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The $SL(K+3,\mathbb{C})$ symmetry of string scatterings from D-branes

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Abstract

By using the solvability of Lauricella function $F_D^{(K)}(\alpha;\beta_1,...,\beta_K;\gamma;x_1,...,x_K)$ with nonpositive integer β_J, we show that each scattering or decay process of string and D-brane states at *arbitrary* mass levels can be expressed in terms of a single Lauricella function. This result extends the previous exact $SL(K+3,\mathbb{C})$ symmetry of tree-level open bosonic string theory to include the D-brane.

1. Introduction

Motivated by the previous calculation of high energy symmetry [1,2] of string scattering amplitudes (SSA) [3–5], [6–10], it was shown [11] recently that all SSA of four arbitrary string states of the open bosonic string theory at all kinematic regimes can be expressed in terms of the D-type Lauricella functions $F_D^{(K)}(\alpha;\beta_1,...,\beta_K;\gamma;x_1,...,x_K)$ with associated exact $SL(K+3,\mathbb{C})$ symmetry. Here the index K counts the number of variety of the oscillators in a string state (see the definition of K in Eq. (3). On the other hand, a class of polarized fermion SSA (PFSSA) at arbitrary mass levels of the R-sector of the fermionic string theory can also be expressed in terms of the D-type Lauricella functions [12]. Indeed, it can be shown [11] that these Lauricella functions form an infinite dimensional representation of the $SL(K+3,\mathbb{C})$ symmetry group.

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Moreover, it was demonstrated that there existed $K+2$ recursion relations among the D-type Lauricella functions. These recursion relations can be used to reproduce the Cartan subalgebra and simple root system of the $SL(K+3,\mathbb{C})$ group with rank $K+2$ and vice versa.

For the cases of nonpositive integer β_J (see Eq. (9)) in the D-type Lauricella functions which correspond to the cases of the SSA or the Lauricella SSA (LSSA) mentioned above, the $SL(K+3,\mathbb{C})$ group or the corresponding $K+2$ stringy Ward identities among the LSSA can be used to solve [13] all the LSSA and express them in terms of one amplitude. These exact Ward identities among the exact LSSA are generalizations of the *linear relations* with *constant coefficients* among SSA in the hard scattering limit conjectured by Gross [1,2] in 1988 and later corrected and proved in [6–10].

More recently, by using the string theory extension [14], [15,16] of the field theory BCFW on-shell recursion relations [17,18], one can show that [19] the residues of all n-point SSA including the Koba-Nielsen (KN) amplitudes can be expressed in terms of the Lauricella functions with nonpositive integer β_J. As a result, the above $SL(K+3,\mathbb{C})$ symmetry group of the 4-point LSSA was extended to the n-point LSSA with arbitrary n [20]. It is thus believed that the $SL(K+3,\mathbb{C})$ symmetry is the fundamental symmetry of the whole bosonic string theory, at least, tree-level of the bosonic string theory.

To justify the conjecture of the fundamental $SL(K+3,\mathbb{C})$ symmetry of the bosonic string theory, one needs to collect more evidences or more SSA to support the proposed exact symmetry. In this paper, we will show that the scattering and decay processes of string and D-brane states at arbitrary mass levels can again be expressed in terms of the D-type Lauricella functions with nonpositive integer β_J. This result extends the previous exact $SL(K+3,\mathbb{C})$ symmetry of tree-level open bosonic string theory to include the D-brane. This is also consistent with the previous results that the linear relations with constant coefficients among SSA in the hard scattering limit persist for the processes of D-brane scatterings [21] and decays [22] as they are all related to the exact $SL(K+3,\mathbb{C})$ symmetry of bosonic string theory. We will see that the calculation will be greatly simplified by using the solvability [13] of the Lauricella functions with nonpositive integer β_J.

2. Review of $SL(K+3,C)$ symmetry

We first review the LSSA of three tachyons and one arbitrary string states in the $26D$ open bosonic string theory and its associated $SL(K+3,\mathbb{C})$ symmetry. The general states are of the following form [11]

$$\left|r_n^T, r_m^P, r_l^L\right\rangle = \prod_{n>0}\left(\alpha_{-n}^T\right)^{r_n^T}\prod_{m>0}\left(\alpha_{-m}^P\right)^{r_m^P}\prod_{l>0}\left(\alpha_{-l}^L\right)^{r_l^L}|0,k\rangle \tag{1}$$

where $e^P=\frac{1}{M_2}(E_2,\mathrm{k}_2,0)=\frac{k_2}{M_2}$ is the momentum polarization, $e^L=\frac{1}{M_2}(\mathrm{k}_2,E_2,0)$ is the longitudinal polarization and $e^T=(0,0,1)$ is the transverse polarization on the $(2+1)$-dimensional scattering plane. In addition to the mass level $M_2^2=2(N-1)$ with

$$N=\sum_{\substack{n,m,l>0\\ \{r_j^X\neq 0\}}}\left(nr_n^T+mr_m^P+lr_l^L\right), \tag{2}$$

we define another important index K for the state in Eq. (1)

$$K = \sum_{\substack{n,m,l>0 \\ \{r_j^X \neq 0\}}} (n+m+l) \tag{3}$$

where $X = (T, P, L)$ and we have put $r_n^T = r_m^P = r_l^L = 1$ in Eq. (2) in the definition of K. Intuitively, K counts the number of variety of the α_{-j}^X oscillators in Eq. (1). For later use, we also define

$$k_j^X \equiv e^X \cdot k_j \text{ for } X = (T, P, L). \tag{4}$$

Note that SSA of three tachyons and one arbitrary string states with polarizations orthogonal to the scattering plane *vanish*.

