Bethe Ansatz for the superconformal index with unequal angular momenta

Ofer Aharony,[1,2] Ohad Mamroud,[1,3,4] Shimon Nowik,[1] and Meir Weissman[1]
[1]*Department of Particle Physics and Astrophysics, Weizmann Institute of Science, Rehovot 7610001, Israel*
[2]*School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540, USA*
[3]*SISSA, Via Bonomea 265, 34136 Trieste, Italy*
[4]*INFN, Sezione di Trieste, Via Valerio 2, 34127 Trieste, Italy*

(Received 15 February 2024; accepted 19 March 2024; published 22 April 2024)

A few years ago it was shown that the superconformal index of the $\mathcal{N}=4$ supersymmetric $SU(N)$ Yang-Mills theory in the large N limit matches with the entropy of 1/16-supersymmetric black holes in type IIB string theory on $\mathrm{AdS}_5 \times S^5$. In some cases, an even more detailed match between the two sides is possible. When the two angular momentum chemical potentials in the index are equal, the superconformal index can be written as a discrete sum of Bethe ansatz solutions, and it was shown that specific terms in this sum are in a one-to-one correspondence to stable black hole solutions, and that the matching can be extended to nonperturbative contributions from wrapped D3-branes. A Bethe ansatz approach to computing the superconformal index exists also when the ratio of the angular momentum chemical potentials is any rational number, but in those cases it involves a sum over a very large number of terms (growing exponentially with N). Benini *et al.* showed that a specific one of these terms matches with the black hole, but the role of the other terms is not clear. In this paper we analyze some of the additional contributions to the index in the Bethe ansatz approach, and we find that their matching to the gravity side is much more complicated than in the case of equal chemical potentials. In particular, we find some contributions that are larger than the one that was found to match the black holes, in which case they must cancel with other large contributions. We give some evidence that cancellations of this type are possible, but we leave a full understanding of how they work to the future.

DOI: 10.1103/PhysRevD.109.085015

I. INTRODUCTION

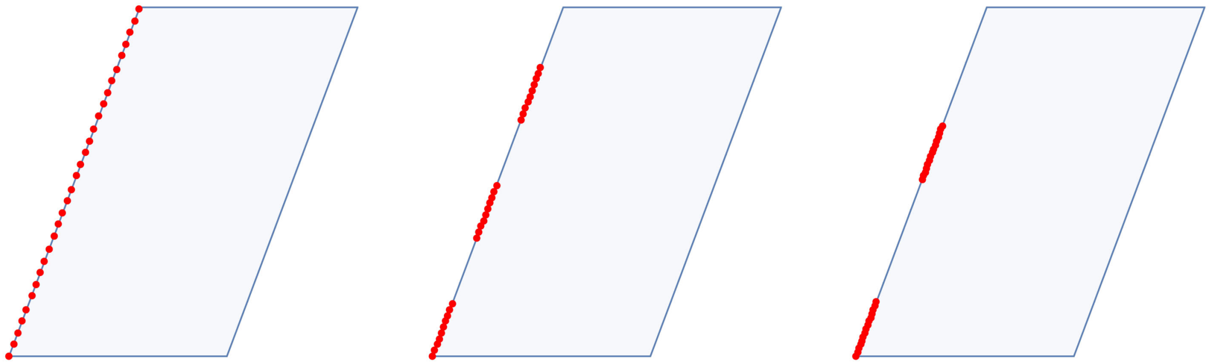
Recent years have seen significant advances in precision holography, where semiclassical bulk information was precisely reproduced by a computation in the dual conformal field theory, mostly in a supersymmetric setting. The first breakthroughs were in counting the microstates of supersymmetric black holes by analyzing various supersymmetric indices, thus reproducing their entropy [1–4]. Further improvements followed, generalizing these results to many different theories and computing more refined data on both sides of the correspondence [5–57].

One particularly convenient setup to study was that of the four-dimensional $\mathcal{N}=4$ $SU(N)$ supersymmetric Yang-Mills theory, where the superconformal index of the theory can be matched to the dual gravitational partition function. While the index is usually expressed via an integral formula, the Bethe ansatz approach [58,59] allows us in some cases to localize the integral and to transform it into a discrete sum (see Sec. II for details and caveats), schematically

$$\mathcal{I} = \sum_{u \in BA} \mathcal{I}_u, \tag{1.1}$$

where the us can be thought of as specific configurations of the complexified holonomies of the $SU(N)$ gauge field that solve some set of transcendental equations (and the sum is over the set of solutions to these equations). In [4,5] a particular family of terms in the sum was analyzed, and at large N each of them contributed to the index at order $e^{\#N^2}$. These turned out to precisely match the contributions of various different gravitational saddle points in the bulk dual in a one-to-one fashion (and including some order-N corrections coming from wrapped D-branes).

Unfortunately, the Bethe ansatz approach can only compute the index for some particular choices of the chemical potentials appearing in the superconformal index, those where the two angular chemical potentials of the theory τ and σ have a rational ratio, $\tau = a\omega$ and $\sigma = b\omega$ for $a, b \in \mathbb{N}$. Each contribution to the index, $\mathcal{I}_u$, is then



(a) (b) (c)

FIG. 1. The eigenvalue distribution for different choices of M (which are divisors of ab) for the basic solution with $ab = 6$, $N = 30$, drawn on a torus with periodicities 1 and $ab\omega$. The $M = 1$ case is the one analyzed in [10]. (a) $M = 1$. (b) $M = 2$. (c) $M = 3$.

described as a sum over $(ab)^{N-1}$ terms, where one shifts the holonomies of the solution u in some prescribed way.

The detailed analysis of the matching to gravity in [5] concerned an even simpler case, that in which $\sigma = \tau$, for which each $\mathcal{I}_u$ is given by exactly one term. In the more general case, [10] analyzed one of the $(ab)^{N-1}$ terms in the sum (for a particular solution to the Bethe ansatz equations) and showed that it reproduces the gravitational action of a specific black hole solution. Another term was analyzed (for more general solutions of the Bethe ansatz equations) in [60] and was also found to reproduce the gravity answers. A natural question is then what happens to the other terms at large N? Do they give negligible contributions, do they cancel amongst themselves, or do they match some other gravitational background?

