

One-loop contributions to the decay $H \rightarrow \nu_l \bar{\nu}_l \gamma$ in the Standard Model revisited

Khiem Hong Phan^{1,2,*} and Dzung Tri Tran³

¹*Institute of Fundamental and Applied Sciences, Duy Tan University, Ho Chi Minh City 700000, Vietnam*

²*Faculty of Natural Sciences, Duy Tan University, Da Nang City 550000, Vietnam*

³*University of Science Ho Chi Minh City, 227 Nguyen Van Cu, District 5, Ho Chi Minh City, Vietnam*

*E-mail: phanhongkiem@duytan.edu.vn

Received November 16, 2021; Revised January 12, 2022; Accepted January 25, 2022; Published January 28, 2022

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One-loop contributions to the decay $H \rightarrow \nu_l \bar{\nu}_l \gamma$ with $l = e, \mu, \tau$ within the Standard Model framework are revisited in this paper. We derive two representations for the form factors in this calculation. As a result, the computations are not only checked numerically by verifying the ultraviolet finiteness of the results but also confirming the ward identity of the amplitude. We find that the results have good stability with varying ultraviolet cutoff parameters as well as satisfying the ward identity. In phenomenological results, we study the partial decay widths for the decay channels in the two cases of the detected photon and invisible photon. Differential decay widths are also generated as a function of the energy of the final photon.
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Subject Index B53, B59

1. Introduction

One of the main purposes of the current experimental program at the High Luminosity Large Hadron Collider (HL-LHC) [1,2] and Future Lepton Colliders [3] is to precisely measure the properties of the Standard-Model-like (SM-like) Higgs boson. In this program, the search for all decay modes of the Higgs boson is a priority task, and they must be probed as precisely as possible, because the partial decay widths of the Higgs boson provide important information for answering the nature of the Higgs sector (for a deeper understanding of the dynamics of electroweak symmetry breaking). Of all the decay channels of the Higgs boson, $H \rightarrow$ invisible particles and $H \rightarrow \gamma$ plus invisible particles [4–10] are of great interest because (i) these processes can be probed at future colliders [4–6,8,10] to test the SM in modern energy regions, and (ii) many new heavy particles such as new extra gauge bosons and charged (and neutral) scalar particles may exchange in the loop diagrams of the aforementioned decay channels. Thus, the decay processes could provide a useful tool for testing the SM and studying new physics.

We know that precise evaluations of the Standard Model background play a crucial role in searching for new physics, which means that detailed theoretical calculations for one-loop contributions to $H \rightarrow \nu_l \bar{\nu}_l \gamma$ are necessary. In Ref. [11], one-loop formulas for $H \rightarrow \nu_l \bar{\nu}_l \gamma$ within the SM framework were provided. In view of the importance of the decay channels, we revisit the evaluations to extend the previous computations and update the numerical predictions

for the processes. The analytic results for the one-loop form factors are expressed in terms of Passarino–Veltman functions (PV-functions), which can be evaluated numerically using the LoopTools package [12]. In comparison with the previous work, the evaluations in the present paper are extended with respect to the following points. Firstly, two representations for the form factors are derived in this calculation. As a result, the computations are not only checked numerically by verifying the ultraviolet finiteness of the results but also by confirming the ward identity of the amplitude. We find that the results have good stability with varying ultraviolet cutoff parameters, as well as following the ward identity of the amplitude. Secondly, we study the partial decay widths for the decay channels in the two cases of the detected photon and invisible photon using the updated input parameter values. Lastly, all the PV-functions are also reduced to scalar one-, two-, three-, and four-point functions in this paper. We also point out that one uses the generalized hypergeometric functions in Refs. [13–15] at general space-time dimensions d for scalar one-loop integrals, so the form factors will be valid in general d . In future, one may consider two-loop and higher-loop contributions to these channels. In this case, the analytic results for the one-loop form factors in this paper can be used to perform higher-power ε expansion.

The paper is organized as follows. In Sect. 2, detailed calculations for the one-loop contributions to $H \rightarrow \nu_l \bar{\nu}_l \gamma$ are presented. Numerical tests for the computations and physical results for the decay processes are also shown. Conclusions and outlook are described in Sect. 3. In the appendices, reduction formulas for scalar PV-functions are presented.

2. Calculation

Detailed calculations for the one-loop contributions to $H \rightarrow \nu_l \bar{\nu}_l \gamma$ are presented in this section. Two different representations for the one-loop form factors are derived. Numerical tests for the computation and physical results for the decay processes are then shown in the following subsections.

2.1 Analytic results

We present the detailed evaluations for the decay channels $H(p) \rightarrow \nu_l(q_1) \bar{\nu}_l(q_2) \gamma(q_3)$ in this subsection. In this article, the evaluations are considered in the unitary gauge, since we have fewer Feynman diagrams than the general R_ξ -gauges. It is known that the results in Refs. [16,17] may have the problem of large numerical cancellations in the unitary gauge, because higher-rank tensor one-loop integrals appear in the diagrams due to W-boson exchange. This problem arises when we derive the one-loop form factor in $d = 4$ dimensions directly. To cure the problem, we first write down the Feynman one-loop amplitude in general d dimensions. All the higher-rank tensor one-loop integrals will then cancel out analytically. The remaining lower-rank tensor one-loop integrals (including only rank 3 for the one-loop two-point and rank 2 for both one-loop triangle and box diagrams) are next reduced to scalar integrals in general d . We finally set $d \rightarrow 4$ for the final results. As a result, we can avoid the large numerical cancellation problem.

Within the Standard Model the decay processes consist of W -bosons and fermions exchanging in one-loop triangle diagrams (see Figs. 1 and 2 respectively), as well as six one-loop box diagrams (shown in Fig. 3) in the unitary gauge.

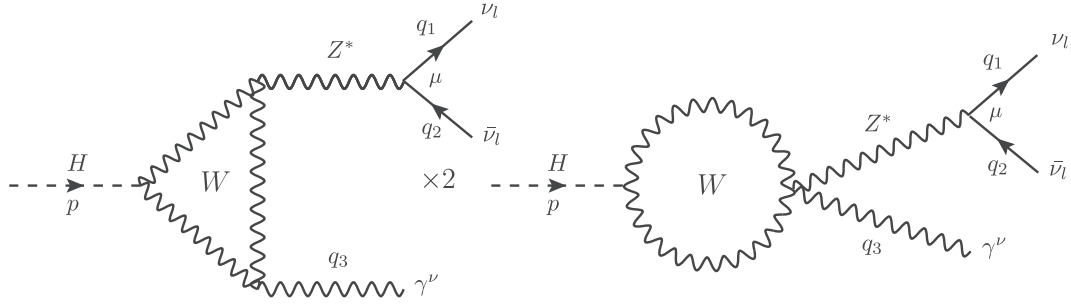


Fig. 1. W -bosons exchanging in the one-loop triangle diagrams of $H \rightarrow \nu_l \bar{\nu}_l \gamma$ in the unitary gauge.

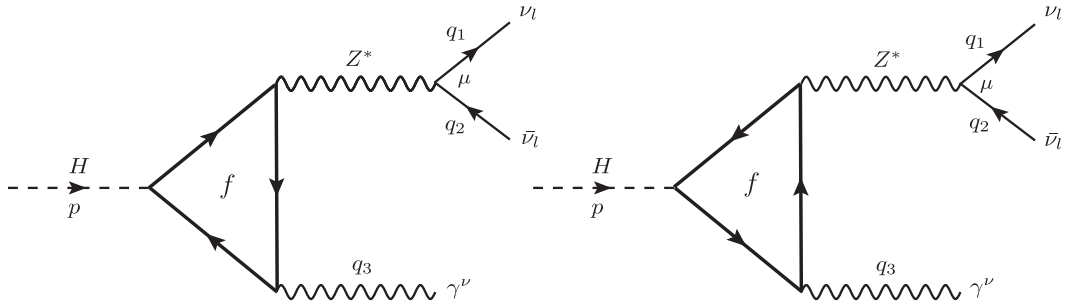


Fig. 2. Fermion f exchanging in the one-loop triangle diagrams of $H \rightarrow \nu_l \bar{\nu}_l \gamma$ in the unitary gauge.

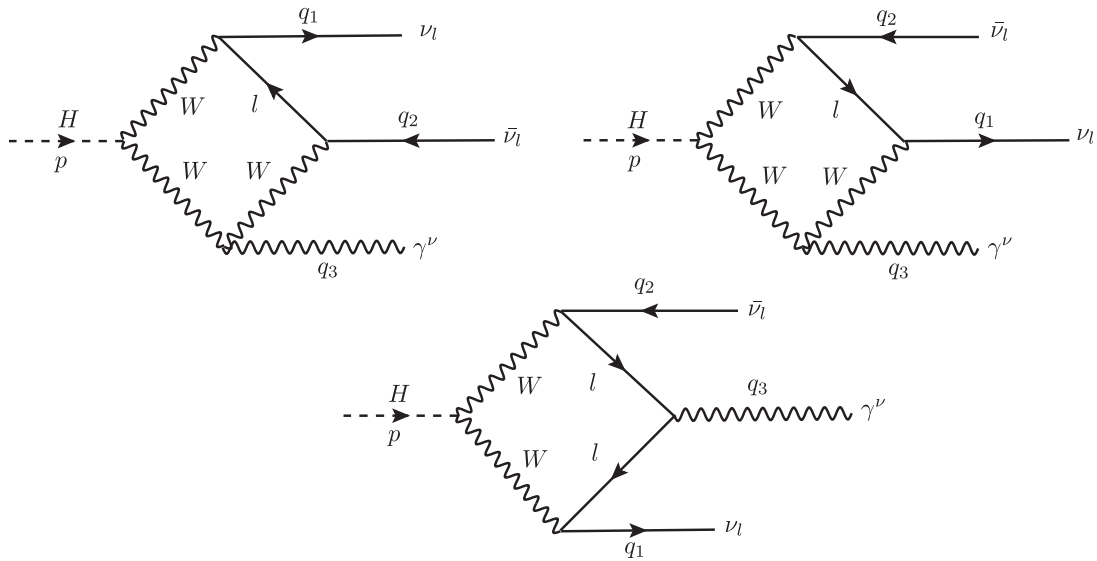


Fig. 3. W -boson and lepton l exchanging in the one-loop box diagrams of $H \rightarrow \nu_l \bar{\nu}_l \gamma$ in the unitary gauge.

In general, the one-loop amplitude of the decay channels can be divided as follows:

$$\mathcal{A} = \mathcal{A}_{\text{tri}}^{(W)} + \mathcal{A}_{\text{tri}}^{(f)} + \mathcal{A}_{\text{box}}^{(W)} + \mathcal{A}_{\text{box}}^{(l)}, \quad (1)$$

where $\mathcal{A}_{\text{tri}}^{(W)}$ represents the contributions of one-loop triangle Feynman diagrams with W -boson internal lines (shown in Fig. 1), and $\mathcal{A}_{\text{tri}}^{(f)}$ is the amplitude of one-loop triangle

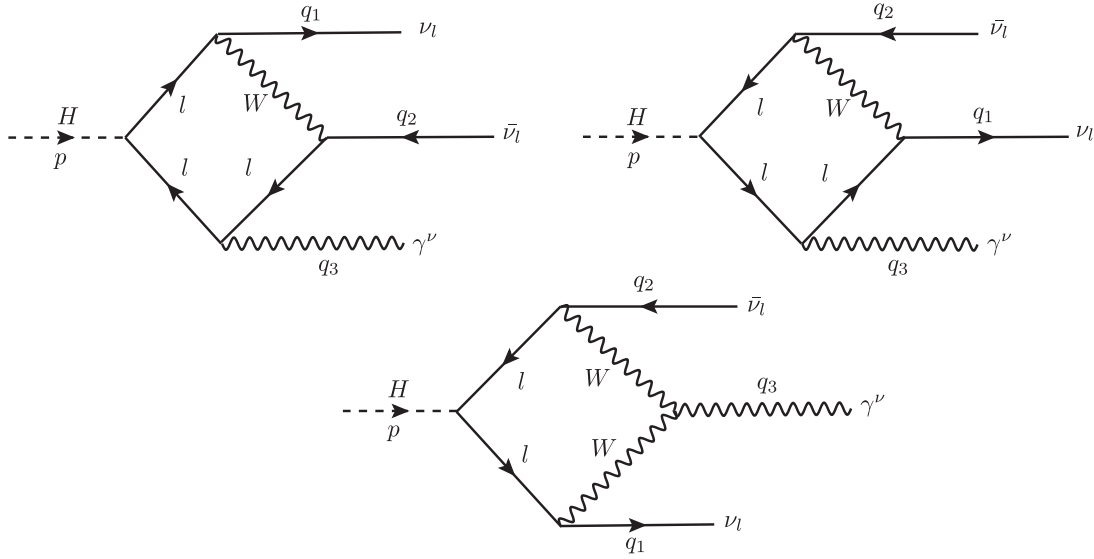


Fig. 4. Lepton l and W -boson exchanging in the one-loop box diagrams of $H \rightarrow \nu_l \bar{\nu}_l \gamma$ in the unitary gauge.

Feynman diagrams with exchanging fermions in the loop (see Fig. 2). In the same notation, $\mathcal{A}_{\text{box}}^{(W)}$ ($\mathcal{A}_{\text{box}}^{(l)}$) are the corresponding amplitudes of the one-loop box diagrams in Fig. 3 (Fig. 4).

For an on-shell external photon, the ward identity is implied. As a result, we have the relation $q_3^\nu \epsilon_\nu^* = 0$, where q_3^ν and ϵ_ν^* correspond to the momentum and polarization vectors of the photon. The kinematic invariant variables involved in this decay process are taken into account:

$$\begin{aligned}
 p^2 &= M_H^2, & q_1^2 &= q_2^2 = q_3^2 = 0, \\
 q_{12} &= (q_1 + q_2)^2 = 2q_1 \cdot q_2, & q_{13} &= 2q_1 \cdot q_3, & q_{23} &= 2q_2 \cdot q_3.
 \end{aligned}
 \tag{2}$$

With this definition, one confirms that $q_{12} + q_{13} + q_{23} = M_H^2$.

We first write down all the Feynman amplitudes for the above diagrams. With the help of Package-X [18], all the Dirac traces and Lorentz contractions in d dimensions are handled. The amplitudes are then decomposed into tensor one-loop integrals. By following the tensor reduction for one-loop integrals in Ref. [19], all the tensor one-loop integrals are expressed in terms of PV-functions (the scalar PV-coefficient functions are called $A_{ijk\dots}$, $B_{ijk\dots}$, $C_{ijk\dots}$, and $D_{ijk\dots}$ for $i, j, k, \dots = 1, 2, 3$ as in Ref. [19]). In this paper, we also reduce the PV-functions to scalar one-loop integrals (denoted A_0 , B_0 , C_0 , and D_0 [19]). Later, these scalar functions A_0 , B_0 , C_0 , and D_0 can be evaluated numerically using LoopTools. Furthermore, one can use the hypergeometric representations in Refs. [13–15] for these scalar functions in general d .

