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Infrared renormalon in $SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$

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We study the infrared renormalon in the gluon condensate in the $SU(N)$ gauge theory with n_W-flavor adjoint Weyl fermions (QCD(adj.)) on $\mathbb{R}^3 \times S^1$ with the $\mathbb{Z}_N$ twisted boundary conditions. We rely on the so-called large-β_0 approximation as a conventional tool to analyze the renormalon, in which only Feynman diagrams that dominate in the large-n_W limit are considered, while the coefficient of the vacuum polarization is set by hand to the one-loop beta function $\beta_0 = 11/3 - 2n_W/3$. In the large N limit within the large-β_0 approximation, the W-boson, which acquires the twisted Kaluza–Klein momentum, produces the renormalon ambiguity corresponding to the Borel singularity at $u = 2$. This provides an example that the system in the compactified space $\mathbb{R}^3 \times S^1$ possesses the renormalon ambiguity identical to that in the uncompactified space $\mathbb{R}^4$. We also discuss the subtle issue that the location of the Borel singularity can change depending on the order of two necessary operations.

Subject Index B00, B06, B32

1. Introduction

In the context of the resurgence program of asymptotically free quantum field theories (for a review, see Ref. [1] and the references cited therein), the interesting possibility has been suggested that the ambiguity in perturbation theory caused by the infrared (IR) renormalon [2,3]—a class of Feynman diagrams whose amplitude grows factorially as a function of the order of perturbation theory—is cancelled by the instability associated with a semi-classical object called a bion [4–7]. This is analogous to the cancellation mechanism between the ambiguity in perturbation theory around the trivial vacuum caused by the proliferation of the number of Feynman diagrams and the instability associated with an instanton–anti-instanton pair [8,9]. This possibility is very intriguing because no one clearly knows what kind of non-perturbative effect cancels the IR renormalon ambiguity. For a fully semi-classical understanding of the physics of asymptotically free quantum field theories along the resurgence program, it appears essential to introduce a certain high-energy scale such as a compactification radius of spacetime (see, for instance, Ref. [10]). Thus, to reinforce the above picture on the IR renormalon, the understanding of the IR renormalon in a compactified space such as $\mathbb{R}^{D-1} \times S^1$ is a basic premise.

The above picture has been examined fairly well in the two-dimensional (2D) supersymmetric $\mathbb{C}P^{N-1}$ model [11] defined on $\mathbb{R} \times S^1$ with the $\mathbb{Z}_N$ twisted boundary conditions [6,7]. In particular, in Ref. [12], one-loop quantum corrections around the bion configuration [13–18] are explicitly computed and the associated ambiguities are obtained, where the integration of the one-loop effective action over quasi-collective coordinates is carried out [19] following the Lefschetz thimble method [20–22]. In a recent paper [23], on the other hand, the IR renormalon ambiguity in the gluon condensate was determined in the leading order of the large-N approximation [24]. A very explicit calculation shows that a Borel singularity at $u = 2$ (see below for this notion), which corresponds to the IR renormalon in $\mathbb{R}^2$, disappears for $\mathbb{R} \times S^1$. Instead of this, in the system on $\mathbb{R} \times S^1$, an unfamiliar renormalon singularity at $u = 3/2$ emerges. This is an unexpected result because the IR renormalon singularity to be cancelled by the semi-classical bion has been considered as the $u = 2$ one. The observation in Ref. [23] thus raises a question in the above semi-classical picture on the IR renormalon.

As indicated in Ref. [23] and further discussed in Ref. [25], the disappearance of the $u = 2$ singularity and the emergence of the $u = 3/2$ singularity can be understood as a "shift" of the renormalon singularity under the compactification $\mathbb{R}^D \to \mathbb{R}^{D-1} \times S^1$. Moreover, it can be seen that this is a very general phenomenon; it generally occurs provided that the *integrand* of the momentum integral in the "renormalon diagram" for $\mathbb{R}^{D-1} \times S^1$ is identical to that for $\mathbb{R}^D$, and that the Kaluza–Klein (KK) loop momentum is not associated with the twisted boundary conditions; see below. The 2D supersymmetric $\mathbb{C}P^{N-1}$ model in the large-N limit satisfies these prerequisites. See also Refs. [26–28] for a related "volume independence" property.

With the above observations, it is natural to repeat a similar analysis in 4D gauge theories, in which the low-energy dynamics has been vigorously studied aiming at a fully semi-classical understanding [10,28–48]. This is the motivation of the present paper. We will study the IR renormalon in the gluon condensate in the $SU(N)$ gauge theory with n_W-flavor adjoint Weyl fermions (QCD(adj.)) on $\mathbb{R}^3 \times S^1$ with the $\mathbb{Z}_N$ twisted boundary conditions. Unlike the 2D $\mathbb{C}P^{N-1}$ model considered in Ref. [23], this system is much difficult to analyze and does not allow a systematic treatment to study the renormalon. So, in this paper we rely on the so-called large-β_0 approximation, a somewhat ad hoc but widely adopted prescription in studies of the renormalon in 4D gauge theories [2,49–51]. In the large-β_0 approximation, only Feynman diagrams that dominate in the large-n_W limit are considered, while the coefficient of the vacuum polarization is set by hand to the one-loop coefficient of the beta function (of the 't Hooft coupling, see below),

$$\beta_0 = \frac{11}{3} - \frac{2}{3}n_W. \tag{1.1}$$

Despite the fact that the large-β_0 approximation is not a systematic approach, this method is considered to be qualitatively reliable in gauge theories on $\mathbb{R}^4$ because the renormalon ambiguity obtained in this approximation often has the same order of magnitude as the expected non-perturbative effects (appearing in the context of the operator product expansion).

In this large-β_0 approximation, we compute the one-loop effective action to the quadratic order in the gauge field in a closed form for general N. From this, we obtain the gauge field propagator and then compute the gluon condensate. The resulting expression is still rather complicated for explicit analyses. Therefore, we further take an $N \to \infty$ limit[1] while the 't Hooft coupling $\lambda = g^2 N$, where g denotes the conventional gauge coupling, and the one-loop dynamical scale,

$$\Lambda \equiv \mu e^{-8\pi^2/(\beta_0 \lambda)}, \tag{1.2}$$

[1] This is not the genuine large-N limit because we are working within the large-β_0 approximation which extracts a portion of the full set of Feynman diagrams.

are kept fixed (here, μ is the renormalization scale). The S^1 radius R is also kept fixed in this limit,

$$\Lambda R = \text{const. as } N \to \infty. \tag{1.3}$$

Then, we can show that terms peculiar to the compactified space $\mathbb{R}^3 \times S^1$ in the gauge field propagator are suppressed. This feature allows simpler analyses.

In this paper we adopt the following definitions in studying a factorially divergent series. For the perturbative series of a quantity $f(\lambda)$ in the form

$$f(\lambda) \sim \lambda \sum_{k=0}^{\infty} f_k \left(\frac{\beta_0 \lambda}{16\pi^2}\right)^k, \tag{1.4}$$

we define the Borel transform by

$$B(u) \equiv \sum_{k=0}^{\infty} \frac{f_k}{k!} u^k. \tag{1.5}$$

Then the Borel sum is defined by[2]

$$f(\lambda) \equiv \frac{16\pi^2}{\beta_0} \int_0^{\infty} du\, B(u)\, e^{-16\pi^2 u/(\beta_0 \lambda)}. \tag{1.6}$$

If the perturbative coefficient f_k in Eq. (1.4) grows factorially, $f_k \sim b^{-k} k!$ as $k \to \infty$, the Borel transform $B(u)$ in Eq. (1.5) develops a singularity at $u = b$. If this singularity is on the positive real u-axis (i.e. $b > 0$), the Borel integral in Eq. (1.6) becomes ill-defined and produces an ambiguity proportional to $\sim e^{-16\pi^2 b/(\beta_0 \lambda)} \propto \Lambda^{2b}$. In this convention, the IR renormalon in the large-β_0 approximation produces Borel singularities at positive integers $u = 1, 2, \ldots$ for $\mathbb{R}^4$. On the other hand, since the classical action of the bion is $16\pi^2/\lambda$ (when the constituent monopole-instanton and anti-monopole-instanton are infinitely separated), the instability associated with the bion configuration would produce singularities at $u = n\beta_0$ with integer n. Although, as it stands, this does not coincide with the renormalon singularity for $\mathbb{R}^4$, it is conjectured [4] that quantum corrections shift the bion contribution to $u = 1, 2, \ldots$

In our analysis of the the gluon condensate in QCD(adj.) on $\mathbb{R}^3 \times S^1$ (with the large-N limit within the large-β_0 approximation, as explained above), we find that the gluon condensate suffers from the IR renormalon corresponding to the Borel singularity at $u = 2$. The position of the singularity is *identical* to that of the system on $\mathbb{R}^4$. Thus, the present system exhibits a completely different property from the systems studied in Refs. [23,25], where the Borel singularity at $u = 2$ for $\mathbb{R}^2$ is shifted to $u = 3/2$ for $\mathbb{R} \times S^1$. This difference from the case of Refs. [23,25] is attributed to the W-boson in the present system, which acquires the twisted KK momentum as a consequence of the twisted boundary condition; the twisted KK momentum is essential to keep the position of the Borel singularity unchanged, as we shall see.

