1}{2},2z\right)}{B(\tau,z)} = \frac{\theta_3^2(\tau,z)}{\theta_3^2(\tau)}.$$

It is more convenient to convert the ratios of theta functions in (3.12) to Jacobi forms A and B. For this we use the identities

$$4\frac{\theta_2(\tau,z)^2}{\theta_2^2(\tau)} = \frac{4}{3}A(\tau,z) - \frac{2}{3}B(\tau,z)\mathcal{E}_2(\tau), \tag{3.14}$$

$$4\frac{\theta_4(\tau,z)^2}{\theta_4^2(\tau)} = \frac{4}{3}A(\tau,z) + \frac{1}{3}B(\tau,z)\mathcal{E}_2(\tau/2),$$

$$4\frac{\theta_3(\tau,z)^2}{\theta_3^2(\tau)} = \frac{4}{3}A(\tau,z) + \frac{1}{3}B(\tau,z)\mathcal{E}_2((\tau+1)/2).$$

---

[9]Let us remark about the notation of [11] in comparison with ours. $4A(\tau,z)_{\mathrm{ours}} = B(\tau,z)_{\mathrm{theirs}}$, $B(\tau,z)_{\mathrm{ours}} = -A(\tau,z)_{\mathrm{theirs}}$.

Thus (3.12) becomes

$$
-g^{(8)}(\tau)\psi_1 = \frac{44}{9}\frac{A^2(2\tau,z)}{B(2\tau,z)} + \frac{8}{3}\frac{A(\tau,z)}{B(2\tau,z)} + \frac{16}{9}A(2\tau,z)\mathcal{E}_2(\tau) + \frac{2}{9}B(2\tau,z)\mathcal{E}_2^2(\tau)
$$
$$
+ \frac{1}{4}E_4(2\tau)B(2\tau,z) + \frac{1}{3}\left[\frac{4}{3}A(2\tau,z) - \frac{2}{3}B(2\tau,z)\mathcal{E}_2(2\tau)\right]\mathcal{E}_2(2\tau)
$$
$$
- \frac{1}{6}\left[\frac{4}{3}A(2\tau,z) + \frac{1}{3}B(2\tau,z)\mathcal{E}_2(\tau+1/2)\right]\mathcal{E}_2(\tau+1/2).
$$

(3.15)

It is clear from this expression that polar terms arise from the meromorphic Jacobi forms $A^2(2\tau,z)/B(2\tau,z)$ and $A(\tau,z)/B(2\tau,z)$ of weight 2 transforming under $\Gamma_0(2)$ with index $1/2$. We can use the following identity derived in [11] to re-write $A^2(2\tau,z)/B(2\tau,z)$ into an Appell-Lerch sum and a mock modular form

$$
-16\frac{A^2(2\tau,z)}{B(2\tau,z)} = 144\sum_{n\in\mathbb{Z}}\frac{q^{2n^2+2n}y^{2n+1}}{(1-q^{2n}y)^2} - E_4(2\tau)B(2\tau,z) - 288\mathcal{H}(2\tau,z).
$$

(3.16)

Here $\mathcal{H}$ is the simplest Jacobi mock modular form related to the generating function of Hurwitz-Kronecker class numbers

$$
\mathcal{H}(\tau,z) = \sum_{n=0}^{\infty}H(4n-j^2)q^n y^l.
$$

(3.17)

The coefficients $H(n)$ are defined by

$$
H(n) = 0 \qquad \text{for } n < 0,
$$

(3.18)

$$
\sum_{n\in\mathbb{Z}}H(n)q^n = -\frac{1}{12} + \frac{1}{3}q^3 + \frac{1}{2}q^4 + q^7 + q^8 + q^{11} + \cdots
$$

(3.19)

What remains is to figure out how to write the meromorphic Jacobi form $A(\tau,z)/B(2\tau,z)$ as a polar part and a finite term. Note that the location of the poles of order 2 lie precisely at the same point as the form $A^2(2\tau,z)/B(2\tau,z)$. Further more some bit of analysis shows that the residue at the double pole and the simple pole of $3\frac{A(\tau,z)}{B(2\tau,z)}$ is precisely equal to the corresponding residues of $A^2(2\tau,z)/B(2\tau,z)$. A bit more study show that we can derive the following identity satisfied by the meromorphic Jacobi forms

$$
\frac{A(\tau,z)}{B(2\tau,z)} = \frac{1}{3}\frac{A^2(2\tau,z)}{B(2\tau,z)} + \frac{1}{12}A(2\tau,z)\mathcal{E}_2(\tau) + \frac{1}{80}B(2\tau,z)E_4(\tau) - \frac{1}{80}B(2\tau,z)E_4(2\tau).
$$

(3.20)

Further more we have the identities

$$
-2\mathcal{E}_2(\tau) + \mathcal{E}_2\left(\frac{\tau}{2}\right) + \mathcal{E}\left(\frac{\tau+1}{2}\right) = 0,
$$

(3.21)

$$
E_4(2\tau) = \frac{-1}{4}E_4(\tau) + \frac{5}{4}\mathcal{E}_2^2(\tau),
$$

$$
\mathcal{E}_2(2\tau)\mathcal{E}_2(\tau+1/2) = \frac{-3}{8}E_4(\tau) + \frac{11}{8}\mathcal{E}_2^2(\tau),
$$

$$
4\mathcal{E}_2^2(2\tau) + \mathcal{E}_2^2(\tau+1/2) = \frac{13}{2}\mathcal{E}_2^2(\tau) - \frac{3}{2}E_4(\tau).
$$
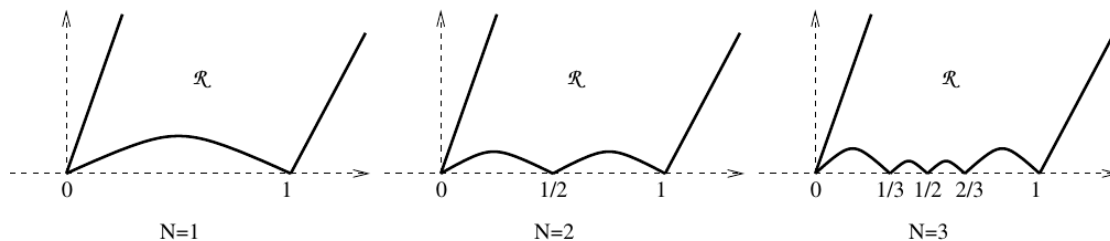
Figure 1. Figure reproduced from [15] showing chamber $\mathcal{R}$ in the upper half axion-dilaton plane, bounded by the walls of marginal stability, for the un-orbifolded theory, the $2A$ and the $3A$ orbifolds.

Now substituting (3.20), (3.21) in (3.15) we obtain

$$-g^{(8)}(\tau)\psi_1 = \frac{52}{9}\frac{A^2(2\tau,z)}{B(2\tau,z)} + \frac{20}{9}A(2\tau,z)\mathcal{E}_2(\tau) + B(2\tau,z)\left(\frac{1}{16}E_4(\tau) + \frac{19}{144}\mathcal{E}_2^2(\tau)\right)$$

Finally using the expansion in (3.16) and the identities in (3.21) we obtain

$$g^{(8)}(\tau)\psi_1^{\mathrm{P}} = 52\sum_{n\in\mathbb{Z}}\frac{q^{2n^2+2n}y^{2n+1}}{(1-q^{2n}y)^2} \tag{3.22}$$

$$g^{(8)}(\tau)\psi_1^{\mathrm{F}} = -\frac{20}{9}A(2\tau,z)\mathcal{E}_2(2\tau) - B(2\tau,z)\left(\frac{13}{36}E_4(2\tau) + \frac{1}{16}E_4(\tau) + \frac{19}{144}\mathcal{E}_2^2(\tau)\right) - 104\mathcal{H}(2\tau,z)$$

From the final expression for the finite part ψ_1^F we see that the relevant mock modular forms is the one constructed from the Hurwitz-Kronecker class numbers. It is the "optimal" choice for the mock modular form. This is because the Fourier coefficients of $q^n y^l$ of the expansion of $\mathcal{H}(q,z)$ grow polynomially in $4n - l^2$. In appendix B we show that $\mathcal{H}(N\tau,z)$ is a Mock Jacobi form of index $1/N$ in $\Gamma_0(N)$.

One important observation from our analysis at $m = 1$ is that there existed a different meromorphic Jacobi form $A(\tau,z)/B(2\tau,z)$ at the intermediate steps of our analysis. However the identity (3.20) related it to the form $A^2(2\tau,z)/B(2\tau,z)$. We could then use the identities obtained by [11] to obtain the polar and finite parts of the Fourier-Jacobi coefficient of $1/\tilde{\Phi}_6(\rho,\sigma,v)$ at $m = 1$. Though we demonstrated this feature only till $m = 1$, our preliminary analysis indicates that this persists in the expansion at higher magnetic charges and the identities obtained by [11] are sufficient to obtain the mock modular forms that determine the polar and the finite parts.

