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Radiative generation of realistic neutrino mixing with $A4$

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Abstract

Radiative generation of realistic mixing in neutrino sector is studied at one-loop level in a scotogenic $A4 \times Z_2$ symmetric framework. A scheme of obtaining non-zero θ_{13} through small mass splitting in right-handed neutrino sector is proposed. The model consists of three right-handed neutrinos, two of which were required to be degenerate in masses to yield the common structure of the left-handed neutrino mass matrix that corresponds to $\theta_{13} = 0$, $\theta_{23} = \pi/4$ and any θ_{12}^0 in particular the choices specific to the Tribimaximal (TBM), Bimaximal (BM) and Golden Ratio (GR) mixings. Non-zero θ_{13}, deviations of θ_{23} from maximality and small corrections to the solar mixing angle θ_{12} can be generated in one stroke by shifting from this degeneracy in the right-handed neutrino sector by a small amount. The lightest among the three Z_2 odd inert $SU(2)_L$ doublet scalars present in the model can be a potential dark matter candidate.
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1. Introduction

Neutrino oscillation observations have clearly demonstrated the massive nature of neutrinos. The mass eigenstates are non-degenerate and distinct from the flavour eigenstates and are connected to each other by the Pontecorvo, Maki, Nakagawa, Sakata – PMNS – matrix usually parametrized as:

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Table 1
θ_{12}^0 for the different popular lepton mixings viz. TBM, BM, and GR mixing.

Model	TBM	BM	GR
θ_{12}^0	35.3°	45.0°	31.7°

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & -s_{13}e^{-i\delta} \\ -c_{23}s_{12} + s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} + s_{23}s_{13}s_{12}e^{i\delta} & -s_{23}c_{13} \\ -s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} + c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{1}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$.

In 2012, the short-baseline reactor anti-neutrino experiments observed non-zero θ_{13}, yet small compared to the other two mixing angles [1]. Prior to this observation, models leading to several structures like the Tribimaximal (TBM), Bimaximal (BM) and Golden Ratio (GR) mixings (which we refer henceforth as popular lepton mixings) were studied all of which were constructed with $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ and varying θ_{12}^0 yielded the different alternatives.

Putting $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ in Eq. (1) can lead to the common structure for all popular mixings:

$$U^0 = \begin{pmatrix} \cos\theta_{12}^0 & \sin\theta_{12}^0 & 0 \\ -\frac{\sin\theta_{12}^0}{\sqrt{2}} & \frac{\cos\theta_{12}^0}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sin\theta_{12}^0}{\sqrt{2}} & \frac{\cos\theta_{12}^0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{2}$$

where θ_{12}^0 for TBM, BM and GR are listed in Table 1.

The present 3σ global fits for the three mixing angles [2,3]:

$$\begin{aligned} \theta_{12} &= (31.42 - 36.05)^\circ, \\ \theta_{23} &= (40.3 - 51.5)^\circ, \\ \theta_{13} &= (8.09 - 8.98)^\circ. \end{aligned} \tag{3}$$

The numbers are from NuFIT3.2 of 2018 [2].

Thus the popular mixings are in disagreement with the observed non-zero θ_{13}. A plethora of activities had been taking place since this discovery to incorporate non-zero θ_{13} in these mixings. Attempts to relate the smallness of solar splitting with that of θ_{13} can be found in [4]. In [5], Δm^2_{atmos} and $\theta_{23} = \pi/4$ were embedded in the dominant component of neutrino masses and mixing and the other oscillation parameters such as θ_{13}, θ_{12}, the deviation of θ_{23} from $\pi/4$, and Δm^2_{solar} were obtained perturbatively from a smaller see-saw [6] contribution.[1] Vanishing θ_{13} can be induced by certain symmetries and models generating non-zero θ_{13} through perturbation to such symmetric structures are also studied [8,9].

In [10,11], discrete flavour symmetries like $A4$, $S3$ were used to devise a two-component Lagrangian formalism at tree-level to ameliorate all the popular mixing patterns in single stroke. The dominant contribution to the Lagrangian was obtained from Type II see-saw mechanism characterized by popular mixing patterns, to which corrections were obtained from a sub-dominant Type I see-saw contribution. In [12] the same scheme was performed for the no solar

[1] Some such earlier models can be found in [7].

mixing (NSM) case i.e., $\theta_{12}^0 = 0$ case with $A4$ symmetry.[2] The basic difference between [11,12] and earlier works with $A4$ [13–15] is that in the earlier works Type II see-saw was used to generate the mass matrices and obtaining TBM was the prime goal. More realistic mixings can be found in recent works [16,17]. In [18] scotogenic models were explored.

In this paper, we intend to generate:

1. the popular mixing structure in Eq. (2) with $\theta_{13} = 0$, $\theta_{23} = \pi/4$ and θ_{12}^0 as listed in Table 1
2. non-zero θ_{13}, deviations of θ_{23} from $\pi/4$ and small corrections to θ_{12}

radiatively[3] with $A4$ flavour symmetry.[4] Precisely, this $A4 \times Z_2$ symmetric model will produce neutrino masses at one-loop level using three right-handed neutrinos that transform as a triplet under $A4$. To get Eq. (2) it is necessary that two of these right-handed neutrinos are degenerate. A little shift from that degeneracy will yield non-zero θ_{13}, deviations of atmospheric mixing from maximality and tinker the solar mixing by a small amount in one go.

In order to accomplish this we also need to introduce a Z_2 odd $A4$ triplet scalar field η, the lightest of which could be a potential dark matter candidate.

There had been ample study [20] of neutrino mass models in the light of the discrete flavour symmetry $A4$, out of which many considers one or some of the aspects discussed above.[5] A common practice in [20] was also to generate non-zero θ_{13}, starting from TBM. However, the attempt of simultaneously addressing all the objectives mentioned above in a scotogenic framework based on $A4 \times Z_2$ i.e., generating the common structure of the neutrino mass matrix that corresponds to $\theta_{13} = 0$, $\theta_{23} = \pi/4$ and θ_{12}^0 of the particular values shown in Table 1 leading to TBM, BM and GR mixing patterns at one-loop level with particles that can be dark matter candidates and using the technique of exploiting tiny mass splitting in the right-handed neutrino sector to generate $\theta_{13} \neq 0$, $\theta_{23} \neq \pi/4$ and small corrections to the solar mixing angle θ_{12} in one stroke to make the mixing angles realistic is specific to our model.

2. The model

The neutrino mass matrix in the mass basis is given by $M_{\nu L}^{mass} = \text{diag}(m_1, m_2, m_3)$. This when expressed in flavour basis using the common structure of U^0 in Eq. (2) for the popular lepton mixings, leads to:

$$M_{\nu L}^{flavour} = U^0 M_{\nu L}^{mass} U^{0T} = \begin{pmatrix} a & c & c \\ c & b & d \\ c & d & b \end{pmatrix} \tag{4}$$

where,

$$a = m_1 \cos^2\theta_{12}^0 + m_2 \sin^2\theta_{12}^0$$
$$b = \frac{1}{2}\left(m_1 \sin^2\theta_{12}^0 + m_2 \cos^2\theta_{12}^0 + m_3\right)$$

[2] The dominant Type II see-saw structure was kept devoid of solar splitting and degenerate perturbation theory was used to obtain large θ_{12}.