[2] In the 2D supersymmetric $\mathbb{C}P^{N-1}$ model, we adopt the convention where the Borel integral is given by [23]

$$f(\lambda) \equiv 4\pi \int_0^{\infty} du\, B(u)\, e^{-4\pi u/\lambda},$$

such that the $u = 2$ renormalon singularity also corresponds to twice the bion action in the two-dimensional spacetime. (We note that $\beta_0 = 1$ for the 2D supersymmetric $\mathbb{C}P^{N-1}$ model.)

To investigate the Borel singularity in the system on $\mathbb{R}^3 \times S^1$, in fact, a careful treatment is required concerning how to take the large-N limit for the perturbative series and the Borel transform. We find that the result is sensitive to the order of the two operations: taking the large-N limit and the construction of the Borel transform. The order of these operations is not commutable: exchanging the order can lead to a completely opposite conclusion, i.e. the emergence of the $u = 3/2$ renormalon and the disappearance of the $u = 2$ renormalon.[3] Since we are interested in the divergence of the perturbative series in the system in the large-N limit, we should first obtain the perturbative coefficients with the large-N limit and then construct the Borel transform. This procedure leads to the above conclusion that the $u = 2$ renormalon exists.

Reference [52] is a preceding study on the IR renormalon in the $SU(N)$ QCD(adj.) in $\mathbb{R}^3 \times S^1$ with the $\mathbb{Z}_N$ twisted boundary conditions. There, for $N = 2$ and $N = 3$, the authors observed that the one-loop vacuum polarization of the photon—the gauge field associated with the Cartan subalgebra—does not have the logarithmic factor $\sim \ln p^2$. Since the IR renormalon is usually attributed to the existence of this logarithmic factor, the authors concluded that there are no IR renormalons in the $SU(N)$ QCD(adj.) in the compactified space $\mathbb{R}^3 \times S^1$ (at least for $N = 2$ and $N = 3$). This is not directly inconsistent with our result in the present paper, because we consider the large-N limit of Eq. (1.3). Since we observe in this paper that the one-loop vacuum polarization acquires the logarithmic factor $\sim \ln p^2$, the contribution of the compactification should depend on N. It must be interesting to clarify how the contribution of the compactification depends on N and how small-N cases and the large-N case are connected.

This paper is organized as follows. In Sect. 2 we compute the one-loop effective action to the quadratic order of the gauge field. We first compute the contribution of the adjoint Weyl fermions. The twisted boundary conditions, or equivalently the presence of a constant gauge potential, give rise to some complications. Then, invoking the large-β_0 approximation, we obtain the one-loop effective action and the gauge field propagator. In Sect. 3, using the gauge field propagator, we compute the gluon condensate. We then determine the perturbative coefficients for the gluon condensate in the large-N limit. The corresponding Borel transform is then constructed. We present these calculations for the photon and the W-boson parts separately. We will also illustrate how the change of ordering of the large-N limit and the construction of the Borel transform completely changes the conclusion. Section 4 is devoted to our conclusions. In Appendix A we summarize our convention on the $SU(N)$ generators, which is required for the computation in Sect. 2. In Appendix B we give a proof of the bounds that are crucial for the above large-N limit.

2. One-loop effective action in the large-β_0 approximation

2.1. *Action and boundary conditions*

We assume that the spacetime is $\mathbb{R}^3 \times S^1$ and the radius of S^1 is R. The coordinates of $\mathbb{R}^3$ are (x_0, x_1, x_2), and that of S^1 is x_3; thus, $0 \leq x_3 < 2\pi R$. The Euclidean action of the $SU(N)$ QCD(adj.) is given by

$$S = -\frac{N}{2\lambda_0}\int d^4x \ \mathrm{tr}\left[\tilde{F}_{\mu\nu}(x)\tilde{F}_{\mu\nu}(x)\right] - 2\int d^4x \ \mathrm{tr}\left\{\bar{\tilde{\psi}}(x)\gamma_\mu\left[\partial_\mu\tilde{\psi}(x) + [\tilde{A}_\mu(x), \tilde{\psi}(x)]\right]\right\}. \quad (2.1)$$

[3] This was the conclusion in the first version of the present paper. We now consider that there are some problems with this conclusion, as discussed in Sect. 3.

(The fields with a tilde are subject to the twisted boundary conditions, as explained shortly.) Here, we have used the matrix notation with which $\tilde{A}_\mu(x) = -i\tilde{A}^a_\mu(x)T^a$, $\tilde{\psi}(x) = -i\tilde{\psi}^a(x)T^a$, and $\tilde{\bar{\psi}}(x) = -i\tilde{\bar{\psi}}^a(x)T^a$, where T^a are Hermitian $SU(N)$ generators in the fundamental representation, assuming the normalization $\mathrm{tr}(T^aT^b) = (1/2)\delta^{ab}$. Our convention for the $SU(N)$ generators is summarized in Appendix A. The field strength is defined by

$$\tilde{F}_{\mu\nu}(x) = \partial_\mu\tilde{A}_\nu(x) - \partial_\nu\tilde{A}_\mu(x) + [\tilde{A}_\mu(x), \tilde{A}_\nu(x)], \tag{2.2}$$

and λ_0 is the bare 't Hooft coupling that is related to the bare gauge coupling g_0 by $\lambda_0 = g_0^2 N$. $\tilde{\psi}(x)$ and $\tilde{\bar{\psi}}(x)$ are n_W-flavor Weyl fermions and the summation over the flavor index is suppressed for simplicity.

We assume that along S^1 the above fields with a tilde obey the following $\mathbb{Z}_N$-invariant twisted boundary conditions:

$$\begin{aligned}
\tilde{\psi}(x_0, x_1, x_2, x_3 + 2\pi R) &= \Omega\tilde{\psi}(x_0, x_1, x_2, x_3)\Omega^{-1}, \\
\tilde{\bar{\psi}}(x_0, x_1, x_2, x_3 + 2\pi R) &= \Omega\tilde{\bar{\psi}}(x_0, x_1, x_2, x_3)\Omega^{-1}, \\
\tilde{A}_\mu(x_0, x_1, x_2, x_3 + 2\pi R) &= \Omega\tilde{A}_\mu(x_0, x_1, x_2, x_3)\Omega^{-1},
\end{aligned} \tag{2.3}$$

where the $SU(N)$ element Ω is defined by

$$\Omega = e^{i\frac{2\pi}{N}\phi\cdot H}. \tag{2.4}$$

H denotes the $SU(N)$ Cartan generator and the vector ϕ is given by

$$\phi_m \equiv 2\sum_{j=1}^{N-1}(\mu^j)_m, \tag{2.5}$$

from the fundamental weights $\mu^j = \sum_{k=1}^{j}\nu^k$ (here, ν^k are weights). See Appendix A. One can confirm that Ω is a diagonal matrix with the diagonal elements

$$e^{i\pi\frac{N+1}{N}}e^{-i\frac{2\pi}{N}j}, \qquad j = 1, 2, \ldots, N. \tag{2.6}$$

Since these diagonal elements are equally placed on the unit circle, the trace of Ω is invariant under the multiplication of the $\mathbb{Z}_N$ center element, i.e.

$$\mathrm{tr}(e^{i\frac{2\pi}{N}}\Omega) = \mathrm{tr}\,\Omega. \tag{2.7}$$

This is the origin of the name of the boundary conditions in Eq. (2.3).

Instead of the above field variables with the twisted boundary conditions, it is often convenient to use field variables which are periodic along S^1. This can be accomplished by substituting

$$\begin{aligned}
\tilde{\psi}(x_0, x_1, x_2, x_3) &= \Omega^{x_3/(2\pi R)}\psi(x_0, x_1, x_2, x_3)\Omega^{-x_3/(2\pi R)}, \\
\tilde{\bar{\psi}}(x_0, x_1, x_2, x_3) &= \Omega^{x_3/(2\pi R)}\bar{\psi}(x_0, x_1, x_2, x_3)\Omega^{-x_3/(2\pi R)}, \\
\tilde{A}_\mu(x_0, x_1, x_2, x_3) &= \Omega^{x_3/(2\pi R)}A_\mu(x_0, x_1, x_2, x_3)\Omega^{-x_3/(2\pi R)},
\end{aligned} \tag{2.8}$$

where the variables on the right-hand side (without a tilde) are periodic along S^1. Note that under this substitution, the derivative acting on the original variables with a tilde is translated into the covariant derivative with respect to a constant gauge potential on the periodic field variables:

$$\partial_\mu \to D_\mu^{(0)} \equiv \partial_\mu + [A_\mu^{(0)}, \,], \qquad A_\mu^{(0)} \equiv i\frac{1}{RN}\phi \cdot H\delta_{\mu 3}. \tag{2.9}$$