Fourier coefficients of $\psi_1^{\mathbf{F}}$ and single centered dyons. The decomposition of ψ_1 given in (3.22) has physical implications. Note that there is no ambiguity in the Fourier expansion of ψ_1^{F}, while the Fourier expansion of ψ_1^{P} depends on the domain in the space (q,y) where the expansion is performed. Let us elaborate on this further. We have defined the degeneracy using the contour defined in (2.2). The ensures that we are in the region $\mathcal{R}$, right of the line that joins 0 and $i\infty$ in the axion-dilaton moduli space The region $\mathcal{R}$ was found in [21] and is shown in figure 1. of [15]. For completeness we have included the figure 1 of [15] below. Using the definitions in (3.1) we see that we can first perform the

expansions in p and therefore obtain the partition function ψ_1 for fixed magnetic charge $P^2 = 2$. Now from the contour defined in (2.2) we see that

$$|q| \ll 1, \quad \frac{1}{|y|} \ll 1, \quad |q^n y| \ll 1, \quad |\frac{q^n}{y}| \ll 1 \tag{3.23}$$

where $n > 0$. This implies that we expand the polar part as

$$g^{(8)}(\tau)\psi_1^{\mathrm{P}} = 52 \left(\cdots \frac{q^4 y^{-3}}{(1 - \frac{q^2}{y})^2} + \frac{1}{y(1 - \frac{1}{y})^2} + \frac{q^4 y^3}{(1 - q^2 y)} + \cdots \right) \tag{3.24}$$

We can then further expand the denominator in each of the terms. It is clear from this expansion, which is determined by the contour in (3.1) that the polar part contributes only to the degeneracies with $Q \cdot P \leq -1$ and $Q \cdot P \geq 3$. Therefore the degeneracies for $Q \cdot P = 0, Q \cdot P = 1, Q \cdot P = 2$ with $P^2 = 2$ and $Q^2 \geq 1/2$ are entirely determined by ψ_1^{F} for the contour choice in (2.2).

Let us now examine the kinematic constraints on charges found by [15] which ensures that the attractor moduli for single centered black holes line in the region $\mathcal{R}$. For the $2A$ orbifold the conditions are given by[10]

$$Q^2 > 0; \qquad P^2 > 0; \qquad Q^2 P^2 > (Q.P)^2; \quad 2Q^2 + P^2 - 3(Q \cdot P)^2 \geq 0; \tag{3.25}$$
$$2Q^2 \geq Q \cdot P; \quad P^2 \geq Q \cdot P; \quad Q \cdot P \geq 0.$$

For $P^2 = 2$, the condition $Q \cdot P \leq P^2$ tell us that $0 \leq Q \cdot P \leq 2$. This fact combined with the discussion in the previous paragraph allows us to conclude that is the Fourier coefficients of ψ_1^{F} which determines the degeneracies of the single centered black hole and ψ_1^{P} does not contribute to these degeneracies. Note however that ψ_1^{F} also contains degeneracies for example $Q^2 = 1/2, Q \cdot P = 2, P^2 = 2$. This however does not satisfy the constraint $(Q \cdot P)^2 < Q^2 P^2$ or the constraint $2Q^2 \geq Q \cdot P$. *Thus we conclude: the degeneracies of the single centered dyons are contained in ψ_1^F, but not all Fourier coefficients of ψ_1^F represent single centered dyons.*

A similar conclusion was also reached in [11] for the un-orbifolded theory by considering the attractor contour. In that work an aprioi definition of ψ_m^{F} was given in terms of a contour originally introduced by [22]. The attractor contour extracts the degeneracies of single centered black holes. Here we have defined ψ_1^{F} in terms of the splitting equation (3.2) and (3.22). The crucial meromorphic form that occurs at this level which is given by $A^2(2\tau, z)/B(2\tau, z)$ is the same as that found in [11] but with the argument $\tau \to 2\tau$. Thus one can indeed carry provide an a priori definition of ψ_m^{F} by following the steps of [11]. We leave this for future work. As we will subsequently see, for the discussions in the rest of the paper, the above analysis indicates that ψ_1^F contains the degeneracy of the single centered black holes is sufficient.

3.2 Other orbifolds g' of $K3 \times T^2$

In this section we study the Fourier-Jacobi coefficients of $1/\tilde{\Phi}_k$ of all the other orbifolds listed in table 1. Our objective is to demonstrate that the mock modular forms which

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[10]See inequalities in equation 3.9 along with the discussion around equation 3.5 of [15].

determine the finite part of the Fourier-Jacobi coefficients till $m = 1$ is that same as that for the un-orbifolded theory. The Fourier-Jacobi coefficients of $1/\tilde{\Phi}_k$ are defined as

$$\frac{1}{\tilde{\Phi}_k(q,p,y)} = \sum_{m=-1}^{\infty} \phi_m(\tau,z)p^{\frac{m}{N}} \tag{3.26}$$

We restrict our analysis to $m = -1, 0, 1$. The first step is to use the product form given in (2.3) to obtain $\psi_m^{(g')}$. We divide our analysis for g' whose order N is odd and g' whose order is even.

g' with order N odd. This case includes the orbifolds pA with $p = 3, 5, 7, 11$ and $15A$ and $15B$. From the product form given in (2.3) we obtain

$$-\psi_{-1} = \frac{1}{B(N\tau,z)g^{(k+2)}(\tau)} \tag{3.27}$$

$$-\psi_0 = \frac{NF^{(1,0)}(N\tau,z)}{B(N\tau,z)g^{(k+2)}(\tau)},$$

$$-\psi_1 = \frac{1}{B(N\tau,z)g^{(k+2)}(\tau)}$$

$$\times \left[\left(\frac{N^2}{2} F^{(1,0)}(N\tau,z) \right)^2 + \frac{N}{2} \left(F^{(1,0)}(2N\tau,2z) + F^{(2,0)}\left(\frac{N\tau}{2}, z \right) + F^{(2,0)}\left(\frac{N(\tau+1)}{2}, z \right) \right) \right].$$

Here the $\Gamma_0(N)$ form $g^{(k+2)}(\tau)$ of weight $k + 2$ for each case can be read out from the table 3. The weights can be read out from table 2. For example for pA it is given by $g^{(k+2)}(\tau) = \eta^{k+2}(\tau)\eta^{k+2}(p\tau)$. Now we can use the from of the twisted elliptic genus in (2.5) to separate out the meromorphic Jacobi form. For $m = 0$ we obtain

$$-g^{(k+2)}\psi_0 = \alpha_{g'}^{(1,0)} \frac{A(N\tau,z)}{B(N\tau,z)} + N\beta_{g'}^{(1,0)}(N\tau). \tag{3.28}$$

Recall the constants $\alpha_{g'}^{(r,s)}$ and the $\Gamma_0(N)$ form $\beta_{g'}^{(r,s)}(\tau)$ are read out from the explicit computation of the twisted elliptic genus for all the orbifolds in table 1 from [9]. Using the identity in (3.10) we see that the polar and the finite parts of ψ_0 are given by

$$g^{(k+2)}\psi_0^P = 3N\alpha^{(1,0)} \sum_{n \in \mathbb{Z}} \frac{q^{Nn}y}{(1 - q^{Nn}y)^2}, \tag{3.29}$$

$$g^{(k+2)}\psi_0^F = \frac{N\alpha^{(1,0)}}{4} E_2(N\tau) - N\beta^{(1,0)}(N\tau).$$

For $m = 1$ we use the identities in (3.13) to isolate the meromorphic Jacobi form of weight 2 and index $1/N$ transforming under $\Gamma_0(N)$. This results in

$$-g^{(k+2)}\psi_1 = \frac{N}{2}\alpha_{g'}^{(1,0)} \left(\frac{N}{2}\alpha_{g'}^{(1,0)} + 1 \right) \frac{A^2(N\tau,z)}{B(N\tau,z)} + \frac{N}{2}\alpha^{(1,0)} \frac{3}{16} E_4(N\tau) B(N\tau,z) + \tag{3.30}$$

$$+ N^2\alpha_{g'}^{(1,0)} A(N\tau,z)\beta^{(1,0)}(N\tau) + \frac{N^2}{2}\beta_{g'}^{(1,0)}(N\tau)^2 B(N\tau,z)$$

$$+ \frac{N}{2} \left[4\frac{\theta_2(N\tau,z)^2}{\theta_2^2(N\tau)}\beta_{g'}^{(1,0)}(2N\tau) + \frac{\theta_3(N\tau,z)^2}{\theta_3^2(N\tau)}\beta_{g'}^{(1,0)}\left(\frac{N(\tau+1)}{2} \right) + \frac{\theta_4(N\tau,z)^2}{\theta_4^2(N\tau)}\beta_{g'}^{(1,0)}\left(\frac{N\tau}{2} \right) \right].$$

From the above expression, we see that the only meromorphic Jacobi form is the term with $A^2(N\tau, z)/B(N\tau, z)$. We can use the identity in (3.16), to write this form as a polar part and a finite term. Also we can see that the rest of the terms in (3.30) can in principle be written in terms of Jacobi forms transforming under $\Gamma_0(N)$. The polar term is given by

$$g^{(k+2)}\psi_1 = \frac{9N}{2}\alpha_{g'}^{(1,0)}\left(\frac{N}{2}\alpha_{g'}^{(1,0)} + 1\right)\sum_{n\in\mathbb{Z}}\frac{q^{Nn^2}y^{Nn+1}}{1 - q^{Nn}y}. \tag{3.31}$$