In this work we analyze some specific additional terms in the sum at large N, as depicted in Fig. 1. As opposed to the cases analyzed in [10,60], here we consider configurations of the holonomies that do not correspond at large N to uniform distributions on cycles of the complex torus with modular parameter $ab\omega$. We find that not only do they contribute at order $e^{\#N^2}$, but they do so in a way that is different from any gravitational background familiar to us. In some cases their contribution is even larger than that of the terms analyzed in [10,60]. However, we did not analyze all the different $(ab)^{N-1}$ terms, and it is possible that they cancel amongst themselves, or with other terms in (1.1).

The paper is organized as follows: in Sec. II we review the Bethe-ansatz approach. In Sec. III we generalize a proof of [61] to show that only solutions to the reduced Bethe ansatz equations contribute to the index, also when the angular chemical potentials are different. In Sec. IV we compute the large N expansion of the aforementioned configurations. In Sec. V we study the case $N = 2$; this case is very far from the large N limit, but in this case we can analyze all the (ab) different terms contributing to $\mathcal{I}_u$, and we show that cancellations between different terms are plausible. In Sec. VI we analyze the known gravitational saddles, and show that they cannot all be explained by a large N limit of one of the known configurations. An Appendix contains some details on the special functions that appear in the computations.

II. REVIEW OF THE BETHE ANSATZ METHOD

The superconformal index can be defined for any four-dimensional $\mathcal{N} = 1$ superconformal field theory. One starts by picking a Poincaré supercharge $\mathcal{Q}$ and its superconformal conjugate $\mathcal{Q}^\dagger$. The superconformal index then counts the difference between bosonic and fermionic supersymmetric operators that are annihilated by this supercharge. One can add fugacities for charges that commute with the relevant supercharge. The superconformal index can then be written as a trace over the entire Hilbert space of states on S^3. For the case of $\mathcal{N} = 4$ super Yang Mills, the index counts 1/16-Bogomol'nyi-Prasad-Sommerfield (BPS) operators and is calculated via[1]

$$\mathcal{I}(y_{1,2}, p, q) = \mathrm{Tr}\left[(-1)^F e^{-\{\mathcal{Q},\mathcal{Q}^\dagger\}} p^{J_1+\frac{1}{2}R_3} q^{J_2+\frac{1}{2}R_3} y_1^{\frac{1}{2}(R_1-R_3)} y_2^{\frac{1}{2}(R_2-R_3)}\right], \tag{2.1}$$

where $R_{1,2,3}$ are in the Cartan of the $SU(4)_R$ R symmetry, with the three complex scalars of the theory having charge 2 under one of them and 0 under the other two, in a symmetric way. $J_{1,2}$ are the half-integer angular momenta quantum numbers of local operators, each rotating an $\mathbb{R}^2 \subset \mathbb{R}^4$. F is the fermion number. $\mathcal{Q}$ is chosen to be the complex supercharge associated with the R-symmetry generator $r = \frac{1}{3}(R_1 + R_2 + R_3)$.

[1]Often the powers of the fugacities are written as $p^{J_1+\frac{1}{2}r} q^{J_2+\frac{1}{2}r}$. Compared to this convention, we have swallowed a power of $(pq)^{1/3}$ into y_1 and y_2, such that the index is a single-valued function. The relation of our variables to those of [62] is $p = t^3 y|_{\mathrm{there}}$, $q = t^3/y|_{\mathrm{there}}$, $y_1 = t^2 v|_{\mathrm{there}}$, $y_2 = t^2 w/v|_{\mathrm{there}}$.

We find it convenient to write the index in terms of chemical potentials instead of fugacities, which are denoted by

$$p = e^{2\pi i\sigma}, \qquad q = e^{2\pi i\tau}, \qquad y_{1,2} = e^{2\pi i\Delta_{1,2}}. \tag{2.2}$$

The index is well defined for $|p|, |q| < 1$, i.e., for $\mathbb{I}\mathrm{m}(\sigma), \mathbb{I}\mathrm{m}(\tau) > 0$.

It has long been known that the index has an integral form [62,63], in which it is expressed in terms of a contour integral over the Cartan subalgebra of the gauge group, in our case $SU(N)$,

$$\mathcal{I}(\Delta_{1,2}, \tau, \sigma) = \kappa_N(\Delta_{1,2}; \tau, \sigma) \oint_{\mathbb{T}^{N-1}} \left(\prod_{i=1}^{N-1} du_i\right) \times \mathcal{Z}(\{u_i\}, \Delta_{1,2}; \tau, \sigma), \tag{2.3}$$

where

$$\mathcal{Z}(\{u_i\},\Delta_{1,2};\tau,\sigma) = \prod_{i\neq j}^{N} \frac{\tilde{\Gamma}(u_{ij}+\Delta_1;\tau,\sigma)\tilde{\Gamma}(u_{ij}+\Delta_2;\tau,\sigma)}{\tilde{\Gamma}(u_{ij};\tau,\sigma)\tilde{\Gamma}(u_{ij}+\Delta_1+\Delta_2;\tau,\sigma)}, \tag{2.4}$$

$$\kappa_N(\Delta_{1,2}; \tau, \sigma) = \frac{1}{N!}\left(\frac{(p;p)_\infty (q;q)_\infty \tilde{\Gamma}(\Delta_1;\tau,\sigma)\tilde{\Gamma}(\Delta_2;\tau,\sigma)}{\tilde{\Gamma}(\Delta_1+\Delta_2;\tau,\sigma)}\right)^{N-1}. \tag{2.5}$$

Here $u_{ij} = u_i - u_j$, $(z;q)_\infty$ is the q-Pochhammer symbol, and $\tilde{\Gamma}$ is the elliptic gamma function, both defined in the Appendix. The variables u_i parametrize the torus $\mathbb{T}^{N-1}$ on which we integrate. u_N is defined such that the $e^{2\pi i u}$s can be thought of as the eigenvalues of a special unitary matrix, i.e., via the constraint

$$\sum_{i=1}^{N} u_i = 0. \tag{2.6}$$

There are various methods for calculating the contour integral (2.3). In this paper we focus on the Bethe ansatz (BA) approach [58,59], which is valid at finite N, and then expand at large N. This formula manipulates the integrand and the contour such that the integral can be written as a sum of residues.

