Published for SISSA by Springer

Received: *September 6, 2021*
Revised: *January 29, 2022*
Accepted: *February 18, 2022*
Published: *March 18, 2022*

Graded Hilbert spaces, quantum distillation and connecting SQCD to QCD

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Abstract: The dimension of the Hilbert space of QFT scales exponentially with the volume of the space in which the theory lives, yet in supersymmetric theories, one can define a graded dimension (such as the supersymmetric index) that counts just the number of bosonic minus fermionic ground states. Can we make this observation useful in non-supersymmetric QFTs in four dimensions? In this work, we construct *symmetry graded state sums* for a variety of non-supersymmetric theories. Among the theories we consider is one that is remarkably close to QCD: Yang-Mills theory with $N_f = N_c$ fundamental Dirac fermions and one adjoint Weyl fermion, QCD(F/adj). This theory can be obtained from SQCD by decoupling scalars and carry exactly the same anomalies. Despite the existence of fundamental fermions, the theory possess an exact 0-form color-flavor center (CFC) symmetry for a particular grading/twist under which Polyakov loop is a genuine order parameters. By a two-loop analysis, we prove that CFC-symmetry remains unbroken at small β due to grading. Chiral symmetry is spontaneously broken within the domain of validity of semi-classics on $\mathbb{R}^3 \times S^1$ in a pattern identical to $N_f = N_c$ SQCD on $\mathbb{R}^4$ and the two regimes are adiabatically connected. The vacuum structures of the theory on $\mathbb{R}^4$ and $\mathbb{R}^3 \times S^1$ are controlled by the same mixed 't Hooft anomaly condition, implying a remarkable persistent order.

Keywords: Nonperturbative Effects, Solitons Monopoles and Instantons, Spontaneous Symmetry Breaking

ArXiv ePrint: 2104.12352

https://doi.org/10.1007/JHEP03(2022)119

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1 The general idea of quantum distillation and summary

This work is a collection of ideas to determine the non-perturbative dynamics and phase structure of a class of non-supersymmetric QFTs, including QCD. The story exploits the mapping between the Hilbert space and Hamiltonian formalisms by constructing fairly exotic looking *symmetry graded state sums*, and their images in the path integral formalism. Despite the fact that the theories of interest here are non-supersymmetric and in the thermo-dynamic limit, our line of thinking can be traced back to two sources which possess qualities opposed to what we desire: supersymmetric indices [1], and standard singularity theorems in the theory of phase transitions [2–4]. Our intention is to construct graded state sums that avoid Lee-Yang-Fisher type singularities in generalized partition functions in non-supersymmetric theories. This class of ideas goes under the umbrella term "adiabatic continuity" [5, 6]. Adiabatic continuity is a stronger version of what is called persistent order in condensed matter physics, or persistent mixed 't Hooft anomaly in QFT.

Thermal phase transitions in quantum field theory are probed with the singularity structure of the partition function

$$\mathcal{Z}(\beta) = \operatorname{tr}[e^{-\beta H}] \tag{1.1}$$

where β is the inverse temperature, and H is the Hamiltonian. Let $\mathcal{H}$ denote the Hilbert space. In the $\beta \to \infty$ limit, $\mathcal{Z}(\beta)$ receives dominant contributions from ground states and low lying states. It is an uncontaminated quantity, but strongly coupled QFTs are not usually amenable to analytic treatment in this limit. As $\beta \to 0$, asymptotically free theories become weakly coupled, but the state sum is extremely contaminated. It essentially receives contributions on the same footing from entire Hilbert space $\mathcal{H}$. In this regime, it is impossible to isolate and understand the role of ground states and low lying states. Furthermore, there is in general a phase transition at some β_c. One may think that this phase transition is due to the growth in the density of states in $\mathcal{H}$.

A well-known way to avoid phase transition in supersymmetric gauge theories is to construct a $\mathbb{Z}_2$-graded state sum [1]:

$$I(\beta) = \operatorname{tr}\left[(-1)^F e^{-\beta H}\right] \tag{1.2}$$

where F is fermion number modulo two, which counts the number of supersymmetric vacua in supersymmetric QFTs . One way to interpret the supersymmetric index (1.2) is as follows: if we were to evaluate $\operatorname{tr}[1]$ over the Hilbert space $\mathcal{H} = \mathcal{B} \oplus \mathcal{F}$, we would in fact count the dimension of the Hilbert space which grows exponentially with the volume of the space in which the theory lives. Yet $\lim_{\beta\to 0} I(\beta)$ may be interpreted as a *graded dimension* of the Hilbert space. If the spectrum of the supersymmetric QFT is rendered discrete, supersymmetry guarantees that all states with $E > 0$ are Bose-Fermi paired and their contribution to the graded sum is zero. Hence, (1.2) is just a pure number independent of β, and merely counts the bosonic minus fermionic ground states up to a sign. Importantly, (1.2) no longer scales with $\dim[\mathcal{H}]$.

In general, we cannot carry over the precise level-by-level Bose-Fermi cancellations pertinent to supersymmetric theory to non-supersymmetric theory.[1] There are no such luxuries in real life. We will, however, state what $(-1)^F$ achieves differently and aim to carry it over to non-supersymmetric theory:

- $(-1)^F$ is a grader over the Hilbert space $\mathcal{H}$ and the graded state sum distills a sub-set of states, such that $\operatorname{tr}\left[(-1)^F e^{-\beta H}\right]$ is an analytic function of β.

The idea of *quantum distillation* of Hilbert space aims to generalize the partition function without changing the Hamiltonian H and Hilbert space $\mathcal{H}$ and obtain new state sums which *i)* reduce the state sum in magnitude due to cancellations between the states, and *ii)* effectively represent a subset of states in the Hilbert space [7, 8]. This is not a "projecting out" procedure, since $\mathcal{H}$ is still the same. Rather, quantum distillation is a useful sign problem over the state sum in the Hamiltonian formulation. We define the subset of states in the Hilbert space which do survive after graded summation as $\text{Distill}[\mathcal{H}]$. This is ultimately tied with the manipulation of singularities of the partition function. Consider a generalized partition function:

$$\mathcal{Z}(\beta, \epsilon_1, \epsilon_2, \ldots) = \operatorname{tr}\left[e^{-\beta H} \prod_a e^{i\epsilon_a Q_a}\right] \tag{1.3}$$

where $Q_a, a = 1, 2, \ldots$ are some charges associated with the QFT, $[H, Q_a] = 0$, and H is Hamiltonian of the theory on $\mathbb{R}^3$. The main question is the following.

> **Main question:** assume that thermal partition function $\mathcal{Z}(\beta) \equiv \mathcal{Z}(\beta, 0\ldots, 0)$ possesses singularities in $\beta \in [0, \infty)$ (which is generically the case). Does there exist a grading $\prod_a e^{i\epsilon_a Q_a}$ over the Hilbert space $\mathcal{H}$ such that $\mathcal{Z}(\beta, \epsilon_1, \ldots, \epsilon_N)$ is an analytic function of β?

[1] There are remarkable exceptions. See first example in the next subsection.

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Our goal is to answer this question in the positive for a broad class of non-supersymmetric quantum field theories intimately related to QCD with fundamental fermions. Three complementary perspectives about the process of quantum distillation are:

- **Grading of Hilbert space, and quantum distillation:** this is the process of drastically reducing the magnitude of the state-sums without changing the Hamiltonian or Hilbert space by using symmetry grading over the Hilbert space.
- **Path integrals with generalized boundary conditions:** (or equivalently, path integrals in the background of global symmetry holonomies). This allows for the evaluation of gauge-holonomy potentials in the presence of global-symmmetry holonomy backgrounds, and sometime admits reliable semi-classical analysis of dynamics and study of phase transitions.
- **Graded-thermodynamics:** thermodynamics is the thermal worth of the Hilbert space $\mathcal{H}$ through the thermal partition function $Z(\beta) = \mathrm{tr}\,[e^{-\beta H}]$. Graded-thermodynamics may be viewed as the thermal worth of the distilled Hilbert space $\mathrm{Distill}[\mathcal{H}]$.

1.1 Trivial and non-trivial examples in 1d, 2d and 4d

Perhaps, with the principle that a picture says thousand words, we can start with a simple system in quantum mechanics and discuss the concept of quantum distillation there, see figure 1. Despite the fact that this is a trivial non-interacting system, extremely similar phenomena does occur in the non-trivial asymptotically free QFT in any dimensions. After this example, we construct two types of perfect quantum distillation in 2d QFT, which will be useful when we build a similar structure in 4d QCD with fundamental and adjoint fermions.

1d QM. Consider N-dimensional simple harmonic oscillator. This system has $\mathrm{U}(N)$ global symmetry, and harmonic level k is in the totally symmetric k-index representations of $\mathrm{SU}(N)$ and have degeneracies given in (1.5). We can construct a symmetry graded state sum

$$\mathcal{Z}_{\Omega_F^0}(\beta) = \mathrm{tr}\left(e^{-\beta H}\prod_{j=1}^{N} e^{i\frac{2\pi}{N}j\widehat{Q}_j}\right), \qquad Q_j = \widehat{a}_j^\dagger \widehat{a}_j \tag{1.4}$$

where Q_j the number operator for the j^{th} oscillator. In the graded state sum, many degenerate states cancel among themselves due to phases attached to them. After cancellations, even arbitrarily large degeneracy factors maps to graded degeneracy factors, which is 0 or 1:

$$\mathrm{deg}(k) = \binom{N+k-1}{k} \longmapsto (\underbrace{1,0,\ldots,0,}_{N}\underbrace{1,0,\ldots,0,}_{N}1,\ldots) \tag{1.5}$$

See figure 1 for $N=3$ case. Clearly, in the $N\to\infty$ limit, the contribution of the whole Hilbert space reduces to just a single state (the ground state) for any finite value of β!

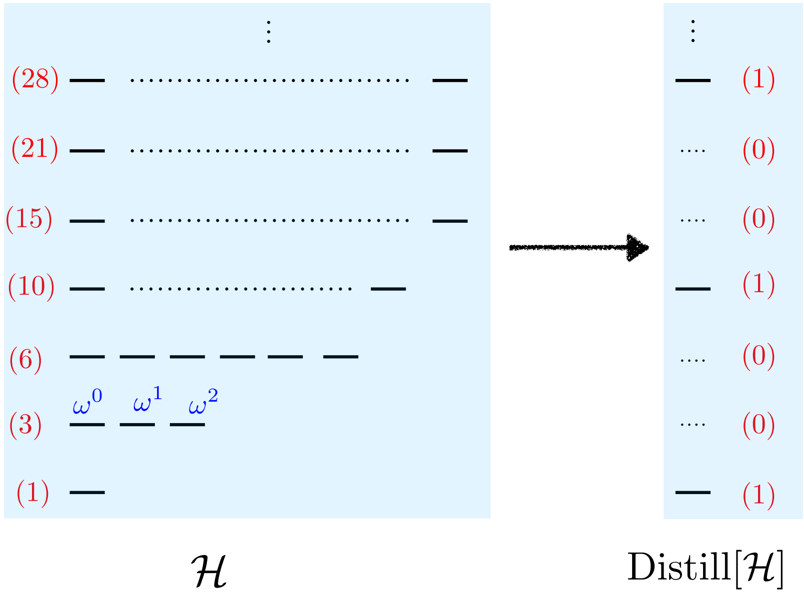

Figure 1. Hilbert space of 3d simple harmonic oscillator and its SU(3) symmetry graded distillation. The terms inside parenthesis indicate degeneracies. Despite its simplicity, this picture morally captures what takes place in 2d sigma models and 4d QCD(F/adj). In the N-dimensional oscillator as $N \to \infty$, only ground state contribute to symmetry graded partition function. This is also the case in the large-N limits of $\mathbb{CP}^{N-1}$ sigma model and principle chiral model, as we review, where symmetry graded state sum exhibits spectral cancellations (similar to Witten index in supersymmetric theories), but now story can take place in purely bosonic theories.

2d QCD(adj). In non-supersymmetric SU(N) QCD(adj) in 2d, there is a mixed anomaly between $(\mathbb{Z}_2)_F$ and $(\mathbb{Z}_2)_\chi$ for N even, and the partition function on the torus with periodic boundary conditions on both cycles vanishes identically,

$$\mathcal{Z}_{++}(\beta, L) = \text{tr}\left[(-1)^F e^{-\beta H_L}\right] = 0. \tag{1.6}$$

at any β and L. This indeed implies that

$$\text{Distill}[\mathcal{H}] = \emptyset \tag{1.7}$$

due to exact pairwise cancellations. In the Hilbert space formulation, this is due to exact level-by-level cancellation between bosonic and fermionic states. The exact bose-fermi degeneracy in this case is protected by mixed anomaly rather than supersymmetry [9]. Generalization and other examples of this phenomena, that exact Bose-Fermi spectral degeneracy can be protected by mixed anomalies involving $(-1)^F$, can be found in recent interesting work [10].[2] Perhaps, more interestingly, for odd N, there is no mixed anomaly

[2]The cancellation we describe in this paper is less perfect than exact Bose-Fermi degeneracy [9, 10] which is protected by mixed anomaly (rather than supersymmetry). In the present case, our construction has Bose-Bose, Bose-Fermi and Fermi-Fermi type cancellations. [10] also provides examples of exact Bose-Fermi degeneracy in 3d examples, but not the 4d ones. In our 4d QCD(F/adj), we demonstrate that the effective density of states (after all the cancellation in the Hilbert space takes place) takes the form of a 2d QFT

and there is no exact Bose-Fermi degeneracy for finite odd values. But smoothness of the large N limit requires spectral degeneracies to emerge in the limit. In this sense, N odd theories is one nice realization of the quantum distillation idea in a strongly coupled QFT where spectral cancellation is not guaranteed by a mixed anomaly but is there nevertheless.

4d QCD(adj). Another example is 4d QCD(adj) in the large-N_c limit [12, 13]. This theory, defined through the graded partition function, $\mathrm{tr}\,[(-1)^F e^{-\beta H}] = Z_{\mathcal{B}}(\beta) - Z_{\mathcal{F}}(\beta)$, avoids all phase transitions as a function of β and satisfies volume independence. For $N_f = 1$, the theory is supersymmetric $\mathcal{N} = 1$ SYM and this is just the supersymmetric index [1]. For $N_f > 1$, where the microscopic theory is non-supersymmetric, ref. [13] showed that powerful Bose-Fermi cancellations over the Hilbert space of the theory take place. In particular, the relative density of states $\rho_B(E) - \rho_F(E)$ for the theory defined on a curved 3-manifold has the scaling of a 2d QFT, identical to supersymmetric theories in a similar set-up [11].

2d sigma models. One may be tempted to think that quantum distillation should not be possible in purely bosonic field theory. However, two powerful and generalizable counter-examples are the following. In the $\mathbb{CP}^{N-1}$ model, a judiciously graded partition function

$$\mathcal{Z}_{\Omega_F^0}(\beta) = \mathrm{tr}\left(e^{-\beta H} \prod_{j=1}^{N} e^{i\frac{2\pi}{N}j\widehat{Q}_j}\right) \tag{1.8}$$

can yield powerful cancellations between all higher states. In the $N \to \infty$ limit of the bosonic $\mathbb{CP}^{N-1}$ model, one can obtain

$$\mathrm{Distilled}[\mathcal{H}] = \{\text{ground state(s)}\} \tag{1.9}$$

at arbitrary θ. (See appendix A.3 for full details.) Another useful example is an asymptotically free matrix model, the principal chiral model in $d = 1 + 1$. In this case, powerful spectral conspiracy can take place with symmetry graded state sums as described in appendix A.4, and all but the ground state cancel. The implication of these exact cancellations is large-N volume independence [6, 14–16].

Perfect quantum distillation. We refer to quantum distillations that can prevent all phase transitions in infinite volume or large-N thermodynamic limits on $\mathbb{R}^{d-1} \times S^1$ as the radius β is dialed as perfect quantum distillations. For $d \leq 2$, to achieve the thermodynamic limit, it is necessary to take $N \to \infty$.

In conceptually simpler cases, quantum distillations may result from supersymmetry or anomalies. Supersymmetric vector spaces are the most obvious graded Hilbert space construction. In non-supersymmetric theories, anomalies are sometimes equally powerful, implying exact Bose-Fermi degeneracy as mentioned above. In general, gauging global symmetries always acts in the direction of diluting the Hilbert space, as one removes non-gauge invariant states from the Hilbert space in the gauged theory. However, the fuller

in large-N limit, same as supersymmetric theories [11]. Our construction in 4d is *not* a perfect spectral cancellation, but a sufficiently good one, which allows to adiabatically continue between small and large $\mathbb{R}^3 \times S^1$ without phase transitions.

story of quantum distillation is not restricted to these special cases. As mentioned above, in a purely bosonic theory, it is capable of generating graded state sums that lead to equally powerful cancellations as in the case of the supersymmetric index, forcing the state sum to just the ground states. On the other hand, what to choose for the symmetry graded sums is not quite obvious and requires some guesswork. Yet, we can actually test whether a given guess works or not via explicit computation. Our construction in 4d QCD(adj/F) is in this category.

1.2 4d SQCD, QCD-like theories and QCD

If we wish to implement the idea of (perfect) quantum distillation in 4d non-supersymmetric gauge theories, a class of theories arises naturally. This is $\mathrm{SU}(N_c)$ gauge theory with N_f fundamental representation Dirac fermions ψ^a and one adjoint representation Weyl fermion, QCD(F/adj) with content (A_μ, ψ^a, λ). A special emphasis is given to $N_f = N_c$ theory for arbitrary N_c. At large N_c, this becomes a slight generalization of the Veneziano limit with $x = \frac{N_f}{N_c}$ fixed [17]. The matter content of this class of theories is the one of N_f-flavor supersymmetric SQCD with decoupled scalars $m_{q^a} \to \infty$ [18, 19].

The QCD(F/adj) theory with mixed representation matter fields allows us to interpolate between different theories as we dial a flavor-symmetric mass $m_\psi \geq 0$ for fundamental fermions ψ_a and a mass $m_\lambda \geq 0$ for the adjoint fermion. With the decoupling of the adjoint fermion, the theory reduces to the flavor symmetric limit of real QCD for $N_f = N_c = 3$. With the decoupling of fundamental fermions, the theory reduces to $\mathcal{N} = 1$ SYM. Decoupling all fermions, the theory reduces to pure YM:

$$\begin{array}{ccccc} \text{SQCD} & \xrightarrow{m_{q^a}\to\infty} & \text{QCD(F/adj)} & \xrightarrow{m_{\psi^a}\to\infty} & \mathcal{N}=1\ \text{SYM} \\ & & \Big\downarrow{\scriptstyle m_\lambda\to\infty} & & \Big\downarrow{\scriptstyle m_\lambda\to\infty} \\ & & \text{QCD(F)} & \xrightarrow{m_{\psi^a}\to\infty} & \text{YM} \end{array} \tag{1.10}$$

We will take advantage of the limits of square (1.10). We construct a graded partition function for QCD(F/adj) by using global symmetries of the theory:

$$\mathcal{Z}(\beta, \epsilon_a) = \mathrm{tr}\left[e^{-\beta H} (-1)^F \prod_{a=1}^{N_f} e^{i\epsilon_a Q_a} \right] \tag{1.11}$$

where $Q_a = \int d^3x\, \overline{\psi}_a \gamma^0 \psi_a$ are charges associated with the maximal abelian subgroup of $\mathrm{U}(N_f)_V$. This object is a mixture of quantum distillations that worked perfectly in 2d QCD(adj) in which we implemented a $(-1)^F$ grading [9] and 2d $\mathbb{CP}^{N-1}$ model where we implemented a flavor-symmetry grading [15, 16]. In path integral formalism, this corresponds to periodic boundary conditions for λ and flavor-twisted boundary conditions on ψ_a. We have

$$\mathcal{Z}(\beta, \epsilon_a) = \int_{\substack{A(\beta) = +A(0) \\ \lambda(\beta) = +\lambda(0) \\ \psi(\beta) = +\psi(0)\overline{\Omega}_F e^{i\pi}}} DA_\mu D\psi D\lambda \;\; e^{-S[A_\mu, \psi, \lambda]} \tag{1.12}$$

There are in principle many choices for boundary conditions Ω_F for fundamental fermions. Restricting to flavor rotations in a vector-like subgroup, $\Omega_F e^{i\pi}$ is a general $\mathrm{U}(N_f)_V$ matrix.

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There are three independent procedures which pick a *unique* boundary condition (up to an over-all $\mathrm{U}(1)_V$ factor), or equivalently grading operator, in Hamiltonian formalism:

- maximizing the graded free energy (or minimizing the graded pressure),
- demanding the presence of color-flavor center symmetry upon compactification on $\mathbb{R}^3 \times S^1$,
- or demanding that certain mixed anomalies present on $\mathbb{R}^4$ persist on $\mathbb{R}^3 \times S^1$

picks a unique configuration for flavor-twist or grading operator:

$$\Omega_F^0 = \mathrm{diag}(1, \omega, \cdots, \omega^{N_f-1}), \qquad \omega = e^{2\pi i/N_f} \tag{1.13}$$

We will explain the reasoning behind all three conditions in detail.

1.3 Color-flavor center symmetry

First, we note an unconventional and fairly recently understood aspect of flavor-twisted boundary conditions. It is well-known that theories with only adjoint representation fields, such as pure YM and $\mathcal{N}=1$ SYM, have $\mathbb{Z}_{N_c}^{[1]}$ one-form center-symmetry. When the theory is compactified on a circle $\mathbb{R}^3 \times S^1$, part of the one-form symmetry becomes a zero-form symmetry, for which the order parameter is the Polyakov loop, i.e., a Wilson line wrapping the S^1 direction.

Once dynamical fundamental fermions are introduced, the one-form center-symmetry is explicitly broken, and in particular, the Polyakov loop is no longer a good order parameter; see [20] or standard texts on thermal field theory [21, 22]. However, it turns out that one can actually salvage the situation [23]: both QCD(F) and QCD(F/adj) can have an **exact** $\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)}^{[0]}$ (gcd stands for greatest common divisor) zero-form center symmetry which acts non-trivially on Polyakov loops, despite the absence of one-form center symmetry in the theory. The $\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)}^{[0]}$ symmetry lives in the center of $\mathrm{SU}(N_c)$ and in a cyclic permutation subgroup $\Gamma_S \subset \mathrm{SU}(N_f)_V$. This was appreciated as a genuine symmetry fairly recently in [23] where it was shown to be an invariance of the partition function, $\mathcal{Z}_{\Omega_F^0} \mapsto \mathcal{Z}_{\Omega_F^0}$. This symmetry is called color-flavor center (CFC) symmetry and Wilson lines around the S^1 circle are charged under it [23]. An earlier lattice study indeed shows a sharp phase transition associated with CFC [24]. For QCD-like theories in (1.10) on $\mathbb{R}^3 \times S^1$, there is a CFC or zero-form part of center-symmetry acting on Polyakov loops as follows:

$$\begin{array}{ccc} \mathbb{Z}_{\mathrm{gcd}(N_f,N_c)}\ \mathrm{CFC} & \xrightarrow{m_\psi \to \infty} & \mathbb{Z}_{N_c} \\ \big\downarrow {\scriptstyle m_\lambda \to \infty} & & \big\downarrow {\scriptstyle m_\lambda \to \infty} \\ \mathbb{Z}_{\mathrm{gcd}(N_f,N_c)}\ \mathrm{CFC} & \xrightarrow{m_\psi \to \infty} & \mathbb{Z}_{N_c} \end{array} \tag{1.14}$$

A Wilson line wrapping the S^1 circle is a good order parameter at any point in the (m_λ, m_ψ) plane provided $\mathrm{gcd}(N_f, N_c) > 1$. Therefore, we can examine the phase structure of these theories according to the CFC or zero-form center-symmetries, and pose questions about

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analyticity of the graded partition function as a function of β. The CFC plays a major role in our construction.

The appearance of CFC is correlated with two other effects. Turning on a background for $\mathrm{SU}(N_f)_V$, or equivalently, a flavor twisted boundary condition Ω_F^0, explicitly breaks the global symmetry of the theory $\mathbf{G}$ given in (2.5) to its maximal abelian subgroup $\mathbf{G}_{\text{max}-\text{ab}}$ given in (2.38). At the same time, this is the origin of the flavor part of the quantum distillation operator in (1.11).

1.4 Color-holonomy potentials in the flavor-holonomy background and surprises

The image of quantum distillation of $\mathcal{H}$ in the path integral formalism is reflected in the flavor-holonomy dependence of the gauge-holonomy potential. Clearly, flavor-holonomy Ω_F is a *choice*; it is intrinsically non-dynamical. It corresponds to boundary conditions for fundamental fermions. On the other hand, gauge-holonomy Ω is dynamical and its vacuum expectation value is determined by the minimum of the gauge-holonomy potential, which in turn determines some properties of the ground states or thermal equilibrium states. It turns out that the minimum of the Wilson line potential is inherently tied to the choice of flavor-holonomy background in pleasantly surprising ways.

We calculate the gauge-holonomy potential for the Polyakov loop:

$$\Omega = e^{i\oint dx_4 a_4} \equiv \mathrm{diag}(e^{iv_1}, \ldots, e^{iv_{N_c}}) \tag{1.15}$$

at two-loop order with the boundary conditions (1.12).[3] To do so, we generalized the tour de force thermal studies of refs. [25, 26] and [27] to incorporate flavor-holonomy backgrounds. The center-breaking gauge boson contribution is completely undone by one adjoint fermion with periodic boundary condition to *all orders* in perturbation theory, similar to $\mathcal{N}=1$ SYM.

$$V^{\text{gauge}} + V^{\lambda} = 0 \qquad \text{all orders in perturbation theory} \tag{1.16}$$

The story therefore boils down to what the fundamental fermions with twisted boundary conditions do. The two-loop result which carries many new insights is:

$$\begin{aligned} V_{\text{1-loop},\Omega_F} + V_{\text{2-loop},\Omega_F} = & \frac{2}{\pi^2\beta^4}\left(1 - \frac{3g^2}{16\pi^2}\left(\frac{N_c^2-1}{N_c}\right)\right)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}[\mathrm{Tr}(\overline{\Omega}_F^n)\mathrm{Tr}(\Omega^n) + \text{c.c.}] \\ & + \frac{g^2 N_f}{8\pi^4\beta^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\left|\mathrm{Tr}(\Omega^n)\right|^2 . \end{aligned} \tag{1.17}$$

where $(g^2)^0$ and $(g^2)^1$ are one- and two-loop terms, respectively. Here is the sharp contrast between two choices of flavor holonomy Ω_F:

- For $\Omega_F = \mathbf{1}_{N_f}$, there are terms like $\mathrm{tr}\,(\Omega^n)$ in the potential that are manifestly non-invariant under center-transformation, hence the center-symmetry is explicitly broken. The minimum of gauge-holonomy potential is at

$$\Omega\big|_{\text{min}} = \mathbf{1}_{N_c}, \tag{1.18}$$

[3] Two-loop order suffices to determine the realization of CFC-symmetry for any flavor-holonomy background.

the theory is in the chirally symmetric high-temperature phase, separated from low temperature chirally broken phase.

- For $\Omega_F = \Omega_F^0$ given in (1.13), the terms in the potential that transform non-trivially under $\mathbb{Z}_{\gcd(N_f,N_c)}$ drop out as they must. Not only is $\mathbb{Z}_{N_c}$ CFC symmetry an exact symmetry of the $N_f = N_c$ theory, it remains unbroken for any value of $\beta\Lambda \lesssim 1$ and for any N_c. The minimum of the gauge-holonomy potential is now at

$$\begin{aligned} \Omega\big|_{\text{min}} &= \text{diag}(1, \omega, \cdots, \omega^{N_c-1}), && N_c \text{ odd}, \\ \Omega\big|_{\text{min}} &= \omega^{1/2}\text{diag}(1, \omega, \cdots, \omega^{N_c-1}), && N_c \text{ even}. \end{aligned} \tag{1.19}$$

  as reported in [7]. The center-stability holds to all loop orders in perturbation theory as well as non-perturbatively as such effects are suppressed compared to our two-loop result and cannot alter the minimum of the potential.

The center-stability also holds away from chiral limit, for $m_\lambda = 0, m_\psi \geq 0$ and $m_\lambda \leq m_\lambda^*$, $m_\psi = 0$, and even in cases where both fermions may have non-zero masses. These limits will be used to access flavor limit of QCD.

Quantum distillation in the Hilbert space of QCD(F/adj). What is happening with the potential is quite striking. Essentially, we changed the small-β limit of QCD(F/adj) from a center-non-invariant configuration given in (1.18) to a center-symmetric configuration given in (1.19) by inserting $(-1)^F \prod_{a=1}^{N_f} e^{i\epsilon_a Q_a}$ into the trace without changing the Hilbert space. One of our goals in this paper is to benefit from the combined thinking of these two-process distillings of Hilbert space and their image in the holonomy potentials as well as corresponding graded thermodynamics.

In the graded state sum, there are cancellations of different nature; bosons cancel against bosons (e.g. mesons), fermions cancel against fermions (e.g. baryons and fermionic mesons), and in certain limits, bosons cancel against fermions (eg. glueballs against gluinoballs). We refer to these cancellations as $\mathcal{BB}, \mathcal{FF}, \mathcal{BF}$ cancellations, respectively.

1.5 Volume independence in Veneziano-type limits

In the Veneziano type large-N_c limit of QCD where N_f scales with N_c [17], it is impossible to satisfy volume (or temperature) independence [28] even in the confined phase. This is because even the contribution of the mesons to the free energy density is of order $\mathcal{F} \approx -N_c^2 T^4$, explicitly violating temperature independence at leading order in N_c. The graded partition function provides the first realization of large-N_c volume independence in this limit. The gauge-holonomy potential (1.17) in the large-N_c Veneziano limit of QCD(F/adj) reduces to:

$$V_{\text{1-loop},\Omega_F} + V_{\text{2-loop},\Omega_F} = +x\frac{(g^2 N_c)}{8\pi^4\beta^4}\sum_{n=1}^{\infty}\frac{\left|\text{Tr}(\Omega^n)\right|^2}{n^4}\,, \qquad x = N_f/N_c \tag{1.20}$$

Clearly, (8.1) has a center-symmetric minimum. Quite strikingly, the fundamental fermions' contribution which is normally of the form $xN_c \sum_{n=1}^{\infty} \frac{1}{n^4} \operatorname{tr} \Omega^n + \text{c.c.}$ turns into a quantity of order $\frac{1}{N_c^3}$ vanishing in the large-N_c limit. One may be tempted to think that all information about the existence of microscopic fundamental fermions is "forgotten", but the truth is subtler. The center-stabilizing double trace operator comes exactly from the fundamental fermions with Ω_F^0 twisted boundary conditions! In other words, fundamental fermions, for the purpose of center-symmetry, act exactly as $x(g^2 N_c)$ many adjoint fermions with periodic boundary conditions, hence stabilizing center-symmetry.

At the $N_c = \infty$ limit, volume independence implies that the graded partition function avoids Hagedorn singularities in all hadronic channels, mesons, glue as well as baryons [29] due to powerful spectral cancellation. This statement can be proven by studying QCD(F/adj) on $S^3 \times S^1$ [13, 30].

1.6 Unified mechanism of chiral symmetry breaking in QCD(F) and $\mathcal{N} = 1$ SYM

Probably, the most important insight that the idea of quantum distillation brings is a unified understanding of chiral symmetry breaking in QCD(F) and $\mathcal{N} = 1$ SYM. Consider $N_f = N_c$ theory with the insertion of the Ω_F^0-twisted boundary condition in path integral. Then, as (1.19) implies, the gauge holonomy in the small circle regime is $\mathbb{Z}_{N_c}$ symmetric. Due to dynamical abelianization, there are N_c types of monopoles each with action $S_0 = \frac{8\pi^2}{g^2 N}$. In QCD(F/adj), each monopole possess four fermion zero modes, two adjoint and two fundamental fermion zero mode due to index theorem for Dirac operator in monopole-background [31, 32]:

$$\mathcal{M}_i \sim e^{-S_i} e^{-\frac{4\pi}{g^2}\alpha_i \cdot \phi + i\alpha_i \cdot \sigma} (\psi_{Ri} \psi_L^i)(\alpha_i \cdot \lambda)^2 \qquad \text{QCD(F/adj)} \tag{1.21}$$

Turning on a mass term for adjoint (fundamental) quark lifts the adjoint (fundamental) zero modes, leaving only two fundamental (adjoint) zero modes per monopole:

$$\begin{aligned} \mathcal{M}_i &\sim e^{-S_i} e^{-\frac{4\pi}{g^2}\alpha_i \cdot \phi + i\alpha_i \cdot \sigma} (\psi_{Ri} \psi_L^i) && \text{QCD(F)} \ + \ \text{massive adjoint} \\ \mathcal{M}_i &\sim e^{-S_i} e^{-\frac{4\pi}{g^2}\alpha_i \cdot \phi + i\alpha_i \cdot \sigma} (\alpha_i \cdot \lambda)^2 && \mathcal{N} = 1 \text{ SYM} \ + \ \text{massive fund.} \end{aligned} \tag{1.22}$$

In the first case, the chiral symmetry which commutes with the quantum distillation operator is a maximal abelian subgroup (or maximal torus) $\mathbf{G}_{\text{max-ab}}$ of the global chiral symmetry $\text{SU}(N_f)_L \times \text{SU}(N_f)_R$ of QCD. In the second case, only $\mathbb{Z}_{2N_c}$ discrete chiral symmetry is present, exactly as in $\mathcal{N} = 1$. The monopole operator is invariant under $\mathbf{G}_{\text{max-ab}}$ and $\mathbb{Z}_{2N_c}$, respectively. Remarkably, the gauge field part of the monopole-operator is capable of soaking up chiral charge (either discrete or continuous) [33]. This is due to intertwining of chiral and topological-shift symmetry which is a generalization of the result of ref. [34] on $\mathbb{R}^3$ to locally four-dimensions on $\mathbb{R}^3 \times S^1$. As a result, an interesting phenomenon occurs. The gauge fluctuations (and flux part of the monopole-operator) acquire chiral charge under continuous and discrete chiral symmetry, respectively.[4]

The spontaneous breaking of the chiral symmetry occurs due to flux part of the monopole operator acquiring a vacuum expectation value. On the vacuum, exactly the same vev

$$\langle \text{VAC} | e^{\alpha_i \cdot z} | \text{VAC} \rangle \neq 0 \tag{1.23}$$

[4] Truth is stranger than fiction.

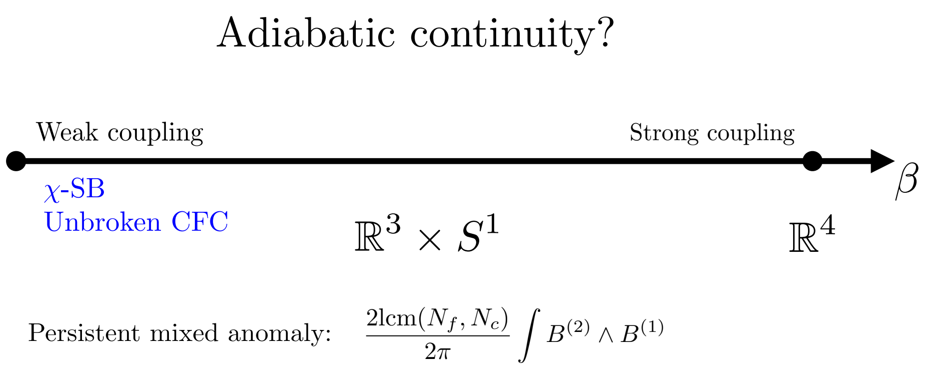

Figure 2. By employing graded partition function (1.11), we prove chiral symmetry breaking at weak coupling for various $\mathrm{SU}(N_c)$ gauge theories with N_f massless fundamental Dirac fermions and one massive adjoint fermion. This is done by using semi-classical methods and index theorem for Dirac operator on $\mathbb{R}^3 \times S^1$ in non-trivial gauge Ω and flavor Ω_F^0 holonomy background. There exists a mixed anomaly polynomial that persists at any compactification scale β (both at weak and strong coupling) if and only if one uses Ω_F^0 background. The combination of the two methods does not suffice to analytically prove χ-SB at strong coupling and $\mathbb{R}^4$.

generates continuous chiral symmetry breaking in QCD(F) and discrete chiral symmetry breaking in $\mathcal{N}=1$ SYM. In the first case, we can actually prove that dual photons remain gapless non-perturbatively, and they are the semi-classical realization of Nambu-Goldstone bosons. In the latter case, this mechanism produces N_c isolated vacua associated with discrete chiral symmetry breaking, and there are domain walls between vacua [35, 36].[5]

1.7 Mixed anomaly (persistent order) vs. adiabatic continuity

What is the implication of spontaneously broken chiral symmetry on small $S^1 \times \mathbb{R}^3$ for the theory on $\mathbb{R}^4$? Is it possible that the theory remains in the same phase at arbitrary radius? This is logically possible, and in fact, for the $N_f = N_c$ theory it is very likely the case. This idea is called adiabatic continuity, and has been achieved via double-trace deformations in Yang-Mills theory [5]. See [37, 38] for lattice tests of this idea. But unfortunately, we cannot prove adiabatic continuity analytically. What we can prove is a weaker (but still non-trivial) statement that the mixed anomaly condition on $\mathbb{R}^3 \times S^1$ and $\mathbb{R}^4$ are the same.

We can say more about the possible vacuum structures of the theory both on $\mathbb{R}^4$ and $\mathbb{R}^3 \times S^1$ by using recent progress on mixed anomalies involving discrete symmetries [39, 40]. There is a mixed anomaly on $\mathbb{R}^4$ between $\mathrm{SU}(N_f)_V/\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)}$ symmetry and $\mathbb{Z}_{2N_f}$ subgroup of the chiral symmetry (2.12). The partition function in the presence of background fields (A, B) corresponding to $\mathrm{SU}(N_f)_V/\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)}$ fails to be invariant under $h \in \mathbb{Z}_{2N_f}$

[5]In the standard interpretation in the literature, χS is asserted to be broken even on small $\mathbb{R}^3 \times S^1$ due to fermion bilinear $\operatorname{tr} \lambda\lambda$ or $(\psi_{Ra}\psi_L^b)$ acquiring a vev. This interpretation is *not* quite correct. It is the vacuum expectation value of magnetic flux part of monopole operator that breaks the symmetry. Once this operator acquires a vev, a chiral symmetry breaking mass term is induced for fermions. See section 5.7 for details.