It is not illuminating to write the finite term explicitly, but it can be written down if needed. We note that the mock modular form that appears in the finite term is essentially the generating function of Hurwitz-Kronecker class numbers $\mathcal{H}(N\tau, z)$.

g' with order N even. Here we deal with the orbifolds belonging to class $4B$, $6A$, $8A$, $14A$, $14B$. Again examining the product representation (2.3) we obtain

$$-\psi_{-1}(q, z) = \frac{1}{B(N\tau, z)g^{(k+2)}(\tau)}, \tag{3.32}$$

$$-\psi_0(q, z) = \frac{NF^{(1,0)}(N\tau, z)}{B(N\tau, z)g^{(k+2)}(\tau)}.$$

Comparing the $m = 0$ expression with that of (3.27) we see that this is same as that when N is odd. Therefore the analysis proceeds identically and we obtain (3.29) for the polar and the finite part. For $m = 1$ all the orbifolds with N even yield different expressions

$$
\begin{aligned}
4B: -g^{(k+2)}(\tau)\psi_1 &= (8F^{(1,0)}(4\tau, z))^2 \\
&\quad + 2\left[F^{(1,0)}(8\tau, 2z) + F^{(2,0)}(2\tau, z) + F^{(2,1)}\left(2\tau + \frac{1}{2}, z\right)\right], \\
6A: -g^{(k+2)}(\tau)\psi_1 &= (18F^{(1,0)}(6\tau, z))^2 \\
&\quad + 3\left[F^{(1,0)}(12\tau, 2z) + F^{(2,0)}(3\tau, z) + F^{(2,3)}\left(3\tau + \frac{3}{2}, z\right)\right], \\
8A: -g^{(k+2)}(\tau)\psi_1 &= (32F^{(1,0)}(8\tau, z))^2 \\
&\quad + 4\left[F^{(1,0)}(16\tau, 2z) + F^{(2,0)}(4\tau, z) + F^{(2,1)}\left(4\tau + \frac{1}{2}, z\right)\right], \\
14A: -g^{(k+2)}(\tau)\psi_1 &= (98F^{(1,0)}(14\tau, z))^2 \\
&\quad + 7\left[F^{(1,0)}(28\tau, 2z) + F^{(2,0)}(7\tau, z) + F^{(2,7)}\left(7\tau + \frac{7}{2}, z\right)\right].
\end{aligned}
$$

Again using the expression for the twisted elliptic genus in (2.5) as well as the identities (3.13) we can isolate the meromorphic Jacobi-form that occurs for each of these cases. We write this as

$$-g^{(k+2)}(\tau)\psi_1 = \frac{N}{2}\alpha_{g'}^{(1,0)}\left(\frac{N}{2}\alpha_{g'}^{(1,0)} + 1\right)\frac{A^2(N\tau, z)}{B(N\tau, z)} + \frac{N}{2}\left(\alpha_{g'}^{(2,0)} - \alpha_{g'}^{(1,0)}\right)\frac{A\left(\frac{N\tau}{2}, z\right)}{B(N\tau, z)} \tag{3.33}$$

$$+ N^2\alpha_{g'}^{(1,0)}A(N\tau, z)\beta_{g'}^{(1,0)}(N\tau) + \frac{N^2}{2}\beta_{g'}^{(1,0)}(N\tau)^2 B(N\tau, z) + \frac{N}{2}\alpha_{g'}^{(1,0)}\frac{3}{16}E_4(N\tau)B(N\tau, z)$$

$$+ \frac{N}{2}\left(\frac{\theta_4(N\tau, z)^2}{\theta_4^2(N\tau)}\beta_{g'}^{(2,0)}\left(\frac{N\tau}{2}\right) + 4\frac{\theta_2(N\tau, z)^2}{\theta_2^2(N\tau)}\beta_{g'}^{(1,0)}(2N\tau)\right) + \phi_{g'}(\tau, z).$$

The Jacobi-form $\phi_{g'}(\tau, z)$ for each of the orbifolds is given by

$$4B: \quad \phi_{g'}(\tau, z) = \frac{2\theta_3^2(4\tau, z)}{\theta_3^2(4\tau)} \beta^{(2,1)}\left(\frac{2\tau + 1}{2}\right), \tag{3.34}$$

$$6A: \quad \phi_{g'}(\tau, z) = \frac{3\theta_3^2(6\tau, z)}{\theta_3^2(6\tau)} \beta^{(2,3)}\left(\frac{3(\tau + 1)}{2}\right),$$

$$8A: \quad \phi_{g'}(\tau, z) = \frac{4\theta_3^2(8\tau, z)}{\theta_3^2(8\tau)} \beta^{(2,1)}\left(\frac{4\tau + 1}{2}\right),$$

$$14A: \quad \phi_{g'}(\tau, z) = \frac{7\theta_3^2(14\tau, z)}{\theta_3^2(14\tau)} \beta^{(2,7)}\left(\frac{7(\tau + 1)}{2}\right).$$

From (3.33) and (3.34) we see that the only meromorphic Jacobi forms that occur in the expansion at $m = 1$ are $A^2(N\tau, z)/B(N\tau, z)$ and $A(N\tau/2, z)/B(N\tau, z)$. Therefore we can use the identity in (3.20) to first convert the form $A(N\tau/2, z)/B(N\tau, z)$ to the form $A^2(N\tau, z)/B(N\tau, z)$ and then use the identity in (3.16) to obtain the polar and the finite term. All these manipulations can be done explicitly if necessary. It is clear that the mock modular form that occurs at $m = 1$ in the finite part is again the generating function Hurwitz-Kronecker class number $\mathcal{H}(N\tau, z)$.

In conclusion we have written the Fourier-Jacobi coefficients that occur at levels $m = 0, m = 1$, that is magnetic charge $P^2 = 0, 2$ as a polar part and a finite part for all the orbifolds in table 1. We have shown that the Mock modular forms that occur in the finite term are $E_2(N\tau)$ and the generating function of Hurwitz-Kronecker class numbers $\mathcal{H}(N\tau, z)$ for these levels. There are identities that allow us to use the results of [11] to obtain the finite terms though we are dealing with forms that meromorphic Jacobi forms that transform under $\Gamma_0(N)$. In appendix B we show that $\mathcal{H}(N\tau, z)$ is a Mock Jacobi form of index $1/N$ in $\Gamma_0(N)$. Finally we mention that just as in the $\mathbb{Z}_2$ case, the degeneracies of single centered black holes can be extracted by examining Fourier coefficients of ψ_1^F. Most likely this pattern persists at higher levels in the magnetic charge expansion. It will be interesting to explore this further.

4 Toroidal orbifolds

In this section we study $\mathcal{N} = 4$ string theories originally constructed by [13]. These involve compactification of type IIB string theory on T^6 with a reflection along 4 of the co-ordinates together with a $1/2$ shift along one of the remaining circles. The type IIA description of the theory is that of a freely acting orbifold with the action of $(-1)^{F_L}$ and a $1/2$ shift along one of the circles of T^6. For details of these two descriptions and the dyon configuration in this theory see [19]. A similar compactification but in which the reflection in the 4 directions along the T^4 in type IIB theory is replaced by $2\pi/3$ rotation along one two dimensional plane of T^4 and a $-2\pi/3$ rotation along the other two dimension place together with a $1/3$ shift along one of the circles of the remaining T^2 was also discussed in [19]. We will call these models $\mathbb{Z}_2$ and $\mathbb{Z}_3$ toroidal orbifolds. The dyon partition function for these model is

given by the same expression as in (2.1) but the Siegel form given by

$$\tilde{\Phi}(\rho, \sigma, v) = e^{2\pi i(\tilde{\rho}+\tilde{v})} \qquad (4.1)$$

$$\prod_{b=0,1} \prod_{r=0}^{N-1} \prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N}, l\in\mathcal{Z}, \\ j\in 2\mathbb{Z}+b \\ k',l\geq 0, \ j<0 \ k'=l=0}} (1 - e^{2\pi i(k'\sigma+l\rho+jv)})^{\sum_{s=0}^{N-1} e^{2\pi i sl/N} c_b^{r,s}(4k'l-j^2)}.$$

Note the difference in the factor on the first line in comparison with (2.3). The coefficients $c^{(r,s)}$ are read out from the following twisted elliptic genus for the $\mathbb{Z}_2$ toroidal orbifold.

$$F^{(0,0)} = 0, \qquad (4.2)$$
$$F^{(0,1)} = \frac{8}{3}A(\tau,z) - \frac{4}{3}B(\tau,z)\mathcal{E}_2(\tau),$$
$$F^{(1,0)} = \frac{8}{3}A(\tau,z) + \frac{2}{3}B(\tau,z)\mathcal{E}_2\left(\frac{\tau}{2}\right),$$
$$F^{(1,1)} = \frac{8}{3}A(\tau,z) + \frac{2}{3}B(\tau,z)\mathcal{E}_2\left(\frac{\tau+1}{2}\right).$$