The BA formula applies when the chemical potentials τ, σ have a rational ratio, i.e., $\tau = a\omega$, $\sigma = b\omega$, for some $a, b \in \mathbb{N}$ such that $\gcd(a,b) = 1$, $\mathbb{I}\mathrm{m}(\omega) > 0$. We also define the fugacity

$$h = e^{2\pi i\omega}, \qquad p = h^b, \qquad q = h^a. \tag{2.7}$$

A. Derivation of the BA formula

Following [58], we can manipulate the contour integral (2.3) by noting that the integrand, $\mathcal{Z}$, is a quasielliptic function,[2]

$$(-1)^{N-1} Q_j(\{u_i\}, \Delta_{1,2}; \omega)\mathcal{Z}(\{u_i\}, \Delta_{1,2}; a\omega, b\omega) = \mathcal{Z}(\{u_i - \delta_{ij}ab\omega\}, \Delta_{1,2}; a\omega, b\omega), \tag{2.8}$$

where for $i = 1, \ldots, N$, and we define

$$Q_i(\{u_j\}, \Delta_{1,2}; \omega) \equiv e^{6\pi i \sum_{j=1}^{N} u_{ij}} \prod_{j=1}^{N} \frac{\theta_0(\Delta_1 + u_{ji}; \omega)\theta_0(\Delta_2 + u_{ji}; \omega)\theta_0(-\Delta_1 - \Delta_2 + u_{ji}; \omega)}{\theta_0(\Delta_1 - u_{ji}; \omega)\theta_0(\Delta_2 - u_{ji}; \omega)\theta_0(-\Delta_1 - \Delta_2 - u_{ji}; \omega)}. \tag{2.9}$$

Note that acting with this operator shifts one of the u_is and therefore breaks the $SU(N)$ constraint (2.6). However, acting with

$$\hat{Q}_i = \frac{Q_i}{Q_N} \tag{2.10}$$

preserves the constraint. We note that the Q_is are elliptic functions in each of the u_is separately, i.e., they are periodic under $u_i \sim u_i + 1 \sim u_i + \omega$.

We can now use (2.8) to modify the contour of integration given in (2.3),

$$\begin{aligned}\mathcal{I}(\Delta_{1,2}; a\omega, b\omega) &= \kappa_N \oint_{\mathbb{T}^{N-1}} \left(\prod_{i=1}^{N-1} du_i\right) \prod_{i=1}^{N-1} \frac{1 - \hat{Q}_i(\{u_j\}, \Delta_{1,2}; \omega)}{1 - \hat{Q}_i(\{u_j\}, \Delta_{1,2}; \omega)} \mathcal{Z}(\{u_i\}, \Delta_{1,2}; a\omega, b\omega),\\ &= \kappa_N \oint_{\mathcal{C}} \left(\prod_{i=1}^{N-1} du_i\right) \frac{\mathcal{Z}(\{u_i\}, \Delta_{1,2}; a\omega, b\omega)}{\prod_{i=1}^{N-1}(1 - \hat{Q}_i(\{u_j\}, \Delta_{1,2}; \omega))},\end{aligned} \tag{2.11}$$

[2]To prove this, use (3.38) of [58] for every term in $\mathcal{Z}$.

where the new integration contour, $\mathcal{C}$, is the contour encircling the annulus

$$\mathcal{A} = \{u_i | 1 < |e^{2\pi i u_i}| < |h|^{-ab}, i = 1, \ldots, N-1\}. \tag{2.12}$$

Applying the residue theorem,[3]

$$\mathcal{I}(\Delta_{1,2}; a\omega, b\omega) = \kappa_N \sum_{\{u_i\} \in \mathrm{BAEs}} \sum_{\{m_i\}=1}^{ab} \mathcal{Z}(\{u_i - m_i\omega\}, \times \Delta_{1,2}; a\omega, b\omega) \cdot H^{-1}(\{u_i\}, \Delta_{1,2}; \omega), \tag{2.13}$$

where the sum is over all solutions to the BA equations[4]

$$\hat{Q}_i(\{u_j\}, \Delta_{1,2}; \omega) = 1, \tag{2.14}$$

and H^{-1} is the Jacobian coming from the change of variables from the u_j to the $\hat{Q}_i$s when we apply the residue theorem to the integral. By $\sum_{\{m_i\}=1}^{ab}$ we mean $\sum_{m_1=1}^{ab} \cdots \sum_{m_{N-1}=1}^{ab}$, with $m_N = -\sum_{k=1}^{N-1} m_k$, coming from all the different shifts of the u_is by ω that are within $\mathcal{A}$, as the BA equations (2.14) are periodic under these shifts. In the Bethe ansatz approach we pick up the poles of the integrand in the region $0 > \mathbb{Im}(u_i) \geq \mathbb{Im}(-ab\omega)$, while every solution to (2.14) can be chosen to be inside the torus with parameter ω [namely, such that for all $i = 1, \ldots, N-1$ we have $\mathbb{Im}(\omega) > \mathbb{Im}(u_i) \geq 0$]. As a consequence, each such solution corresponds to $(ab)^{N-1}$ poles of the integrand, where we shift each u_i by $(-m_i\omega)$ for $i = 1, \ldots, N-1$, $m_i = 1, \ldots, ab$, with a compensating shift for u_N.