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chiral transformations. As dictated by the anomaly polynomial,

$$\begin{aligned}\mathcal{Z}(h(A,B)) &= \exp\left[-i\frac{2\mathrm{lcm}(N_f,N_c)}{4\pi}\int B\wedge B\right]\mathcal{Z}((A,B))\\ &= \exp\left[-i2\pi\frac{2\mathrm{lcm}(N_f,N_c)}{(\gcd(N_f,N_c))^2}\right]\mathcal{Z}((A,B))\end{aligned}\tag{1.24}$$

which means that the ground state cannot be a unique, gapped (i.e. trivial) state provided $\frac{2\mathrm{lcm}(N_f,N_c)}{(\gcd(N_f,N_c))^2}\in\mathbb{Q}\backslash\mathbb{Z}$. Possibilities include spontaneous (chiral) symmetry breaking or a CFT on $\mathbb{R}^4$.

Remarkably, the same anomaly condition persists on $\mathbb{R}^3\times S^1$ *if and only if* one uses the twisted boundary conditions (1.13) for fundamental fermions. For similar discussions in QCD(adj) and QCD(F), see [41–44]. This gives a triple mixed anomaly between shift symmetry $\Gamma_S\subset \mathrm{SU}(N_f)_V$, abelianized flavor symmetry $\mathrm{U}(1)_V^{N_f-1}/\mathbb{Z}_{\gcd(N_f,N_c)}$, and the discrete chiral symmetry $\mathbb{Z}_{2N_f}$ as in provided $\mathbb{R}^4$. In this case, spontaneous symmetry breaking scenario includes chiral symmetry and color-flavor center symmetry. We cannot have a phase in which none of the symmetries is unbroken. This notion is referred to as persistent order.

Therefore, the anomaly conditions that constrain the vacuum structure on $\mathbb{R}^4$ and $\mathbb{R}^3\times S^1$ are one and the same. The chirally broken phase is a realization of an anomaly permitted phase. This is certainly not as strong as the adiabatic continuity conjecture which states that only one mode of the mixed anomaly permitted phase is operative at any radius. However, it is also encouraging that the ground states are to be chosen by dynamics among just a few anomaly-permitted possibilities.

1.8 Reading guide

Section 2 is a review of mostly known facts about SQCD and QCD(F/adj) on $\mathbb{R}^4$.[6] It pays extra attention to faithful global symmetries, and factoring out gauge redundancies, which is important in our discussion. The matching of mixed anomalies between SQCD and QCD(F/adj) is obvious, since their fermionic content and global zero form symmetries are the same. This part also reviews the color-flavor center symmetry, which is relatively recent.

Section 3 studies the gauge holonomy potentials in the presence of background flavor holonomies. Standard one and two-loop potential in thermal field theory are well-known, see for one-loop [20, 53] and two-loop orders [25, 26] and [27]. The incorporation of the flavor background and the study of the potential for gauge holonomy, and their extremization are new.

Section 4 examines the effect of background flavor holonomy and $(-1)^F$ from the perspective of Hilbert space. This section uses inputs from QCD phenomenology and explicitly exhibits the cancellations between either same spin or opposite spin particles related to each other via symmetries.

Section 5 studies non-perturbative dynamics of QCD(F/adj) and its mass deformations on $\mathbb{R}^3\times S^1$. It includes some background on the topological excitations and index

[6] Thanks to the referee for suggesting a reading guide for the paper. This gives us an opportunity to state which parts are review and background, and which parts are new materials.

theorems for Dirac operator on $\mathbb{R}^3 \times S^1$. But specific application of these techniques to QCD(F/adj) is new. The main achievement of this section is that chiral symmetry breaking in QCD(F/adj) and QCD(F) can actually take place at arbitrarily weak coupling just like $\mathcal{N} = 1$ SYM on $\mathbb{R}^3 \times S^1$.

Section 6 reviews the mixed anomaly and the conditions under which it persists on $\mathbb{R}^3 \times S^1$ following [42]. It shows that the same conditions also apply to QCD(F/adj).

Section 7 examines phase transitions which respect persistent mixed anomaly. Some phase transitions are analytically calculable. This material is new.

Section 8 starts with a reminder of volume independence idea, and shows that with the flavor and $(-1)^F$ twisted boundary conditions, it is possible to achieve large-N volume (on $\mathbb{R}^3 \times S^1$ temperature) independence even in QCD with fundamental flavors in the Veneziano type limits. Without these twists, this is impossible.

Section A provides pedagogical examples of quantum distillation interpretation over the Hilbert space. Two of these applications are new (Hydrogen and principal chiral model in $d = 2$) and two are borrowed from past work (N-dimensional simple harmonic oscillator and $\mathbb{CP}^{N-1}$ in $d = 2$) for pedagogical completeness.

2 QCD(F/adj) and SQCD: general considerations

Consider QCD with SU(N_c) gauge group, N_f flavors of fundamental massless Dirac fermions ψ^a, $a = 1, \cdots, N_f$ and one adjoint Weyl fermion λ with Euclidean Lagrangian:[7]

$$\mathcal{L} = \frac{1}{2g^2}\,\mathrm{tr}\,F_{\mu\nu}^2 + \sum_{a=1}^{N_f} \overline{\psi}_a \gamma_\mu D_\mu \psi^a + 2\,\mathrm{tr}\,\overline{\lambda}\overline{\sigma}_\mu D_\mu \lambda \tag{2.1}$$

where $D_\mu \psi^a = \partial_\mu \psi^a + i a_\mu \psi^a$ and $D_\mu \lambda = \partial_\mu \lambda + i[a_\mu, \lambda]$. These theories are one-parameter family deformations of the general SQCD [46] with a supersymmetry breaking mass term for the scalar quark field (squark, the super-partner of ψ_a fundamental fermion) m_{Φ_a} turned on, and taken to decoupling limit [19]. However, we investigate (2.1) without any reference to supersymmetry, with semi-classical and mixed anomaly tools that usefully apply to non-supersymmetric theories.

We consider both massless theory (2.1) as well as it its mass deformations, with a common mass $m_{\psi_1} = \ldots = m_{\psi_{N_f}} \equiv m_\psi \geq 0$ and with $m_\lambda \geq 0$. If both mass terms are turned on, then the θ angle also becomes a physical parameter in the Lagrangian. Hence,

$$\delta\mathcal{L} = m_\psi \overline{\psi}_a \psi^a + m_\lambda(\lambda\lambda + \overline{\lambda\lambda}) + i\frac{\theta}{16\pi^2}\,\mathrm{tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{2.2}$$

We refer to this theory as QCD(F/adj). When the adjoint fermion decouples and $m_\psi = 0$, it corresponds to the chiral limit of QCD(F). We will see that this theory carries many new insights into non-perturbative dynamics of QCD(F). It also possesses calculable examples of chiral and color-flavor center symmetry changing phase transitions in thermodynamic limit. We will study the dynamics of this theory on $\mathbb{R}^3 \times S^1$ as a function of $(m_\lambda, m_\psi, \beta)$ parameters. First, let us discuss the global symmetries of this theory.

[7] We use Wess and Bagger convention with spinors which is commonly used in supersymmetry literature [45]. So, our Dirac spinor is $\psi = \binom{\psi_L}{\overline{\psi}_R}$, and mass term is $\overline{\psi}\psi = \psi_R\psi_L + \overline{\psi}_L\overline{\psi}_R$.

2.1 Global symmetries

In the chiral limit, $(m_\lambda, m_\psi) = (0,0)$, the classical theory possesses the global symmetry

$$\mathbf{G}_{\text{classical}} = \frac{\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_V \times \mathrm{U}(1)_{A_\psi} \times \mathrm{U}(1)_{A_\lambda}}{\mathbb{Z}_{N_c} \times (\mathbb{Z}_{N_f})_L \times (\mathbb{Z}_{N_f})_R \times (\mathbb{Z}_2)_\psi}. \tag{2.3}$$

where $\mathbb{Z}_{N_c}$ is the center of gauge group $\mathrm{SU}(N_c)$, which is not a global symmetry but just local gauge structure which has to be removed. The other discrete groups in the denominator are there to prevent double counting of the symmetries. However, due to ABJ anomaly, the classical abelian axial symmetry is reduced:

$$\mathrm{U}(1)_{A_\psi} \times \mathrm{U}(1)_{A_\lambda} \longrightarrow \mathrm{U}(1)_{A_D} \times \mathbb{Z}_{2\mathrm{gcd}(N_c,N_f)} \tag{2.4}$$

The symmetry of the quantum theory is

$$\mathbf{G} = \frac{\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_V \times \mathrm{U}(1)_{A_D} \times \mathbb{Z}_{2\mathrm{gcd}(N_c,N_f)}}{\mathbb{Z}_{N_c} \times (\mathbb{Z}_{N_f})_L \times (\mathbb{Z}_{N_f})_R \times (\mathbb{Z}_2)_\psi}. \tag{2.5}$$

To see this, note that the classical axial $\mathrm{U}(1)_{A_\psi}$ and $\mathrm{U}(1)_{A_\lambda}$ currents are non-conserved as:

$$\partial_\mu J_\psi^{\mu 5} = 2N_f \frac{1}{16\pi^2} \operatorname{tr} F_{\mu\nu}\tilde{F}^{\mu\nu}, \qquad \partial_\mu J_\lambda^{\mu 5} = 2N_c \frac{1}{16\pi^2} \operatorname{tr} F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{2.6}$$

and a diagonal subgroup of $\mathrm{U}(1)_{A_\psi} \times \mathrm{U}(1)_{A_\lambda}$ (call it $\mathrm{U}(1)_{A_D}$) survives in the quantum theory.

$$\partial_\mu \left(N_c J_\psi^{\mu 5} - N_f J_\lambda^{\mu 5} \right) = 0 \tag{2.7}$$

There is also a discrete remnant. To determine it, consider another linear combination of classical axial currents $k_1 J_\psi^{\mu 5} + k_2 J_\lambda^{\mu 5}$. The charge associated with this current is conserved modulo $k_1(2N_f) + k_2(2N_c)$. According to Bezout identity in elementary number theory, there exists $k_1, k_2 \in \mathbb{Z}$ such that

$$N_f k_1 + N_c k_2 = \mathrm{gcd}(N_f, N_c) \tag{2.8}$$

Therefore, the charge associated with this current is conserved modulo $2\mathrm{gcd}(N_f, N_c)$. This is the discrete chiral symmetry which cannot be undone by continuous chiral rotations $\mathrm{U}(1)_{A_D}$. The action of $\mathrm{U}(1)_{A_D} \times \mathbb{Z}_{2\mathrm{gcd}(N_f,N_c)}$ on the fermions is given by

$$\begin{aligned} \mathrm{U}(1)_{A_D} \times \mathbb{Z}_{2\mathrm{gcd}(N_f,N_c)} : &\qquad \psi_{L/R} \to e^{-i\frac{N_c}{N_f}\gamma}\ e^{\frac{2\pi i}{2\mathrm{gcd}(N_f,N_c)}k_1}\ \psi_{L/R} \\ : &\qquad \lambda \longrightarrow\ e^{+i\gamma}\ e^{\frac{2\pi i}{2\mathrm{gcd}(N_f,N_c)}k_2}\ \lambda \end{aligned} \tag{2.9}$$

The ABJ anomaly also manifests itself in the instanton amplitude [47], which for the QCD(F/adj) takes the form:

$$\mathcal{I}_{4d} \sim e^{-\frac{8\pi^2}{g^2}} (\operatorname{tr}\lambda\lambda)^{N_c} \det_{a,b=1}^{N_f} \left[\psi_{Ra}\psi_L^b\right] \tag{2.10}$$

To summarize, the transformation properties of the microscopic fermions under gauge structure and continuous global symmetry is given by:[8]

	$\mathrm{SU}(N_c)$	$\mathrm{SU}(N_f)$	$\mathrm{SU}(N_f)$	$\mathrm{U}(1)_V$	$\mathrm{U}(1)_{A_D}$
ψ_L^a	$\square$	$\overline{\square}$	$\mathbf{1}$	$+1$	$-\frac{N_c}{N_f}$
ψ_{Ra}	$\overline{\square}$	$\mathbf{1}$	$\square$	-1	$-\frac{N_c}{N_f}$
λ	**adj**	$\mathbf{1}$	$\mathbf{1}$	0	$+1$

(2.11)

The global symmetry (2.5) coincides with the bosonic symmetry of the $\mathrm{SU}(N_c)$ SQCD with N_f quarks [18], see also [48] for a pedagogical introduction.

Turning on masses for fermions and global symmetry. We will consider turning on two types of mass terms in QCD(F/adj): correspondingly, global 0-form symmetries reduce to:

$$\begin{aligned}
(m_\psi = 0, m_\lambda > 0): &\qquad \mathbf{G} = \frac{\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_V \times (\mathbb{Z}_{2N_f})_\psi}{\mathbb{Z}_{N_c} \times (\mathbb{Z}_{N_f})_R \times (\mathbb{Z}_{N_f})_L \times \mathbb{Z}_2} \\
(m_\psi > 0, m_\lambda = 0): &\qquad \mathbf{G} = \frac{\mathrm{SU}(N_f)_V \times \mathrm{U}(1)_V}{\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}} \times (\mathbb{Z}_{2N_c})_\lambda \\
(m_\psi > 0, m_\lambda > 0): &\qquad \mathbf{G} = \frac{\mathrm{SU}(N_f)_V \times \mathrm{U}(1)_V}{\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}} \\
(m_\psi = \infty, m_\lambda = 0): &\qquad \mathbf{G} = (\mathbb{Z}_{2N_c})_\lambda
\end{aligned} \tag{2.12}$$

The first one of these is the correct global symmetry of massless QCD(F), and the last is the one of $\mathcal{N} = 1$ SYM. In each case, $\mathbf{G}$ has the faithful representation on the physical Hilbert space $\mathcal{H}$ of corresponding theory [49].

2.2 Expectations on $\mathbb{R}^4$ in $N_f = N_c$ theory and relation to SQCD

In this section, we focus our attention to $N_f = N_c$ QCD(F/adj). There are two physically well motivated possibilities for the behaviour of $N_f = N_c$ QCD(F/adj) theory on $\mathbb{R}^4$ concerning the realization of continuous global symmetry (2.5).

1) The chiral global symmetry (2.5) can be broken down to vector-like subgroup:

$$\mathbf{G} \to \frac{\mathrm{SU}(N_f)_V \times \mathrm{U}(1)_V}{\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}} \tag{2.13}$$

This is QCD-type behaviour where fermion bilinears condense

$$\langle \psi_{Ra} \psi_L^b \rangle = c_1 \Lambda^3 \delta_b^a, \qquad \langle \operatorname{tr} \lambda\lambda \rangle = c_2 \Lambda^3 \tag{2.14}$$

[8] In (2.9), (2.11), we chose $\mathrm{U}(1)_{A_D}$ charges of $\lambda, \psi_{Ra}, \psi_L^a$ to match SQCD literature. In the context of SQCD, $\mathrm{U}(1)_{A_D}$ is called $\mathrm{U}(1)_R$ symmetry.

2) The chiral global symmetry can be broken down to a subgroup which possess a chiral $\mathrm{U}(1)_{A_D}$ part

$$\mathbf{G} \to \frac{\mathrm{SU}(N_f)_V \times \mathrm{U}(1)_V \times \mathrm{U}(1)_{A_D}}{\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f} \times \mathbb{Z}_2} \tag{2.15}$$

Unbroken $\mathrm{U}(1)_{A_D}$ along with broken non-abelian chiral symmetry $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \to \mathrm{SU}(N_f)_V$ is possible if fermion bilinears do not acquire a vev (because they are charged under both chiral symmetries), but only if a $\mathrm{U}(1)_A$ singlet four-fermion operator acquires a vev:

$$\langle \psi_{Rb} \psi_L^a \rangle = 0, \qquad \langle \operatorname{tr} \lambda\lambda \rangle = 0, \qquad \langle \psi_{Rb} \psi_L^a \operatorname{tr} \lambda\lambda \rangle = c_3 \delta_b^a \Lambda^6 \tag{2.16}$$

In the first case, there will be N_f^2 gapless NG-bosons, instead of $N_f^2 - 1$ as in QCD(F) since there is one extra axial charge generator. Therefore, the IR theory will also possess an exactly gapless η' boson, and will be described by the chiral Lagrangian of N_f^2 NG-bosons. In particular, since the symmetry breaks to vector-like subgroup, there are no massless fermions in the spectrum just like chiral limit of QCD(F).

The second case is strongly motivated by SQCD as described in [19]. Since a part of the chiral symmetry remains unbroken, the IR theory must possess both exactly massless bosons and composite fermions, and it must satisfy standard (zero form) 't Hooft anomaly matching conditions between UV and IR degrees of freedom [50]. The reason that the second possibility is not ruled out immediately is because it satisfies non-trivial 't Hooft anomaly matching conditions in the non-supersymmetric QCD(F/adj) just like supersymmetric SQCD.

Although chiral symmetry breaking without quark bilinear condensate is ruled out in QCD(F) by Tanizaki [49], this is still a perfectly viable (mixed anomaly allowed) option in QCD(F/adj).

Anomalies in $N_f = N_c$ SQCD and QCD(F/adj) are the same. Let us first describe why the second option is as viable as first option starting with SQCD. The space of ground states of $N_f = N_c$ SQCD theory is a quantum moduli space, described in terms of composite superfields, mesons M_b^a and baryon B, obeying [46].

$$\det M - B\overline{B} = \Lambda^{2N_c}. \tag{2.17}$$

At the point $B = \overline{B} = 0$, we have $\det M = \Lambda^{2N_c}$, corresponding to

$$M_b^a = Q_{Ra} Q_L^b = \Lambda^2 \delta_a^b \tag{2.18}$$

and the chiral symmetry is broken as in (2.15). At this point, gaugino condensate for adjoint quark vanishes, $\langle \operatorname{tr} \lambda\lambda \rangle = 0$. Ref. [19] showed that with a supersymmetry breaking soft mass for the scalar field (squark, the super-partner of ψ_a fundamental fermion) m_{q_a} turned on, this pattern persists. With a small m_{q_a}, $N_f = N_c$ SQCD breaks its non-abelian chiral symmetry, but not the $\mathrm{U}(1)_{A_D}$ part. Increasing m_{q_a}, there are two possibilities. Unbroken $\mathrm{U}(1)_{A_D}$ may persist to decoupling limit or there may be a $\mathrm{U}(1)_{A_D}$ breaking phase transition at some critical value of $m_{q_a}^{\mathrm{cr}}$. If unbroken $\mathrm{U}(1)_{A_D}$ persists, the pattern of the chiral condensate in the decoupling limit must be given by (2.16). In this case, infrared physics can be described in terms of $N_f^2 - 1$ NG bosons, and composite massless fermions which saturate the anomalies associated with unbroken $\mathrm{U}(1)_{A_D}$.

Since QCD(F/adj) has a fermionic matter content identical to SQCD, the UV anomalies coincide precisely. In the IR, we can construct fermionic mesons and baryons in QCD(F/adj) by using Fradkin-Shenker complementarity [51] to SQCD. To see this, denote supersymmetric chiral matter multiplets as:

$$
\begin{aligned}
Q_L^a &= q_L^a + \theta \psi_L^a + \dots \\
Q_{Rb} &= q_{Rb} + \theta \psi_{Rb} + \dots
\end{aligned}
\tag{2.19}
$$

The fermionic component of the composite meson multiplet can be expressed as:

$$
M_b^a = Q_{Ra} Q_L^b = \dots + \theta (q_{Ra} \psi_L^b + \psi_{Ra} q_L^b) + \dots \tag{2.20}
$$

The gauge and global quantum numbers of the q_{Ra} scalar and $(\psi_{Ra}\lambda)$ composite agree with each other. Therefore, we can view the mesino in QCD(F/adj) as continuation of mesino in SQCD and use the replacement:

$$
q_{Ra} \longleftrightarrow (\psi_{Ra}\lambda), \qquad q_L^b \longleftrightarrow \lambda \psi_L^b \tag{2.21}
$$

resulting in the fermionic meson (or mesino) ψ_{Ma}^b given by

$$
\psi_{Ma}{}^b = \psi_{Ra} \lambda \psi_L^b \qquad a, b = 1, \dots, N_f \tag{2.22}
$$

Note that the natural continuation of scalar-quark bilinear (2.18) to the QCD-like regime where scalar decouples is given by four-fermi operators:

$$
q_{Ra} q_L^b \longleftrightarrow \langle \psi_{Rb} \psi_L^a \operatorname{tr} \lambda\lambda \rangle \tag{2.23}
$$

the expression given in (2.16).

Similarly, the baryon multiplet in $N_f = N_c$ SQCD and its fermionic component are given by

$$
B_L = \epsilon_{a_1 \dots a_{N_f}} Q_L^{a_1} \dots Q_L^{a_{N_f}} = \dots + \theta \epsilon_{a_1 \dots a_{N_f}} q_L^{a_1} \dots q_L^{a_{N_f - 1}} \psi_L^{a_{N_f}} + \dots \tag{2.24}
$$

Therefore, the fermionic baryon ψ_{B_L} (and ψ_{B_R}) can be written as[9]

$$
\begin{aligned}
\psi_{B_L} &= \epsilon_{a_1 \dots a_{N_f}} (\lambda \psi_L^{a_1}) \dots (\lambda \psi_L^{a_{N_f - 1}}) \psi_L^{a_{N_f}} \\
\psi_{B_R} &= \epsilon^{a_1 \dots a_{N_f}} (\lambda \psi_{Ra_1}) \dots (\lambda \psi_{Ra_{N_f - 1}}) \psi_{Ra_{N_f}}
\end{aligned}
\tag{2.25}
$$

The quantum numbers of these composite fermions under unbroken symmetry (2.15) are given by

	$\mathrm{SU}(N_f)_V$	$\mathrm{U}(1)_V$	$\mathrm{U}(1)_{A_D}$
$\psi_{Ma}{}^b$	**adj**	0	-1
ψ_{B_L}	**1**	N_c	-1
ψ_{B_R}	**1**	$-N_c$	-1

(2.26)

[9] Note that in QCD(F), baryons are fermionic for $N_c =$ odd and bosonic for $N_c =$ even. In QCD(F/adj), we can have fermionic baryons for either choice of N_c.

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The non-trivial anomalies in the UV and IR are (set $N_c = N_f$ in all formulas:)

$$\begin{aligned} \mathrm{U}(1)_{A_D}: & \quad -2N_fN_c + (N_c^2-1) = -(N_f^2-1) - 2 \\ \mathrm{U}(1)^3_{A_D}: & \quad -2N_fN_c + (N_c^2-1) = -(N_f^2-1) - 2 \\ \mathrm{U}(1)^2_V \times \mathrm{U}(1)_{A_D}: & \quad -2N_fN_c = -2N_c^2 \\ \mathrm{SU}(N_f)^2_V \times \mathrm{U}(1)_{A_D}: & \quad -N_f d^{(2)}(\square) - N_f d^{(2)}(\overline{\square}) = -d^{(2)}(\mathbf{adj}) \end{aligned} \tag{2.27}$$

where $d^{(2)}(\square) = \frac{1}{2}, d^{(2)}(\mathbf{adj}) = N_f$ are the corresponding quadratic $\mathrm{SU}(N_f)$ Casimir operators. Needless to say, these are the anomaly matching conditions for $N_f = N_c$ SQCD as well [46] as they must be, because we can obtain QCD(F/adj) by decoupling the scalar in supersymmetric theory. This point is also emphasized in the analogous discussion of $\mathcal{N} = 2$ SYM theory and its non-supersymmetric deformation to QCD(adj) with $n_f = 2$ flavors [52].

The 0-form 't Hooft anomalies of the UV theory are matched by the massless composite IR fermions, $N_f^2 - 1$ fermionic meson and massless baryon. This suggests that apart from the NG-bosons and the composite fermions (2.26), there should not be any other gapless degrees of freedom.

Note one crucial distinction from QCD(F). Since both adjoint as well as fundamental fermion bilinears are charged under $\mathrm{U}(1)_{A_D}$, a constituent quark mass cannot be created with the chiral symmetry breaking pattern (2.15) in QCD(F/adj).

2.3 Exact color-flavor center symmetry on $\mathbb{R}^3 \times S^1$

It is well-known that introducing fundamental fermions in $\mathrm{SU}(N_c)$ gauge theory breaks 1-form $\mathbb{Z}_{N_c}^{[1]}$ center-symmetry explicitly and completely, as all Wilson lines become endable on quarks. Obviously, in the theory compactified on a circle $\mathbb{R}^3 \times S^1$, it also breaks the 0-form center-symmetry that acts on the Wilson line (Polyakov loop) wrapping S^1 circle. It is recently understood that in QCD(F) with N_f fermions, it is actually possible to preserve a $\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)}$ sub-group of 0-form center-symmetry acting non-trivially on Polyakov loop by paying an appropriate price. This procedure does not restore a 1-form center symmetry on $\mathbb{R}^3$. Below, we describe the appearance of 0-form center-symmetry in compactified theory.

The center-symmetry in pure Yang-Mills theory is a 1-form symmetry acting on Wilson line operators on $\mathbb{R}^4$. On the theory compactified on $\mathbb{R}^3 \times S^1$, it decompose into a 0-form symmetry acting non-trivially on Polyakov loops (which becomes a point operator from the $\mathbb{R}^3$ point of view), and 1-form symmetry acting on line operators on $\mathbb{R}^3$. Traditional way to think about 0-form center-symmetry is sufficient for our purpose. 0-form center-symmetry may be associated with gauge transformations $g(x_4)$ aperiodic up to an element of center group, $g(x_4 + \beta) = \omega^{-1} g(x_4)$, $\omega^{N_c} = 1$. Polyakov loop transforms under it as $\mathrm{tr}\, e^{i\int_0^\beta a_4\, dx_4} \to \mathrm{tr}\,(g(0) e^{i\int_0^\beta a_4\, dx_4} g^\dagger(\beta)) = \omega\, \mathrm{tr}\, e^{i\int_0^\beta a_4\, dx_4}$. Therefore,

$$\mathbb{Z}_{N_c}: \ \ \mathrm{tr}\, \Omega(\mathbf{x}) \equiv \ \mathrm{tr}\, e^{i\int_0^\beta a_4\, dx_4} \mapsto \omega\, \mathrm{tr}\, \Omega(\mathbf{x}) \tag{2.28}$$

For example, the holonomy potential for pure Yang-Mills theory that one obtains at small-β by integrating out heavy modes is composed of the terms like $|\,\mathrm{tr}\,(\Omega^n)|^2, n = 1, 2, \ldots$ and are manifestly invariant under this symmetry [20].

Let us now describe how the 0-form center-symmetry is violated in the presence of fundamental fermions. Start with anti-periodic (or any other flavor independent) boundary condition for fermions, $\psi(x_4+\beta) = -\psi(x_4)$. Now, consider a gauge transformation of this condition, $\psi^G(x_4+\beta) = -\psi^G(x_4)$, where G is via a transformation $g(x_4)$ aperiodic up to an element of center group, $g(x_4+\beta) = \omega g(x_4)$. Then, $\psi(x_4+\beta) = -\omega\psi(x_4)$. Therefore, the center-symmetry transformation does not respect the original boundary conditions, and leads to explicit breaking of center symmetry. For example, the one-loop potential one obtains at small-β by integrating out fermions induce terms of the form $\operatorname{tr}(\Omega^n), n = 1, 2, \ldots$, which does explicitly break the center symmetry [20].

Surprisingly, the 0-form center symmetry can actually be "rescued" even in the presence of fundamental fermions. This is appreciated as a genuine symmetry and called color-flavor center (CFC) symmetry in [23]. Also see earlier related work [24, 33, 42]. Assume $N_f = N_c$ momentarily. This assumption will be relaxed. Impose an $\mathrm{U}(N_f)_V$ flavor twisted boundary condition on fermions, where we choose the flavor twist to be

$$\Omega_F^0 = \operatorname{diag}(1, \omega, \cdots, \omega^{N_f-1}), \qquad \omega = e^{2\pi i/N_f}. \tag{2.29}$$

Now, fermion boundary conditions are

$$\psi(x_4+\beta) = -\psi(x_4)\overline{\Omega}_F^0 \tag{2.30}$$

and under a gauge transformation aperiodic up to an element of the center, $\psi^G(x_4+\beta) = -\psi^G(x_4)\overline{\Omega}_F^0$. This amounts to changing the boundary conditions into $\psi(x_4+\beta) = -\psi(x_4)\omega^{-1}\overline{\Omega}_F^0$. Since the boundary conditions are different, in general, this is again a non-invariance and explicitly breaks center-symmetry. However, $\omega\Omega_F^0$ is a cyclic permutation of Ω_F^0 and can be brought to the original boundary conditions by using a flavor rotation generated by the shift matrix $(S)_{a,b} = \delta_{a+1,b}$:

$$S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & & 0 \end{bmatrix} \tag{2.31}$$

obeying the algebra:

$$S\Omega_F^0 S^{-1} = \omega\Omega_F^0, \qquad S \in \Gamma_S \subset \mathrm{SU}(N_f)_V \tag{2.32}$$

where $\Gamma_S \subset \mathrm{SU}(N_f)_V$ is the cyclic permutation subgroup of the vector-symmetry. Therefore, the theory with $N_f = N_c$ possesses an exact diagonal 0-form center symmetry,

$$\mathbb{Z}_{N_c} \text{ CFC-symmetry}: \qquad \operatorname{tr}\Omega(\mathbf{x}) \mapsto \omega \operatorname{tr}\Omega(\mathbf{x}), \qquad \psi_a \mapsto \psi_{a+1} \tag{2.33}$$

which is an admixture of center of gauge group and Γ_S subgroup of flavor rotation. Since CFC symmetry intertwines 0-form part of the center symmetry on $\mathbb{R}^3 \times S^1$, and flavor transformations, it has both local and extended order parameters. The order parameters are Polyakov loops and Fourier transforms of fermion bilinears with respect to flavor index:

$$\widehat{(\psi_R\psi_L)}_p :\equiv \frac{1}{\sqrt{N_c}}\sum_{a=1}^{n_\mathrm{f}} \omega^{-ap}\, \psi_{Ra}\psi_L^a : \tag{2.34}$$

which transform non-trivially under CFC-symmetry:

$$\mathbb{Z}_{N_c}: \qquad \mathrm{tr}\,\Omega^p \mapsto \omega^p\ \mathrm{tr}\,\Omega^p\,, \tag{2.35}$$

$$\widehat{(\psi_R\psi_L)}_p \mapsto \omega^p\ \widehat{(\psi_R\psi_L)}_p \tag{2.36}$$

Therefore, $\mathrm{tr}\,\Omega^p$ is the natural order parameter for CFC symmetry, just like Polyakov loop is the order parameter of center-symmetry in pure Yang-Mills theory.

The CFC 0-form symmetry is present for any $m_\psi \geq 0$. In the limit $m_\psi \to \infty$, the CFC symmetry is part of genuine $\mathbb{Z}_{N_c}^{[1]}$ center-symmetry in pure Yang-Mills theory. For general N_f, CFC 0-form symmetry becomes:

$$\text{CFC-symmetry} : \mathbb{Z}_{\gcd(N_f,N_c)} \qquad \text{general } N_f \tag{2.37}$$

The exact CFC symmetry will also be important and manifest itself in beautiful ways in our discussion of one- and two-loop potential for gauge holonomy in QCD(F/adj) where it will be a manifest symmetry as described in subsection 3.1. More importantly, the realization of the CFC symmetry at small $\mathbb{R}^3\times S^1$ will be important for the idea of adibatic continuity in QCD(F/adj), and in the realization of Nye-Singer index theorem for fermions as described in section 5.

The price one pays for keeping an exact 0-form center-symmetry appears as explicit reduction of the non-abelian chiral symmetry to its maximal abelian torus. This is because Ω_F^0 only commutes with the Cartan generators of $\mathrm{SU}(N_f)_V$. Therefore, with twisted boundary conditions, the global symmetry is explicitly broken down to maximal torus and the global symmetry of the compactified theory becomes

$$\mathbf{G}_{\mathrm{max-ab}} = \frac{\mathrm{U}(1)_L^{N_f-1}\times \mathrm{U}(1)_R^{N_f-1}\times \mathrm{U}(1)_V\times \mathrm{U}(1)_{A_D}\times \mathbb{Z}_{2\gcd(N_c,N_f)}}{\mathbb{Z}_{N_c}\times(\mathbb{Z}_{N_f})_L\times(\mathbb{Z}_{N_f})_R\times(\mathbb{Z}_2)_\psi}. \tag{2.38}$$

With these boundary conditions, only N_f Nambu-Goldstone bosons remain gapless at large-S^1, while the off-Cartan NG-bosons acquire masses of order $\frac{2\pi}{\beta N_f}$. The chiral Lagrangian reduce to a non-linear sigma model on the maximal torus, as described in subsection 5.6.

3 Frustration, collapse and a new governance: gauge-holonomy potentials in the presence of flavor-holonomies

In this section, we describe some implications of the grading over the Hilbert space via $(-1)^F e^{i\pi Q_0}\prod_{a=1}^{N_f} e^{i\frac{2\pi a}{N_f}Q_a}$ in terms of gauge holonomy potentials. We will show that the grading and consequent quantum distillation maps to an intriguing phenomenon in gauge holonomy potentials. Gauge holonomy potential is a function of $\mathrm{tr}_\mathcal{R}(\Omega^n)$ for representations

$\mathcal{R}$ appearing in the microscopic theory. We demonstrate explicitly that some reprepresentations appearing in holonomy potential are prone to frustrations and their effects collapse to zero, while some others are immune and their effect dictates the new ground states and/or thermal equilibrium states. This is a fairly entertaining story, as pointed in the section title and depicted in figure 3.

As described in the Introduction, gauge holonomy Ω is dynamical and its value at small-β is determined by the potential $V[\Omega]$. In thermal gauge theory, this type of potentials are well understood at one-loop [20, 53] and two-loop orders [25, 26] and [27] and in some supersymmetric gauge theories where perturbative-loop potential vanishes, it is understood non-perturbatively [54, 55].

It is easy to implement the non-dynamical Ω_F flavor backgrounds and examine their physical effects on the gauge holonomy potentials. This is the image of quantum distillation of Hilbert space in path integral formalism. We will obtain quite surprising results in QCD(F/adj) as well as QCD(F) for these potentials and their extrema.

One goal is to determine the realization of the color-flavor center symmetry (2.33) at small-β regime for SU(N_c) gauge theory. For $N_c \geq 3$, we show that in the presence of the flavor twist Ω_F^0, there is an exponentially increasing number of degenerate minima at one-loop order in contradistinction with thermal case, where there is a unique deconfined minimum at $\Omega = 1$. The degeneracy at one-loop order becomes continuous in the Veneziano-type large-N_c limit where the one-loop potential in QCD(F/adj) vanishes. This is surprising considering that the theories we are dealing with are non-supersymmetric, and characteristic scaling of the potential must be of order N_c^2. At two-loop order, we will show that for the grading that achieves perfect quantum distillation of Hilbert space (Ω_F^0) leads to a center-symmetric minimum for the holonomy potential!

3.1 Frustration and collapse of one-loop Gross-Pisarski-Yaffe potential

Before turning on the Ω_F twist, it is useful to remind the thermal one-loop potentials. In the small-circle limit where the theories are weakly coupled, the one-loop potential for the gauge holonomy (1.15) receives contributions from the weakly coupled (A_μ, λ, ψ^a) fields:

$$\begin{aligned} V_{1-\text{loop,thermal}}(\Omega) &= \frac{2}{\beta}\int \frac{d^3p}{(2\pi)^3}\Bigg[+\sum_{i,j}\log(1-e^{-\beta p+iv_{ij}})-\sum_{i,j}\log(1+e^{-\beta p+iv_{ij}}) \\ &\qquad -N_f\sum_i\Big(\log(1+e^{-\beta p+iv_i})+\text{c.c.}\Big)\Bigg] \\ &= \frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\left[(-1+(-1)^n)\frac{1}{n^4}|\operatorname{tr}(\Omega^n)|^2+N_f\frac{(-1)^n}{n^4}(\operatorname{tr}(\Omega^n)+\text{c.c.})\right] \end{aligned} \tag{3.1}$$

As the center-symmetry is explicitly broken by the fundamental fermions with anti-periodic boundary conditions, this potential has terms of the form $\operatorname{tr}(\Omega^n), n = 1, 2, \ldots$ which indeed violates center symmetry. The global minimum of the potential is located at $\Omega = 1$. Although this is a result of one-loop calculation, it is true to all orders in perturbation theory and non-perturbatively [20] in small-β domain.

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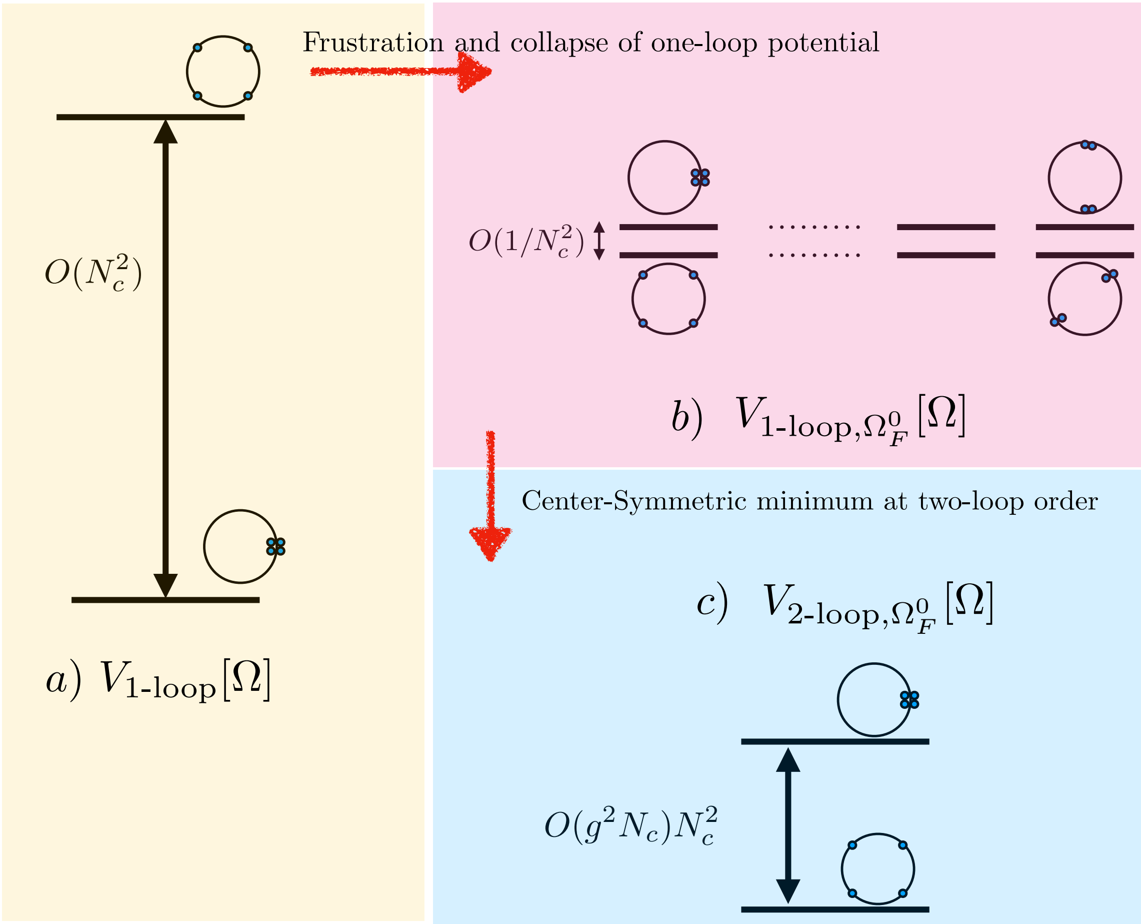

Figure 3. The figure describes the image of Hilbert space distillation in path integral and gauge holonomy potential description. a) This is the standard implication of Gross-Pisarski-Yaffe analysis. Center-broken configuration is the minimum and governs ground state (thermal equilibrium state properties.) In particular, the center-symmetric configuration is extremely disfavored. The potential gap between min and max is order N_c^2. b) The image of distillation of $\mathcal{H}$ is the frustration of GPY potential at one-loop. Now, there are exponentially increasing number of min and max. In particular, the center-symmetric and center-broken configurations are almost on the same footing as the gap between min and max is order $1/N_c^2$. c) At two-loop, there is a center-stabilizing double-trace term in the potential which is resistant to frustration. That term decides the new ground state of the theory, which is now center-symmetric. This holds to all orders in perturbation theory and non-perturbatively.