Since this twisted elliptic genus is closely related to the $2A$ orbifold given in (3.4) with $F^{(0,0}$ set to zero and the remaining indices multiplied by a factor of 2. The corresponding Siegel form is of weight $k=2$ and can be written as

$$\tilde{\Phi}_2(\rho, \sigma, v) = \frac{\tilde{\Phi}_6^2(\rho, \sigma, v)}{\tilde{\Phi}_{10}(\rho, \sigma, v)}. \qquad (4.3)$$

Here $\tilde{\Phi}_6$ is the weight 6 Siegel modular form associated with the $2A$ orbifold. For the $\mathbb{Z}_3$ toroidal orbifold the twisted elliptic genus is given by

$$F^{(0,0)} = 0 \qquad (4.4)$$
$$F^{(0,s)} = A(\tau,z) - \frac{3}{4}B(\tau,z)\mathcal{E}_3(\tau)$$
$$F^{(r,rk)} = A(\tau,z) + \frac{1}{4}B(\tau,z)\mathcal{E}_3\left(\frac{\tau+k}{3}\right), \quad r=1,2.$$

Note that $F^{(r,s)}$ are defined with $r,s \bmod 3$. The Siegel modular form associated with the $\mathbb{Z}_3$ toroidal orbifold is of weight $k=1$ and is given by

$$\tilde{\Phi}_1(\rho, \sigma, v) = \frac{\tilde{\Phi}_4^{3/4}(\rho, \sigma, v)}{\tilde{\Phi}_{10}^{1/2}(\rho, \sigma, v)}. \qquad (4.5)$$

In [15] it was argued that the index $d(Q,P)$ as defined in (2.1) must be positive for single centered black holes.[11] The argument relied on the fact that single centered black holes are spherically symmetric and therefore carry zero angular momentum. The only source of signs in the index $d(Q,P)$ for single centered black holes then arise only from

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[11]In [15] $d(Q,P)$ was refered to as the index $-B_6$.

fermionic zero modes associated with a 1/4 BPS state. This results in the $d(Q,P)$ being positive. This conjecture was verified for the orbifolds pA with $p = 2, 3, 5, 7$ in [15]. In [9] the conjecture was verified for all the orbifolds listed in table 1 which includes the CHL orbifolds $4B, 6A, 8A$ as well as the non-geometric orbifolds associated with the class $23A/B$ and classes $2B, 3B$ which lie in the Mathieu group M_{24}. However it was noticed in [9] that Siegel modular forms associated with certain twisted elliptic genera written down in [27] did not satisfy the conjecture[12] These twisted elliptic genera satisfied the property

$$\sum_{r,s=0}^{N-1} F^{(r,s)} = 0 \tag{4.6}$$

At present there are no known string compactification which results in the twisted elliptic genera written down by [27][13]

In this section we show that even the toroidal $\mathbb{Z}_2, \mathbb{Z}_3$ orbifolds defined above which admit a string compactification as well as a construction of dyons do not satisfy the positivity conjecture for the index $d(Q,P)$ for single centered dyons at low values of charges. This suggests that these single centered dyons are not spherically symmetric and possibly admit hair modes which contain additional fermionic zero modes. We identify the single centered dyon by both the constraints given in [15] as well as subtracting the polar part in the meromorphic Jacobi-form that occurs at a definite magnetic charge.

4.1 $\mathbb{Z}_2$ toroidal orbifold

The $\mathbb{Z}_2$ orbifold compactification involves a $1/2$ shift on one of the circles of T^2. Thus the $SL(2,\mathbb{Z})$ symmetry of the torus is broken down to $\Gamma_0(2)$, which is now the duality symmetry of the dyon system. In [15] the positivity property of $d(Q,P)$ was studied for the $2A$ orbifold was studied and the constraints on charges for the single centered black hole was identified. To arrive at these conditions only the $\Gamma_0(2)$ duality symmetry of the dyon system was used. This symmetry was used to map the wall of marginal stability that runs from 0 to $i\infty$ in the axion-dilaton moduli space to all the walls which border the region $\mathcal{R}$,[14] (figure 1). Since the duality symmetry of the $\mathbb{Z}_2$ toroidal orbifold is also $\Gamma_0(2)$, the conditions will remain same. Given the fact that the Fourier coefficients are extracted using the contour (2.2), the conditions on charges that the attractor moduli lie in $\mathcal{R}$ are given by

$$Q^2 > 0; \qquad P^2 > 0; \qquad Q^2 P^2 > (Q.P)^2; \quad 2Q^2 + P^2 - 3(Q \cdot P)^2 \geq 0; \tag{4.7}$$
$$2Q^2 \geq Q \cdot P; \quad P^2 \geq Q \cdot P; \quad Q \cdot P \geq 0.$$

Later in this section we will also identify single centered dyons by subtracting out the polar part in the meromorphic Jacobi form that occurs at a definite magnetic charge. As discussed earlier, the degeneracies of single centered black holes can be obtained by

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[12]See tables XVI, XVII of the Phys. Rev. D version of [9].

[13]See equations 3.20, 3.22 of Phys. Rev. D version of [9] for the explicit form of the twisted elliptic genera.

[14]As discussed earlier the region $\mathcal{R}$ in the axion-dilaton moduli space corresponds to the contour given in equation (2.2).

$Q^2 \backslash P^2$	0	2	4	6	8
0	2	64	816	6912	45584
1	-12	**-224**	**-1248**	1728	95104
2	48	1152	18240	233984	2432544
3	-168	**-3392**	**-10320**	542976	12103360
4	528	-11520	200736	4575744	86712256
5	-1512	**-30336**	**-55424**	12914944	412163328
6	4032	83968	1544832	61928448	2013023104
7	-10176	**-202560**	**-179022**	175358304	8292093664
8	24528	496512	9480000	638922240	32998944096
9	-56796	**-1118496**	**-155232**	1735394112	119618619520
10	127008	2521600	49523328	5364983808	415768863360
11	-275544	**-5374656**	2684560	13858160960	1359548367552
12	581952	11389440	228872064	38347445760	4277873003392
13	-1199688	**-23194176**	24502656	94345755264	12874682948352
14	2419584	46824960	959446272	240772494336	37480253184000
15	-4783968	**-91770432**	142728318	566613885216	105389524965472
16	9288528	178117376	3712290336	1358448247296	288023853905856
17	-17735256	**-337839744**	678230784	3072125756544	765208401512448
18	33343344	634494592	13426540992	7004675317248	1983801614528672
19	-61794600	**-1169806144**	2834120592	15289076372544	5022356020513856
20	113002848	2136181248	45830851200	33439408301056	12447769083229056
21	-204081024	**-3841753664**	10783524096	7071452719680	30229570751178240
22	364274496	6846494720	148756097664	149298142934016	72059338059045504
23	-643092768	**-12044893632**	38123432260	306899147706368	168747648892043328

Table 17. The index $d(Q, P)$ for the $\mathbb{Z}_2$ toroidal orbifold some low lying values of Q^2, P^2 with $Q \cdot P = 0$.

examining the Fourier coefficients of the finite part of the meromorphic Jacobi form. We will also need to impost the conditions in (4.7) for the Fourier coefficients so that they correspond to single centered dyons.

We extract the Fourier coefficients from the finite part of the meromorphic part of the magnetic charge expansion, or more directly using the contour in (2.2) with the constraints in (4.7) which ensure that we are studying single centered dyons. We show that there are cases in which the positivity conjecture of [15] is violated.

Let us first list the some of the violations. Tables 17, 18 19 list out the degeneracy $d(Q, P)$ for small values of Q^2, P^2 with $Q \cdot P = 0, 1, 2$ respectively. The charges corresponding to single centered dyons in the range (3.25) and violating the positivity conjecture are marked in bold.

Let us again confirm that the charges as well as the degeneracies which are violating the positivity conjecture for $d(Q, P)$ is that of the single centered black hole. For this we study the Fourier-Jacobi coefficients that occur at for low but fixed magnetic charges but arbitrary electric charge and angular momentum. Thus we look for the expansion

$$\frac{1}{\widetilde{\Phi}_2(q, p, y)} = \sum_{m=0}^{\infty} \psi_m(\tau, z) p^{\frac{m}{2}} \,. \tag{4.8}$$

$Q^2 \setminus P^2$	0	2	4	6	8
0	0	0	-8	-128	-1160
1	1	96	1968	22528	190047
2	-8	**-256**	840	70912	1127672
3	37	1376	34656	728256	11046139
4	-136	**-3840**	16632	2497408	61486056
5	439	13152	343152	13144832	348876305
6	-1288	**-33536**	171152	42058240	1603241304
7	3503	92928	2476752	162898624	7016918625
8	-8968	**-220672**	1265256	480911872	27503872048
9	21854	540416	14545584	1556561664	102315259287
10	-51080	**-1204992**	7558560	4271142656	354800345088
11	115154	2711616	73540080	12261114752	1175752005781
12	-251528	**-5741824**	38736600	31586749312	3705255587616
13	534304	12144096	331284816	83106163712	11241057088056
14	-1107080	**-24613888**	176485368	202830655232	32810366529704
15	2242936	49597408	1360242048	499048223424	92762004787995
16	-4452488	**-96865536**	731764656	1162636791680	254219096542800
17	8675803	187681920	5172820416	2710918677760	678135519966520
18	-16618760	**-355014144**	2806978216	6065899132672	1762706150153656
19	31335779	665705664	18435647328	13529566137472	4476930500026908
20	-58228616	**-1224694784**	10082072832	29223048194432	11122701903357048
21	106740533	2233279616	62133135120	62776998234368	27083291897745248
22	-193201800	**-4009231104**	34221009384	131432145572096	64699862426642976
23	345565877	7135993088	199430638848	273349419121472	151855990384385978

Table 18. The index $d(Q, P)$ for the $\mathbb{Z}_2$ toroidal orbifold some low lying values of Q^2, P^2 with $Q \cdot P = 1$.