2.1.1 *First expressions for form factors.* We first arrive at one-loop three-point Feynman diagrams with the W -boson at internal lines (as shown in Fig. 1). The corresponding

amplitude for these Feynman diagrams is decomposed into a Lorentz-invariant structure as follows:

$$\mathcal{A}_{\text{tri}}^{(W)} = \left[F_1^{(W)} q_3^\mu q^\nu + F_2^{(W)} g^{\mu\nu} \right] \bar{u}(q_1) \gamma^\mu P_L \nu(q_2) \epsilon_\nu^*(q_3), \quad (3)$$

where we have used $P_L = \frac{1}{2}(1 - \gamma_5)$. Applying the tensor reduction method in Ref. [19], all the form factors in this formula are computed. The analytic results for the form factors are expressed in terms of the B_0 and C_0 functions as follows:

$$\begin{aligned} F_1^{(W)} &= \frac{\alpha^2}{2M_W^3 s_W^3 (d-2) (M_H^2 - q_{12})^2} \\ &\times \frac{1}{q_{12} - M_Z^2 + i\Gamma_Z M_Z} \\ &\times \left\{ [M_H^2 (2M_W^2 - q_{12}) + 4M_W^4 (d-1) - 2M_W^2 q_{12}] \right. \\ &\times [(M_H^2 (d-4) + (2-d)q_{12}) B_0(M_H^2, M_W^2, M_W^2) + 2q_{12} B_0(q_{12}, M_W^2, M_W^2)] \\ &+ 4M_W^2 (M_H^2 - q_{12}) [2M_W^2 (M_H^2 - q_{12}) (2d-5) - 4M_W^4 (d-1) \\ &+ q_{12}^2 (d-2) - M_H^2 q_{12} (d-3)] \\ &\left. \times C_0(0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2) \right\}. \quad (4) \end{aligned}$$

Here, $s_W = \sin \theta_W$, where θ_W is the weak mixing angle, and $d = 4 - 2\epsilon$ is the space-time dimension. In this paper, we also compute the form factor $F_2^{(W)}$; its analytical result is shown in the next subsection. In general, $F_2^{(W)}$ may take a different form than $F_1^{(W)}$. Following the ward identity, one confirms that

$$F_2^{(W)} = \frac{q_{12} - M_H^2}{2} F_1^{(W)}. \quad (5)$$

The relation in Eq. (5) can be tested analytically by performing ϵ expansion for the form factors. The expressions for $F_1^{(W)}$ and $F_2^{(W)}$ both give same result (see Appendix B). We note that the relation can also be verified numerically in the following subsections.

We next consider the Feynman amplitude for one-loop triangle diagrams with fermions f at internal lines. Following the same procedure, the amplitude is cast into the form

$$\mathcal{A}_{\text{tri}}^{(f)} = \left[F_1^{(f)} q_3^\mu q^\nu + F_2^{(f)} g^{\mu\nu} \right] \bar{u}(q_1) \gamma^\mu P_L \nu(q_2) \epsilon_\nu^*(q_3). \quad (6)$$

All the form factors in this equation are evaluated by using the tensor reduction method in Ref. [19]. The results read:

$$\begin{aligned} F_1^{(f)} &= \frac{\alpha^2 N_C^f Q_f m_f^2 (T_3^f - 2Q_f s_W^2)}{M_W c_W^2 s_W^3 (d-2) (M_H^2 - q_{12})^2} \frac{1}{q_{12} - M_Z^2 + i\Gamma_Z M_Z} \\ &\times \left\{ [2M_H^2 (d-4) - 2q_{12} (d-2)] B_0(M_H^2, m_f^2, m_f^2) + 4q_{12} B_0(q_{12}, m_f^2, m_f^2) \right. \\ &\left. + [(d-2) (M_H^2 - q_{12})^2 - 8m_f^2 (M_H^2 - q_{12})] C_0(0, q_{12}, M_H^2, m_f^2, m_f^2, m_f^2) \right\}. \quad (7) \end{aligned}$$

Here, N_C^f and Q_f are the number of colors and the charge quantum number

of fermion f , respectively, while T_{3f} is defined as the hypercharge of fermion f . The form factor $F_2^{(f)}$ is shown in the next subsection. For all fermions f at internal lines, their masses are kept non-zero. Therefore, there are no infrared divergences in this calculation.

We turn our attention to one-loop four-point Feynman diagrams contributing to these decay processes. We first consider one-loop box diagrams where the Higgs couples directly to W -bosons (as shown in Fig. 3). The corresponding amplitude is expressed in a Lorentz-invariant structure as follows:

$$\mathcal{A}_{\text{box}}^{(W)} = \bar{u}(q_1) \left\{ F_3^{(W)} q_1^\nu q_3^\nu + F_4^{(W)} q_2^\nu q_3^\nu + F_5^{(W)} \gamma^\nu \right\} P_L v(q_2) \epsilon_\nu^*(q_3), \quad (8)$$

where $F_5^{(W)} = -F_3^{(W)}(q_1 \cdot q_3) - F_4^{(W)}(q_2 \cdot q_3)$. The analytic results for all form factors in terms of PV-functions are:

$$\begin{aligned} F_3^{(W)} = & \frac{\alpha^2}{2M_W^5 s_W^3} \\ & \times \left\{ (M_H^2 M_W^2 + 2M_W^4) [(C_{22} + C_{12} + C_2)(0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2) \right. \\ & + (C_{22} + C_{12} + C_2)(q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2)] \\ & - M_W^2 m_l^2 [(C_{22} + C_{12} + C_2)(0, 0, q_{13}, m_l^2, m_l^2, M_W^2) \\ & + (C_{22} + C_{12} + C_2)(0, 0, q_{13}, m_l^2, M_W^2, M_W^2)] \\ & + 2M_W^4 m_l^2 [D_0(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, m_l^2, M_W^2, M_W^2) \\ & + D_0(0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, m_l^2, M_W^2) \\ & + D_0(0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, m_l^2, m_l^2, M_W^2)] \\ & + [M_W^2 m_l^2 (M_H^2 + 2M_W^2) + 2(d-2)M_W^6] \\ & \times [(D_{33} + D_{23})(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, m_l^2, M_W^2, M_W^2) \\ & + (D_{33} + D_{23})(0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, m_l^2, m_l^2, M_W^2) \\ & + (D_{33} + D_{23} + D_{13})(0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, m_l^2, M_W^2)] \\ & + [M_W^2 m_l^2 (M_H^2 + 2M_W^2) + 2(d-4)M_W^6] \\ & \times [D_3(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, m_l^2, M_W^2, M_W^2) \\ & + D_3(0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, m_l^2, M_W^2) \\ & + D_3(0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, m_l^2, m_l^2, M_W^2)] \\ & + 4M_W^4 [C_0(0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2) \\ & \left. - M_W^2 D_2(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, m_l^2, M_W^2, M_W^2) \right\}, \quad (9) \end{aligned}$$

$$\begin{aligned}
 F_4^{(W)} = & \frac{\alpha^2}{2M_W^5 \delta_W^3} \\
 & \times \left\{ (M_H^2 M_W^2 + 2M_W^4) [(C_{22} + C_{12} + C_2) (0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2) \right. \\
 & + (C_{22} + C_{12} + C_2) (q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2)] \\
 & - M_W^2 m_l^2 [(C_{22} + C_{12} + C_2) (0, 0, q_{23}, M_W^2, m_l^2, m_l^2) \\
 & + (C_{22} + C_{12} + C_2) (0, 0, q_{23}, M_W^2, M_W^2, m_l^2)] \\
 & + 2M_W^4 m_l^2 [D_0 (0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, m_l^2, M_W^2, M_W^2) \\
 & + D_0 (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, m_l^2, M_W^2) \\
 & + D_0 (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, m_l^2, m_l^2, M_W^2)] \\
 & + [2M_W^2 m_l^2 (M_H^2 + 2M_W^2) + 4(d - 2) M_W^6] \\
 & \times [D_{23} (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, m_l^2, M_W^2) \\
 & + D_{23} (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, m_l^2, m_l^2, M_W^2)] \\
 & + [M_W^2 m_l^2 (M_H^2 + 2M_W^2) + 2(d - 2) M_W^6] \\
 & \times [(D_{33} + D_{23} + D_{13}) (0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, m_l^2, M_W^2, M_W^2) \\
 & + (D_{33} + D_{22} + D_{13} + D_{12}) (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, m_l^2, M_W^2) \\
 & + (D_{33} + D_{22} + D_{13} + D_{12}) (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, m_l^2, m_l^2, M_W^2)] \\
 & + [M_W^2 m_l^2 (M_H^2 + 2M_W^2) + 2d M_W^6] \\
 & \times [D_3 (0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, m_l^2, M_W^2, M_W^2) \\
 & + (D_3 + D_2) (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, m_l^2, M_W^2) \\
 & + (D_3 + D_2) (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, m_l^2, m_l^2, M_W^2)] \\
 & + 4M_W^6 [D_0 (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, m_l^2, M_W^2) \\
 & + (D_0 + D_1) (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, m_l^2, m_l^2, M_W^2) \\
 & + (D_2 + D_1 + D_0) (0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, m_l^2, M_W^2, M_W^2)] \\
 & \left. + 4M_W^4 C_0 (0, q_{12}, M_H^2; M_W^2, M_W^2, M_W^2) \right\}. \tag{10}
 \end{aligned}$$

We have different kinds of D -coefficient functions relating to tensor reduction for box diagrams appearing in all the above form factors. In principle, tensor one-loop box integrals with rank $P \geq 4$ appear in the amplitude of each diagram. However, these tensor integrals are cancelled out. As a result, we have up to D_{ij} (for $i, j = 1, \dots, 3$) functions contributing to the aforementioned form factors.

We finally consider all the one-loop box diagrams in Fig. 4 in which the Higgs couples directly to leptons. The amplitudes for the diagrams are given in the same form as Eq. (8):

$$\mathcal{A}_{\text{box}}^{(l)} = \bar{u}(q_1) \left\{ F_3^{(l)} q_1^\nu q_3 + F_4^{(l)} q_2^\nu q_3 + F_5^{(l)} \gamma^\nu \right\} P_L v(q_2) \epsilon_\nu^*(q_3), \tag{11}$$

with $F_5^{(l)} = -F_3^{(l)}(q_1 \cdot q_3) - F_4^{(l)}(q_2 \cdot q_3)$. All the related form factors read as follows:

$$\begin{aligned}
 F_3^{(l)} = & \frac{\alpha^2}{2M_W^3 s_W^3} m_l^2 \\
 & \times \left\{ (C_{22} + C_{12} + C_2) (0, 0, q_{13}, M_W^2, m_l^2, m_l^2) \right. \\
 & + (C_{22} + C_{12} + C_2) (0, 0, q_{13}, M_W^2, M_W^2, m_l^2) \\
 & + [(6 - d) M_W^2 - m_l^2] D_2 (0, 0, 0, M_H^2; q_{12}, q_{13}; m_l^2, M_W^2, m_l^2, m_l^2) \\
 & + [2m_l^2 + 2(d - 2) M_W^2] [(D_{33} + D_{23}) (0, 0, 0, M_H^2; q_{12}, q_{13}; m_l^2, M_W^2, m_l^2, m_l^2) \\
 & + (D_{33} + D_{23}) (0, 0, 0, M_H^2; q_{23}, q_{13}; m_l^2, M_W^2, M_W^2, m_l^2) \\
 & + (D_{33} + D_{23} + D_{13}) (0, 0, 0, M_H^2; q_{23}, q_{12}; m_l^2, m_l^2, M_W^2, m_l^2)] \\
 & + [(d + 2) M_W^2 + m_l^2] [D_3 (0, 0, 0, M_H^2; q_{12}, q_{13}; m_l^2, M_W^2, m_l^2, m_l^2) \\
 & + D_3 (0, 0, 0, M_H^2; q_{23}, q_{12}; m_l^2, m_l^2, M_W^2, m_l^2) \\
 & + D_3 (0, 0, 0, M_H^2; q_{23}, q_{13}; m_l^2, M_W^2, M_W^2, m_l^2)] \\
 & + 2M_W^2 [D_0 (0, 0, 0, M_H^2; q_{12}, q_{13}; m_l^2, M_W^2, m_l^2, m_l^2) \\
 & + D_0 (0, 0, 0, M_H^2; q_{23}, q_{12}; m_l^2, m_l^2, M_W^2, m_l^2) \\
 & + D_0 (0, 0, 0, M_H^2; q_{23}, q_{13}; m_l^2, M_W^2, M_W^2, m_l^2)] \left. \right\}, \tag{12}
 \end{aligned}$$

$$\begin{aligned}
 F_4^{(l)} = & \frac{\alpha^2}{2M_W^3 s_W^3} m_l^2 \times \\
 & \times \left\{ (C_{22} + C_{12} + C_2) (0, 0, q_{23}; m_l^2, m_l^2, M_W^2) \right. \\
 & + (C_{22} + C_{12} + C_2) (0, 0, q_{23}; m_l^2, M_W^2, M_W^2) \\
 & + [4m_l^2 + 4(d - 2)M_W^2] [D_{23} (0, 0, 0, M_H^2; q_{23}, q_{12}; m_l^2, m_l^2, M_W^2, m_l^2) \\
 & + D_{23} (0, 0, 0, M_H^2; q_{23}, q_{13}; m_l^2, M_W^2, M_W^2, m_l^2)] \\
 & + [2m_l^2 + 2(d - 2)M_W^2] [(D_{33} + D_{23} + D_{13}) (0, 0, 0, M_H^2; q_{12}, q_{13}; m_l^2, M_W^2, m_l^2, m_l^2) \\
 & + (D_{33} + D_{22} + D_{13} + D_{12}) (0, 0, 0, M_H^2; q_{23}, q_{12}; m_l^2, m_l^2, M_W^2, m_l^2) \\
 & + (D_{33} + D_{22} + D_{13} + D_{12}) (0, 0, 0, M_H^2; q_{23}, q_{13}; m_l^2, M_W^2, M_W^2, m_l^2)] \\
 & + [m_l^2 + (d - 4)M_W^2] [D_0 (0, 0, 0, M_H^2; q_{12}, q_{13}; m_l^2, M_W^2, m_l^2, m_l^2) \\
 & + D_0 (0, 0, 0, M_H^2; q_{23}, q_{12}; m_l^2, m_l^2, M_W^2, m_l^2) \\
 & + D_0 (0, 0, 0, M_H^2; q_{23}, q_{13}; m_l^2, M_W^2, M_W^2, m_l^2)] \\
 & + [3m_l^2 + (3d - 10)M_W^2] [D_3 (0, 0, 0, M_H^2; q_{12}, q_{13}; m_l^2, M_W^2, m_l^2, m_l^2) \\
 & + (D_3 + D_2) (0, 0, 0, M_H^2; q_{23}, q_{12}; m_l^2, m_l^2, M_W^2, m_l^2) \\
 & + (D_3 + D_2) (0, 0, 0, M_H^2; q_{23}, q_{13}; m_l^2, M_W^2, M_W^2, m_l^2)] \\
 & + [m_l^2 + (d - 6)M_W^2] [D_1 (0, 0, 0, M_H^2; q_{23}, q_{13}; m_l^2, M_W^2, M_W^2, m_l^2) \\
 & + (D_2 + D_1) (0, 0, 0, M_H^2; q_{12}, q_{13}; m_l^2, M_W^2, m_l^2, m_l^2)] \left. \right\}. \tag{13}
 \end{aligned}$$

We find that this contribution is much smaller than other attributions because it is proportional m_τ^2 . It is enough to take the τ -lepton for this contribution.