For $\sigma = \tau$, $a = b = 1$ and the sum over $\{m_i\}$ is trivial. However, for any other case $ab > 1$, and the number of terms in the sum, $(ab)^{N-1}$, is exponential in N. We stress that the dependence on a, b and the m_i is present only in $\mathcal{Z}$, and not in κ_N or H^{-1}, so this complication will only affect the ability to calculate $\mathcal{Z}$.

B. Hong-Liu solutions

While the full set of solutions to the Bethe Ansatz Equations (BAEs) for $N > 2$ is unknown, a specific set of solutions, named the Hong-Liu (HL) solutions, was identified [65]. They correspond to symmetric configurations of the us on the $(1, \omega)$ torus, and are denoted by three integers, $\{m, n, r\}$, such that $N = m \cdot n$, and $r = 0, \ldots, n-1$. Their explicit form is

$$u_j \equiv u_{\hat{j}\hat{k}} = \bar{u} + \frac{\hat{j}}{m} + \frac{\hat{k}}{n}\left(\omega + \frac{r}{m}\right), \tag{2.15}$$

such that $\hat{j} = 0, \ldots, m-1$ and $\hat{k} = 0, \ldots, n-1$, and $\bar{u}$ is a constant chosen to satisfy the $SU(N)$ constraint (2.6). The $\{1, N, 0\}$ solution is sometimes named the "basic solution," and it is given by

$$u_j = \bar{u} + \frac{j\omega}{N}. \tag{2.16}$$

Different HL solutions and different series of $\{m_j\}$, have a different degree of dominance in the sum (2.13), and most of these contributions have not been calculated yet.

III. THE REDUCED BETHE ANSATZ SUFFICES

As discussed in [61], in addition to the shifts of u_i by $(-m_i\omega)$ discussed above, every BA solution is related to N^2 additional solutions by shifting all $u_i \to u_i + \frac{\alpha+\beta\omega}{N}$ $(i = 1, \ldots, N-1)$, and $u_N \to u_N + (1-N)\frac{\alpha+\beta\omega}{N}$, where $\alpha, \beta = 0, \ldots, N-1$. The differences u_{ij} are affected if and only if either $i = N$ or $j = N$, and the u_{iN} are just shifted by $(\alpha + \beta\omega)$, which leaves the Bethe ansatz equations invariant.[5]

If a shift by some $(\frac{\beta}{N} - m_i)\omega$ takes us outside of the integration region, then we can shift for that specific shifted solution $u_i \to u_i - ab\omega$ and $u_N \to u_N + ab\omega$ to return back into it. As mentioned in the previous section, this leaves the Jacobian invariant and multiplies $\mathcal{Z}$ by $\hat{Q}_i$. Since we started from a solution to the BAE and since $\hat{Q}_i$ has periodicity ω, $\hat{Q}_i = 1$ in our case and the contribution to the index is invariant under this shift. Thus, in computing the index we can keep the original shifts $u_i \to u_i + \frac{\alpha+\beta\omega}{N}$ and we do not need to worry about this issue. In fact, by the same arguments, we can consider such shifts with $\beta = 0, 1, \ldots, abN-1$. The extra shifts we added by integer multiples of ω do not give new solutions but rather different m_i shifts of the same solutions, so adding these extra shifts just multiplies the result by ab, but it will be useful to write the index as a sum over these shifts for the argument below.

Note that since $\mathcal{Z}$ only depends on the differences of the u_i and does not depend on the constraint $\sum_{i=1}^N u_i = 0$, for computing $\mathcal{Z}$ we can equally well describe these shifts as acting only on u_N, namely

[3]As shown in [58], no other poles in the integrand will contribute, since they are either canceled by poles of the denominator of a high enough degree, or they are outside the new contour of integration.

[4]This is a bit of a simplification, since we assumed that all solutions are discrete and not part of a continuum. That is known not to be the case for $N \geq 3$. The contribution of continuous families of solutions will be discussed in [64]. It does not seem to qualitatively change the behavior discussed in this paper.

[5]Note that in some cases the shift may take a solution to itself up to a permutation of the eigenvalues, but this will not affect the arguments below.

$$u_i \to \begin{cases} u_i & i \neq N \\ u_N - \alpha - \beta\omega & i = N \end{cases}. \tag{3.1}$$

Since both $\mathcal{Z}$ and the Jacobian H^{-1} are invariant under integer shifts of the u_{ij}s, summing over the α shifts will just give a multiplicative factor of N. In the case $\tau = \sigma$ the authors of [61] showed that due to the β shifts, solutions for the Bethe ansatz equations $\{\hat{Q}_i = 1; i = 1, \ldots, N-1\}$, which are not solutions to the reduced Bethe ansatz equation $\{Q_i = (-1)^{N-1}; i = 1, \ldots, N\}$, cancel each other, such that the index only gets contributions from solutions to the reduced BAE. We will now argue that the same is true also for $\tau \neq \sigma$.

Picking a solution $\{u_i\}$ to the BAE, as described above we can sum over the shifts $\beta = 0, \ldots, abN-1$ (in addition to the m_j shifts). The Jacobian H^{-1} is invariant under $\omega\mathbb{Z}$ shifts of the us, so we will just need to evaluate the effect of summing over the $\beta\omega$ shifts in $\mathcal{Z}$. The contribution of the solution (and its shifted relatives) to the index is therefore

$$\mathcal{I}_{\{u\}} = \frac{N}{ab} \cdot \kappa_N \cdot H^{-1}(\{u_i\}, \Delta_{1,2}; \omega) \sum_{\{m_j\}=1}^{ab} \sum_{\beta=0}^{abN-1} \mathcal{Z}\left(u_i - m_i\omega, u_N + \sum_{i=1}^{N-1} m_i\omega - \beta\omega; \Delta_{1,2}, a\omega, b\omega\right). \tag{3.2}$$