2.2. *Action in terms of component fields*

In what follows, we first extensively use the field variables that are periodic in S^1, the field variables on the right-hand side of Eq. (2.8). We decompose these field variables in the Cartan–Weyl basis as

$$\begin{aligned}
\psi(x) &= -i\sum_{\ell=1}^{N-1}\psi^\ell(x)H_\ell - i\sum_{m\neq n}\psi^{mn}(x)E_{mn},\\
\bar{\psi}(x) &= -i\sum_{\ell=1}^{N-1}\bar{\psi}^\ell(x)H_\ell - i\sum_{m\neq n}\bar{\psi}^{mn}(x)E_{mn},\\
A_\mu(x) &= -i\sum_{\ell=1}^{N-1}A_\mu^\ell(x)H_\ell - i\sum_{m\neq n}A_\mu^{mn}(x)E_{mn}.
\end{aligned} \tag{2.10}$$

See Appendix A for our convention on the $SU(N)$ generators. Throughout this paper, the gauge field $A_\mu^\ell(x)$ is referred to as the "photon" and $A_\mu^{mn}(x)$ as the "W-boson." Then, by using the relations in Eqs. (A.3), (A.5), and

$$\phi \cdot (\nu^m - \nu^n) = 2\sum_{j=1}^{N-1}(\mu^j)\cdot(\nu^m - \nu^n) = -(m-n), \tag{2.11}$$

which follows from Eq. (A.4), the action in Eq. (2.1) in terms of the periodic component fields is given by

$$\begin{aligned}
S = \frac{N}{4\lambda_0}\int d^4x \Bigg\{&\Big[\partial_\mu A_\nu^\ell - \partial_\nu A_\mu^\ell - iA_\mu^{mn}A_\nu^{nm}(\nu^m - \nu^n)_\ell\Big]\\
&\qquad\times\Big[\partial_\mu A_\nu^\ell - \partial_\nu A_\mu^\ell - iA_\mu^{pq}A_\nu^{qp}(\nu^p - \nu^q)_\ell\Big]\\
&+\Bigg[\left(\partial_\mu - i\delta_{\mu 3}\frac{m-n}{RN}\right)A_\nu^{mn} - \left(\partial_\nu - i\delta_{\nu 3}\frac{m-n}{RN}\right)A_\mu^{mn}\\
&\quad - i(A_\mu^\ell A_\nu^{mn} - A_\mu^{mn}A_\nu^\ell)(\nu^m - \nu^n)_\ell - i\frac{1}{\sqrt{2}}(A_\mu^{m\ell}A_\nu^{\ell n} - A_\mu^{\ell n}A_\nu^{m\ell})\Bigg]\\
&\quad\times\Bigg[\left(\partial_\mu - i\delta_{\mu 3}\frac{n-m}{RN}\right)A_\nu^{nm} - \left(\partial_\nu - i\delta_{\nu 3}\frac{n-m}{RN}\right)A_\mu^{nm}\\
&\qquad - i(A_\mu^p A_\nu^{nm} - A_\mu^{nm}A_\nu^p)(\nu^n - \nu^m)_p - i\frac{1}{\sqrt{2}}(A_\mu^{np}A_\nu^{pm} - A_\mu^{pm}A_\nu^{np})\Bigg]\Bigg\}\\
+\int d^4x \Bigg\{&\bar{\psi}^m \partial\!\!\!/ \psi^m + \bar{\psi}^{mn}\left(\partial\!\!\!/ - i\gamma_3\frac{n-m}{RN}\right)\psi^{nm}\\
&- i\bar{\psi}^{mn}\left[-A\!\!\!/^\ell(\nu^m - \nu^n)_\ell\delta^{mq}\delta^{np} + \frac{1}{\sqrt{2}}(A\!\!\!/^{np}\delta^{mq} - A\!\!\!/^{qm}\delta^{np})\right]\psi^{pq}\\
&- i\left[\bar{\psi}^{mn}A\!\!\!/^{nm}\psi^\ell(\nu^m - \nu^n)_\ell + \bar{\psi}^\ell A\!\!\!/^{mn}\psi^{nm}(\nu^m - \nu^n)_\ell\right]\Bigg\},
\end{aligned} \tag{2.12}$$

where we have taken the shift in Eq. (2.9) into account. To this gauge-invariant action we add the gauge-fixing term,

$$S_{\rm gf} = -\frac{N\xi_0}{\lambda_0}\int d^4x\ \mathrm{tr}\left[D_\mu^{(0)}A_\mu(x)D_\nu^{(0)}A_\nu(x)\right]$$
$$= \frac{N\xi_0}{2\lambda_0}\int d^4x\left[\partial_\mu A_\mu^\ell\partial_\nu A_\nu^\ell + \left(\partial_\mu - i\delta_{\mu 3}\frac{m-n}{RN}\right)A_\mu^{mn}\left(\partial_\nu - i\delta_{\nu 3}\frac{n-m}{RN}\right)A_\nu^{nm}\right], \qquad (2.13)$$

where ξ_0 is the bare gauge-fixing parameter and $D_\mu^{(0)}$ is the covariant derivative in Eq. (2.9). Then, from the quadratic part of the action $S + S_{\rm gf}$, we have free propagators of the periodic fields,

$$\left\langle A_\mu^m(x)A_\nu^n(y)\right\rangle_0 = \frac{\lambda_0}{N}\delta^{mn}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}e^{ip(x-y)}\frac{1}{(p^2)^2}\left[(\delta_{\mu\nu}p^2 - p_\mu p_\nu) + \frac{1}{\xi_0}p_\mu p_\nu\right],$$

$$\left\langle A_\mu^{mn}(x)A_\nu^{pq}(y)\right\rangle_0$$
$$= \frac{\lambda_0}{N}\delta^{mq}\delta^{np}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}e^{ip(x-y)}\frac{1}{(p_{mn}^2)^2}\left[(\delta_{\mu\nu}p_{mn}^2 - p_{mn,\mu}p_{mn,\nu}) + \frac{1}{\xi_0}p_{mn,\mu}p_{mn,\nu}\right], \qquad (2.14)$$

and

$$\left\langle\psi^m(x)\bar\psi^n(y)\right\rangle_0 = \delta^{mn}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}e^{ip(x-y)}\frac{1}{i\not p},$$
$$\left\langle\psi^{mn}(x)\bar\psi^{pq}(y)\right\rangle_0 = \delta^{mq}\delta^{np}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}e^{ip(x-y)}\frac{1}{i\not p_{mn}}. \qquad (2.15)$$

In these expressions, p_3 denotes the KK momentum along S^1, and thus

$$p_3 = \frac{n}{R}, \qquad n\in\mathbb{Z}. \qquad (2.16)$$

Also, we have introduced the *twisted momentum*,

$$p_{mn,\mu} \equiv p_\mu - \delta_{\mu 3}\frac{m-n}{RN}. \qquad (2.17)$$

Note that the field components corresponding to the Cartan subalgebra do not refer to the twisted momentum.

2.3. *One-loop effective action*

We now compute the vacuum polarization arising from one-loop radiative corrections of the adjoint Weyl fermions.[4] This amounts to the computation of the one-loop effective action of the gauge field to the quadratic order arising from the Gaussian integration of the adjoint fermions.

Let us start with computing the part of the one-loop effective action that contains $A_\mu^\ell(x)A_\nu^r(y)$. From the interaction terms in Eq. (2.12) and the free propagator in Eq. (2.15), we have

$$\boldsymbol{\Gamma}^{(1)} = -\frac{1}{2}\int d^4x\,d^4y\,A_\mu^\ell(x)A_\nu^r(y)\sum_{\substack{m\neq n\\1\leq m,n\leq N}}(\nu^m-\nu^n)_\ell(\nu^m-\nu^n)_r\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}e^{-ip(x-y)}$$

[4] This calculation is required to construct the large-β_0 approximation.

$$\times \frac{n_W}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\pi R}\sum_{k_3}\mathrm{tr}\Bigg[\frac{1}{i\not k - i\gamma_3(n-m)/(RN)}\gamma_\mu \times \frac{1}{i(\not k-\not p)-i\gamma_3(n-m)/(RN)}\gamma_\nu\Bigg]+\cdots. \tag{2.18}$$

To this, we apply the identity

$$\sum_{j=-\infty}^{\infty} e^{ik_3 2\pi Rj} = \frac{1}{R}\sum_{j=-\infty}^{\infty}\delta(k_3 - j/R), \tag{2.19}$$

or

$$\frac{1}{2\pi R}\sum_{j=-\infty}^{\infty}F(j/R) = \sum_{j=-\infty}^{\infty}\int\frac{dk_3}{2\pi}e^{ik_3 2\pi Rj}F(k_3), \tag{2.20}$$

to make the sum over k_3 the integrals $\int dk_3$. After this, we can shift the momentum variable as $k_3 \to k_3 + (n-m)/(RN)$. Then, the trace over the Dirac indices yields

$$\begin{aligned}\boldsymbol{\Gamma}^{(1)} = n_W\int d^4x\,d^4y\,A^\ell_\mu(x)A^r_\nu(y)\sum_{\substack{m\neq n\\1\le m,n\le N}}(\nu^m-\nu^n)_\ell(\nu^m-\nu^n)_r\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}e^{-ip(x-y)}\\ \times\sum_{j=-\infty}^{\infty}e^{i(n-m)2\pi j/N}\int\frac{d^4k}{(2\pi)^4}e^{ik_3 2\pi Rj}\\ \times\int_0^1 dx\,\frac{1}{(k^2-2xkp+xp^2)^2}\left[2k_\mu k_\nu - k_\mu p_\nu - p_\mu k_\nu - k(k-p)\delta_{\mu\nu}\right]+\cdots.\end{aligned} \tag{2.21}$$