2.1.2 *Second expression for form factors.* Another expression for all the form factors appearing in this paper is:

$$\begin{aligned}
F_2^{(W)} = & \frac{\alpha^2}{2M_W^3 s_W^3 (q_{12} - M_Z^2 + i\Gamma_Z M_Z)} \\
& \times \left\{ \frac{q_{12}}{(d-1)} [2M_W^2 B_0(0, M_W^2, M_W^2) - (d-2) A_0(M_W^2)] \right. \\
& + \frac{q_{12}}{(d-2)(M_H^2 - q_{12})} [M_H^2 (q_{12} - 2M_W^2) - 4(d-1) M_W^4 + 2M_W^2 q_{12}] \\
& \times B_0(q_{12}, M_W^2, M_W^2) \\
& + \frac{1}{2(2-d)(M_H^2 - q_{12})} \\
& \times \left\{ (d-4) M_H^4 (2M_W^2 - q_{12}) - 2M_W^2 q_{12} (d-2) [2M_W^2 (d-1) - q_{12}] \right. \\
& + M_H^2 [4M_W^4 (d-1)(d-4) - 4M_W^2 q_{12} (d-3) + (d-2) q_{12}^2] \left. \right\} \\
& \times B_0(M_H^2, M_W^2, M_W^2) \\
& + \frac{2M_W^2}{(d-2)} \left\{ M_H^2 [(10-4d) M_W^2 + (d-3) q_{12}] \right. \\
& + 4M_W^4 (d-1) + 2M_W^2 q_{12} (2d-5) - (d-2) q_{12}^2 \left. \right\} \\
& \times C_0(0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2) \left. \right\}. \tag{14}
\end{aligned}$$

For fermion particles exchanging in one-loop triangle Feynman diagrams, we also have the following form factor:

$$\begin{aligned}
F_2^{(f)} = & \frac{\alpha^2 N_C^f Q_f m_f^2 (T_3^f - 2Q_f s_W^2)}{2M_W c_W^2 s_W^3 (d-2)(M_H^2 - q_{12})(q_{12} - M_Z^2 + i\Gamma_Z M_Z)} \\
& \times \left\{ [4M_H^2 - 2(d-2)(M_H^2 - q_{12})] B_0(M_H^2, m_f^2, m_f^2) - 4q_{12} B_0(q_{12}, m_f^2, m_f^2) \right. \\
& + (M_H^2 - q_{12}) [8m_f^2 - (d-2)(M_H^2 - q_{12})] C_0(0, q_{12}, M_H^2, m_f^2, m_f^2, m_f^2) \left. \right\}. \tag{15}
\end{aligned}$$

The form factor $F_5^{(W)}$ of the box diagrams in Fig. (3) is expressed as:

$$\begin{aligned}
 F_5^{(W)} = & \frac{\alpha^2}{4M_W^5 s_W^3} \\
 & \times \left\{ 2M_W^2 A_0(M_W^2) - 4M_W^2 B_{00}(0, M_W^2, M_W^2) - M_H^2 M_W^2 B_0(M_H^2, M_W^2, M_W^2) \right. \\
 & + M_W^2 (2M_W^2 + m_l^2) B_0(0, M_W^2, m_l^2) + M_W^2 (m_l^2 - 4M_W^2) B_0(q_{13}, m_l^2, M_W^2) \\
 & - M_W^2 m_l^2 q_{23} (C_2 + C_1 + C_0)(0, 0, q_{23}, M_W^2, m_l^2, m_l^2) \\
 & - 2M_W^2 m_l^2 [C_{00}(0, 0, q_{13}, m_l^2, m_l^2, M_W^2) + C_{00}(0, 0, q_{13}, m_l^2, M_W^2, M_W^2) \\
 & + C_{00}(0, 0, q_{23}, M_W^2, m_l^2, m_l^2) + C_{00}(0, 0, q_{23}, M_W^2, M_W^2, m_l^2)] \\
 & + [2M_W^6 (1 - d) - 2M_H^2 M_W^4 - M_H^2 m_l^2 M_W^2] C_0(0, q_{13}, M_H^2, M_W^2, m_l^2, M_W^2) \\
 & + 2M_W^4 \left\{ [(q_{13} - 3M_H^2) C_2 + (q_{13} - M_H^2) C_1] (0, q_{13}, M_H^2, M_W^2, m_l^2, M_W^2) \right. \\
 & \left. + 2(q_{12} - M_H^2) C_0(0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2) \right\} \\
 & + 2M_W^2 (2M_W^2 + M_H^2) [C_{00}(0, q_{12}, M_H^2; M_W^2, M_W^2, M_W^2) \\
 & + C_{00}(q_{12}, 0, M_H^2; M_W^2, M_W^2, M_W^2)] \\
 & - [2M_W^4 q_{13} (2M_W^2 + m_l^2) + 2M_W^4 q_{23} (4M_W^2 + m_l^2)] \\
 & \times D_0(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, m_l^2, M_W^2, M_W^2) \\
 & - [2M_W^4 q_{13} m_l^2 + 2M_W^4 q_{23} (2M_W^2 + m_l^2)] \\
 & \times D_0(0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, m_l^2, M_W^2) \\
 & + \{ M_W^2 q_{23} [M_H^2 m_l^2 + 2M_W^4 (d - 6)] - 2M_W^4 q_{13} m_l^2 \} \\
 & \times D_0(0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, m_l^2, m_l^2, M_W^2) \\
 & + M_W^2 q_{23} [2M_W^4 (d - 6) + m_l^2 (2M_W^2 + M_H^2)] \\
 & \times (D_3 + D_2 + D_1)(0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, m_l^2, m_l^2, M_W^2) \\
 & + [4M_W^6 (d - 2) + 2M_W^2 m_l^2 (M_H^2 + 2M_W^2)] \\
 & \times [D_{00}(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, m_l^2, M_W^2, M_W^2) \\
 & + D_{00}(0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, m_l^2, M_W^2) \\
 & + D_{00}(0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, m_l^2, m_l^2, M_W^2)] \\
 & - 8M_W^6 q_{23} [(D_3 + D_2 + D_1)(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, m_l^2, M_W^2, M_W^2) \\
 & \left. + (D_3 + D_2)(0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, m_l^2, M_W^2)] \right\}. \tag{16}
 \end{aligned}$$

Similarly, the form factor $F_5^{(l)}$ of one-loop box diagrams in Fig. 4 is as follows:

$$\begin{aligned}
F_5^{(l)} = & \frac{\alpha^2}{4M_W^3 s_W^3} m_l^2 \\
& \times \left\{ B_0(M_H^2, m_l^2, m_l^2) - 3B_0(q_{13}, M_W^2, m_l^2) - q_{23}C_2(0, 0, q_{23}, m_l^2, m_l^2, M_W^2) \right. \\
& + [(q_{13} - 3M_H^2)C_2 + (q_{13} - M_H^2)C_1](0, q_{13}, M_H^2, m_l^2, M_W^2, m_l^2) \\
& + [(5 - 2d)M_W^2 - M_H^2 - 3m_l^2]C_0(0, q_{13}, M_H^2, m_l^2, M_W^2, m_l^2) \\
& + 2C_{00}(0, 0, q_{13}, M_W^2, m_l^2, m_l^2) + 2C_{00}(0, 0, q_{13}, M_W^2, M_W^2, m_l^2) \\
& + 2C_{00}(0, 0, q_{23}, m_l^2, m_l^2, M_W^2) + 2C_{00}(0, 0, q_{23}, m_l^2, M_W^2, M_W^2) \\
& + [4M_W^2(d - 2) + 4m_l^2][D_{00}(0, 0, 0, M_H^2; q_{12}, q_{13}; m_l^2, M_W^2, m_l^2, m_l^2) \\
& + D_{00}(0, 0, 0, M_H^2; q_{23}, q_{12}; m_l^2, m_l^2, M_W^2, m_l^2) \\
& + D_{00}(0, 0, 0, M_H^2; q_{23}, q_{13}; m_l^2, M_W^2, M_W^2, m_l^2)] \\
& + q_{13}[(4 - d)M_W^2 - m_l^2]D_0(0, 0, 0, M_H^2; q_{12}, q_{13}; m_l^2, M_W^2, m_l^2, m_l^2) \\
& + q_{23}[(10 - 2d)M_W^2 - 2m_l^2]D_0(0, 0, 0, M_H^2; q_{12}, q_{13}; m_l^2, M_W^2, m_l^2, m_l^2) \\
& + [(4 - d)M_W^2 q_{23} - m_l^2 q_{23} - 2M_W^2 q_{13}]D_0(0, 0, 0, M_H^2; q_{23}, q_{12}; m_l^2, m_l^2, M_W^2, m_l^2) \\
& + 2q_{23}[(6 - d)M_W^2 - m_l^2][(D_3 + D_2 + D_1)(0, 0, 0, M_H^2; q_{12}, q_{13}; m_l^2, M_W^2, m_l^2, m_l^2) \\
& + (D_3 + D_2)(0, 0, 0, M_H^2; q_{23}, q_{12}; m_l^2, m_l^2, M_W^2, m_l^2)] \\
& + 8M_W^2 q_{23}(D_3 + D_2 + D_1)(0, 0, 0, M_H^2; q_{23}, q_{13}; m_l^2, M_W^2, M_W^2, m_l^2) \\
& \left. + 2M_W^2(3q_{23} - q_{13})D_0(0, 0, 0, M_H^2; q_{23}, q_{13}; m_l^2, M_W^2, M_W^2, m_l^2) \right\}. \tag{17}
\end{aligned}$$

Having the form factors, we can now calculate the decay widths. The differential decay width is derived in detail in Appendix C. Several decay width formulas are presented in terms of the above form factors. In this subsection we present one of the differential decay width expressions, which is written in terms of F_1 , F_3 , and F_4 :

$$\begin{aligned}
\frac{d^2\Gamma(H \rightarrow \nu_l \bar{\nu}_l \gamma)}{dq_{12}dq_{13}} = & \frac{q_{12}}{512\pi^3 M_H^3} \{ (|F_1|^2 + |F_3|^2 + 2\text{Re}[F_1 F_3^*]) q_{13}^2 \\
& + (|F_1|^2 + |F_4|^2 + 2\text{Re}[F_1 F_4^*]) q_{23}^2 \}, \tag{18}
\end{aligned}$$

where

$$F_i = \sum_f F_i^{(f)} + F_i^{(W)}, \quad F_j = \sum_{l=e,\mu,\tau} F_j^{(l)} + F_j^{(W)} \quad \text{for } i = 1, 2, \quad j = 3, 4, 5, \tag{19}$$

and the integration region is

$$0 \leq q_{12} \leq M_H^2, \quad 0 \leq q_{13} \leq M_H^2 - q_{12}. \tag{20}$$

Other forms for the differential decay width expressed in terms of F_2 , F_3 (or F_4), and F_5 are shown in Appendix C. It is important to note that all the form factors appearing in this paper are used for computing the decay widths.

Table 1. Numerical checks for $F_1^{(W)}$ of triangle diagrams with exchanging W bosons.

Diagrams	(C_{UV}, μ^2)	
	(0, 1)	$(10^5, 10^7)$
Diagram 1	$3.822254271421409 \cdot 10^{-9}$ $+ 1.1328167325068205 \cdot 10^{-10} i$	-0.000025848850122985293 $- 7.660926734863565 \cdot 10^{-7} i$
Diagram 2	$-4.284907193808991 \cdot 10^{-9}$ $- 1.2699350230775137 \cdot 10^{-10} i$	0.00005170105984731962 $+ 1.5322849168169064 \cdot 10^{-6} i$
$2 \times$ Diagram 1 $+$ Diagram 2	$3.3596013490338273 \cdot 10^{-9}$ $+ 9.956984419361279 \cdot 10^{-11} i$	$3.3596013490338273 \cdot 10^{-9}$ $+ 9.956984419361279 \cdot 10^{-11} i$

In Appendix B, all the reduction formulas for PV-functions expressed in terms of the scalar one-loop integrals A_0 , B_0 , C_0 , and D_0 are derived in space-time dimensions d . Analytical results for these scalar integrals are well known in $d = 4 - 2\varepsilon$ at ε^0 expansion, and they can be evaluated numerically by using LoopTools [12].

We end the analytical calculations with an interesting point for future prospects. Recently, we derived hypergeometric representations for scalar one-loop integrals in general space-time dimensions d in Refs. [13–15]. By expressing the scalar one-loop integrals A_0 , B_0 , C_0 , and D_0 in terms of the hypergeometric functions in Refs. [13–15], we guarantee that all form factors in this paper will be valid in general d . In future, one may consider two-loop and higher-loop integrals contributing to these channels. In this case, the analytic results for the one-loop form factors in this paper can be used to perform higher-power ε expansion.

2.2 Phenomenological results

All the physical results for the decay channels are examined with the updated input parameters in Ref. [20]. In detail, we use the following input parameters: $\alpha = 1/137.035999084$, $M_Z = 91.1876$ GeV, $\Gamma_Z = 2.4952$ GeV, $M_W = 80.379$ GeV, $M_H = 125.1$ GeV, $m_\tau = 1.77686$ GeV, $m_t = 172.76$ GeV, $m_b = 4.18$ GeV, $m_s = 0.93$ GeV, and $m_c = 1.27$ GeV.

Before we discuss the phenomenological results for these processes, numerical checks for the computations are first performed. The results must be independent of the ultraviolet cutoff ($C_{UV} = 1/\varepsilon$) and μ^2 parameters (μ^2 plays the role of a renormalization scale [12]). We take the form factors $F_1^{(W)}$, $F_3^{(W)}$, and $F_4^{(W)}$ as typical examples. The numerical results are shown at arbitrary sampling points in the physical region in Tables 1, 2, and 3.

After verifying the ultraviolet finiteness and μ^2 independence of the form factors, we perform a further test for the results: checking the ward identity for the amplitudes. As we mentioned in previous sections, two representations for the form factors are derived in this paper. Their relations are tested numerically. The numerical results are shown at arbitrary sampling points in the physical region:

$$\begin{aligned}
 F_1^{(W)} &= \left(\frac{2}{q_{12} - M_H^2} \right) F_2^{(W)} \\
 &= 4.417241820666953 \cdot 10^{-8} + 5.8717660992932236 \cdot 10^{-8} i,
 \end{aligned} \tag{21}$$

Table 2. Numerical checks for $F_3^{(W)}$ box diagrams with exchanging W bosons.

Diagrams	(C_{UV}, μ^2)	
	(0, 1)	$(10^5, 10^7)$
Diagram 1	$-1.510623096811763 \cdot 10^{-10}$ $-1.703281388817785 \cdot 10^{-24} i$	$-1.510623096811763 \cdot 10^{-10}$ $-1.703281388817785 \cdot 10^{-24} i$
Diagram 2	$6.682548276674295 \cdot 10^{-10}$ $+6.582748337547062 \cdot 10^{-10} i$	$6.682548276674295 \cdot 10^{-10}$ $+6.582748337547062 \cdot 10^{-10} i$
Diagram 3	$-1.5802922737417574 \cdot 10^{-10}$ $+1.680875383453354 \cdot 10^{-10} i$	$-1.5802922737417574 \cdot 10^{-10}$ $+1.680875383453354 \cdot 10^{-10} i$
Sum	$3.591632904379816 \cdot 10^{-10}$ $+8.263623719173174 \cdot 10^{-10} i$	$3.591632904379816 \cdot 10^{-10}$ $+8.263623719173174 \cdot 10^{-10} i$

Table 3. Numerical checks for $F_4^{(W)}$ box diagrams with exchanging W bosons.