[3] For review of radiative neutrino mass models see [19].

[4] For a brief discussion on $A4$ see Appendix A of the paper.

[5] We have tried to include many works in this direction available in literature. Nonetheless the list in [20] is not completely exhaustive.

$$c = \frac{1}{2\sqrt{2}} \sin 2\theta_{12}^0 (m_2 - m_1)$$
$$d = \frac{1}{2}\left(m_1 \sin^2\theta_{12}^0 + m_2 \cos^2\theta_{12}^0 - m_3\right) \tag{5}$$

Equivalently,

$$\tan 2\theta_{12}^0 = \frac{2\sqrt{2}c}{b+d-a} \tag{6}$$

For non-degenerate realistic neutrino masses a, b, c and d are non-zero.

Our objective is to obtain the structure of the matrix shown in Eq. (4) at one-loop level. For that we assign specific $A4 \times Z_2$ charges to the scalars and fermions in our model. This model has three right-handed neutrino fields. As we will see in course of the discussion that in order to obtain the structure in Eq. (4), two of these right-handed neutrino states will require to be degenerate in masses. Once the structure in Eq. (4) is produced, we will exploit small relaxation of this degenerate feature in right-handed neutrino sector to yield the realistic neutrino mixings, namely non-zero θ_{13}.

In this model, apart from the three $SU(2)_L$ lepton doublets we have three right-handed neutrinos, $N_{\alpha R}$, $(\alpha = 1, 2, 3)$ invariant under the standard model (SM) gauge group. Under $A4$ these three right-handed neutrinos transform as a triplet and so does the three $SU(2)_L$ lepton doublets. In the scalar sector we have two $A4$ symmetric triplet fields Φ and η each of which comprises of three $SU(2)_L$ doublet fields[6] $\Phi_i \equiv (\phi_i^+, \phi_i^0)^T$ and $\eta_j \equiv (\eta_j^+, \eta_j^0)^T$, $(i, j = 1, 2, 3)$. In addition to $A4$ we have an unbroken Z_2 under which all the fields are even except the scalar field η and the right-handed neutrinos. Thus the scalars η_j do not acquire vacuum expectation value (vev) after spontaneous symmetry breaking (SSB), whereas the fields ϕ_i do. All the fields along with their quantum numbers are listed in Table 2. Here we restrict ourselves to the neutrino sector only.[7] We work in a basis in which the charged lepton mass matrix is diagonal and the mixing is entirely from the neutrino sector.[8]

With these fields one can generate neutrino mass at one-loop level as shown in Fig. 1. The relevant part of the scalar potential from the four-point scalar vertex contributing to the neutrino mass matrix is given by[9]:

6 Thus the scalars involved in our model simultaneously transform both under $SU(2)_L$ and $A4$. Our approach is similar to that in [13]. This is distinctly different from the scheme adopted in [14] where $SU(2)_L$ and $A4$ breaking are treated separately. In [14], the $SU(2)_L$ is broken spontaneously when the usual doublets or triplets of $SU(2)_L$ that are invariant under $A4$ acquire their vevs. The $A4$ is broken when the $SU(2)_L$ singlet scalars with non-trivial $A4$ transformation behaviours called 'flavons' acquire vevs. However, while simultaneously connecting the fermions with these two types of scalars in order to construct the mass term, one ends up writing dimension-5 operators causing the theory to be an effective one.

7 This model differs from [21] in terms of the particle content. Unlike [21], here we consider all the popular mixings viz. TBM, BM and GR and also generate non-zero θ_{13}, deviations of θ_{23} from maximality and small corrections to θ_{12} simultaneously through small mass splitting in the right-handed neutrino sector.

8 In some earlier works, the neutrino mixing matrix were taken of TBM, BM, GR mixing patterns and realistic mixings were yielded by applying subsequent rotations through the charged lepton mixing. Some such studies also gave rise to mixing sum rules [22]. Our approach is different as in our case the basis that we work in has the charged lepton mass matrix diagonal and the whole leptonic mixing is solely coming from the neutrino sector.

9 Note at the four-point scalar vertex, both the ϕ fields are annihilated and both the η fields are created. So terms of the scalar potential of $(\eta^\dagger\phi)(\eta^\dagger\phi)$ kind will contribute to the neutrino mass matrix and are therefore relevant. For the total scalar potential consisting of all the allowed terms see Appendix B.

Table 2
Fields and their quantum numbers. Here we are concerned with the neutrino sector only.

Leptons	$SU(2)_L$	$A4$	Z_2
$L_p \equiv \begin{pmatrix} \nu_e & e^- \\ \nu_\mu & \mu^- \\ \nu_\tau & \tau^- \end{pmatrix}$	2	3	1
$N_{\alpha R} \equiv \begin{pmatrix} N_{1R} \\ N_{2R} \\ N_{3R} \end{pmatrix}$	1	3	−1

Scalars	$SU(2)_L$	$A4$	Z_2
$\Phi \equiv \begin{pmatrix} \phi_1^+ & \phi_1^0 \\ \phi_2^+ & \phi_2^0 \\ \phi_3^+ & \phi_3^0 \end{pmatrix}$	2	3	1
$\eta \equiv \begin{pmatrix} \eta_1^+ & \eta_1^0 \\ \eta_2^+ & \eta_2^0 \\ \eta_3^+ & \eta_3^0 \end{pmatrix}$	2	3	−1

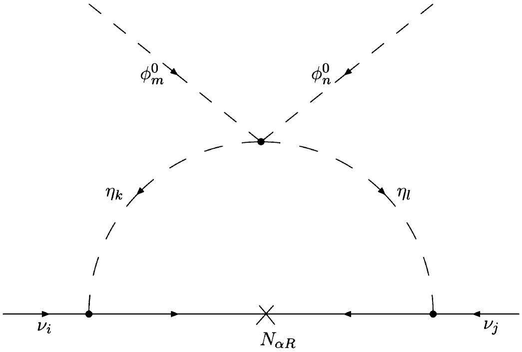

Fig. 1. Neutrino mass generation at one-loop level.

$$\begin{aligned} V_{relevant} \supset\ & \lambda_1 \left[\left\{ (\eta_1^\dagger \phi_1 + \eta_2^\dagger \phi_2 + \eta_3^\dagger \phi_3)^2 \right\} + h.c. \right] \\ & + \lambda_2 \left[\left\{ (\eta_1^\dagger \phi_1 + \omega \eta_2^\dagger \phi_2 + \omega^2 \eta_3^\dagger \phi_3)(\eta_1^\dagger \phi_1 + \omega^2 \eta_2^\dagger \phi_2 + \omega \eta_3^\dagger \phi_3) \right\} + h.c. \right] \\ & + \lambda_3 \left[\left\{ (\eta_2^\dagger \phi_3)^2 + (\eta_3^\dagger \phi_2)^2 + (\eta_3^\dagger \phi_1)^2 + (\eta_1^\dagger \phi_3)^2 + (\eta_1^\dagger \phi_2)^2 + (\eta_2^\dagger \phi_1)^2 \right\} + h.c. \right] \\ & + \lambda_4 \left[\left\{ (\eta_2^\dagger \phi_3)(\eta_3^\dagger \phi_2) + (\eta_3^\dagger \phi_1)(\eta_1^\dagger \phi_3) + (\eta_1^\dagger \phi_2)(\eta_2^\dagger \phi_1) \right\} + h.c. \right], \end{aligned} \tag{7}$$

where all the quartic couplings λ_i $(i = 1, 2, 3, 4)$ are considered to be real.