Note that the expansion in terms of the magnetic variable y can take half integral values but cannot be negative. Here also m is related to the magnetic charge by $P^2 = 2m$. From the product form of $\tilde{\Phi}_2$ given in (2.3) constructed using the twisted elliptic genus in (4.2) we obtain

$$
\left[\frac{\eta^{16}(2\tau)}{\eta^8(\tau)} \right] \psi_0 = -\frac{F^{(1,0)}(2\tau, z)}{8B(2\tau, z)},
$$
$$
\left[\frac{\eta^{16}(2\tau)}{\eta^8(\tau)} \right] \psi_1 = -\frac{F^{(1,0)}(2\tau, z)^2}{4B(2\tau, z)}.
$$
(4.9)

We define

$$
g^{(4)}(\tau) = \left[\frac{\eta^{16}(2\tau)}{\eta^8(\tau)} \right].
$$
(4.10)

We re-write the meromorphic Jacobi form ψ_0 and ψ_1 corresponding to magnetic charges $P^2 = 0, P^2 = 2$ as a polar part and finite term. We perform this decomposition for $m = 0$ and $m = 1$.

$Q^2 \setminus P^2$	0	2	4	6	8
0	0	0	0	0	16
1	0	0	−12	−224	−1248
2	0	64	2592	43264	491904
3	−2	**−224**	2432	191168	3805600
4	16	1152	43392	1440256	30853488
5	−72	**−3392**	33720	5363680	171782688
6	256	11520	414336	24533248	893029504
7	−806	**−30336**	302400	80281536	3963098880
8	2320	83968	2926080	287831552	16432262672
9	−6200	**−202560**	2049968	851816352	62214237440
10	15616	496512	16919712	2627695616	222752294016
11	−37508	**−1118496**	11568000	7176834368	750069187008
12	86544	2521600	84554880	19942216704	2414262572768
13	−192800	**−5374656**	56838432	51008186976	7425202332576
14	416512	11389440	377428608	131082715648	22009992439296
15	−875808	**−23194176**	250745920	317429798336	62951326894880
16	1797648	46824960	1538196480	767174552576	174613994718000
17	−3610064	**−91770432**	1013176056	1773519888864	470403008967552
18	7107328	178117376	5813224704	4077368575488	1234828601424128
19	−13741542	**−337839744**	3805021440	9056713382272	3162966840870720
20	26130192	634494592	20608359552	19969539018240	7923569863533760
21	−48930016	**−1169806144**	13425820256	42839061178880	19436689033887616
22	90327040	2136181248	69137356032	91147913531648	46764751712533632
23	−164551050	**−3841753664**	44883305472	189628816546240	110476832098945280

Table 19. The index $d(Q, P)$ for the $\mathbb{Z}_2$ toroidal orbifold some low lying values of Q^2, P^2 with $Q \cdot P = 2$.

$P^2 = 0, m = 0$. Now we examine the case $m = 0$. Using the expression for the twisted elliptic genus in (4.2) we see that it reduces to

$$-g^{(4)}(\tau)\psi_0 = \frac{1}{3}\frac{A(2\tau, z)}{B(2\tau, z)} + \frac{1}{12}\mathcal{E}_2(\tau). \tag{4.11}$$

Combining (3.9) and (4.11) we see that the polar and the finite part is given by

$$g^{(4)}(\tau)\psi_0^{\mathrm{P}} = \sum_{s \in \mathbb{Z}} \frac{q^{2s}y}{(1 - q^{2s}y)^2}, \tag{4.12}$$

$$\psi_0^{\mathrm{F}} = \frac{1}{12g^{(4)}(\tau)}\left(E_2(2\tau) - \mathcal{E}_2(\tau)\right).$$

The finite part is certainly not of definite sign. Examining the numerator we see that each of the terms are of the same sign. This is because

$$E_2(2\tau) - \mathcal{E}_2(\tau) = -24\sum_{n=1}^{\infty}(\sigma(n)q^n - \sigma(n)q^{2n}) \tag{4.13}$$

where $\sigma(n)$ is the divisor function of n. However due to the presence $g^{(4)}(\tau)$ in the denominator the signs of the terms in the q expansion are not definite. We can contrast this situation with the $2A$ orbifold. The finite part for the meromorphic Jacobi form for the $2A$ orbifold at $m = 0$ is given by (3.11) which by the same analysis is manifestly of a definite sign.

As discussed for the $2A$ orbifold, let us also examine what is the effect of the subtraction of the polar part to the degeneracy $d(Q, P)$. For this recall the degeneracy is evaluated by extracting out the Fourier coefficients using the contour (2.2). This implies that we expand the polar part as follows

$$
g^{(4)}(\tau)\psi_0^{\mathrm{P}} = \frac{q^2 y}{(1 - q^2 y)^2} + \frac{q^4}{(1 - q^4 y)^2} + \cdots \tag{4.14}
$$
$$
+ \frac{1}{y(1 - \frac{1}{y})^2} + \frac{q^2}{y(1 - \frac{q^2}{y})^2} + \cdots
$$

This is because from the contour in (2.2) and using the definitions in (3.1) we see that $|q| \ll 1$, $1/|y| \ll 1$ and $|q^n y| \ll 1$, $|q^n/y| \ll 1$. In each of the above lines we can further expand the denominators in q. Thus we see that subtraction of the polar part at $m = 0$ affects only the $d(Q, P)$ with $Q \cdot P \geq 1$ or $Q \cdot P \leq -1$ with $P^2 = 0$. Thus the degeneracies with $Q \cdot P = 0, P^2 = 0$ are not affected by the subtraction of the polar part. This implies that the list of degeneracies in the first column of table 17 is that can be obtained from the q expansion of the finite term Ψ_0^{F}. This can be verified explicitly and indeed the terms have the sign which is given by $(-1)^{Q^2}$.

However note that according to the domain of charges given in (4.7) the charges which satisfy $Q \cdot P = 0$ are not single centered. Therefore we see that the domain (3.25) is a more restrictive definition of single centered black holes. However the Fourier coefficients of ψ_0^F is not of the same sign for the $\mathbb{Z}_2$ toroidal orbifold.

$\boldsymbol{P^2 = 2, m = 1.}$ Using the expression of the twisted elliptic genus (4.2) we can write the meromorphic Jacobi from that occurs at $m = 1$ in (4.9). This results in

$$
-g^{(4)}(\tau)\psi_1 = \frac{16}{9}\frac{A^2(2\tau, z)}{B(2\tau, z)} + \frac{8}{9}A(2\tau, z)\mathcal{E}_2(\tau) + \frac{1}{9}B(2\tau, z)\mathcal{E}_2^2(\tau).
$$

We can use the identity in (3.16) to extract out the polar and the finite parts of ψ_1. This results in

$$
g^{(4)}(\tau)\psi_1^{\mathrm{P}} = 16 \sum_{n \in \mathbb{Z}} \frac{q^{2n^2 + 2n} y^{2n+1}}{(1 - q^{2n} y)^2}, \tag{4.15}
$$
$$
g^{(4)}(\tau)\psi_1^{\mathrm{F}} = -\frac{8}{9}A(2\tau, z)\mathcal{E}_2(\tau) - \frac{1}{9}B(2\tau, z)\left[E_4(2\tau) + \mathcal{E}_2^2(\tau)\right] - 32\mathcal{H}(2\tau, z).
$$

Note again the appearance of the function $\mathcal{H}$. Let us examine if the sign of the Fourier coefficients of the finite part ψ_1^{F} is the same. For this let again go through the analysis of what is the effect of the subtraction of the polar part ψ_1^{P}. Since we are evaluating the

degeneracies in the domain (2.2) we expand the polar part as

$$g^{(4)}(\tau)\psi_1^{\mathrm{P}} = \frac{16q^4y^3}{(1-q^2y)^2} + \frac{16q^{10}y^5}{(1-q^4y)^2} + \cdots$$

$$+ \frac{16}{y(1-\frac{1}{y})^2} + \frac{16q^4}{y^3(1-\frac{q^2}{y})^2} + \cdots \tag{4.16}$$

Therefore subtraction of the polar part at $m=1$ affects $d(Q,P)$ with $P^2=2$ and $Q\cdot P \geq 3$ or $Q\cdot P \leq -1$. Thus the degeneracies in listed in the second column of tables 17, 18, 19 are unchanged by the subtraction, and therefore equal to that given by the finite part ψ_1^{F}. We have verified this explicitly, in fact this is within the domain given in (3.25). Also in [11] it was argued that the finite part captures the degeneracies of the single centered black hole. It is clear from the second column of the tables 17, 18, 19 that the signs violate the positivity conjecture.