A quixotic quest. Assume momentarily $N_f = N_c$. Clearly, all three contributions in (3.1) works against what we would like to achieve, and all three terms prefer $\Omega \propto 1$ as the global minimum. Moreover, all the terms in the fundamental fermion contribution, except the ones of the form $\operatorname{tr}(\Omega^{N_c k})$ explicitly break center-symmetry. We would like to achieve two things.

- Make the potential CFC-symmetric by using appropriate Ω_F.
- Make sure CFC-symmetry is unbroken and $\Omega \propto \operatorname{diag}(1, \omega, \cdots, \omega^{N_c-1})$ is the global minimum of the potential.

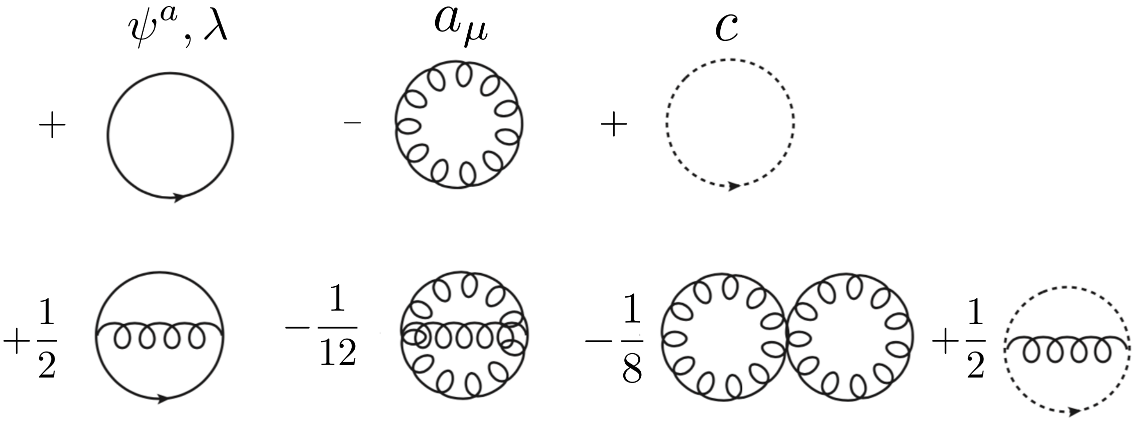

Figure 4. Feynman diagrams for the gauge holonomy potential at one- and two-loop order for N_f-Dirac fundamental fermions and one-adjoint Majorana ($N_{\rm adj}^{\rm Dirac} = \frac{1}{2}$) fermion. Diagrams are reproduced from [25, 27] and slightly generalized according to new matter content. Up to two-loop order, the adjoint and fundamental fermions are decoupled in their contributions. In the graded case, where we insert $(-1)^F \prod_{a=1}^{N_f} e^{i\epsilon_a Q_a}$ into the partition function, we determine the potential for the dynamical gauge holonomy Ω in the fixed background of Ω_F flavor holonomy.

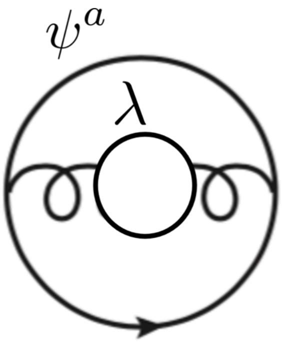

Figure 5. In the determination of gauge holonomy potential, adjoint and fundamental fermions only start to couple at three-loop order in diagrams as above. However, as explained in the text, two-loop result will suffice for the determination of CFC-symmetry.

Obviously, this goal sounds surreal. But it is crucial for the realization of the adiabatic continuity idea. However, the thermal potential for center-symmetric configuration is of order $O(N_c^{-2})$, while the potential for the global minimum is $-O(N_c^2)$, and it is not a priori obvious how to get rid of terms like $N_f \,\mathrm{tr}\,(\Omega^n), n \neq N_c k$ in the potential. It may seem almost impossible to overcome these difficulties and achieve a center-symmetric minimum for QCD(F/adj) or QCD(F) with heavy adjoint fermion. On the other hand, we know that the holonomy potential at its minimum corresponds to free energy of the QFT, and quantum distillation must alter free energy drastically. Therefore, if the quantum distillation idea ought to be operative at the path integral level, it must alter the potential in useful ways. In other words, in the space of graded partition functions, there may be interesting opportunities and indeed, there are!

In order to capture the graded partition function, we impose somehow exotic looking boundary conditions on fermions. The implication of these boundary conditions on gauge invariant states in the Hilbert space (such as mesons, baryons etc) will be discussed in section 4. In fact, by using the idea of quantum distillation, we will *derive* the most efficient boundary conditions as an extremum of twist free energy.

The boundary conditions on fermions in the path integral formulation, which is the mapping of the (1.11) in operator formalism, are:

$$\begin{aligned} \lambda(x_4+\beta) &= (+)\lambda(x_4), \\ \psi(x_4+\beta) &= (+)\psi(x_4)e^{i\pi\overline{\Omega}_F} \end{aligned} \tag{3.2}$$

where $(+)$ is due to $(-1)^F$ insertion into the state-sum, and $e^{i\pi}\Omega_F$ is a $\mathrm{U}(N_f)_V$ twist. The boundary condition on gauge connection is periodic.

The one-loop potential for gauge holonomy Ω in the presence of the (non-dynamical) Ω_F background and $(-1)^F$ is given by

$$\begin{aligned} V_{1-\mathrm{loop},\Omega_\mathrm{F}}(\Omega) = \frac{2}{\beta}\int\frac{d^3p}{(2\pi)^3}\Bigg[&+\sum_{i,j}\log(1-e^{-\beta p+iv_{ij}}) - \sum_{i,j}\log(1+e^{-\beta p+iv_{ij}+i\pi}) \\ &-\sum_{a=1}^{N_f}\sum_i\Big(\log(1+e^{-\beta p+iv_i-i\epsilon_a})+\text{c.c.}\Big)\Bigg] \end{aligned} \tag{3.3}$$

The first sum is center-destabilizing gauge boson contribution, the second is center-stabilizing adjoint fermion contribution and the third one is twisted boundary condition fundamental fermion contribution. In terms of gauge-invariant Wilson lines, the potential can be written as:

$$\begin{aligned} V_{1-\mathrm{loop},\Omega_\mathrm{F}} &= V^{\mathrm{gauge}}_{1-\mathrm{loop}} + V^{\lambda}_{1-\mathrm{loop}} + V^{\psi}_{1-\mathrm{loop},\Omega_\mathrm{F}}\,, \\ V^{\mathrm{gauge}}_{1-\mathrm{loop}} + V^{\lambda}_{1-\mathrm{loop}} &= (-1+1)\frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\frac{1}{n^4}|\operatorname{tr}(\Omega^n)|^2 = 0\,, \\ V^{\psi}_{1-\mathrm{loop},\Omega_\mathrm{F}} &= \frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\Big[\operatorname{tr}(\Omega^n)\operatorname{tr}(\overline{\Omega}_F^n)+\text{c.c.}\Big]\,. \end{aligned} \tag{3.4}$$

The gauge boson and one-adjoint fermion (with periodic boundary condition) contributions cancel each other out $V^\lambda + V^{\mathrm{gauge}} = 0$ as in the $\mathcal{N}=1$ supersymmetric Yang-Mills theory, not only at one-loop level but to all-loop orders in perturbation theory.[10] The center-symmetry breaking effect of gauge fluctuations are undone by the quantum fluctuations of periodic adjoint fermions. The final decision about the presence/absence of CFC symmetry and its realization is given by fundamental fermions. In (3.4), $\operatorname{tr}(\Omega^n)$ is dynamical, determined by extremizing the potential, and $\operatorname{tr}(\overline{\Omega}_F^n)$ is non-dynamical. It is

[10]More precisely, $V^\lambda + V^{\mathrm{gauge}} = 0$ in the $N_f = 0$ theory. For general N_f, the subset of diagrams involving gauge fluctuations and *only* adjoint fermion loops must cancel each other exactly as in $\mathcal{N}=1$ SYM. At 3-loop order and higher, there are diagrams involving both adjoint and fundamental fermion loops, such as figure 5. These diagrams will not cancel against gauge fluctuations, but they will not cause any difficulty either.

the boundary condition we choose. Clearly, the behavior of the small-β theory at one-loop order crucially depends on our choice of Ω_F.[11]

Now comes one of the main points. For a generic flavor holonomy backgroud Ω_F, the potential explicitly breaks $\mathbb{Z}_{N_c}^{[0]}$ center-symmetry, which acts on Polyakov loop $\mathrm{tr}\,(\Omega^n) \to \omega^n\,\mathrm{tr}\,(\Omega^n)$. For special values of Ω_F, the one-loop potential is invariant under various subgroups of $\mathbb{Z}_{N_c}$. Of course, this is not an accident and in fact, holds to all-loop orders and non-perturbatively. It is just manifestation of the relevant subgroup of $\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)}$ CFC symmetry as described in section 2.3 and also (1.14).

The twisted boundary condition fundamental fermion induced potential having terms of the form $\mathrm{tr}\,(\Omega^n)\,\mathrm{tr}\,(\overline{\Omega}_F^n)$ is just the manifestation of the fact that ψ transforms as fundamental under color and anti-fundamental under flavor:

$$\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)_V: \quad \psi(x) \to g_c(x)\psi(x)g_f^\dagger \tag{3.5}$$

where the first one is gauge structure and latter is vector-like flavor symmetry. If flavor symmetry were to be gauged, this would indeed be the holonomy potential for fermions in QCD with bi-fundamental fermions, see e.g. [56, 57]. But in our current discussion, $\mathrm{SU}(N_f)_V$ is not gauged, only a background is turned on.

For $\Omega_F = \Omega_F^0$ given in (1.13), the potential becomes invariant under the full $\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)}$ CFC-symmetry. Indeed, in the $V^{\psi}_{1-\mathrm{loop},\Omega^0_{\mathrm{F}}}$, all terms which transform non-trivially under the CFC vanish identically:

$$\begin{aligned} \mathrm{tr}\,(\Omega_F^0)^n &= 0 \qquad \text{for } \; n \neq 0 \mod N_f \\ \mathrm{tr}\,(\Omega_F^0)^{N_f k} &= N_f \end{aligned} \tag{3.6}$$

Consequently, $V^{\psi}_{1-\mathrm{loop},\Omega^0_{\mathrm{F}}}$ becomes a sum of the terms of the type $\mathrm{tr}\,(\Omega^{N_f k})$ and the full one-loop potential takes the simple form:

$$V_{\text{1-loop},\Omega_F^0} = \frac{2}{\pi^2 N_f^3 \beta^4} \sum_{k=1}^{\infty} \frac{(-1)^{N_f k}}{k^4} \left[\mathrm{tr}\,(\Omega^{N_f k}) + \text{c.c.} \right] \tag{3.7}$$

This is a quite peculiar potential whose number of degenerate minima (and maxima) is exponentially increasing for $N_f \sim N_c$.[12] The characteristic size of the potential (3.7) is of order $O(\frac{N_c}{N_f^3})$, much smaller compared to thermal gauge holonomy potential, which is of order $O(N_f N_c)$ induced by thermal fundamental fermion induced term in (3.1).

3.2 Counting the min-max and exponential degeneracy at one-loop

The one-loop potential (3.7) is valid for general N_f and N_c. Below, we describe the minima and maxima of this potential for the case $N_f = N_c = N$.[13] We dropped the subscript to

[11] The flavor holonomy Ω_F is indeed a choice at this stage, and we are free to choose any boundary condition. However, not all choices are on equal footing. It is meaningful to ask which choice corresponds to having a CFC-symmetry, or a persistent mixed anomaly or is the most efficient quantum distillation. From the last point of view, there is an extremization problem, despite Ω_F being non-dynamical! One can write down a twist free-energy in confined phase for hadrons and extremize it. Its maximum correspond to $\Omega_F = \Omega_F^0$.

[12] The vacuum structure for $N_c = N_f = 3$ was analyzed numerically in [58].

[13] We do determine both minima and maxima, because in the graded partition function, by turning on a $\mathrm{U}(1)_V$-twist, we can switch the minima with maxima.

lessen the clutter. The main contribution to the one-loop potential (3.7) comes from the $k = 1$ term. This amounts to finding the extremum of the potential

$$V(v_i) = \sum_{i=1}^{N} (-1)^N \cos(N v_i), \qquad \sum_{i=1}^{N} v_i = 0 \mod 2\pi \tag{3.8}$$

within the Weyl-cell of the SU(N). The Weyl group of SU(N) is the permutation group S_N. Therefore, configurations $(v_1, \ldots v_N)$ related to one another by Weyl permutations are gauge equivalent and should not be counted independently. This is the process of removing gauge redundancy by using the quotient with S_N. The minima of the potential is thus given by Ω such that $\Omega^N = \mathbb{1}$ for odd N and $\Omega^N = -\mathbb{1}$ for even N. Furthermore, $\det \Omega = 1$. There are four distinct cases.

Minima set conditions, *N*-odd.

1. $\Omega = \mathrm{diag}(e^{i\frac{2\pi}{N}a_1}, \ldots, e^{i\frac{2\pi}{N}a_N})$
2. Each a_i takes a number from 0 to $N-1$.
3. $a_1 \leq a_2 \leq \cdots \leq a_N$.
4. $\sum_{i=1}^{N} a_i = 0 \pmod{N}$.

Maxima set conditions, *N*-odd.

1. $\Omega = \mathrm{diag}(e^{i\frac{2\pi}{N^2}a_1}, \ldots, e^{i\frac{2\pi}{N^2}a_N})$
2. Each a_i takes a number from the set of N positive-integers $\frac{N-1}{2} + 2Nk,\ k = 0, N-1$ in the range 0 to N^2.
3. Choose N a_i's such that $a_1 \leq a_2 \leq \cdots \leq a_N$.
4. $\sum_{i=1}^{N} a_i = 0 \pmod{N^2}$.
5. Duplicate the same process with the set $\frac{N+1}{2} + 2Nk,\ k = 0, N-1$.

Minima set conditions, *N*-even.

1. $\Omega = \mathrm{diag}(e^{i\frac{2\pi}{2N}a_1}, \ldots, e^{i\frac{2\pi}{2N}a_N})$
2. Each a_i takes an odd number from 0 to $2N-1$.
3. $a_1 \leq a_2 \leq \cdots \leq a_N$.
4. $\sum_{i=1}^{N} a_i = 0 \pmod{2N}$.

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Maxima set conditions, N-even.

1. $\Omega = \mathrm{diag}(e^{i\frac{2\pi}{N}a_1}, \dots, e^{i\frac{2\pi}{N}a_N})$
2. Each a_i takes a number from 0 to $N-1$.
3. $a_1 \leq a_2 \leq \cdots \leq a_N$.
4. $\sum_{i=1}^{N} a_i = 0 \pmod N$.

For $N=2$, there is a unique minimum given by $(a_1, a_2) = (1,3) \mod 4$, which corresponds to $\Omega = \mathrm{diag}(e^{i\pi/2}, e^{i3\pi/2})$, and center-symmetry is already stable at one-loop level. This is an exceptional case and with increasing N, vacuum degeneracy increases quickly. In table 1, we list the Minima-Set and Maxima-Set for $N = 2,3,4,5$. Note that the value of the toy potential (3.8) for both N-even and N-odd is $V_{\min} = -N$, the maximum for N-even is at $V_{\max} = +N$, while for N-odd, it is located at $V_{\max} = N\cos\frac{\pi(N\pm1)}{N} = N\cos\frac{\pi}{N}$. This is the reason that the number of maxima for N-odd being roughly twice as many as the number of minima.

The exact formula for the number of gauge inequivalent minima and maxima of the one-loop potential as described above maps to a problem in additive number theory.[14] The number of degenerate minima are given by:

$$\mathfrak{N}_{\min}(N) = \frac{1}{2N}\sum_{d|N}(-1)^{N+d}\,\phi\left(\frac{N}{d}\right)\binom{2d}{d} \tag{3.9}$$

where the sum runs over all positive divisors of N, $\phi(\cdot)$ is the Euler totient function, and the last term is binomial coefficient. The number of degenerate maxima are given by:

$$\mathfrak{N}_{\max}(N) = \begin{cases} \frac{1}{2N}\sum\limits_{d|N}\phi\left(\frac{N}{d}\right)\binom{2d}{d} & N \text{ even} \\ \\ \frac{1}{N}\sum\limits_{d|N}\mu(d)\binom{2N/d}{N/d} & N \text{ odd} \end{cases} \tag{3.10}$$

where $\mu(d)$ is Möbius function.

Let us derive an explicit growth in the number of minima within the asymptotic approximation. First, the number of sequences obeying the conditions 2 and 3 for minima set is given by combinatorial expression $\binom{2N-1}{N}$. Statistically, the condition 4 is met with probability $\sim 1/N$, hence the rough number of minima of one-loop potential is $\sim \frac{1}{N}\binom{2N-1}{N}$. This is nothing but the dominant $d=N$ term in the summation (3.9). Asymptotically we have an exponential growth in the number of gauge inequivalent minima of the one-loop potential:

$$\mathfrak{N}_{\min}(N) \approx \frac{(2N-1)!}{(N!)^2} \underbrace{\longrightarrow}_{\text{large}-N} \frac{2^{2N-1}N^{-3/2}}{\sqrt{\pi}}\,. \tag{3.11}$$

[14] All of the above problems fall into a set of combinatorial problems studied by Erdős, Ginzburg and Ziv [59]. The minima set follows the OEIS entry https://oeis.org/A145855.

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Min-SU(2) Mod 4	Max-SU(2) Mod 2
13	00
	11

Min-SU(3) Mod 3	Max-SU(3) Mod 9
000	117
012	144
111	477
222	225
	288
	588

Min-SU(4) Mod 8	Max-SU(4) Mod 4
1115	0000
1133	0013
1177	0022
1357	0112
1555	0233
3337	1111
3355	1133
3777	1223
5577	2222
	3333

Min-SU(5) Mod 5	Max-SU(5) Mod 25
00000	(2, 2, 2, 2, 17)
00014	(2, 2, 2, 7, 12)
00023	(2, 2, 2, 22, 22)
00113	(2, 2, 7, 7, 7)
00122	(2, 2, 7, 17, 22)
00244	(2, 2, 12, 12, 22)
00334	(2, 2, 12, 17, 17)
01112	(2, 7, 7, 12, 22)
01144	(2, 7, 7, 17, 17)
01234	(2, 7, 12, 12, 17)
01333	(2, 7, 22, 22, 22)
02224	(2, 12, 12, 12, 12)
02233	(2, 12, 17, 22, 22)
03444	(2, 17, 17, 17, 22)
11111	(7, 7, 7, 7, 22)
11134	(7, 7, 7, 12, 17)
11224	(7, 7, 12, 12, 12)
11233	(7, 7, 17, 22, 22)
12223	(7, 12, 12, 22, 22)
12444	(7, 12, 17, 17, 22)
13344	(7, 17, 17, 17, 17)
22222	(12, 12, 12, 17, 22)
22344	(12, 12, 17, 17, 17)
23334	(12, 22, 22, 22, 22)
33333	(17, 17, 22, 22, 22)
44444	(3, 3, 3, 3, 13)
	(3, 3, 3, 8, 8)
	(3, 3, 3, 18, 23)
	(3, 3, 8, 13, 23)
	(3, 3, 8, 18, 18)
	(3, 3, 13, 13, 18)
	(3, 3, 23, 23, 23)
	(3, 8, 8, 8, 23)
	(3, 8, 8, 13, 18)
	(3, 8, 13, 13, 13)
	(3, 8, 18, 23, 23)
	(3, 13, 13, 23, 23)
	(3, 13, 18, 18, 23)
	(3, 18, 18, 18, 18)
	(8, 8, 8, 8, 18)
	(8, 8, 8, 13, 13)
	(8, 8, 13, 23, 23)
	(8, 8, 18, 18, 23)
	(8, 13, 13, 18, 23)
	(8, 13, 18, 18, 18)
	(8, 23, 23, 23, 23)
	(13, 13, 13, 13, 23)
	(13, 13, 13, 18, 18)
	(13, 18, 23, 23, 23)
	(18, 18, 18, 23, 23)

Table 1. The min and max set for one-loop $V_{\text{1-loop},\Omega_F^0}$. In the thermal case, there is a unique minimum.

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Comparison of this formula with the exact number (see [60] sequence A145855) below shows that the approximate formula is rather accurate.

N	2	3	4	5	6	7	8	9	10	11	12
Minima Exact	1	4	9	26	76	246	809	2704	9226	32066	112716
Approx.	1.5	3.33	8.75	25.2	77	245.14	804.38	2701.11	9237.8	32065.1	112673.16
Maxima Exact	2	6	10	50	80	490	810	5400	9252	64130	112720

(3.12)

There are two important physical points to be made:

1) Exponential degeneracy. Recall that typical one-loop gauge holonomy potentials [61] for non-supersymmetric SU(N) gauge theory has 1, 2 or at most N minima or maxima. For the Ω_F^0 twist, which seems to lead to the most efficient distillation on the Hilbert space, the corresponding one-loop potential has exponentially increasing number of minima and maxima.

2) Collapse of the one-loop potential. In the thermal case where there is no twist, the gap between the minima and maxima of the one-loop potential

$$\Delta V_{1-\text{loop}} \equiv V_{1-\text{loop,max}} - V_{1-\text{loop,min}} \sim O(N^2) \tag{3.13}$$

Since the one-loop potential of Ω_F^0-twisted fermions is suppressed by a factor of $\frac{1}{N^4}$ compared to the case with no twist, the gap between the maximum and minimum of the potential becomes

$$\Delta V_{1-\text{loop},\Omega_F^0} \sim O(N^{-2}). \tag{3.14}$$

In fact, if we take large-N_c limit with $N_f = N_c$ or more general Veneziano large-N_c limit, the one-loop potential (3.7) vanishes as $\frac{1}{N_c^2}$ and we obtain a extremely dramatic result for a non-supersymmetric QFT:

$$V_{\text{1-loop},\Omega_F^0}[\Omega] = 0, \qquad N_c = \infty. \tag{3.15}$$

Namely, in large-N_c limit, there is a moduli space at one-loop order in perturbation theory. In supersymmetric theories, the moduli space for gauge holonomy persists to all orders in perturbation theory (assuming supersymmetry preserving boundary conditions) both at finite and large N_c, and it may only be lifted non-perturbatively. Below, we show that, in QCD(F/adj) the degeneracy is lifted at two-loop order, and center-symmetry is stabilized completely.

3.3 Frustration tolerant operators and center-stability at two-loop order

The presence of a plethora of degenerate minima of one-loop potential for $N > 2$ prompts us to ask if they are stable or not with the inclusion of higher order contributions. In the usual thermal QCD, the two-loop potential for the holonomy has been computed in [25–27]. See formula (17) in [26] and formula (5.11) in [27]. These two formulas at first sight looks quite different, the fist one is in terms of products of Bernoulli polynomials B_1, B_2, B_3 whose arguments are either eigenvalues of gauge holonomy or eigenvalue differences, while

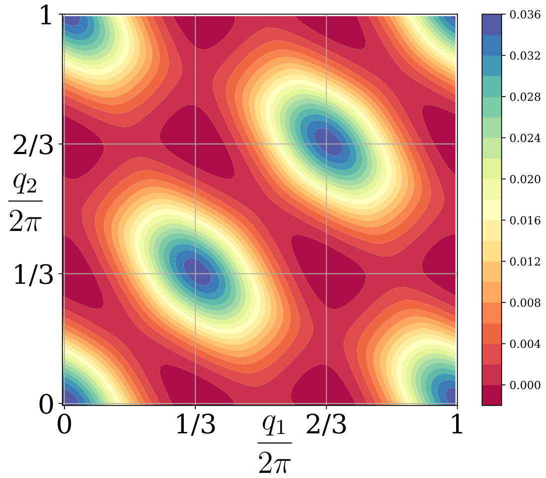

Figure 6. The two-loop potential of twisted fundamental fermions $g^{-2}\beta^4 V^{\psi}_{2-\text{loop},\Omega^0_{\text{F}}}$ for $N_c = 3$ as a function of the eigenvalues of the holonomy $\Omega = \text{diag}(e^{iq_1}, e^{iq_2}, e^{iq_3})$.

the second one is only in terms of B_4 polynomial. Due to non-trivial Bernoulli polynomial identities, these two formulas are actually identical!

Remarkably, the full two-loop potential of the whole QCD(F/adj) can be calculated by just using the two-loop result based on the fundamental fermions. This is due to the fact that the adjoint and fundamental fermion loops are decoupled from each other at one- and two-loop order level. The one-loop order diagrams are individual fermion bubble diagrams, and at two-loop, we have fermion bubble diagrams with the gluon propagator insertion, see figure 4. At three loop order, there will be mixing effects between adjoint and fundamental fermions, such as fermion bubble diagram with gluon propagator with adjoint fermion bubble insertions, see figure 5, or vice versa. But these effects are of order $O((g^2N_c)^3)$ compared to classical action and in order to determine CFC realization in the full theory, we will need only $O((g^2N_c)^2)$.

Re-expressing the formula (5.11) from [27] in terms of gauge invariant Polyakov loops, the two-loop potential induced by N_f flavors of massless fundamental fermions obeying the thermal (anti-periodic) boundary condition is given by

$$V^{\psi}_{\text{2-loop,thermal}} = \frac{g^2 N_f}{\beta^4}\frac{3}{\pi^4}\left\{-\frac{N_c^2-1}{8N_c}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}[\text{Tr}(\Omega^n)+\text{c.c.}]+\frac{1}{24}\sum_{n=1}^{\infty}\frac{|\text{Tr}(\Omega^n)|^2}{n^4}\right\} \tag{3.16}$$

What we need is the two-loop potential in the presence of the Ω_F flavor-twisted boundary conditions (3.2). With the appropriate insertion of Ω_F flavor-holonomy eigenvalues into the analysis of [27], we find:

$$V^{\psi}_{\text{2-loop},\Omega_F} = \frac{g^2}{\beta^4}\frac{3}{\pi^4}\left\{-\frac{N_c^2-1}{8N_c}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}[\text{Tr}(\overline{\Omega}_F^n)\text{Tr}(\Omega^n)+\text{c.c.}]+\frac{N_f}{24}\sum_{n=1}^{\infty}\frac{|\text{Tr}(\Omega^n)|^2}{n^4}\right\} \tag{3.17}$$

Employing the flavor-twist $\Omega_F = \Omega_F^0$ given in (1.13), the two-loop potential takes the form:

$$V^{\psi}_{\text{2-loop},\Omega_F^0} \to \frac{g^2}{\beta^4}\frac{3}{\pi^4}\left\{-\frac{N_c^2-1}{8N_cN_f^3}\sum_{n=1}^{\infty}\frac{(-1)^{N_fk}}{k^4}[\mathrm{Tr}(\Omega^{N_fk})+\text{c.c.}]+\frac{N_f}{24}\sum_{n=1}^{\infty}\frac{\left|\mathrm{Tr}(\Omega^n)\right|^2}{n^4}\right\} \quad (3.18)$$

The first term is proportional to the one-loop potential. For $N_f = N_c$, it represents an $\mathcal{O}(g^2N_c)$ correction to (3.7) by the $\mathbb{Z}_{N_c}$ symmetric single-trace operators $\mathrm{Tr}(\Omega^{N_ck})$. As such, it does not alter the exponentially large degeneracy of the one-loop potential. By contrast, the second term is quite different. In particular, this double-trace operator is independent of the flavor-twisted boundary condition, and we refer to it as *frustration tolerant*. Much like adjoint fermions endowed with periodic boundary conditions, (3.18) is minimized at the center-symmetric minimum:

$$\begin{aligned}
\Omega &= \mathrm{diag}(1,\omega,\cdots,\omega^{N_c-1}), && N_c \text{ odd},\\
\Omega &= \omega^{1/2}\mathrm{diag}(1,\omega,\cdots,\omega^{N_c-1}), && N_c \text{ even}.
\end{aligned} \quad (3.19)$$

As a result, the exponential degeneracy of the one-loop potential is lifted and the total two-loop potential picks out the center-symmetric vacuum (3.19) as a global minimum.

This result cannot be altered by three- or higher- loop orders or non-perturbatively due to weakness of the coupling constant at the scale of compactification, thanks to asymptotic freedom. Therefore, the center-symmetry is stable in QCD(F/adj) with $N_f = N_c$ at small-β.

In figure 6 the potential at two loop order $V^{\psi}_{\text{2-loop},\Omega_F^0}$ is depicted for $N_c = 3$. The minima are located at $\Omega = (1,\omega,\omega^2)$ and its gauge-equivalent copies. On the other hand, the CFC-breaking holonomies $\Omega = \mathbb{1}, \omega\mathbb{1}$ and $\omega^2\mathbb{1}$ are maxima of the potential and hence unstable. This structure holds for all N_c. Since the color-flavor center symmetry is stable for both small β and large β we have a good chance to have unbroken CFC symmetry for all β, at least for sufficiently light m_λ.

3.4 (Non)decoupling of massive adjoint fermion

Our ultimate goal in this work is to find a method to study ground state structure, and low-lying states in QCD(F). In (3.18), we showed that the ground state of the holonomy potential for the $N_f = N_c$ QCD(F/adj) theory is at a center-symmetric point (3.19). Ultimately, we would like to turn on a mass term for adjoint fermion m_λ and take $m_\lambda \gg \Lambda$ so that adjoint fermion decouples from the IR physics. In this way, IR-physics reduce to the one of QCD(F). At the same time, for center-symmetry stability, sufficiently light adjoint fermions are necessary. Can both of these requests be satisfied simultaneously?

Turning on a soft adjoint fermion mass m_λ, the balance between the gauge fluctuations and adjoint fermion fluctuations shown in penultimate line in (3.4) breaks down. However, the massless fundamental fermions still generate a CFC-stabilizing hololonomy potential at two-loop order. The one-loop gauge plus adjoint fermion contribution is soft, and is of order $m_\lambda^2\beta^2 O((g^2N_c)^0)$, and the second order contribution is further suppressed, of order $-m_\lambda^2\beta^2(g^2N_c)$, and can be neglected below. The combined potential which determines

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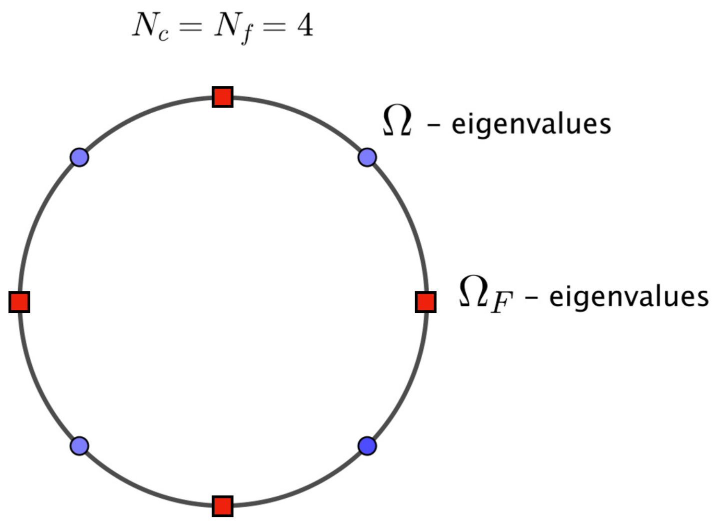

Figure 7. The flavor-holonomy is non-dynamical, and gauge holonomy is dynamical. But the value of flavor-holonomy determines the gauge holonomy potential and its minimum. In quantum distillation discussion over the Hilbert space, we establish Ω_F^0 (1.13) as an extremum of the twist-free energy for flavor holonomy. Despite Ω_F being non-dynamical, the extremization of flavor-holonomy potentials are physically meaningful. For $\Omega_F = 1$, the pressure is maximized, for Ω_F shown in figure, the pressure is minimized. This means as if less degrees of freedom are contributing to the state sum. In this way, we will be able to circumnavigate all phase transitions in certain 4d QFTs.

the center-symmetry realization (which picks the minimum among the states described in section 3.2) is given by

$$\begin{aligned} V^{\text{gauge}}_{1-\text{loop}} + V^{\lambda}_{1-\text{loop}} + V^{\psi}_{2-\text{loop},\Omega_\text{F}} &= \frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\frac{\overbrace{\left[-1+\frac{1}{2}(n\beta m_\lambda)^2K_2(n\beta m_\lambda)+\frac{g^2N_c}{16\pi^2}\right]}^{a_n}}{n^4}|\operatorname{tr}(\Omega^n)|^2 \\ &\approx \frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\frac{\left[-\frac{1}{4}(n\beta m_\lambda)^2+\frac{g^2N_c}{16\pi^2}\right]}{n^4}|\operatorname{tr}(\Omega^n)|^2 \end{aligned} \tag{3.20}$$

In the second line, we used small z asymptotic of the Bessel function, $K_2(z) \sim \frac{2}{z^2} - \frac{1}{2} + O(z^2)$, at fixed m_λ and sufficiently small β. The first line of the potential is reliable provided $\beta\Lambda \ll 1$. The $\mathbb{Z}_{N_c}$ CFC-symmetry will remain intact provided the coefficients of first $\lfloor \frac{N}{2} \rfloor$ Polyakov loop operators remains positive [5], giving the condition $(\beta m_\lambda) < \frac{(g^2N_c)^{1/2}}{N_c\pi}$. Therefore, in the window,

$$\Lambda \ll m_\lambda < \frac{(g^2N_c)^{1/2}}{\beta N_c \pi} \tag{3.21}$$

adjoint fermion exhibits the following striking properties:

- It decouples from the long distance physics (chiral Lagrangian etc.) because $\Lambda \ll m_\lambda$ so that the IR theory is essentially QCD(F) both on $\mathbb{R}^4$ as well as on $\mathbb{R}^3 \times S^1$.

- It does not decouple from the holonomy potential, rather, for that purpose, it acts in the same way as light (or even massless) adjoint fermion because $m_\lambda < \frac{(g^2 N_c)^{1/2}}{\beta N_c \pi}$.

These conditions accommodate adjoint fermions *heavy* compared to strong scale, but *light* compared to compactification scale provided $N_c \sim 3, 4, \ldots$. When $N_c \to \infty$, the above condition cannot be satisfied, and m_λ must be light compared to strong scale to keep the full $\mathbb{Z}_{N_c}$ center-symmetry intact.

4 Quantum distillation as reduction of effective density of states

Motivation. Consider thermal partition function:

$$\mathcal{Z}(\beta) = \mathrm{tr}\left[e^{-\beta H}\right] = \sum_{n \in \mathcal{B}} e^{-\beta E_n} \deg(n) + \sum_{n \in \mathcal{F}} e^{-\beta E_n} \deg(n) \tag{4.1}$$

which correspond to a state sum over the Hilbert space of the QFT. In QCD(F), the states are mesons, baryons, glueballs, resonances and alike. In the low-temperature limit, $\beta \to \infty$, the thermal partition function is dominated by low-lying states in the Hilbert space. In the high-temperature limit, $\beta \to 0$, all the states contribute to the state sum on a similar footing.

The phase transitions are associated with the singularities in the free-energy or other thermodynamic observables. We will consider any non-analytic point of $\mathcal{F}_{\text{thermal}}(\beta)$ taking place for real positive $\beta \in [0, \infty)$ as a phase transition point. Since H is Hamiltonian of QFT on $\mathbb{R}^3$, we are already in thermodynamic limit, hence, according to Lee-Yang analysis, one expects singularities in $\mathcal{F}_{\text{thermal}}(\beta)$ as β varies. It is in general very hard to determine the Lee-Yang singularities analytically.

The idea of quantum distillation of Hilbert space is ultimately tied with the manipulation of the singularities of partition function. Clearly, each term in (4.1) is positive definite and analytic in β. However, in thermodynamic limit, the sum need not be analytic, it generically has singularities. On the other hand, it is also clear that the insertion of phase factors associated with global symmetries $\mathbf{G}$ into the thermal partition function, hence obtaining a generalization (1.11), necessarily reduces its magnitude:

$$|\mathcal{Z}(\beta, \epsilon_1, \ldots, \epsilon_N)| \leq \mathcal{Z}(\beta) \tag{4.2}$$

Clearly, this manipulation does not alter the Hilbert space $\mathcal{H}$ of H, because $\mathcal{Z}(\beta, \epsilon_1, \ldots, \epsilon_N)$ is a symmetry graded partition function. The intention is to show that $\mathcal{Z}(\beta, \epsilon_1, \ldots, \epsilon_N)$ can be tame enough such that it does not possess singularities in $\beta \in [0, \infty)$ while $\mathcal{Z}(\beta)$ does. In order to provide one perspective on the quantum distillation idea, we need to first explain the density of states in a generic QFT, especially for asymptotically high-energy states.

Density of states. The density of states is the inverse Laplace transform to the partition function. Although we cannot calculate the partition function at arbitrary temperatures, we can easily calculate its leading order behavior at high temperature for asymptotically free theories. From there, we can infer the density of hadronic states, by applying an inverse Laplace transform.