As discussed earlier, after SSB, ϕ_i^0 will get vevs whereas the η_j^0 will not owing to the Z_2 assignments. Let $\langle \Phi_i \rangle = v_i$ where $(i = 1, 2, 3)$. In [24,25], it has been shown that for $A4$ symmetric three-Higgs-doublets the four vev configurations for which the scalar potential acquires the global minima[10] are:

$$\langle\Phi\rangle_{case1} = v\begin{pmatrix}0 & 1\\ 0 & 0\\ 0 & 0\end{pmatrix}, \ \langle\Phi\rangle_{case2} = v\begin{pmatrix}0 & 1\\ 0 & e^{i\alpha}\\ 0 & 0\end{pmatrix},$$
$$\langle\Phi\rangle_{case3} = v\begin{pmatrix}0 & 1\\ 0 & 1\\ 0 & 1\end{pmatrix}, \ \langle\Phi\rangle_{case4} = v\begin{pmatrix}0 & 1\\ 0 & \omega\\ 0 & \omega^2\end{pmatrix}. \tag{8}$$

To show this the technique of geometric minimization of the scalar potential was used in [24]. In course of our discussion we will see, out of the four alternatives in Eq. (8), only case 3 i.e., $v(1, 1, 1)$ can serve our purpose[11] At this point, let us have a look at the Dirac Yukawa vertices[12] in Fig. 1. The corresponding terms in the Lagrangian are given by:

$$\mathscr{L}_{Yukawa} = h_{p\beta q}\,\bar{N}_{\beta R}\nu_p\eta_q^0 + h.c. \tag{9}$$

Note that $A4$ symmetry forbids any two of the three indices viz. p, β and q from being equal for the Yukawa coupling $h_{p\beta q} \neq 0$ in Eq. (9). This can be readily inferred from Eqs. (A.2), (A.3) and (A.4).

Our model has three right-handed neutrinos, $N_{\alpha R}$ ($\alpha = 1, 2, 3$). The direct mass term for the right-handed neutrinos allowed by the $A4 \times Z_2$ symmetry:

$$\mathscr{L}_{\text{right-handed neutrinos}} = \frac{1}{2}M_1\left[N_{1R}^T C^{-1}N_{1R} + N_{2R}^T C^{-1}N_{2R} + N_{3R}^T C^{-1}N_{3R}\right]. \tag{10}$$

This leads to the right-handed neutrino mass matrix (M_{N_R}) proportional to identity i.e., $M_{N_R} = M_1\mathbb{I}$. In other words, we get all the three right-handed neutrinos degenerate in mass from Eq. (10). As we will see later it is necessary for our purpose to split this degeneracy in the right-handed neutrino sector in such a manner that the right-handed neutrino masses will be close but not exactly degenerate. This can be achieved by obtaining a right-handed neutrino mass matrix which is diagonal but not proportional to identity by breaking $A4$ softly[13] at the mass scale of the right-handed neutrinos i.e., by adding soft $A4$ breaking terms:

$$\mathscr{L}_{soft} = \frac{1}{2}\kappa\left[N_{2R}^T C^{-1}N_{2R}\right] + \frac{1}{2}\kappa'\left[N_{3R}^T C^{-1}N_{3R}\right]. \tag{11}$$

Here κ and κ' are constants with mass dimension one. If $\kappa \neq \kappa'$, we get the right-handed neutrino mass matrix as $M_{N_R} = diag(M_1, M_2, M_3)$ with all three right-handed neutrinos non-degenerate in mass and we will consider $\kappa \neq \kappa'$ in our analysis. Needless to mention $\kappa = \kappa'$ gives $M_{N_R} = diag(M_1, M_2, M_2)$ for which degeneracy between the masses of N_{2R} and N_{3R} still prevail. In

[10] In [26], it has been shown that alignment follows as a natural consequence of the discrete symmetry $A4$ in $A4$ symmetric three-Higgs-doublet model for all the four global minima configurations mentioned in Eq. (8).

[11] In one of our earlier works [26], a scenario consisting of three $SU(2)_L$ doublet scalars transforming as a triplet under $A4$ i.e., $A4$-symmetric three-Higgs-doublet model (3HDM) was considered. The physical scalar mass square matrices were calculated explicitly and diagonalized to obtain the massless Goldstones modes as well as other physical scalar mass eigenstates for all the four global minima configurations of the vevs shown in Eq. (8). The results were in agreement with positivity and unitarity conditions.

[12] All symmetries under consideration are preserved at each of the three vertices in Fig. 1.

[13] Such soft $A4$ breaking terms are similar to the first term present in Eq. (16) of the pioneering work in this direction [13].

course of our discussion we will see, $\kappa = \kappa'$ will give the structure of the left-handed neutrino mass matrix shown in Eq. (4) whereas $\kappa \neq \kappa'$ will yield realistic mixings viz. non-zero θ_{13}, deviation of θ_{23} from $\pi/4$ and small corrections to the solar mixing.

The presence of the Z_2 symmetry is predictive of dark matter candidates in the model since discrete symmetries like Z_2 are often used to prevent dark matter candidates from decaying. The inert $SU(2)_L$ doublet scalars η_j, $(j = 1, 2, 3)$ and the right-handed neutrinos $N_{\alpha R}$, $(\alpha = 1, 2, 3)$ are odd under the Z_2 symmetry, among which the scalars η_j are lighter than the $N_{\alpha R}$. From the m_η^2 term of the scalar potential as shown in Eq. (B.1), the η_j fields seem to be degenerate in masses as dictated by the $A4$ symmetry. However $A4$ is softly broken at the right-handed neutrino mass scale making it possible for the scalars η_j, $(j = 1, 2, 3)$ to have non-degenerate masses. The lightest among the η_j fields can serve as a viable dark matter candidate.