Further more note that from the tables for $P^2=2$ and $Q\cdot P \leq 2$ we see that the sign is $d(Q,P)$ is given by $(-1)^{Q^2}$. This can be seen also analytically by writing ψ_1 in terms of its product form. Using the form of ψ_1 given in (4.9) and writing all the modular forms in terms of their product representation we obtain

$$\psi_1 = \frac{1}{(q_\infty^2)^4 qy(1-1/y)^2} \frac{\prod_{m=1}^{\infty}(1-q^{2m-1}y)^4(1-q^{2m-1}/y)^4}{\prod_{m=1}^{\infty}(1-q^{2m}y)^2(1-q^{2m}/y)^2}. \tag{4.17}$$

Here $q_\infty^2 = \prod_{n=1}^{\infty}(1-q^{2n})$. From the above equation it is evident that the sign of the coefficient of odd or even powers of q is independent of the power of z. This implies that no matter what is the value of $Q\cdot P$, the sign of the coefficient of q^n is given by $(-1)^n$. This is what is seen in the tables 17, 18, 19 for $Q\cdot P = 0,1,2$ respectively. In fact this analysis shows that the positivity conjecture is violated for infinite values of Q^2 in with $Q\cdot P = 0,1,2$ and $P^2 = 2$. Note that these charges satisfy the condition (3.25). As argued above they are also counted in ψ_1^{F} and therefore correspond to single centered black holes.

The violation of the sign of $d(Q,P)$ with $P^2 = 2,4$ for the case of single centered dyons suggest that these states might have fermionic zero modes as hairs. This was one of the options provided in [15] if the positivity conjecture fails. As further evidence of possible other degrees of freedom, we evaluate the leading saddle point statistical entropy at one loop for charges satisfying the condition (3.25) and therefore single centered. We then compare it to the exact degeneracy obtained by evaluating the Fourier coefficient $d(Q,P)$. Going through a similar analysis as in section 2 for the $\mathbb{Z}_2$ toroidal orbifold, we find that the statistical entropy at one loop is given by

$$S_{\mathrm{stat}}^{(1)} = \frac{\pi}{2\tau_2}|\frac{P}{\sqrt{2}} - \sqrt{2}\tau Q|^2 - \ln g^{(4)}(\tau) - \ln g^{(4)}(-\bar{\tau}) - 4\ln(2\tau_2) - 9\ln 2 + O(Q^{-2}, P^{-2}),$$

$$\tau_1 = \frac{Q\cdot P}{2Q^2}, \qquad \tau_2 = \frac{1}{2Q^2}\sqrt{Q^2P^2 - (Q\cdot P)^2} \tag{4.18}$$

The reason that there is a replacement of $Q \to P/\sqrt{2}$ and $P \to Q\sqrt{2}$ compared to (2.42) is that Siegel modular form $\tilde{\Phi}_2$ does not posses the symmetry in (2.13) which is obeyed

$(Q^2,\ P^2\ Q.P)$	d^{stat}	S^{stat}	S^1_{stat}	δ
$(2, 4, 0)$	18240	9.81137	9.95979	-1.5
$(4, 4, 0)$	200736	12.2097	11.9331	2.2
$(4, 8, 2)$	30853488	17.2448	17.5761	-1.92
$(6, 8, 1)$	1603241304	21.1953	21.5658	-1.75
$(8, 6, 0)$	638922240	20.2753	20.5197	-1.2
$(11, 4, 0)$	2684560	14.80	18.12	-22.43
$(3, 4, 2)$	2432	7.79	9.21	-18.23
$(2, 4, 1)$	840	6.73	9.26	-37.6
$(2, 8, 1)$	16632	9.72	11.6	19.34
$(1, 6, 0)$	1728	7.45	13.26	-77.98

Table 20. Comparison of the statistical entropy and the statistical entropy at one loop for the toroidal $\mathbb{Z}_2$ orbifold.

by the modular form corresponding to the orbifolds of $K3 \times T^2$ Note that we have also evaluated the constant C_1 in the statistical entropy function. This was done using the relation obeyed by $\tilde{\Phi}_2$ in (4.3).

We now compare the one loop statistical entropy in (4.18) to the exact degeneracy which is obtained from extracting the Fourier coefficients using (2.1) in table 20. The charges chosen are in the range (3.25). Therefore they are single centered, they also obey the property that $d(Q, P)$ is positive. We note that there is a set of charges for which the one loop statistical entropy is off from the exact degeneracy by over 75%. There is also a set of charges for which the statistical entropy agrees with the exact degeneracy to within 2%. The fact that there is a set of small charges for which the deviation from the one loop statistical entropy is high certainly indicates that we need to understand the geometric description for dyons in these theories better.

4.2 $\mathbb{Z}_3$ toroidal orbifold

In this section we briefly discuss the $\mathbb{Z}_3$ orbifold, We first list out the violations of the positivity conjecture in table 21. The violations are indicated in bold face. Again let us mention that by performing the Fourier expansion using the contour in (2.2) we in the region $\mathcal{R}$ in the axion-dilaton moduli. Now demanding that the attractor moduli lie in the region $\mathcal{R}$ so that the dyon is single centered results in following constraints on the charges [15].

The condition for the dyon to be single centered are given by [15].

$$Q^2 > 0, P^2 > 0; \qquad Q^2 P^2 > (Q \cdot P)^2, Q \cdot P \geq 0, Q \cdot P \leq 3Q^2, Q \cdot P \leq P^2, \quad (4.19)$$
$$5Q \cdot P \leq 6Q^2 + P^2, \quad 5Q \cdot P \leq 3Q^2 + P^2, \quad 7Q \cdot P \leq 6Q^2 + 2P^2$$

The single centered charges that violate the positivity conjecture are given in the table 21.

$(Q^2,\ P^2)\setminus Q\cdot P$	-1	0	1	2	3	4
$(2/3, 2)$	0	18	9	0	0	0
$(4/3, 2)$	-252	$\mathbf{-36}$	45	0	0	0
$(4/3, 4)$	-1458	540	864	54	0	0
$(4/3, 8)$	18378	93816	72099	9846	45	0
$(4/3, 12)$	-119502	3522240	2436363	447606	9243	$\mathbf{-6}$

Table 21. Some results for the index $-B_6$ for the torus order 3 orbifold of T^4 for different values of Q^2, P^2 and $Q\cdot P$.

For completeness we provide the meromorphic Jacobi forms that occur in the expansion of the inverse Siegel modular form $\tilde{\Phi}_1$ Again we define

$$\frac{1}{\Phi_1(q,y,z)} = \sum_{m=0}^{\infty} \psi_m(q,z)y^{(m)/3}, \tag{4.20}$$

where $\phi_m(q,z)$ is a modular function of 2 variables.

$$\psi_0 = \frac{F^{(1,0)}}{3B(3\tau,z)}\frac{\eta^3(\tau)}{\eta^9(3\tau)}, \tag{4.21}$$
$$\psi_1 = \psi_0\,(3F^{(1,0)}).$$

Rewriting these in terms in the Jacobi forms A, B we obtain

$$\psi_0 = \frac{1}{3}\frac{\eta^3(\tau)}{\eta^9(3\tau)}\left(\frac{A(3\tau,z)}{B(3\tau,z)} + \frac{1}{12}\frac{\eta^3(\tau)}{\eta^9(3\tau)}\mathcal{E}_3(\tau)\right), \tag{4.22}$$
$$\psi_1 = \frac{\eta^3(\tau)}{\eta^9(3\tau)}\left(\frac{A^2(3\tau,z)}{B(3\tau,z)} + \frac{1}{2}A(3\tau,z)\mathcal{E}_3(\tau) + \frac{1}{16}B(3\tau,z)\mathcal{E}_3^2(\tau)\right)$$

Note again since only the meromorphic forms $A(3\tau,z)/B(3\tau,z)$ and $A^2(3\tau,z)/B(3\tau,z)$ occur. We can use the identities (3.10) and (3.16) to obtain the polar and the finite part for these ratios respectively.

5 Conclusions

We have observed three properties of 1/4 BPS dyons in $\mathcal{N} = 4$ theories at low charges. We have seen that the constant C_1 contributes crucially to the entropy at low charges. As we have discussed in section 2.4 reproducing this constant using the method of localization proposed in [25, 26] will be interesting to pursue. In fact all the present works in this direction do not address the $\mathbb{Z}_N$ orbifolds of $K3 \times T^2$.

We have extended the observations of [11]. We showed that the for all the orbifolds considered in this paper, we can decompose the meromorphic Jacobi form that occurs in the Fourier-Jacobi decomposition of the inverse of Siegel modular forms in to a polar part and a finite part which involves a mock modular form. This has been done to magnetic charge $P^2 = 2$. We have seen that the mock modular form that occurs is same as that

occurred in the un-orbifolded theory to this order in magnetic charge. It will be interesting to go to higher levels in magnetic charge and see if this is always the case.

Finally we have observed a infinite set of violations of the positivity conjecture for the Fourier coefficient of the inverse Siegel modular form [15]. This was seen for the case of $\mathbb{Z}_2$ toroidal orbifold. We have demonstrated that the charges which violate the conjecture are single centered. Therefore according to the arguments of [15] it is possible that these violations might be due to the presence of hair. It will be interesting to understand this more precisely. Violations of the positivity conjecture was also seen for the $\mathbb{Z}_3$ toroidal orbifold.