The summation over m and n in this expression can be carried out by using Eq. (A.1) as

$$\begin{aligned}(\sigma_{j,N})_{\ell r} &\equiv \frac{1}{N}\sum_{\substack{m\neq n\\1\le m,n\le N}}(\nu^m-\nu^n)_\ell(\nu^m-\nu^n)_r e^{i(n-m)2\pi j/N}\\ &=\begin{cases}\delta_{\ell r}, & \text{for } j=0 \bmod N,\\ -\frac{1}{N}\frac{1}{\sqrt{\ell(\ell+1)r(r+1)}}\mathrm{Re}\left[\left(\frac{e^{-i\ell 2\pi j/N}-1}{e^{-i2\pi j/N}-1}-\ell e^{-i\ell 2\pi j/N}\right)\left(\frac{e^{ir2\pi j/N}-1}{e^{i2\pi j/N}-1}-re^{ir2\pi j/N}\right)\right], & \text{for } j\neq 0 \bmod N.\end{cases}\end{aligned} \tag{2.22}$$

In Eq. (2.21), the term with $j=0$ is ultraviolet (UV) divergent while the terms with $j\neq 0$ are Fourier transforms and UV finite. We apply dimensional regularization to the former by setting $4\to D\equiv 4-2\varepsilon$. Then the result of the momentum integrations is

$$\begin{aligned}\boldsymbol{\Gamma}^{(1)} = \frac{1}{2}\frac{N}{16\pi^2}\frac{2}{3}n_W\int d^4x\,d^4y\,A^\ell_\mu(x)A^r_\nu(y)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}e^{-ip(x-y)}\\ \times\left\{\delta_{\ell r}(p^2\delta_{\mu\nu}-p_\mu p_\nu)\left[\frac{1}{\varepsilon}+\ln(4\pi e^{-\gamma_E})+\ln\left(\frac{e^{5/3}}{p^2}\right)\right]\right.\end{aligned}$$

$$
\begin{aligned}
&+12\sum_{j\neq 0}(\sigma_{j,N})_{\ell r}\int_0^1 dx\, e^{ixp_3 2\pi Rj}x(1-x)\\
&\times\left[(p^2\delta_{\mu\nu}-p_\mu p_\nu)K_0(z)-(p^2\delta_{\mu 3}\delta_{\nu 3}-p_\mu p_3\delta_{\nu 3}-p_\nu p_3\delta_{\mu 3}+p_3^2\delta_{\mu\nu})K_2(z)\right]\Bigg\}\\
&+\cdots, \qquad (2.23)
\end{aligned}
$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind[5] and

$$
z\equiv\sqrt{x(1-x)p^2}2\pi R|j|. \qquad (2.24)
$$

We can repeat a similar calculation for the term of the effective action containing the combination $A_\mu^{mn}(x)A_\nu^{pq}(y)$. After some calculation, using $(\nu^m-\nu^n)^2=1$ for any fixed $m\neq n$, we have

$$
\begin{aligned}
&\boldsymbol{\Gamma}^{(1)}\\
&=\frac{1}{2}\frac{N}{16\pi^2}\frac{2}{3}n_W\int d^4x\,d^4y\,A_\mu^\ell(x)A_\nu^r(y)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}e^{-ip(x-y)}\\
&\quad\times\left\{\delta_{\ell r}(p^2\delta_{\mu\nu}-p_\mu p_\nu)\left[\frac{1}{\varepsilon}+\ln(4\pi e^{-\gamma_E})+\ln\left(\frac{e^{5/3}}{p^2}\right)\right]\right.\\
&\qquad+12\sum_{j\neq 0}(\sigma_{j,N})_{\ell r}\int_0^1 dx\, e^{ixp_3 2\pi Rj}x(1-x)\\
&\qquad\times\left.\left[(p^2\delta_{\mu\nu}-p_\mu p_\nu)K_0(z)-(p^2\delta_{\mu 3}\delta_{\nu 3}-p_\mu p_3\delta_{\nu 3}-p_\nu p_3\delta_{\mu 3}+p_3^2\delta_{\mu\nu})K_2(z)\right]\right\}\\
&\quad+\frac{1}{2}\frac{N}{16\pi^2}\frac{2}{3}n_W\int d^4x\,d^4y\,A_\mu^{mn}(x)A_\nu^{nm}(y)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}e^{-ip(x-y)}\\
&\quad\times\left\{(p^2\delta_{\mu\nu}-p_\mu p_\nu)\left[\frac{1}{\varepsilon}+\ln(4\pi e^{-\gamma_E})+\ln\left(\frac{e^{5/3}}{p^2}\right)\right]\right.\\
&\qquad+12\sum_{j\neq 0, j=0 \bmod N}\int_0^1 dx\, e^{ixp_3 2\pi Rj}x(1-x)\\
&\qquad\times\left.\left[(p^2\delta_{\mu\nu}-p_\mu p_\nu)K_0(z)-(p^2\delta_{\mu 3}\delta_{\nu 3}-p_\mu p_3\delta_{\nu 3}-p_\nu p_3\delta_{\mu 3}+p_3^2\delta_{\mu\nu})K_2(z)\right]\right\}_{p\to p_{mn}}\\
&\quad+O(A^3). \qquad (2.25)
\end{aligned}
$$

[5] In obtaining Eq. (2.23), one may use the relations

$$
K_0'(z)=-K_1(z),\qquad K_0(z)-K_2(z)=-\frac{2}{z}K_1(z),
$$

and the relation following from integration by parts such as

$$
\int_0^1 dx\, e^{ixp_3 2\pi Rj}\left[2ip_3\frac{\sqrt{x(1-x)}}{\sqrt{p^2}2\pi R|j|}K_1(z)-\frac{1-2x}{2\pi Rj}K_0(z)\right]=0,
$$

because of $zK_1'(z)+K_1(z)=-zK_0(z)$.

The explicit form of this expression depends on the assignment of the loop momentum because the original integral in the $j = 0$ term is UV divergent. The different form corresponds to different regularization, and the difference can be removed by a local counterterm. In Eq. (2.25), we adopted a particular loop momentum assignment which leads to the simplest form. Note that inside the last parentheses in Eq. (2.25), the momentum p is replaced by p_{mn}, the twisted momentum defined by Eq. (2.17). A further calculation shows that no mixing term containing $A^{\ell}_{\mu}(x)A^{mn}_{\nu}(y)$ arises. Equation (2.25) thus gives the part of the effective action arising from one-loop radiative corrections of n_W Weyl fermions to the quadratic order in the gauge potential.

We now consider the large-flavor limit $n_W \to \infty$ (with the combination $g^2 n_W \propto \lambda n_W$ fixed), which is required as an intermediate step in constructing the large-β_0 approximation. In this approximation it is sufficient to consider only the fermion contribution to the effective action of the gauge field as above, because radiative corrections of the gauge field are subleading. In this way, we obtain the leading-order result of the one-loop effective action in the large-n_W limit as Eq. (2.25), whose result is gauge invariant. Furthermore, Eq. (2.25) is regarded as the same order as the classical action [the sum of Eqs. (2.12) and (2.13)] for the gauge field due to $\lambda n_W = O(1)$. From Eqs. (2.12), (2.13), and (2.25), we see that the effective action in this large-n_W limit,

$$S + S_{\mathrm{gf}} + \boldsymbol{\Gamma}^{(1)}, \tag{2.26}$$

is made finite by the following parameter renormalizations (in the $\overline{\mathrm{MS}}$ scheme):

$$\lambda_0 = \lambda\mu^{2\varepsilon}(4\pi e^{-\gamma_E})^{-\varepsilon}\mathcal{Z}^{-1}, \qquad \xi_0 = \xi\mathcal{Z}^{-1}, \qquad \mathcal{Z} = 1 + \frac{\lambda}{16\pi^2}\left(-\frac{2}{3}n_W\right)\frac{1}{\varepsilon}, \tag{2.27}$$

where μ is the renormalization scale and λ denotes the renormalized coupling at μ, i.e. $\lambda = \lambda(\mu)$.