Let η_{Rj} and η_{Ij} be the real and imaginary parts of η_j^0 respectively. Splitting among the masses of η_{Rj} and η_{Ij} is proportional to λv_j and is expected to be small. Here λ stands for the quartic couplings λ_1, λ_2, λ_3 and λ_4 in Eq. (7). Also the mass splittings between η_j $(j = 1, 2, 3)$ constituting the $A4$ triplet are neglected and m_0 is their common mass. The right-handed neutrino mass is represented by M_α. In the limit $M_\alpha^2 >> m_0^2$, the diagram in Fig. 1 leads to neutrino mass of the kind [23]:

$$(M_{\nu_L}^{flavour})_{ij} = \lambda \frac{v_m v_n}{8\pi^2} \sum_{\alpha,k,l \neq (i,j)} \frac{h_{i\alpha k} h_{j\alpha l}}{M_\alpha} [\ln z_\alpha - 1], \tag{12}$$

where $z_\alpha \equiv \frac{M_\alpha^2}{m_0^2}$. The vevs of ϕ_m^0 and ϕ_n^0 are given by v_m and v_n respectively. The Yukawa couplings $h_{i\alpha k}$ and $h_{j\alpha l}$ are determined by the $A4$ invariance which in its turn governs the structure of the neutrino mass matrix. As already mentioned, owing to the $A4$ symmetry any two of the three indices of the Yukawa couplings $h_{i\alpha k}$ and $h_{j\alpha l}$ in Eq. (12) cannot be equal. Incorporating all these features and choosing appropriate limits as discussed above, the expression in Eq. (12) was obtained.[14] Since the logarithm is a slowly varying function and the heavy right handed neutrino masses M_α $(\alpha = 1, 2, 3)$ are expected to be close to each other, the M_α dependence of z_α in the RHS of Eq. (12) can be neglected. Leaving the vevs v_m, v_n and the quartic couplings λ, let us denote the contribution to left-handed neutrino mass matrix $(M_{\nu_L}^{flavour})_{ij}$ from everything else in Eq. (12) by loop contributing factors $r_\alpha \propto \frac{1}{M_\alpha}$.

In terms of the right-handed neutrino loop contributing factors r_α we have the contribution coming from right-handed neutrino sector as $diag(r_1, r_2, r_3)$ with $r_1 \neq r_2 \neq r_3 \neq r_1$ when $\kappa \neq \kappa'$ in Eq. (11). Also note $\kappa = \kappa'$ in Eq. (11) will correspond to $r_1 \neq r_2 = r_3 = r$. Using Eqs. (12) and (7), the left-handed neutrino mass matrix that arises from Fig. 1 is given by[15]:

$$M_{\nu_L}^{flavour} = \begin{pmatrix} \chi_1 & \chi_4 & \chi_5 \\ \chi_4 & \chi_2 & \chi_6 \\ \chi_5 & \chi_6 & \chi_3 \end{pmatrix} \tag{13}$$

where,

$$\chi_1 \equiv (\lambda_1 + \lambda_2)(r_3 v_2^2 + r_2 v_3^2) + \lambda_3[r_2(v_1^2 + v_2^2) + r_3(v_1^2 + v_3^2)]$$
$$\chi_2 \equiv (\lambda_1 + \lambda_2)(r_1 v_3^2 + r_3 v_1^2) + \lambda_3[r_1(v_1^2 + v_2^2) + r_3(v_2^2 + v_3^2)]$$

[14] In [23], the loop calculations has been discussed in details for an analogous $SU(2)_L \times U(1)_Y \times Z_2$ model.

[15] All the symmetries under consideration were conserved at each of the three vertices in Fig. 1.

$$\chi_3 \equiv (\lambda_1+\lambda_2)(r_1 v_2^2 + r_2 v_1^2) + \lambda_3[r_2(v_2^2+v_3^2) + r_1(v_1^2+v_3^2)]$$
$$\chi_4 \equiv r_3[\lambda_4 + 2\lambda_1 - \lambda_2]v_1 v_2$$
$$\chi_5 \equiv r_2[\lambda_4 + 2\lambda_1 - \lambda_2]v_1 v_3$$
$$\chi_6 \equiv r_1[\lambda_4 + 2\lambda_1 - \lambda_2]v_2 v_3. \tag{14}$$

In order to obtain the neutrino mass matrix of the form of Eq. (4) from Eq. (13), one will simultaneously require $\chi_1 \neq \chi_2 = \chi_3$ and $\chi_4 = \chi_5$. Let us now try each of the vev configurations in Eq. (8) and find out the one suitable to obtain this feature along with the constraints put on to r_1, r_2 and r_3.

1. Choice A: For $(v_1, v_2, v_3) = v(1, 0, 0)$, irrespective of the choices for r_1, r_2 and r_3, the off-diagonal entries in Eq. (13) will vanish and one cannot obtain mixing in the neutrino sector.
2. Choice B: For $(v_1, v_2, v_3) = v(1, e^{i\alpha}, 0)$, two of the three off-diagonal entries in Eq. (13) will vanish for any r_1, r_2 and r_3, and one cannot obtain the structure in Eq. (4).
3. Choice C: For $(v_1, v_2, v_3) = v(1, \omega, \omega^2)$, one cannot achieve $\chi_2 = \chi_3$ as required to obtain the structure of the mass matrix in Eq. (4) from Eq. (13), whatever may be the choices for r_1, r_2, r_3.
4. Choice D: For $(v_1, v_2, v_3) = v(1, 1, 1)$, note first that $r_1 = r_2 = r_3$ implies all the diagonal terms to be equal to each other and the off-diagonal entries are equal among themselves. This leads to two left-handed degenerate states and only TBM is admissible. We will not consider that choice. However the form in Eq. (4) starting from Eq. (13) is achieved for $r_1 \neq r_2 = r_3 = r$ when $(v_1, v_2, v_3) = v(1, 1, 1)$, which we refer to as choice D from now onwards. This choice allows all three mixings viz. TBM, BM, GR and all three left-handed neutrinos to be non-degenerate. Hence we will consider this case for further analysis. As already mentioned, such choice of $r_1 \neq r_2 = r_3 = r$ is achieved when $\kappa = \kappa'$ in Eq. (11) i.e., the right-handed neutrinos N_{2R} and N_{3R} are degenerate in masses.

Putting choice D i.e., $v_1 = v_2 = v_3 = v$ and $r_1 \neq r_2 = r_3 = r$ in Eq. (13) one gets the following form of the left-handed neutrino mass matrix in the flavour basis:

$$M_{\nu_L}^{flavour} = \begin{pmatrix} \lambda_{123}(2rv^2) & \lambda_{124} r v^2 & \lambda_{124} r v^2 \\ \lambda_{124} r v^2 & \lambda_{123}(r+r_1)v^2 & \lambda_{124} r_1 v^2 \\ \lambda_{124} r v^2 & \lambda_{124} r_1 v^2 & \lambda_{123}(r+r_1)v^2 \end{pmatrix} \tag{15}$$

where, $\lambda_{123} = \lambda_1 + \lambda_2 + 2\lambda_3$ and $\lambda_{124} = \lambda_4 + 2\lambda_1 - \lambda_2$. Thus the neutrino mass matrix generated at one-loop level as shown in Fig. 1 can produce the form of $M_{\nu_L}^{flavour}$ as in Eq. (4) that corresponds to $\theta_{13} = 0$, $\theta_{23} = \pi/4$ and θ_{12}^0 of the popular mixing alternatives, with the vevs and right-handed neutrino masses as specified in choice D. This follows from the identifications:

$$a \equiv \lambda_{123}(2r_2 v^2) = (\lambda_1 + \lambda_2 + 2\lambda_3)(2r_2 v^2)$$
$$b \equiv \lambda_{123}(r + r_1)v^2 = (\lambda_1 + \lambda_2 + 2\lambda_3)(r + r_1)v^2$$
$$c \equiv \lambda_{124} r v^2 = (\lambda_4 + 2\lambda_1 - \lambda_2) r v^2$$
$$d \equiv \lambda_{124} r_1 v^2 = (\lambda_4 + 2\lambda_1 - \lambda_2) r_1 v^2 \tag{16}$$