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In small-β limit where the theory becomes weakly coupled at the scale β, a way to calculate the asymptotic form of the partition function is to calculate the free energy of the system. At leading order, the free energy of the QCD(F/adj) is given by the Stefan-Boltzmann law, see standard texts [21, 22, 62], which is equal to one-loop potential (3.1) evaluated at its minimum:

$$\mathcal{F}_{\text{thermal}}(\beta) = -\frac{\pi^2}{90}\frac{V_3}{\beta^4}\left[\underbrace{2(N_c^2-1)}_{\text{gluons}} + \frac{7}{8}\times \underbrace{2(N_c^2-1)}_{\text{adj.Weyl ferm.}} + \frac{7}{8}\times \underbrace{4N_f N_c}_{\text{fund.D.ferm.}}\right] \qquad \beta\to 0 \tag{4.3}$$

This is the sum of (A_μ, λ, ψ^a) contributions. The $\frac{7}{8}$ factor arises due to Dirac-Fermi distribution corresponding to fermions with thermal boundary conditions.

Quark-hadron duality. The free energy has at least two interpretations, and the two are related via the quark-hadron duality [63, 64]

- **Micro interpretation:** one interpretation is in terms of microscopic constituents, gluons and quarks. The numbers appearing in free energy such as (N_c^2-1), (N_c^2-1), $4N_f N_c$ count respectively, the number of microscopic bosonic and fermionic degrees of freedom in the QFT.

- **Macro interpretation:** the other interpretation is in terms of macroscopic states, the hadrons in physical Hilbert space $\mathcal{H}$. The inverse Laplace transform of the partition function is the density of states of hadrons:

$$\mathcal{Z}(\beta) \sim e^{-\beta\mathcal{F}_{\text{thermal}}} \sim e^{+aN_c^2V_3T^3} \qquad \Longleftrightarrow \qquad \rho_{SB}(E) \sim e^{E^{3/4}N_c^{1/2}(aV_3)^{1/4}} \tag{4.4}$$

  where $\rho_{SB}(E)$ is the growth in correspondence with Stefan-Boltzmann(SB) law, and a is a pure order one number. SB growth is special in the sense that it is the largest asymptotic growth in a local finite-N_c QFT. Only at $N_c = \infty$ and string theory, one can obtain a Hagedorn-growth.

Let us now consider the graded partition function and associated graded free energy and density of states. The translation of the center-symmetry preserving boundary conditions (3.2) to operator formalism is:

$$\mathcal{Z}(\beta) = \operatorname{tr}\left[e^{-\beta H}(-1)^F e^{i\pi Q_0}\prod_{a=1}^{N_f} e^{i\frac{2\pi a}{N_f}Q_a}\right], \tag{4.5}$$

where $Q_0 \equiv \int d^3x \sum_a \psi_a^\dagger\psi^a$ and $Q_a \equiv \int d^3x\ \psi_a^\dagger\psi^a$.

The twist free energy associated with this partition function can be found by calculating the holonomy potential at its minimum. We have the holonomy potential at two loop order at our disposal, and the global minimum of it is stable to all orders in perturbation theory and non-perturbatively. Plugging the global minimum (3.19) to one-loop potential (to make comparison with thermal result clearer), yields the twist free energy $\mathcal{F}_{\Omega_F^0}(\beta)$:

$$\begin{aligned}\mathcal{F}_{\Omega_F^0}(\beta) &= -\frac{\pi^2}{90}\frac{V_3}{\beta\beta^4}\left[\underbrace{2(N_c^2-1)}_{\text{gluons}}\ \underbrace{-2(N_c^2-1)}_{\text{adj.Weyl ferm.pbc.}}\ +\frac{7}{2}\ \underbrace{\frac{1}{N_c^2}}_{\text{fund.D.ferm.tbc.}}\right]\\ &= -\frac{\pi^2}{90}\frac{V_3}{\beta^4}\left[\frac{7}{2}\frac{1}{N_c^2}\right] \xrightarrow{N_c\to\infty} 0 \end{aligned} \tag{4.6}$$

This is a quite striking result. It is as if there is merely $\frac{1}{N_c^2}$ quark degree of freedom in the system instead of $\sim N_c^2$ bosonic and $\sim N_c^2$ fermionic degrees of freedom, similar to the description in [13] in large-N_c QCD(adj). The twist free energy behaves as if the QFT at hand does not even have a single particle worthy degree of freedom in 4d. Yet we did not touch the Hamiltonian and the Hilbert space. The corresponding density of states for the growth of the hadronic states (after quantum distillation) takes the form

$$\mathcal{Z}_{\Omega_F^0}(\beta) \sim e^{-\beta\mathcal{F}_{\Omega_F^0}(\beta)} \sim e^{+a\frac{1}{N_c^2}V_3/\beta^3} \qquad \Longleftrightarrow \qquad \rho_{\Omega_F^0}(E) \sim e^{\frac{1}{N_c^{1/2}}E^{3/4}(aV_3)^{1/4}} \tag{4.7}$$

in sharp contrast with the Stefan-Boltzmann growth (4.4). This implies that a dramatic amount of cancellation ought to occur in the graded state sum over the Hilbert space $\mathcal{H}$ of QFT among high energy physical states in $\mathcal{H}$.

$$\text{distillation at high-}E\text{ spectrum}: \quad \rho_{\Omega_F^0}(E) \ll \rho_{SB}(E) \tag{4.8}$$

The mapping of the quantum distillation of the Hilbert space $\mathcal{H}$ to thermodynamics is following:

> Despite the fact that the state sum using $\rho_{SB}(E)$ leads to a CFC broken and chirally restored phase at small-β and phase transitions in $\mathcal{Z}(\beta)$, the state sum with $\rho_{\Omega_F^0}(E)$ leads to CFC unbroken, and chiral symmetry broken phase at small-β adiabatically connected to large-β. In particular, possible phase transitions can be avoided as the theory interpolates from large to small β by using $\mathcal{Z}_{\Omega_F^0}(\beta)$.

A remark is on the large-N limit following the rationale of [13]. If we consider our theory on a three dimensional spatial manifold $\mathcal{M}_3$ with a characteristic curvature scale ℓ and volume V, it is possible to show that

$$\rho_{\Omega_F^0}(E) \sim e^{\sqrt{\ell E}}, \tag{4.9}$$

exhibiting a two-dimensional scaling. In particular, Hagedorn growth $e^{\beta_H E}$ expected in large-N theories and string theory, as well as Stefan-Boltzmann growth $e^{V^{1/4}E^{3/4}}$ which is natural in four-dimensional theory in spatial volume V completely disappears. In the sense of graded density of states, our non-supersymmetric theory acts in a similar way to supersymmetric theories on curved spaces [11], similar to [13].

Similar effects are also present in [13, 30, 65–67]. In string theory, this has parallels to [65] where despite the fact that bosonic and fermionic sector of the theory has tacyonic instabilities, the $(-1)^F$ graded sum is tachyon-free and stable.

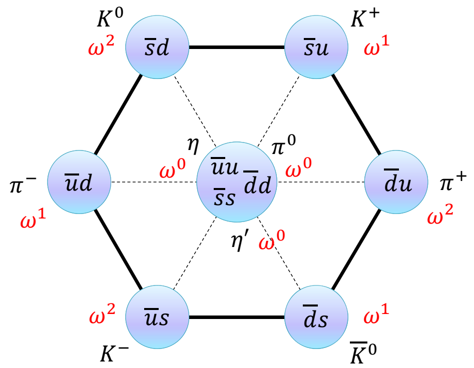

Figure 8. Grading and quantum distillation in the spin-zero meson octet sector. Grading assigns phases $\omega^k = e^{\frac{i2\pi k}{3}}$ to different mesons in such a way that the graded sum is zero. This means spin-zero meson contribution disappear in the graded sum.

4.1 Distillation of mesons: cancelling bosons against bosons

Below, we describe the quantum distillation in the Hilbert space explicitly in terms of low lying hadronic states. This discussion is aimed to provide simple pedagogical insights into the process.

Consider $m_{\psi_1} = m_{\psi_2} = \ldots = m_{\psi_{N_f}} \geq 0$ in QCD(F/adj), a positive semidefinite equal mass for the N_f-flavors. If $m_\psi > 0$, then the global symmetry of the theory on $\mathbb{R}^4$ that acts on fundamental fermions is given in (2.12), roughly $\mathrm{U}(N_f)_V$. In QCD(F) phenomenology corresponding to $N_f = 3$, this is called flavor-SU(3) limit, see eg. [68].

In QCD(F/adj), as discussed in section 2.2, there are two possible patterns of chiral symmetry breaking, down to a vector-like subgroup (2.13) or to a subgroup which has a chiral $\mathrm{U}(1)_{A_D}$ factor (2.15). If the first is true behavior of the theory on $\mathbb{R}^4$, the IR theory is described in terms of N_f^2 Nambu-Goldstone bosons, whereas if the second is true behavior, it is described in terms of N_f^2-1 Nambu-Goldstone bosons, (N_f^2-1) massless fermionic mesons and two massless baryons, tabulated in (2.26). These IR-degrees of freedom satisfy rather non-trivial collection of 't Hooft anomalies, and hence, we view both as logical possibilities.

As an example, consider the first pattern (2.13), in which IR theory has N_f^2 Nambu-Goldstone bosons, with quantum numbers

$$M_a^b = \overline{\psi}_a \psi^b, \qquad a, b = 1, \ldots, N_f \tag{4.10}$$

In thermal case, these states would contribute to the thermal free energy as $\mathcal{F}_{\text{mesons}}(\beta) \sim -\frac{\pi^2}{90}\frac{V_3}{\beta^4}N_f^2$. However, in the state sum, the presence of the $\prod_{a=1}^{N_f} e^{i\epsilon_a Q_a}$ operator amounts to grading the states with phases, which are roots of unity. See figure 8 for the assignment of phases to mesons in theory with $N_f = 3$ flavors. The state M_a^b is assigned a phase

$$M_a^b \mapsto e^{-2\pi i(b-a)/N_f} M_a^b \tag{4.11}$$

modifying the terms in the state sum into

$$N_f^2 e^{-\beta E_\pi} \mapsto \sum_{a,b=1}^{N_f} e^{-2\pi i(b-a)/N_f} e^{-\beta E_\pi} = 0 \tag{4.12}$$

Namely, in the graded partition function, the contribution of the scalar mesons to the state sum (4.5) vanishes and corresponding free energy for this sector is mapped to zero $\mathcal{F}_{\text{mesons},\Omega_F^0}(\beta) = 0$ as if there are no mesons in the spectrum at all!

If we turn on a mass term for adjoint fermion m_λ, then, as described in section 2, $\mathrm{U}(1)_{A_D}$ reduces to $\mathbb{Z}_{2N_f}$ and η' is no longer degenerate with the rest of the pions. In this limit, the state sum is modified as:

$$(N_f^2 - 1)e^{-\beta E_\pi} + 1e^{-\beta E_{\eta'}} \mapsto (-1)e^{-\beta E_\pi} + 1e^{-\beta E_{\eta'}} \tag{4.13}$$

which means that the contribution of this sector can even be negative.

In QCD(F/adj), there are also fermionic mesons such as $\overline{\psi}_a \lambda \psi^b \equiv {\psi_M}_a{}^b$, see (2.22). If (2.15) of chiral symmetry breaking is realized, these are gapless modes. Even if a mass term is turned on for quarks, the masses of $(N_f^2 - 1)$ of these fermionic mesons are equal due to flavor symmetry. This is so because ${\psi_M}_a{}^b$ decompose as $\mathbf{adj} \oplus \mathbf{1}$ under $\mathrm{SU}(N_f)_V$. Their contribution is essentially diminished by grading in the same way as mesons (4.13).

The meson states in the spectrum transform in the singlet, adjoint or product of adjoint representations under flavor symmetry. This comes from the fact that in the global symmetry, we must mode out the gauge redundancies. For $N_f = N_c$, the global symmetry is not $\mathrm{SU}(N_f)_V$, but $\mathrm{PSU}(N_f) = \mathrm{SU}(N_f)/\mathbb{Z}_{N_f}$. This aspect is exactly parallel to $\mathbb{CP}^{N-1}$ model where a perfect quantum distillation takes place in the large-N limit, as explained in [8, 16] in order to explain the path integral with twisted boundary conditions [15]. We refer the distillation in bosonic large-N $\mathbb{CP}^{N-1}$ as *perfect distillation*, because the graded sum cancels all states in the Hilbert space, but the ground state(s). In this respect, despite the theory being purely bosonic, the graded partition function can be engineered to emulate the supersymmetric index in a supersymmetric gauge theory, which counts just the ground states [1].

4.2 Distillation of baryons: fermions against fermions

Baryons in $\mathrm{SU}(N_c)$ QCD(F) are fermions for odd N_c and bosons for even N_c. Below, we assume N_c is odd, where we will describe cancellation of fermions against fermions. The discussion can straightforwardly be generalized to N_c even.

For simplicity, consider SU(3) QCD with $N_f = 3$ fermions. Lightest baryons decompose into $S = \frac{1}{2}$ octet and $S = \frac{3}{2}$ decuplet. In the SU(3)-flavor limit where all quark masses are equal, the decuplet masses are equal to each other and the octet masses are equal to each other, but $m_{\text{decuplet}} \neq m_{\text{octet}}$.

The action of grading on the decuplet baryons is as follows. The spin-3/2 wavefunction in the $m = 3/2$ state is just $|\uparrow\uparrow\uparrow\rangle$ and is completely symmetric, and so are the other m states. The flavor wave function must also be completely symmetric. (We suppress completely anti-symmetric color structure.) Then, a baryon in decuplet is

$$B^{abc} = \psi^{\{a}\psi^b\psi^{c\}}, \qquad a,b,c = 1,\ldots,3 \tag{4.14}$$

In thermal state sum, the decuplet baryons would contribute as $40e^{-\beta E_{\text{decuplet}}}$. However, in the graded state sum, similar to the discussion of mesons, the presence of the $\prod_{a=1}^{N_f} e^{i\epsilon_a Q_a}$ operator amounts to phases for B_{abc} of the form:

$$B_{abc} \to e^{-2\pi i(a+b+c)/3} B_{abc} \tag{4.15}$$

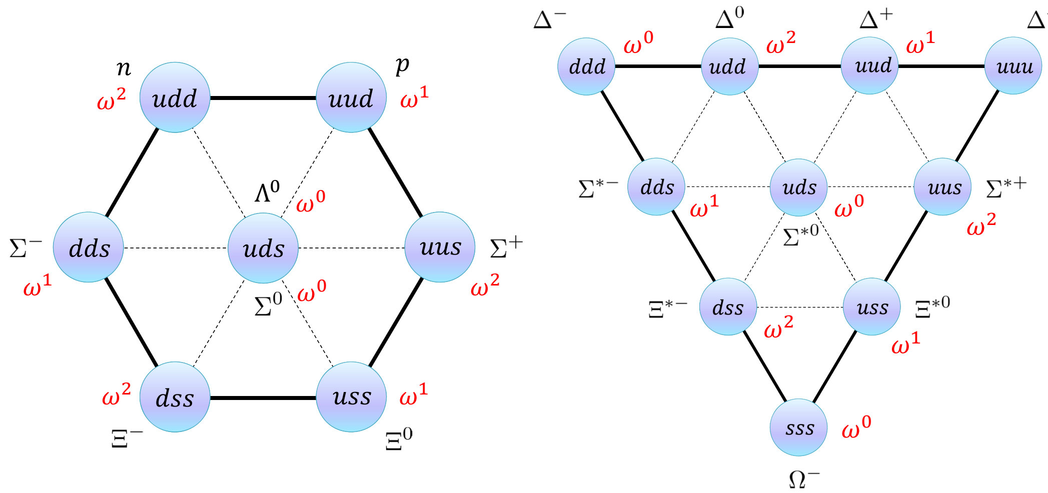

Figure 9. Grading and quantum distillation in the baryon spin-$\frac{1}{2}$ octet and spin-$\frac{3}{2}$ decuplet sectors.

The action of the grading into the state sum for the $S=\frac{1}{2}$ octet and $S=\frac{3}{2}$ decuplet are:

$$\begin{aligned} 16e^{-\beta E_{\text{octet}}} &\underbrace{\rightarrow}_{\Omega_F^0} (-2)e^{-\beta E_{\text{octet}}} \\ 40e^{-\beta E_{\text{decuplet}}} &\underbrace{\rightarrow}_{\Omega_F^0} (+4)e^{-\beta E_{\text{decuplet}}} \end{aligned} \tag{4.16}$$

Clearly, the contribution of baryons to graded partition function and graded thermodynamics will be rather small. We will momentarily quantify how small this contribution is.

If we take masses of quarks larger than strong scale, we can assume quark are sufficiently non-relativistic. The non-relativistic quark model has an enhanced SU(6) spin-flavor symmetry, under which spin up and down and 3-flavor indices combine to **6** dimensional fundamental representation of SU(6), see [68] for a review. The lightest baryons transform as the symmetric product of three **6**'s of SU(6), which is the **56** dimensional representation. In this case, the effect of the quantum grading is the reduction

$$56e^{-\beta E} \underbrace{\rightarrow}_{\Omega_F^0} (+2)e^{-\beta E} \tag{4.17}$$

The contribution of 56 lightest baryonic states reduce to the one of 2 states only.

In the case of general N and heavy quark limit, the spin-flavor symmetry group is SU$(2N)$ and the baryons transform in the N-index symmetric representation. Its dimension $\binom{3N-1}{N}$ grows exponentially as $\sim (27/4)^N/\sqrt{3\pi N}$. For $N=2,3,4,5,6,7,8,9$ we verified numerically that the quantum grading reduces this to just 2 states only.[15] Hence, we have the reduction

$$\binom{3N-1}{N} e^{-\beta E_{\text{baryon}}} \rightarrow (+2)e^{-\beta E_{\text{baryon}}}. \tag{4.18}$$

[15]Thanks to Takuya Kanazawa for demonstrating this.

Also note that at large-N limit, the masses of baryons scale as N. As such, at finite-β, their contribution vanishes either in thermal state sum or in graded state sum. Therefore, at least for the large-N theory, the baryon effects are doubly suppressed, both due to quantum distillation and due to the fact that they gradually decouple from the spectrum.

4.3 Graded thermodynamics of QCD(F) ≈ thermodynamics of YM

Consider graded thermodynamics of the QCD(F) defined through the generalized partition function:

$$\mathcal{Z}(\beta) = \text{tr}\left[e^{-\beta H}\prod_{a=1}^{N_f} e^{i\frac{2\pi a}{N_f}Q_a}\right], \tag{4.19}$$

where for N_f odd, the twist lives in $\text{SU}(N_F)_V$ and for N_f even, it lives in $\text{U}(N_F)_V$.

We claim that the graded thermodynamics of QCD(F) should be approximately equivalent to regular thermodynamics of the pure Yang-Mills theory *both* in the confined and deconfined phases provided a physical scale (deconfinement temperature or strong scale) are matched between the two theories. We provide some evidence for this claim and use it to explain lattice simulation results of ref. [24].

First consider the thermal and graded free energy of QCD(F) in comparison with thermal free energy in YM in the very high temperature regime, $\beta \to 0$. Assume $N_f = N_c$.

$$\begin{aligned}
\mathcal{F}^{\text{QCD(F)}}_{\text{thermal}} &= -\frac{\pi^2}{90}\frac{V_3}{\beta^4}\Bigg[\underbrace{2(N_c^2-1)}_{\text{gluons}} + \frac{7}{8}\times \underbrace{4N_fN_c}_{\text{fund.D.ferm.}}\Bigg] \\
\mathcal{F}^{\text{QCD(F)}}_{\Omega_F^0} &= -\frac{\pi^2}{90}\frac{V_3}{\beta^4}\Bigg[\underbrace{2(N_c^2-1)}_{\text{gluons}} + \frac{7}{2}\underbrace{\frac{1}{N_c^2}}_{\text{fund.D.ferm.tbc.}}\Bigg] \\
&= \mathcal{F}^{\text{YM}}\left(1 + O(1/N_c^4)\right)
\end{aligned} \tag{4.20}$$

The thermal free energy of QCD(F) in the Ω_F^0 background is essentially same as pure YM theory. It only differs from it by an $O(1/N_c^4)$ correction.

As discussed in the context of quark-hadron duality, this result has two related interpretations. In the microscopic picture, it implies that the contribution of quarks is suppressed by a factor $1/N_c^4$. Instead of having $N_fN_c \sim N_c^2$ quark degree of freedom, one has effectively $N_fN_c/N_f^3 \sim 1/N_c^2$ of them contributing to the graded-thermodynamics of QCD(F). In the language of hadrons, this result means that the asymptotic growth of the non-glueball hadrons in Hilbert space at high energies is reduced from the Stefan-Boltzmann growth $\rho_{SB}(E) \sim e^{E^{3/4}N_c^{1/2}(aV_3)^{1/4}}$ down to $\rho_{\Omega_F^0}(E) \sim e^{E^{3/4}N_c^{-1/2}(aV_3)^{1/4}}$.

The distillation of Hilbert space does not mean that the quarks are projected out from the theory, despite the fact that their effect in $\mathcal{F}^{\text{QCD(F)}}_{\Omega_F^0}$ disappear in the large-N_c limit. The quarks are present in the Lagrangian, and similarly, the baryonic or mesonic hadrons they build are present in the physical spectrum of the theory. In particular, the renormalization group β function of the graded QCD(F) is not changed at all. The first coefficient of β function is still $\beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f$, and the theory runs in the same way as the regular

QCD, with a strong-scale $\Lambda_{\rm QCD} = \mu \exp[-\frac{8\pi^2}{g^2(\mu)\beta_0}]$. However, the graded thermodynamics associated with the QCD(F) approximates the thermodynamics of pure Yang-Mills theory at very high temperatures. Below, we argue that the matching of the thermodynamics must also be valid at low temperature strong coupling regime.

Quantum distillation of flavor channels. Quantum distillation acts on degrees of freedom which transform under flavor (such as vector mesons, other resonances and baryons), but not on the glueball sector. Using the rationale described around (4.7), the density of states coming from the mesons and baryons reduce so much so that no single 4d degrees of freedom coming from the flavor sector contribute to the graded-thermodynamics of QCD(F). This is completely analogous to the discussion in 2d large-N bosonic $\mathbb{CP}^{N-1}$ model where all states except the ground state cancel among themselves [15] in the Ω_F twisted partition function.

Therefore, we expect the graded free energy of QCD(F) to be approximately the thermal free energy of YM

$$\mathcal{F}^{\rm QCD(F)}_{\Omega_F^0}(\beta) \approx \mathcal{F}^{\rm YM}(\beta), \qquad \forall \beta \in [0, \infty] \tag{4.21}$$

both in the confined and deconfined phases. In deconfined phase, using the one and two-loop potentials of the fundamental fermion given in (3.7) and (3.18) for $N_f \sim N_c$ theory, and the one of the pure gauge sector, it is straightforward to show that

$$\mathcal{F}^{\rm QCD(F)}_{\Omega_F^0}(\beta) = \mathcal{F}^{\rm YM}(\beta) + O(N_c^{-2}) + O(g^2 N_c) \tag{4.22}$$

i.e. the difference in free energy is either suppressed with the weak coupling at small β and by factors of N_c.

As emphasized earlier around (4.4), this has implications for the asymptotic growth of the density of states of hadrons in the Hilbert space $\mathcal{H}$. In particular, the (4.22) imply that leading order effective density of states in QCD(F) with Ω_F^0-graded Hilbert space and the usual density of states in Yang-Mills theory are equal,

$$\rho^{\rm QCD(F)}_{\Omega_F^0}(E) = \rho^{\rm YM}_{SB}(E), \qquad \text{leading order, } E\text{−large} \tag{4.23}$$

Furthermore, at the low-energy end of QCD(F) with Ω_F^0-grading, there are tremendous cancellations. For example, as described in (4.13), the contribution of $N_f^2 - 1$ NG bosons and one η' to the partition function becomes $(-1)e^{-\beta E_\pi} + 1 e^{-\beta E_{\eta'}}$, which is $O(N_f^0)$ instead of $O(N_f^2)$. It is highly conceivable that the contribution of low-lying hadrons (which are not glueballs and alike) cancel to a great extend. In the low energy limit, $O(N_c^2)$ spectral density of the hadrons just reduce to $O(N_c^0)$, just like glueball spectral density.

$$\rho^{\rm QCD(F)}_{\Omega_F^0}(E) \sim \rho^{\rm YM}_{SB}(E) \sim O(N_c^0) \qquad E\text{−small} \tag{4.24}$$

Provided a physical scale matching is done, for example the CFC breaking temperature $T_d[\Omega_F^0]$ in QCD(F) and deconfinement temperature $T_d^{\rm YM}$ at which the $\mathbb{Z}_{N_c}$ center symmetry realization changes in YM are set equal, $T_d[\Omega_F^0] \equiv T_d^{\rm YM}$, we expect that all common observables related to thermodynamic properties, such as free energies, pressure, entropy to behave in a similar manner in these two theories. Quantum distillation of Hilbert space, and in particular (4.23) and (4.23) is the physical explanation for the intriguing simulation results that appeared in ref. [24].

5 Topology of field space and chiral symmetry

So far, we have shown that with the $N_f = N_c$ QCD(F/adj), the CFC symmetry can be made stable at small-β with a judicious choice of grading over Hilbert space. This regime (or its small perturbations) is expected to be adiabatically connected to large-β strongly coupled regime. In this section, we examine the non-perturbative aspects of the small-S^1 regime with unbroken CFC symmetry.

5.1 Monopole-operators and index theorem

At small β, a_4 acts as an compact adjoint Higgs field. The two-loop holonomy potential (3.18) leads to a CFC-stabilizing $\mathbb{Z}_{N_c}$ symmetric minimum (3.19), see figure 7 as well. The gauge structure of the theory abelianizes at the length scales larger than the inverse lightest W-boson mass $(m_W)^{-1} = \frac{\beta N_c}{2\pi}$.

$$\mathrm{SU}(N_c) \to \mathrm{U}(1)^{N_c-1}\,. \tag{5.1}$$

Small $S^1 \times \mathbb{R}^3$ regime can be described by an abelian 3D effective field theory. Note that the microscopic theory is always four dimensional; the symmetry and ABJ anomaly of the theory are the ones of the 4d theory.

The long distance theory is different in different parts of parameter space (m_λ, m_ψ). In the vicinity of $(m_\lambda, m_\psi) = (0,0)$, Cartan subalgebra gluons a^i_μ $(i = 1, \dots, N_c - 1$, $\mu = 1,2,3)$ remain gapless to all orders in perturbation theory. a^i_4 modes acquire masses due to two-loop potential covering the range

$$m_{a^i_4} \in \left[\frac{(g^2 N_c)}{(\beta N_c)}, \frac{(g^2 N_c)}{\beta}\right]\,. \tag{5.2}$$

Therefore,[16] we can ignore a^i_4 fluctuations in the description of physics at distances larger than $(\min(m_{a^i_4}))^{-1}$ and not include at all in the long-distance effective Lagrangian. However, in the vicinity of $m_\lambda = 0, m_\psi = \infty$, QCD(F/adj) reduces to supersymmetric $\mathcal{N} = 1$ SYM. There, a^i_4 field remains gapless to all orders in perturbation theory with the rest of supersymmetric multiplet and must be included in the long-distance Lagrangian. Therefore, we will include a^i_4 in the long-distance EFT despite the fact that it is not needed everywhere of the phase diagram for capturing long-distance EFT.

[16] Note that this is a mass induced by two-loop center-stabilizing potential. This scale is different from the electric mass scale $m_{\mathrm{el.}} \sim gT$ induced by one-loop center-destabilizing potential and magnetic mass scale $m_{\mathrm{mag.}} \sim g^2 T \equiv g_3^2$ which is the strong scale of long-distance 3d YM theory. It is similar in structure but still different from the gap of the holonomy field in QCD(adj) which is induced at one-loop level by a center-stabilizing potential and taking values in the range $m_{a^i_4} \in \left[\frac{\sqrt{(g^2 N_c)}}{(\beta N_c)}, \frac{\sqrt{(g^2 N_c)}}{\beta}\right]$. See [69] for the latter and [22] for thermal case.

In order to describe topological defects, it is more convenient to use Abelian duality transformation [70]:

$$F^i_{\mu\nu} = g^2/(2\pi\beta)\epsilon_{\mu\nu\alpha}\partial^\alpha\sigma^i, \tag{5.3}$$

relating the Cartan gluons to 3d scalars σ^i, which is called "dual photons". Although there are $N_c - 1$ dual photons, we will use a sligthly redundant description, and take $i = 1, \dots N_c$. One extra mode decouples from dynamics.[17] Define the dimensionless combination

$$a_4^i\beta \equiv v^i + \phi^i, \qquad i = 1, \dots N_c \tag{5.4}$$

where the v^i is minimum of the holonomy potential, and ϕ^i indicates the fluctuations of the fields around it. It is also convenient to combine the two-scalars into a complex scalar:[18]

$$z^i = -\frac{4\pi}{g^2}v^i - \frac{4\pi}{g^2}\phi^i + i\sigma^i, \qquad z \equiv -\frac{4\pi}{g^2}v - \frac{4\pi}{g^2}\phi + i\sigma, \tag{5.5}$$

With these conventions, the kinetic term for the scalars can be written as

$$\mathcal{L}_{kin.} = \frac{g^2}{16\pi^2\beta}|\partial_\mu z|^2 = \frac{g^2}{16\pi^2\beta}\left((\partial_\mu\phi)^2 + (\partial_\mu\sigma)^2\right) , \tag{5.6}$$

where $g^2 = g^2(m_W)$.

In the weak coupling regime where the dynamics abelianize, there are saddle-points of the classical action, solution to classical Euclidean monopole-instanton equation, which are just the dimensional reduction of the self-duality equation:

$$Da_4 = *_3F \tag{5.7}$$

The solutions are fully classified. There are N_c types of monopole-instantons [71, 72] associated with the affine root system of Lie algebra $\mathfrak{su}(N_c)$. The action of these configurations is given by the distance between the consecutive eigenvalues of Wilson line Ω.

$$S_i = \frac{4\pi}{g^2}v\cdot\alpha_i = \frac{4\pi}{g^2}(v_{i+1} - v_i), \qquad i = 1, \dots, N_c \tag{5.8}$$

In the center-symmetric minimum of two-loop potential (3.19), $(v_{i+1} - v_i) = \frac{2\pi}{N_c}$ and all of the N_c monopole-instantons have identical Euclidean actions given by

$$S_0 = \frac{8\pi^2}{g^2N_c} \equiv \frac{S_\mathcal{I}}{N_c} \tag{5.9}$$

where $S_\mathcal{I} = \frac{8\pi^2}{g^2}$ is the 4d instanton action.

In the absence of the fermionic zero modes, the leading monopole amplitudes would be given by

$$\mathcal{M}_i = e^{-S_0}e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}, \ i = 1, \dots N_c \tag{5.10}$$

[17] This would be trivially true if all matter was in adjoint representation. Showing this in the presence of fundamental matter requires some care. We will return to this issue at the end of this section.

[18] This definition is sufficient for our purpose. If desired, θ-angle can be incorporated to this definition as in [35, 54].

similar to Polyakov model [70]. However, in the presence of massless fermions, the monopole-instantons also pick up a number of fermionic zero modes as dictated by index theorem for Dirac operator on $\mathbb{R}^3 \times S^1$ [31, 32].

With periodic boundary conditions for adjoint fermions, each monopole has two adjoint zero modes [34, 35, 73–75]. For the fundamental fermions, the story is quite different as explored in [33]. For $N_f = N_c$ theory, if no flavor twisting is used i.e. $\Omega_F = 1$, all $2N_c$ zero modes would be localized to one type of monopole. If one imposes a $\mathrm{U}(1)_V$ twist into the fermions, $\psi(x_4 + \beta) = e^{i\delta}\psi(x_4)$, the zero mode is localized to the monopole with charge α_i if $\delta \in (v_i, v_{i+1})$. As δ crosses an eigenvalue of gauge holonomy, the fundamental fermion zero mode jumps from one-type of monopole to the neighboring one [76, 77], see [32] for a discussion of the jumping phenomenon for general representations, and [78] for a brane interpretation of the jumping phenomena. Ref. [76] shows this explicitly by using ADHM construction. The jumping phenomenon is dictated by the structure of index theorem, and interplay between gauge holonomy and boundary conditions on fermions [32, 79, 80]. With the center-symmetric flavor holonomy Ω_F^0, the $2N_c$ zero modes fractionalize into N_c groups of two and each monopole acquires two fundamental zero modes, mimicking exactly adjoint fermions (modulo some minor caveats described below).

For a general (abelianizing) gauge holonomy and arbitrary flavor holonomy of the form

$$\begin{aligned}\Omega &= \mathrm{Diag}\left(e^{iv_1}, e^{iv_2}, \ldots, e^{iv_{N_c}}\right)\\ \Omega_F &= \mathrm{Diag}\left(e^{i\epsilon_1}, e^{i\epsilon_2}, \ldots, e^{i\epsilon_{N_f}}\right)\end{aligned} \tag{5.11}$$

the number of fermionic zero modes localized at monopole-instanton α_i for periodic adjoint fermion λ and the N_f fundamental fermions $\psi^a, a = 1, \ldots, N_f$ are given by:

$$\begin{aligned}\mathcal{I}_{\alpha_i} &= 2 \qquad \text{adjoint}\\ \mathcal{I}_{\alpha_i} &= \sum_{a=1}^{N_f} \left(\mathrm{sign}[\epsilon_a - v_i] - \mathrm{sign}[\epsilon_a - v_{i+1}]\right)\end{aligned} \tag{5.12}$$

We have few remarks. Index jumps each time ϵ_a crosses a v_i. For $v_{i-1} < \epsilon_a < v_i$, the zero mode is localized to monopole with charge α_{i-1}. The fermion zero mode wave functions decays exponentially as $e^{-|\epsilon_a - v_i|r}$ and decays algebraically for $|\epsilon_a - v_i| = 0$. For $v_i < \epsilon_a < v_{i+1}$, the zero mode do get localized to monopole α_i. For $|\epsilon_a - v_i| = 0$ i.e., crossing the boundary, the index is not well-defined.

For $N_f = N_c$ =even, Ω_F^0 eigenvalues given in (1.13) are interspersed between the gauge holonomy eigenvalues Ω (3.19), see figure 7. However, for $N_f = N_c$ =odd, Ω_F^0 eigenvalues coincide precisely with the gauge holonomy eigenvalues Ω. One can actually generate a gap between the two set of eigenvalues by imposing slightly more general boundary condition, twisted by $e^{i\alpha}\Omega_F^0 \in \mathrm{U}(N_f)_V$. From now on, we will treat the $N_f = N_c$ odd case on the same footing with $N_f = N_c$ even case, and declare that the flavor holonomy eigenvalues are interspersed between gauge holonomy eigenvalues as evenly as possible.

For $N_f = N_c$ QCD(F/adj), the N_c-tuple of the indices is given by:

$$\left[\mathcal{I}_{\alpha_1}, \mathcal{I}_{\alpha_2}, \ldots, \mathcal{I}_{\alpha_{N_c}}\right] = \underbrace{[2, 2, \ldots, 2]}_{\text{adj. fermion}} + \underbrace{[2, 2, \ldots, 2]}_{\text{fund. fermion}} \tag{5.13}$$

Consequently, the monopole-instanton amplitudes in the $(m_\lambda, m_\psi) = (0,0)$ theory are given by:

$$\mathcal{M}_i = e^{\alpha_i \cdot z}(\psi_{Ri}\psi_L^i)(\alpha_i \cdot \lambda)^2, \qquad i = 1, \dots, N_c. \tag{5.14}$$

and possess a total of 4 fermi zero modes. In the $N_f = N_c$ theory in the (5.11) background, adjoint and fundamental fermion zero modes appear exactly on the same footing, and this will help us when we decouple one or the other.

Note the usual relation between the 4d instanton amplitude (2.10) and monopole-instanton amplitudes:

$$\mathcal{I}_{4d} \sim \prod_{i=1}^{N_c} \mathcal{M}_i \sim e^{-\frac{8\pi^2}{g^2}} \prod_{i=1}^{N_c} (\psi_{Ri}\psi_L^i)(\alpha_i \cdot \lambda)^2, \tag{5.15}$$

following from the fact$\sum_{i=1}^{N_c} \alpha_i = 0$. The 4d instanton does couple neither to holonomy field ϕ nor to dual photons σ. It is also exponentially suppressed with respect to monopole-instantons. However, they are topological configurations which comply with the ABJ anomaly and determine the anomaly free quantum symmetry of the theory given in (2.9) or anomaly free discrete remnants once a mass term is turned either for λ or ψ. Also note that the $4N_c$ bosonic zero modes of the 4d instanton and its $2N_c + 2N_c$ fermionic zero modes are matched by the N_c types of monopole-instantons.

Introducing a soft mass term either for adjoint fermion or fundamental fermion, some or all fermi zero modes of the monopoles can be soaked up. As such, the dynamical role of monopole operators, as well as bion operators, does change in different regions in the $(m_\lambda,\ m_\psi)$ plane. Below we list the possible forms of the monopole operators in different regime in the parameter space $(m_\lambda,\ m_\psi)$.

$$\mathcal{M}_i = \begin{cases} e^{-S_i} e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\psi_{Ri}\psi_L^i)(\alpha_i\cdot\lambda)^2, & m_\lambda = 0,\ m_\psi = 0 \\ e^{-S_i} f_\lambda\, e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\psi_{Ri}\psi_L^i), & m_\lambda > 0,\ m_\psi = 0 \\ e^{-S_i} f_\psi e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\alpha_i\cdot\lambda)^2, & m_\lambda = 0,\ m_\psi > 0 \\ e^{-S_i} f_{\lambda\psi} e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}, & m_\lambda > 0,\ m_\psi > 0 \end{cases} \tag{5.16}$$

where $f_\psi = \frac{\det(\gamma_\mu D_\mu\big|_{\text{monopole}} + m_\psi)}{\lim_{m_\psi\to\infty}\det(\gamma_\mu D_\mu\big|_{\text{monopole}} + m_\psi)}$ etc. For small parameters, the functions $f_\lambda \sim m_\lambda$ and $f_\psi \sim m_\psi$ and $f_{\lambda\psi} \sim m_\lambda m_\psi$ as can be inferred by soaking up the fermion zero modes to soft mass term. In the limit where the masses decouple, e.g. $m_\psi/m_W \gg 1$, the corresponding functions approach to a constant. See section 2.1.2 of [44]. We will drop $f_\lambda,\ f_\psi,\ f_{\lambda\psi}$ unless it is not strictly necessary to keep them, in order to lessen the clutter.

There are other correlated topological configurations, but the first two-orders in semi-classics (for this particular problem) suffice to capture the interesting features of non-perturbative dynamics.