We can write $\beta = ab\beta_1 + \beta_2$ with $\beta_1 = 0, \ldots, N-1$ and $\beta_2 = 0, \ldots, ab-1$,

$$\mathcal{I}_{\{u\}} = \frac{N}{ab} \cdot \kappa_N \cdot H^{-1} \cdot \sum_{\{m_j\}=1}^{ab} \sum_{\beta_2=0}^{ab-1} \sum_{\beta_1=0}^{N-1} \mathcal{Z}\left(u_i - m_i\omega, u_N + \sum_{i=1}^{N-1} m_i\omega - \beta_2\omega - \beta_1 ab\omega; \Delta_{1,2}, a\omega, b\omega\right). \tag{3.3}$$

But since shifting only $u_N \to u_N - ab\omega$ amounts to multiplying $\mathcal{Z}$ by $(-1)^{N-1}Q_N$, as in (2.8), and since Q_N has ω periodicity in each u_i separately, this is the same as

$$\mathcal{I}_{\{u\}} = \frac{N}{ab} \cdot \kappa_N \cdot H^{-1} \cdot \sum_{\{m_j\}=1}^{ab} \sum_{\beta_2=0}^{ab-1} \sum_{\beta_1=0}^{N-1} ((-1)^{N-1} Q_N(u_i))^{\beta_1} \mathcal{Z}\left(u_i - m_i\omega, u_N + \sum_{i=1}^{N-1} m_i\omega - \beta_2\omega; \Delta_{1,2}, a\omega, b\omega\right). \tag{3.4}$$

Since $\hat{Q}_i$ solves the Bethe ansatz equations and $\prod_{i=1}^{N} Q_i = 1$, Q_N is an Nth root of unity, and so the sum over β_1 vanishes unless $Q_N = (-1)^{N-1}$. Since all the Q_is are equal to each other, we see that only solutions to the reduced Bethe ansatz equation

$$Q_i = (-1)^{N-1} \tag{3.5}$$

contribute to the index.

IV. LARGE N EXPANSION

Here we will be interested in the large N contribution of the Hong-Liu solutions to the $SU(N)$ index, and in particular the contribution of the basic solution (2.16). We will concentrate on the leading order in N terms that are exponential in N^2, whose log is $O(N^2)$. As explained previously, there are $(ab)^{N-1}$ different terms in this contribution, coming from the possible shifts by $\{m_i\omega\}_{i=1}^{N-1}$ of the basic solution. Therefore, an exponential in N^2 dependence must come from individual shifts, and not from the summation over them. Each choice for the ms gives

$$\log\left(\mathcal{I}_{\{u\}}(\{m_i\})\right) = \log(\kappa_N) + \log(H^{-1}) + \log\left(\mathcal{Z}\left(\{u_i - m_i\omega\}, u_N + \sum_{i=1}^{N-1} m_i\omega; \Delta, a\omega, b\omega\right)\right). \tag{4.1}$$

The first term does not contribute at order $O(N^2)$, as is obvious from (2.5). The second term was shown to be $O(1)$ in [4], and was exactly computed in [47] for the HL solutions with $m = 0$. So we are left with evaluating the N dependence of the third term, $\log(\mathcal{Z})$.

Evaluating the last term depends significantly on the choice of ms. In [10], the authors evaluated it for the choice $m_j = j \bmod ab$ and showed that[6]

$$\log\left[\mathcal{I}_{\{1,N,0\}}(\{m_j = j\})\right] = -\pi i N^2 \frac{[\Delta_1]_\omega [\Delta_2]_\omega [\Delta_3]_\omega}{ab\omega^2} + O(N\log(N)), \tag{4.2}$$

which agrees with the on-shell action of dual black holes, where $[\Delta_3]_\omega = \tau + \sigma - [\Delta_1]_\omega - [\Delta_2]_\omega - 1$ and the

[6]In some regimes of the chemical potentials Δ we need to replace $[\Delta]_\omega$ by $[\Delta]'_\omega = [\Delta]_\omega + 1$.

function $[\Delta]_\omega$ simply shifts Δ to a strip, such that it satisfies

$$[\Delta]_\omega = \Delta \mod 1, \text{ such that } -\mathbb{Im}\left(\frac{1}{\omega}\right) > \mathbb{Im}\left(\frac{[\Delta]_\omega}{\omega}\right) > 0. \tag{4.3}$$

It is not clear how to evaluate $\log(Z)$ for generic choices of the m_j. In this section, we will generalize the result (4.2) to the choice

$$m_j = M \cdot j \mod ab, \tag{4.4}$$

for any integer M. At large N we expect the leading order to only depend on the (shifted) eigenvalue distribution, and one can show that this depends on M only through $\gcd(M, ab)$. We will show this momentarily for the leading order in N. When $M = 1$ the (shifted) eigenvalues are uniformly distributed along the $ab\omega$ cycle of the torus, and otherwise they are distributed as a chain of step functions along it, covering $\frac{1}{M}$ of the cycle, as in Fig. 1.

In order to calculate $\mathcal{Z}$ for the basic solution (2.16), it is useful to define the building block

$$\begin{aligned}
\tilde{\Psi}_\Delta &\equiv \sum_{i\neq j}^{N} \log\left[\tilde{\Gamma}\left(\Delta + \omega\frac{j-i}{N} + \omega(m_j - m_i); a\omega, b\omega\right)\right], \\
&= \sum_{i,j=1}^{N} \log\left[\tilde{\Gamma}\left(\Delta + \omega\frac{j-i}{N} + \omega(m_j - m_i); a\omega, b\omega\right)\right] - N\log[\tilde{\Gamma}(\Delta; a\omega, b\omega)], \\
&\equiv \Psi_\Delta - N\log[\tilde{\Gamma}(\Delta; a\omega, b\omega)],
\end{aligned} \tag{4.5}$$

which allows us to write $\mathcal{Z}$ in (2.4) as

$$\log(\mathcal{Z}) = \tilde{\Psi}_{\Delta_1} + \tilde{\Psi}_{\Delta_2} - \tilde{\Psi}_{\Delta_1+\Delta_2} - \tilde{\Psi}_0. \tag{4.6}$$