Note that to achieve BRST invariance or physical state conditions in the old covariant quantization scheme for the state in Eq. (1), one needs to add polarizations and put on the Virasoro constraints. As an example, let's calculate the case of symmetric spin 3 state of mass level $M_2^2 = 4$. We first note that the three momentum polarizations defined on the scattering plane above satisfy the completeness relation

$$\eta^{\mu\nu} = \sum_{\alpha,\beta} e_\alpha^\mu e_\beta^\nu \eta^{\alpha\beta} \tag{5}$$

where $\mu, \nu = 0, 1, 2$ and $\alpha, \beta = P, L, T$. $Diag\ \eta^{\mu\nu} = (-1, 1, 1)$. We can use Eq. (5) to transform all μ, ν coordinates to coordinates α, β on the scattering plane. One gauge choice of the symmetric spin 3 state with Virasoro constraints can be calculated to be

$$\epsilon_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda} |0, k\rangle ; k^\mu \epsilon_{\mu\nu\lambda} = 0, \eta^{\mu\nu}\epsilon_{\mu\nu\lambda} = 0. \tag{6}$$

We can then use the helicity decomposition and writing $\epsilon_{\mu\nu\lambda} = \Sigma_{\mu,\nu,\lambda} e_\mu^\alpha e_\nu^\beta e_\lambda^\delta u_{\alpha\beta\delta}; \alpha, \beta, \delta = P, L, T$ to get

$$\epsilon_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda} |0, k\rangle = [u_{TTL}(3\alpha_{-1}^{TTL} - \alpha_{-1}^{LLL}) + u_{TTT}(\alpha_{-1}^{TTT} - 3\alpha_{-1}^{LLT})] |0, k\rangle . \tag{7}$$

It is now easy to see from Eq. (7) that to achieve BRST invariance the spin 3 state can be written as a linear combination of states in Eq. (1) with coefficients u_{TTL} and u_{TTT}.

The 4-point LSSA of three tachyons and one string state in Eq. (1) can be calculated to be [11]

$$\begin{aligned} A_4 =& B\left(-\frac{t}{2}-1, -\frac{s}{2}-1\right) F_D^{(K)}\left(-\frac{t}{2}-1; R_n^X; \frac{u}{2}+2-N; \tilde{Z}_n^X\right) \\ &\times \prod_X \left(\prod_{n=1} \left[-(n-1)! k_3^X\right]^{r_n^X}\right) \end{aligned} \tag{8}$$

where $B(a, b)$ is the Beta function with (s, t) being the usual Mandelstam variables, k_i^X is the momentum of the ith string state projected on the X polarization. In Eq. (8), we have defined

$$R_l^X \equiv \left\{-r_1^X\right\}^1, \cdots, \left\{-r_l^X\right\}^l \text{ with } \{a\}^n = \underbrace{a, a, \cdots, a}_{n}, \tag{9}$$

for the β_J in the Lauricella function and

$$Z_l^X \equiv \left[z_1^X\right], \cdots, \left[z_l^X\right] \quad \text{with} \quad \left[z_l^X\right] = z_{l0}^X, \cdots, z_{l(l-1)}^X, \tag{10}$$

where in Eq. (10), we have defined

$$z_l^X = \left| \left(-\frac{k_1^X}{k_3^X} \right)^{\frac{1}{k}} \right|, \; z_{ll'}^X = z_l^X e^{\frac{2\pi i l'}{l}}, \; \tilde{z}_{ll'}^X \equiv 1 - z_{ll'}^X \quad \text{for} \quad l' = 0, \cdots, l-1. \tag{11}$$

It is important to note that all β_j of $F_D^{(K)}$ in Eq. (8) are nonpositive integer. The D-type Lauricella function $F_D^{(K)}$ in Eq. (8) is defined to be

$$\begin{aligned} &F_D^{(K)}(\alpha; \beta_1, ..., \beta_K; \gamma; x_1, ..., x_K) \\ &= \sum_{n_1, \cdots, n_K = 0}^{\infty} \frac{(\alpha)_{n_1+\cdots+n_K}}{(\gamma)_{n_1+\cdots+n_K}} \frac{(\beta_1)_{n_1} \cdots (\beta_K)_{n_K}}{n_1! \cdots n_K!} x_1^{n_1} \cdots x_K^{n_K} \end{aligned} \tag{12}$$

where $(\alpha)_n = \alpha \cdot (\alpha+1) \cdots (\alpha+n-1)$ is the Pochhammer symbol. The result in Eq. (8) can be generalized to LSSA of four arbitrary string states, and to those of n arbitrary string states (see Eq. (25)) below [19,20].

For illustration, we calculate the Lauricella functions which correspond to the LSSA for levels $K = 1, 2$. For $K = 1$, there are three type of LSSA ($\alpha = -\frac{t}{2} - 1, \gamma = \frac{u}{2} + 2$)

$$(\alpha_{-1}^T)^{p_1}, \, F_D^{(1)}(\alpha, -p_1, \gamma - p_1, 1), \, N = p_1, \tag{13}$$

$$(\alpha_{-1}^P)^{q_1}, \, F_D^{(1)}(\alpha, -q_1, \gamma - q_1, \left[\tilde{z}_1^P\right]), \, N = q_1, \tag{14}$$

$$(\alpha_{-1}^L)^{r_1}, \, F_D^{(1)}(\alpha, -r_1, \gamma - r_1, \left[\tilde{z}_1^L\right]), \, N = r_1. \tag{15}$$