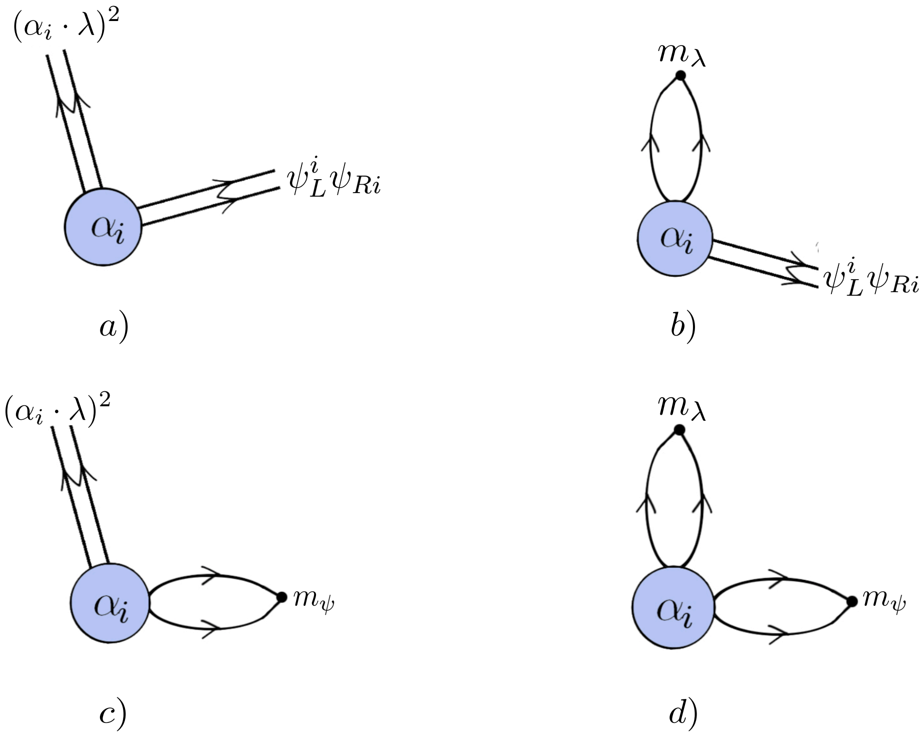

Figure 10. a)Monopole operators in $N_f = N_c$ QCD(F/adj) with Ω_F^0 twisted boundary conditions b) with soft m_λ. c)with soft m_ψ d) with soft m_λ and m_ψ. In the limits where either of the quark masses is taken large, the effect of the corresponding mass on the monopole operator disappears.

5.2 Grand canonical ensemble and EFT

In the semi-classical regime, the long-distance theory at scales larger than m_W^{-1} is described in terms of proliferation of topological 1-defects, 2-defects, 3-defects etc. We may view the Euclidean vacuum as a grand canonical ensemble of these configurations. These events are:

- Monopole-instantons $\mathcal{M}_i,\ i = 1\dots, N_c$ which are saddle points (critical points) of the classical action with characteristic size $r_\mathcal{M} \sim m_W^{-1}$ and density e^{-S_i}.
- Neutral bions $\mathcal{B}_{ii} = [\mathcal{M}_i\overline{\mathcal{M}}_i]$ and magnetic bions $\mathcal{B}_{i,i\pm1} = [\mathcal{M}_i\overline{\mathcal{M}}_{i\pm1}]$ [57, 81] with characteristic size $r_\mathcal{B} \sim r_\mathcal{M}/g^2$ and densities e^{-2S_i} and $e^{-S_i-S_{i+1}}$. These contributions should be interpreted as exact critical points at infinity in the context of Picard-Lefschetz theory [82]. The characteristic size $r_\mathcal{B}$ arises because Lefschetz thimbles associated with such critical points are non-Gaussian due to quasi-zero mode (QZM) directions. The characteristic size emerges as the point on the thimble where the integration is dominated. There is also a crucial hidden topological angle associated with neutral bion, such that $\arg \mathcal{B}_{ii} = \arg \mathcal{B}_{i,i+1} + \pi$, due to the fact that the QZM thimble of $\mathcal{B}_{ii}$ makes a journey to complex domain.[19]

[19] In $\mathcal{N} = 1$ SYM, for example, the relative phase is crucial to explain the vanishing of the gluon condensate, $\langle \mathrm{tr}\, F_{\mu\nu}^2 \rangle$, which receives positive contribution from $\mathcal{B}_{i,i+1}$ and negative contribution from $\mathcal{B}_{ii}$. This is because in vacuum $\langle \mathcal{B}_{i,i+1} \rangle = e^{i\pi} \langle \mathcal{B}_{ii} \rangle$. This observation resolved an old puzzle why the gluon condensate can ever vanish in a vector-like gauge theory with a positive-definite path integral measure [83].

- The higher order terms in the semi-classical expansion can be incorporated via the cluster (virial) expansion for an interacting gas of monopole-instantons. But the qualitatively important behavior of the theory is dominated by first two-order in semi-classics, while higher orders are needed for resurgent cancellations.

Let $\mathcal{T}$ denote the set of all topological defects, 1-defects, 2-defects, etc as described above.

$$\mathcal{T} = \left\{ \mathcal{M}_i,\ \ \overline{\mathcal{M}}_i,\ \ [\mathcal{M}_i\overline{\mathcal{M}}_{i\pm1}],\ \ [\mathcal{M}_i\overline{\mathcal{M}}_i],\ \ [\mathcal{M}_i\overline{\mathcal{M}}_j\mathcal{M}_k], \ldots \right\} \tag{5.17}$$

The ordering is according to the action (S) or density (e^{-S}) of the corresponding topological event. The leading ones are rare, their density is proportional is e^{-S_i} and subleading ones are rarer, their density is proportional is $e^{-S_i - S_{i+1}}$, but nevertheless all are present. It is also crucial that not all the terms come with the same sign due to hidden topological angles [83], in other words, the densities are real, but fugacities may be complex.

The sum over all events is equivalent to adding all operators induced by these topological configurations to the action (5.6):

$$\begin{aligned}
&\prod_{\mathcal{T}} \sum_{n_{\mathcal{T}}=0}^{\infty} \frac{1}{n_{\mathcal{T}}!} \left[\int d^3x\ \mathcal{T} \right]^{n_{\mathcal{T}}} \\
&\quad = \prod_{i=1}^{N_c} \left(\sum_{n_{\mathcal{M}_i}=0}^{\infty} \frac{1}{n_{\mathcal{M}_i}!} \left[\int d^3x\ \mathcal{M}_i \right]^{n_{\mathcal{M}_i}} \right) \left(\sum_{n_{\overline{\mathcal{M}}_i}=0}^{\infty} \frac{1}{n_{\overline{\mathcal{M}}_i}!} \left[\int d^3x\ \overline{\mathcal{M}}_i \right]^{n_{\overline{\mathcal{M}}_i}} \right) \ldots \\
&\quad = \exp \left[\sum_{i=1}^{N_c} \int d^3x\ \left(\mathcal{M}_i + \overline{\mathcal{M}}_i + \mathcal{B}_{ii} + \mathcal{B}_{i,i+1} \ldots \right) \right]
\end{aligned} \tag{5.18}$$

resulting in an EFT which is valid provided the theory is at sufficiently weak coupling. The range of physical phenomena that this effective EFT explains is quite diverse in various limits of QCD(F/adj), and the rest of the paper is devoted to explanation of these effects. This is hardly surprising because the EFT is derived within the domain of applicability of semi-classical methods.

Before proceeding, we remind that the existence of monopole operator in the effective Lagrangian does not imply the existence of mass gap for gauge fluctuations, despite the fact that monopole operator has both holonomy and dual photon dependence. When at least one of the fermion type, F or adj, is exactly massless, the proliferation of monopoles do not induce a mass gap for gauge fluctuations, as evident from the monopole operators (5.16). Only when both fermions are massive, the monopole operators themselves may induce a mass gap for gauge fluctuations as in [70] and [5].

5.3 Can gluons acquire a chiral charge?

Topological shift symmetry. In the absence of monopole instantons, the dual formulation has an $[\mathrm{U}(1)_J]^{N_c-1}$ topological shift symmetry, which protects the gaplessness of the dual photon to all orders in perturbation theory. The symmetry and its Noether current are:

$$[\mathrm{U}(1)_J]^{N_c-1} : \sigma \to \sigma + \varepsilon, \qquad \mathcal{J}_\mu = \partial_\mu \sigma \tag{5.19}$$

By abelian duality relation, the current is the (euclidean) magnetic field, B_μ, and current conservation is the statement of the absence of monopoles:

$$\partial_\mu \mathcal{J}_\mu = \partial_\mu B_\mu = 0. \tag{5.20}$$

If there are monopoles in the theory, (and no massless fermions), then, of course, current is no longer conserved:

$$\partial_\mu \mathcal{J}_\mu = \partial_\mu B_\mu = \rho_m(x) \neq 0 \tag{5.21}$$

where ρ_m is the magnetic charge density. This implies that the topological shift symmetry (5.19) is not present at non-perturbative level. Indeed, absence of the shift symmetry is synonymous with the proliferation of the monopole-instantons, and non-perturbatively $[\mathrm{U}(1)_J]^{N_c-1}$ violating monopole-operators are induced. The effective long distance Lagrangian exhibits both confinement of electric charge with finite string tensions and a non-perturbative mass gap for the gauge fluctuations, see [70] on $\mathbb{R}^3$ and [5] for its generalization to locally four-dimension on $\mathbb{R}^3 \times S^1$.

Mixing of topological shift symmetry with chiral symmetry. With the inclusion of massless fermions, the story is different and not less interesting, as it captures other non-perturbative phenomena. What happens precisely depends on the theory. However, overall story develops as follows.

- Unlike the 4d instanton amplitude which is singlet under the chiral symmetries of the theory $\mathbf{G}_{\text{max}-\text{ab}} \subset \mathbf{G}$ given in (2.38), the fermion zero mode structure of the monopole operator is generically not a singlet under the chiral symmetry.

- Since the corresponding symmetry is a genuine *non-anomalous* symmetry of the QFT, the monopole operator as a whole *must be* singlet under it.

- What happens is that a subgroup (or whole) of $[\mathrm{U}(1)_J]^{N_c-1}$ intertwines with $\mathbf{G}_{\text{max}-\text{ab}}$. This mixing guarantee that $\mathbf{G}_{\text{max}-\text{ab}}$ is respected by monopole-operators.

- This is a strange phenomenon, however. It is usually believed that in theories with massless fermions, only microscopic fermions and their composites are charged under chiral symmetry. A priori, gauge field has "nothing" to do with chiral charges associated with (2.38). Yet, the intertwining of emergent shift symmetry and microscopic chiral symmetry tells us that the gauge fluctuations **must** acquire a chiral charge, and there is no other option! This type effect first appeared in a gauge theory on $\mathbb{R}^3$ in the work of Affleck, Harvey and Witten [34]. Its generality and usefulness in gauge theories on $\mathbb{R}^3 \times S^1$ is newer [33, 81]. It actually provides an example of analytically calculable chiral symmetry breaking in both QCD(F/adj), QCD(F) and $\mathcal{N}=1$ SYM, a phenomenon that is believed to be an incalculable strong coupling effect is realized in weak coupling calculable domain.

Below, we explain this qualitative picture quantitatively.

What is conserved and what is violated at the monopole-instanton event? The charge associated with the topological current is magnetic charge $\mathbf{Q}_m = \int d^2x\ \mathcal{J}_0 = \int d^2x\ \partial_0\sigma$ and at the monopole-event, it is clearly violated. Consider a collection of n_i monopoles of type α_i for $i = 1, \ldots, N_c$ sprinkled in between two asymptotic time slice. Then, the magnetic charge non-conservation is

$$
\begin{aligned}
\Delta\mathbf{Q}_m = \mathbf{Q}_m(t=\infty) - \mathbf{Q}_m(t=-\infty) &= \int d^2x F_{12}\Big|_{t=-\infty}^{t=+\infty} \\
&= \int_{S^2_\infty} F_{12} \\
&= \frac{4\pi}{g}\sum_{i=1}^{N_c} n_i\alpha_i \\
&= \frac{4\pi}{g}\left(n_1 - n_{N_c}, n_2 - n_1, n_3 - n_2, \ldots, n_{N_c-1} - n_{N_c}\right)
\end{aligned}
\tag{5.22}
$$

These charges violate *emergent* $[\mathrm{U}(1)_J]^{N_c-1}$ explicitly and completely. However, the non-conservation of magnetic charge is not whole story in theories with dynamical fermions.

In the presence of massless fermions, each monopole-event carries certain number of fermionic zero modes [31, 32]. The zero modes are charged under maximal torus of *microscopic non-abelian* axial symmetry, $\mathbf{G}_{\max-\mathrm{ab}}$ given in (2.38). However, unlike the $\mathrm{U}(1)_A$ which is anomalous in QCD(F), the non-abelian chiral symmetry (2.5) is non-anomalous, and any of its subgroups cannot be violated by the non-perturbative events! The resolution of this puzzle is the key piece to understand the chiral symmetry breaking and the fact that gauge fields in gauge theories may and do acquire a chiral charge.

The axial current associated with non-abelian chiral symmetry in 4d can be written as $J^{5A}_\mu = \overline{\psi}\gamma_\mu\gamma_5 T^A\psi$ where $T^A, A = 1, \ldots, N_f^2 - 1$ are generators of $\mathrm{SU}(N_f)$. This current is conserved, and the corresponding charges are: $\mathbf{Q}^{5A} = \int d^3x\ \psi^\dagger\gamma_5 T^A\psi$. Namely, the charge commutes with the Hamiltonian $[H, \mathbf{Q}^{5A}] = 0$ for all A. However, in the graded partition function (1.11), we have $\mathrm{SU}(N_f)_V$ charges along Cartan sub-algebra, and the operator $H' = H - i\sum_{a\in\mathrm{Cartan}} \frac{\epsilon_a}{\beta} Q_a$ only commutes with the Cartan generators of the axial charges, $[H', \mathbf{Q}^{5A}] = 0$ for all $A \in$ Cartan. Let us use an explicit basis for Cartan generators:

$$
(H^{A=a}) = \frac{1}{2}\mathrm{Diag}(0, \ldots, 0\underbrace{1}_{a}, \underbrace{-1}_{a+1}, 0, \ldots, 0)
\tag{5.23}
$$

In this case, we can express the chiral charge as:

$$
\mathbf{Q}^{5,A=a} = \frac{1}{2}(Q^{5,a} - Q^{5,a+1}) = \alpha_a \cdot Q^5, \qquad A = 1, \ldots N_f - 1
\tag{5.24}
$$

where $Q^{5,a} = \int d^3x\ \psi^{\dagger,a}\gamma_5\psi_a$ and all $\mathbf{Q}^{5,A=a}$ commutes with Hamiltonian and H'. We can augment this list with an affine-charge, which is conserved because it is a linear combination of the other conserved charges:

$$
\mathbf{Q}^{5,A=N_f} \equiv -\sum_{a=1}^{N_f-1} \alpha_a \cdot Q^5 = \frac{1}{2}(Q^{5,N_f} - Q^{5,1}) = \alpha_{N_f} \cdot Q^5
\tag{5.25}
$$

Here is the main point: despite the fact that $\mathbf{Q}^{5,A=a}$ commutes with the Hamiltonian as well as grading, it appears to be violated by the fermionic zero mode structure of the monopole-instantons:

$$\mathbf{Q}^{5,A=a} : (\psi_{Ri}\psi_L^i) \to (\delta_{ai} - \delta_{a+1i})(\psi_{Ri}\psi_L^i) \tag{5.26}$$

How can a symmetry be simultaneously non-anomalous and also "apparently" violated by monopole-instanton amplitude? Of course, this is not possible. A non-anomalous symmetry of the theory must be respected by all topological configurations.

Consider again collection of n_i monopoles of type α_i for $i = 1, \dots, N_c$ sprinkled in between two asymptotic time slice. The background will have n_1 many $(\psi_{R1}\psi_L^1)$ zero modes, n_2 many $(\psi_{R2}\psi_L^2)$ zero modes and n_{N_f} many $(\psi_{RN_f}\psi_L^{N_f})$ zero modes. Then, the apparent axial charge non-conservation will be

$$\begin{aligned}\Delta\mathbf{Q}^5 &= \mathbf{Q}^5(t=\infty) - \mathbf{Q}^5(t=-\infty)\\ &= \sum_{A=1}^{N_f} n_A \alpha_A\\ &= 2\left(n_1 - n_{N_f}, n_2 - n_1, n_3 - n_2, \dots, n_{N_f-1} - n_{N_f}\right)\end{aligned} \tag{5.27}$$

Assume momentarily $N_f = N_c$. In this case, it is clear that magnetic charge non-conservation and chiral charge non-conservation are exactly proportional to each other for any background. In fact, we can construct a linear combination of these two-charges which is respected by all non-perturbative and topological configurations:

$$\tilde{\mathbf{Q}} = \frac{g}{4\pi}\mathbf{Q}_m - \mathbf{Q}^5, \qquad \text{such that} \qquad \Delta\tilde{\mathbf{Q}} = 0 \tag{5.28}$$

What does this mean? Here, $\mathbf{Q}^5$ microscopic chiral charge is associated with Cartan subgroup $[\mathrm{U}(1)_A]^{N_f-1}$ of the full non-abelian chiral symmetry, while $\mathbf{Q}_m$ is an emergent symmetry in EFT, but it is only valid to all orders in perturbation theory. So, the genuine microscopic symmetry here is only chiral symmetry, and this whole mechanism is present so that the chiral charge of the fermion bilinear can be transferred to gauge fluctuations! We can also equivalently say that the diagonal of the $\mathbf{G}_{\text{max}-\text{ab}}$ and topological shift symmetry

$$[\mathrm{U}(1)^{N_c-1}]_{AJ} = \mathrm{Diag}\left([\mathrm{U}(1)^{N_c-1}]_A \times [\mathrm{U}(1)^{N_c-1}]_J\right) \tag{5.29}$$

is the symmetry of the long-distance effective field theory.

The conserved charge (5.28) is the global content of the local current conservation, $\partial_\mu\left(\frac{g}{4\pi}\mathcal{J}_\mu - J_\mu^5\right) = 0$. Naively, the fermionic zero mode part of the monopole operator would induce an anomaly, because it is proportional to $\psi_{Ri}\psi_L^i$. But this is always accompanied by a magnetic flux event, and a correlated change in the magnetic flux compensates the change in the fermion number. The combination in (5.28) is conserved in any perturbative or non-perturbative process. As described below, the choice of the vacuum breaks this symmetry spontaneously.

The mechanism described above can be generalized to all QCD(F) with $N_f \leq N_c$ and $1 \leq N_f \leq N_c - 1$ QCD(F/adj) in abelianizing gauge holonomy background. We will show that in those cases, it accounts for all Nambu-Goldstone bosons in the theory with twisted boundary condition.

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Chiral symmetry order parameters. Because of the topological shift and chiral symmetry mixing, in gauge theories in general there are two types of chiral order parameters:

$$\begin{aligned} \text{Monopole (magnetic flux) operators}: &\qquad e^{\alpha_i\cdot z} \\ \text{Fermion bilinears, multilinears}: &\qquad \psi_{Ra}\psi_L^b, \quad \operatorname{tr}\lambda\lambda, \quad (\psi_{Ra}\psi_L^b \operatorname{tr}\lambda\lambda) \end{aligned} \tag{5.30}$$

In all calculable examples in semi-classical domain on $\mathbb{R}^3\times S^1$, χSB occurs due to condensation of the magnetic flux operators. In most interesting cases, $(m_\lambda=0, m_\psi=0), (m_\lambda>0, m_\psi=0), (m_\lambda=0, m_\psi>0)$, the realization of this scenario differs in crucial ways. Below, we describe each in some detail.

5.4 QCD(F/adj)

$\boldsymbol{m_\lambda = 0, m_\psi = 0.}$ What happens in the presence of massless fermions and twisted boundary conditions? As described in section 2, although the microscopic theory has non-abelian chiral symmetry $\mathbf{G}$ given in (2.5), the twisted boundary conditions explicitly reduces the global symmetry down to its maximal abelian subgroup $\mathbf{G}_{\text{max-ab}}\subset\mathbf{G}$ (2.38).

The action of the axial subgroup of $\mathbf{G}_{\text{max-ab}}$ on fermion bi-linears is, for $N_f=N_c$:

$$\begin{aligned} [\mathrm{U}(1)_A]^{N_f-1}: &\qquad (\psi_{Ri}\psi_L^i)\to e^{i\varepsilon_i}(\psi_{Ri}\psi_L^i), &\qquad \lambda\lambda\to\lambda\lambda \\ \mathrm{U}(1)_{A_D}: &\qquad (\psi_{Ri}\psi_L^i)\to e^{-2i\gamma}(\psi_{Ri}\psi_L^i), &\qquad \lambda\lambda\to e^{2i\gamma}\lambda\lambda \end{aligned} \tag{5.31}$$

such that there is one constraint among N_c continuous variables ε_i.

$$\sum_{i=1}^{N_c}\varepsilon_i = 0, \tag{5.32}$$

This means the fermion zero mode part of the monopole operator (5.14) transform non-trivially under $[\mathrm{U}(1)_A]^{N_c-1}$. In order for it to be invariant under the continuous chiral symmetries, the pure flux part of the monopole operator must transform as

$$e^{\alpha_i\cdot z}\to e^{-i\varepsilon_i}e^{\alpha_i\cdot z}, \qquad \varepsilon_i=\alpha_i\cdot\varepsilon \tag{5.33}$$

which is nothing but the shift symmetry (5.19) described above and ε_i satisfy the constraint (5.32). The axial-emergent topological symmetry mixing is the mechanism that the monopole operators respect invariance under the non-anomalous chiral symmetry and this is an explicit realization of the $[\mathrm{U}(1)^{N_c-1}]_{AJ}$ given in (5.29). Note that both the flux operator as well as four-fermi operator in (5.14) are singlet under $\mathrm{U}(1)_{A_D}$.

Unlike the discrete shift symmetry, the continuous shift symmetry (5.29) forbids a mass term for the dual photon at any non-perturbative order. To appreciate this contrast, recall that in $\mathcal{N}=1$ SYM, we only have topological shift symmetry intertwining with the discrete chiral $\mathbb{Z}_{2N_c}$ symmetry. Although mass term for gauge fluctuations cannot be induced at first order in semi-classics, at second order, semi-classical magnetic bion effects induce a mass term for gauge fluctuations [81]. But the magnetic bions in QCD(F/adj) has fermionic zero modes which cannot be contracted. The magnetic bion amplitudes in these two cases are:

$$\mathcal{B}_{i,i+1} \sim e^{-2S_0} e^{-\frac{4\pi}{g^2}(\alpha_i+\vec{\alpha}_{i+1})\cdot\phi} e^{i(\alpha_i-\vec{\alpha}_{i+1})\cdot\sigma}, \qquad \mathcal{N}=1 \text{ SYM},$$
$$\mathcal{B}_{i,i+1} \sim e^{-2S_0} e^{-\frac{4\pi}{g^2}(\alpha_i+\vec{\alpha}_{i+1})\cdot\phi} e^{i(\alpha_i-\vec{\alpha}_{i+1})\cdot\sigma} (\psi_{Ri}\psi^i_L)(\overline{\psi}^{i+1}_R\overline{\psi}_{L,i+1}) \qquad \text{QCD(F/adj)} \tag{5.34}$$

Therefore, in $\mathcal{N}=1$ SYM, magnetic bions induce a mass term for σ fluctuations and in QCD(F/adj), the continuous shift symmetry forbids formation of any potential for the dual photon.

The chiral symmetry is spontaneously broken with a magnetic flux order parameter acquiring a vev. In center-symmetric minimum,

$$\begin{aligned}&\langle \text{VAC}|e^{-\alpha_i\cdot z}|\text{VAC}\rangle = e^{-\frac{4\pi}{g^2}(v_{i+1}-v_i)}\langle e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi+i\alpha_i\cdot\sigma}\rangle = e^{-S_0}e^{i\delta_i},\\ &\text{Diag}[e^{i\delta_1},\dots,e^{i\delta_{N_f}}] \in \mathbf{T}^{N_f-1}\end{aligned} \tag{5.35}$$

leading to spontaneous breaking of chiral symmetry:

$$[\text{U}(1)_A]^{N_f-1}\times \text{U}(1)_{A_D} \longrightarrow \text{U}(1)_{A_D} \tag{5.36}$$

Therefore, the N_c-1 dual photons (which can be viewed as fluctuations around constant phases $e^{i\delta_i}$) are identified with the N_c-1 Nambu-Goldstone bosons of the spontaneously broken maximal abelian chiral symmetry. These NG-bosons live in the maximal torus $\mathbf{T}^{N_f-1}$ of the $\text{SU}(N_f)_A$, where $N_f=N_c$, and constitute the fields that make the chiral Lagrangian.

Fermion multi-linear condensate in statistical interpretation. In the vacuum, we can set the flux part of the monopole-operator to its vev (5.35) following a similar rationale in [35, 36]. Therefore, at leading order in semi-classics, the vacuum is a dilute gas of N types of monopole-instantons each with complex fugacity

$$\zeta_i = e^{-S_0}e^{i\delta_i}, \qquad i=1,\dots,N_c \tag{5.37}$$

Physically, the magnitude of fugacity $|\zeta_i|$ is the density of monopole of type-i: $\mathcal{N}_{\mathcal{M}_i}/V_{\mathbb{R}^3\times S^1}$ where $\mathcal{N}_{\mathcal{M}_i}$ is the number of type-i monopoles in volume $V_{\mathbb{R}^3\times S^1}$. In the statistical interpretation, we can think of fermion multi-linear condensate $\langle \psi_{Ri}\psi^j_L \operatorname{tr}\lambda\lambda\rangle$ as follows. The multi-linear order parameter will pick up contributions only from the support of the monopole-cores. The zero modes can be saturated by the zero modes of the monopole. Depending on the monopole type, each species has a different complex phase. In this statistical interpretation, $\langle \psi_{Ri}\psi^j_L \operatorname{tr}\lambda\lambda\rangle \sim \delta^i_j e^{-S_0}\beta^{-6}e^{i\delta_i}$ in the semi-classical domain where $\beta N_c\Lambda \lesssim 1$, and Λ is the strong scale of QCD(F/adj). In this regime, there is no reason for the remaining $\text{U}(1)_{A_D}$ to break as

$$\Delta\mathcal{L} \sim e^{-S_0}\langle e^{i\alpha_i\cdot\sigma}\rangle \sum_{i=1}^{N} \psi_{Ri}\psi^i_L(\alpha_i\cdots\lambda)^2 + \text{h.c.} \tag{5.38}$$

is quite weak, and the fermion bilinears are not capable of producing a vev. As a result,

$$\langle \psi_{Ri}\psi^j_L\rangle = \langle \operatorname{tr}\lambda\lambda\rangle = 0, \qquad \langle \psi_{Ri}\psi^j_L \operatorname{tr}\lambda\lambda\rangle = \delta^j_i\Lambda^6(\Lambda\beta N_c)^{-11/3}e^{i\delta_i}, \qquad \beta N_c\Lambda \lesssim 1 \tag{5.39}$$

Two possibilities at large $S^1 \times \mathbb{R}^3$ and adiabatic continuity. In the semiclassical regime, $\mathrm{U}(1)_{A_D}$ axial chiral symmetry remains unbroken, because the flux part of the monopole-operator is not charged under this symmetry and the four-fermi operator obtained upon condensation of flux operator is very weak. On the other hand, in the strong coupling domain, as we described earlier, two patterns of chiral symmetry breaking are plausible, (2.13) and (2.15).

If (2.15) and (2.16) is realized in $\mathbb{R}^4$, then, QCD(F/adj) exhibits adiabatic continuity, i.e. there are no singularities of the graded partition function (4.5) for any value of $\beta \in (0, \infty)$. Then, $\mathbb{Z}_{N_c}$ CFC-symmetry remains unbroken, and $[\mathrm{U}(1)_A]^{N_c-1}$ is broken and $\mathrm{U}(1)_{A_D}$ is unbroken for any value of $\beta \in (0, \infty)$. This option is not completely unreasonable because there is a large-amount of non-trivial anomaly matching that works out on $\mathbb{R}^4$ (2.27).

If (2.13) and (2.14) is realized in $\mathbb{R}^4$, and chiral symmetry is broken to vector-like subgroup, as the radius is varied from small-β to large-β, there must exist a chiral transition associated with a change in the realization of $\mathrm{U}(1)_{A_D}$. Note that at small-β, we have shown that $\mathbb{Z}_{N_c}$ remains unbroken, and $[\mathrm{U}(1)_A]^{N_c-1}$ is broken, and we expect these symmetry realizations to be exactly the same at large-β strong coupling. Therefore, the only symmetry that can clash with analyticity is the $\mathrm{U}(1)_{A_D}$.

On dynamical grounds, the four-fermi operators induced by condensation of flux operators can in fact generate fermion bilinear condensates if its coefficient becomes sufficiently strong. However, strong coupling, assuming that it sets in (which is reasonable), will take place at $\beta N_c \Lambda \gtrsim 1$ and as such, we may expect a phase transition at the non-'t Hooftian scale $\beta_c \sim \Lambda^{-1}/N_c$. However, this is parametrically at the boundary of the region of validity of EFT. As such, it is not completely reliable, yet, we find more reasonable than the first option above.

At this stage, despite the fact that $\mathbb{Z}_{N_c} \times [\mathrm{U}(1)_A]^{N_c-1}$ symmetry realizations are same in small- and large-β, $\mathrm{U}(1)_{A_D}$ realization may change. As a result, adiabatic continuity in QCD(F/adj) may or may not work as a function of S^1 radius just because of a single $\mathrm{U}(1)_{A_D}$ factor. See figure 11.

5.5 SQCD with $N_f = N_c$ on $\mathbb{R}^4$ and QCD(F/adj) on $\mathbb{R}^3 \times S^1$: adiabatic continuity

Seiberg showed that the ground states of $N_f = N_c$ (quantum moduli space) SQCD is parametrized by chiral meson and baryon superfields obeying $\det M - B\overline{B} = \Lambda^{2N_c}$ [46]. At the point $B = \overline{B} = 0$, we have $\det M = \Lambda^{2N_c}$, and the chiral symmetry is broken as in (2.15), $\mathbf{G} \to \mathbf{G_V} \times \mathrm{U}(\mathbf{1})_{\mathbf{A_D}}$ to a subgroup which has a chiral component. As described around (2.16), at this point in moduli space, fermion bilinear condensate cannot form, as they are charged under the $\mathrm{U}(1)_{A_D}$. Instead the operator the scalar component of the meson superfield acquires a vev: $M^a_b = Q_{Ra} Q^b_L = \Lambda^2 \delta^b_a$. This is singlet und

Aharony et al. [19] showed that this patterns continues to hold with a supersymmetry breaking soft mass for the scalar quark field (squark). When the squark mass infinite, and the theory reduces to QCD(F/adj), as explained in ref. [19] and reviewed in section 2.2, there are two physically well-motivated possibilities,

$$\mathbf{G} \to \mathbf{G}_{\mathrm{V}} \qquad \text{or} \qquad \mathbf{G} \to \mathbf{G}_{\mathrm{V}} \times \mathrm{U}(\mathbf{1})_{\mathrm{A_D}} \tag{5.40}$$

As we explained in section 2.2, the second option is also reasonable because it large amount of zero-form mixed anomalies can be matched by massless composite fermions (2.27). Ref. [19] argued, as the squark mass is increased, the theory may or may not have a phase transition, i.e. SQCD may or may not be adiabatically connected to QCD(F/adj) on $\mathbb{R}^4$. This question is still unsettled and we do not know the answer either. However, we will provide an adiabatic continuity between SQCD on arbitrarily large $\mathbb{R}^3 \times S^1$ and QCD(F/adj) on small $\mathbb{R}^3 \times S^1$.

First, consider SQCD with Ω_F^0 twisted boundary condition on large $S^1 \times \mathbb{R}^3$:

$$\begin{aligned} W_\alpha(\beta) &= W_\alpha(0), \\ Q_{R/L}(\beta) &= Q_{R/L}(0)\overline{\Omega}_F^0 e^{i\pi} \end{aligned} \tag{5.41}$$

where W_α is field strength multiplet and $Q_{R/L}$ are $N_f = N_c$ chiral multiplets. Since these boundary conditions are implemented at the level of superfields, they respect $\mathcal{N} = 1$ supersymmetry.

With the use of Ω_F^0 twisted boundary condition on large $S^1 \times \mathbb{R}^3$, the chiral symmetry explicitly reduces to its maximal torus at the scale of compactification. By twisted compactification, the pattern of the chiral symmetry breaking becomes $\mathbf{G}_{\mathrm{max-ab}} \to \mathbf{G}_{\mathrm{V,max-ab}} \times \mathrm{U}(\mathbf{1})_{\mathrm{A_D}}$ in the compactified supersymmetric theory at $B = \overline{B} = 0$. This also holds for sufficiently soft squark mass. Note that with this explicit breaking, the $N_f^2 - 1$ Nambu-Goldstone bosons reduce to $N_f - 1$ NG bosons, and $N_f^2 - N_f$ modes acquire masses are in the range $[\frac{2\pi}{\beta N_f}, \frac{2\pi}{\beta}]$. These massive modes become gapless in the $\beta \to \infty$ limit. Turning on a soft mass for squark field, $m_{q_a} \neq 0$, this pattern holds for sufficiently small mass perturbation.

The moral behind this boundary condition is that, just like our non-supersymmetric example, the Polyakov loop wrapping the S^1 cicrle becomes a genuine order parameter under $\mathbb{Z}_N$ color-flavor center symmetry, see section 2.3. In the non-supersymmetric QCD(F/adj) case, these boundary condition make sure that the gauge holonomy potential is minimized at the center-symmetric point (3.19). In supersymmetric theory, the gauge holonomy potential is zero to all orders in perturbation theory even in the presence of Ω_F^0 background, as it respects supersymmetry. The contributions of component fields in W_α cancel exactly, and the contributions of scalar and fermionic contribution in $Q_{R/L}$ cancel exactly. Therefore, since the effect of fermionic component of $Q_{R/L}$ is in favor of stabilization of CFC symmetry, (3.18), the effect of the scalar must be other way around. Lifting the scalars (whose effect is to undo center-stabilizing contribution of fundamental fermions with Ω_F^0 boundary conditions), the $\mathcal{N} = 0$ deformation of the susy theory will land on the CFC symmetric phase on $S^1 \times \mathbb{R}^3$.

It is intriguing to note that in QCD(F/adj) on small $\mathbb{R}^3 \times S^1$, we find ourself in exactly the same situation. In the calculable weak coupling domain, the chiral symmetry is broken

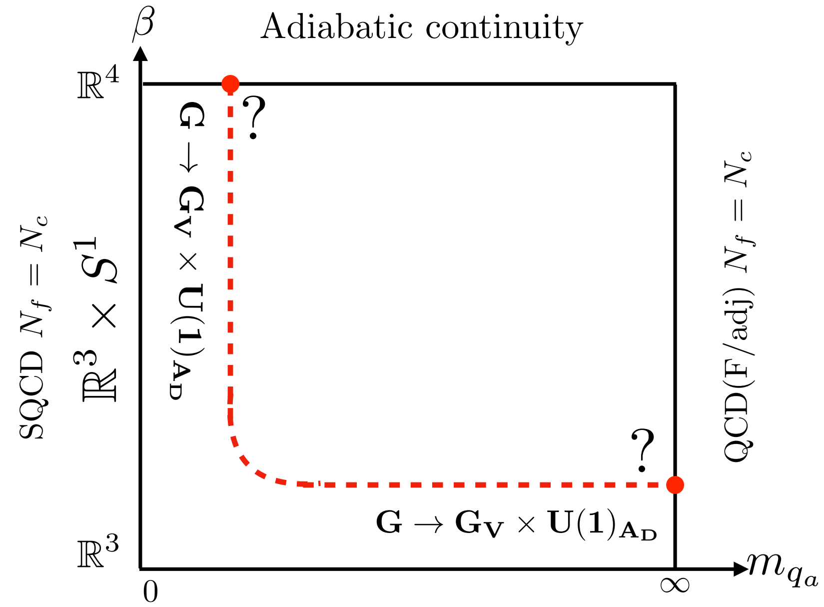

Figure 11. There exists an adiabatic continuity between $N_f = N_c$ SQCD and its soft supersymmetry breaking scalar mass deformation m_{q_a} at strong coupling on large $S^1 \times \mathbb{R}^3$ and non-supersymmetric QCD(F/adj) at weak coupling on small $S^1 \times \mathbb{R}^3$. There is very likely a phase transition associated with $\mathrm{U}(1)_{A_D}$ crossing red-dotted line, but absence of phase transition is also consistent with mixed anomalies.

as $\mathbf{G}_{\text{max-ab}} \to \mathbf{G}_{\text{V,max-ab}} \times \mathrm{U}(\mathbf{1})_{\mathrm{A_D}}$ by the condensation of monopole flux operators (5.35), but there may be a $\mathrm{U}(1)_{A_D}$ changing phase transition as β is dialed.

On the other hand, there seems to be an adiabatic continuity between the strongly coupled SQCD with scalar mass deformation and QCD(F/adj) in the weak coupling domain (both endowed with flavor twisted boundary conditions with Ω_F^0) provided

$$\max(m_{q_a}\Lambda^{-1}, \beta\Lambda) \lesssim 1 \tag{5.42}$$

in the phase diagram in the (m_{q_a}, β) plane. (We assumed N_c is fixed and small here.)[20]

5.6 QCD(F) and adiabatic continuity

Let us first remind some aspects of the chiral Lagrangian in QCD(F). At $\mathbb{R}^4$, the continuous chiral symmetry $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ is spontaneously broken to vector-like subgroup and the infrared physics is described by chiral Lagrangian $S = \int_{\mathbb{R}^4} \left[\frac{f_\pi^2}{4} \operatorname{tr} |\partial_\mu \Sigma|^2 + \dots\right]$ where chiral field $\Sigma(x)$ captures $N_f^2 - 1$ NG bosons. Turning on a background vector-like flavor field amounts to $\partial_\mu \to D_\mu = \partial_\mu + i[A_\mu, \ ]$. Hence, the effect of the flavor homonomy in the compactified direction is to turn on

$$A_\mu = \delta_{\mu 4} A_4 \equiv \delta_{\mu 4} \frac{1}{\beta} \mathrm{Diag}\left(\epsilon_1, \epsilon_2, \dots, \epsilon_{N_f}\right), \tag{5.43}$$

[20] A related interesting problem is the $\mathcal{N} = 2$ SYM with scalar mass m_Φ breaking it to $\mathcal{N} = 0$ [52, 84], and its compactication to $\mathbb{R}^3 \times S^1$. The (m_Φ, β) plane admits a calculable decoupling limit $m_\Phi \to \infty$ provided $\beta\Lambda N \lesssim 1$. This work is ongoing in collaboration with M. Anber.