As a result, terms that are linear or constant in Δ will cancel between the different $\tilde{\Psi}$ s when evaluating the index. In order to simplify the form of the building block, we utilize the identity [66]

$$\tilde{\Gamma}(u; \tau, \sigma) = \prod_{r=0}^{a-1}\prod_{s=0}^{b-1} \tilde{\Gamma}\left(u + (r\tau + s\sigma); a\tau, b\sigma\right), \tag{4.7}$$

to write Ψ_Δ as

$$\Psi_\Delta = \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{i,j=1}^{N} \log\left[\tilde{\Gamma}\left(\Delta + \omega\frac{j-i}{N} + \omega(m_j - m_i + as + br); ab\omega, ab\omega\right)\right]. \tag{4.8}$$

We can now make two simplifications. The first is to assume that $ab|N$ and denote $\tilde{N} = \frac{N}{ab}$. The justification is that [10] showed that the effect of ignoring the residue of $N/(ab)$ is subleading at large N, when $N \gg ab$. Let us set $i = \gamma ab + c$, $j = \delta ab + d$, with $\gamma, \delta = 0, \ldots, \tilde{N} - 1$ and $c, d = 1, \ldots, ab$. The building block for

$$m_j = M \cdot j \mod ab \tag{4.9}$$

will then take the form

$$\begin{aligned}
\Psi_\Delta &= \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{\gamma,\delta=0}^{\tilde{N}-1}\sum_{c,d=1}^{ab} \log\left[\tilde{\Gamma}\left(\Delta + \omega\frac{\delta-\gamma}{\tilde{N}} + \omega\frac{d-c}{N} + \omega(\eta_{dc} + as + br); ab\omega, ab\omega\right)\right], \\
\eta_{dc} &= (Md \mod ab) - (Mc \mod ab).
\end{aligned} \tag{4.10}$$

A second simplification can be made by dropping the term $\omega\frac{d-c}{N}$, which also does not affect the result at leading order in N.[7]

We see that indeed at this point the M dependence enters only through η_{dc}. Because[8] of the summation over c, d, this depends only on $\gcd(M, ab)$, and so without loss of generality from now on we will assume $M = \gcd(M, ab)$. Using $\sum_{c=1}^{ab} f(Mc \bmod ab) = M\sum_{c=1}^{\frac{ab}{M}} f(Mc)$ we have

[7]In Appendix A of [10] this is shown for the case $M = 1$, and with minor modifications the proof can be generalized to any $M \in \mathbb{N}$.

[8]Write $M = \mu x$ where $\gcd(x, ab) = 1$ and therefore x has a modular inverse modulo ab. Then $\sum_{c=1}^{ab} f(Mc \bmod ab) = \sum_{c=1}^{ab} f(\mu x c \bmod ab) = \sum_{c=1}^{ab} f(\mu c \bmod ab)$, and so the sum only depends on $\mu = \gcd(M, ab)$.

$$\begin{aligned}\Psi_\Delta &= M^2\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{\gamma,\delta=0}^{\tilde N-1}\sum_{c,d=1}^{ab/M}\log\left[\tilde\Gamma\left(\Delta+\omega\frac{\delta-\gamma}{\tilde N}+\omega(Md-Mc+as+br);ab\omega,ab\omega\right)\right],\\ &= M^2\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{\gamma,\delta=0}^{\tilde N-1}\sum_{c,d=0}^{ab/M-1}\log\left[\tilde\Gamma\left(\Delta+\omega\frac{\delta-\gamma}{\tilde N}+M\omega(c+d)+\omega(M-ab+as+br);ab\omega,ab\omega\right)\right],\end{aligned}\tag{4.11}$$

and using (4.7) we have

$$\Psi_\Delta = M^2\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{\gamma,\delta=0}^{\tilde N-1}\log\left[\tilde\Gamma\left(\Delta+\omega\frac{\delta-\gamma}{\tilde N}+\omega(M-ab+as+br);M\omega,M\omega\right)\right].\tag{4.12}$$

Now we use the modular formula (A17) to rewrite this as

$$\Psi_\Delta = M^2\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{\gamma,\delta=1}^{\tilde N}\left[-\pi i\mathcal{Q}(u;M\omega,M\omega)-\log\left[\theta_0\left(\frac{u}{M\omega},-\frac{1}{M\omega}\right)\right]+\sum_{k=0}^{\infty}\log\left[\frac{\psi\left(\frac{k+1+u}{M\omega}\right)}{\psi\left(\frac{k-u}{M\omega}\right)}\right]\right],\tag{4.13}$$

where

$$u=[\Delta]_\omega+\omega\frac{\delta-\gamma}{\tilde N}+\omega(M-ab+as+br),\tag{4.14}$$

and the functions $\mathcal{Q}$, θ_0, and ψ are defined in the Appendix. As Ψ_Δ was originally invariant under shifting $\Delta\to\Delta+1$, we shift $\Delta\to[\Delta]_\omega$, which is exactly the domain where the plethystic expansion for $\theta_0(\frac{u}{\omega},-\frac{1}{\omega})$ is valid, and where the sum over the ψ functions converges. Note that $[\Delta]_\omega=[\Delta]_{M\omega}$.

A. Evaluating $\mathcal{Q}$

After plugging in the expression for the polynomial $\mathcal{Q}$ from (A18) and summing over γ, δ, r, s, the first term in (4.13) becomes

$$\begin{aligned}&-\pi i\frac{N^2}{3ab\omega^2}B_3\left([\Delta]_\omega+1-\frac{a+b}{2}\omega\right)\\ &-\pi i\frac{N^2(1-\frac{a^2}{2}-\frac{b^2}{2}+a^2b^2-M^2)-a^2b^2}{6ab}B_1\left([\Delta]_\omega+1-\frac{a+b}{2}\omega\right),\end{aligned}\tag{4.15}$$

where $B_1(x)=x-\frac{1}{2}$ and $B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x$ are Bernoulli polynomials. The second term cancels in (4.6), while the first term is the same as in the $M=1$ case analyzed in [10].