For $K = 2$, there are six type of LSSA

$$(\alpha_{-1}^T)^{p_1}(\alpha_{-1}^P)^{q_1}, \, F_D^{(2)}(\alpha, -p_1, -q_1, \gamma - p_1 - q_1, 1, \left[\tilde{z}_1^P\right]), N = p_1 + q_1, \tag{16}$$

$$(\alpha_{-1}^T)^{p_1}(\alpha_{-1}^L)^{r_1}, \, F_D^{(2)}(\alpha, -p_1, -r_1, \gamma - p_1 - r_1, 1, \left[\tilde{z}_1^L\right]), N = p_1 + r_1, \tag{17}$$

$$(\alpha_{-1}^P)^{q_1}(\alpha_{-1}^L)^{r_1}, \, F_D^{(2)}(\alpha, -q_1, -r_1, \gamma - q_1 - r_1, \left[\tilde{z}_1^P\right], \left[\tilde{z}_1^L\right]), N = q_1 + r_1, \tag{18}$$

$$(\alpha_{-2}^T)^{p_2}, \, F_D^{(2)}(\alpha, -p_2, -p_2, \gamma - 2p_2, 1, 1), \, N = 2p_2, \tag{19}$$

$$(\alpha_{-2}^P)^{q_2}, \, F_D^{(2)}(\alpha, -q_2, -q_2, \gamma - 2q_2, 1 - Z_2^P, 1 - \omega Z_2^P), \, N = 2q_2, \tag{20}$$

$$(\alpha_{-2}^L)^{r_2}, \, F_D^{(2)}(\alpha, -r_2, -r_2, \gamma - 2r_2, 1 - Z_2^L, 1 - \omega Z_2^L), \, N = 2r_2. \tag{21}$$

It is important to note that for a given K, there are infinite number of string states with arbitrary higher mass levels. Moreover, each string state was assigned a particular value of integer K, and its associated LSSA is a basis of the $SL(K+3, \mathbb{C})$ group representation.

To demonstrate the $SL(K+3, \mathbb{C})$ symmetry of the LSSA, one first defines the basis functions [23]

$$\begin{aligned} &f_{ac}^{b_1 \cdots b_K}(\alpha; \beta_1, \cdots, \beta_K; \gamma; x_1, \cdots, x_K) \\ &= B(\gamma - \alpha, \alpha) \, F_D^{(K)}(\alpha; \beta_1, \cdots, \beta_K; \gamma; x_1, \cdots, x_K) \, a^\alpha b_1^{\beta_1} \cdots b_K^{\beta_K} c^\gamma, \end{aligned} \tag{22}$$

so that the LSSA in Eq. (8) can be rewritten as [24]

$$A_4 = f_{11}^{-(n-1)! k_3^X} \left(-\frac{t}{2} - 1; \, R_n^X, ; \, \frac{u}{2} + 2 - N; \, \tilde{Z}_n^X \right). \tag{23}$$

One can then introduce the $(K+3)^2-1$ generators $\mathcal{E}_{ij}$ of $SL(K+3,\mathbb{C})$ group [23,24]

$$\left[\mathcal{E}_{ij},\mathcal{E}_{kl}\right]=\delta_{jk}\mathcal{E}_{il}-\delta_{li}\mathcal{E}_{kj};\quad 1\leqslant i,j\leqslant K+3 \tag{24}$$

to operate on the basis functions in Eq. (22). These are 1 E^{α}, K $E^{\beta_k}(k=1,2\cdots K)$, 1 E^{γ}, 1 $E^{\alpha\gamma}$, K $E^{\beta_k\gamma}$ and K $E^{\alpha\beta_k\gamma}$ which sum up to $3K+3$ raising generators. There are also $3K+3$ corresponding lowering operators. In addition, there are $K(K-1)$ $E^{\beta_k}_{\beta_p}$ and $K+2$ $\left\{J_\alpha, J_{\beta_k}, J_\gamma\right\}$ generators, the Cartan subalgebra. In sum, the total number of generators are $2(3K+3)+K(K-1)+K+2=(K+3)^2-1$ [24].

For the general 4-point LSSA, it is straightforward to calculate the operation of these generators on the basis functions and show the $SL(K+3,\mathbb{C})$ symmetry [24]. For the cases of higher point ($n\geq 5$) LSSA, one encounters the operation on the sum of products of the Lauricella functions [19]

$$\text{Residue of } n\text{-point } LSSA\sim\sum\text{coefficient}\prod(\text{single tensor 4-point } LSSA). \tag{25}$$

Therefore, one needs to deal with product representations of $SL(K+3,\mathbb{C})$.

Indeed, we have recently applied the string theory extension of field theory BCFW on-shell recursion relations [17,18] to show that the $SL(K+3,C)$ symmetry group of the 4-point LSSA persists for general n-point SSA with arbitrary higher point couplings in string theory [20]. We thus have shown that the $SL(K+3,C)$ symmetry is an exact symmetry of the whole bosonic string theory, and that all n-point SSA of the bosonic string theory form an infinite dimensional representation of the $SL(K+3,C)$ group. Moreover, all residues of SSA in the string theory on-shell recursion prescription can be expressed in terms of the four-point LSSA.

There is an interesting issue of the stringy on-shell Ward identities or decoupling of zero-norm states associated with the $SL(K+3,C)$ symmetry. For the n-point Ward identities with $n\geq 5$, one can either write down the linear on-shell Ward identities in terms of n-point functions or, through reduction of stringy BCFW recursion, calculate the non-linear on-shell Ward identities in terms of 4-point functions. For the latter case, we conjecture that the non-linear Ward identities can be reduced to the equivalent linear 4-point Ward identities since both forms of Ward identities are associated with the same $SL(K+3,C)$ group.