In terms of these renormalized parameters, the effective action reads

$$\begin{aligned}
S + S_{\mathrm{gf}} + \boldsymbol{\Gamma}^{(1)} &= \frac{N}{2\lambda}\int d^4x\, d^4y\, A^{\ell}_{\mu}(x)A^{r}_{\nu}(y)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3} e^{-ip(x-y)} \\
&\times\Bigg\{\delta_{\ell r}(p^2\delta_{\mu\nu} - p_\mu p_\nu)\left[1 + \frac{\lambda}{16\pi^2}\frac{2}{3}n_W\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right)\right] + \delta_{\ell r}\xi p_\mu p_\nu \\
&\quad + \frac{\lambda}{16\pi^2}\frac{2}{3}n_W 12\sum_{j\neq 0}(\sigma_{j,N})_{\ell r}\int_0^1 dx\, e^{ixp_3 2\pi Rj}x(1-x) \\
&\quad \times\left[(p^2\delta_{\mu\nu} - p_\mu p_\nu)K_0(z) - (p^2\delta_{\mu 3}\delta_{\nu 3} - p_\mu p_3\delta_{\nu 3} - p_\nu p_3\delta_{\mu 3} + p_3^2\delta_{\mu\nu})K_2(z)\right]\Bigg\} \\
&+ \frac{N}{2\lambda}\int d^4x\, d^4y\, A^{mn}_{\mu}(x)A^{nm}_{\nu}(y)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3} e^{-ip(x-y)} \\
&\times\Bigg\{(p^2\delta_{\mu\nu} - p_\mu p_\nu)\left[1 + \frac{\lambda}{16\pi^2}\frac{2}{3}n_W\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right)\right] + \xi p_\mu p_\nu \\
&\quad + \frac{\lambda}{16\pi^2}\frac{2}{3}n_W 12\sum_{j\neq 0, j=0 \bmod N}\int_0^1 dx\, e^{ixp_3 2\pi Rj}x(1-x) \\
&\times\left[(p^2\delta_{\mu\nu} - p_\mu p_\nu)K_0(z) - (p^2\delta_{\mu 3}\delta_{\nu 3} - p_\mu p_3\delta_{\nu 3} - p_\nu p_3\delta_{\mu 3} + p_3^2\delta_{\mu\nu})K_2(z)\right]\Bigg\}_{p\to p_{mn}} \\
&+ O(A^3).
\end{aligned} \tag{2.28}$$

2.4. Large-β_0 approximation

Now, we consider the large-β_0 approximation, which is a somewhat ad hoc way to include radiative corrections of the gauge field. In this approximation, we first consider the large-n_W limit as above. By this, we obtained the gauge-invariant result of the one-loop effective action for the gauge field. However, the large-n_W limit breaks the asymptotic freedom and makes the contribution of the gauge field to the vacuum polarization sub-dominant. To remedy these points, the coefficient of the vacuum polarization is set by hand to the one-loop coefficient of the beta function of the 't Hooft coupling,

$$-\frac{2}{3}n_W \to \beta_0 = \frac{11}{3} - \frac{2}{3}n_W. \tag{2.29}$$

In this way, some part of the radiative corrections due to the gauge field is supposed to be included.[6]

Under Eq. (2.29), the action given in Eq. (2.28) is changed to

$$\begin{aligned}
&S + S_{\mathrm{gf}} + \boldsymbol{\Gamma}^{(1)} \\
&= \frac{N}{2\lambda}\int d^4x\, d^4y\, A^{\ell}_{\mu}(x)A^{r}_{\nu}(y)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3} e^{-ip(x-y)} \\
&\quad\times\Bigg\{\delta_{\ell r}(p^2\delta_{\mu\nu} - p_\mu p_\nu)\left[1 - \frac{\beta_0\lambda}{16\pi^2}\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right)\right] + \delta_{\ell r}\xi p_\mu p_\nu \\
&\qquad - \frac{\beta_0\lambda}{16\pi^2}12\sum_{j\neq 0}(\sigma_{j,N})_{\ell r}\int_0^1 dx\, e^{ixp_3 2\pi R j}x(1-x) \\
&\qquad \times\left[(p^2\delta_{\mu\nu} - p_\mu p_\nu)K_0(z) - (p^2\delta_{\mu 3}\delta_{\nu 3} - p_\mu p_3\delta_{\nu 3} - p_\nu p_3\delta_{\mu 3} + p_3^2\delta_{\mu\nu})K_2(z)\right]\Bigg\} \\
&\quad + \frac{N}{2\lambda}\int d^4x\, d^4y\, A^{mn}_{\mu}(x)A^{nm}_{\nu}(y)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3} e^{-ip(x-y)} \\
&\quad\times\Bigg\{(p^2\delta_{\mu\nu} - p_\mu p_\nu)\left[1 - \frac{\beta_0\lambda}{16\pi^2}\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right)\right] + \xi p_\mu p_\nu \\
&\qquad - \frac{\beta_0\lambda}{16\pi^2}12\sum_{j\neq 0, j=0 \bmod N}\int_0^1 dx\, e^{ixp_3 2\pi R j}x(1-x) \\
&\qquad \times\left[(p^2\delta_{\mu\nu} - p_\mu p_\nu)K_0(z) - (p^2\delta_{\mu 3}\delta_{\nu 3} - p_\mu p_3\delta_{\nu 3} - p_\nu p_3\delta_{\mu 3} + p_3^2\delta_{\mu\nu})K_2(z)\right]\Bigg\}_{p\to p_{mn}} \\
&\quad + O(A^3).
\end{aligned} \tag{2.30}$$

This is the effective action in the large-β_0 approximation.

2.5. Gauge field propagator in the large-β_0 approximation

We next obtain the gauge field propagator from the effective action in Eq. (2.30). For this, it is convenient to introduce the projection operators $\mathcal{P}^T_{\mu\nu}$ and $\mathcal{P}^L_{\mu\nu}$ by [52]

$$\mathcal{P}^T_{ij} \equiv \delta_{ij} - \frac{p_i p_j}{p^2 - p_3^2}, \qquad \mathcal{P}^T_{i3} = \mathcal{P}^T_{3i} = \mathcal{P}^T_{33} \equiv 0,$$

[6] It is worth noting that, in the large-β_0 approximation, the leading logarithmic part of the perturbative series is correctly obtained.

$$\mathcal{P}^L_{\mu\nu} \equiv \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \mathcal{P}^T_{\mu\nu}, \tag{2.31}$$

where the Roman letters $i, j, \ldots,$ run only over 0, 1, and 2. These satisfy $p_\mu \mathcal{P}^T_{\mu\nu} = \mathcal{P}^T_{\mu\nu} p_\nu = p_\mu \mathcal{P}^L_{\mu\nu} = \mathcal{P}^L_{\mu\nu} p_\nu = 0$ and, suppressing Lorentz indices,

$$\mathcal{P}^T\mathcal{P}^T = \mathcal{P}^T, \qquad \mathcal{P}^L\mathcal{P}^L = \mathcal{P}^L, \qquad \mathcal{P}^T\mathcal{P}^L = \mathcal{P}^L\mathcal{P}^T = 0. \tag{2.32}$$

In terms of these projection operators, the effective action in Eq. (2.30) is expressed as

$$\begin{aligned} S + S_{\mathrm{gf}} + \boldsymbol{\Gamma}^{(1)} = \frac{N}{2\lambda} \int d^4x\, d^4y\, A^\ell_\mu(x) A^r_\nu(y) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{p_3} e^{-ip(x-y)} \\ \times \left[p^2 \mathcal{P}^L_{\mu\nu}(\delta^{\ell r} - L^{\ell r}) + p^2 \mathcal{P}^T_{\mu\nu}(\delta^{\ell r} - T^{\ell r}) + \delta^{\ell r} \xi p_\mu p_\nu \right] \\ + \frac{N}{2\lambda} \int d^4x\, d^4y\, A^{mn}_\mu(x) A^{nm}_\nu(y) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{p_3} e^{-ip(x-y)} \\ \times \left[p^2 \mathcal{P}^L_{\mu\nu}(1 - L) + p^2 \mathcal{P}^T_{\mu\nu}(1 - T) + \xi p_\mu p_\nu \right]_{p \to p_{mn}} \\ + O(A^3), \end{aligned} \tag{2.33}$$

with

$$\begin{aligned} L^{\ell r} &\equiv \frac{\beta_0 \lambda}{16\pi^2} \Bigg\{ \delta^{\ell r} \ln\left(\frac{e^{5/3}\mu^2}{p^2}\right) \\ &\quad + 12 \sum_{j \neq 0} (\sigma_{j,N})_{\ell r} \int_0^1 dx\, e^{ixp_3 2\pi R j} x(1-x) \left[K_0(z) - K_2(z)\right] \Bigg\}, \\ T^{\ell r} &\equiv \frac{\beta_0 \lambda}{16\pi^2} \Bigg\{ \delta^{\ell r} \ln\left(\frac{e^{5/3}\mu^2}{p^2}\right) \\ &\quad + 12 \sum_{j \neq 0} (\sigma_{j,N})_{\ell r} \int_0^1 dx\, e^{ixp_3 2\pi R j} x(1-x) \left[K_0(z) - \frac{p_3^2}{p^2} K_2(z)\right] \Bigg\}, \\ L &\equiv \frac{\beta_0 \lambda}{16\pi^2} \Bigg\{ \ln\left(\frac{e^{5/3}\mu^2}{p^2}\right) \\ &\quad + 12 \sum_{j \neq 0, j = 0 \bmod N} \int_0^1 dx\, e^{ixp_3 2\pi R j} x(1-x) \left[K_0(z) - K_2(z)\right] \Bigg\}, \\ T &\equiv \frac{\beta_0 \lambda}{16\pi^2} \Bigg\{ \ln\left(\frac{e^{5/3}\mu^2}{p^2}\right) \\ &\quad + 12 \sum_{j \neq 0, j = 0 \bmod N} \int_0^1 dx\, e^{ixp_3 2\pi R j} x(1-x) \left[K_0(z) - \frac{p_3^2}{p^2} K_2(z)\right] \Bigg\}, \end{aligned} \tag{2.34}$$