Having achieved this, next we concentrate on generation of realistic neutrino mixing i.e., non-zero θ_{13}, deviations of θ_{23} from maximality and small corrections in the solar mixing θ_{12}. For that one has to deviate from the r_α ($\alpha = 1, 2, 3$) of choice D. Being more specific, one has to shift from the choice $\kappa = \kappa'$ in Eq. (11) and consider the more general scenario of $\kappa \neq \kappa'$ in place of it. Thus now we have to split the degeneracy in masses of the right handed neutrinos N_{2R} and N_{3R} by a small amount ϵ i.e., consider $r_3 = r_2 + \epsilon$ and $r_1 \neq r_2 \neq r_3 \neq r_1$, keeping the vevs still to be $v_1 = v_2 = v_3 = v$. With such a choice one is expected to get a dominant contribution of the form of $M_{\nu_L}^{flavour}$ as was achieved in Eq. (15), say M^0, together with small shift from it, M', proportional to ϵ. Thus,

$$M_{\nu_L}^{flavour} = M^0 + M' \tag{17}$$

where,

$$M^0 = \begin{pmatrix} \lambda_{123}(2r_2v^2) & \lambda_{124}r_2v^2 & \lambda_{124}r_2v^2 \\ \lambda_{124}r_2v^2 & \lambda_{123}(r_1+r_2)v^2 & \lambda_{124}r_1v^2 \\ \lambda_{124}r_2v^2 & \lambda_{124}r_1v^2 & \lambda_{123}(r_1+r_2)v^2 \end{pmatrix} \text{ and } M' = \epsilon \begin{pmatrix} x & y & 0 \\ y & x & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{18}$$

where $x = \lambda_{123}v^2$ and $y = \lambda_{124}v^2$. Here M^0 is the form of the $M_{\nu_L}^{flavour}$ required for $\theta_{13} = 0$, $\theta_{23} = \pi/4$ and θ_{12}^0 of the popular mixings. Thus in analogy to Eq. (16), one can identify the followings[16]:

$$\begin{aligned} a' &\equiv \lambda_{123}(2r_2v^2) = (\lambda_1 + \lambda_2 + 2\lambda_3)(2r_2v^2) \\ b' &\equiv \lambda_{123}(r_1+r_2)v^2 = (\lambda_1 + \lambda_2 + 2\lambda_3)(r_1+r_2)v^2 \\ c' &\equiv \lambda_{124}r_2v^2 = (\lambda_4 + 2\lambda_1 - \lambda_2)r_2v^2 \\ d' &\equiv \lambda_{124}r_1v^2 = (\lambda_4 + 2\lambda_1 - \lambda_2)r_1v^2 \end{aligned} \tag{19}$$

It is straightforward to incorporate the corrections offered by M' to M^0 using the non-degenerate perturbation theory. Columns of U^0 in Eq. (2) is the unperturbed flavour basis. From Eq. (19) one can define:

$$\gamma \equiv (b' - 3d' - a') \;\text{ and }\; \rho \equiv \sqrt{a'^2 + b'^2 + 8c'^2 + d'^2 - 2a'b' - 2a'd' + 2b'd'} \tag{20}$$

The third first order corrected ket is then given by:

$$|\psi_3\rangle = \begin{pmatrix} \frac{\epsilon}{\gamma^2-\rho^2}\left[\rho(\sqrt{2}y\cos 2\theta_{12}^0 - x\sin 2\theta_{12}^0) - \gamma\sqrt{2}y\right] \\ -\frac{1}{\sqrt{2}}[1+\xi\epsilon] \\ \frac{1}{\sqrt{2}}[1-\xi\epsilon] \end{pmatrix}, \tag{21}$$

where,

$$\xi \equiv [\gamma x + \rho(x\cos 2\theta_{12}^0 + \sqrt{2}y\sin 2\theta_{12}^0)]/(\gamma^2 - \rho^2). \tag{22}$$

Thus one can write[17]:

$$\sin\theta_{13} = \frac{\epsilon}{\gamma^2 - \rho^2}\left[\rho(\sqrt{2}y\cos 2\theta_{12}^0 - x\sin 2\theta_{12}^0) - \gamma\sqrt{2}y\right]. \tag{23}$$

Using Eqs. (19), (20) and (23), one can easily read off non-zero θ_{13} in terms of the model parameters, namely, ϵ, the quartic couplings and the vevs. Throughout our discussion we have assumed

[16] To distinguish from $r_2 = r_3$ case, let us use a primed notation.

[17] Here we restrict ourselves to no CP-violation.

r_α ($\alpha = 1, 2, 3$) are real and restricted ourselves to a CP-conserving scenario. In principle, the right-handed neutrino masses can have Majorana phases causing these r_α to be complex. Then one can have a complex ϵ, from which one can generate CP-violation in the lepton sector.

From Eq. (21) the deviation of atmospheric mixing from maximality is given by:

$$\tan\varphi \equiv \tan(\theta_{23} - \pi/4) = \xi\epsilon. \tag{24}$$

Similarly, one can obtain small corrections to θ_{12} from the corrections of the first and second kets. The solar mixing angle after receiving first order corrections is given by:

$$\tan\theta_{12} = \frac{\sin\theta_{12}^0 + \epsilon\zeta\cos\theta_{12}^0}{\cos\theta_{12}^0 - \epsilon\zeta\sin\theta_{12}^0} \tag{25}$$

where,

$$\zeta \equiv \frac{\left[\frac{y}{\sqrt{2}}\cos 2\theta_{12}^0 + \frac{x}{4}\sin 2\theta_{12}^0\right]}{\rho} \tag{26}$$

The corrections to the solar mixing and deviations of atmospheric mixing from θ_{23} in Eq. (25) and (24) respectively can be expressed in terms of the model parameters using Eqs. (19), (20), (22) and (26).

From Eqs. (22), (23), (24), (25) and (26), the following interrelationships between the mixing angles emerge:

$$\sin\theta_{13} = \tan(\theta_{23} - \pi/4)\left[\frac{\rho(\sqrt{2}y\cos 2\theta_{12}^0 - x\sin 2\theta_{12}^0) - \gamma\sqrt{2}y}{\gamma x + \rho(x\cos 2\theta_{12}^0 + \sqrt{2}y\sin 2\theta_{12}^0)}\right] \tag{27}$$

and

$$\tan\theta_{12} = \tan(\theta_{12}^0 + \widetilde{\sigma}) \tag{28}$$

where,

$$\tan\widetilde{\sigma} \equiv \frac{1}{2\rho^2}\left[\sin\theta_{13}(\gamma^2 - \rho^2) + \epsilon\left(\frac{3}{2}x\rho\sin 2\theta_{12}^0 + \gamma\sqrt{2}y\right)\right]. \tag{29}$$

In order to illustrate the above interrelationships between the mixing angles, θ_{13} has been plotted as a function of the atmospheric mixing angle θ_{23} using Eq. (27) for normal (inverted) ordering in left (right) panel of Fig. 2 when the best fit values of the solar and atmospheric mass splittings [2] were taken as input. The pink, brown and green curves of all kinds represent TBM, BM and GR mixing patterns respectively. Two representative values of the lightest neutrino mass (m_0) have been taken into account viz. 5×10^{-5} eV and 0.2 eV denoted by solid and dashed lines respectively. The dotted horizontal lines mark the 3σ range of θ_{13} in all the plots. The line conventions for all the plots in both the left and right panels are same.