3. Solvability of LSSA

There exist $K+2$ recurrence relations for the D-type Lauricella functions [24]. Moreover, these recurrence relations can be used to reproduce the Cartan subalgebra and simple root system of the $SL(K+3,\mathbb{C})$ group with rank $K+2$ [24]. With the Cartan subalgebra and the simple roots, one can easily write down the whole Lie algebra of the $SL(K+3,\mathbb{C})$ group. So one can construct the $SL(K+3,\mathbb{C})$ Lie algebra from the recurrence relations and vice versa.

On the other hand, one can use the $K+2$ recurrence relations to deduce the following key recurrence relation [13]

$$\begin{aligned}&x_j F_D^{(K)}(\alpha;\beta_1,.,\beta_i-1,..,\beta_K;\gamma;x_1,...,x_K)\\&-x_i F_D^{(K)}\left(\alpha;\beta_1,.,\beta_j-1,..,\beta_K;\gamma;x_1,...,x_K\right)\\&+\left(x_i-x_j\right)F_D^{(K)}(\alpha;\beta_1,...,\beta_K;\gamma;x_1,...,x_K)=0,\end{aligned} \tag{26}$$

which, for the case of nonpositive β_j, can be repeatedly used to decrease the value of K and reduce all the Lauricella functions $F_D^{(K)}$ in the LSSA to the Gauss hypergeometric functions $F_D^{(1)}=$

${}_2F_1(\alpha, \beta, \gamma, x)$. Indeed, one can repeatedly apply Eq. (26) to the Lauricella functions in Eq. (8) and express $F_D^{(K)}(\alpha; \beta_1, ..., \beta_K; \gamma; x_1, ..., x_K)$ in terms of $F_D^{(K-1)}(\beta_1, ..\beta_{i-1}, \beta_{i+1}...\beta'_j, ...\beta_K)$ with $\beta'_j = \beta_j, \beta_j - 1, ..., \beta_j - |\beta_i|$ or $F_D^{(K-1)}(\beta_1, ...\beta'_i, ...\beta_{j-1}, \beta_{j+1}, ...\beta_K)$ with $\beta'_i = \beta_i, \beta_i - 1, ..., \beta_i - \left|\beta_j\right|$ (assuming $i < j$).

For example, for say $K = 2$, Eq. (26) reduces to

$$x_2 F_D^{(2)}(\alpha; \beta_1 - 1, \beta_2; \gamma; x_1, x_2) - x_1 F_D^{(2)}(\alpha; \beta_1, \beta_2 - 1; \gamma; x_1, x_2) + (x_1 - x_2) F_D^{(2)}(\alpha; \beta_1, \beta_2; \gamma; x_1, x_2) = 0. \tag{27}$$

For say $\beta_1 = 0$ and $\beta_2 = -1$, we get

$$\begin{aligned} & x_2 F_D^{(2)}(\alpha; -1, -1; \gamma; x_1, x_2) \\ & = x_1 F_D^{(2)}(\alpha; 0, -2; \gamma; x_1, x_2) - (x_1 - x_2) F_D^{(2)}(\alpha; 0, -1; \gamma; x_1, x_2) \\ & = x_1 F_D^{(1)}(\alpha; -2; \gamma; x_2) - (x_1 - x_2) F_D^{(1)}(\alpha; -1; \gamma; x_2) \end{aligned} \tag{28}$$

which express the Lauricella function with $K = 2$ in terms of those of $K = 1$. We can repeat similar process to decrease the value of K.

Moreover, one can further reduce the Gauss hypergeometric functions by deriving a multiplication theorem for them, and then solve [13] all the LSSA in terms of one single amplitude. This solvability is crucial to show that all scattering and decay processes of string and D-brane states at arbitrary mass levels can be expressed in terms of the Lauricella function and thus its associated $SL(K + 3, \mathbb{C})$ symmetry.

4. Closed string scattered off D-brane

In this paper, we will consider scattering and decay processes of string and D-brane states at arbitrary mass levels. These are three classes of processes [25], [26–30]: (A). Closed string scattered off D-brane, (B). Closed string decays into two open strings on the brane and (C). Four open string scattering on the brane. The calculation of process (C) is similar to that of four open string scattering without D-brane, and thus can be expressed in terms of the Lauricella function.

In this section, we first consider process (A). In [25], the calculation was done only for the massless string states. Here we will consider scatterings of arbitrary massive string states for the bosonic string. The standard propagators of the left and right moving fields are $\langle X^\mu(z) X^\nu(w)\rangle = -\eta^{\mu\nu} \log(z - w)$, $\left\langle \tilde{X}^\mu(\bar{z}) \tilde{X}^\nu(\bar{w})\right\rangle = -\eta^{\mu\nu} \log(\bar{z} - \bar{w})$. In addition, there are nontrivial correlator as well [25]

$$\left\langle X^\mu(z) \tilde{X}^\nu(\bar{w})\right\rangle = -D^{\mu\nu} \log(z - \bar{w}) \tag{29}$$

as a result of the Dirichlet boundary condition at the real axis. The diagonal matrix D in Eq. (29) reverses the sign for fields satisfying Dirichlet boundary condition. That is, there are $p + 1$ Neumann and $25 - p$ Dirichlet for a Dp-brane. We will follow the standard notation and make the following replacement [25]

$$\tilde{X}^\mu(\bar{z}) \to D^\mu{}_\nu X^\nu(\bar{z}) \tag{30}$$

which allows us to use the standard correlators throughout our calculations. As a warm up exercise, we first consider tachyon to tachyon scattering [21]

$$\begin{aligned} A_{tach} &= \int d^2z_1 d^2z_2 \left\langle V_1\left(z_1,\bar{z}_1\right) V_2\left(z_2,\bar{z}_2\right)\right\rangle \\ &= \int d^2z_1 d^2z_2 \left(z_1-\bar{z}_1\right)^{k_1\cdot D\cdot k_1} \left(z_2-\bar{z}_2\right)^{k_2\cdot D\cdot k_2} \left|z_1-z_2\right|^{2k_1\cdot k_2} \left|z_1-\bar{z}_2\right|^{2k_1\cdot D\cdot k_2}. \end{aligned} \tag{31}$$