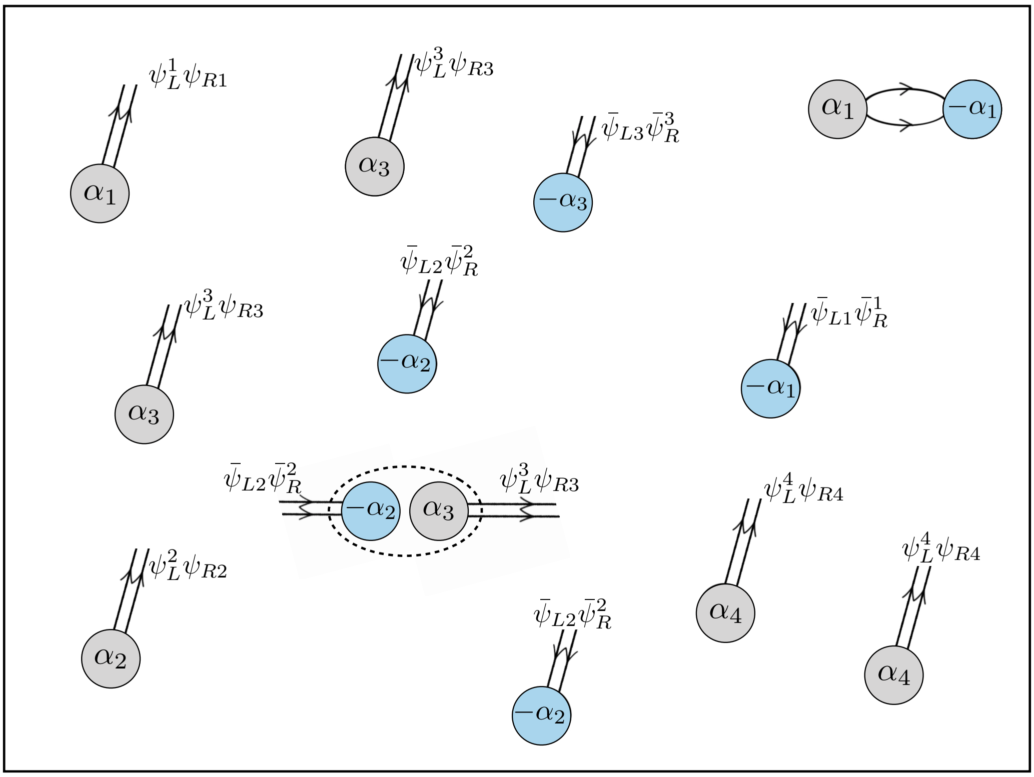

Figure 12. Dilute gas of topological configurations in the weak coupling regime obtained by using graded partition function. Due to Ω_F^0 background, each species of monopole-instanton has exactly two fermionic zero modes. Naively, each monopole-event violates the maximal abelian chiral symmetry $\mathrm{U}(1)^{N_f-1}$. However, this is not the case because of the intertwining of chiral symmetry with topological shift symmetry, i.e. gauge fluctuations carry a chiral charge. Once a chiral order parameter acquires a vacuum expectation value $\langle \mathrm{VAC}|e^{-\alpha_i\cdot z}|\mathrm{VAC}\rangle = e^{-S_0}e^{i\delta_i}$, the vacuum looks as if it is populated by $(\psi_{Ri}\psi^i_L)$ pairs. In a given vacuum, $\langle\psi_{Ri}\psi^j_L\rangle = \delta^j_i\Lambda^3$ where Λ is strong scale. This is the chiral symmetry breaking vacuum of QCD. Note that this is a mechanism taking place at weak coupling. Historically, it is believed that the chiral symmetry breaking in QCD-like theories is inherently a strong coupling phenomenon and incalculable.

The chiral Lagrangian on $\mathbb{R}^3\times S^1$ in the flavor holonomy background can be written as

$$S_{\Omega_F} = \int_{\mathbb{R}^3\times S^1}\left[\frac{f_\pi^2}{4}\,\mathrm{tr}\,|D_\mu\Sigma|^2\right] \tag{5.44}$$

The background flavor holonomy gives a mass to the off-Cartan components of the meson field of the order of $\frac{2\pi}{\beta N_f}$. Therefore, only N_f-1 mesons remain exactly massless in the Ω_F^0-twisted background.

$\boldsymbol{m_\lambda > 0, m_\psi = 0.}$ Once a mass term for adjoint fermion is added, the theory resembles more closely to the flavor-limit of QCD. In this case, one looses the $\mathrm{U}(1)_{A_\lambda}$ to begin with and ABJ anomaly reduces $\mathrm{U}(1)_{A_\psi}$ down to $\mathbb{Z}_{2N_f}$. With the Ω_F^0-twisted boundary condition,

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the continuous global symmetry of the QCD(F) is:

$$\mathbf{G}_{\text{max-ab}} = \frac{\mathrm{U}(1)_L^{N_f-1} \times \mathrm{U}(1)_R^{N_f-1} \times \mathrm{U}(1)_V \times \mathbb{Z}_{2N_f}}{\mathbb{Z}_{N_c} \times (\mathbb{Z}_{N_f})_L \times (\mathbb{Z}_{N_f})_R \times (\mathbb{Z}_2)_\psi}. \tag{5.45}$$

The action of the axial chiral symmetry on fundamental fermion bilinears is given in (5.31). Again, the invariance of the monopole operator leads to intertwining of topological shift symmetry with the $[\mathrm{U}(1)_A]^{N_f-1}$ symmetry, and the gauge fluctuations acquire a chiral charge as described after (5.31).

In QCD(F/adj) as well as QCD(F) obtained by decoupling adjoint fermion (while keeping the CFC-symmetry intact), the chiral symmetry breaking occurs at weak coupling. To see this and its consequences, note that the proliferation of the monopoles induces the terms in the effective long-distance Lagrangian:

$$\mathcal{L}^m \supset -\sum_{i=1}^{N_f=N_c} e^{\alpha_i \cdot z}(\psi_{Ri}\psi_L^i) + \text{h.c.} \tag{5.46}$$

Similar to QCD(F/adj), $[\mathrm{U}(1)_A]^{N_c-1}$ intertwining with the topological shift symmetry forbids a mass generation for the dual photon field σ. (5.46) is responsible for the transmutation of the chiral charge of fermion bilinear into gauge fluctuations σ. The spontaneous symmetry breaking occurs by the condensation of the pure flux part of the monopole operator (5.35), and the NG-field is valued in the maximal torus $\mathbf{T}^{N_c-1}$. In this case, since there are only two fermion zero mode per monopole instanton, the chiral symmetry breaking yields a chirally non-invariant constituent mass for the microscopic quarks. Indeed, setting the vev for the flux part of the monopole operator (5.35), we find:

$$\mathcal{L}^m \sim -\sum_{i=1}^{N_f=N_c} e^{-S_0} e^{i\delta_i}(\psi_{L,i}\psi_{R,i}) + \text{h.c.} \tag{5.47}$$

All fermion species acquire a chirally non-invariant dynamical mass apart from the real mass which appears due to combined effect of gauge holonomy Ω and flavor holonomy Ω_F. On small $\mathbb{R}^3 \times S^1$, this provides a reliable mechanism of the chiral symmetry breaking and a mechanism to induce the constituent quark mass for fermions.

We can combine the pure flux part of the monopole operators into a matrix field, which can be interpreted as the *chiral field* of the chiral Lagrangian:

$$\Sigma(x) = \begin{bmatrix} e^{i\alpha_1 \cdot \sigma} & 0 & & \\ 0 & e^{i\alpha_2 \cdot \sigma} & & \\ & & \ddots & \\ & & & e^{i\alpha_{N_f} \cdot \sigma} \end{bmatrix} \tag{5.48}$$

and the effective field theory at arbitrary long distances takes the form:

$$S = \int_{\mathbb{R}^3 \times S^1} \frac{f_\pi^2}{4} \operatorname{tr} |\partial_\mu \Sigma|^2 \tag{5.49}$$

which is nothing but the chiral Lagrangian at small $\mathbb{R}^3 \times S^1$, and $\Sigma \in \mathbf{T}^{N_f-1}$, the maximal torus. Note that the effect of quarks is to render $\Sigma(x)$ chirally charged, by transmuting the chiral charge of the fermions into dual photons. In this sense, the effect of the fermions are present in EFT, but the fermions themselves acquire a real mass in the Ω, Ω_F background and do not appear in long distance EFT.

Turning on a soft chiral symmetry breaking mass for quarks, M_ψ in the matrix form, lifts the zero modes of the monopole-operators (5.16) and induce a mass gap for the system. The long distance EFT is described as

$$S = \int_{\mathbb{R}^3 \times S^1} \left[\frac{f_\pi^2}{4} \operatorname{tr} |\partial_\mu \Sigma|^2 - c \operatorname{tr} (M_\psi^\dagger \Sigma + \text{h.c.}) \right] \tag{5.50}$$

This is again nothing but chiral Lagrangian with a soft mass term for pion fields. At this stage, our original promise is fully realized and we have three comments.

- Recall that in $\mathcal{N}=1$ SYM and many other SQCD theories, one can get access to the ground state structure of the theory on $\mathbb{R}^4$ via supersymmetry preserving $\mathbb{R}^3 \times S^1$ compactifications through graded partition function $\operatorname{tr}(-1)^F e^{-\beta H}$ [35, 36, 73–75]. The merit of the graded partition function (supersymmetric index) is that the state sum is not contaminated by higher states even at arbitrarily small β thanks to Bose-Fermi symmetry. Supersymmetry guarantees the absence of phase transitions. In QCD(F/adj) with $m_\lambda \gg \Lambda$ for which the infrared physics is same as QCD(F), we showed that one can determine the ground state structure of the theory on $\mathbb{R}^4$ via $\mathbb{R}^3 \times S^1$ compactification through graded partition function $\operatorname{tr}\left[e^{-\beta H}(-1)^F e^{i\epsilon_0 Q_0} \prod_{a=1}^{N_f} e^{i\epsilon_a Q_a}\right]$. This provides a realization of the adiabatic continuity idea in QCD(F).[21]

- Despite the lack of supersymmetry in our QFT, we were able to generate powerful enough global symmetry induced cancellations in the graded state sum over the Hilbert space, such that Distill$[\mathcal{H}]$ does not seem to lead to any phase transition as β is dialed. More precisely, at small-β, the long distance EFT is the chiral Lagrangian (5.49), which is nothing but dimensional reduction of the QCD-chiral lagrangian described at large-β in the Ω_F^0 flavor holonomy background, providing a Hilbert space interpretation of our earlier result [33].

- We will prove in the next section that the mixed anomalies that control the ground state structure of the theory on $\mathbb{R}^4$ (6.19) and on $\mathbb{R}^3 \times S^1$ (with Ω_F^0 background) given in (6.24) are the same.

[21] Adiabatic continuity between weak coupling confined phase and strong coupling confined phase cannot currently be proven analytically. However, it can be shown numerically in some non-trivial QFTs. These include lattice simulations of $\mathcal{N}=1$ SYM [85] and deformed Yang-Mills theory [37]. In QCD(adj) with massive fermions with periodic boundary conditions, one can show the existence of the weak coupling confined phase at small $\mathbb{R}^3 \times S^1$ in QCD(adj) [86] with an intermediate deconfined regime as β is increased.

Fermion bi-linear condensate in statistical interpretation. At leading order in semi-classics, the vacuum is a dilute gas of N_c types of monopole-instantons each with complex fugacity

$$\zeta_i = e^{-S_0} e^{i\delta_i}, i = 1, \dots, N_c \tag{5.51}$$

The magnitude of the fugacity is the density of monopole-instanton events. In statistical interpretation, the vacuum expectation value $\langle \psi_{Ri} \psi_L^j \rangle$ is the average of the condensate over the space $V_{\mathbb{R}^3 \times S^1}$. This receives contributions from the lumps of monopole events. Therefore, the chiral condensate will be proportional to e^{-S_0}.

In the large m_λ limit while staying center-symmetric (3.21), the theory is analytic continuation of CFC symmetric QCD(F) to small-β regime. In this case, m_λ can be made large-enough so that it disappears from the renormalization group β-function. Adjoint fermion λ does not enter to the RG-flow of the coupling constant at energy scales smaller than $\frac{(g^2 N_c)^{1/2}}{\pi \beta N_c}$. However, λ still enters to the holonomy potential which is valid for $\beta\Lambda \lesssim 1$, and for the purpose of holonomy potential, m_λ is light provided $(\beta m_\lambda N_c) \lesssim 1$. This is a strange state of affairs described around (3.21), which exhibits decoupling from IR-physics and non-decoupling from holonomy potential.

There are two striking similarities between $N_f = N_c$ QCD(F), and $\mathcal{N} = 1$ SYM, which allows us to determine the chiral condensate in $N_f = N_c$ QCD(F). First, there is a kinematic accident that takes place between QCD(F) with $N_f = N_c$, and $\mathcal{N} = 1$ SYM concerning renormalization group β-function, explained below. Furthermore, in the Ω_F^0-background flavor holonomy, the fermionic zero mode structure of the monopole-instantons is identical in these two theories, see (5.13). Each monopole has exactly two-fermionic zero mode, $(\psi_{Ri} \psi_\beta^i)$ in QCD(F) and $(\alpha_i \cdot \lambda)^2$ in $\mathcal{N} = 1$ SYM.

Concerning the renormalization group β-functions, the universal two-loop β function coefficients are:

$$\begin{aligned} \text{QCD(F)}, \quad N_f = N_c: \qquad & \beta_0 = 3N_c, \qquad \beta_1 = 7N_c^2 \\ \mathcal{N} = 1 \text{ SYM}: \qquad & \beta_0 = 3N_c, \qquad \beta_1 = 6N_c^2 \end{aligned} \tag{5.52}$$

The two-loop strong scale of a gauge theory is given by $\Lambda = \mu e^{-\frac{8\pi^2}{g^2 \beta_0}} (g^2)^{-\frac{\beta_1}{2\beta_0^2}}$, which implies that

$$\begin{aligned} \text{QCD(F)}, \quad N_f = N_c: \qquad & \Lambda_{\text{QCD}} = \mu \left(\frac{16\pi^2}{3N_c g^2} \right)^{7/18} e^{-\frac{8\pi^2}{3g^2 N_c}} \\ \mathcal{N} = 1 \text{ SYM}: \qquad & \Lambda_{\text{SYM}} = \mu \left(\frac{16\pi^2}{3N_c g^2} \right)^{1/3} e^{-\frac{8\pi^2}{3g^2 N_c}} \end{aligned} \tag{5.53}$$

In other words, the one-loop strong scales of the two theories are precisely the same, but there is a small difference at two-loop order.

In the center-symmetric holonomy background, the flux-part of the monopole operator condense as (5.35), spontaneously breaking chiral symmetry to its vector-like subgroup: in the vacuum of the theory, we can evaluate the fermion bilinear condensate $\langle \psi_{Ri} \psi_L^j \rangle$ as follows. First, since the fermi zero mode structure of the monopole-zero modes is diagonal in flavor, and the zero modes can absorb the bilinear, the non-zero vev will be proportional to δ_i^j. The condensate will receive contributions from monopole-instanton lumps and hence, its value is proportional to the density of monopole-instantons e^{-S_0}. Therefore, the chiral condensate in QCD(F) with $N_f = N_c$ is given by

$$\langle \psi_L^i \psi_{Rj} \rangle \sim \delta_j^i \beta^{-3} e^{-S_0} e^{i\delta_i} = \delta_j^i \Lambda^3 e^{i\delta_i} \tag{5.54}$$

Similar to $\mathcal{N}=1$ SYM, we expect this quantity to agree between the small and large-β theory.

In other words, we claim that the mechanism of chiral symmetry breaking, and the way fermion bilinear condensate forms is same in $\mathcal{N}=1$ SYM and QCD(F) with Ω_F^0-twist. In fact, the density of monopole-instantons are also equal in the vacuum of these two theories on $\mathbb{R}^3 \times S^1$. Despite this almost identical nature of the two, the fluctuations around the respective condensates are quite different. The fluctuations in QCD(F) are described by gapless NG modes, and the fluctuations in $\mathcal{N}=1$ SYM are gapped. In other words, the magnetic bions in $\mathcal{N}=1$ do induce a mass gap for dual photon, while they do not generate any potential or mass gap for gauge fluctuations in QCD(F) as described around (5.34)

The maximal abelian subgroup (5.45) is an exact symmetry at any radius, and within the semi-classical description in the weak coupling regime, we proved that it is broken down to its vector-like subgroup.

$$[\mathrm{U}(1)_V \times \mathrm{U}(1)_A]^{N_f - 1} \times \mathrm{U}(1)_V \longrightarrow [\mathrm{U}(1)_V]^{N_f} \tag{5.55}$$

This is indeed the expected behavior at large-β that comes out from chiral Lagrangian in the background of the flavor holonomy Ω_F^0. Therefore, it is reasonable to expect that the weak coupling and strong coupling regimes are continuously connected in the sense of realization of global symmetries, i.e, chiral and CFC symmetries.

5.7 $\mathcal{N}=1$ SYM: what breaks chiral symmetry on $\mathbb{R}^3 \times S^1$?

$\mathbf{m}_\lambda = \mathbf{0}, \mathbf{m}_\psi \in (0, \infty]$: Once a mass term for fundamental fermion is added, the theory resembles more closely to the $\mathcal{N}=1$ SYM. In the decoupling limit $m_\psi \to \infty$, it reduces to the supersymmetric theory. In this case, one looses all the continuous axial chiral symmetries, and ABJ anomaly reduces $\mathrm{U}(1)_{A_\lambda}$ down to $\mathbb{Z}_{2N_c}$.

$$\mathbf{G} = \mathbb{Z}_{2N_c}, \tag{5.56}$$

This is also manifest in the instanton amplitude $\mathcal{I}_{4d} \sim e^{-\frac{8\pi^2}{g^2}} (\operatorname{tr} \lambda\lambda)^{N_c}$ which is only invariant $\mathbb{Z}_{2N_c}$ discrete subgroup of $\mathrm{U}(1)_{A_\lambda}$. In $\mathcal{N}=1$ SYM, on $\mathbb{R}^4$, it is believed that the discrete χS is broken down to $\mathbb{Z}_2$ by fermion bilinear condensate $\langle \operatorname{tr} \lambda\lambda \rangle$ and there are N_c vacua. Indeed, the analysis on $\mathbb{R}^3 \times S^1$ demonstrates analytically the discrete chiral symmetry breaking [35, 36]. Recently, similar techniques are also used to understand broader class of vector-like and chiral supersymmetric gauge theories on $\mathbb{R}^3 \times S^1$ [87–97].

Although the chiral symmetry breaking in $\mathcal{N}=1$ SYM is well-known in literature both on $\mathbb{R}^4$ as well as $\mathbb{R}^3 \times S^1$, there is a subtle issue that we would like to highlight concerning the mechanism of chiral symmetry breaking, that is not clear in earlier works.

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On $\mathbb{R}^3 \times S^1$, since each monopole operator ((5.16) third line) has two-fermi zero modes and $\mathbb{Z}_{2N_c}$ is anomaly free, there is again intertwining of the discrete chiral symmetry with a discrete subgroup of topological shift symmetry:

$$\mathbb{Z}_{2N_c}: \begin{cases} (\alpha_i \cdot \lambda)^2 \to e^{i\frac{2\pi k}{N_c}} (\alpha_i \cdot \lambda)^2, \\ e^{\alpha_i \cdot z} \quad\;\; \to e^{-i\frac{2\pi k}{N_c}} e^{\alpha_i \cdot z} \end{cases} \tag{5.57}$$

Therefore, on $\mathbb{R}^3 \times S^1$, as it is the case in QCD(F), QCD(F/adj), there are two-types of order parameters for the chiral symmetry: the magnetic flux part of monopole operator $e^{\alpha_i \cdot z}$ and fermion bilinear $\operatorname{tr} \lambda\lambda$.

Standard interpretation in literature is that χS is broken on small $\mathbb{R}^3 \times S^1$ due to chiral condensate $\operatorname{tr} \lambda\lambda$ acquiring a vev. This story is actually more subtle than often stated. On $\mathbb{R}^4$, $\operatorname{tr} \lambda\lambda$ can indeed acquire a vev and break chiral symmetry without breaking supersymmetry. This is because it is the lowest component of chiral multiplet,

$$\epsilon^{\alpha\beta} \operatorname{tr} W_\alpha W_\beta = \operatorname{tr} \lambda\lambda + \theta \dots \tag{5.58}$$

hence it cannot be expressed as

$$\operatorname{tr} \lambda\lambda \neq \{Q, \cdot\} \qquad \text{(microscopic theory)} \tag{5.59}$$

Therefore, $\operatorname{tr} \lambda\lambda$ can acquire a vev without clashing with supersymmetry [98].

On $\mathbb{R}^3 \times S^1$, Cartan components of λ_α are *not* lowest component of the supersymmetric multiplet in long-distance EFT, rather 3d $\mathcal{N}=2$ supersymmetric multiplets that enters to EFT can be written as:

$$Z = z + \theta\lambda + \dots, \tag{5.60}$$

where the lowest component of the multiplet is z, given in (5.5). However, in supersymmetric EFT with the scalar multiplet,

$$z \neq \{Q, \cdot\} \qquad \lambda = \{Q, z\} \qquad \text{EFT on } \mathbb{R}^3 \times S^1 \tag{5.61}$$

Therefore, in the EFT based on monopoles, it is in fact only z which can obtain a vacuum expectation value without breaking supersymmetry! The essence of this argument, without this particular application, is explained in ref. [98]. Hence, $e^{\alpha_i \cdot z}$ can acquire a vev without breaking supersymmetry, and indeed, this is what is dictated by the affine Toda superpotential (5.63), as described below. In the chirally broken vacuum

$$\langle \text{VAC}_k | e^{\alpha_i \cdot z} | \text{VAC}_k \rangle = e^{-\frac{8\pi^2}{g^2 N_c} + i\frac{\theta}{N_c}} e^{i\frac{2\pi k}{N_c}}, \qquad k = 1, \dots, N_c \tag{5.62}$$

and these are the super-selection sectors for EFT on $\mathbb{R}^3 \times S^1$.

At this level, discrete chiral symmetry is *already* broken by the condensation of magnetic flux operator (5.62). This is also the interpretation that is consistent with QCD(F) and QCD(F/adj) in the non-supersymmetric cases. We can also see the same effect by studying the EFT in the small $\mathbb{R}^3 \times S^1$ theory.

In $\mathcal{N}=1$ SYM theory, since each monopole-instanton has two fermi zero modes, it can induce a superpotential $\mathcal{W}$ [34–36, 73, 74], which can be expressed in terms of 3d $\mathcal{N}=2$ supersymmetric multiplet Z given in (5.60). The super-potential is given by

$$\mathcal{W}_{\mathbb{R}^3\times S^1}(Z) = \sum_{i=1}^{N_c-1} e^{\alpha_i\cdot Z} + e^{2\pi i\tau} e^{\alpha_N\cdot Z}, \tag{5.63}$$

where $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$ and $e^{2\pi i\tau} = e^{-\frac{8\pi^2}{g^2}+i\theta}$. The long-distance EFT on $\mathbb{R}^3\times S^1$ is given by:

$$\mathcal{L} = \frac{1}{2}|\partial_\mu z|^2 + i\bar{\lambda}\sigma_\mu\partial_\mu\lambda + \frac{1}{2}\sum_i \left|\frac{\partial W}{\partial z_i}\right|^2 + \sum_{i,j}\left(\frac{\partial^2 W}{\partial z_i \partial z_j}\lambda_i\lambda_j + \text{c.c.}\right) \tag{5.64}$$

The supersymmetric vacua of the theory are located at the minimum of the bosonic potential at which $\partial W/\partial z_i = 0$. There are N_c isolated minima, at which

$$\langle e^{\alpha_i\cdot z}\rangle = e^{-\frac{8\pi^2}{g^2 N_c}+i\frac{\theta}{N_c}} e^{i\frac{2\pi k}{N_c}}, \qquad k=1,\ldots,N_c \tag{5.65}$$

Again, EFT tells us that, at this stage, the discrete chiral symmetry is already broken because the magnetic flux part of the monopole operator acquires a vev. It is an independent question if the fermion bi-linear acquires a vev in these vacua or not. For example, in the QCD(F/adj) with $N_f = N_c$, four-fermi operator acquires a vev but not the fermion bilinears as described around (5.39). In $\mathcal{N}=1$ SYM, the fermion bilinear acquires a vev in a given vacuum

$$\langle \text{VAC}_k|\frac{1}{N_c}\,\text{tr}\,\lambda\lambda|\text{VAC}_k\rangle = \Lambda^3 e^{i\frac{2\pi k+\theta}{N_c}} \tag{5.66}$$

5.8 Why the quark condensate can form at weak coupling?

Two standard beliefs concerning how quark condensate forms in QCD are following: a) The vacuum of space must be populated by quark-anti-quark pairs. b) Quark condensate in QCD cannot occur in weakly coupled theory. See Tong's lectures on QFT for an up to date review of chiral symmetry breaking.[22] The relation between these beliefs and our analytic construction is as follows.[23]

The graded partition function demonstrates that the vacuum of the theory is indeed populated by quark-anti-quark pairs in the small $\mathbb{R}^3\times S^1$ regime, see figure 12 in the calculable regime. Yet, quark-antiquark pairs are all chirally charged, either under continuous or discrete (non-anomalous) chiral symmetry. An aspect that did not appear at all in the older literature in QCD is that the chiral charges of fermions can actually be absorbed by gauge fluctuations, i.e, magnetic flux part of monopole operators is capable to soak up the chiral charge, hence, the combination is chirally neutral. Therefore, gauge fluctuations also end up transforming under chiral symmetry. It is the condensation of the monopole-flux operators that induces chiral symmetry breaking mass terms in QCD as well as $\mathcal{N}=1$ SYM.

[22]http://www.damtp.cam.ac.uk/user/tong/gaugetheory/5chisb.pdf.

[23]Another analytic approach to chiral symmetry breaking appears in [99] by employing anomaly mediated supersymmetry breaking of SQCD.

The statement that the formation of quark condensate cannot occur at weak coupling is not correct, as we have seen in detail in this work. The standard (and incorrect) lore behind this is that the condensate itself is proportional to $\Lambda^3_{\rm QCD}$, where $\Lambda_{\rm QCD}$ is the strong length scale of QCD. Hence in QCD, chiral symmetry breaking must be a strong coupling phenomenon. and it is impossible to study the formation of chiral condensate at weak coupling.

On $\mathbb{R}^3 \times S^1$, at small-β, the theory is indeed weakly coupled per asymptotic freedom at the scale of compactification, where CFC symmetry is unbroken. Yet, the density of monopoles $\zeta_i = \mathcal{N}_i / V_{\mathbb{R}^3\times S^1}$ of type i, which sources chiral condensate is:

$$\zeta_i = \beta^{-3} e^{-S_0} \equiv \beta^{-3} e^{-\frac{8\pi^2}{(g^2(\beta) N_c)}} \tag{5.67}$$

which is non-perturbative due to appearance of $e^{-8\pi^2/(g^2 N_c)}$ factor. It is one of those lucky moments that one realizes

$$\beta^{-3} e^{-\frac{8\pi^2}{(g^2(\beta) N_c)}} = \Lambda^3_{\rm QCD} \tag{5.68}$$

using the one-loop dimensional transmutation. This is due to the fact that the one-loop beta function of $N_f = N_c$ QCD(F) is $\beta_0 = 3N$, just like $\mathcal{N} = 1$ SYM theory. This is nothing but the non-perturbative density of quark-anti-quark pairs in vacuum! In other words, we learn that the quark condensate of order $\Lambda^3_{\rm QCD}$ can arise from weak coupling semi-classical construction naturally.

6 Persistent mixed anomalies vs. quantum distillations

The idea of quantum distillation of Hilbert space, and corresponding path integrals in the flavor holonomy background is related in interesting ways to mixed persistent anomalies (when the latter exists). However, they are certainly not the same concepts. Roughly, quantum distillation describe (mostly imperfect) cancellations within a given superselection sector and persistent mixed anomalies are tied with exact cancellations among superselection sectors.

It is recently understood that mixed anomalies involving discrete 1-form symmetries continues to persist upon compactification [39, 40, 100]. Even if the theory does not have a 1-form symmetry, as it is the case in QCD(F/adj) and QCD(F), it is possible to prove the following remarkable result: [42–44, 101, 102].

> **Persistent mixed anomaly in the absence of 1-form symmetry:** assume the theory has only 0-form symmetry (but not a 1-form symmetry) $G_1 \times G_2$ where $G_1 = \widetilde{G}_1/\Gamma$ and gauging $\widetilde{G}_1$ turns Γ into a 1-form symmetry. Then, a triple-mixed anomaly that is present on $\mathbb{R}^4$ persists upon compactification on $\mathbb{R}^3 \times S^1$, and imposes equally powerful constraints on long distance physics of compactified theory.

Below, we will take advantage of this result.

6.1 Mixed anomalies in QCD(F) and SU(N_c) $\times$ SU(N_f) quiver theory on $\mathbb{R}^4$

Consider turning on mass for adjoint fermion $m_\lambda > 0$ in QCD(F/adj). If desired, the mass can be taken large and the theory reduces to QCD(F). The faithful vector-like symmetry

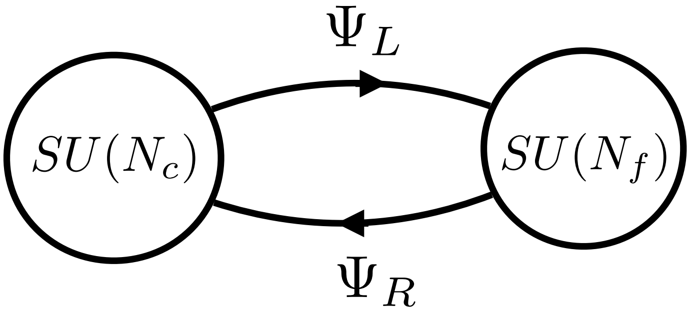

Figure 13. Upon gauging $\mathrm{SU}(N_f)_V$, QCD(F) becomes a two site quiver theory (QCD(BF)) with 1-Dirac fermion, $\mathbb{Z}_{2\mathrm{gcd}(N_f,N_c)}$ chiral and $\mathbb{Z}^{[1]}_{\mathrm{gcd}(N_f,N_c)}$ 1-form symmetry. We must gauge $\mathbb{Z}^{[1]}_{\mathrm{gcd}(N_f,N_c)}$ 1-form symmetry in order to turn on a background or gauge the faithful symmetry of the microscopic theory. There are intimate relations between the anomalies of the original multi-flavor QCD(F) theory and auxiliary QCD(BF) theory.

of the theory has a subgroup:

$$\mathrm{SU}(N_f)_V/\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)} \tag{6.1}$$

where $\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)}$ part of the $\mathrm{SU}(N_f)_V$ is a gauge redundancy and is not a genuine symmetry. Therefore, it is modded out. Under an $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)_V$ gauge $\times$ global transformation, the quarks transform as

$$\psi \to g_c(x)\psi g_f^\dagger \tag{6.2}$$

and hence, a global flavor transformation residing in $\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)} \subset \mathbb{Z}_{N_f} \subset \mathrm{SU}(N_f)$ can be undone by a gauge transformation $\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)} \subset \mathbb{Z}_{N_c} \subset \mathrm{SU}(N_c)$. We will use this symmetry to describe a mixed anomaly of the theory on $\mathbb{R}^4$ and $\mathbb{R}^3 \times S^1$ following and slightly generalizing [42].

Recall that QCD(F) has a $\mathbb{Z}_{2N_f}$ chiral symmetry, as can be seen from the non-invariance of the fermion integration measure. Under a $\mathrm{U}(1)_A$ action, the measure transforms as:

$$d\mu^{\mathrm{QCD(F)}}_{\mathrm{fermion}} \to e^{i\alpha 2N_f \times \frac{1}{8\pi^2}\int \mathrm{tr}\,(F\wedge F)}\, d\mu^{\mathrm{QCD(F)}}_{\mathrm{fermion}} \tag{6.3}$$

Therefore, this is a symmetry only when $\alpha = \frac{2\pi}{2N_F}k$ and $\mathrm{U}(1)_A$ axial is explicitly broken down to $\mathbb{Z}_{2N_f}$. The $\mathbb{Z}_{N_f}$ subgroup of $\mathbb{Z}_{2N_f}$ is actually a part of the continuous chiral symmetry $\mathbf{G}$. Consider $\psi_{Ra}\psi_L^b$ which rotates under a $\mathbb{Z}_{2N_f}$ discrete chiral rotation into $e^{2\pi ik/N_f}\psi_{Ra}\psi_L^b$. The same discrete transformation can be achieved with a continuous rotation by an $\mathrm{SU}(N_f)_L$ matrix or a transformation in its maximal torus $(\mathrm{U}(1)_L)^{N_f-1}$: $U_L(\delta) = \mathrm{diag}\left(e^{i\delta}, \ldots, e^{i\delta}, e^{-i(N_f-1)\delta}\right)$, by continuously varying $\delta \in [0, 2\pi ik/N_f]$. Therefore, the $\mathbb{Z}_{2N_f}$ transformation is part of a continuous symmetry $\mathrm{SU}(N_f)_A$ or its maximal torus, i.e.,

$$\mathbb{Z}_{N_f} \subset \mathbf{G}_{\mathrm{max-ab}} \subset \mathbf{G}_{\mathrm{non-ab}} = \mathrm{SU}(N_f)_A \tag{6.4}$$

Therefore, in QCD(F), any order parameter that is charged under discrete chiral symmetry (which is not an independent symmetry) will automatically be charged under continuous chiral symmetry, and the breaking of the former implies the breaking of the latter.[24]

Below, we first describe the mixed anomaly between $\mathrm{SU}(N_f)_V/\mathbb{Z}_{\gcd(N_f,N_c)}$ and $\mathbb{Z}_{2N_f}$ on $\mathbb{R}^4$ generalizing ref. [42] to arbitrary (N_f, N_c) for which $\gcd(N_f, N_c)$ is non-trivial and then, find the conditions under which the mixed anomaly persists under compactification.

Quiver theory. First, gauge $\mathrm{SU}(N_f)_V$ flavor symmetry. We obtain a two-site quiver gauge theory, in which N_f fundamental Dirac fermions ψ_a turn into *one* bi-fundamental Ψ which transform under $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)$ gauge transformation as

$$\Psi \to g_c(x)\Psi g_f^\dagger(x) \tag{6.5}$$

This is a local gauged version of (6.2). The covariant derivative takes the form

$$D_\mu(a, A)\Psi = \partial_\mu\Psi + ia_\mu\Psi - i\Psi A_\mu \qquad a_\mu \in \mathrm{SU}(N_c),\ \ A_\mu \in \mathrm{SU}(N_f) \tag{6.6}$$

We call this theory with product gauge group structure the quiver theory. Such non-supersymmetric quiver theories are examined in the context of large-N_c orbifold equivalence as well as mixed anomalies, see e.g. [56, 101, 103–105].

The $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)$ quiver gauge theory has an *exact* $\mathbb{Z}^{[1]}_{\gcd(N_f,N_c)}$ 1-form center-symmetry and $\mathbb{Z}_{2\gcd(N_f,N_c)}$ 0-form chiral symmetry. The gauge redundancy that has been modded out in (6.1) became an exact 1-form symmetry upon the gauging of $\mathrm{SU}(N_f)_V$. Needless to say, this is also exactly the color-flavor-center (CFC) symmetry (2.33), (2.37) which is promoted to a full 1-form center-symmetry in the quiver theory. The 0-form symmetry $\mathbb{Z}_{2\gcd(N_f,N_c)}$ lives in the $\mathbb{Z}_{2N_f}$ axial symmetry of QCD(F) and we must gauge 1-form center-symmetry to determine the anomaly structure.

Let us first determine how the $\mathbb{Z}_{2\gcd(N_f,N_c)}$ chiral symmetry arises in the 2-site quiver theory. Since one-Dirac fermion is in bi-fundamental of $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)$, the non-invariance of the fermion integration measure can now be phrased as:

$$d\mu^{\text{quiver}}_{\text{fermion}} \to \exp\left\{ i\Upsilon\left[(2N_f)\times\frac{1}{8\pi^2}\int \mathrm{tr}\,(F_1\wedge F_1) + (2N_c)\times\frac{1}{8\pi^2}\int \mathrm{tr}\,(F_2\wedge F_2)\right]\right\} d\mu^{\text{quiver}}_{\text{fermion}} \tag{6.7}$$

where F_1, F_2 are gauge field strengths for $\mathrm{SU}(N_c)\times\mathrm{SU}(N_f)$. The measure can be rewritten as:

$$d\mu^{\text{quiver}}_{\text{fermion}} \to \exp\left\{ i2\Upsilon\left[N_f m_1 + N_c m_2\right]\right\}\ d\mu^{\text{quiver}}_{\text{fermion}} \tag{6.8}$$

[24]Note that reverse statement is not true. Any order parameter that is charged under the continuous chiral symmetry is not necessarily charged under discrete chiral symmetry. An example is four fermi operator that appears in the discussion of Stern phases [49]. More precisely, if a Euclidean QFT on $\mathbb{R}^d$ is compactified to $\mathbb{R}^{d-1} \times S^1$ where $d-1 > 2$, then, discrete chiral symmetry (which is a subgroup of continuous chiral symmetry) breaking implies continuous chiral symmetry breaking. However, in a theory compactified to 1+0 QFT (quantum mechanics) or 1+1 QFT, since it is not possible to break continuous global symmetries in $d \leq 2$ dimensions, the theory may have multiple minima due to mixed anomalies involving discrete chiral symmetry without breaking continuous global symmetry.

where $m_1, m_2 \in \mathbb{Z}$ are topological charges in the corresponding gauge group factors. To find the discrete remnant of $\mathrm{U}(1)_A$ chiral symmetry, use Bezout identity in elementary number theory which states that there exists $m_1, m_2 \in \mathbb{Z}$ such that

$$N_f m_1 + N_c m_2 = \gcd(N_f, N_c) \tag{6.9}$$

Therefore,

$$d\mu_{\text{fermion}}^{\text{quiver}} \to \exp\left\{i2\Upsilon \gcd(N_f, N_c)\right\}\, d\mu_{\text{fermion}}^{\text{quiver}} \tag{6.10}$$

and this is a symmetry only when

$$\Upsilon = \frac{2\pi}{2\gcd(N_f, N_c)} k \tag{6.11}$$

and the discrete chiral symmetry of the auxiliary quiver theory is $\mathbb{Z}_{2\gcd(N_f,N_c)}$.