For a particular ordering of neutrino masses, the values of γ and ρ are completely determined for a specific value of m_0 for a given value[18] of θ_{12}^0. It was found that the variation of γ and ρ for a given m_0 for all the three mixing patterns for both the orderings were very small. As a result the slopes of the straight lines resulting from Eq. (27) in Fig. 2 are very close causing the lines to overlap often. This is clearly demonstrated in the left panel by the lines for $m_0 = 0.2$

[18] For different mixing patterns θ_{12}^0 is given in Table 1.

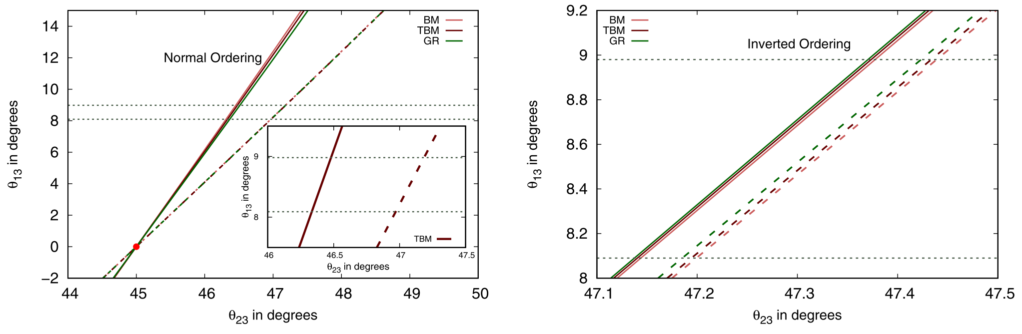

Fig. 2. In the left (right) panel θ_{13} is plotted as a function of θ_{23} for normal (inverted) ordering. The solid and dotted lines are for $m_0 = 5 \times 10^{-5}$ eV and 0.2 eV respectively. The pink, brown and green lines of all types represent TBM, BM and GR mixing scenarios respectively. The line conventions in all the plots are the same. The red dot in the left panel marks maximal atmospheric mixing and $\theta_{13} = 0$. The dotted horizontal line gives the 3σ range of θ_{13} allowed by the data in all the plots. For normal ordering, this 3σ range of θ_{13} has been shown in the inset of the left panel only for TBM mixing to achieve a magnified view. In the right panel θ_{13} -vs- θ_{23} plots for the observed 3σ range of θ_{13} are shown for inverted ordering for all the three mixing patterns. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

eV where the lines resulting from the three mixing patterns viz. TBM, BM and GR are almost indistinguishable. The relevant part of the θ_{13}-vs-θ_{23} plane allowed by our model consistent with the data corresponds to the region marked by the dotted horizontal lines denoting the observed 3σ range of θ_{13}. Thus this region has been magnified and plotted in the inset of the left panel for both the values of m_0 viz. 5×10^{-5} eV and 0.2 eV only for TBM. As already noted, since the lines for the three mixing patterns significantly overlap, the other two mixing patterns are not shown in the inset. It is worth noting that from Eq. (27), it immediately follows that for any value of m_0 when atmospheric mixing is maximal, $\theta_{13} = 0$ irrespective of the mixing pattern and nature of the ordering of neutrino mass. As a result all the lines pass through the point corresponding to vanishing θ_{13} and maximal θ_{23}, marked by the red dot in the left panel of Fig. 2. The lines resulting from Eq. (27) for inverted ordering have been plotted in the right panel of Fig. 2 for all three mixing patterns. Since the lines for the different mixing scenarios, for both the values of m_0 immensely overlap and are practically inseparable, only the region allowed by 3σ range of θ_{13} has been considered for a magnified view for all the three mixing scenarios.

Neutrino mass models with $\mu - \tau$ symmetry are extensively studied in literature. The $\mu - \tau$ symmetry leads to $\theta_{23} = \pi/4$. Therefore, deviation from the $\mu - \tau$ symmetry can yield non-zero $(\theta_{23} - 45°)$ that can be related to non-zero θ_{13} which have been presented in scatter plots in [27]. However, the one to one correspondence of these two mixing angles viz. θ_{13} and θ_{23} with m_0 and θ_{12}^0 for a particular mass ordering as presented in Fig. (2) is specific to our model.

Let $m_1^{(0)}$, $m_2^{(0)}$, $m_3^{(0)}$ be the unperturbed mass eigenvalues of M^0 shown in Eq. (18), the sum of which is given by the trace of the unperturbed mass matrix M^0 in Eq. (18). In terms of the model parameters it can be expressed as:

$$s^{(0)} \equiv m_1^{(0)} + m_2^{(0)} + m_3^{(0)} = \lambda_{123} v^2 (2r_1 + 4r_2) \tag{30}$$

where, $\lambda_{123} = \lambda_1 + \lambda_2 + 2\lambda_3$ as mentioned earlier.

Using M' in Eq. (18) one can calculate the first order corrections to the mass eigenvalues viz. $m_1^{(1)}$, $m_2^{(1)}$ and $m_3^{(1)}$ by applying non-degenerate perturbation theory:

$$m_1^{(1)} = \epsilon\left[\frac{x}{2} + \frac{x}{2}\cos^2\theta_{12}^0 - \frac{y}{\sqrt{2}}\sin 2\theta_{12}^0\right]$$
$$m_2^{(1)} = \epsilon\left[\frac{x}{2} + \frac{x}{2}\sin^2\theta_{12}^0 + \frac{y}{\sqrt{2}}\sin 2\theta_{12}^0\right]$$
$$m_3^{(1)} = \frac{\epsilon x}{2} \tag{31}$$

with $x = \lambda_{123}v^2$ and $y = \lambda_{124}v^2$ as defined earlier. Thus,

$$s^{(1)} \equiv m_1^{(1)} + m_2^{(1)} + m_3^{(1)} = 2\epsilon x \tag{32}$$

Needless to mention that $s^{(1)}$ in Eq. (32) is the trace of M' in Eq. (18).