To fix the $SL\,(2, R)$ invariance, we set $z_1 = iy$ and $z_2 = i$ and, for the contribution of the (0, 1) interval, we obtain [21]

$$\begin{aligned} A_{tach}^{(0,1)} &= 4\,(2i)^{2a_0} \int_0^1 dy\; y^{a_0}\,(1-y)^{b_0}\,(1+y)^{c_0} \\ &= 4\,(2i)^{2a_0}\, 2^{-2a_0-1-N} \int_0^1 dx\; x^{b_0}\,(1-x)^{a_0}\,(1+x)^{a_0+N}\,. \end{aligned} \tag{32}$$

In the above calculations, we have defined

$$a_0 = k_1\cdot D\cdot k_1 = k_2\cdot D\cdot k_2, \tag{33}$$

$$b_0 = 2k_1\cdot k_2 + 1, \tag{34}$$

$$c_0 = 2k_1\cdot D\cdot k_2 + 1, \tag{35}$$

so that

$$2a_0 + b_0 + c_0 + 2 = 4N_1 \equiv -N, \tag{36}$$

and $-k_1^2 = M^2 \equiv \frac{\alpha'_{closed} M^2_{closed}}{2} = 2(N_1 - 1)$, $N_1 = 0$ for tachyon. The momentum conservation on the D-brane

$$D\cdot k_1 + k_1 + D\cdot k_2 + k_2 = 0 \tag{37}$$

is crucial to get the final result Eq. (32). Similarly, for the contribution of the $(1, \infty)$ interval, we end up with

$$\begin{aligned} A_{tach}^{(1,\infty)} &= 4\,(2i)^{2a_0} \int_1^\infty dy\; y^{a_0}\,(y-1)^{b_0}\,(1+y)^{c_0} \\ &= 4\,(2i)^{2a_0}\, 2^{-2a_0-1-N} \int_0^1 dx\; x^{b_0}\,(1-x)^{a_0+N}\,(1+x)^{a_0}\,. \end{aligned} \tag{38}$$

For the general massive tensor to another massive tensor scattering, the calculation will be very complicated as there are many new contraction terms. We will use the *solvability* of the LSSA discussed above to simplify the calculation. The strategy is as follows: We can simply calculate a typical term of a given process. If the result turns out to be a Lauricella function with nonpositive β_j, we can then use the solvability property to argue that the final amplitude after summing up all typical terms of the process is a LSSA. To do the calculation, we first define

$$a = k_1\cdot D\cdot k_1 + n_a, \tag{39}$$

$$b = 2k_1\cdot k_2 + 1 + n_b, \tag{40}$$

$$c = 2k_1\cdot D\cdot k_2 + 1 + n_c, \tag{41}$$

where n_a, n_b and n_c are integer and then define $N' = -(2n_a + n_b + n_c)$, so that

$$2a + b + c + 2 + N' = 4N_1 \Longrightarrow 2a + b + c + 2 = 4N_1 - N' \equiv -N \tag{42}$$

where $k_1^2 = 2(N_1 - 1)$ and N_1 is the mass level of k_1. It is easy to see that a typical term in the general tensor to tensor scattering can be calculated to be [21]

$$\begin{aligned} I_{(0,1)} &= \int_0^1 dt\ y^a (1-y)^b (1+y)^c\,, \\ &= 2^{-2a-1-N} \int_0^1 dx\ x^b (1-x)^a (1+x)^{a+N}\,. \end{aligned} \tag{43}$$

Similarly, for the $(1, \infty)$ interval, one gets

$$\begin{aligned} I_{(1,\infty)} &= \int_1^\infty dy\ y^a (y-1)^b (1+y)^c \\ &= 2^{-2a-1-N} \int_0^1 dx\ x^b (1-x)^{a+N} (1+x)^a\,. \end{aligned} \tag{44}$$

The sum of the two channels gives [21]

$$\begin{aligned} I &= 2^{-2a-2-N} \sum_{m=0}^{N} \left[1 + (-1)^m\right] \binom{N}{m} \cdot \frac{\Gamma(a+1)\,\Gamma\left(\frac{b+1}{2} + \frac{m}{2}\right)}{\Gamma\left(a + \frac{b+3}{2} + \frac{m}{2}\right)} \\ &= 2^{-2a-1-N} \frac{\Gamma(a+1)\,\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(a + \frac{b+3}{2}\right)} \sum_{n=0}^{\left[\frac{N}{2}\right]} \binom{N}{2n} \frac{\left(\frac{b+1}{2}\right)_n}{\left(a + \frac{b+3}{2}\right)_n} \\ &= 2^{-2a-1-N} \cdot B\left(a+1, \frac{b+1}{2}\right) \cdot {}_3F_2\left(\frac{b+1}{2}, -\left[\frac{N}{2}\right], \frac{1}{2} - \left[\frac{N}{2}\right]; a + \frac{b+3}{2}, \frac{1}{2}; 1\right) \end{aligned} \tag{45}$$

where ${}_3F_2$ is a generalized hypergeometric function. For the special arguments of ${}_3F_2$ in Eq. (45), the hypergeometric function terminates to a finite sum and, as a result, the whole scattering amplitudes consistently reduce to the usual beta function. In calculating Eq. (45), we have used the identity

$$\sum_{n=0}^{\left[\frac{N}{2}\right]} \binom{N}{2n} \frac{(A)_n}{(C)_n} = {}_3F_2\left(A, -\left[\frac{N}{2}\right], \frac{1}{2} - \left[\frac{N}{2}\right]; C, \frac{1}{2}; 1\right), \tag{46}$$

which can be easily proved.