Diagrams	(C_{UV}, μ^2)	
	(0, 1)	$(10^5, 10^7)$
Diagram 1	$-2.8890520180530153 \cdot 10^{-10}$ $+3.09465553621708 \cdot 10^{-25} i$	$-2.8890520180530153 \cdot 10^{-10}$ $+3.09465553621708 \cdot 10^{-25} i$
Diagram 2	$5.932051711362282 \cdot 10^{-10}$ $+9.380790178780355 \cdot 10^{-10} i$	$5.932051711362282 \cdot 10^{-10}$ $+9.380790178780355 \cdot 10^{-10} i$
Diagram 3	$7.503951368583933 \cdot 10^{-11}$ $+3.476774032077318 \cdot 10^{-10} i$	$7.503951368583933 \cdot 10^{-11}$ $+3.476774032077318 \cdot 10^{-10} i$
Sum	$3.793394830167515 \cdot 10^{-10}$ $+1.2857564210858306 \cdot 10^{-9} i$	$3.793394830167515 \cdot 10^{-10}$ $+1.2857564210858306 \cdot 10^{-9} i$

$$\begin{aligned}
F_1^{(\tau)} &= \left(\frac{2}{q_{12} - M_H^2} \right) F_2^{(\tau)} \\
&= -2.906788516027605 \cdot 10^{-12} - 1.660128209352433 \cdot 10^{-12} i, \quad (22)
\end{aligned}$$

$$\begin{aligned}
F_5^{(W)} &= -F_3^{(W)}(q_1 \cdot q_3) - F_4^{(W)}(q_2 \cdot q_3) \\
&= -8.31386693767833 \cdot 10^{-7} + 2.076359788449464 \cdot 10^{-20} i, \quad (23)
\end{aligned}$$

$$\begin{aligned}
F_5^{(\tau)} &= -F_3^{(\tau)}(q_1 \cdot q_3) - F_4^{(\tau)}(q_2 \cdot q_3) \\
&= 9.43776046706387 \cdot 10^{-9} - 3.509321200912556 \cdot 10^{-9} i. \quad (24)
\end{aligned}$$

We find that the results show good stability with varying C_{UV}, μ^2 parameters as well as following the ward identity checks.

The computations are confirmed successfully by numerical checks. The phenomenological results for the decay channels are studied. In the present paper the events of Higgs decay to invisible particles and to a photon plus invisible particles are computed. We first arrive at the case of a photon that may or may not be tested. In this case, we do not apply any cut for the

Table 4. Decay widths in the case of a photon that can be tested.

Γ [keV]	E_γ^{cut} [GeV]		
	5	30	50
$\Gamma_{H \rightarrow \gamma \nu_l \bar{\nu}_l}^{\text{Trig}}$	0.464684(7)	0.171761(3)	0.004632(7)
$\Gamma_{H \rightarrow \gamma \nu_l \bar{\nu}_l}^{\text{Trig} \times \text{W-box}}$	0.006064(2)	0.005928(6)	0.001541(7)
$\Gamma_{H \rightarrow \gamma \nu_l \bar{\nu}_l}^{\text{Trig} \times \text{lepton-box}}$	0.000042(4)	0.000006(2)	0.000000(1)
$\Gamma_{H \rightarrow \gamma \nu_l \bar{\nu}_l}^{\text{tot}}$	0.471108(1)	0.178727(1)	0.006398(1)

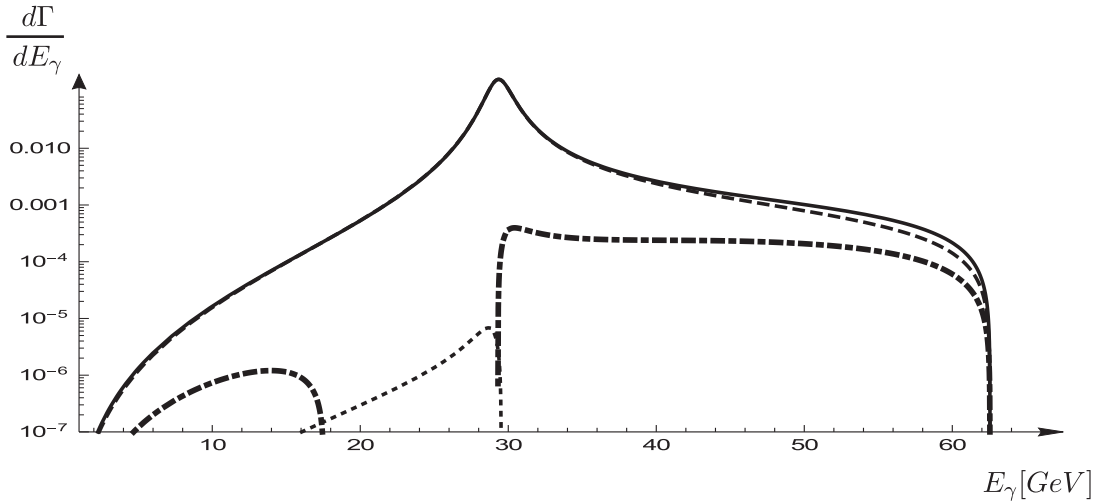


Fig. 5. Differential decay width as a function of E_γ .

final photon. The partial decay width for $H \rightarrow \nu_l \bar{\nu}_l \gamma$ in which l can be one of e , μ , or τ is:

$$\begin{aligned} \Gamma_{H \rightarrow \gamma \nu_l \bar{\nu}_l}^{\text{tot}} &= 0.471109(5) \text{ keV}, \\ \Gamma_{H \rightarrow \gamma \nu_l \bar{\nu}_l}^{\text{Trig}} &= 0.464686(1) \text{ keV}, \\ \Gamma_{H \rightarrow \gamma \nu_l \bar{\nu}_l}^{\text{Trig} \times \text{W-box}} &= 0.006064(4) \text{ keV}, \\ \Gamma_{H \rightarrow \gamma \nu_l \bar{\nu}_l}^{\text{Trig} \times \text{lepton-box}} &= 0.000042(5) \text{ keV}. \end{aligned}$$

$\Gamma_{H \rightarrow \gamma \nu_l \bar{\nu}_l}^{\text{tot}}$ is for the case of taking all Feynman diagrams. It gives perfect agreement with the result in Ref. [11]. $\Gamma_{H \rightarrow \gamma \nu_l \bar{\nu}_l}^{\text{Trig}}$ is for the contributions of all triangle diagrams. This gives dominant contributions in comparison with other parts. The contributions of the interference between three-point diagrams and box diagrams, $\Gamma_{H \rightarrow \gamma \nu_l \bar{\nu}_l}^{\text{Trig} \times \text{W-box}}$ (for the box diagrams in Fig. 3) and $\Gamma_{H \rightarrow \gamma \nu_l \bar{\nu}_l}^{\text{Trig} \times \text{lepton-box}}$ (for the box diagrams in Fig. 4), are shown in these equations. These contributions are much smaller than the results for the triangle diagrams.

We are also concerned with the case of a photon that can be tested. In this case, we apply energy cuts for the final photon. The results are shown in Table 4.

Figure 5 shows the differential decay widths as functions of E_γ . The solid line denotes the contributions of all diagrams, the dashed line shows the contributions of triangle diagrams, while the dot-dashed line presents the interference between the three-point diagrams and the box diagrams in Fig. 3 and the dotted line the amplitude of the interference between the three-

point diagrams and the box diagrams in Fig. 4. We find that the dominant contributions are from triangle diagrams. We observe a peak that corresponds to the Z -pole where Z bosons decay to neutrinos. The position of this peak is at $E_\gamma = \frac{M_H^2 - M_Z^2}{2M_H} = 29.3159$ GeV. Beyond this peak, the contributions of the interference between three-point diagrams and box diagrams are visible. We note that all the results presented here are for a family of neutrinos in the final state. For all neutrinos, we simply multiply by a factor of 3 the above results. The physical results are important for testing the Standard Model at modern energy and evaluating precisely the Standard Model background.

Furthermore, we stress that general results for one-loop contributions to the mentioned decay processes in arbitrary settings beyond the Standard Model have been presented in Ref. [21].

As mentioned in this report, the decay processes are related to two kinds of events: Higgs decays to invisible particles, and to a single photon plus missing energy (ME). Both processes have been sought at the LHC [4–6,8,10,22]; updated results have been reported in Ref. [23]. At future Lepton Colliders [3,24], following a recoil mass method in the Higgs-strahlung process $e^-e^+ \rightarrow ZH$, the invisible decays of the Higgs boson can be probed in a model-independent way. In Ref. [25], the author studied the search ability for invisible decays of the Higgs boson at the International Linear Collider (ILC). Applying the same method, dark matter may be probed via the invisible decays of the Higgs boson in Ref. [26]. With the high-luminosity scenario, an upper bound on the branching fraction of the Higgs decaying to an invisible final state may be achieved at a level of $\sim 0.26\%$ [25].

Following the results in this paper, it will be interesting to search for Higgs decays to photon plus ME via the photon energy distribution (see Fig. 5), since the decay rates are dominant around the recoil photon energy at the Z -pole (peak of photon energy). Moreover, by assuming general $\nu_l \bar{\nu}_l$ in the final state, we can extract new physics using Higgs decays to photon plus ME around the peak of photon energy.

3. Conclusions

One-loop contributions to the decay $H \rightarrow \nu_l \bar{\nu}_l \gamma$ with $l = e, \mu, \tau$ within the SM framework have revisited in this paper, and two representations for the form factors have been computed in this calculation. As a result, the calculations are not only checked numerically by verifying their ultraviolet finiteness but also by confirming the ward identity of the amplitude. We find that the results satisfy the ward identity and possess good stability with varying ultraviolet cutoff parameters. For the phenomenological results, all the physical results have examined with the updated input parameters. In detail, we have studied the partial decay widths for the decay channels in both cases of a detected photon and an invisible photon. Differential decay widths have also been generated as a function of the energy of the final photon. The physical results are important for testing the SM at modern energy and evaluating precisely the SM background. We have also pointed out that if one applies hypergeometric functions for scalar one-loop integrals at general space-time dimensions d , the results in this paper can be performed for higher powers of ϵ expansion that may be taken into account in two-loop and higher-loop contributions to the mentioned channels.

Acknowledgments

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 103.01-2019.346.

Funding

Open Access funding: SCOAP³.

Appendix A. Reduction for scalar PV-functions

In this appendix we present all the reduction formulas for scalar PV-functions expressed in terms of the scalar one-loop integrals A_0 , B_0 , C_0 , and D_0 . We first show the reduction for all A , B -functions:

$$A_{00}(M_W^2) = \frac{M_W^2}{d} A_0(M_W^2), \quad (\text{A1})$$

$$B_1(M_H^2, M_W^2, M_W^2) = -\frac{1}{2} B_0(M_H^2, M_W^2, M_W^2), \quad (\text{A2})$$

$$B_{00}(p^2, M_W^2, M_W^2) = \frac{1}{4(d-1)} \{2A_0(M_W^2) - (p^2 - 4M_W^2) B_0(p^2, M_W^2, M_W^2)\}, \quad (\text{A3})$$

$$B_{11}(M_H^2, M_W^2, M_W^2) = \frac{1}{4M_H^2(d-1)} \{(dM_H^2 - 4M_W^2) \times B_0(M_H^2, M_W^2, M_W^2) + 2(d-2)A_0(M_W^2)\}, \quad (\text{A4})$$

$$B_{001}(M_H^2, M_W^2, M_W^2) = \frac{1}{8(d-1)} \times \{(M_H^2 - 4M_W^2) B_0(M_H^2, M_W^2, M_W^2) - 2A_0(M_W^2)\}, \quad (\text{A5})$$

$$B_{111}(M_H^2, M_W^2, M_W^2) = \frac{1}{8M_H^2(1-d)} \{[(d+2)M_H^2 - 12M_W^2] \times B_0(M_H^2, M_W^2, M_W^2) + 6(d-2)A_0(M_W^2)\}. \quad (\text{A6})$$

Here, $p^2 = M_H^2, 0$.

We next have the following relations for the C -functions:

$$C_1(0, q_{13}, M_H^2, M_W^2, 0, M_W^2) = \frac{1}{(M_H^2 - q_{13})^2} \left[(M_H^2 - q_{13}) B_0(0, M_W^2, 0) + (M_H^2 + q_{13}) B_0(q_{13}, 0, M_W^2) - 2M_H^2 B_0(M_H^2, M_W^2, M_W^2) + M_H^2 (2M_W^2 - M_H^2 + q_{13}) \times C_0(0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \right], \quad (\text{A7})$$

$$C_2(0, q_{13}, M_H^2, M_W^2, 0, M_W^2) = \frac{1}{M_H^2 - q_{13}} \left[B_0(M_H^2, M_W^2, M_W^2) - B_0(q_{13}, 0, M_W^2) - M_W^2 C_0(0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \right], \quad (\text{A8})$$

$$\begin{aligned}
 & C_2(q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2) \\
 &= \frac{1}{(M_H^2 - q_{12})^2} \left\{ (q_{12} - M_H^2) B_0(0, M_W^2, M_W^2) + (M_H^2 + q_{12}) B_0(M_H^2, M_W^2, M_W^2) \right. \\
 &\quad \left. - 2q_{12} B_0(q_{12}, M_W^2, M_W^2) + q_{12} (M_H^2 - q_{12}) C_0(q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2) \right\}, \tag{A9}
 \end{aligned}$$

$$\begin{aligned}
 & C_{00}(0, q_{12}, M_H^2, M^2, M^2, M^2) \\
 &= \frac{1}{2(d-2)(M_H^2 - q_{12})} \left\{ M_H^2 B_0(M_H^2, M^2, M^2) - q_{12} B_0(q_{12}, M^2, M^2) \right. \\
 &\quad \left. + 2M^2 (M_H^2 - q_{12}) C_0(0, q_{12}, M_H^2, M^2, M^2, M^2) \right\}, \tag{A10}
 \end{aligned}$$

$$C_{00}(q_{12}, 0, M_H^2; M^2, M^2, M^2) = C_{00}(0, q_{12}, M_H^2; M^2, M^2, M^2), \tag{A11}$$

$$\begin{aligned}
 & C_{12}(0, q_{12}, M_H^2, M^2, M^2, M^2) \\
 &= \frac{1}{2(d-2)(M_H^2 - q_{12})^2} \left\{ -2M_H^2 B_0(M_H^2, M^2, M^2) \right. \\
 &\quad + [(d-2)M_H^2 - (d-4)q_{12}] B_0(q_{12}, M^2, M^2) \\
 &\quad \left. + 4M^2 (q_{12} - M_H^2) C_0(0, q_{12}, M_H^2, M^2, M^2, M^2) \right\}, \tag{A12}
 \end{aligned}$$

$$\begin{aligned}
 C_{22}(0, q_{12}, M_H^2, M^2, M^2, M^2) &= \frac{1}{2(M_H^2 - q_{12})} \\
 &\quad \times \{ B_0(q_{12}, M^2, M^2) - B_0(M_H^2, M^2, M^2) \}, \tag{A13}
 \end{aligned}$$

where $M^2 = M_W^2, m_f^2$. Further, we also have

$$\begin{aligned}
 & C_{12}(q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2) \\
 &= \frac{1}{(d-2)(M_H^2 - q_{12})^3} \left\{ (d-2)(M_H^2 - q_{12}) A_0(M_W) \right. \\
 &\quad + (d-2)(M_H^2 + M_W^2)(M_H^2 - q_{12}) B_0(0, M_W^2, M_W^2) \\
 &\quad + [(3-2d)M_H^2 q_{12} - M_H^4] B_0(M_H^2, M_W^2, M_W^2) \\
 &\quad + [(2d-3)M_H^2 q_{12} + q_{12}^2] B_0(q_{12}, M_W^2, M_W^2) \\
 &\quad + (q_{12} - M_H^2) [2M_H^2 M_W^2 + M_H^2 q_{12} (d-2) + 2M_W^2 q_{12}] \\
 &\quad \left. \times C_0(q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2) \right\}, \tag{A14}
 \end{aligned}$$

$$\begin{aligned}
& C_{22}(q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2) \\
&= \frac{1}{2(d-2)(M_H^2 - q_{12})^3} \left\{ (4-2d)(M_H^2 - q_{12}) A_0(M_W^2) \right. \\
&+ (4-2d)(M_H^2 - q_{12})(M_W^2 + q_{12}) B_0(0, M_W^2, M_W^2) \\
&+ 4q_{12}^2(1-d) B_0(q_{12}, M_W^2, M_W^2) \\
&+ [(2-d)(M_H^4 - 3q_{12}^2) + 2dM_H^2q_{12}] B_0(M_H^2, M_W^2, M_W^2) \\
&\left. + 2q_{12}(M_H^2 - q_{12}) [4M_W^2 + (d-2)q_{12}] C_0(q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2) \right\}. \quad (\text{A15})
\end{aligned}$$