Acknowledgments

A Details on obtaining C_1 from the threshold integral

In this appendix we provide the details on how to obtain the constant C_1 by performing the threshold integrals $\hat{\mathcal{I}}$ in (2.33) and $\tilde{\mathcal{I}}$ given in (2.26). The un-folding technique is used to perform these integrals. We can do the integral $\tilde{\mathcal{I}}$ following [19] and we obtain the result (2.30). The constant arises in the integral $\hat{\mathcal{I}}$. Here we outline the steps in the integration which gives rise to the constant C_1. The first step to do the integral $\hat{\mathcal{I}}$ in (2.33) is to perform the Poisson sum on m_1, m_2. This results in

$$\hat{\mathcal{I}}_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{Y}{U_2} \sum_{r,s} \sum_{n_1,n_2,k_1,k_2 \in \mathbb{Z}} e^{-2\pi i s n_2/N} e^{2\pi i k_2 r/N} \exp(\mathcal{G}(n_1, n_2, k_1, k_2)) h_l^{r,s}, \quad (A.1)$$

where $\mathcal{G}(n_1, n_2, k_1, k_2)$ is given as,

$$\mathcal{G}(n_1, n_2, k_1, k_2) = -\frac{\pi Y}{U_2^2 \tau_2} |\mathcal{A}|^2 - 2\pi i \det(A) T \quad (A.2)$$
$$+ \frac{\pi b}{U_2}(V\tilde{\mathcal{A}} - \bar{V}\mathcal{A}) - \frac{\pi n_2}{U_2}(V^2 \tilde{\mathcal{A}} - \bar{V}^2 \mathcal{A}) 2\pi i \frac{V_2^2}{U_2^2}(n_1 + n_2 \bar{U})\mathcal{A} + \frac{2\pi i \tau b^2}{4},$$

and

$$A = \begin{pmatrix} n_1 & k_1 \\ n_2 & k_2 \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} 1 & U \end{pmatrix} A \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \qquad \tilde{\mathcal{A}} = \begin{pmatrix} 1 & \bar{U} \end{pmatrix} A \begin{pmatrix} \tau \\ 1 \end{pmatrix} \quad (A.3)$$

Using the unfolding technique as in [19], the integration splits into the zero orbit, degenerate orbit and the non-degenerate orbit. What we need to keep track of is the constants and that too the difference of the constants that occur in the integral $\tilde{\mathcal{I}}$ of (2.26) and the integral $\hat{\mathcal{I}}$. Constants can arise in the zero orbit and the degenerate orbit. In the zero orbit here we have $A = 0$ therefore

$$\mathcal{I}_{\text{zero orbit}} = \frac{Y}{U_2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{r,s} F^{(r,s)}(\tau, 0).$$

Performing this integral, for all the orbifolds in table 1 we see that the constant that arises here is same as that of the un orbifolded $K3$ and equal to the constant that arises in the zero orbit of the $\tilde{\mathcal{I}}$ integral. Lets now examine the degenerate orbit. Here the matrix A is given by

$$A = \begin{pmatrix} 0 & k_1 \\ 0 & k_2 \end{pmatrix}, \qquad k_1, k_2 \in \mathbb{Z}, \quad (k_1, k_2) \neq (0, 0). \tag{A.4}$$

The integration region in the degenerate orbit is the strip given by

$$-\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}, \qquad \tau_2 \geq 0 \tag{A.5}$$

Apart from the moduli dependent terms and the constant $-2k \ln \kappa$ as in the integral $\tilde{\mathcal{I}}$ there is an additional constant due to the contribution from the twisted sector. The twisted sector does not contribute in the zero orbit of the $\tilde{\mathcal{I}}$ integral. Furthermore from (A.2) the only τ_1 dependence arises from the q expansion of the twisted elliptic genus. Note that since $n_2 = 0$, we obtain a sum over s. Then it can be seen that the twisted elliptic genus obeys the property

$$\sum_{s=0}^{N-1} F^{(r,s)}(\tau + 1, z) = \sum_{s=0}^{N-1} F^{(r,s)}(\tau, z) \tag{A.6}$$

Due to this periodicity in τ, only the coefficient of q^0 contributes on performing the τ_1 integral in the domain (A.5). Then doing the τ_2 integral in the twisted sectors we are left with the additional constant term[15]

$$C = \sum_s c^{(r,s)}(0) \sum_{k_2 \in \mathbb{Z}, k_2 > 0} \frac{2}{k_2} e^{2\pi i k_2 r/N}. \tag{A.7}$$

From the explicit evaluation of the twisted elliptic genus in [9] we list the values of $\sum_{s=0}^{N-1} c^{(r,s)}(0)$ for each twisted sector in different orbifolds of $K3$ given in table 1. For a prime N we have,

$$\sum_s c^{(r,s)}(0) \sum_{k_2 \in \mathbb{Z}, k_2 > 0, r \neq 0} \frac{2}{k_2} e^{2\pi i k_2 r/N} = 2(k+2) \sum_{k_2 \in \mathbb{Z}, k_2 > 0} \frac{1}{k_2} e^{2\pi i k_2 r/N}. \tag{A.8}$$

We have the identity

$$\sum_{k_2 \in \mathbb{Z}, k_2 > 0} \frac{1}{k_2} e^{2\pi i k_2 r/N} = -\ln(1 - e^{2\pi i r/N}). \tag{A.9}$$

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[15]In principle there can be another moduli dependent term in this sector if $\sum_s c^{(r,s)}(-1)$ be non-zero. However for all orbifolds g' in table 1 this vanishes.

Now summing this on the required range of r ie, $0 < r < n$ we get,

$$\sum_{r=1}^{N-1} \sum_{k_2 \in \mathbb{Z}, k_2 > 0} \frac{1}{k_2} e^{2\pi i k_2 r/N} = -\log\left(\prod_{r=1}^{N-1}(1 - e^{2\pi i r/N})\right) = -\ln N. \tag{A.10}$$

Thus, we have the result

$$C = \sum_{r=1}^{N-1} \sum_s c^{(r,s)}(0) \sum_{k_2 \in \mathbb{Z}, k_2 > 0} \frac{2}{k_2} e^{2\pi i k_2 r/N} = -2(k+2)\ln(N). \tag{A.11}$$

Again from the list given in table 22 we perform the sums in C for every other orbifolds listed in table 1. For $4B$ we have,

$$\begin{aligned}
C &= \sum_{r=1}^{3} \sum_s 4 \sum_{k_2 \in \mathbb{Z}, k_2 > 0, r \neq 0} \frac{2}{k_2} e^{2\pi i k_2 r/4} + 2 \sum_{k_2 \in \mathbb{Z}, k_2 > 0} \frac{2}{k_2} e^{2\pi i k_2/2} \\
&= (-8\ln 4 + 4\ln 2) = -10\log(4) \\
&= -2(k+2)\ln 4.
\end{aligned} \tag{A.12}$$

In case of $6A$ we have,

$$\begin{aligned}
C &= \sum_{r=1}^{5} \sum_s 2 \sum_{k_2 \in \mathbb{Z}, k_2 > 0, r \neq 0} \frac{2}{k_2} e^{2\pi i k_2 r/6} + 2 \sum_{k_2 \in \mathbb{Z}, k_2 > 0} \frac{2}{k_2} e^{2\pi i k_2/3} + 2 \sum_{k_2 \in \mathbb{Z}, k_2 > 0} \frac{2}{k_2} e^{2\pi i k_2/2} \\
&= -(4\ln 6 + 4\ln 3 + 4\ln 2) = -8\ln(6) \\
&= -2(k+2)\ln 6.
\end{aligned} \tag{A.13}$$

Similarly for $8A$ orbifold we get,

$$\begin{aligned}
C &= \sum_{r=1}^{7} \sum_s 2 \sum_{k_2 \in \mathbb{Z}, k_2 > 0, r \neq 0} \frac{2}{k_2} e^{2\pi i k_2 r/6} + \sum_{k_2 \in \mathbb{Z}, k_2 > 0} \frac{2}{k_2} e^{2\pi i k_2/4} + \sum_{k_2 \in \mathbb{Z}, k_2 > 0} \frac{2}{k_2} e^{2\pi i k_2/2} \\
&= -(4\ln 8 + 2\ln 4 + 2\ln 2) = -6\ln(8) \\
&= -2(k+2)\ln 8.
\end{aligned} \tag{A.14}$$

Similarly one can perform this computation for the orbifolds $14A$ and $15A$ resulting in $C = -2(k+2)\ln 14$ and $C = -2(k+2)\ln 15$ respectively. It is interesting to note that though the sums initially seems different for all the different orbifolds, they all yield $C = -2(k+2)\ln N$. This concludes our derivation of the additional constant C in (2.35).

B   Mock modular forms

In this appendix we define the Mock modular forms taking the definitions from [11] as follows:

A weakly holomorphic pure mock modular form of weight $k \in \mathbb{Z}/2$ $h(\tau)$ is defined as:

1. $h(\tau)$ is a holomorphic function in $\mathbb{H}$ with at most exponential growth at all cusps.