At this point, one might think that the amplitude calculated in Eq. (45) is not a LSSA, and the $SL(K+3, \mathbb{C})$ group may be just a subgroup of an unknown larger symmetry group $G \supseteq SL(K+3, \mathbb{C})$ of the bosonic string theory. However, we will see that this is not the case. To show that ${}_3F_2$ in the amplitude Eq. (45) is a LSSA, we first do a change of variable $y = x^2$ to get

$$
\begin{aligned}
I &= 2^{-2a-1-N} \int_0^1 dx\; x^b (1-x)^a (1+x)^a \left[(1+x)^N + (1-x)^N\right] \\
&= 2^{-2a-1-N} \int_0^1 dy\; y^{\frac{b-1}{2}} (1-y)^a \, G(y)
\end{aligned}
\tag{47}
$$

where

$$
G(y) = G\left(x^2\right) = \frac{1}{2}\left[(1+x)^N + (1-x)^N\right]. \tag{48}
$$

We can solve

$$
G\left(x^2\right) = \frac{1}{2}\left[(1+x)^N + (1-x)^N\right] = 0 \tag{49}
$$

to get

$$
x = x_k = \frac{(-1)^{\frac{1}{N}} - 1}{(-1)^{\frac{1}{N}} + 1} = \frac{w_{N,k} - 1}{w_{N,k} + 1} = i \tan\frac{\theta_k}{2} \tag{50}
$$

where

$$
w_{N,k} = e^{\frac{i\pi}{N} + \frac{2i\pi k}{N}} = e^{i\theta_k}, \theta_k = \frac{\pi}{N} + \frac{2\pi k}{N}, k = 1, 2 \cdots, N. \tag{51}
$$

We can now do the following factorization

$$
\begin{aligned}
G\left(x^2\right) &= \prod_{k=1}^{N}\left(1 - \frac{x}{x_k}\right) = \prod_{k=1}^{\left[\frac{N}{2}\right]}\left(1 - \frac{x}{x_k}\right)\left(1 - \frac{x}{\bar{x}_k}\right) = \prod_{k=1}^{\left[\frac{N}{2}\right]}\left(1 - \left(\frac{x}{x_k} + \frac{x}{\bar{x}_k}\right) + \frac{x^2}{x_k \bar{x}_k}\right) \\
&= \prod_{k=1}^{\left[\frac{N}{2}\right]}\left(1 + \frac{x^2}{x_k \bar{x}_k}\right) = \prod_{k=1}^{\left[\frac{N}{2}\right]}\left(1 - \frac{x^2}{x_k^2}\right)
\end{aligned}
\tag{52}
$$

to obtain

$$
G(y) = \prod_{k=1}^{\left[\frac{N}{2}\right]}\left(1 - \frac{y}{x_k^2}\right) = \prod_{k=1}^{\left[\frac{N}{2}\right]}\left(1 + \frac{y}{\left(\tan\frac{\theta_k}{2}\right)^2}\right) = \prod_{k=1}^{\left[\frac{N}{2}\right]}\left(1 + \frac{y}{\left(\tan\left(\frac{\pi}{2N} + \frac{\pi k}{N}\right)\right)^2}\right). \tag{53}
$$

Finally, we can use

$$
\begin{aligned}
&\int_0^1 dy\; y^{\frac{b-1}{2}} (1-y)^a \, G(y) \\
&= \int_0^1 dy\; y^{\frac{b-1}{2}} (1-y)^a \prod_{k=1}^{\left[\frac{N}{2}\right]}\left(1 - \frac{y}{x_k^2}\right)
\end{aligned}
$$

$$= B\left(a+1,\frac{b+1}{2}\right)\cdot F_D^{\left(\left[\frac{N}{2}\right]\right)}\left(\frac{b+1}{2};\underbrace{-1,...,-1}_{\left[\frac{N}{2}\right]};a+\frac{b+3}{2};\frac{1}{x_1^2},...,\frac{1}{x_{\left[\frac{N}{2}\right]}^2}\right) \tag{54}$$

to obtain the identification

$$\begin{aligned}&{}_3F_2\left(\frac{b+1}{2},-\left[\frac{N}{2}\right],\frac{1}{2}-\left[\frac{N}{2}\right];a+\frac{b+3}{2},\frac{1}{2};1\right)\\&=F_D^{\left(\left[\frac{N}{2}\right]\right)}\left(\frac{b+1}{2};\underbrace{-1,...,-1}_{\left[\frac{N}{2}\right]};a+\frac{b+3}{2};\frac{1}{x_1^2},...,\frac{1}{x_{\left[\frac{N}{2}\right]}^2}\right)\end{aligned} \tag{55}$$

where x_k is defined in Eq. (50). We believe that the identity in Eq. (55) derived from string theory was not known previously in the literature [31]. In conclusion, we have shown that each amplitude of process (A) can be expressed in terms of a single Lauricella function with nonpositive integer β_j and thus is a LSSA. As a result, all scattering of string at arbitrary mass levels from D-brane calculated in (A) form a part of an infinite dimensional representation of the exact $SL(K+3,\mathbb{C})$ symmetry of the bosonic string theory.