We finally present all the relations for the D -functions. In this paper, we consider reduction formulas for all D -functions involving the decay processes in the limit $m_l \rightarrow 0$. In detail, the relations are:

$$\begin{aligned}
& D_1(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2) \\
&= \frac{1}{2q_{12}q_{13}q_{23}} \left\{ q_{13}(M_H^2 - q_{12}) C_0(0, 0, q_{13}; 0, M_W^2, M_W^2) \right. \\
&+ q_{12}(q_{13} - q_{23}) C_0(0, 0, q_{12}, M_W^2, 0, M_W^2) \\
&+ [q_{12}(q_{12} + q_{23}) - M_H^2(q_{12} - q_{23})] C_0(0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \\
&- (M_H^2 - q_{12})^2 C_0(q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2) \\
&+ (M_H^2 - q_{12}) [M_H^2(M_W^2 - q_{12}) + q_{12}(-M_W^2 + q_{12} + q_{23})] \\
&\left. \times D_0(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2) \right\}, \quad (\text{A16})
\end{aligned}$$

$$\begin{aligned}
& D_1(0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2) \\
&= \frac{1}{2q_{12}q_{13}q_{23}} \left\{ q_{13}(q_{12} + q_{13}) C_0(0, 0, q_{13}, 0, 0, M_W^2) \right. \\
&+ q_{23}(q_{13} - q_{12}) C_0(0, 0, q_{23}, M_W^2, 0, 0) \\
&+ (M_H^2q_{12} - q_{13}q_{23}) C_0(0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \\
&- (q_{12} + q_{13})^2 C_0(q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) \\
&- \{M_H^4(q_{23} - M_W^2) + M_H^2[M_W^2(q_{12} + q_{23}) - q_{23}(q_{12} + 2q_{23})] \\
&+ q_{23}[M_W^2q_{12} + q_{23}(q_{12} + q_{23})]\} \\
&\left. \times D_0(0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2) \right\}, \quad (\text{A17})
\end{aligned}$$

$$\begin{aligned}
 & D_2(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2) \\
 &= \frac{1}{2q_{12}q_{13}q_{23}} \left\{ q_{13}(q_{12} - q_{23}) C_0(0, 0, q_{13}; 0, M_W^2, M_W^2) \right. \\
 &\quad - (q_{12} + q_{23})^2 C_0(0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \\
 &\quad + [q_{12}(q_{12} + q_{23}) - M_H^2(q_{12} - q_{23})] C_0(q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2) \\
 &\quad + q_{12}(q_{12} + q_{23}) C_0(0, 0, q_{12}, M_W^2, 0, M_W^2) \\
 &\quad + [M_H^2 M_W^2 (q_{12} - q_{23}) - M_H^2 q_{12} (q_{12} + q_{23}) + q_{12} (q_{12} + q_{23}) (-M_W^2 + q_{12} + q_{23})] \\
 &\quad \left. \times D_0(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2) \right\}, \tag{A18}
 \end{aligned}$$

$$\begin{aligned}
 & D_2(0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2) \\
 &= \frac{1}{2q_{12}q_{13}q_{23}} \left\{ - (q_{12} + q_{23})^2 C_0(0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \right. \\
 &\quad - q_{13}(q_{12} - q_{23}) C_0(0, 0, q_{13}, 0, 0, M_W^2) \\
 &\quad + (M_H^2 q_{12} - q_{13} q_{23}) C_0(q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) + q_{23}(q_{12} + q_{23}) C_0(0, 0, q_{23}, M_W^2, 0, 0) \\
 &\quad + \{ (q_{12} + q_{23}) [M_W^2 q_{12} + q_{23}(q_{12} + q_{23})] - M_H^2 [M_W^2 (q_{12} - q_{23}) + q_{23}(q_{12} + q_{23})] \} \\
 &\quad \left. \times D_0(0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2) \right\}, \tag{A19}
 \end{aligned}$$

$$\begin{aligned}
 & D_2(0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2) \\
 &= -\frac{1}{2q_{12}q_{13}q_{23}} \left\{ q_{12}(q_{13} - q_{23}) C_0(0, 0, q_{12}, M_W^2, 0, M_W^2) \right. \\
 &\quad + q_{23}(q_{12} - M_H^2) C_0(0, 0, q_{23}, M_W^2, M_W^2, 0) \\
 &\quad + (M_H^2 - q_{12})^2 C_0(0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2) \\
 &\quad + (q_{12}q_{23} - M_H^2 q_{13}) C_0(q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) \\
 &\quad + (q_{12} - M_H^2) [M_H^2 M_W^2 - q_{12}(M_W^2 + q_{23})] \\
 &\quad \left. \times D_0(0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2) \right\}, \tag{A20}
 \end{aligned}$$

$$\begin{aligned}
 & D_3(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2) \\
 &= \frac{1}{2q_{13}q_{23}} \left\{ (q_{12} + q_{23}) C_0(0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \right. \\
 &\quad - q_{13} C_0(0, 0, q_{13}, 0, M_W^2, M_W^2) \\
 &\quad + (q_{13} - q_{23}) C_0(q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2) - q_{12} C_0(0, 0, q_{12}, M_W^2, 0, M_W^2) \\
 &\quad + [M_W^2 (q_{12} + 2q_{23}) - M_H^2 (M_W^2 - q_{12}) - 2q_{12} (q_{12} + q_{23})] \\
 &\quad \left. \times D_0(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2) \right\}, \tag{A21}
 \end{aligned}$$

$$\begin{aligned}
 & D_3(0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2) \\
 &= -\frac{1}{2q_{12}q_{13}} \left\{ (q_{13} - q_{12}) C_0(q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) \right. \\
 &\quad + q_{12} C_0(0, 0, q_{12}, M_W^2, 0, M_W^2) + q_{23} C_0(0, 0, q_{23}, M_W^2, M_W^2, 0) \\
 &\quad - (M_H^2 - q_{12}) C_0(0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2) \\
 &\quad \left. + [M_W^2 (M_H^2 - q_{12}) - q_{12}q_{23}] D_0(0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2) \right\}, \tag{A22}
 \end{aligned}$$

$$\begin{aligned}
 & D_3(0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2) \\
 &= -\frac{1}{2q_{12}q_{13}} \left\{ q_{13} C_0(0, 0, q_{13}, 0, 0, M_W^2) \right. \\
 &\quad - (q_{12} + q_{23}) C_0(0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \\
 &\quad + (q_{12} - q_{13}) C_0(q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) + q_{23} C_0(0, 0, q_{23}, M_W^2, 0, 0) \\
 &\quad \left. + [M_W^2 (q_{13} + q_{23}) - q_{23}q_{13}] D_0(0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2) \right\}. \tag{A23}
 \end{aligned}$$

For one-loop tensor four-point diagrams with rank $P = 2$, we have:

$$\begin{aligned}
 & D_{00}(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2) \\
 &= \frac{1}{4(3-d)q_{12}q_{13}q_{23}} \left\{ [M_W^2 q_{12} (q_{13} - q_{23}) \right. \\
 &\quad - M_W^2 q_{23} (q_{13} + q_{23}) - q_{12}q_{13} (q_{12} + q_{23})] \\
 &\quad \times C_0(0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \\
 &\quad + (q_{13} + q_{23}) [M_W^2 (q_{13} + q_{23}) - q_{12}q_{13}] C_0(q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2) \\
 &\quad + q_{12} [M_W^2 (q_{23} - q_{13}) + q_{12}q_{13}] C_0(0, 0, q_{12}, M_W^2, 0, M_W^2) \\
 &\quad + q_{13} [q_{12}q_{13} - M_W^2 (q_{13} + q_{23})] C_0(0, 0, q_{13}, 0, M_W^2, M_W^2) \\
 &\quad - [M_W^4 (q_{13} + q_{23})^2 + 2M_W^2 q_{12}q_{13} (q_{23} - q_{13}) + q_{12}^2 q_{13}^2] \\
 &\quad \left. \times D_0(0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2) \right\}, \tag{A24}
 \end{aligned}$$

$$\begin{aligned}
& D_{00} (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2) \\
&= \frac{1}{4(3-d)q_{12}q_{13}q_{23}} \left\{ (q_{13} + q_{23}) [M_W^2 (q_{13} + q_{23}) - q_{12}q_{23}] \right. \\
&\quad \times C_0 (0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2) \\
&\quad - [M_W^2 q_{12} (q_{13} - q_{23}) + M_W^2 q_{13} (q_{13} + q_{23}) + q_{12}q_{23} (q_{12} + q_{13})] \\
&\quad \times C_0 (q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) \\
&\quad + q_{12} [M_W^2 (q_{13} - q_{23}) + q_{12}q_{23}] C_0 (0, 0, q_{12}, M_W^2, 0, M_W^2) \\
&\quad + q_{23} [q_{12}q_{23} - M_W^2 (q_{13} + q_{23})] C_0 (0, 0, q_{23}, M_W^2, M_W^2, 0) \\
&\quad - [M_W^4 (q_{13} + q_{23})^2 + 2M_W^2 q_{12}q_{23} (q_{13} - q_{23}) + q_{12}^2 q_{23}^2] \\
&\quad \left. \times D_0 (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2) \right\}, \tag{A25}
\end{aligned}$$

$$\begin{aligned}
& D_{00} (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2) \\
&= \frac{1}{4(3-d)q_{12}q_{13}q_{23}} \left\{ [M_W^2 q_{12} (q_{23} - q_{13}) \right. \\
&\quad + M_W^2 q_{23} (q_{13} + q_{23}) - q_{13}q_{23} (q_{12} + q_{23})] \\
&\quad \times C_0 (0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \\
&\quad + \{M_W^2 [q_{12} (q_{13} - q_{23}) + q_{13} (q_{13} + q_{23})] - q_{13}q_{23} (q_{12} + q_{13})\} \\
&\quad \times C_0 (q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) \\
&\quad + q_{13} [q_{13}q_{23} - M_W^2 (q_{13} + q_{23})] C_0 (0, 0, q_{13}, 0, 0, M_W^2) \\
&\quad + q_{23} [q_{13}q_{23} - M_W^2 (q_{13} + q_{23})] C_0 (0, 0, q_{23}, M_W^2, 0, 0) \\
&\quad \left. - [q_{13}q_{23} - M_W^2 (q_{13} + q_{23})]^2 D_0 (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2) \right\}, \tag{A26}
\end{aligned}$$

$$\begin{aligned}
 & D_{12} (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2) \\
 &= \frac{M_H^2}{q_{12}q_{13}q_{23}} B_0 (M_H^2, M_W^2, M_W^2) - \frac{(q_{12} + q_{13})}{q_{12}q_{13}q_{23}} B_0 (q_{13}, 0, M_W^2) \\
 &\quad - \frac{(q_{12} + q_{23})}{q_{12}q_{13}q_{23}} B_0 (q_{23}, M_W^2, 0) + \frac{1}{4(3-d)q_{12}^2q_{13}q_{23}^2} \\
 &\quad \times \left\{ M_W^2 q_{12}^2 [(d-2)q_{13} + (10-3d)q_{23}] + M_W^2 q_{12} [(d-2)(q_{13}^2 + q_{23}^2) - 2q_{13}q_{23}(d-4)] \right. \\
 &\quad \left. - (d-2)M_W^2 q_{13}q_{23}(q_{13} + q_{23}) + q_{13}q_{23} [(d-4)M_H^2 q_{12} + (d-2)q_{13}q_{23}] \right\} \\
 &\quad \times C_0 (0, 0, q_{13}, 0, 0, M_W^2) + \frac{1}{4(3-d)q_{12}^2q_{13}^2q_{23}} \\
 &\quad \times \left\{ M_W^2 q_{12}^2 [(d-2)q_{23} + (10-3d)q_{13}] + M_W^2 q_{12} [(d-2)(q_{13}^2 + q_{23}^2) - 2q_{13}q_{23}(d-4)] \right. \\
 &\quad \left. - (d-2)M_W^2 q_{13}q_{23}(q_{13} + q_{23}) + q_{13}q_{23} [(d-4)M_H^2 q_{12} + (d-2)q_{13}q_{23}] \right\} \\
 &\quad \times C_0 (0, 0, q_{23}, M_W^2, 0, 0) + \frac{1}{4(d-3)q_{12}^2q_{13}^2q_{23}^2} \\
 &\quad \times \left\{ q_{13}q_{23}(q_{12} + q_{23}) [(d-4)M_H^2 q_{12} + (d-2)q_{13}q_{23}] - M_W^2 q_{12}^3 (q_{13} - q_{23})(d-2) \right. \\
 &\quad \left. - M_W^2 q_{12}^2 [(d-2)(q_{13}^2 - 2q_{23}^2) + (3d-10)q_{13}q_{23}] \right. \\
 &\quad \left. - M_W^2 q_{12}q_{23} [2q_{13}^2 (d-4) + (3d-10)q_{13}q_{23} - (d-2)q_{23}^2] \right. \\
 &\quad \left. - M_W^2 q_{13}q_{23}^2 (q_{13} + q_{23})(d-2) \right\} C_0 (0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \\
 &\quad + \frac{1}{4(d-3)q_{12}^2q_{13}^2q_{23}^2} \\
 &\quad \times \left\{ (d-2)M_W^2 q_{12}q_{13}(q_{12} + q_{13})^2 - M_H^2 M_W^2 q_{23} [(d-2)(q_{12}^2 + q_{13}^2) + 2q_{12}q_{13}(d-4)] \right. \\
 &\quad \left. + q_{13}q_{23}(q_{12} + q_{13}) [(d-4)M_H^2 q_{12} + (d-2)q_{13}q_{23}] \right\} C_0 (q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) \\
 &\quad + \frac{1}{4(d-3)q_{12}^2q_{13}^2q_{23}^2} \\
 &\quad \times \left\{ (d-2)M_W^4 q_{13}q_{23}(q_{13} + q_{23})^2 - M_H^2 M_W^4 q_{12} [(d-2)(q_{13}^2 + q_{23}^2) - 2q_{13}q_{23}(d-4)] \right. \\
 &\quad \left. + 2M_W^2 q_{13}q_{23} [q_{12}^2 (q_{13} + q_{23}) + q_{12} (q_{13}^2 - 2(d-4)q_{13}q_{23} + q_{23}^2) \right. \\
 &\quad \left. - (d-2)q_{13}q_{23}(q_{13} + q_{23})] \right. \\
 &\quad \left. + q_{13}^2 q_{23}^2 [(d-4)M_H^2 q_{12} + (d-2)q_{13}q_{23}] \right\} \\
 &\quad \times D_0 (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2), \tag{A27}
 \end{aligned}$$