B. Evaluating θ_0

Remembering that $M|ab$, the second term in (4.13) is

$$-M^2\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{\gamma,\delta=1}^{\tilde N}\log\left[\theta_0\left(\frac{[\Delta]_\omega}{M\omega}+\frac{\delta-\gamma}{M\tilde N}+\frac{as+br}{M},-\frac{1}{M\omega}\right)\right].\tag{4.16}$$

After using the Plethystic expansion (A8) for θ_0 it takes the form

$$M^2\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{\gamma,\delta=1}^{\tilde N}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\frac{\tilde y^\ell\zeta_{M\tilde N}^{\ell(\delta-\gamma)}\zeta_M^{\ell(as+br)}+\tilde h^\ell\tilde y^{-\ell}\zeta_{M\tilde N}^{\ell(\gamma-\delta)}\zeta_M^{-\ell(as+br)}}{1-\tilde h^\ell},\tag{4.17}$$

where

$$\zeta_v \equiv e^{2\pi i/v}, \qquad \tilde{y} \equiv e^{\frac{2\pi i[\Delta]_\omega}{M\omega}}, \qquad \tilde{h} \equiv e^{-\frac{2\pi i}{M\omega}}, \tag{4.18}$$

and so the second term of (4.13) is

$$\begin{aligned} &abM \sum_{\gamma,\delta=1}^{\tilde{N}} \sum_{\ell=1}^{\infty} \frac{1}{\ell} \frac{\tilde{y}^{M\ell}\zeta_{\tilde{N}}^{\ell(\delta-\gamma)} + \tilde{h}^{M\ell}\tilde{y}^{-M\ell}\zeta_{\tilde{N}}^{\ell(\gamma-\delta)}}{1-\tilde{h}^{M\ell}} \\ &= abM\tilde{N} \sum_{\ell=1}^{\infty} \frac{1}{\ell} \frac{\tilde{y}^{M\tilde{N}\ell} + \tilde{h}^{M\tilde{N}\ell}\tilde{y}^{-M\tilde{N}\ell}}{1-\tilde{h}^{M\tilde{N}\ell}} \\ &= -abM\tilde{N} \log\left[\theta_0\left(\frac{\tilde{N}[\Delta]_\omega}{\omega}; -\frac{\tilde{N}}{\omega}\right)\right], \end{aligned} \tag{4.19}$$

which is exponentially suppressed in N.

C. Evaluating ψ

Last but not least, we have the third term in (4.13). We'll expand it using the plethystic expansion for the ψ function, $\log[\psi(t)] = -\sum_{\ell=1}^{\infty} (\frac{t}{\ell} + \frac{1}{2\pi i \ell^2}) e^{-2\pi i \ell t}$, and then sum over γ, δ, r, s. When $M|\ell$ none of the terms in the plethystic expansion vanishes. Otherwise, the second term always vanishes in the sum over r, s, while the first term vanishes unless $M|a\ell$ and we are looking at the term with br multiplying the exponent, or $M|b\ell$ and we are looking at as multiplying it. The ℓ s that satisfy $M \nmid \ell$ but $M|b\ell$ are of the form $\ell \to M\ell + \frac{M}{M_b}t$, with $t = 1, \ldots, M_b - 1$ and $M_b = \gcd(M, b)$. Similarly for $M \nmid \ell$ but $M|a\ell$ we have $\ell \to M\ell + \frac{M}{M_a}t$ with $t = 1, \ldots, M_a - 1$ and $M_a = \gcd(M, a)$. Thus, we can rewrite the numerator of the third term in (4.13) as

$$\begin{aligned} &-M^2 \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \sum_{\gamma,\delta=1}^{\tilde{N}} \sum_{k=0}^{\infty} \left[\sum_{\ell=1}^{\infty} \left(\frac{\frac{k+1}{M\omega} + \frac{[\Delta]_\omega}{M\omega} + \frac{\delta-\gamma}{M\tilde{N}} + 1 - \frac{ab}{M} + \frac{as+br}{M}}{M\ell} + \frac{1}{2\pi i M^2 \ell^2}\right) e^{-2\pi i M\ell v}\right] \\ &-\sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \sum_{\gamma,\delta=1}^{\tilde{N}} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \left[\sum_{t=1}^{M_b-1} \frac{asM_b}{M_b\ell + t} e^{-2\pi i (M\ell + \frac{M}{M_b}t)v} + \sum_{t=1}^{M_a-1} \frac{brM_a}{M_a\ell + t} e^{-2\pi i (M\ell + \frac{M}{M_a}t)v}\right], \end{aligned} \tag{4.20}$$

where the first line comes from those ℓ s that satisfy $M|\ell$, while the first term in the second line comes from the ℓ s that satisfy $M|b\ell$ and $M\nmid\ell$, and the second from those that have $M\nmid\ell$ but $M|a\ell$, and v is the argument of the ψ function,

$$v = \frac{k+1}{M\omega} + \frac{[\Delta]_\omega}{M\omega} + \frac{\delta-\gamma}{M\tilde{N}} + 1 - \frac{ab}{M} + \frac{as+br}{M}. \tag{4.21}$$