where the variable z is defined by Eq. (2.24). From this expression, we see that the propagators of the gauge field *with the twisted boundary conditions in Eq.* (2.3) are given by[7]

$$\begin{aligned}
&\left\langle \tilde{A}^{\ell}_{\mu}(x)\tilde{A}^{r}_{\nu}(y)\right\rangle \\
&= \frac{\lambda}{N}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3} \\
&\qquad \times e^{ip(x-y)}\frac{1}{(p^2)^2}\left\{\left[(1-L)^{-1}\right]^{\ell r}p^2\mathcal{P}^L_{\mu\nu}+\left[(1-T)^{-1}\right]^{\ell r}p^2\mathcal{P}^T_{\mu\nu}+\delta^{\ell r}\frac{1}{\xi}p_\mu p_\nu\right\}, \\
&\left\langle \tilde{A}^{mn}_{\mu}(x)\tilde{A}^{pq}_{\nu}(y)\right\rangle \\
&= \frac{\lambda}{N}\delta^{mq}\delta^{np}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3} \\
&\qquad \times \left\{e^{ip(x-y)}\frac{1}{(p^2)^2}\left[(1-L)^{-1}p^2\mathcal{P}^L_{\mu\nu}+(1-T)^{-1}p^2\mathcal{P}^T_{\mu\nu}+\frac{1}{\xi}p_\mu p_\nu\right]\right\}_{p\to p_{mn}}. \qquad (2.35)
\end{aligned}$$

In the last expression, the twisted momentum p_{mn} is substituted, which is defined by Eq. (2.17).

3. Borel singularity in the gluon condensate in the large-N limit

In this section we compute the gluon condensate in the large-β_0 approximation by using Eq. (2.35), and determine the perturbative coefficients for the gluon condensate under the large-N limit of Eq. (1.3). We then construct the corresponding Borel transform. The IR renormalon ambiguity associated with the gluon condensate in $\mathbb{R}^4$ has been studied in Refs. [53–58].

In the large-β_0 approximation, the gluon condensate is computed as (see Fig. 1)

$$\begin{aligned}
&\left\langle \mathrm{tr}(\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu})\right\rangle \\
&= -\frac{1}{2}\left\langle(\partial_\mu\tilde{A}^\ell_\nu-\partial_\nu\tilde{A}^\ell_\mu)^2\right\rangle-\frac{1}{2}\left\langle(\partial_\mu\tilde{A}^{mn}_\nu-\partial_\nu\tilde{A}^{mn}_\mu)(\partial_\mu\tilde{A}^{nm}_\nu-\partial_\nu\tilde{A}^{nm}_\mu)\right\rangle \\
&= -\frac{\lambda}{N}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}\sum_{\ell=1}^{N-1}\left\{\left[(1-L)^{-1}\right]^{\ell\ell}+2\left[(1-T)^{-1}\right]^{\ell\ell}\right\} \\
&\quad -\frac{\lambda}{N}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}\sum_{\substack{m\neq n\\ 1\leq m,n\leq N}}\left[(1-L)^{-1}+2(1-T)^{-1}\right]_{p\to p_{mn}}. \qquad (3.1)
\end{aligned}$$

In the last expression, the first line corresponds to the contribution of the photon, the gauge field associated with the Cartan subalgebra, whereas the second line does to the W-boson which acquires the twisted momentum p_{mn} due to the twisted boundary conditions. We treat these contributions separately.

[7] For this, we have to recall the relation in Eq. (2.8).

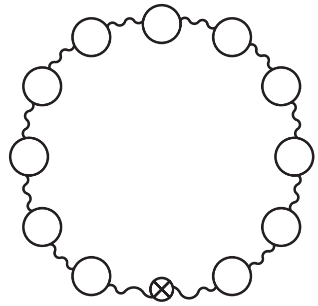

Fig. 1. The Feynman diagram dominating the gluon condensate in Eq. (3.1) in the large-β_0 approximation. The gauge field propagators in Eq. (2.35), which are given by a chain of vacuum polarizations, are used to contract two gauge fields in $\mathrm{tr}(\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu})$.

3.1. *Contribution of the photon*

Since the functions $L^{\ell r}$ and $T^{\ell r}$ in Eq. (2.34) are $O(\lambda)$, we obtain the perturbative expansion of the gluon condensate for the photon part as

$$\left\langle \mathrm{tr}(\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu})\right\rangle_{\text{photon}} = -\frac{\lambda}{N}\sum_{k=0}^{\infty}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}\sum_{\ell=1}^{N-1}\left[(L^k)^{\ell\ell} + 2(T^k)^{\ell\ell}\right], \tag{3.2}$$

from which the kth perturbative coefficient can be read off. In Appendix B we show that, in the large-N limit of Eq. (1.3),

$$\sum_{j\neq 0}\sigma_{j,N}\int dx\, e^{ixp_3 2\pi Rj}x(1-x)K_\nu(z) = O(1/N), \tag{3.3}$$

for $\nu = 0$ or 2. Therefore, the second terms in $L^{\ell r}$ and in $T^{\ell r}$ of Eq. (2.34) give only sub-dominant contributions in the large-N limit.[8] Thus, the kth perturbative coefficient f_k [defined as in Eq. (1.4)] in the large-N limit is given by

$$(f_k)_{\text{photon}} = -3\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}\left[\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right)\right]^k, \tag{3.4}$$

which is $O(N^0)$.

We construct the corresponding Borel transform [cf. Eq. (1.5)] to investigate the large-order behavior:

$$\begin{aligned} B(u)_{\text{photon}} &= -3\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}\left(\frac{e^{5/3}\mu^2}{p^2}\right)^u \\ &= -3\sum_{j=-\infty}^{\infty}\int \frac{d^4p}{(2\pi)^4}\, e^{ip_3 2\pi Rj}\left(\frac{e^{5/3}\mu^2}{p^2}\right)^u, \end{aligned} \tag{3.5}$$

[8] The bounds for the finite volume corrections get larger in the lower-energy region, as shown in Eqs. (B.7)–(B.10). Thus, the finite volume corrections may cause IR divergences in perturbative coefficients. However, we regard such divergence as the subleading effect in terms of large N. The same identification is applied to the calculation of the contribution from the W-boson.

where we have used Eq. (2.20). In this expression, the term with $j = 0$ is UV divergent and we introduce a UV cutoff $q > 0$ to the momentum integral, $|p| \leq q$. The $j \neq 0$ terms are Fourier transforms and UV convergent. Then, the momentum integration yields

$$B(u)_{\text{photon}} = \frac{3}{16\pi^2}(e^{5/3}\mu^2)^u \left[(q^2)^{2-u}\frac{1}{u-2} - 2(\pi^2 R^2)^{u-2}\frac{\boldsymbol{\Gamma}(2-u)}{\boldsymbol{\Gamma}(u)}\zeta(4-2u) \right]. \tag{3.6}$$

The only singularity of this function is given by the simple pole at $u = 3/2$:

$$B(u)_{\text{photon}} \overset{u\sim 3/2}{\sim} \frac{3}{16\pi^2}(e^{5/3}\mu^2)^{3/2} 2(\pi^2 R^2)^{-1/2}\frac{1}{u-3/2}. \tag{3.7}$$

We again note that this is $O(N^0)$. In fact, the photon part satisfies the prerequisites for the analysis of Ref. [25], and the general argument therein indicates the singularity at $u = 3/2$, as a consequence of the shift of the singularity by $-1/2$ in the Borel u-plane.

3.2. *Contribution of the W-boson*

The perturbative expansion of the W-boson part is given by

$$\left\langle \operatorname{tr}(\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}) \right\rangle_{\text{W-boson}} = -\frac{\lambda}{N}\sum_{k=0}^{\infty}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}\sum_{\substack{m\neq n\\ 1\leq m,n\leq N}} (L^k + 2T^k)_{p\to p_{mn}}. \tag{3.8}$$

Since we can show that (Appendix B)

$$\sum_{j\neq 0, j=0 \bmod N} \int dx\, e^{ixp_3 2\pi Rj} x(1-x)K_\nu(z) = O(1/N^3), \tag{3.9}$$

the second terms of L and T in Eq. (2.34) again give only sub-dominant contribution in the large-N limit. Hence, in the large-N limit we obtain the kth perturbative coefficient f_k [defined as in Eq. (1.4)] as

$$\begin{aligned}
(f_k)_{\text{W-boson}} &= -3\frac{1}{N}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3}\sum_{\substack{m\neq n\\ 1\leq m,n\leq N}} \left[\ln\left(\frac{e^{5/3}\mu^2}{p_{mn}^2}\right)\right]^k \\
&= -3\sum_{j=-\infty}^{\infty}\int \frac{d^4p}{(2\pi)^4} e^{ip_3 2\pi Rj}\frac{1}{N}\sum_{\substack{m\neq n\\ 1\leq m,n\leq N}} e^{i(m-n)2\pi j/N}\left[\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right)\right]^k,
\end{aligned} \tag{3.10}$$

where we have used Eqs. (2.20) and (2.17), and shifted the momentum $p_3 \to p_3 + (m-n)/(RN)$. Now, we note that

$$\frac{1}{N}\sum_{\substack{m\neq n\\ 1\leq m,n\leq N}} e^{i(m-n)2\pi j/N} = \begin{cases} N-1, & \text{for } j = 0 \mod N, \\ -1, & \text{for } j \neq 0 \mod N, \end{cases} \tag{3.11}$$

and thus

$$(f_k)_{\text{W-boson}} = -3\sum_{j=-\infty}^{\infty}\int \frac{d^4p}{(2\pi)^4} e^{ip_3 2\pi RNj}\left[(N-1)-(-1)\right]\left[\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right)\right]^k$$

$$- 3 \sum_{j=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} e^{ip_3 2\pi Rj}(-1) \left[\ln \left(\frac{e^{5/3}\mu^2}{p^2} \right) \right]^k . \tag{3.12}$$