$$\begin{aligned}
& D_{12} (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2) \\
&= \frac{1}{q_{12}q_{23}} B_0 (0, M_W^2, 0) - \frac{(q_{13} + q_{23})}{q_{12}q_{13}q_{23}} B_0 (q_{23}, M_W^2, 0) \\
&+ \frac{M_H^2}{q_{12}q_{13}q_{23}} B_0 (M_H^2, M_W^2, M_W^2) \\
&- \frac{(q_{12} + q_{13})}{q_{12}q_{13}q_{23}} B_0 (q_{12}, M_W^2, M_W^2) + \frac{1}{4(d-3)q_{12}q_{13}^2q_{23}^2} \\
&\times \left\{ q_{12}M_W^2 [(d-2)(q_{13}^2 + q_{23}^2) + 2q_{13}q_{23}(d-4)] - q_{12}^2q_{23} [(d-4)q_{13} + (d-2)q_{23}] \right. \\
&- (d-4)q_{12}q_{23}q_{13}(q_{13} + q_{23}) + q_{13}M_W^2 [(d-2)(q_{13}^2 - q_{23}^2) + 4q_{13}q_{23}(d-3)] \left. \right\} \\
&\times C_0 (0, 0, q_{12}, M_W^2, 0, M_W^2) + \frac{1}{4(3-d)q_{12}^2q_{13}^2q_{23}} \\
&\times \left\{ (d-2)M_W^2 (q_{13} + q_{23}) [q_{12}(q_{13} - q_{23}) + q_{13}(q_{13} + q_{23})] \right. \\
&+ q_{12}^2q_{23} [(d-4)q_{13} + (d-2)q_{23}] + (d-4)q_{12}q_{13}q_{23}(q_{13} + q_{23}) \left. \right\} \\
&\times C_0 (0, 0, q_{23}, M_W^2, M_W^2, 0) + \frac{(q_{13} + q_{23})}{4(d-3)q_{12}^2q_{13}^2q_{23}^2} \\
&\times \left\{ (d-2)M_W^2 (q_{13} + q_{23}) [q_{12}(q_{13} - q_{23}) + q_{13}(q_{13} + q_{23})] \right. \\
&+ q_{12}q_{23} [(d-4)q_{13} + (d-2)q_{23}] + (d-4)q_{13}q_{12}q_{23}(q_{13} + q_{23}) \left. \right\} \\
&\times C_0 (0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2) + \frac{1}{4(d-3)q_{12}^2q_{13}^2q_{23}^2} \\
&\times \left\{ q_{12}q_{23}(q_{12} + q_{13}) [q_{12} \{ (d-4)q_{13} + (d-2)q_{23} \} + (d-4)q_{13}(q_{13} + q_{23})] \right. \\
&- M_W^2 \left\{ q_{12}^2 [(d-2)(q_{13}^2 + q_{23}^2) + 2(d-4)q_{13}q_{23}] + (d-2)q_{13}^2 (q_{13} + q_{23})^2 \right. \\
&+ 2q_{12}q_{13}(q_{13} + q_{23}) [(d-2)q_{13} + (d-4)q_{23}] \left. \right\} \left. \right\} \\
&\times C_0 (q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) + \frac{1}{4(d-3)q_{12}^2q_{13}^2q_{23}^2} \\
&\times \left\{ (2-d)M_W^4 (q_{13} + q_{23})^2 [q_{12}(q_{13} - q_{23}) + q_{13}(q_{13} + q_{23})] \right. \\
&- 2M_W^2 q_{12}q_{23} \left\{ q_{12} [q_{13}^2 + (d-5)q_{13}q_{23} + (d-2)q_{23}^2] + q_{13}(q_{13}^2 - q_{23}^2) \right\} \\
&+ q_{12}^2q_{23}^2 \left\{ q_{12} [(d-4)q_{13} + (d-2)q_{23}] + (d-4)q_{13}(q_{13} + q_{23}) \right\} \left. \right\} \\
&\times D_0 (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2), \tag{A28}
\end{aligned}$$

$$\begin{aligned}
& D_{13} (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2) \\
&= -\frac{M_H^2}{q_{12}q_{13}(q_{12} + q_{23})} B_0 (M_H^2, M_W^2, M_W^2) \\
&+ \frac{1}{q_{12}(q_{12} + q_{23})} B_0 (q_{13}, 0, M_W^2) + \frac{1}{q_{12}q_{13}} B_0 (q_{23}, M_W^2, 0) \\
&+ \frac{1}{4(d-3)q_{12}^2q_{13}q_{23}} \\
&\times \left\{ q_{13}q_{23} [(d-4)q_{12} + (d-2)q_{13}] - M_W^2q_{12} [(d-4)q_{13} - (d-2)q_{23}] \right. \\
&- \left. M_W^2q_{13}(q_{13} + q_{23})(d-2) \right\} C_0 (0, 0, q_{13}, 0, 0, M_W^2) + \frac{1}{4(d-3)q_{12}^2q_{13}^2} \\
&\times \left\{ q_{13}q_{23} [(d-4)q_{12} + (d-2)q_{13}] - M_W^2q_{12} [(d-4)q_{13} - (d-2)q_{23}] \right. \\
&- \left. M_W^2q_{13}(q_{13} + q_{23})(d-2) \right\} C_0 (0, 0, q_{23}, M_W^2, 0, 0) \\
&+ \frac{1}{4(3-d)q_{12}^2q_{13}^2q_{23}(q_{12} + q_{23})} \\
&\times \left\{ M_W^2q_{12}^3 [(d-4)q_{13} + (d-2)q_{23}] + (d-2)M_W^2q_{12}^2 [q_{13}^2 - q_{13}q_{23} + 2q_{23}^2] \right. \\
&- (d-2)M_W^2q_{13}q_{23}^2 (q_{13} + q_{23}) \\
&+ M_W^2q_{12}q_{23} [4q_{13}^2(3-d) + (8-3d)q_{13}q_{23} + (d-2)q_{23}^2] \\
&+ \left. q_{13}q_{23}(q_{12} + q_{23})^2 [(d-4)q_{12} + (d-2)q_{13}] \right\} C_0 (0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \\
&+ \frac{1}{4(d-3)q_{12}^2q_{13}^2q_{23}} \\
&\times \left\{ M_W^2q_{12}^2 [(d-4)q_{13} + (d-2)q_{23}] + 2M_W^2q_{12}q_{13} [(d-3)q_{13} + (d-4)q_{23}] \right. \\
&+ (d-2)M_W^2q_{13}^2 (q_{13} + q_{23}) - q_{13}q_{23}(q_{12} + q_{13}) [(d-4)q_{12} + (d-2)q_{13}] \left. \right\} \\
&\times C_0 (q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) + \frac{q_{13}q_{23} - M_W^2(q_{13} + q_{23})}{4(d-3)q_{12}^2q_{13}^2q_{23}} \\
&\times \left\{ M_W^2q_{12} [(d-4)q_{13} - (d-2)q_{23}] + (d-2)M_W^2q_{13}(q_{13} + q_{23}) \right. \\
&- \left. q_{13}q_{23} [(d-4)q_{12} + (d-2)q_{13}] \right\} \\
&\times D_0 (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2), \tag{A29}
\end{aligned}$$

$$\begin{aligned}
 & D_{13} (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2) \\
 &= -\frac{M_H^2}{q_{12}q_{13} (q_{13} + q_{23})} B_0 (M_H^2, M_W^2, M_W^2) + \frac{1}{q_{13} (q_{13} + q_{23})} B_0 (q_{12}, M_W^2, M_W^2) \\
 &+ \frac{1}{q_{12}q_{13}} B_0 (q_{23}, M_W^2, 0) + \frac{1}{4 (d - 3) q_{12}q_{13}^2q_{23}} \\
 &\times \left\{ q_{12}q_{23} [(d - 2) q_{12} + (d - 4) q_{13}] - M_W^2 q_{12} [(d - 4) q_{13} + (d - 2) q_{23}] \right. \\
 &\left. - (d - 2) M_W^2 q_{13} (q_{13} - q_{23}) \right\} C_0 (0, 0, q_{12}, M_W^2, 0, M_W^2) \\
 &+ \frac{1}{4 (d - 3) q_{12}^2q_{13}^2} \\
 &\times \left\{ M_W^2 q_{12} [(d - 4) q_{13} - (d - 2) q_{23}] + (d - 2) M_W^2 q_{13} (q_{13} + q_{23}) \right. \\
 &\left. + q_{12}q_{23} [(d - 2) q_{12} + (d - 4) q_{13}] \right\} C_0 (0, 0, q_{23}, M_W^2, M_W^2, 0) \\
 &+ \frac{q_{13} + q_{23}}{4 (3 - d) q_{12}^2q_{13}^2q_{23}} \\
 &\times \left\{ M_W^2 q_{12} [(d - 4) q_{13} - (d - 2) q_{23}] + (d - 2) M_W^2 q_{13} (q_{13} + q_{23}) \right. \\
 &\left. + q_{12}q_{23} [(d - 2) q_{12} + (d - 4) q_{13}] \right\} C_0 (0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2) \\
 &+ \frac{1}{4 (d - 3) q_{12}^2q_{13}^2q_{23}} \\
 &\times \left\{ M_W^2 q_{12}^2 [(d - 4) q_{13} + (d - 2) q_{23}] + 2M_W^2 q_{12}q_{13} [(d - 3) q_{13} + (d - 4) q_{23}] \right. \\
 &\left. - q_{12}q_{23} (q_{12} + q_{13}) [(d - 2) q_{12} + (d - 4) q_{13}] + (d - 2) M_W^2 q_{13}^2 (q_{13} + q_{23}) \right\} \\
 &\times C_0 (q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) \frac{1}{4 (d - 3) q_{12}^2q_{13}^2q_{23}} \\
 &\times \left\{ M_W^4 (q_{13} + q_{23}) \{q_{12} [(d - 4) q_{13} - (d - 2) q_{23}] + (d - 2) q_{13} (q_{13} + q_{23})\} \right. \\
 &\left. - 2M_W^2 q_{12}q_{23} \{q_{12} [q_{13} - (d - 2) q_{23}] + q_{13} (q_{23} - q_{13})\} \right. \\
 &\left. - q_{12}^2q_{23}^2 [(d - 2) q_{12} + (d - 4) q_{13}] \right\} \\
 &\times D_0 (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2), \tag{A30}
 \end{aligned}$$

$$\begin{aligned}
& D_{13} (0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2) \\
&= -\frac{M_H^2}{q_{13}q_{23}(q_{12} + q_{23})} B_0 (M_H^2, M_W^2, M_W^2) + \frac{1}{q_{13}q_{23}} B_0 (q_{12}, M_W^2, M_W^2) \\
&+ \frac{1}{q_{23}(q_{12} + q_{23})} B_0 (q_{13}, 0, M_W^2) \\
&+ \frac{1}{4(d-3)q_{13}^2q_{23}^2} \{ (d-2) [q_{13}^2(q_{12} - M_W^2) - M_W^2q_{23}^2] + (d-4)q_{13}q_{23}(q_{12} - 2M_W^2) \} \\
&\times C_0 (0, 0, q_{12}, M_W^2, 0, M_W^2) \\
&+ \frac{1}{4(d-3)q_{12}q_{13}q_{23}^2} \{ q_{12}q_{13} [(d-2)q_{13} + (d-4)q_{23}] - (d-2)M_W^2(q_{13}^2 - q_{23}^2) \} \\
&\times C_0 (0, 0, q_{13}, 0, M_W^2, M_W^2) + \frac{1}{4(d-3)q_{12}q_{13}^2q_{23}^2(q_{12} + q_{23})} \\
&\times \left\{ M_W^2 \left[q_{12}^2 [(d-2)(q_{13}^2 + q_{23}^2) + 2q_{13}q_{23}(d-4)] \right. \right. \\
&+ 2q_{12}q_{23} [2q_{13}^2(d-3) + (d-4)q_{13}q_{23} + (d-2)q_{23}^2] + (d-2)q_{23}^2(q_{23}^2 - q_{13}^2) \left. \right] \\
&- q_{12}q_{13}(q_{12} + q_{23})^2 [(d-2)q_{13} + (d-4)q_{23}] \left. \right\} C_0 (0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \\
&+ \frac{1}{4(d-3)q_{12}q_{13}^2q_{23}^2} \\
&\times \left\{ (q_{13} + q_{23}) [(d-2)M_W^2(q_{13}^2 - q_{23}^2) - q_{12}q_{13} [(d-2)q_{13} + (d-4)q_{23}]] \right\} \\
&\times C_0 (q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2) + \frac{1}{4(3-d)q_{12}q_{13}^2q_{23}^2} \\
&\times \left\{ (d-2)M_W^4(q_{13} - q_{23})(q_{13} + q_{23})^2 - 2M_W^2q_{12}q_{13} [(d-2)q_{13}^2 + (d-5)q_{13}q_{23} + q_{23}^2] \right. \\
&+ q_{12}^2q_{13}^2 [(d-2)q_{13} + (d-4)q_{23}] \left. \right\} \\
&\times D_0 (0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2), \tag{A31}
\end{aligned}$$

$$\begin{aligned}
 & D_{22} (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2) \\
 &= \frac{1}{q_{12}(q_{12} + q_{13})} B_0 (0, 0, M_W^2) - \frac{1}{q_{12}q_{23}} B_0 (0, M_W^2, 0) \\
 &+ \frac{(q_{13} - q_{23})}{q_{12}q_{13}q_{23}} B_0 (q_{12}, M_W^2, M_W^2) \\
 &- \frac{1}{q_{12}q_{13}q_{23}(q_{12} + q_{13})^2} \left\{ q_{12}^2 (q_{13} - q_{23}) + q_{12} [2q_{13}^2 + q_{13}q_{23} - q_{23}^2] + q_{13} (q_{13} + q_{23})^2 \right\} \\
 &\times B_0 (M_H^2, M_W^2, M_W^2) \\
 &+ \frac{1}{q_{12}q_{13}q_{23}(q_{12} + q_{13})^2} \left\{ q_{12}^2 q_{13} + q_{12} [2q_{13}^2 + 2q_{13}q_{23} - q_{23}^2] + q_{13} (q_{13} + q_{23})^2 \right\} \\
 &\times B_0 (q_{23}, M_W^2, 0) + \frac{1}{4(d-3)q_{12}q_{13}^2q_{23}^2} \\
 &\times \left\{ q_{12}q_{23} [(3d-10)q_{13}^2 - (d-2)q_{23}(2q_{13} - q_{23})] \right. \\
 &\left. - M_W^2 (q_{13} - q_{23}) [(d-2)(q_{13}^2 + q_{23}^2) + 2q_{13}q_{23}(3d-8)] \right\} \\
 &\times C_0 (0, 0, q_{12}, M_W^2, 0, M_W^2) + \frac{(d-2)(q_{13} + q_{23})^2}{4(d-3)q_{12}^2q_{13}^2q_{23}^2} [M_W^2 (q_{13} + q_{23}) - q_{12}q_{23}] \\
 &\times [C_0 (0, 0, q_{23}, M_W^2, M_W^2, 0) - C_0 (0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2)] \\
 &+ \frac{1}{4(d-3)q_{12}^2q_{13}^2q_{23}^2(q_{12} + q_{13})^2} \\
 &\times \left\{ M_W^2 [q_{12}(q_{13} - q_{23}) + q_{13}(q_{13} + q_{23})] \right. \\
 &\times \left\{ q_{12}^2 [(d-2)(q_{13}^2 + q_{23}^2) + 2q_{13}q_{23}(3d-8)] \right. \\
 &+ 2q_{12}q_{13}(q_{13} + q_{23})[(d-2)q_{13} + (3d-8)q_{23}] + (d-2)q_{13}^2(q_{13} + q_{23})^2 \left. \right\} \\
 &- q_{12}q_{23}(q_{12} + q_{13}) \left\{ q_{12}^2 [(3d-10)q_{13}^2 - (d-2)q_{23}(2q_{13} - q_{23})] \right. \\
 &+ (3d-10)q_{13}^2(q_{13} + q_{23})^2 \\
 &+ 2q_{12}q_{13}(q_{13} + q_{23})[(3d-10)q_{13} - (d-2)q_{23}] \left. \right\} \left. \right\} \\
 &\times C_0 (q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) + \frac{(q_{13} + q_{23})^2}{4(d-3)q_{12}^2q_{13}^2q_{23}^2} \\
 &\times \left\{ (d-2) \left[M_W^4 (q_{13} + q_{23})^2 + q_{12}^2 q_{23}^2 \right] - 2M_W^2 q_{12}q_{23} [(d-4)q_{13} + (d-2)q_{23}] \right\} \\
 &\times D_0 (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2), \tag{A32}
 \end{aligned}$$