5. Closed string decays into two open string

In this section, we consider process (B), namely, closed string decays into two open strings on the brane. We will adapt the same strategy used in the last section and calculate only a typical term of a given process. We begin with the kinematics of the decay process. The momentum conservation on the D-brane reads

$$\frac{1}{2}(k_c+D\cdot k_c)+k_1+k_2=0 \tag{56}$$

where k_c is the momentum of the closed string state. In the usual three-point amplitudes, momentum conservation completely constrains the kinematics. In the presence of D-brane, the non-conservation of momentum in the directions transverse to the D-brane gives precisely one kinematic variable which can be defined to be

$$t=-(k_1+k_2)^2\,. \tag{57}$$

By using Eq. (56) and Eq. (57), one easily gets

$$\begin{aligned}k_1\cdot k_c&=k_1\cdot D\cdot k_c=\frac{t+M_1^2-M_2^2}{2},\\k_2\cdot k_c&=k_2\cdot D\cdot k_c=\frac{t+M_2^2-M_1^2}{2},\end{aligned} \tag{58}$$

which give

$$t=k_1\cdot k_c+k_2\cdot D\cdot k_c=k_2\cdot k_c+k_1\cdot D\cdot k_c. \tag{59}$$

We first calculate the amplitude of a closed string tachyon to decay into two open string tachyons

$$
\begin{aligned}
A_{tach} &= \int dx_1 dx_2 d^2 z \left\langle e^{ik_1\cdot X(x_1)} e^{ik_2\cdot X(x_2)} e^{ik_c\cdot X(z)} e^{ik_c\cdot \bar{X}(\bar{z})} \right\rangle \\
&= \int dx_1 dx_2 d^2 z \cdot (x_1 - x_2)^{k_1\cdot k_2} (z-\bar{z})^{k_c\cdot D\cdot k_c} (x_1 - z)^{k_1\cdot k_c} \\
&\cdot (x_1-\bar{z})^{k_1\cdot D\cdot k_c} (x_2 - z)^{k_2\cdot k_c} (x_2-\bar{z})^{k_2\cdot D\cdot k_c} .
\end{aligned}
\tag{60}
$$

The next step is to use $\{z_1, z_2, z_3, \bar{z}_3\} = \{-x, x, i, -i\}$ to fix the $SL(2, R)$ invariance and obtain

$$
\begin{aligned}
A_{tach} &= \int dx\ x^{k_1\cdot k_2} (x+i)^{k_1\cdot k_c + k_2\cdot D\cdot k_c} (x-i)^{k_2\cdot k_c + k_1\cdot D\cdot k_c} \\
&= \int dx\ x^{k_1\cdot k_2} (x+i)^{t} (x-i)^{t} \\
&= \int dx\ x^{b_0} \left(1+x^2\right)^{a_0}
\end{aligned}
\tag{61}
$$

where we have used Eq. (59) and defined

$$
a_0 = t, b_0 = k_1 \cdot k_2. \tag{62}
$$

We now turn to the general mass level case. We will again use the *solvability* of the LSSA discussed above to simplify the calculation as before. A typical term of an arbitrary massive closed string state decays into two arbitrary massive open string states can be written as

$$
A = \int dx\ x^{k_1\cdot k_2 + n_b} (x+i)^{t+n_a} (x-i)^{t+n_c} \tag{63}
$$

where n_a, n_b and n_c are related to mass levels of k_c, k_1 and k_2. At this point, we expect after summing up all terms in the calculation, a real amplitude will be obtained. So we are going to calculate only the real part of A

$$
\begin{aligned}
A + \bar{A} &= \int_{-\infty}^{+\infty} dx\ x^{k_1\cdot k_2 + n_b} \left(x^2+1\right)^{t} \left[(x+i)^{n_a} (x-i)^{n_c} + (x-i)^{n_a} (x+i)^{n_c}\right] \\
&= \int_{-\infty}^{+\infty} dx\ x^{k_1\cdot k_2 + n_b + N} \left(x^2+1\right)^{t+\min\{n_a,n_c\}} \left[\left(1-\frac{i}{x}\right)^N + \left(1+\frac{i}{x}\right)^N\right]
\end{aligned}
\tag{64}
$$

where $N \equiv |n_c - n_a|$, and see whether the final answer is a Lauricella function. Eq. (64) can be further reduced to

$$
\begin{aligned}
A + \bar{A} &= \int_{-\infty}^{+\infty} dx\ x^{b} \left(x^2+1\right)^{a} \sum_{m=0}^{N} \binom{N}{m} \left[1+(-1)^m\right] \left(\frac{i}{x}\right)^m \\
&= 2\sum_{n=0}^{\left[\frac{N}{2}\right]} \binom{N}{2n} (-1)^n \int_{-\infty}^{+\infty} dx\ x^{b-2n} \left(x^2+1\right)^{a}
\end{aligned}
\tag{65}
$$

where we have defined

$$
\begin{aligned}
a &= t + \min\{n_a, n_c\}, \\
b &= k_1 \cdot k_2 + n_b + N,
\end{aligned}
\tag{66}
$$

which are higher mass level generalization of Eq. (62). We can use the change of variable $y = \frac{x^2}{x^2+1}$ to perform the integral in Eq. (65)

$$\begin{aligned}
&\int_{-\infty}^{+\infty} dx\, x^b \left(1+x^2\right)^a \\
&= \frac{1}{2}\left[1+(-1)^b\right] \int_0^1 dy\, y^{\frac{b}{2}-\frac{1}{2}} (1-y)^{-a-\frac{b}{2}-\frac{3}{2}} \\
&= \frac{1}{2}\left[1+(-1)^b\right] \frac{\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(-a-\frac{b}{2}-\frac{1}{2}\right)}{\Gamma(-a)}.
\end{aligned} \tag{67}$$