Orbifold	Order	k	r	$\sum_{s=0}^{N-1} c^{(r,s)}(0)$
pA	p (prime)	$\frac{24}{p+1} - 2$	$r \neq 0$	$k+2$
4B	4	3	$r = 1,3$	4
			$r = 2$	6
6A	6	2	$r = 1,5$	2
			$r = 2,4$	4
			$r = 3$	4
8A	8	1	$r = 1,3,5,7$	2
			$r = 2,6$	3
			$r = 4$	4
14A	14	0	$r = 1,3,5,9,11,13$	1
			$r = 2,4,6,8,10,12$	2
			$r = 7$	2
15A	15	0	$r = 1,2,4,7,8,11,13,14$	1
			$r = 3,6,9,12$	2
			$r = 5,10$	2

Table 22. List of $c^{(r,s)}(0)$.

2. The function $g(\tau)$, called the shadow of h, is a holomorphic modular form of weight $2 - k$, and

3. the sum $\hat{h} := h + g^*$ is called the completion of h, which transforms like a holomorphic modular form of weight k for some congruent subgroup of $\mathrm{SL}(2, \mathbb{Z})$.

Here $g^*(\tau)$ is the non-holomorphic Eichler integral which is the solution of the differential equation

$$(4\pi\tau_2)^k \partial_{\bar{\tau}} g^*(\tau) = -2\pi i \overline{g(\tau)} \tag{B.1}$$

The simplest examples we encounter in the Fourier-Jacobi expansions of Siegel modular forms is $E_2(\tau)$ which is mock modular and its completion $\hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi\tau_2}$. In this case we have

$$h(\tau) = E_2(\tau), \qquad g^*(\tau) = -\frac{3}{\pi\tau_2}, \qquad \hat{h}(\tau) = \hat{E}_2(\tau). \tag{B.2}$$

Computing $g(\tau)$ for $E_2(\tau)$ we get -12 ie, a constant. Now if we replace τ with $N\tau$ we would get,

$$h(N\tau) = E_2(N\tau), \qquad g^*(N\tau) = -\frac{3}{\pi N\tau_2}, \qquad \hat{h}(N\tau) = \hat{E}_2(N\tau). \tag{B.3}$$

Now since $\hat{h}(\tau)$ is a modular form of $\mathrm{SL}(2, \mathbb{Z})$ we have under a $\Gamma_0(N)$ transformation:

$$\hat{h}(N\tau) \to \hat{h}\left(N\frac{a\tau + b}{cN\tau + d}\right) \quad \text{with } ad - bcN = 1, \tag{B.4}$$

where $a, b, c, d \in \mathbb{Z}$.

$$\hat{h}\left(N\frac{a\tau + b}{cN\tau + d}\right) = \hat{h}\left(\frac{aN\tau + bN}{cN\tau + d}\right) = \hat{h}\left(\frac{a\tau' + bN}{c\tau' + d}\right), \tag{B.5}$$
$$= (c\tau' + d)^k \hat{h}(\tau') = (cN\tau + d)^k \hat{h}(N\tau),$$

with $ad - bcN = 1$, $\tau' = N\tau$. Therefore $\hat{h}(N\tau)$ is a modular form under $\Gamma_0(N)$. This observation can be easily generalized if $\hat{h}(\tau)$ itself is a modular form under subgroups of SL$(2, \mathbb{Z})$. For instance if $\hat{h}(\tau)$ is a modular form under $\Gamma_0(n)$, then $\hat{h}(\tau)$ is a modular form under $\Gamma_0(nN)$. The second example we deal with in this paper is the generating function of Hurwitz-Kronecker class numbers $H(\tau)$ which is a mock modular form under $\Gamma_0(4)$. Its shadow function is given by $\theta_3(2\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}$. Therefore $H(N\tau)$ is a mock modular form under $\Gamma_0(4N)$ and its shadow function is $\theta_3(2N\tau)$.

Jacobi forms. A Jacobi form $F(\tau, z)$ of weight k and index m is defined with the following transformation properties:

$$F\left(\frac{a\tau + b}{c\tau + d}, \frac{\nu}{c\tau + d}\right) = (c\tau + d)^k \, e^{\frac{2\pi imc\nu}{c\tau + d}} F(\tau, \nu), \tag{B.6}$$

$$F(\tau, \nu + \lambda\tau + \mu) = e^{-2\pi im(\lambda^2\tau + 2\lambda\nu)} F(\tau, \nu), \tag{B.7}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ SL$(2, \mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z}^2$

Mock Jacobi forms. By a (pure) mock Jacobi form of weight k and index m we denote a holomorphic function ϕ on $\mathbb{H} \times C$ that satisfies the elliptic transformation property (B.7). A few specific properties follow for the Jacobi forms. If F be a Jacobi form of weight k and index m then we have,

$$F(\tau, z) = \sum_{n,r} c(4mn - r^2) q^n z^r \tag{B.8}$$

A weakly holomorphic Jacobi form satisfies $c(4mn - r^2) = 0$, if $4mn - r^2 < n_0$, if $m = 1$ this is -1. The Jacobi forms also satisfy:

$$F(\tau, z) = \sum_{l \in \mathbb{Z}} q^{l^2/4m} h_l(\tau) z^l \tag{B.9}$$

Also this can be written as:

$$F(\tau, z) = \sum_{l \in \mathbb{Z}/2m\mathbb{Z}} h_l(\tau) \theta_{m,l}(\tau, z), \tag{B.10}$$

where $\theta_{m,l}(\tau, z) = \sum_{\substack{r \in \mathbb{Z} \\ r = l \mod 2m}} q^{r^2/4m} z^r$. For a mock Jacobi form $\phi(\tau, z)$ the modular property (B.6) is weakened. However the completed function $\hat{\phi}$ satisfies (B.6).

$$\hat{\phi}(\tau, z) = \phi(\tau, z) + \sum_{l \in \mathbb{Z}/2m\mathbb{Z}} g_l^*(\tau) \theta_{m,l}(\tau, z), \tag{B.11}$$

with g_l^* being the corresponding Eichler integral.

Since $\hat{\phi}(\tau, z)$ satisfies (B.6) for the full $\text{SL}(2, \mathbb{Z})$ it is easy to show that $\hat{\phi}(N\tau, z)$ obeys the transformation (B.6) for $\Gamma_0(N)$. This is proved as follows. Under a modular transformation of $\Gamma_0(N)$

$$\hat{\phi}(N\tau, \nu) \to \hat{\phi}\left(N\frac{a\tau + b}{cN\tau + d}, \frac{\nu}{cN\tau + d}\right) \tag{B.12}$$

where $ad - bcN = 1$. Now we know, from (B.6) that if $a'd' - b'c' = 1$, $a', b', c', d' \in \mathbb{Z}$ then,

$$\hat{\phi}\left(\frac{a'\tau' + b'}{c'\tau' + d'}, \frac{\nu}{c'\tau' + d'}\right) = (c'\tau' + d')^k e^{\frac{2\pi i m' c' \nu}{c'\tau' + d'}} \hat{\phi}(\tau', \nu). \tag{B.13}$$

Putting $\tau' = N\tau$ in the above equation and choosing $a' = a, b' = bN, c' = c, d' = d$ the above equation becomes:

$$\hat{\phi}\left(\frac{aN\tau + bN}{cN\tau + d}, \frac{\nu}{cN\tau + d}\right) = (cN\tau + d)^k e^{\frac{2\pi i m' c \nu}{cN\tau + d}} \hat{\phi}(N\tau, \nu) \tag{B.14}$$

$$= (cN\tau + d)^k e^{\frac{2\pi i m N c \nu}{cN\tau + d'}} \hat{\phi}(N\tau, \nu), \quad \text{if } m' = Nm.$$

Therefore we conclude that $\hat{\phi}(N\tau, \nu)$ is a Jacobi form of index m'/N and weight k under $\Gamma_0(N)$ if $\hat{\phi}(\tau, \nu)$ is a Jacobi form of index m' and weight k.

This implies that completion of $\mathcal{H}(N\tau, z)$ denoted by $\hat{\mathcal{H}}(N\tau, z)$ will transforms as:

$$\hat{\mathcal{H}}\left(N\frac{a\tau + b}{cN\tau + d}, \frac{\nu}{cN\tau + d}\right) = (cN\tau + d)^2 e^{2\pi i \frac{c\nu}{cN\tau + d}} \hat{\mathcal{H}}(N\tau), \tag{B.15}$$

where, $ad - bcN = 1$ and $a, b, c, d \in \mathbb{Z}$. So it is a Jacobi form under $\Gamma_0(N)$ with weight 2 and index $1/N$. The shadow for $\mathcal{H}(N\tau, z)$ will be the same as for $\mathcal{H}(\tau, z)$, with $\tau \to N\tau$. The shadow of $\mathcal{H}(\tau, z)$ is given in [11].

Remarks. The above analysis both in case of Mock modular and Mock Jacobi forms would go through if $b \in \mathbb{Z}/N$ and $ad - bcN = 1$ and $a'd' - b'c'N = 1$ and also with further restrictions on b, b' being integer multiples of M/N and c, c' being integer multiples of some integer M.

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