The second line on the right-hand side precisely cancels the contribution of the photon in Eq. (3.4). For the first line, we apply Eq. (2.20) in an opposite way (cf. Ref. [28]):

$$(f_k)_{\text{W-boson}} = -3N \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi RN} \sum_{p_3=n/(RN)} \left[\ln \left(\frac{e^{5/3}\mu^2}{p^2} \right) \right]^k$$
$$- 3 \sum_{j=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} e^{ip_3 2\pi Rj}(-1) \left[\ln \left(\frac{e^{5/3}\mu^2}{p^2} \right) \right]^k . \tag{3.13}$$

Remarkably, in the first term of the right-hand side, the effective radius of the compactified direction becomes RN as a consequence of the twisted boundary conditions. Hence, S^1 is effectively decompactified in the large-N limit, and the first term is reduced to the expression in the uncompactified $\mathbb{R}^4$:

$$(f_k)_{\text{W-boson}} = -3N \int \frac{d^4p}{(2\pi)^4} \left[\ln \left(\frac{e^{5/3}\mu^2}{p^2} \right) \right]^k$$
$$- 3 \sum_{j=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} e^{ip_3 2\pi Rj}(-1) \left[\ln \left(\frac{e^{5/3}\mu^2}{p^2} \right) \right]^k , \tag{3.14}$$

where the sum over the KK momentum has been replaced by an integral in the $N \to \infty$ limit [28].

From these perturbative coefficients, we obtain the Borel transform [cf. Eq. (1.5)] as

$$B(u)_{\text{W-boson}} = -3N \int \frac{d^4p}{(2\pi)^4} \left(\frac{e^{5/3}\mu^2}{p^2} \right)^u - 3 \sum_{j=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} e^{ip_3 2\pi Rj}(-1) \left(\frac{e^{5/3}\mu^2}{p^2} \right)^u$$
$$= \frac{3N}{16\pi^2} (e^{5/3}\mu^2)^u (q^2)^{2-u} \frac{1}{u-2} - B(u)_{\text{photon}}. \tag{3.15}$$

This has the singularity at $u = 2$; this position coincides with that of the uncompactified spacetime $\mathbb{R}^4$. We note that the contribution from the W-boson is of $O(N)$.

3.3. *IR renormalon in the gluon condensate*

As the sum of Eqs. (3.6) and (3.15), we have

$$B(u) = B(u)_{\text{photon}} + B(u)_{\text{W-boson}} = \frac{3N}{16\pi^2} (e^{5/3}\mu^2)^u (q^2)^{2-u} \frac{1}{u-2}. \tag{3.16}$$

Therefore, the gluon condensate in the present system suffers from the IR renormalon ambiguity corresponding to $u = 2$. Through the Borel sum of Eq. (1.6), this pole singularity produces the ambiguity,

$$\frac{3N}{\beta_0} (e^{5/3}\mu^2)^2 e^{-32\pi^2/(\beta_0\lambda)} (\mp\pi i) = \frac{3N}{\beta_0} e^{10/3} \Lambda^4 (\mp\pi i). \tag{3.17}$$

This is the main result of this paper.

Some remarks are in order. First, in the large-N limit, the contribution of the W-boson dominates the Borel singularity, i.e. the IR renormalon. This can be seen from the fact that the contribution of the W-boson, Eq. (3.15), is $O(N)$, while that of the photon in Eq. (3.6) is $O(N^0)$. This result is in contrast to the argument in Ref. [52] for small N that the W-boson does not contribute to the IR renormalon at all.

Secondly, we note that the following calculation leads us to a completely different conclusion. If we construct the Borel transform from the perturbative coefficient for the W-boson of Eq. (3.12), where the large-N limit is not taken for each perturbative coefficient,[9] we obtain

$$\begin{aligned}&\widetilde{B}(u)_{\text{W-boson}}\\&= -3N\sum_{j=-\infty}^{\infty}\int\frac{d^4p}{(2\pi)^4}e^{ip_3 2\pi RNj}\left(\frac{e^{5/3}\mu^2}{p^2}\right)^u - 3\sum_{j=-\infty}^{\infty}\int\frac{d^4p}{(2\pi)^4}e^{ip_3 2\pi Rj}(-1)\left(\frac{e^{5/3}\mu^2}{p^2}\right)^u\\&= \frac{3N}{16\pi^2}(e^{5/3}\mu^2)^u\left[(q^2)^{2-u}\frac{1}{u-2} - 2(\pi^2R^2N^2)^{u-2}\frac{\boldsymbol{\Gamma}(2-u)}{\boldsymbol{\Gamma}(u)}\zeta(4-2u)\right] - B(u)_{\text{photon}}.\end{aligned} \tag{3.18}$$

In this Borel transform, one can see that the pole singularity at $u = 2$ disappears, but a pole singularity at $u = 3/2$ emerges instead. This is a conclusion completely opposite to the one following from Eq. (3.15), which shows the presence of the singularity at $u = 2$. This peculiar situation indicates that the large-N limit and the construction of the Borel transform are not commutable:[10]

$$\sum_{k=0}^{\infty}(\lim_{N\to\infty} f_k)\frac{u^k}{k!} \neq \lim_{N\to\infty}\left(\sum_{k=0}^{\infty} f_k\frac{u^k}{k!}\right). \tag{3.19}$$

[9] At this stage, the large-N limit is taken only for the loop integrands, L^k and T^k, but this limit is not considered after the loop integral, which gives the additional N dependence. This treatment is not systematic because only part of the subleading effects is considered. (In fact, the $j \neq 0$ terms in the first line of Eq. (3.12) are subleading compared to the $j = 0$ term there, as can be seen from the fact that the first term in Eq. (3.14) is exactly the same as the $j = 0$ term.) However, we demonstrate here that even when the integrand is exactly given by the logarithmic factor, as in Eq. (3.12), there is a subtle issue on how to take the large-N limit.

[10] The following remark may be useful. From Eq. (1.5), the coefficients of the perturbative series f_k corresponding to the Borel transform in Eq. (3.18) are given by

$$f_k = \frac{\partial^k}{\partial u^k}\widetilde{B}(u)_{\text{W-boson}}\bigg|_{u=0}.$$

Therefore, the contribution of the $j \neq 0$ terms in Eq. (3.18) to f_k is

$$\frac{3N}{16\pi^2}(-2)(\pi^2R^2N^2)^{-2}\ \frac{\partial^k}{\partial u^k}e^{2u\ln(e^{5/6}\mu\pi RN)}g(u)\bigg|_{u=0},$$

where we defined the N-independent function $g(u)$ by

$$g(u) \equiv \frac{\boldsymbol{\Gamma}(2-u)}{\boldsymbol{\Gamma}(u)}\zeta(4-2u),$$

$g(0) = 0$, and $g'(0) = \pi^4/90$. Then, the $j \neq 0$ terms give the following leading-N dependence to the perturbative coefficient f_k with *the order k kept fixed*:

$$\frac{3N}{16\pi^2}(-2)(\pi^2R^2N^2)^{-2}k[2\ln(e^{5/6}\mu\pi RN)]^{k-1}\frac{\pi^4}{90} = O(N^{-3}(\ln N)^{k-1}).$$

This is sub-dominant compared to the $j = 0$ term in Eq. (3.18) that is $O(N)$. On the other hand, if one considers the large-order behavior of f_k as $k \to \infty$ *with N kept fixed*, the terms of $j \neq 0$ produce the Borel singularity at $u = 2$, as shown by the second term in the square parentheses of Eq. (3.18), which is $O(N)$.

We see that this inequality holds especially around $u = 3/2$ and 2.[11] In the present paper, we find the Borel singularity at $u = 2$ under the procedure where we first determine the perturbative coefficients in the large-N limit and then construct the Borel transform. From the perspective of our original subject of how the perturbative series diverges in the large-N theory, we should adopt this ordering of operations.[12] We emphasize that this subtlety is peculiar to the W-boson, which acquires the twisted momentum.[13]

We finally make a comment on an example of the UV-finite quantity which possesses the renormalon ambiguity corresponding to the Borel singularity at $u = 2$. The gluon condensate is quartically divergent, as seen from Eq. (3.1), and it may not be regarded as a physical observable. However, we may consider (as in Ref. [59]) the gluon condensate of the gauge field defined by the Yang–Mills gradient flow [60,61]. We can repeat the above analysis for this perfectly UV-finite quantity, and obtain the same renormalon ambiguity as the gluon condensate investigated above.