Thus, the sum of the neutrino masses (Σm) in terms of the model parameters is given by:

$$\Sigma m = s^{(0)} + s^{(1)} = \lambda_{123}v^2[(2r_1 + 4r_2) + 2\epsilon] \tag{33}$$

It is noteworthy that Σm is independent of θ_{12}^0 i.e., the particular mixing pattern such as TBM, BM, GR.[19]

Before concluding, let us have a brief discussion about the flavour violation in the charged lepton sector. The Yukawa term in the Lagrangian that can give rise to charged lepton flavour violation (LFV) at tree level, can be written in analogy to Eq. (9) as:

$$\mathscr{L}_{\text{LFV}} = h_{p\beta q}\,\bar{N}_{\beta R} l_p^- \eta_q^+ + h.c., \tag{34}$$

where, $l_p^- = e^-, \mu^-, \tau^-$ for $p = 1, 2, 3$ respectively. At one-loop, LFV decays can arise through processes as shown in Fig. 3. It can be seen from Eq. (34), the kinematically feasible processes $\mu^- \to e^-\gamma$, $\tau^- \to \mu^-\gamma$ and $\tau^- \to e^-\gamma$ at one-loop as shown in Fig. 3 are prohibited owing to the $A4$ symmetry in the model. This is because the properties of the $A4$ group as shown in Eqs. (A.2), (A.3) and (A.4) will compel the Yukawa coupling $h_{p\beta q}$ to vanish if any of the two indices among p, q and β are equal. The same η_q and $N_{\beta R}$ that couples to l_p^- at one of the Yukawa vertices in Fig. 3 cannot couple to $l_{p'}^-$ at the other Yukawa vertex.[20] Hence at one-loop LFV decays as in Fig. 3 are forbidden in the model due to the $A4$ symmetry.[21]

Summing up, a scotogenic $A4 \times Z_2$ symmetric model of radiatively obtaining realistic neutrino mixing is proposed. Among others, the model comprises of three gauge singlet right-handed neutrino fields $N_{\alpha R}$, $(\alpha = 1, 2, 3)$. If N_{2R} and N_{3R} are degenerate in masses, one can obtain the common structure of the left-handed neutrino mass matrix required by $\theta_{13} = 0$, $\theta_{23} = \pi/4$ and θ_{12}^0 of the particular choices leading to popular lepton mixing scenarios viz. TBM, BM, GR at one-loop level. A slight shift from this degeneracy of right-handed neutrino masses could generate realistic mixing viz. non-zero θ_{13}, deviations of θ_{23} from $\pi/4$ and also tweak θ_{12} by a small amount. The model has three inert $SU(2)_L$ doublet scalars η, odd under the unbroken Z_2, the lightest of which can be a dark matter candidate.

19 The model parameters in Eq. (33) can be suitably adjusted to satisfy cosmological bounds on Σm. A detailed analysis is beyond the scope of this paper.

20 Note for LFV decays, $p \neq p'$.

21 W boson mediated LFV processes are negligible due to the smallness of the neutrino mass.

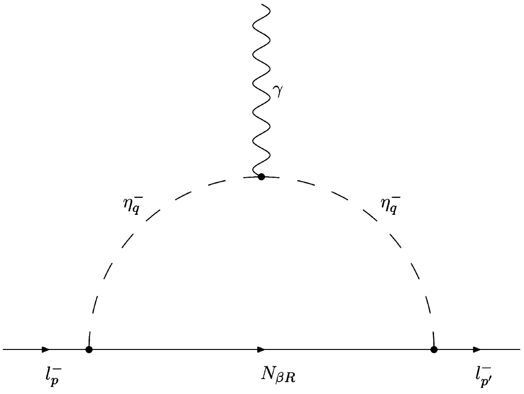

Fig. 3. Flavour violating charged lepton decay processes at one-loop level. For p, $p' = 1, 2, 3$ we have $l_p^- = e^-, \mu^-, \tau^-$ respectively. Charged lepton flavour violation (LFV) occurs when $p \neq p'$. Though only $\mu^- \to e^-\gamma$, $\tau^- \to \mu^-\gamma$ and $\tau^- \to e^-\gamma$ can take place kinematically, at one-loop level all these LFV decays are forbidden by the $A4$ symmetry in this model.

CRediT authorship contribution statement

Soumita Pramanick: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. The discrete group $A4$

$A4$ being the group of even permutations of four objects has 12 elements. The group $A4$ has two generators S and T. These generators satisfy $S^2 = T^3 = (ST)^3 = \mathbb{I}$. The inequivalent irreducible representations for $A4$ are four in number out of which three are 1-dimensional viz. 1, $1'$ and $1''$ and one is 3-dimensional. The 1-dimensional representations transform as 1, ω, and ω^2 under[22] T but are invariant under S. Thus, $1' \times 1'' = 1$. The generators are represented by,

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} . \tag{A.1}$$

Below is the combination rule for two $A4$ triplets:

[22] Here ω is a cube root of 1.

$$3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3 \oplus 3 \ . \tag{A.2}$$

Let us have two $A4$ triplet fields, $3_a \equiv a_i$ and $3_b \equiv b_i$, where $i = 1, 2, 3$, and combine them according to Eq. (A.2). The triplets that we get can be written as $3_c \equiv c_i$ and $3_d \equiv d_i$ where,

$$\begin{aligned} c_i &= \left(\frac{a_2 b_3 + a_3 b_2}{2}, \frac{a_3 b_1 + a_1 b_3}{2}, \frac{a_1 b_2 + a_2 b_1}{2}\right) \ , \text{ or, } c_i \equiv \alpha_{ijk} a_j b_k \ , \\ d_i &= \left(\frac{a_2 b_3 - a_3 b_2}{2}, \frac{a_3 b_1 - a_1 b_3}{2}, \frac{a_1 b_2 - a_2 b_1}{2}\right) \ , \text{ or, } d_i \equiv \beta_{ijk} a_j b_k \ , \\ &\qquad\qquad\qquad\qquad\qquad\qquad (i, j, k, \text{are cyclic}) \ . \end{aligned} \tag{A.3}$$

The 1, $1'$ and $1''$ in this case are:

$$\begin{aligned} 1 &= a_1 b_1 + a_2 b_2 + a_3 b_3 \equiv \rho_{1ij} a_i b_j \ , \\ 1' &= a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \equiv \rho_{3ij} a_i b_j \ , \\ 1'' &= a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 \equiv \rho_{2ij} a_i b_j \ . \end{aligned} \tag{A.4}$$

The group was studied in context of neutrino mass and mixings in the pioneering works [13,14].

Appendix B. The potential

The model contains three $SU(2)_L$ doublet scalars, $\Phi_i \equiv (\phi_i^+, \phi_i^0)^T$, $(i = 1, 2, 3)$ as well as three inert $SU(2)_L$ doublet scalars, $\eta_j \equiv (\eta_j^+, \eta_j^0)^T$, $(j = 1, 2, 3)$ denoted by Φ and η respectively. Both Φ and η transform as a triplet under $A4$. Under Z_2, Φ is even and η is odd. Hence after SSB, ϕ_i^0 obtain vev v_i, $(i = 1, 2, 3)$ but η_j^0 do not. The scalar content of the model is shown in Table 2. Allowing all terms in the scalar potential that conserves the SM gauge symmetry and the $A4 \times Z_2$ symmetry we get:

$$\begin{aligned} V_{total} = {} & m_\Phi^2 \left(\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2 + \Phi_3^\dagger \Phi_3\right) + m_\eta^2 \left(\eta_1^\dagger \eta_1 + \eta_2^\dagger \eta_2 + \eta_3^\dagger \eta_3\right) \\ & + \frac{\widetilde{\lambda}_1}{2} \left(\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2 + \Phi_3^\dagger \Phi_3\right)^2 + \frac{\widetilde{\lambda}_2}{2} \left(\eta_1^\dagger \eta_1 + \eta_2^\dagger \eta_2 + \eta_3^\dagger \eta_3\right)^2 \\ & + \frac{\widetilde{\lambda}_3}{2} \left[\left(\Phi_1^\dagger \Phi_1 + \omega \Phi_2^\dagger \Phi_2 + \omega^2 \Phi_3^\dagger \Phi_3\right) \left(\Phi_1^\dagger \Phi_1 + \omega^2 \Phi_2^\dagger \Phi_2 + \omega \Phi_3^\dagger \Phi_3\right)\right] \\ & + \frac{\widetilde{\lambda}_4}{2} \left[\left(\eta_1^\dagger \eta_1 + \omega \eta_2^\dagger \eta_2 + \omega^2 \eta_3^\dagger \eta_3\right) \left(\eta_1^\dagger \eta_1 + \omega^2 \eta_2^\dagger \eta_2 + \omega \eta_3^\dagger \eta_3\right)\right] \\ & + \frac{\widetilde{\lambda}_5}{2} \left[\left(\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2 + \Phi_3^\dagger \Phi_3\right) \left(\eta_1^\dagger \eta_1 + \eta_2^\dagger \eta_2 + \eta_3^\dagger \eta_3\right)\right] \\ & + \widetilde{\lambda}_6 \left[\left\{\left(\Phi_1^\dagger \Phi_1 + \omega \Phi_2^\dagger \Phi_2 + \omega^2 \Phi_3^\dagger \Phi_3\right) \left(\eta_1^\dagger \eta_1 + \omega^2 \eta_2^\dagger \eta_2 + \omega \eta_3^\dagger \eta_3\right)\right\} + h.c.\right] \\ & + \frac{\widetilde{\lambda}_7}{2} \Big[\left(\Phi_2^\dagger \Phi_3\right)^2 + \left(\Phi_3^\dagger \Phi_2\right)^2 + \left(\Phi_3^\dagger \Phi_1\right)^2 + \left(\Phi_1^\dagger \Phi_3\right)^2 + \left(\Phi_1^\dagger \Phi_2\right)^2 + \left(\Phi_2^\dagger \Phi_1\right)^2 \\ & \quad + \left\{\left(\Phi_2^\dagger \Phi_3\right)\left(\Phi_3^\dagger \Phi_2\right)\right\} + \left\{\left(\Phi_1^\dagger \Phi_3\right)\left(\Phi_3^\dagger \Phi_1\right)\right\} + \left\{\left(\Phi_1^\dagger \Phi_2\right)\left(\Phi_2^\dagger \Phi_1\right)\right\}\Big] \\ & + \frac{\widetilde{\lambda}_8}{2} \Big[\left(\eta_2^\dagger \eta_3\right)^2 + \left(\eta_3^\dagger \eta_2\right)^2 + \left(\eta_3^\dagger \eta_1\right)^2 + \left(\eta_1^\dagger \eta_3\right)^2 + \left(\eta_1^\dagger \eta_2\right)^2 + \left(\eta_2^\dagger \eta_1\right)^2 \\ & \quad + \left\{\left(\eta_2^\dagger \eta_3\right)\left(\eta_3^\dagger \eta_2\right)\right\} + \left\{\left(\eta_1^\dagger \eta_3\right)\left(\eta_3^\dagger \eta_1\right)\right\} + \left\{\left(\eta_1^\dagger \eta_2\right)\left(\eta_2^\dagger \eta_1\right)\right\}\Big] \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\widetilde{\lambda}_9}{2}\Big[\Big\{\Big(\Phi_2^\dagger\Phi_3\Big)\Big(\eta_2^\dagger\eta_3\Big)\Big\} + \Big\{\Big(\Phi_3^\dagger\Phi_2\Big)\Big(\eta_3^\dagger\eta_2\Big)\Big\} + \Big\{\Big(\Phi_3^\dagger\Phi_1\Big)\Big(\eta_3^\dagger\eta_1\Big)\Big\} \\
&\quad + \Big\{\Big(\Phi_1^\dagger\Phi_3\Big)\Big(\eta_1^\dagger\eta_3\Big)\Big\} + \Big\{\Big(\Phi_1^\dagger\Phi_2\Big)\Big(\eta_1^\dagger\eta_2\Big)\Big\} + \Big\{\Big(\Phi_2^\dagger\Phi_1\Big)\Big(\eta_2^\dagger\eta_1\Big)\Big\}\Big] \\
&+ \frac{\widetilde{\lambda}_{10}}{2}\Big[\Big\{\Big(\Phi_2^\dagger\Phi_3\Big)\Big(\eta_3^\dagger\eta_2\Big)\Big\} + \Big\{\Big(\Phi_3^\dagger\Phi_2\Big)\Big(\eta_2^\dagger\eta_3\Big)\Big\} + \Big\{\Big(\Phi_3^\dagger\Phi_1\Big)\Big(\eta_1^\dagger\eta_3\Big)\Big\} \\
&\quad + \Big\{\Big(\Phi_1^\dagger\Phi_3\Big)\Big(\eta_3^\dagger\eta_1\Big)\Big\} + \Big\{\Big(\Phi_1^\dagger\Phi_2\Big)\Big(\eta_2^\dagger\eta_1\Big)\Big\} + \Big\{\Big(\Phi_2^\dagger\Phi_1\Big)\Big(\eta_1^\dagger\eta_2\Big)\Big\}\Big] \\
&+ V_{relevant}\,. \qquad \text{(B.1)}
\end{aligned}
$$

Here,

$$
\begin{aligned}
V_{relevant} &= \lambda_1\Big[\Big\{(\eta_1^\dagger\phi_1 + \eta_2^\dagger\phi_2 + \eta_3^\dagger\phi_3)^2\Big\} + h.c.\Big] \\
&+ \lambda_2\Big[\Big\{(\eta_1^\dagger\phi_1 + \omega\eta_2^\dagger\phi_2 + \omega^2\eta_3^\dagger\phi_3)(\eta_1^\dagger\phi_1 + \omega^2\eta_2^\dagger\phi_2 + \omega\eta_3^\dagger\phi_3)\Big\} + h.c.\Big] \\
&+ \lambda_3\Big[\Big\{(\eta_2^\dagger\phi_3)^2 + (\eta_3^\dagger\phi_2)^2 + (\eta_3^\dagger\phi_1)^2 + (\eta_1^\dagger\phi_3)^2 + (\eta_1^\dagger\phi_2)^2 + (\eta_2^\dagger\phi_1)^2\Big\} + h.c.\Big] \\
&+ \lambda_4\Big[\Big\{(\eta_2^\dagger\phi_3)(\eta_3^\dagger\phi_2) + (\eta_3^\dagger\phi_1)(\eta_1^\dagger\phi_3) + (\eta_1^\dagger\phi_2)(\eta_2^\dagger\phi_1)\Big\} + h.c.\Big]. \qquad \text{(B.2)}
\end{aligned}
$$

At the scalar four point vertex of Fig. 1, two η are created and two ϕ are annihilated. Thus for the neutrino mass matrix only the $(\eta^\dagger\phi)(\eta^\dagger\phi)$ terms are relevant. These terms are given by $V_{relevant}$ in Eq. (B.2). The quartic couplings λ_i, $(i = 1, 2, 3, 4)$, in Eq. (B.2) are real.

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