To summarize, in quiver theory, we have the global symmetries:

$$\text{quiver theory symmetries}: \begin{cases} \frac{\mathrm{U}(1)_V \times \mathbb{Z}_{2\gcd(N_f,N_c)}}{\mathbb{Z}_2}, & 0-\text{form} \\ \\ \mathbb{Z}^{[1]}_{\gcd(N_f,N_c)}, & 1-\text{form, center} \end{cases} \tag{6.12}$$

Gauging center. Now, we can gauge $\mathbb{Z}^{[1]}_{\gcd(N_f,N_c)}$ 1-form center on $\mathbb{R}^4$ and describe, under what conditions we loose parts of $\mathbb{Z}_{2\gcd(N_f,N_c)}$ 0-form chiral symmetry. It is also possible to proceed oppositely, gauge $\mathbb{Z}_{2\gcd(N_f,N_c)}$ 0-form symmetry and show that we loose the 1-form symmetry partially. Gauging $\mathbb{Z}_{\gcd(N_f,N_c)}$ amounts to introducing a (B, C) pair where B is a 2-form U(1) gauge field, C is a 1-form gauge field, and the pair must satisfy [39, 106]

$$\gcd(N_f, N_c) B + dC = 0. \tag{6.13}$$

The constraint obeys the gauge invariance,

$$B \to B - d\lambda, \qquad C \to C + \gcd(N_f, N_c)\lambda \tag{6.14}$$

where λ is a 1-form gauge transformation. We now define the $\mathrm{U}(N_c)$ and $\mathrm{U}(N_f)$ gauge fields by using a common U(1) 1-form gauge field C. Express

$$\begin{aligned} \widetilde{a} &= a + \frac{1}{\gcd(N_f, N_c)} C \mathbf{1}_{N_c}, \\ \widetilde{A} &= A + \frac{1}{\gcd(N_f, N_c)} C \mathbf{1}_{N_f}, \end{aligned} \tag{6.15}$$

along with their respective gauge field strengths

$$F'_c = d\widetilde{a} + i\widetilde{a} \wedge \widetilde{a}, \qquad F'_f = d\widetilde{A} + i\widetilde{A} \wedge \widetilde{A} \tag{6.16}$$

Clearly, F'_c and F'_f are not gauge invariant under the 1-form gauge transformation because of their common C components, $F'_c \to F'_c + d\lambda$ and $F'_f \to F'_f + d\lambda$. But the combinations

$$F'_c + B\mathbf{1}_{N_c}, \qquad F'_f + B\mathbf{1}_{N_f} \tag{6.17}$$

are gauge-invariant under one-form gauge transformations (6.14).

Under an $h \in \mathbb{Z}_{2\text{gcd}(N_f,N_c)}$ discrete chiral transformation, (which used to be an invariance of the theory before the $\mathbb{Z}^{[1]}_{\text{gcd}(N_f,N_c)}$ 1-form center-symmetry was gauged), using (6.7) and (6.11), one observes that the fermion measure under the background 2-form field B fields transforms into

$$\begin{aligned} d\mu^{\text{quiver}}_{\text{fermion}} \to \exp i \Bigg[&\frac{(2N_f)}{2\text{gcd}(N_f,N_c)} \frac{1}{4\pi} \int \text{tr}\left((F'_c + B\mathbf{1}_{N_c}) \wedge (F'_c + B\mathbf{1}_{N_c})\right) \\ &+ \frac{(2N_c)}{2\text{gcd}(N_f,N_c)} \frac{1}{4\pi} \int \text{tr}\left((F'_f + B\mathbf{1}_{N_f}) \wedge (F'_f + B\mathbf{1}_{N_f})\right) \Bigg] d\mu^{\text{quiver}}_{\text{fermion}} \end{aligned} \tag{6.18}$$

Therefore, the non-invariance of partition function in the (A, B) background under a chiral transformation can be expressed as

$$\begin{aligned} \mathbb{Z}_{2\text{gcd}(N_f,N_c)}: \;\; \mathcal{Z}(A,B)) \mapsto &\exp\left[-i \frac{(4N_f N_c)}{2\text{gcd}(N_f,N_c)} \frac{1}{4\pi} \int B \wedge B\right] \mathcal{Z}((A,B)) \\ &= \exp\left[-i \frac{2\text{lcm}(N_f,N_c)}{4\pi} \int B \wedge B\right] \mathcal{Z}((A,B)) \\ &= \exp\left[-i2\pi \frac{2\text{lcm}(N_f,N_c)}{(\text{gcd}(N_f,N_c))^2}\right] \mathcal{Z}((A,B)) \end{aligned} \tag{6.19}$$

We used $\int B \wedge B = \left(\frac{1}{\text{gcd}(N_f,N_c)}\right)^2 \int dC \wedge dC = 8\pi^2 \left(\frac{1}{\text{gcd}(N_f,N_c)}\right)^2$ in the last step. The phase is non-trivial (mod 2π) provided

$$\frac{2\text{lcm}(N_f,N_c)}{(\text{gcd}(N_f,N_c))^2} \in \mathbb{Q}\backslash\mathbb{Z}, \tag{6.20}$$

For example, for $N_f = N_c$, there is a mixed anomaly provided $\frac{2}{N_c} \in \mathbb{Q}\backslash\mathbb{Z}$. This agrees with the existence of mixed anomaly for all $N_c = N_f \geq 3$ theories [42].[25]

Assuming (6.20) is satisfied and anomaly exists (clearly, there are many such cases), it has implications for two related class of theories:

- **SU(N_c) × SU(N_f) quiver theory:** there is a mixed anomaly between $\mathbb{Z}^{[1]}_{\text{gcd}(N_f,N_c)}$ 1-form center-symmetry and $\mathbb{Z}_{2\text{gcd}(N_f,N_c)}$ 0-form chiral-symmetry.
- **SU(N_c) QCD(F) with N_f flavors of massless fermions:** there is a mixed anomaly between $\text{SU}(N_f)/\mathbb{Z}_{\text{gcd}(N_f,N_c)}$ symmetry and $\mathbb{Z}_{2N_f}$ chiral symmetry.

The anomaly polynomial corresponding to both cases is given by (6.19). This implies that a unique gapped (trivial) ground state in SU(N_c) QCD(F) and SU(N_c) × SU(N_f) quiver theory are impossible. We list the possibilities for ground states after discussing the conditions under which this anomaly persists upon compactification.

[25]By turning on an extra $\text{U}(1)_V$ background, it is also possible to make the non-invariance of the action $\mathcal{Z}(A_1, A_2, B) \mapsto \exp\left[-i\frac{2\pi}{N_c}\right] \mathcal{Z}((A_1, A_2.B))$ in the $N_f = N_c$ case. Then, an anomaly will also persist in the $N_c = 2$ theory, and also impose stronger constraint on the $N_c = N_f \geq 3$ theories. The general idea of turning on all possible backgrounds to obtain a stronger constraint on IR-physics is discussed in detail [107].

6.2 Conditions for persistence of 0-form mixed anomalies on $\mathbb{R}^3 \times S^1$

A mixed anomaly between a 1-form symmetry and 0-form symmetry persists upon compactification, e.g. $\mathbb{R}^3 \times S^1$, see [39] for center and time reversal mixed anomaly at $\theta = \pi$ and center and discrete chiral in QCD(adj) in [41, 43]. This concept is sometimes called persistent order.

As we stated at the beginning of this section, if the theory does not possess a 1-form symmetry, then, a mixed anomaly involving two 0-form symmetries $G_1 \times G_2$ does not impose a constraint on IR-physics in an obvious way. However, if the 0-form symmetries are of the form $G_1 \times G_2$, where $G_1 = \widetilde{G}_1/\Gamma$ and gauging $\widetilde{G}_1$ turns Γ into a 1-form symmetry, then a triple-mixed anomaly may persist upon compactification [42].

Let us now investigate the condition under which the anomaly persists. This will bring $\mathrm{SU}(N_f)_V$ flavor twist that we called Ω_F^0 as the hero of the story.

Consider the partition function in the Ω_F^0 background $\mathcal{Z}_{\Omega_F^0}$. This is equivalent to imposing Ω_F^0 twisted boundary conditions on fermions (2.30) as described in section 2.3. Under an aperiodic center-transformation, $\Omega_F^0 \mapsto \omega\Omega_F^0$ and partition function maps to $\mathcal{Z}_{\omega\Omega_F^0}$, hence $\mathcal{Z}_{\Omega_F^0}$ is not invariant. However, the partition function can be made invariant by invoking a transformation, $S \in \Gamma_S \subset \mathrm{SU}(N_f)_V$, provided it obeys

$$S\Omega_F^0 S^{-1} = \omega\Omega_F^0 \tag{6.21}$$

This is the symmetry of $\mathcal{Z}_{\Omega_F^0}$. The solution to this algebra is unique up to conjugations. As described around (2.38), the choice of Ω_F^0 introduces a 0-form color-flavor center (CFC)-symmetry under which Polyakov loop is charged. But at the same time, it explicitly breaks the flavor symmetry down to maximal Abelian subgroup $\boldsymbol{G}_{\text{max}-\text{ab}}$. Crucially, the faithful vector-like flavor symmetry of compactified theory becomes

$$K = \widetilde{K}/\Gamma \equiv \mathrm{U}(1)^{N_f-1}/\mathbb{Z}_{\gcd(N_f,N_c)}. \tag{6.22}$$

Introducing the background gauge field for $K = \mathrm{U}(1)^{N_f-1}$, $\mathbb{Z}_{\gcd(N_f,N_c)}$ emerges as a 1-form symmetry. We also introduce a 2-form field $B^{(2)}$ and 1-form field associated with 1-form and 0-form part of center-symmetry $\mathbb{Z}_{\gcd(N_f,N_c)}$, and decompose the 2-form field B on

$$B = B^{(2)} + B^{(1)} \wedge \beta^{-1} dx^4 \tag{6.23}$$

The partition function in the $(A_K, B^{(2)}, B^{(1)})$ background is not invariant under a discrete chiral transformation $h \in \mathbb{Z}_{2\gcd(N_f,N_c)}$ and the anomaly polynomial on $\mathbb{R}^3 \times S^1$ can be obtained as:

$$\begin{aligned} \mathcal{Z}_{\Omega_F^0}(h(A_K, B^{(2)}, B^{(1)})) &= \exp\left[-i\frac{2\mathrm{lcm}(N_f,N_c)}{2\pi}\int B^{(2)} \wedge B^{(1)}\right]\mathcal{Z}_{\Omega_F^0}(A_K, B^{(2)}, B^{(1)}) \\ &= \exp\left[-i2\pi\frac{2\mathrm{lcm}(N_f,N_c)}{(\gcd(N_f,N_c))^2}\right]\mathcal{Z}_{\Omega_F^0}(A_K, B^{(2)}, B^{(1)}) \end{aligned} \tag{6.24}$$

The anomaly polynomial can also be deduced from (6.19) with the substitution (6.23) Therefore, there is a triple mixed anomaly between shift symmetry $\Gamma_S \subset \mathrm{SU}(N_f)_V$, abelianized flavor symmetry $\mathrm{U}(1)^{N_f-1}/\mathbb{Z}_{\gcd(N_f,N_c)}$, and the discrete chiral symmetry $\mathbb{Z}_{2N_f}$ provided (6.20) holds. This is indeed the same condition as in $\mathbb{R}^4$.

6.3 Implication of mixed anomalies on $\mathbb{R}^4$ and $\mathbb{R}^3 \times S^1$

The mixed anomaly on $\mathbb{R}^4$ is between $\mathrm{SU}(N_f)_V/\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)}$ symmetry and $\mathbb{Z}_{2N_f}$. We make two remarks. **1)** There exists no order parameter which is charged under the discrete χS $\mathbb{Z}_{2N_f}$ but not under continuous χS.[26] Therefore, the spontaneous breaking of the discrete symmetry implies spontaneous breaking of $\mathrm{SU}(N_f)_A$ [44]. **2)** By a theorem in ref. [108], in vector-like theories, vector-like global symmetries cannot be spontaneously broken as long as one assures positivity of the path integral measure (which is the case in our theory with Ω_F^0 twist.)

The existence of the mixed anomaly implies that the ground state of QCD(F) on $\mathbb{R}^4$ cannot be unique, gapped (trivial) state. In the light of above statements, there are two options on $\mathbb{R}^4$:

- $\mathrm{SU}(N_f)_A$ chiral symmetry is spontaneously broken and there are massless NG-bosons.
- Low energy theory is a CFT.

On $\mathbb{R}^3 \times S^1$, the triple mixed anomaly is between shift symmetry $\Gamma_S \subset \mathrm{SU}(N_f)_V$, abelianized flavor symmetry $\mathrm{U}(1)_V^{N_f-1}/\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)}$, and the discrete χS $\mathbb{Z}_{2N_f} \subset \mathrm{U}(1)_A^{N_f-1}$. The options that can saturate the anomaly are:

- $\mathrm{U}(1)_A^{N_f-1}$ chiral symmetry is spontaneously broken and there are NG-bosons.
- The color-flavor center (CFC) symmetry $\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)}$ is spontaneously broken.
- Both are spontaneously broken.
- Low energy theory is a CFT.

By using rigorous semi-classics on small $\mathbb{R}^3 \times S^1$ and provided $m_\lambda < m_\lambda^*$, we showed that the first option, spontaneous breaking of $\mathrm{U}(1)_A^{N_f-1}$ chiral symmetry is realized on small circle. At large-S^1, we also expect to see spontaneous breaking of chiral symmetry. However, anomaly allows other mixed anomaly respecting intermediate phases or anomaly respecting phase transitions which is not associated with the change in symmetry realization.[27] This is a realization of the persistent order idea in QCD. Indeed, in the next section, we will show that for $m_\lambda > m_\lambda^*$, there exists intermediate phases in which only CFC symmetry is spontaneously broken.

For $0 < m_\lambda < m_\lambda^*$, there is strong reasons to believe that the theory exhibits adiabatic continuity in the (β, m_λ) plane, i.e, small-circle $\mathbf{G}_{\mathrm{max-ab}}$ broken phase is continuously connected to the strong coupling $\mathbf{G}_{\mathrm{max-ab}}$ broken phase. In this sense, adiabatic continuity

[26] Opposite statement is not true. See Footnote 24.

[27] As discussed around (5.40), for $m_\lambda = 0$ theory, $\mathrm{U}(1)_A^{N-1}$ and CFC realization in small and large circle are certainly the same, but $\mathrm{U}(1)_{A_D}$ realization may change. Let us assume that there exists a phase transition associated with $\mathrm{U}(1)_{A_D}$, and it is broken at large-β. Turning on a soft $m_\lambda > 0$ breaks $\mathrm{U}(1)_{A_D}$ symmetry explicitly. If the transition at $m_\lambda = 0$ is first order, it will continue to persists even for small $m_\lambda > 0$, but the transition will not be associated with a change in symmetry realization. Furthermore, the first order line may end up with a second order critical point.

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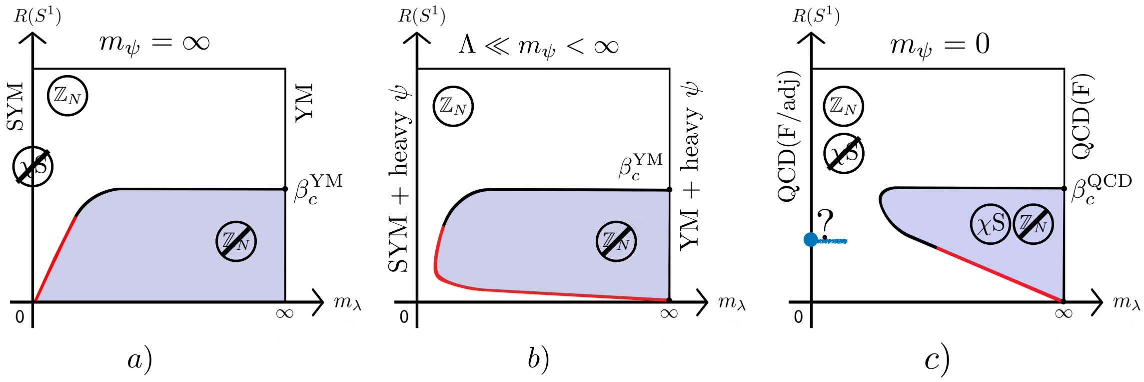

Figure 14. Analytically calculable (red) and incalculable (black) $\mathbb{Z}_N$ center or CFC symmetry changing phase transitions in gauge theories on $\mathbb{R}^3 \times S^1$ for various values of m_ψ on (β, m_λ) plane. a) $\mathbb{Z}_{N_c}$ center-symmetry changing phase transition in mass deformed SYM. This is studied in detail in [55] and presented here for completeness. b) For heavy fundamental quark with Ω_F^0-twist, the $\mathbb{Z}_{N_c}$ CFC symmetry is stabilized for any $m_\lambda < m_\lambda^*$. This theory exhibits adiabatic continuity for Yang-Mills theory (with very heavy adjoint and fundamental fermions) between small and large S^1 regimes. c) This theory also exhibits adiabatic continuity for QCD(F) (with very heavy adjoint fermions) between small and large S^1 regimes. See text concerning the blue line.

is a refined version of the persistent order or mixed 't Hooft anomaly in which only one mode of the anomaly constraint is realized at arbitrary β. We believe providing a proof of this statement would be a tremendous progress in the understanding of strongly coupled QCD.

7 Mixed anomaly permitted phase transitions on $\mathbb{R}^3 \times S^1$

As described in section 6.3, the theory can only be in a phase that is compliant with mixed anomaly. Phase transitions are possible, but different phases must be a realization of a mode of the anomaly constraint. This is called persistent order. The ground state is never a trivial gapped phase.

Throughout this section, we consider the phases of $N_f = N_c$ QCD(F/adj) as a function of the parameters $(m_\lambda, m_\psi, \beta)$. For $m_\psi = \infty$, $\Lambda \ll m_\psi < \infty$, $m_\psi = 0$, we determine phase diagram in the (m_λ, β) plane. There are both calculable and incalculable phase transitions. The main outcome is shown in figure 14 which we discuss below.

7.1 Phases of $m_\psi = 0$ theory in the (m_λ, β) plane

Turning on a soft mass term for adjoint fermion, the balance between the center-destabilizing gauge fluctuations and center-stabilizing adjoint fermion breaks down. Hence, at one-loop order, a center-destabilizing one-loop potential do get induced [54, 109]

$$
\begin{aligned}
V^{\text{gauge}}_{1-\text{loop}} + V^{\lambda}_{1-\text{loop}} &= \frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\frac{\left[-1+\frac{1}{2}(n\beta m_\lambda)^2 K_2(n\beta m_\lambda)\right]}{n^4}|\operatorname{tr}(\Omega^n)|^2 \\
&= -\frac{m_\lambda^2}{2\pi^2\beta^2}\sum_{n=1}^{\infty}\frac{1}{n^2}|\operatorname{tr}(\Omega^n)|^2 + O(m_\lambda^4)
\end{aligned}
\tag{7.1}
$$

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where K_2 is modified Bessel function. However, this potential is parametrically small compared to generic one-loop potentials by a factor $(m_\lambda \beta)^2$. At two-loop order, there exists a center-stabilizing term due to fundamental fermions (with Ω_F^0-boundary conditions) given in (3.18). Despite the fact that the two-loop potential is suppressed compared to *generic* one-loop terms with a parameter $(g^2 N_c)$, it can compete with (7.1) which is already parametrically suppressed.

Non-commutativity of limits-1. For sufficiently small $(m_\lambda \beta)^2 \ll g^2 N_c$, the center-stabilizing two loop potential dominates. For sufficiently small $g^2 N_c \ll (m_\lambda \beta)^2$, the center-destabilizing one-loop potential dominates. Therefore, the two limits are non-commuting and we obtain:

$$\begin{Bmatrix} m_\lambda^{-1} \to 0 & \beta = \text{fixed} & \mathbb{Z}_{N_c} \ \text{broken} \\ \beta \to 0, & m_\lambda^{-1} = \text{fixed} & \mathbb{Z}_{N_c} \ \text{symmetric} \end{Bmatrix} \tag{7.2}$$

In the weak coupling regime, this implies that the center-symmetry changing phase transition can be studied analytically along a line emanating from $(m_\lambda^{-1}, \beta) = (0,0)$ point, shown in figure 14c.

In this regime, center-symmetry changing phase transition is due to a competition between a parametrically suppressed one-loop potential (with parameter $m_\lambda \beta$) and two-loop potential. The two contribution become parametrically comparable for

$$\beta_c \sim \frac{(g^2 N_c)^{1/2}}{m_\lambda} \tag{7.3}$$

where a phase transition is expected. This will be further quantified below.

As described earlier, the combination of the one- and two-loop potentials for fundamental fermions ψ^a with twisted boundary condition has both single-trace terms of the form $\operatorname{tr} \Omega^{N_c k}$ which do not play important role in center-symmetry realization and double trace terms which prefer center-symmetric vacuum, see (3.18). The combined potential which determines the center-symmetry realization is:

$$\begin{aligned} V^{\text{gauge}}_{1-\text{loop}} + V^{\lambda}_{1-\text{loop}} + V^{\psi}_{2-\text{loop},\Omega^0_{\text{F}}} &= \frac{2}{\pi^2 \beta^4} \sum_{n=1}^{\infty} \frac{\overbrace{\left[-1 + \frac{1}{2}(n\beta m_\lambda)^2 K_2(n\beta m_\lambda) + \frac{g^2 N_c}{16\pi^2}\right]}^{a_n}}{n^4} |\operatorname{tr}(\Omega^n)|^2 \\ &\approx \frac{2}{\pi^2 \beta^4} \sum_{n=1}^{\infty} \frac{\left[-\frac{1}{4}(n\beta m_\lambda)^2 + \frac{g^2 N_c}{16\pi^2}\right]}{n^4} |\operatorname{tr}(\Omega^n)|^2 \end{aligned} \tag{7.4}$$

where in the second formula, we used small-z asymptotic of the Bessel function $K_2(z)$. a_n is the effective mass square m_n^2 for the winding number n Polyakov loop $\operatorname{tr} \Omega^n$. If the effective mass is positive for all $n \leq \lfloor \frac{N_c}{2} \rfloor$, the minimum of the effective potential lies at a center-symmetric point, invariant under $\mathbb{Z}_{N_c}$ transformation:

$$\mathbb{Z}_{N_c} \ \text{stability} : a_1 > 0, \ a_2 > 0, \ \ldots, a_{N_c/2} > 0 :\Longrightarrow \qquad (\beta m_\lambda) < \frac{(g^2 N_c)^{1/2}}{N_c \pi} \tag{7.5}$$

Since m_n^2 is monotonically decreasing function of its argument, if a_1 is negative, then so are $a_n, n \geq 2$. In this case, $\mathbb{Z}_{N_c}$ is completely broken:

$$\mathbb{Z}_{N_c} \text{ fully broken} : a_1 < 0 :\Longrightarrow \qquad (\beta m_\lambda) > \frac{(g^2 N_c)^{1/2}}{2\pi} \tag{7.6}$$

If N_c is large, there exists intermediate ranges where center symmetry is partially broken. The instability point for the Wilson line with winding number k is, $a_k = 0$, corresponding to

$$\beta_k = \frac{(g^2 N_c)^{1/2}}{2\pi m_\lambda} \frac{1}{k} \equiv \beta_1 \frac{1}{k} \tag{7.7}$$

When the compactification radius lies between $\beta \in [\beta_k, \beta_{k-1}]$, the partial center symmetry that is preserved is $\mathbb{Z}_k$, corresponding to a configuration of eigenvalues in which eigenvalues form k-clump each of which possess N_c/k coincident eigenvalues.

So, in the range $\beta < \frac{(g^2 N_c)^{1/2}}{N_c \pi m_\lambda}$, $\mathbb{Z}_{N_c}$ is fully restored. This range shrinks to zero as N_c increases. We also expect that for $\beta \gtrsim c\Lambda^{-1}$ where c is a pure number, the center to be fully stabilized. This is shown in figure 14c but intermediate phases with partial center breaking are not detailed in there.

$$\left\{ \begin{array}{lll} \beta < \beta_{N_c/2} & \mathbb{Z}_{N_c} \text{ symmetric} \\ \beta_k < \beta < \beta_{k-1} & (\approx \text{or exactly}) \;\; \mathbb{Z}_k \text{ symmetric} \\ \beta_1 < \beta < \beta^* = c\Lambda^{-1} & \mathbb{Z}_{N_c} \text{ fully broken} \\ \beta > \beta^* & \mathbb{Z}_{N_c} \text{ symmetric} \end{array} \right\} \tag{7.8}$$

Note that this analysis is almost identical to the one loop analysis of the calculable phase transition in QCD(adj) with $1 \ll N_f \leq N_f^*$ flavors in the (m_λ, β) plane, see [109]. The reason is that in QCD(F/adj), fundamental fermions, in their contribution to holonomy potential, behave as $xO(g^2 N_c)$ many adjoint fermions. Hence, the system behaves as if it has $1 + xO(g^2 N_c)$ adjoint fermions, where $x = N_f/N_c$ is finite.

Finally, we remark on the blue line in figure 14c. As described around (5.40), the theory for $m_\lambda = 0$ on small circle exhibits $G \to G_{\rm V} \times {\rm U}(1)_{\rm A_D}$, but on large S_1 , both $G \to G_{\rm V} \times {\rm U}(1)_{\rm A_D}$ and $G \to G_{\rm V}$ are reasonable possibilities according to anomaly consideration. As explained around (5.39), the second possibility seems more likely if we trust semi-classical EFT at the boundary of its region of validity. If true, there has to be a phase transition associated with the change of ${\rm U}(1)_{\rm A_D}$. The rest of axial symmetry is already broken both at large and small circle. Turning on a small m_λ, it is plausible that the phase transition, which is now not associated with any exact symmetry, may persist for a while. If the transition at $m_\lambda = 0$ is first order, this is certainly expected to be the case. But ultimately, the phase transition line is expected to end by a second order critical point. The interesting thing about this phase transition is that the symmetry realization is same both above and below the phase transition line: $G \to G_{\rm V}$ and $\mathbb{Z}_N$ CFC is unbroken. It is again conceivable that these two regimes are adiabatically connected.[28]

[28] Thanks to Ofer Aharony for explanations at this point.

7.2 Phases of $m_\psi = \infty$ theory in the (m_λ, β) plane

The $m_\psi = \infty$ limit of QCD(adj/F) is $\mathcal{N} = 1$ SYM theory. The phase diagram of this theory in the (m_λ, β) plane is investigated in depth in the literature [54, 55, 109]. In fact, this example provides the first semi-classically calculable realization of center-symmetry changing phase transitions in gauge theory on $\mathbb{R}^3 \times S^1_\beta$. For completeness of the phase diagram in the $(m_\lambda, m_\psi, \beta)$-domain, we briefly remind the phase transition in this limit.

Turning on a soft mass term for adjoint fermion, as discussed earlier (7.1), leads to a center-destabilizing one-loop potential. In the $m_\psi = \infty$ limit, since the fundamental fermions decouple, their help to center-stabilization is lost. In fact, in the $m_\lambda = 0$ theory, the holonomy direction is a moduli-space classically and to all orders in perturbation theory due to $\mathcal{N} = 1$ supersymmetry. At this level, the eigenvalues of Wilson line do not interact.

Due to non-perturbative effects, the moduli space is lifted in favor of a center-symmetric minimum. This happens at second order is semi-classics. $\mathcal{N} = 1$ SYM has center stabilizing neutral bions with amplitudes $[\mathcal{M}_i \overline{\mathcal{M}}_i] = e^{-2\times \frac{4\pi}{g^2}\alpha_i\cdot(v+\phi)} \sim e^{-2\times \frac{4\pi}{g^2}(v_{i+1}-v_i)}$ where in the latter form, we set the fluctuations to zero. The effect of the neutral bions is to induce a repulsive potential between the eigenvalues v_i of Wilson line. It should be noted that in the presence of $m_\lambda > 0$ fermions, the fermionic zero modes of the monopoles do get lifted, and hence, it becomes possible for the monopoles to contribute to holonomy potential as well. But they do not contribute to holonomy potential at $m_\lambda = 0$. So, there are three competing effects:

- $O(m_\lambda^2 \beta^2)$: perturbative one-loop center-destabilizing,
- $O(m_\lambda \beta e^{-S_0})$: NP, semi-classic first order, center-destabilizing,
- $O(e^{-2S_0})$: NP, semi-classic second order, center-stabilizing,

where $e^{-S_0} = (\Lambda\beta)^3$ in $\mathcal{N} = 1$ SYM. Below, we briefly review this phase transition, because momentarily, we will show that this phase structure changes in crucial ways once massive fundamental fermions with $m_\psi < \infty$ and Ω_F^0 twisted boundary condition is introduced.

Non-commutativity of limits-2 and phase transition. First, let us parametrize the holonomy potential as:

$$V[\Omega] = \frac{1}{\beta^4} \sum_{n=1}^{\lfloor \frac{N_c}{2} \rfloor} M_n^2 \frac{1}{n^4} \operatorname{tr} |\Omega^n|^2 \tag{7.9}$$

The (dimensionless) mass square M_n^2 for the Wilson line with winding number one (or few) receives contributions from perturbative fluctuations (7.1), monopole-instantons and neutral bions, of the form $-O(m_\lambda^2\beta^2)$, $-O(m_\lambda\beta(\Lambda\beta)^3)$ and $+O((\Lambda\beta)^6)$. In the vicinity of the phase transition scale, the perturbative term is down by three powers of $\log(\frac{1}{\beta N_c \Lambda})$ and can be neglected. Therefore, the center-symmetry changing phase transition boils down to a competition between neutral bion effect and monopole-instanton effect. The mass square for the Wilson line around the phase transition scale

$$M_1^2\big|_{m_\psi=\infty} \sim (-(m_\lambda\beta)(\Lambda\beta)^3 + (\Lambda\beta)^6) \qquad \text{for } \beta \sim \beta_{c1} \sim \Lambda^{-1}\sqrt{\frac{m_\lambda}{\Lambda}} \tag{7.10}$$

where

$$\beta_{c1} \sim \Lambda^{-1}\sqrt{\frac{m_\lambda}{\Lambda}} \tag{7.11}$$

is the parametric value of the center-symmetry changing phase transition.

For sufficiently small mass $m_\lambda \beta \ll e^{-S_0} = (\Lambda\beta)^3$, the center-stabilizing neutral bion potential dominates. For $e^{-S_0} \ll m_\lambda \beta$, the center-destabilizing monopole-instantons dominate. As $\beta \to 0$, the mass square is dominated by the perturbative $-O(m_\lambda^2 \beta^2)$ term.

$$M_1^2|_{m_\psi=\infty} \sim -(m_\lambda^2 \beta^2) < 0 \qquad \beta \to 0 \tag{7.12}$$

and center symmetry is broken for all $0 < \beta < \beta_{c1}$. Therefore, the theory may land on two different phases in the vicinity of $(m_\lambda, \beta) = (0, 0)$:

$$\begin{Bmatrix} m_\lambda \to 0 & \beta = \text{fixed} & \mathbb{Z}_{N_c} \ \text{symmetric} \\ \beta \to 0, & m_\lambda = \text{fixed} & \mathbb{Z}_{N_c} \ \text{broken} \end{Bmatrix} \tag{7.13}$$

The mass square is always negative definite for $\beta < \beta_{c1}$. A sketch of the mass-square for Wilson line is shown in figure 15, left panel.

7.3 Absence of phase transition for $m_\lambda = 0, 0 \leq m_\psi < \infty$

In this domain, center-destabilizing effect of the gauge fluctuations is cancelled by the massless adjoint fermion to all orders in perturbation theory. There are two-center stabilizing effects. One is perturbative two-loop potential due to Ω_F^0-twisted boundary conditions for fundamental fermions (3.18) and the other is non-perturbative neutral bion effects $[\mathcal{M}_i\overline{\mathcal{M}}_i] = e^{-2\times\frac{4\pi}{g^2}(v_{i+1}-v_i)}$.

In the whole $m_\lambda = 0$ subspace of the $(m_\lambda, \beta, m_\psi)$ space, CFC-symmetry is always stable. At small m_ψ, stability is due to two-loop fundamental fermion contribution and in the $m_\psi = \infty$, it is due to neutral bion effects in $\mathcal{N} = 1$ SYM. The QCD(F/adj) with $N_f = N_c$ in the $m_\lambda = 0$ plane is free of any center-symmetry changing phase transition, and for $0 < m_\psi < \infty, 0 < \beta < \infty$, it is free of any phase transitions assuming expected behavior on $\mathbb{R}^4$.

7.4 Phases of large m_ψ theory: two calculable phase transitions

There is one extremely interesting corner of the phase diagram, in which one can analytically show that:

$$\begin{aligned} &\text{for } \ m_\lambda < m_\lambda^* : && \Big\{ 0 < \beta < \infty && \mathbb{Z}_{N_c} \ \text{symmetric} \\ &\text{for } \ m_\lambda > m_\lambda^* : && \begin{cases} \beta > \beta_{c1} & \mathbb{Z}_{N_c} \ \text{symmetric} \\ \beta_{c1} > \beta > \beta_{c2} & \mathbb{Z}_{N_c} \ \text{broken} \\ \beta_{c2} > \beta & \mathbb{Z}_{N_c} \ \text{symmetric} \end{cases} \end{aligned} \tag{7.14}$$

Namely, there exists a critical m_λ^* such that for $m_\lambda \in [(0, m_\lambda^*)$, the small-circle regime is $\mathbb{Z}_{N_c}$ center-symmetric. and chiral symmetry (2.38) is broken to vector-like subgroup. If we make standard assumptions about the dynamics concerning $\mathbb{R}^4$, this regime realizes the same symmetry realization as large S^1 and is likely continuously connected to it.

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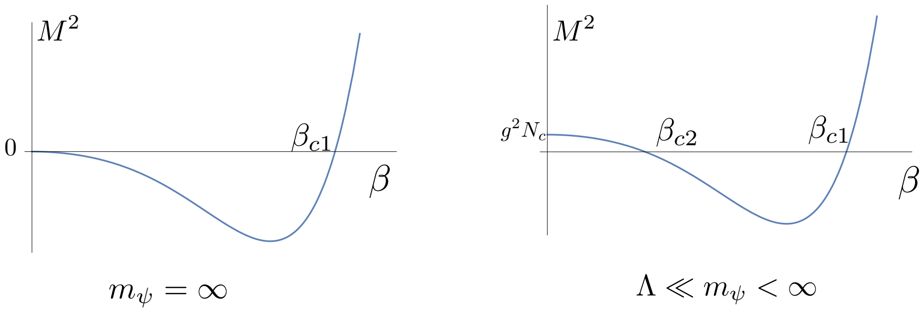

Figure 15. The mass square for gauge holonomy. $m_\psi = \infty$, $0 < m_\lambda \ll \Lambda$ of QCD(F/adj) is $\mathcal{N} = 1$ SYM with soft supersymmetry breaking mass term. The center-symmetry changing phase transition is analytically calculable, and takes place at β_{c1}. When $m_\psi < \infty$, the fundamental fermions with Ω_F^0 twist always gives a center-stabilizing contribution, and the center-symmetry always restores at arbitrarily small-β. This gives an example of two calculable phase transitions.

For $m_\lambda > m_\lambda^*$ there are two center-symmetry changing phase transitions. If m_λ is slightly above m_λ^*, then both of these phase transitions are analytically calculable, see figure 14b. If $m_\lambda > \Lambda_{QCD}$, the phase transition at β_{c1} turn into a semi-classically incalculable phase transition.

There can be partially broken phases for $\beta_{c2} < \beta < \beta_{c1}$, but our primary concern is the points where center is fully restored. The realization of center-symmetry is determined by the first $\lfloor \frac{N_c}{2} \rfloor$ terms in the holonomy potential. Below, we would like to treat N_c as an order one number, and give order of magnitude estimates for the phase transition scales.

Non-commutativity of limits-3. Does heavy fermion always decouple from dynamics? Assume $m_\psi < \infty$ and large. Normally, one would think that the heavy fermion should decouple from the dynamics. Indeed, in the one- and two-loop potentials $V^\psi_{1-\text{loop},\Omega_\text{F}^0} + V^\psi_{2-\text{loop},\Omega_\text{F}^0}$, m_ψ appear through the combinations such as $\frac{1}{2}(m_\psi \beta n)^2 K_2(m_\psi \beta n)$. At finite-$\beta$, as $m_\psi \to \infty$, the effect of the fundamental fermion will disappear exponentially $e^{-m_\psi \beta n}$ by the large-argument asymptotic of Bessel function. However, at finite- m_ψ, and as $\beta \to 0$, despite the fact that fermion is heavy, it will behave as if it is massless, and will follow small-argument asymptotic of Bessel function. This has interesting impacts on phase transitions.

Consider the $(m_\lambda, \beta, m_\psi) = (0, 0, \infty)$ corner of phase diagram. Move off the corner slightly in a generic direction. There are four competing effects which can determine the realization of center symmetry.

- $O(m_\lambda^2 \beta^2)$: perturbative one-loop center-destabilizing from $V^\text{gauge}_{1-\text{loop}} + V^\lambda_{1-\text{loop}}$
- $O(m_\lambda \beta e^{-S_0})$: NP, semi-classic first order (monopole-instantons) center-destabilizing,
- $O(e^{-2S_0})$: NP, semi-classic second order (neutral bions), center-stabilizing,
- $O((g^2 N_c) K_2(m_\psi \beta))$: perturbative two-loop center-stabilizing from $V^\psi_{2-\text{loop},\Omega_\text{F}}$

$\boldsymbol{m_\psi < \infty}$. As long as $m_\psi \beta_{c1} \gg 1$, the $V^{\psi}_{2-\text{loop},\Omega_\text{F}}$ term has no effect on the phase transition taking place at $\beta = \beta_{c1}$. However, if we take $\beta \to 0$ limit such that $m_\psi \beta \lesssim 1$, the fundamental fermion induces $V^{\psi}_{2-\text{loop},\Omega^0_\text{F}}$ term, in which the contribution to mass-square for the Wilson lines is always positive definite!

Therefore, with the inclusion of finite mass fundamental fermion, the mass-square in the $\beta \to 0$ limit, as per (3.18), becomes positive:

$$M_1^2|_{m_\psi<\infty} \to +\frac{(g^2 N_c)}{8\pi^4} > 0 \qquad \beta \to 0 \tag{7.15}$$

which implies that the center-symmetry must be stabilized as $\beta \to 0$ limit. This shows that there must exist an $\beta_{c2} \in (0, \beta_{c1})$ such that Wilson line is non-tachyonic for $\beta < \beta_{c2}$ and center symmetry is restored. A sketch of the mass-square for Wilson line for this case is shown in figure 15, right panel.

The phase transition at β_{c2} is a result of the competition between $O(m_\lambda^2 \beta^2)$ one-loop center-destabilizing effects from $V^{\text{gauge}}_{1-\text{loop}} + V^\lambda$ and perturbative two-loop center-stabilizing effects from $V^{\psi}_{2-\text{loop},\Omega^0_\text{F}}$. We can use this to determine the position of β_{c2}. However, currently, we do not have a closed form expression for $V^{\psi}_{2-\text{loop},\Omega^0_\text{F}}$ for massive fundamental fermions. Note that this does not interfere with the argument above about the existence of center-symmetric phase for $\beta < \beta_{c2}$ and the positivity of the mass term for Wilson line in the $\beta \to 0$ limit. To prove this positivity, we needed the expression in the $m_\psi \beta \to 0$ limit, which we could deduce analytically.