4. Conclusion

In this paper we have studied the IR renormalon ambiguity in the gluon condensate in the $SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$ with the $\mathbb{Z}_N$ twisted boundary conditions. In the large-N limit within the the large-β_0 approximation, we showed that the Borel transform develops a pole singularity at $u = 2$. This provides an example that the system in the compactified space $\mathbb{R}^3 \times S^1$ possesses the renormalon ambiguity identical to that in the uncompactified space $\mathbb{R}^4$. This situation is caused by the W-boson—the gauge field which acquires the twisted KK momentum due to the twisted boundary conditions—and this is quite different from the $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$. We hope that the observation made in this paper can be of relevance to the conjectured cancellation of the renormalon ambiguity by the instability associated with the semi-classical bion solution.

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[11] The equality holds in the vicinity of $u = 0$. Thus, if one defines the Borel transform of the right-hand side by the analytic continuation of the result around $u = 0$ in the large-N limit, the second term inside the square brackets of Eq. (3.18) vanishes, and the singularity at $u = 2$ follows.

[12] The first version of the present paper concluded the singularity at $u = 3/2$ based on the calculation leading to Eq. (3.18). However, we consider that there are some problems with this treatment: as noted in footnote 9, we keep a part of subleading effects in this calculation, which is not well justified.

[13] If momentum is not twisted, the order counting in $1/N$ is straightforward (like in the photon case), and such a subtlety does not arise. We also note that this subtlety is irrelevant to Refs. [23,25], where the loop momentum of the renormalon diagram is not twisted.

Appendix A. *SU*(*N*) generators

We follow the convention in Chap. 13 of Ref. [62]. The Cartan generators in the fundamental representation are taken as

$$(H_m)_{ij} = \frac{1}{\sqrt{2m(m+1)}}\left(\sum_{k=1}^{m}\delta_{ik}\delta_{jk} - m\delta_{i,m+1}\delta_{j,m+1}\right), \qquad m = 1,\ldots,N-1, \tag{A.1}$$

whereas $(N-1)N$ raising and lowering generators are taken as (here m, n, … run from 1 to N)

$$(E_{mn})_{ij} = \frac{1}{\sqrt{2}}\delta_{im}\delta_{jn}, \qquad m \neq n, \qquad E^{\dagger}_{mn} = E_{nm}. \tag{A.2}$$

In terms of these generators, the $SU(N)$ algebra reads

$$\begin{aligned}
&[H_m, H_n] = 0,\\
&[H_\ell, E_{mn}] = (\nu^m - \nu^n)_\ell E_{mn},\\
&[E_{mn}, E_{pq}] = \begin{cases} (\nu^m - \nu^n)\cdot H, & \text{when } m = q \text{ and } n = p,\\ -\frac{1}{\sqrt{2}}E_{pn}, & \text{when } m = q \text{ and } n \neq p,\\ \frac{1}{\sqrt{2}}E_{mq}, & \text{when } m \neq q \text{ and } n = p,\\ 0, & \text{otherwise,}\end{cases}
\end{aligned} \tag{A.3}$$

where the ν^m denote the weights ($(\nu^m)_i = (H_i)_{mm}$; here, no sum is taken over m) and thus $\nu^m - \nu^n$ are the roots. We note that

$$\nu^i \cdot \nu^j = -\frac{1}{2N} + \frac{1}{2}\delta_{ij}. \tag{A.4}$$

The above generators are normalized such that

$$\mathrm{tr}(H_m H_n) = \frac{1}{2}\delta_{mn}, \qquad \mathrm{tr}(E_{mn}E_{pq}) = \frac{1}{2}\delta_{mq}\delta_{np}, \qquad \mathrm{tr}(H_\ell E_{mn}) = 0. \tag{A.5}$$

Appendix B. Proofs of Eqs. (3.3) and (3.9)

We start with

$$\left|\sum_{j\neq 0}\sigma_{j,N}\int_0^1 dx\, e^{ixp_3 2\pi Rj}x(1-x)K_\nu(z)\right| < \sum_{j\neq 0}|\sigma_{j,N}|\int_0^1 dx\, x(1-x)K_\nu(z). \tag{B.1}$$

We first note that

$$|\sigma_{j,N}| \leq \begin{cases} 1, & \text{for } j = 0 \bmod N,\\ \frac{4}{N}, & \text{for } j \neq 0 \bmod N,\end{cases} \tag{B.2}$$

from Eqs. (2.22) and

$$\left|\frac{e^{-i\ell 2\pi j/N} - 1}{e^{-i2\pi j/N} - 1} - \ell e^{-i\ell 2\pi j/N}\right| = \left|\sum_{n=0}^{\ell-1}\left(e^{-i2\pi j/N}\right)^n - \ell e^{-i\ell 2\pi j/N}\right|$$

$$< \sum_{n=0}^{\ell-1} 1 + \ell = 2\ell. \tag{B.3}$$

Next, as explained in Appendix B of Ref. [23], one can show, for the bounds, that

$$K_0(z) < \frac{2}{z} e^{-z/2}, \qquad K_1(z) < \frac{2}{z} e^{-z/2}, \qquad \text{for } z > 0. \tag{B.4}$$

From these, using $K_2(z) = K_0(z) + \frac{2}{z} K_1(z)$, we have

$$K_2(z) < \left[\frac{2}{z} + \left(\frac{2}{z} \right)^2 \right] e^{-z/2}, \qquad \text{for } z > 0. \tag{B.5}$$

Now, using the above relations, we can proceed as, for instance,

$$\begin{aligned} &\left| \sum_{j\neq 0, j=0 \bmod N} \sigma_{j,N} \int_0^1 dx\, e^{ixp_3 2\pi R j} x(1-x) K_0(z) \right| \\ &< \frac{2}{(p^2)^{1/2} \pi R} \sum_{j\neq 0, j=0 \bmod N} \frac{1}{|j|} \int_0^{1/2} dx\, \sqrt{x(1-x)} e^{-\sqrt{x(1-x)p^2}\pi R |j|}, \end{aligned} \tag{B.6}$$

where we have used $z = \sqrt{x(1-x)p^2} 2\pi R|j|$ in Eq. (2.24). Noting that $x/2 \leq x(1-x) \leq 1/4$ for $0 \leq x \leq 1/2$, we have the further bounds

$$\begin{aligned} &< \frac{1}{(p^2)^{1/2} \pi R} \sum_{j\neq 0, j=0 \bmod N} \frac{1}{|j|} \int_0^{1/2} dx\, e^{-\sqrt{xp^2/2}\pi R |j|} \\ &< \frac{4}{(p^2)^{3/2} (\pi R N)^3} \sum_{k=1}^{\infty} \frac{1}{k^3} \int_0^{\infty} dx\, e^{-\sqrt{x}} \\ &= \frac{8}{(p^2)^{3/2} (\pi R N)^3} \zeta(3) = O(1/N^3). \end{aligned} \tag{B.7}$$

Similarly,

$$\begin{aligned} &\left| \sum_{j\neq 0 \bmod N} \sigma_{j,N} \int_0^1 dx\, e^{ixp_3 2\pi R j} x(1-x) K_0(z) \right| \\ &< \frac{8}{N (p^2)^{1/2} \pi R} \sum_{j\neq 0 \bmod N} \frac{1}{|j|} \int_0^{1/2} dx\, \sqrt{x(1-x)} e^{-\sqrt{x(1-x)p^2}\pi R |j|} \\ &< \frac{4}{N (p^2)^{1/2} \pi R} \sum_{j\neq 0 \bmod N} \frac{1}{|j|} \int_0^{1/2} dx\, e^{-\sqrt{xp^2/2}\pi R |j|} \\ &< \frac{16}{N (p^2)^{3/2} (\pi R)^3} \sum_{j=1}^{\infty} \frac{1}{j^3} \int_0^{\infty} dx\, e^{-\sqrt{x}} \\ &= \frac{32}{N (p^2)^{3/2} (\pi R)^3} \zeta(3) = O(1/N). \end{aligned} \tag{B.8}$$

Equations (B.7) and (B.8) imply Eq. (3.3) for $\nu = 0$.

In a similar manner, noting Eq. (B.5), we have

$$\left| \sum_{j\neq 0, j=0 \bmod N} \sigma_{j,N} \int_0^1 dx\, e^{ixp_3 2\pi R j} x(1-x) K_2(z) \right| < \frac{8}{(p^2)^{3/2}(\pi R N)^3}\zeta(3) + \frac{16}{(p^2)^2(\pi R N)^4}\zeta(4) = O(1/N^3), \tag{B.9}$$

and

$$\left| \sum_{j\neq 0 \bmod N} \sigma_{j,N} \int_0^1 dx\, e^{ixp_3 2\pi R j} x(1-x) K_2(z) \right| < \frac{32}{N(p^2)^{3/2}(\pi R)^3}\zeta(3) + \frac{64}{N(p^2)^2(\pi R)^4}\zeta(4) = O(1/N). \tag{B.10}$$

These imply Eq. (3.3) for $\nu = 2$. Noting Eq. (B.2), Eqs. (B.7) and (B.9) imply Eq. (3.9).

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