Finally, we obtain

$$A+\bar{A} = 2\sum_{n=0}^{\left[\frac{N}{2}\right]} \binom{N}{2n} (-1)^n \frac{1}{2}\left[1+(-1)^b\right] \frac{\Gamma\left(\frac{b}{2}+\frac{1}{2}+n\right)\Gamma\left(-a-\frac{b}{2}-\frac{1}{2}-n\right)}{\Gamma(-a)}. \tag{68}$$

To derive a Lauricella function in Eq. (68), we note that

$$\frac{\Gamma\left(\frac{b}{2}+\frac{1}{2}+n\right)\Gamma\left(-a-\frac{b}{2}-\frac{1}{2}-n\right)}{\Gamma(-a)} = \frac{\left(\frac{b}{2}+\frac{1}{2}\right)_n \Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(-a-\frac{b}{2}-\frac{1}{2}\right)}{(-1)^n \left(a+\frac{b}{2}+\frac{3}{2}\right)_n \Gamma(-a)}. \tag{69}$$

So Eq. (68) can be written as

$$\begin{aligned}
A+\bar{A} &= 2\left[1+(-1)^b\right] \frac{\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(-a-\frac{b}{2}-\frac{1}{2}\right)}{\Gamma(-a)} \sum_{n=0}^{\left[\frac{N}{2}\right]} \binom{N}{2n} \frac{\left(\frac{b}{2}+\frac{1}{2}\right)_n}{\left(a+\frac{b}{2}+\frac{3}{2}\right)_n} \\
&= 2\left[1+(-1)^b\right] B\left(-a-\frac{b+1}{2}, \frac{b+1}{2}\right) \\
&\times {}_3F_2\left(\frac{b+1}{2}, -\left[\frac{N}{2}\right], \frac{1}{2}-\left[\frac{N}{2}\right]; a+\frac{b+3}{2}; \frac{1}{2}; 1\right) \\
&= 2\left[1+(-1)^b\right] B\left(-a-\frac{b+1}{2}, \frac{b+1}{2}\right) \\
&\times F_D^{\left(\left[\frac{N}{2}\right]\right)}\left(\frac{b+1}{2}; \underbrace{-1, \ldots, -1}_{\left[\frac{N}{2}\right]}; a+\frac{b+3}{2}; \frac{1}{x_1^2}, \ldots, \frac{1}{x_{\left[\frac{N}{2}\right]}^2}\right)
\end{aligned} \tag{70}$$

where we have used the identities in Eq. (46) and Eq. (55). Finally, we can use the solvability property of $F_D^{\left(\left[\frac{N}{2}\right]\right)}$ with nonpositive β_j to argue that the final amplitude after summing up all typical terms of the decay process is a LSSA.

Incidentally, we note that the factor in the first line of Eq. (70)

$$\frac{1}{2}\left[1+(-1)^b\right] \frac{\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(-a-\frac{b}{2}-\frac{1}{2}\right)}{\Gamma(-a)} = \int_{-\infty}^{+\infty} dx\, x^b \left(1+x^2\right)^a \tag{71}$$

with $a = 2t - 1, b = -2t$ can be calculated to be

$$\frac{1}{2}\left[1+(1)^{-t}\right]\frac{\Gamma\left(-t+\frac{1}{2}\right)\Gamma\left(-2t+1+t-\frac{1}{2}\right)}{\Gamma\left(-2t+1\right)}=\frac{\Gamma^{2}\left(-t+\frac{1}{2}\right)}{\Gamma\left(-2t+1\right)}. \tag{72}$$

By using the duplication formula for the gamma function

$$\Gamma\left(-t+\frac{1}{2}\right)=\frac{2^{1+2t}\sqrt{\pi}\,\Gamma\left(-2t\right)}{\Gamma\left(-t\right)}, \tag{73}$$

the result in Eq. (72) can be further reduced to

$$\frac{\Gamma^{2}\left(-t+\frac{1}{2}\right)}{\Gamma\left(-2t+1\right)}=\frac{2^{2+4t}\pi\,\Gamma\left(-2t\right)}{-2t\Gamma^{2}\left(-t\right)}=\frac{4\cdot 16^{t}t\pi\,\Gamma\left(-2t\right)}{-2\left(-t\Gamma\left(-t\right)\right)^{2}}=-2t\cdot 16^{t}\frac{\pi\,\Gamma\left(-2t\right)}{\Gamma^{2}\left(-t+1\right)}. \tag{74}$$

The factor $\frac{\Gamma(-2t)}{\Gamma^{2}(-t+1)}$ can also be found in [25,27] for the massless string/D-brane decay process.

6. Conclusion

In conclusion, in this paper we have shown that each amplitude of processes (A) and (B) for arbitrary massive string/D-brane states can be expressed in terms of a single Lauricella function with nonpositive integer β_j and thus is a LSSA. To obtain the final results, we have used the *solvability* of the LSSA with nonpositive integer β_j to simplify the calculation.

In addition to the scattering processes calculated in (A), all decay amplitudes of string/D-brane states at arbitrary mass levels calculated in (B) also form a part of an infinite dimensional representation of the exact $SL(K+3,\mathbb{C})$ symmetry of the bosonic string theory. The results in this paper extends the previous exact $SL(K+3,\mathbb{C})$ symmetry of tree-level open bosonic string theory to include the D-brane.

CRediT authorship contribution statement

Sheng-hong Lai, Jen-chi Lee and Yi Yang: the authors contribute equally.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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