$$\begin{aligned}
 & D_{22} (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2) \\
 &= \frac{1}{q_{13} (q_{12} + q_{13})} B_0 (0, 0, M_W^2) + \frac{(q_{12} - q_{23})}{q_{12} q_{13} q_{23}} B_0 (q_{13}, 0, M_W^2) \\
 &+ \frac{M_H^2 (q_{13} q_{23} - M_H^2 q_{12})}{q_{12} q_{13} q_{23} (q_{12} + q_{13})^2} B_0 (M_H^2, M_W^2, M_W^2) \\
 &+ \frac{(M_H^4 q_{12} - q_{13} q_{23}^2)}{q_{12} q_{13} q_{23} (q_{12} + q_{13})^2} B_0 (q_{23}, M_W^2, 0) + \frac{1}{4 (d - 3) q_{12}^2 q_{13} q_{23}^2} \\
 &\times \left\{ M_W^2 q_{12}^2 [(d - 2) q_{13} + (10 - 3d) q_{23}] + 2 M_W^2 q_{12} q_{23} [(d - 2) q_{23} - (d - 4) q_{13}] \right. \\
 &+ q_{13} q_{23} [(3d - 10) q_{12}^2 - (d - 2) q_{23} (2q_{12} + q_{23})] + (d - 2) M_W^2 q_{23}^2 (q_{13} + q_{23}) \left. \right\} \\
 &\times C_0 (0, 0, q_{13}, 0, 0, M_W^2) + \frac{(q_{12} + q_{23})}{4 (d - 3) q_{12}^2 q_{13}^2 q_{23}} \\
 &\times \left\{ M_W^2 q_{12} [(10 - 3d) q_{13} + (d - 2) q_{23}] + (d - 2) q_{23} [M_W^2 (q_{13} + q_{23}) - q_{13} (q_{12} + q_{23})] \right\} \\
 &\times C_0 (0, 0, q_{23}, M_W^2, 0, 0) \\
 &+ \frac{(d - 2) (q_{12} + q_{23})^2}{4 (d - 3) q_{12}^2 q_{13}^2 q_{23}^2} \left\{ M_W^2 [q_{12} (q_{13} - q_{23}) - q_{23} (q_{13} + q_{23})] + q_{13} q_{23} (q_{12} + q_{23}) \right\} \\
 &\times C_0 (0, q_{13}, M_H^2, M_W^2, 0, M_W^2) + \frac{1}{4 (3 - d) q_{12}^2 q_{13}^2 q_{23}^2 (q_{12} + q_{13})^2} \\
 &\times \left\{ M_W^2 q_{23}^3 (q_{13} - q_{12}) [(d - 2) (q_{12}^2 + q_{13}^2) + 2q_{12} q_{13} (3d - 8)] \right. \\
 &- M_W^2 q_{12} q_{23} (q_{12} + q_{13})^2 [(d - 2) q_{12}^2 + (5d - 18) q_{12} q_{13} + 2q_{13}^2 (d - 4)] \\
 &- M_W^2 q_{23}^2 (q_{12} + q_{13}) [(d - 2) (2q_{12}^3 - 4q_{12} q_{13}^2 - q_{13}^3) + (11d - 34) q_{12}^2 q_{13}] \\
 &+ q_{13} q_{23}^3 (q_{12} + q_{13}) [(3d - 10) q_{12}^2 - (d - 2) q_{13} (2q_{12} - q_{13})] \\
 &+ 2q_{12} q_{13} q_{23}^2 (q_{12} + q_{13})^2 [(3d - 10) q_{12} - (d - 2) q_{13}] \\
 &+ (d - 2) M_W^2 q_{12}^2 q_{13} (q_{12} + q_{13})^3 \\
 &+ q_{12}^2 q_{13} q_{23} (q_{12} + q_{13})^3 (3d - 10) \left. \right\} C_0 (q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) \\
 &+ \frac{1}{4 (d - 3) q_{12}^2 q_{13}^2 q_{23}^2} \\
 &\times \left\{ M_W^4 q_{12}^2 [(d - 2) (q_{13}^2 + q_{23}^2) - 2q_{13} q_{23} (d - 4)] \right. \\
 &+ 2M_W^4 q_{12} q_{23} (q_{13} + q_{23}) [(d - 2) q_{23} - (d - 4) q_{13}] \\
 &+ 2M_W^2 q_{13} q_{23} (q_{12} + q_{23}) [(d - 4) q_{12} q_{13} - M_H^2 q_{23} (d - 2)] \\
 &+ (d - 2) q_{23}^2 \left[M_W^4 (q_{13} + q_{23})^2 + q_{13}^2 (q_{12} + q_{23})^2 \right] \left. \right\} \\
 &\times D_0 (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2), \tag{A33}
 \end{aligned}$$

$$\begin{aligned}
& D_{23} (0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2) \\
&= -\frac{1}{q_{13}(q_{13} + q_{23})} B_0 (0, M_W^2, M_W^2) + \frac{1}{q_{13}q_{23}} B_0 (q_{13}, 0, M_W^2) \\
&+ \frac{(q_{23} - q_{13}) M_H^2}{q_{13}q_{23}(q_{13} + q_{23})^2} B_0 (M_H^2, M_W^2, M_W^2) \\
&+ \frac{q_{12}(q_{13} - q_{23}) - q_{23}(q_{13} + q_{23})}{q_{13}q_{23}(q_{13} + q_{23})^2} B_0 (q_{12}, M_W^2, M_W^2) \\
&+ \frac{1}{4(d-3)q_{13}^2q_{23}^2} \\
&\times \left\{ M_W^2 q_{23} q_{13} (d-4) + (d-2) [M_W^2 q_{23}^2 - M_W^2 q_{12} (q_{13} - q_{23}) + q_{12} q_{13} (q_{12} + q_{23})] \right\} \\
&\times C_0 (0, 0, q_{12}, M_W^2, 0, M_W^2) + \frac{1}{4(d-3)q_{12}q_{13}q_{23}^2} \\
&\times \left\{ M_W^2 q_{12} [(2-d)q_{13} + (3d-10)q_{23}] + M_W^2 q_{23} [(d-4)q_{13} - (d-2)q_{23}] \right. \\
&+ \left. q_{12} q_{13} (q_{12} + q_{23}) (d-2) \right\} C_0 (0, 0, q_{13}, 0, M_W^2, M_W^2) \\
&+ \frac{(q_{12} + q_{23})}{4(3-d)q_{12}q_{13}^2q_{23}^2} \\
&\times \left\{ M_W^2 q_{23} [(d-2)q_{23} - (d-4)q_{13}] + (2-d) [M_W^2 q_{12} (q_{13} - q_{23}) - q_{12} q_{13} (q_{12} + q_{23})] \right\} \\
&\times C_0 (0, q_{13}, M_H^2, M_W^2, 0, M_W^2) + \frac{1}{4(d-3)q_{12}q_{13}^2q_{23}^2(q_{13} + q_{23})} \\
&\times \left\{ M_W^2 (q_{13} + q_{23}) [(d-2)[q_{12}(q_{13}^2 + q_{23}^2) + q_{23}^3] - (d-4)q_{13}q_{23}(2q_{12} + q_{13}) + 2q_{13}q_{23}^2] \right. \\
&- \left. q_{12}q_{13} [q_{12}[(d-2)q_{13}(q_{13} + 2q_{23}) + (10-3d)q_{23}^2] \right. \\
&+ \left. q_{23}(q_{13} + q_{23}) [(d-2)q_{13} + (10-3d)q_{23}] \right\} C_0 (q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2) \\
&+ \frac{1}{4(3-d)q_{12}q_{13}^2q_{23}^2} \left\{ M_W^4 [(d-2)q_{12}q_{13}^2 + q_{23}^2 [(d-2)q_{12} + 2q_{13}] \right. \\
&- \left. (d-4)q_{13}q_{23}(2q_{12} + q_{13}) + (d-2)q_{23}^3] \right. \\
&- \left. 2M_W^2 q_{12}q_{13} [(d-2)q_{12}(q_{13} - q_{23}) + q_{23}[q_{13} + (2-d)q_{23}] \right. \\
&+ \left. (d-2)q_{12}^2q_{13}^2(q_{12} + q_{23}) \right\} D_0 (0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2), \quad (A34)
\end{aligned}$$

$$\begin{aligned}
 & D_{23} (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2) \\
 &= -\frac{1}{q_{13}(q_{12} + q_{13})} B_0 (0, 0, M_W^2) + \frac{1}{q_{12}q_{13}} B_0 (q_{13}, 0, M_W^2) \\
 &+ \frac{M_H^2 (q_{12} - q_{13})}{q_{12}q_{13}(q_{12} + q_{13})^2} B_0 (M_H^2, M_W^2, M_W^2) \\
 &+ \frac{(q_{13}q_{23} - M_H^2 q_{12})}{q_{12}q_{13}(q_{12} + q_{13})^2} B_0 (q_{23}, M_W^2, 0) + \frac{1}{4(d-3)q_{12}^2 q_{13} q_{23}} \\
 &\times \left\{ M_W^2 q_{12} [(d-4)q_{13} - (d-2)q_{23}] - (d-2)q_{23} [M_W^2 (q_{13} + q_{23}) - q_{13}(q_{12} + q_{23})] \right\} \\
 &\times C_0 (0, 0, q_{13}, 0, 0, M_W^2) + \frac{1}{4(d-3)q_{12}^2 q_{13}^2} \\
 &\times \left\{ M_W^2 q_{12} [(d-4)q_{13} - (d-2)q_{23}] - (d-2)q_{23} [M_W^2 (q_{13} + q_{23}) - q_{13}(q_{12} + q_{23})] \right\} \\
 &\times C_0 (0, 0, q_{23}, M_W^2, 0, 0) + \frac{(q_{12} + q_{23})}{4(d-3)q_{12}^2 q_{13}^2 q_{23}} \\
 &\times \left\{ M_W^2 q_{12} [(d-4)q_{13} + (d-2)q_{23}] + (d-2)q_{23} [M_W^2 (q_{13} + q_{23}) - q_{13}(q_{12} + q_{23})] \right\} \\
 &\times C_0 (0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \\
 &+ \frac{1}{4(3-d)q_{12}^2 q_{13}^2 q_{23} (q_{12} + q_{13})^2} \\
 &\times \left\{ (d-4)M_W^2 q_{12} q_{13} (q_{12} + q_{13})^3 \right. \\
 &+ M_W^2 q_{23}^2 (q_{12} - q_{13}) [(d-2)(q_{12}^2 + q_{13}^2) + 2q_{12}q_{13}(3d-8)] \\
 &+ M_W^2 q_{23} (q_{12} + q_{13}) [(d-2)(q_{12}^3 - 3q_{12}q_{13}^2 - q_{13}^3) + (7d-22)q_{12}^2 q_{13}] \\
 &+ q_{13}q_{23}^2 (q_{12} + q_{13}) [(10-3d)q_{12}^2 + (d-2)q_{13}(2q_{12} + q_{13})] \\
 &\left. + q_{12}q_{13}q_{23} (q_{12} + q_{13})^2 [(10-3d)q_{12} + (d-2)q_{13}] \right\} \\
 &\times C_0 (q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) + \frac{q_{13}q_{23} - M_W^2 (q_{13} + q_{23})}{4(d-3)q_{12}^2 q_{13}^2 q_{23}} \\
 &\times \left\{ M_W^2 q_{12} [(d-2)q_{23} - (d-4)q_{13}] + (d-2)q_{23} [M_W^2 (q_{13} + q_{23}) - q_{13}(q_{12} + q_{23})] \right\} \\
 &\times D_0 (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2), \tag{A35}
 \end{aligned}$$

$$\begin{aligned}
 & D_{23} (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2) \\
 &= -\frac{1}{q_{12}(q_{12} + q_{13})} B_0 (0, 0, M_W^2) + \frac{1}{q_{12}q_{13}} B_0 (q_{12}, M_W^2, M_W^2) \\
 &+ \frac{M_H^2 (q_{13} - q_{12})}{q_{12}q_{13} (q_{12} + q_{13})^2} B_0 (M_H^2, M_W^2, M_W^2) \\
 &- \frac{q_{12} (q_{13} - q_{23}) + q_{13} (q_{13} + q_{23})}{q_{12}q_{13} (q_{12} + q_{13})^2} B_0 (q_{23}, M_W^2, 0) + \frac{1}{4(d-3)q_{12}q_{13}^2q_{23}} \\
 &\times \left\{ M_W^2 [(d-2)(q_{13}^2 - q_{23}^2) - 4q_{13}q_{23}(d-3)] + (d-2)q_{12}q_{23}(q_{13} + q_{23}) \right\} \\
 &\times C_0 (0, 0, q_{12}, M_W^2, 0, M_W^2) \\
 &+ \frac{(d-2)(q_{13} + q_{23})}{4(3-d)q_{12}^2q_{13}^2} [M_W^2 (q_{13} + q_{23}) - q_{12}q_{23}] C_0 (0, 0, q_{23}, M_W^2, M_W^2, 0) \\
 &+ \frac{(d-2)(q_{13} + q_{23})^2}{4(d-3)q_{12}^2q_{13}^2q_{23}} [M_W^2 (q_{13} + q_{23}) - q_{12}q_{23}] C_0 (0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2) \\
 &+ \frac{1}{4(3-d)q_{12}^2q_{13}^2q_{23}(q_{12} + q_{13})^2} \\
 &\times \left\{ M_W^2 q_{12}^3 [(d-2)(q_{13}^2 - q_{23}^2) - 4q_{13}q_{23}(d-3)] + (d-2)M_W^2 q_{13}^3 (q_{13} + q_{23})^2 \right. \\
 &+ M_W^2 q_{12}^2 q_{13} [(d-2)q_{13}(3q_{13} + 2q_{23}) + (14-5d)q_{23}^2] \\
 &+ M_W^2 q_{12}q_{13}^2 (q_{13} + q_{23}) [3q_{13}(d-2) + (5d-14)q_{23}] \\
 &+ q_{12}q_{23}(q_{12} + q_{13}) \left[(d-2)q_{12}^2 (q_{13} + q_{23}) + 2q_{12}q_{13} [(d-2)q_{23} - (d-4)q_{13}] \right. \\
 &\left. \left. - (3d-10)q_{13}^2 (q_{13} + q_{23}) \right] \right\} C_0 (q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) \\
 &+ \frac{(q_{13} + q_{23})}{4(3-d)q_{12}^2q_{13}^2q_{23}} \\
 &\times \left\{ (d-2)M_W^4 (q_{13} + q_{23})^2 - 2M_W^2 q_{12}q_{23} [(d-4)q_{13} + (d-2)q_{23}] \right. \\
 &\left. + (d-2)q_{12}^2q_{23}^2 \right\} D_0 (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2), \tag{A36}
 \end{aligned}$$