If we would like to estimate the parametric form of β_{c2}, we need the holonomy potential (3.18) for Ω^0_F twisted massive fundamental fermions, in particular, the center-stabilizing double trace operator. In perturbation theory, a holonomy dependence involving double-trace term arises when one gluon or adjoint quark propagator goes around the S^1 circle. However, we are considering (3.18), which is sourced by fundamental quarks. At two loop order, a quark and anti-quark that goes around the circle S^1 can emulate an adjoint matter and induce a double-trace term. This term is roughly: $\frac{(g^2N_c)}{\beta^4 8\pi^4} \sum_{n=1}^\infty \frac{1}{2}(n\beta(2m_\psi))^2 K_2(n\beta(2m_\psi)) \frac{1}{n^4} |\operatorname{tr}\Omega^n|^2$. Therefore, to estimate the phase transition point closer to the $\beta \to 0$ limit, we inspect the mass square for the Polyakov loop order parameter:

$$M_1^2|_{m_\psi<\infty} \sim -\frac{1}{2\pi^2} m_\lambda^2 \beta^2 + \frac{(g^2 N_c)}{8\pi^4}(1 - m_\psi^2 \beta^2) \qquad \text{for } \beta \lesssim \beta_{c2} \tag{7.16}$$

Hence, the critical radius parametrically takes the form:

$$\beta_{c2} \sim \frac{\frac{(g^2N_c)^{1/2}}{2\pi}}{\left(m_\lambda^2 + \frac{(g^2N_c)}{4\pi^2} m_\psi^2\right)^{1/2}} \tag{7.17}$$

For $m_\psi = \infty$, the $\beta_{c2} \to 0$ and the center symmetry is broken for all $\beta < \beta_{c1}$, which is the result for $\mathcal{N} = 1$ SYM. For $m_\psi = 0$, we obtain $\beta_{c2} \sim \frac{(g^2N_c)^{1/2}}{2\pi m_\lambda}$, in agreement with (7.3).

8 Large-N_c volume independence in the Veneziano type limit

In this section, we describe briefly the relation between the quantum distillation, Ω_F^0 twisted boundary conditions and large-N_c volume independence in QCD(F/adj).

A sub-class of large-N_c gauge theories, when studied in toroidal compactification of $\mathbb{R}^d$ (or its latticy version) have properties independent of compactification radius. This property is called large-N_c volume independence [28], which is a special case of large-N orbifold equivalence. The extreme version of volume independence, where a space-time lattice L^d is reduced to one-site 1^d lattice is called Eguchi-Kawai reduction or large-N_c reduction [110–112]. The necessary and sufficient conditions for the validity of the volume independence are

- Translation symmetry of lattice L^d is not spontaneously broken.
- $(\mathbb{Z}_{N_c})^d$ center symmetry is not spontaneously broken.

Volume independence applies to the expectation values and connected correlators of topologically trivial Wilson loops at leading order in N_c. The sector of the theory neutral under center transformations is called *neutral sector*. For example, string tensions, mass gap, spectrum of the theory, free energies, pressures are in the neutral sector and volume independence applies to them. Polyakov loop expectation values and correlators are part of the non-neutral sector observables.

The original proposal [110] actually fails to satisfy volume independence below certain critical size due to center symmetry breaking. But there are other versions or theories in which it works. The cleanest examples to volume independence are new versions of twisted Eguchi-Kawai (TEK) models [113, 114] and QCD with N_f adjoint Weyl fermions endowed with periodic boundary conditions [28]. QCD(adj), if studied with periodic boundary conditions for fermions in path integral formulation or equivalently via $\mathrm{tr}\,[(-1)^F e^{-\beta H}]$ in Hamiltonian formulation, obeys volume independence [28]. The fact that volume independence works means the theory avoids all possible phase transition and Hagedorn singularities. An important point that started to emerge fairly recently is that at the root of working versions of volume independence, there must be profound spectral cancellations [12, 13, 16] or an extreme version of quantum distillation. Otherwise, generically, there will be phase transitions. In QCD(adj), this manifests itself as Bose-Fermi cancellations in the absence of supersymmetry discussed in [12, 13]. The spectral cancellation and quantum distillation of Hilbert space must also occur in the twisted Eguchi-Kawai model [111, 113, 114].

Volume independence on $\mathbb{R}^3\times S^1$ translates to temperature independence of the neutral sector observables in the leading order in large-N_c limit. In SU(N_c) Yang-Mills theory, large N_c volume independence holds as long as all compactification radii are larger than a critical radius $\beta_c \sim \Lambda^{-1}$ and fails below β_c due to a center-symmetry breaking phase transition [115]. In QCD with $N_f \sim N_c$ fundamental fermions (namely, the Veneziano limit), one does not even discuss volume independence, because there is not even a center-symmetry to begin with.

In fact, it is historically believed that QCD(F) with $N_f \sim N_c$ fundamental fermions manifestly disobeys volume (or temperature) independence due to lack of center-symmetry.

For example, assuming chiral symmetry is broken, this theory has $(N_f^2 - 1)$ NG bosons and the pressure is $p(T) = \frac{\pi^2}{45}(N_c^2 - 1)T^4$ at leading order in large-N_c for $N_f = N_c$. Clearly, this is temperature dependent at leading order and large-N_c temperature independence fails to hold.

However, as described in section 2.3, the theories with $N_f = kN_c$ possess an exact color-flavor center symmetry, which is in the diagonal of center of gauge group and cyclic permutations $\Gamma_S \in \mathrm{SU}(N_f)$ living in flavor rotations. The Polyakov loops winding around the S^1 circle are charged under the $\mathbb{Z}_{N_c}$ CFC symmetry. Therefore, one can meaningfully talk about the realization of this CFC-symmetry.

The existence of $\mathbb{Z}_{N_c}$ CFC symmetry manifests itself in the loop potential for holonomy. The potential is only a function of $\mathrm{tr}\,(\Omega^{N_c k})$ and $\left|\mathrm{Tr}(\Omega^k)\right|^2, k \in \mathbb{Z}$ which are singlet under the CFC symmetry. In particular, terms such as $\mathrm{tr}\,(\Omega^q), q \neq 0 \pmod{N_c}$, which explicitly break the CFC symmetry does not appear in the holonomy potential. Whether the $\mathbb{Z}_{N_c}$ symmetry is spontaneously broken or not depends on the dynamics. In QCD(F), CFC is unbroken for $\beta > \beta_c \sim \Lambda^{-1}$ and is broken for $\beta < \beta_c$. Therefore, volume and temperature independence hold in QCD(F) for $\beta > \beta_c \sim \Lambda^{-1}$ provided it is studied via the graded partition function (4.5) or equivalently, Ω_F^0-twisted boundary conditions.

In QCD(F/adj) with $N_f = kN_c$, which is a slight generalization of Veneziano limit to mixed representation matter, the situation is very intriguing. At large-N_c limit, the combination of one and two loop potential for QCD(F/adj) acquires a simple form. Remarkably, the one-loop potential actually vanishes. So, in perturbation theory at one-loop order, this class of theories has a moduli-space, which allows all possible realizations of center-symmetry. At two-loop order, the theory prefers a center-symmetric minimum. At $N_c = \infty$,

$$
\begin{aligned}
V_{\text{1-loop}} &= V^{\text{gauge}}_{1-\text{loop}} + V^{\lambda}_{1-\text{loop}} + V^{\psi}_{1-\text{loop},\Omega_{\mathrm{F}}} = 0 \\
V_{\text{2-loop},\Omega_F^0} &= +x\frac{(g^2N_c)}{8\pi^4\beta^4}\sum_{n=1}^{\infty}\frac{\left|\mathrm{Tr}(\Omega^n)\right|^2}{n^4}\,, \qquad x = \frac{N_f}{N_c}
\end{aligned}
\tag{8.1}
$$

First of all, the $\mathbb{Z}_{N_c}$ CFC symmetry is stable at large-N_c. (It is in fact stable at any N_c). The N_f fundamental Dirac fermions with flavor twisted boundary conditions (1.13) behaves as if $(g^2N_c)x$ adjoint Weyl fermions with periodic boundary conditions for the purpose of holonomy potential!

The two-loop stability of center-symmetry cannot be altered by three or higher loop orders or non-perturbative contributions. The three or higher loop appear at order $(g^2N_c)^p, p = 2, 3, \ldots$ respectively and at weak coupling, it cannot alter the implications of two-loop order.[29]

One intriguing implication of volume-independence in the Veneziano-type limit is emergent CFC symmetry. For finite N_f, N_c, the exact CFC symmetry is $\mathbb{Z}_{\gcd(N_f,N_c)}$. If N_f and

[29] Once the realization of center-symmetry is determined at p loop order, the $(p+1)$ and higher loop orders as well as non-perturbative corrections cannot alter this result. $(p+1)^{\text{th}}$ order is needed if and only if $(p)^{\text{th}}$ order exhibits degeneracy. In our analysis, there is degeneracy at one-loop order $p = 1$, hence we need $p = 2$. In supersymmetric QFTs, all orders in perturbation theory do not lift degeneracy, hence one needs non-perturbative terms to determine center-symmetry realization.

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N_c are co-prime, formally there is no center-symmetry. However, despite being formally correct, this does not reflect the truth sufficiently well. For example, with the center-symmetric Ω_F^0 and in the Veneziano limit, all single trace terms in the potential of the form $\text{tr}\,(\Omega^{N_f k})$ disappear. Normally, these terms transform non-trivially under $\mathbb{Z}_{N_c}$ and breaks it explicitly to $\mathbb{Z}_1$. However, the prefactor of these terms is $\frac{1}{N_f^3}$ and they vanish in the Veneziano large-N_c limit. In the two-loop expression, the only term that is left is the $\mathbb{Z}_{N_c}$ center-symmetric double-trace term. And hence, for co-prime N_f, N_c, in the Veneziaono limit, despite the fact that center-symmetry is formally $\mathbb{Z}_1$, the minimum of the holonomy potential is still at a $\mathbb{Z}_{N_c}$ center-symmetric point. This is similar to emergent center-symmetry previously appeared in the context of two-index representation fermions in the large-N_c limit [116]. For fundamental fermions in 't Hooft limit, it is a trivial result as the fermionic degrees of freedom are suppressed as $\frac{N_f N_c}{N_c^2} \sim \frac{N_f}{N_c}$. For fundamental fermions in the Veneziano limit with Ω_F^0 twist, it is a non-trivial result. Fermion effects on the holonomy potential is suppressed by quantum distillation with a factor $\frac{N_c/N_f^3}{N_c^2} \sim \frac{1}{N_c^4}$ as can be deduced from (3.18), more so than the loop suppression of the quarks in the 't Hooft limit.

The fact that the theory satisfies volume independence implies that there are very powerful Bose-Fermi, Fermi-Fermi and Bose-Bose cancellation over the state sum. These cancellations follow a very similar pattern to pheonomenological description given in section 4, and powerful enough to demolish both the Hagedorn growth as well as standard Boltzmann growth of the density of states. As conjectured in section 4, the relative density of states on a curved 3-manifold grow similar to the one of 2d QFT. Non-supersymmetric QCD(F/adj) theory acts in a similar way to supersymmetric theories on curved spaces [11], and QCD(adj) [13].

8.1 Color-flavor-momentum transmutation

If a gauge theory is compactified on a circle with size β in the presence of trivial gauge holonomy, the Kaluza-Klein modes of the periodic fermion fields ψ gets quantized in units of $\frac{2\pi}{\beta}$. The KK-decomposition of the fermions take the form

$$\overline{\psi}_{aj}(n)\left(\gamma_i\partial_i + \gamma_4\frac{2\pi n}{\beta}\right)\psi^{aj}(-n), \qquad j=1,\ldots,N_c, \qquad a=1,\ldots,N_f, \tag{8.2}$$

where j is color, a is flavor and $n \in \mathbb{Z}$ is Kaluza-Klein momentum index. Clearly, the KK-decomposition does not quite care about the color and flavor structure. In contradistinction, if the dynamical gauge holonomy background Ω and non-dynamical flavor holonomy Ω_F^0 are center-symmetric, the fermion kinetic term and KK-decomposition is refined as:

$$\begin{aligned}&\overline{\psi}_{aj}(n)\left(\gamma_i\partial_i + \gamma_4\left(\frac{2\pi j}{\beta N_c} + \frac{2\pi a}{\beta N_f} + \frac{2\pi n}{\beta}\right)\right)\psi^{aj}(-n)\\ &\quad = \overline{\psi}_{aj}(n)\left(\gamma_i\partial_i + \gamma_4\frac{2\pi q}{\text{lcm}(N_c,N_f)l}\right)\psi^{aj}(-n)\end{aligned} \tag{8.3}$$

where

$$q(j,a,n) = \frac{\text{lcm}(N_c,N_f)}{N_c}j + \frac{\text{lcm}(N_c,N_f)}{N_f}a + \text{lcm}(N_c,N_f)n \in \mathbb{Z} \tag{8.4}$$

and lcm stands for least common multiple. We refer to this effect as *color-flavor-momentum transmutation*. It is a generalization of Gross-Kitazawa color-momentum transmutation which incorporates flavor [61], and has some similarity to ref. [117]. However, unlike refs. [61, 117], center-symmetry is dynamically stable in our construction in the Ω_F^0 flavor holonomy background, and volume independence is valid even at arbitrarily small-β. The KK modes for fermions are now quantized in units of

$$\frac{2\pi q}{\mathrm{lcm}(N_c, N_f)\beta} \tag{8.5}$$

For $N_f = N_c$ theory, this is quantization of KK-momentum modes in units of $\frac{2\pi}{\beta N_c}$, which is an imprint of large-N_c volume independence. In other words, for quarks, it is as if the effective space size is $\mathrm{lcm}(N_c, N_f)\beta$. For gauge bosons in center-symmetric background, of course, the space size is effectively, βN_c, and same for the full theory.

In the standard Kaluza-Klein decomposition, infinite volume can only be captured by taking $\beta\Lambda \gg 1$, hence there are many KK-modes below the strong scale Λ, $\frac{1}{\beta} \ll \Lambda$. This is how lattice gauge theory formulated in a finite volume can capture the properties of the QFT on $\mathbb{R}^4$.

With the Ω_F^0 twisted boundary conditions and graded partition function, in the $N_c \to \infty$ limit, regardless of value of β, the refined KK modes form a continuum as if the theory is on $\mathbb{R}^4$. In perturbation theory, this is how infinite volume limit is captured at arbitrarily small-β in the large-N_c limit, similar to the discussions in [61, 109, 111].

For example, one can derive the renormalization group β function of the theory on $\mathbb{R}^4$ even by studying with the theory at $\mathbb{R}^3 \times S^1$, as Gross and Kitazawa did with the matrix model reduction [61]. It is easy to show that the reduced 3d theory produces, to all orders in perturbation theory, the standard Feynman diagrams for invariant Green functions of the theory on $\mathbb{R}^4$. Of course, this equivalence is not only restricted to perturbation theory, and all the neutral sector observables must agree between the reduced theory and the theory on $\mathbb{R}^4$ non-perturbatively.

Lattice simulation of the QCD(F/adj) is possible [118]. In particular, the flavor twist Ω_F does not induce a sign problem in Euclidean path integral formulation. Such a construction may be useful to learn further about non-perturbative properties of the theory and test adiabatic continuity (at finite N_c) and volume independence (at large N_c) in the strong coupling domain.

9 Summary

Color-flavor center symmetry. The presence of fundamental matter fields explicitly breaks one-form center-symmetry $\mathbb{Z}_N^{[1]}$. Thus, one may be tempted to think, as it is commonly accepted, that Polyakov loops on $\mathbb{R}^3 \times S^1$ can never be genuine order parameters in theories with fundamental matter. In [23], it was realized that a diagonal subgroup of the center of $\mathrm{SU}(N_c)$ and a cyclic permutation subgroup of $\mathrm{SU}(N_f)_V$, $(\mathbb{Z}_{\mathrm{gcd}(N_f,N_c)})_D$, the color-flavor center (CFC) symmetry, can remain as a true symmetry of the theory, and Polyakov loops can in fact be good order parameters provided $\mathrm{gcd}(N_f, N_c) > 1$.

The existence of CFC symmetry explains the sharp phase transition observed in lattice simulations with flavor-twisted background [24]. In this work, we provided an interpretation for the results of [24] from the viewpoint of quantum distillation of Hilbert space. We argued that the distilled Hilbert space of QCD(F) with Ω_F^0 distillation carries characteristic features of the Hilbert space of pure Yang-Mills theory as described in (4.23) and (4.24). Ref. [24] indeed showed that CFC is spontaneously broken in QCD(F) at small-β and restored at large-β. The reason for the breaking of the CFC in QCD(F) is in essence the same as in Yang-Mills theory. The growth of the density of flavor singlet hadronic states is powerful enough to induce a phase transition.

Preservation of the CFC-symmetry at small-β. The story takes an even more interesting form if we consider QCD(F/adj). Fundamental fermions with the Ω_F^0 twisted boundary conditions in fact favor a center-symmetric minimum, just like adjoint fermions with periodic boundary conditions! In QCD(F/adj) with $N_f = N_c$ where the center-breaking effect of gauge fluctuation is undone by one adjoint fermion, the ultimate decision is given by fundamental fermion, which lead to the stability of $\mathbb{Z}_{N_c}$ color-flavor center symmetry. This opens the prospect of adiabatic continuity in QCD(F/adj) and QCD(F) between $\mathbb{R}^4$ where these theories are strongly coupled and $\mathbb{R}^3 \times S^1$ where they become weakly coupled and calculable.

Power of quantum distillation. The Hilbert space distillation in QCD(F/adj) is induced by the insertion of $(-1)^F \prod_{a=0}^{N_f} e^{\frac{2\pi i}{N_f} Q_k}$ in the operator formalism. What is left from the Hilbert space of QCD(F/adj) $\mathcal{H}_{\text{QCD(F/adj)}}$ after all the cancellation is quite small, the effective density of states after all cancellations grows as the one of 2d QFT in the large-N_c limit, in a similar way to supersymmetric theories on curved spaces [11]. Although the cancellation is milder than what supersymmetric index achieves on flat space, e.g. $\text{Distill}[\mathcal{H}_{\mathcal{N}=1\text{SYM}}] = \{\text{Ground states}\}$, it is powerful enough to avoid Lee-Yang singularities and phase transitions in certain cases, providing generalized partition functions which are smooth functions of β.

Chiral symmetry breaking by monopole-operators at weak coupling on $\mathbb{R}^3 \times S^1$. Perfect quantum distillation tells us that the graded partition function is saturated by only a few states, so that as one interpolates between large S^1 and small S^1, the ground state may remain adiabatically connected. In QCD(F/adj) with one heavy adjoint fermion and $N_f = N_c$ massless fundamental fermions in the CFC symmetric regime, the continuous chiral symmetry must be spontaneously broken even at weak coupling [33]. The most interesting and unconventional outcome of this analysis is that the chiral field $\Sigma(x)$ that appears in the chiral Lagrangian is the collection of the monopole-flux operators $\Sigma(x) = \text{Diag}\left(\mathrm{e}^{\mathrm{i}\alpha_1\cdot\sigma}, \mathrm{e}^{\mathrm{i}\alpha_2\cdot\sigma}, \dots, \mathrm{e}^{\mathrm{i}\alpha_{\mathrm{N_f}}\cdot\sigma}\right)$, where σ is the dual photon associated with gauge fluctuations. Due to Nye-Singer index theorem for Dirac operators in monopole-backgrounds [31, 32], and the fact that the Cartan subgroup of axial chiral symmetry is non-anomalous, the dual photons must and does acquire a chiral charge. The condensation of Σ, rather than condensation of the fermion bilinear, leads to chiral symmetry breaking in a weak coupling regime of QCD(F), and dual photons acquire an interpretation as Nambu-Goldstone bosons. In a given superselection sector, determined by the VEV of monopole operators, we can calculate the expectation value of the fermion bilinear condensate. Remarkably, it is $\langle \psi_R \psi_L \rangle = \Lambda^3_{\text{QCD}}$, and the strong scale emerges naturally as described around (5.54), almost exactly as in $\mathcal{N} = 1$ SYM. The condensation of the monopole-flux operators induce a chiral symmetry breaking mass term for massless fermions.

Persistent anomaly upon compactification on $\mathbb{R}^3 \times S^1$. The mixed anomaly which dictates possible ground state structures on $\mathbb{R}^4$ persists upon compactification on $\mathbb{R}^3 \times S^1$ provided $\gcd(N_f, N_c) > 1$ and an appropriate Ω_F twisted background for $\text{SU}(N_f)_V$ is used. Phase transitions exist between various anomaly respecting phases. Remarkably, a number of these phase transitions are calculable on $\mathbb{R}^3 \times S^1$.

Adiabatic continuity. We showed that $N_f = N_c$ QCD(F) with one massive adjoint fermion on small $\mathbb{R}^3 \times S^1$ does not break CFC and the Polyakov loop expectation value is strictly zero. But it does break continuous chiral symmetry (2.38) at weak coupling. This is the expected behavior of this theory on large $\mathbb{R}^3 \times S^1$ at strong coupling. It seems highly plausible that these two regimes are continuously connected, but we cannot prove this statement. The best we can do is to prove that the possible ground state structures are controlled by the same mixed 't Hooft anomalies, given in (6.19) and (6.24), which is still a remarkable persistent order, but not a proof of adiabatic continuity.

Summary of general construction.

Hamiltonian $\mathbf{H}$	Hamiltonian $\mathbf{H}$ + grading operator
Hilbert space $\mathcal{H}$	$\text{Distill}[\mathcal{H}]$
Thermal state sum over $\mathcal{H}$ $\mathcal{Z}(\beta) = \text{tr}\left[e^{-\beta H}\right]$	Graded state sum over $\mathcal{H}$ $\mathcal{Z}_{\Omega_0}(\beta) = \text{tr}\left[e^{-\beta H}(-1)^F e^{i\pi Q_0} e^{-i\frac{\pi}{N_f}Q_0} \prod_{a=1}^{N_f} e^{i\frac{2\pi a}{N_f}Q_a}\right]$,
Path integral with thermal b.c. Gauge-holonomy potential $V[\Omega]$	Path integral with $(-1)^F$ and Ω_F^0 flavor twisted b.c. Gauge-holonomy potential in the presence of flavor holonomy $V_{\Omega_F}[\Omega]$
Mixed anomaly on $\mathbb{R}^4$ between $\text{SU}(N_f)/\mathbb{Z}_{N_c}$ and $(\mathbb{Z}_{2N_f})_A$ Does not persists on $\mathbb{R}^3 \times S^1$ with thermal compactification.	Mixed anomaly on $\mathbb{R}^4$ is persistent on $\mathbb{R}^3 \times S^1$. Does persists on $\mathbb{R}^3 \times S^1$ with Ω_F^0 compactification.
Thermodynamics: thermodynamic worth of $\mathcal{H}$ Free energy Pressure	Graded Thermodynamics: thermodynamic worth of $\text{Distill}[\mathcal{H}]$ Graded free energy (or twist free-energy or flavor-holonomy potential) Graded Pressure
All incalculable phase transitions No anomaly constraint on phase transitions	Adibatic continuity and/or persistent order Many calculable phase transitions, mixed anomaly respected
$\rho(E)$, density of hadronic states	$\rho_{\text{distill}}(E)$, density of hadronic states corresponding to $\text{Distill}[\mathcal{H}]$
Volume dependence at large-N_c	Volume independence at large-N_c

(9.1)

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Remarks

Not all sign problems are bad. The insertion of the operator $\prod_{a=0}^{N_f} e^{\frac{2\pi i}{N_f} Q_k}$ into the trace induces a "sign problem" in the state sum based on the Hamiltonian formulation. In other words, quantum distillation is actually a sign problem, albeit a good one! The tremendous cancellations in the graded state sum are a consequence of destructive interference due to concerted phases attached to physical states. Interestingly, these phases do not induce a sign problem in the Euclidean path integral formulation.[30] The sign problem in the state sum over Hilbert space may lead to sufficient cancellations which in turn lead to the absence of phase transitions as β is dialed.

SQCD vs. QCD(F/adj). Consider $N_f = N_c$ SQCD with a soft mass for the fundamental scalar. Ref. [19] showed that with a soft mass m_{q_a}, the chiral symmetry breaking pattern (2.15) holds where the $\mathrm{U}(1)_{A_D}$ part of the chiral symmetry is unbroken. It is not known if this phase persists in the decoupling limit $m_{q_a} = \infty$, where SQCD reduce to QCD(F/adj). Remarkably, in QCD(F/adj) with $m_\lambda = 0$ we demonstrated at small $\mathbb{R}^3 \times S^1$ that the chiral symmetry breaking pattern that takes place is same as in supersymmetric theory with soft SUSY breaking term (2.15). Therefore, it seems very likely that provided

$$\mathrm{Max}(\beta\Lambda, m_{q_a}\Lambda^{-1}) \ll 1 \tag{9.2}$$

there will be adiabatic continuity between SQCD and QCD(F/adj). In particular, the $m_{q_a}\Lambda^{-1} \ll 1, \beta\Lambda = \infty$ limit seems to be adiabatically connected to the $m_{q_a}\Lambda^{-1} = \infty, \beta\Lambda \ll 1$ regime, as depicted in figure 11. Turning on a small mass for adjoint fermion, the continuity may persist beyond the domain (9.2) and to the whole (m_{q_a}, β) plane.

Possibility of dualities and interesting IR behaviors in QCD(F/adj). We have shown on $\mathbb{R}^4$ that all the interesting mixed anomalies of SQCD for $N_f = N_c$ theory are also satisfied by QCD(F/adj). Of course, this is not an accident. What enters into the traditional 't Hooft anomalies [50] of SQCD are just the currents of global symmetries that act on fermions. And the microscopic fermionic content and associated currents of these theories are the same. Hence, the UV anomalies of SQCD and QCD(F/adj) are the same for any suitable N_f. The interesting thing is in the IR, Fradkin-Shenker complementarity [51] and the fact that the existence of just one adjoint fermion changes the story of QCD(F/adj) dramatically. Thanks to complementarity (2.21) which substitutes elementary scalars with adjoint/fundamental fermion bilinears with identical local and global symmetry quantum numbers, we can express all the fermionic components of composite mesons and baryons in the same way as in SQCD, mimicking Seiberg's analysis [46, 119]. Therefore, we believe that non-trivial anomaly matching that occurs in the context of SQCD (either in dual formulations or as interesting IR-behaviors) must have an image in QCD(F/adj).

It is becoming clear that QCD(F/adj) is a class of non-supersymmetric theories that is intermediate between QCD and supersymmetric QCD. It is in this unfamiliar realm that we find new possibilities.

[30] The situation is reversed for the inclusion of a chemical potential, which does not induce a sign problem in the Hamiltonian formulation but does induce one in the path integral formulation.

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Acknowledgments

I am thankful to O. Aharony, P. Argyres, P. Draper, M. Nguyen, D. Gross, A. Behtash, T. Sulejmanpasic, A. Cherman, Y. Tanizaki, Y. Guo, and S. Gukov for discussions. I am especially grateful to T. Kanazawa for discussions, collaboration in an earlier related work, his help with some of the figures, and numerical extremization of the holonomy potential. This work was supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics under Award Number DE-FG02-03ER41260.

A Elementary examples of symmetry graded state sums

First, we provide two simple examples of symmetry graded state sums and cancellations in quantum mechanics. These example are presented for pedagogical reasons. Then, we will use exactly the same symmetry grading in non-trivial asymptotically free quantum field theories in 2d, a vector model $\mathbb{CP}^{N-1}$ and a matrix model, the principle chiral model. In the bulk of this paper, we implemented these types of symmetry grading to QCD(F/adj) in 4d.

A.1 N-dimensional harmonic oscillator

Consider N-dimensional simple harmonic oscillator with Hamiltonian $H = \sum_{j=1}^{N} \omega \widehat{a}_j^\dagger \widehat{a}_j$. The global symmetry is $\mathrm{U}(N)$. States are totally symmetric representations of $\mathrm{SU}(N)$. The partition function of this system is

$$\mathcal{Z}(\beta) = \operatorname{tr}\left(e^{-\beta H}\right) = \sum_{k=0}^{\infty} \deg(k) \mathrm{e}^{-\beta\omega k}, \qquad \deg(k) = \binom{N+k-1}{k} \tag{A.1}$$

where the degeneracy increases extremely rapidly.

Now, construct the symmetry graded state sum

$$\mathcal{Z}_{\Omega_F^0}(\beta) = \operatorname{tr}\left(e^{-\beta H} \prod_{j=1}^{N} e^{i\frac{2\pi}{N} j \widehat{Q}_j}\right) \tag{A.2}$$

where $Q_j = \widehat{a}_j^\dagger \widehat{a}_j$ is the number operator for the j^{th} oscillator. In the graded state sum, many degenerate states cancel among themselves due to phases attached to them. After cancellations, graded degeneracy factors reduce into

$$\deg(k) \mapsto \begin{cases} 1 & k = 0 \mod N \\ 0 & k = 0 \mod N \end{cases} \tag{A.3}$$

instead of the growth (A.1). As a result, the graded partition function of the N-dimensional oscillator with frequency ω is equivalent to thermal partition function of a 1-dimensional oscillator with frequency $N\omega$ (or 1-dimensional oscillator with frequency ω but inverse temperature βN)

$$\mathcal{Z}_{\Omega_F^0}(\beta) = \prod_{j=1}^{N} \left(\frac{1}{1 - e^{-\beta\omega + i\frac{2\pi}{N} j}}\right) = \frac{1}{1 - e^{-\beta N\omega}} \tag{A.4}$$

This indicates dramatic reduction in the density of states. In particular, in the $N \to \infty$ limit,

$$\lim_{N\to\infty} \mathcal{Z}_{\Omega_F^0}(\beta) = 1 \tag{A.5}$$

indicating that only the ground state contributes to the state sum.

At this stage, the graded state sum in purely bosonic theory achieves something quite remarkable, similar to what supersymmetric index achieves in supersymmetric quantum mechanics [1]. The distilled Hilbert space is just the ground state of the bosonic $N \to \infty$ dimensional oscillator and only one-state contributes to the state sum.

The reader may think that this is a trivial non-interacting system in quantum mechanics. Remarkably, the same phenomena does occur in the non-trivial asymptotically free QFT in 2d and higher dimensions.

A.2 Hydrogen atom

Consider the hydrogen atom and ignore the spin of the electron. The discrete part of the spectrum is $E_n = -E_1/n^2$ and the degeneracy factor is $\deg(n) = n^2$. The simplest way to understand the degeneracy is to note that the global symmetry of the hydrogen atom is

$$G = \frac{\mathrm{SU}(2) \times \mathrm{SU}(2)}{\mathbb{Z}_2} \tag{A.6}$$

which are ultimately related to angular momentum and Laplace-Runge-Lenz vector. The states in the Hilbert space fill the bi-product of spin-j irreducible representation with its conjugate, where $j = (n-1)/2, n = 1, 2, \ldots$ and the degeneracy factor is given by

$$j \otimes j \in G \qquad\qquad \deg[(j \otimes j)] = (2j+1)^2 = n^2 \tag{A.7}$$

Now, we can construct a symmetry graded state sum by the insertion of the operator $e^{i\epsilon(J_{1z}+J_{2z})}$ into the trace. This modifies the partition function as

$$\sum_{n=1}^{\infty} \deg(n) \mathrm{e}^{-\beta E_n} \Rightarrow \sum_{n=1}^{\infty} (\chi_{\frac{n-1}{2}}(\epsilon))^2 \mathrm{e}^{-\beta E_n} \tag{A.8}$$

where $\chi_j(\epsilon) = \frac{\sin\frac{(2j+1)\epsilon}{2}}{\sin\frac{\epsilon}{2}}$. The best quantum distillation is achieved at $\epsilon = \pi$ and hence, a useful object to consider is $\operatorname{tr}(e^{-\beta H} e^{i\pi(J_{1z}+J_{2z})})$, leading to modification in the state sum

$$\deg(n) = n^2 \mapsto \begin{cases} 1 & n = 1, 3, 5, \ldots \\ 0 & n = 2, 4, 6, \ldots \end{cases} \tag{A.9}$$

The state sum is modified as

$$\sum_{n=1}^{\infty} n^2 \, \mathrm{e}^{-\beta E_n} \mapsto \sum_{n=1,3,\ldots}^{\infty} 1 \, \mathrm{e}^{-\beta E_n} \tag{A.10}$$

A.3 $\mathbb{CP}^{N-1}$ model in 2d

The global symmetry of the $\mathbb{CP}^{N-1}$ model is

$$G = \mathrm{PSU}(N) = \frac{\mathrm{SU}(N)}{\mathbb{Z}_N}. \tag{A.11}$$

This is because $\mathbb{Z}_N$ is part of the U(1) gauge redundancy of the theory and G is the symmetry that acts faithfully in the Hilbert space $\mathcal{H}$. This means only the representations of $\mathrm{SU}(N)$ that also happen to be representations of $\mathrm{PSU}(N)$ appear in the Hilbert space. This amounts to adjoints and products thereof and singlets. If we denote the elementary $\mathbb{CP}^{N-1}$ field as $z_a(x)$, the states in the Hilbert space are generated by operators such as

$$\bar{z}^a(x) z_b(x), \qquad \bar{z}^b(x) e^{i\int_x^y a} z_b(y), \ldots \tag{A.12}$$

which are faithful realizations of the symmetry.

In the large-N limit, with the use of regular thermal partition function, and periodic boundary conditions,

$$z_a(x_1, x_2 + \beta) = z_a(x_1, x_2) \tag{A.13}$$

$\mathbb{CP}^{N-1}$ is known to have a zero temperature ($\beta = \infty$) phase transition shown by Affleck [120]. With the Ω_F^0 twisted boundary condition using flavor rotation (2.29)

$$z_a(x_1, x_2 + \beta) = \Omega^0_{ab} z_b(x_1, x_2) \tag{A.14}$$

in path formulation, the story is opposite. There are no phase transition at any finite β. The theory satisfies volume independence at $N = \infty$ and adiabatic continuity at finite-N [121]. Ref. [16] interpreted this result of the path integral formalism in terms of state sums. The Hilbert space interpretation of the twisted boundary condition is analogous with what we have written down for N-dimensional oscillator, $\mathcal{Z}_{\Omega_F^0}(\beta) = \mathrm{tr}\,(e^{-\beta H} \prod_{j=1}^N e^{i\frac{2\pi}{N} j \widehat{Q}_j})$ where now $\widehat{Q}_j$ are charges associated with Cartan subgroup of G. The implication of this insertion is that it assigns phases $e^{i\frac{2\pi(a-b)}{N}}$ to adjoint representation states, and modifies the state sum as $(N^2-1)e^{-\beta E_{\mathrm{adj}}} \to (-1)e^{-\beta E_{\mathrm{adj}}}$. At large-$N$ limit, since the singlet $\bar{z}^a(x)e^{i\int_x^y a} z_a(y)$ is degenerate with the adjoint [122], the symmetry graded state sum leads to cancellation among the lightest N^2 physical particles in the spectrum.

$$(N^2-1)e^{-\beta E_{\mathrm{adj}}} + 1e^{-\beta E_{\mathrm{sing.}}} \to (-1)e^{-\beta E_{\mathrm{adj}}} + e^{-\beta(E_{\mathrm{adj}} + O(1/N^2))} \underbrace{\to}_{N\to\infty} 0 \tag{A.15}$$

This process continues for the other states as well, and in the large-N limit, the only states that contribute to the graded state sum is the ground state.

In Affleck's analysis of partition function, it is the large multiplicity of the first excited state $N^2 e^{-\beta E_{\mathrm{adj}}}$ that forces the system to a phase transition immediately at $\beta = \infty$ [120]. In our graded partition function, the fact that the first excited state contribute $0 \times e^{-\beta E_{\mathrm{adj}}}$ (similarly for higher states) is the reason that there is no phase transition all the way down to $\beta = 0$ [121]. In contrast to simple N-dimensional oscillator example, this is a non-trivial strongly coupled quantum field theory and yet,

$$\mathrm{Distilled}[\mathcal{H}] = \{\text{ground state(s)}\} \tag{A.16}$$

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The quantum distillation in this purely bosonic QFT achieves what the supersymmetric index achieves in supersymmetric QFTs. This is ultimately the reason for the adiabatic continuity and volume independence observed in ref. [121]. In particular, in the $N \to \infty$ limit, only the ground state survives in the graded partition function for $\theta \neq \pi$ and two-ground states survive for $\theta = \pi$:

$$\begin{aligned}\lim_{N\to\infty} \mathcal{Z}_{\Omega^0}(\beta,\theta) &= 1 e^{-\beta E_0(\theta)}, \qquad \theta \neq \pi \\ \lim_{N\to\infty} \mathcal{Z}_{\Omega^0}(\beta,\theta=\pi) &= 2 e^{-\beta E_0(\theta=\pi)},\end{aligned} \tag{A.17}$$

A.4 Principle chiral model in 2d

The bosonic PCM is an asymptotically free matrix field theory in $d = 2$ dimensions. Let $\mathrm{U}(x)$ denote the principal chiral field. The global symmetry of the theory is

$$G = \frac{\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R}{\mathbb{Z}_N} \tag{A.18}$$

The states in the Hilbert space fill the bi-product of k-index antisymmetric irreducible representations with its conjugate, and the degeneracy factor is. given by

$$V_k \otimes \overline{V}_k \in G \qquad \deg[(k,\overline{k})] = \left[\binom{N}{k}\right]^2 \tag{A.19}$$

Imposing

$$\mathrm{U}(x_1, x_2+\beta) = \Omega^0 \mathrm{U}(x_1,x_2)\overline{\Omega}^0 \tag{A.20}$$

boundary conditions in path integral correspond to $\mathcal{Z}_{\Omega^0}(\beta) = \mathrm{tr}\,(e^{-\beta H}\prod_{j=1}^{N} e^{i\frac{2\pi}{N}j\widehat{Q}_j})$ in path integral where $\widehat{Q}_j$ are charges associated with Cartan subgroup of the vector-like symmetry subgroup $\mathrm{SU}(N)_V$. Clearly, this assignment forces the contributions of all excited states with $k \neq 0 \mod N$ to vanish, i.e., in the state sum, we have the replacement

$$\deg[(k,\overline{k})] \Rightarrow 0, \qquad \forall k \neq 0 \mod N \tag{A.21}$$

In the large-N limit, it is again only the ground state that contributes to the state sums (A.16). This is again the reason for volume independence for $N = \infty$ limit and adiabatic continuity for finite N [14].

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