$$\begin{aligned}
 & D_{33} (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2) \\
 &= \frac{1}{q_{12} (q_{12} + q_{13})} B_0 (0, 0, M_W^2) - \frac{1}{q_{12} q_{13}} B_0 (q_{12}, M_W^2, M_W^2) \\
 &+ \frac{(M_H^2 q_{12} - q_{13} q_{23})}{q_{12} q_{13} (q_{12} + q_{13})^2} B_0 (M_H^2, M_W^2, M_W^2) - \frac{q_{23} (q_{12} - q_{13})}{q_{12} q_{13} (q_{12} + q_{13})^2} B_0 (q_{23}, M_W^2, 0) \\
 &+ \frac{1}{4 (d - 3) q_{12} q_{13}^2} \{ M_W^2 [(3d - 10) q_{13} + (d - 2) q_{23}] - (d - 2) q_{12} q_{23} \} \\
 &\times C_0 (0, 0, q_{12}, M_W^2, 0, M_W^2) \\
 &+ \frac{(d - 2) q_{23}}{4 (d - 3) q_{12}^2 q_{13}^2} [M_W^2 (q_{13} + q_{23}) - q_{12} q_{23}] C_0 (0, 0, q_{23}, M_W^2, M_W^2, 0) \\
 &+ \frac{(d - 2) (q_{13} + q_{23})}{4 (3 - d) q_{12}^2 q_{13}^2} [M_W^2 (q_{13} + q_{23}) - q_{12} q_{23}] C_0 (0, q_{12}, M_H^2, M_W^2, M_W^2, M_W^2) \\
 &+ \frac{1}{4 (d - 3) q_{12}^2 q_{13}^2 (q_{12} + q_{13})^2} \\
 &\times \left\{ M_W^2 q_{12}^3 [(10 - 3d) q_{13} - (d - 2) q_{23}] + M_W^2 q_{12}^2 q_{13} [(14 - 5d) q_{23} - (d - 6) q_{13}] \right. \\
 &+ M_W^2 q_{12} q_{13}^2 [3q_{13} (d - 2) + (5d - 14) q_{23}] + (d - 2) M_W^2 q_{13}^3 (q_{13} + q_{23}) \\
 &\left. + q_{12} q_{23} (q_{12} + q_{13}) [(d - 2) q_{12} (q_{12} + 2q_{13}) + (10 - 3d) q_{13}^2] \right\} \\
 &\times C_0 (q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) + \frac{1}{4 (d - 3) q_{12}^2 q_{13}^2} \\
 &\times \left\{ (d - 2) [M_W^4 (q_{13} + q_{23})^2 + q_{12}^2 q_{23}^2] - 2 M_W^2 q_{12} q_{23} [(d - 4) q_{13} + (d - 2) q_{23}] \right\} \\
 &\times D_0 (0, 0, 0, M_H^2; q_{23}, q_{12}; M_W^2, M_W^2, 0, M_W^2), \tag{A37}
 \end{aligned}$$

$$\begin{aligned}
 & D_{33} (0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2) \\
 &= \frac{1}{q_{13} (q_{13} + q_{23})} B_0 (0, M_W^2, M_W^2) - \frac{1}{q_{13} q_{23}} B_0 (q_{13}, 0, M_W^2) \\
 &+ \frac{q_{12} (q_{13} - q_{23}) + q_{13} (q_{13} + q_{23})}{q_{13} q_{23} (q_{13} + q_{23})^2} B_0 (M_H^2, M_W^2, M_W^2) \\
 &+ \frac{q_{12} (q_{23} - q_{13})}{q_{13} q_{23} (q_{13} + q_{23})^2} B_0 (q_{12}, M_W^2, M_W^2) \\
 &+ \frac{q_{12} (2 - d) [M_W^2 (q_{23} - q_{13}) + q_{12} q_{13}]}{4 (d - 3) q_{13}^2 q_{23}^2} C_0 (0, 0, q_{12}, M_W^2, 0, M_W^2) \\
 &+ \frac{q_{13} (M_W^2 - q_{12}) (d - 2) + M_W^2 q_{23} (10 - 3d)}{4 (d - 3) q_{13} q_{23}^2} C_0 (0, 0, q_{13}, 0, M_W^2, M_W^2) \\
 &+ \frac{1}{4 (d - 3) q_{13}^2 q_{23}^2} \\
 &\times \left\{ M_W^2 q_{13} q_{23} (10 - 3d) + (d - 2) [M_W^2 q_{23}^2 - M_W^2 q_{12} (q_{13} - q_{23}) + q_{12} q_{13} (q_{12} + q_{23})] \right\} \\
 &\times C_0 (0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \\
 &+ \frac{1}{4 (d - 3) q_{13}^2 q_{23}^2 (q_{13} + q_{23})} \\
 &\times \left\{ q_{12} q_{13} [q_{13} (q_{13} + 2q_{23}) (d - 2) + (10 - 3d) q_{23}^2] \right. \\
 &\left. - M_W^2 (q_{13} + q_{23}) [(d - 2) (q_{13}^2 + q_{23}^2) - 2 (d - 4) q_{13} q_{23}] \right\} \\
 &\times C_0 (q_{12}, 0, M_H^2, M_W^2, M_W^2, M_W^2) \\
 &+ \frac{1}{4 (d - 3) q_{13}^2 q_{23}^2} \\
 &\times \left\{ (d - 2) [M_W^4 q_{23}^2 + q_{13}^2 (M_W^2 - q_{12})^2] + 2 M_W^2 q_{13} q_{23} [(d - 2) q_{12} - (d - 4) M_W^2] \right\} \\
 &\times D_0 (0, 0, 0, M_H^2; q_{12}, q_{13}; M_W^2, 0, M_W^2, M_W^2), \tag{A38}
 \end{aligned}$$

$$\begin{aligned}
& D_{33} (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2) \\
&= \frac{1}{q_{13}(q_{12} + q_{13})} B_0 (0, 0, M_W^2) - \frac{1}{q_{12}q_{13}} B_0 (q_{13}, 0, M_W^2) \\
&+ \frac{1}{q_{12}q_{13}(q_{12} + q_{13})^2} [q_{12}(q_{13} - q_{23}) + q_{13}(q_{13} + q_{23})] B_0 (M_H^2, M_W^2, M_W^2) \\
&+ \frac{q_{23}(q_{12} - q_{13})}{q_{12}q_{13}(q_{12} + q_{13})^2} B_0 (q_{23}, M_W^2, 0) \\
&+ \frac{(d-2)}{4(d-3)q_{12}^2q_{13}} [M_W^2(q_{13} + q_{23}) - q_{13}q_{23}] C_0 (0, 0, q_{13}, 0, 0, M_W^2) \\
&+ \frac{(d-2)q_{23}}{4(d-3)q_{12}^2q_{13}^2} [M_W^2(q_{13} + q_{23}) - q_{13}q_{23}] C_0 (0, 0, q_{23}, M_W^2, 0, 0) \\
&+ \frac{1}{4(d-3)q_{12}^2q_{13}^2} \\
&\times \left\{ (d-2)q_{13}q_{23}(q_{12} + q_{23}) - M_W^2q_{12}[(3d-10)q_{13} + (d-2)q_{23}] \right. \\
&\left. - M_W^2q_{23}(q_{13} + q_{23})(d-2) \right\} C_0 (0, q_{13}, M_H^2, M_W^2, 0, M_W^2) \\
&+ \frac{1}{4(d-3)q_{12}^2q_{13}^2(q_{12} + q_{13})^2} \\
&\times \left\{ M_W^2q_{12}^3[(3d-10)q_{13} + (d-2)q_{23}] + M_W^2q_{12}^2q_{13}[(d-6)q_{13} + (5d-14)q_{23}] \right. \\
&+ M_W^2q_{12}q_{13}^2[(14-5d)q_{23} - 3(d-2)q_{13}] - (d-2)M_W^2q_{13}^3(q_{13} + q_{23}) \\
&\left. - q_{13}q_{23}(q_{12} + q_{13})[(3d-10)q_{12}^2 - 2(d-2)q_{12}q_{13} - (d-2)q_{13}^2] \right\} \\
&\times C_0 (q_{23}, 0, M_H^2, M_W^2, 0, M_W^2) \\
&+ \frac{(d-2)}{4(d-3)q_{12}^2q_{13}^2} [q_{13}q_{23} - M_W^2(q_{13} + q_{23})]^2 \\
&\times D_0 (0, 0, 0, M_H^2; q_{23}, q_{13}; M_W^2, 0, 0, M_W^2). \tag{A39}
\end{aligned}$$

Appendix B. Expansion for form factors of $\mathcal{A}_{\text{tri}}^{(W)}$ and $\mathcal{A}_{\text{tri}}^{(f)}$

Given the dominant contributions of triangle diagrams with W boson and fermion at internal lines to the decay channels, it is worth performing the ε expansion for the form factors of $\mathcal{A}_{\text{tri}}^{(W)}$ and $\mathcal{A}_{\text{tri}}^{(f)}$. Both representations of the form factors give the same results:

$$\begin{aligned}
\mathcal{A}_{\text{tri}}^{(W)} &= \frac{\alpha^2 [(M_H^2 - q_{12}) g^{\mu\nu} - 2q^\nu q_3^\mu]}{4M_H^2 M_W^3 s_W^3 (M_H^2 - q_{12})^2 (q_{12} - M_Z^2 + i\Gamma_Z M_Z)} \\
&\times \left\{ M_H^2 M_W^2 [M_H^2 (q_{12} - 6M_W^2) + 12M_W^4 + 6M_W^2 q_{12} - 2q_{12}^2] \right. \\
&\times \ln^2 \left[\frac{-M_H^2 + \sqrt{M_H^4 - 4M_H^2 M_W^2 + 2M_W^2}}{2M_W^2} \right] \\
&+ M_H^2 [2M_H^4 M_W^2 - M_H^4 q_{12} + 12M_H^2 M_W^4 - 4M_H^2 M_W^2 q_{12} + M_H^2 q_{12}^2 \\
&- 12M_W^4 q_{12} + 2M_W^2 q_{12}^2] \\
&+ M_H^2 \sqrt{q_{12}^2 - 4M_W^2 q_{12}} [M_H^2 (q_{12} - 2M_W^2) + 2M_W^2 (q_{12} - 6M_W^2)] \\
&\times \ln \left[\frac{-q_{12} + \sqrt{q_{12}^2 - 4M_W^2 q_{12} + 2M_W^2}}{2M_W^2} \right] \\
&- M_W^2 M_H^2 [M_H^2 (q_{12} - 6M_W^2) + 12M_W^4 + 6M_W^2 q_{12} - 2q_{12}^2] \\
&\times \ln^2 \left[\frac{-q_{12} + \sqrt{q_{12}^2 - 4M_W^2 q_{12} + 2M_W^2}}{2M_W^2} \right] \\
&+ q_{12} \sqrt{M_H^4 - 4M_H^2 M_W^2} [M_H^2 (2M_W^2 - q_{12}) + 12M_W^4 - 2M_W^2 q_{12}] \\
&\times \ln \left[\frac{-M_H^2 + \sqrt{M_H^4 - 4M_H^2 M_W^2 + 2M_W^2}}{2M_W^2} \right] \left. \right\} \bar{u}(q_1) \gamma^\mu P_L \nu(q_2) \epsilon_\nu^*(q_3), \\
\mathcal{A}_{\text{tri}}^{(f)} &= \frac{\alpha^2 m_f^2 N_C^f Q_f [2q^\nu q_3^\mu + (q_{12} - M_H^2) g^{\mu\nu}] (T_f^3 - 2Q_f s_W^2)}{4M_H^2 M_W c_W^2 s_W^3 (M_H^2 - q_{12})^2 (q_{12} - M_Z^2 + i\Gamma_Z M_Z)} \\
&\times \left\{ M_H^2 \left[(4m_f^2 - M_H^2 + q_{12}) \ln^2 \left[\frac{-q_{12} + \sqrt{q_{12}^2 - 4q_{12} m_f^2 + 2m_f^2}}{2m_f^2} \right] \right. \right. \\
&+ 4\sqrt{q_{12}^2 - 4q_{12} m_f^2} \ln \left[\frac{-q_{12} + \sqrt{q_{12}^2 - 4q_{12} m_f^2 + 2m_f^2}}{2m_f^2} \right] - 4M_H^2 + 4q_{12} \left. \right] \\
&+ M_H^2 (M_H^2 - 4m_f^2 - q_{12}) \ln^2 \left[\frac{-M_H^2 + \sqrt{M_H^4 - 4m_f^2 M_H^2 + 2m_f^2}}{2m_f^2} \right] \\
&- 4q_{12} \sqrt{M_H^4 - 4M_H^2 m_f^2} \ln \left[\frac{-M_H^2 + \sqrt{M_H^4 - 4m_f^2 M_H^2 + 2m_f^2}}{2m_f^2} \right] \left. \right\} \\
&\times \bar{u}(q_1) \gamma^\mu P_L \nu(q_2) \epsilon_\nu^*(q_3). \tag{B1}
\end{aligned}$$

For W bosons (fermions) exchanging in the loop, their masses are included in Feynman's prescription as $M_W^2 - i\rho$ ($m_f^2 - i\rho$). As a result, all the logarithmic functions in the above equations are well defined in the complex plane.

Appendix C. Decay rate

The differential decay width is given by

$$\frac{d^2\Gamma(H \rightarrow \nu_l \bar{\nu}_l \gamma)}{dq_{12} dq_{13}} = \frac{1}{256\pi^3 M_H^3} \sum_{\text{spin}} |\mathcal{A}_{\text{tri}} + \mathcal{A}_{\text{box}}|^2. \quad (\text{C1})$$

We have used the following notations:

$$\mathcal{A}_{\text{tri}} = [F_1 q_3^\mu q^\nu + F_2 g^{\mu\nu}] \bar{u}(q_1) \gamma^\mu P_L v(q_2) \epsilon_\nu^*(q_3), \quad (\text{C2})$$

$$\mathcal{A}_{\text{box}} = \bar{u}(q_1) \{F_3 q_1^\nu q_3 + F_4 q_2^\nu q_3 + F_5 \gamma^\nu\} P_L v(q_2) \epsilon_\nu^*(q_3). \quad (\text{C3})$$

After performing the spin-sum for the above amplitude-squared, one gets several representations (together with Eq. (18)) for the differential decay width as follows:

$$\begin{aligned} \frac{d^2\Gamma(H \rightarrow \nu_l \bar{\nu}_l \gamma)}{dq_{12} dq_{13}} &= \frac{q_{12}}{256\pi^3 M_H^3 (q_{13} + q_{23})^2} \\ &\times \left\{ 2(q_{13}^2 + q_{23}^2) |F_2|^2 + q_{13}^2 (q_{13} + q_{23})^2 |F_3|^2 + 2(q_{13} + q_{23})^2 |F_5|^2 \right. \\ &+ 4q_{23} (q_{13} + q_{23}) \text{Re}[F_2 F_5^*] + 2q_{13} (q_{23}^2 - q_{13}^2) \text{Re}[F_2 F_3^*] \\ &\left. + 2q_{13} (q_{13} + q_{23})^2 \text{Re}[F_3 F_5^*] \right\}, \quad (\text{C4}) \end{aligned}$$

$$\begin{aligned} \frac{d^2\Gamma(H \rightarrow \nu_l \bar{\nu}_l \gamma)}{dq_{12} dq_{13}} &= \frac{q_{12}}{256\pi^3 M_H^3 (q_{13} + q_{23})^2} \\ &\times \left\{ 2(q_{13}^2 + q_{23}^2) |F_2|^2 + q_{23}^2 (q_{13} + q_{23})^2 |F_4|^2 + 2(q_{13} + q_{23})^2 |F_5|^2 \right. \\ &+ 4q_{13} (q_{13} + q_{23}) \text{Re}[F_2 F_5^*] + 2q_{23} (q_{13}^2 - q_{23}^2) \text{Re}[F_2 F_4^*] \\ &\left. + 2q_{23} (q_{13} + q_{23})^2 \text{Re}[F_4 F_5^*] \right\}. \quad (\text{C5}) \end{aligned}$$

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