For the first line of (4.20) we can sum over δ, γ, r, s to find

$$\begin{aligned} &-ab \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \left[\left(\frac{\frac{k+1}{\omega} + \frac{[\Delta]_\omega}{\omega}}{\ell}\tilde{N} + \frac{1}{2\pi i \ell^2}\right) e^{-2\pi i \tilde{N}\ell(\frac{k+1}{\omega} + \frac{[\Delta]_\omega}{\omega})}\right] \\ &-N\left(M - \frac{a}{2} - \frac{b}{2}\right) \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \left[\frac{1}{\ell} e^{-2\pi i \tilde{N}\ell(\frac{k+1}{\omega} + \frac{[\Delta]_\omega}{\omega})}\right], \end{aligned} \tag{4.22}$$

and similarly, from the denominator of the third term in (4.13) we have

$$\begin{aligned} &ab \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \left[\left(\frac{\frac{k}{\omega} - \frac{[\Delta]_\omega}{\omega}}{\ell}\tilde{N} + \frac{1}{2\pi i \ell^2}\right) e^{-2\pi i \tilde{N}\ell(\frac{k}{\omega} - \frac{[\Delta]_\omega}{\omega})}\right] \\ &-N\left(M - \frac{a}{2} - \frac{b}{2}\right) \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \left[\frac{1}{\ell} e^{-2\pi i \tilde{N}\ell(\frac{k}{\omega} - \frac{[\Delta]_\omega}{\omega})}\right], \end{aligned} \tag{4.23}$$

which are all exponentially suppressed[9] in N.

We now move to the second line of (4.20). One uses the identities

$$\begin{aligned} &\sum_{n=0}^{N-1} e^{\lambda n} = \frac{1-e^{\lambda N}}{1-e^{\lambda}}, \\ &\sum_{n=0}^{N-1} n e^{\lambda n} = \frac{d}{d\lambda} \sum_{n=0}^{N-1} e^{\lambda n} = \frac{(N-1)e^{\lambda(N+1)} - Ne^{\lambda N} + e^{\lambda}}{(1-e^{\lambda})^2}, \end{aligned} \tag{4.25}$$

to find that

$$\begin{aligned} &\sum_{\gamma,\delta=1}^{\tilde{N}} e^{2\pi i x \frac{(\gamma-\delta)}{M\tilde{N}}} = \frac{\sin^2\left(\frac{\pi x}{M}\right)}{\sin^2\left(\frac{\pi x}{M\tilde{N}}\right)}, \\ &\sum_{s=0}^{b-1} as \exp\left[\frac{2\pi i\left(M\ell + \frac{M}{M_b}t\right)}{M} as\right] = -\frac{ab}{1-e^{\frac{2\pi i t a}{M_b}}}, \end{aligned} \tag{4.26}$$

[9] We note that these two infinite sums can be resummed into

$$\begin{aligned} &ab \sum_{k=0}^{\infty} \log\left[\frac{\psi\left(\frac{\tilde{N}(k+1+[\Delta]_\omega)}{\omega}\right)}{\psi\left(\frac{\tilde{N}(k-[\Delta]_\omega)}{\omega}\right)}\right] \\ &- N\left(M - \frac{a}{2} - \frac{b}{2}\right) \log\left[\theta_0\left(\frac{\tilde{N}[\Delta]_\omega}{\omega}; -\frac{\tilde{N}}{\omega}\right)\right]. \end{aligned} \tag{4.24}$$

which allows us to sum over r, s, γ, δ, resulting in

$$ab\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{t=1}^{M_b-1}\frac{aM_b}{1-e^{\frac{2\pi ita}{M_b}}}\frac{1}{M_b\ell+t}\frac{\sin^2\left(\frac{\pi t}{M_b}\right)}{\sin^2\left(\frac{\pi\ell}{\tilde{N}}+\frac{\pi t}{M_b\tilde{N}}\right)}e^{-2\pi i(\ell+\frac{t}{M_b})(\frac{k+1}{\omega}+\frac{[\Delta]_\omega}{\omega})}+(a\leftrightarrow b). \tag{4.27}$$

Similarly, the analogous term coming from the denominator of the third term in (4.13) takes the form

$$ab\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{t=1}^{M_b-1}\frac{aM_b}{1-e^{-\frac{2\pi iat}{M_b}}}\frac{1}{M_b\ell+t}\frac{\sin^2(\frac{\pi t}{M_b})}{\sin^2(\frac{\pi\ell}{\tilde{N}}+\frac{\pi t}{M_b\tilde{N}})}e^{-2\pi i(\ell+\frac{t}{M_b})(\frac{k}{\omega}-\frac{[\Delta]_\omega}{\omega})}+(a\leftrightarrow b). \tag{4.28}$$

At large N the sine in the denominator of these two expressions gives an $O(N^2)$ dependence. Moreover, the sum over k can be easily done, leaving us with an overall contribution of the second line of (4.20)

$$\begin{aligned}ab\tilde{N}^2\Bigg[&\sum_{\ell=0}^{\infty}\sum_{t=1}^{M_b-1}\frac{a}{1-e^{\frac{2\pi iat}{M_b}}}\frac{M_b^3\sin^2(\frac{\pi t}{M_b})}{\pi^2(M_b\ell+t)^3}\frac{e^{2\pi i(\ell+\frac{t}{M_b})(-\frac{1}{\omega}-\frac{[\Delta]_\omega}{\omega})}}{1-e^{-\frac{2\pi i}{\omega}(\ell+\frac{t}{M_b})}}+(a\leftrightarrow b)\\&+\sum_{\ell=0}^{\infty}\sum_{t=1}^{M_b-1}\frac{a}{1-e^{-\frac{2\pi iat}{M_b}}}\frac{M_b^3\sin^2(\frac{\pi t}{M_b})}{\pi^2(M_b\ell+t)^3}\frac{e^{2\pi i(\ell+\frac{t}{M_b})\frac{[\Delta]_\omega}{\omega}}}{1-e^{-\frac{2\pi i}{\omega}(\ell+\frac{t}{M_b})}}+(a\leftrightarrow b)\Bigg]+O(N).\end{aligned} \tag{4.29}$$

When $M=1$ the sum over t contains no terms, so these contributions are irrelevant. We also note that in the Cardy limit, $\omega\to 0$ with fixed a, b, these terms are exponentially suppressed.

D. Overall contribution

Let us now compute $\mathcal{Z}$, which to leading order in N is[10]

$$\mathcal{Z}=\Psi_{\Delta_1}+\Psi_{\Delta_2}-\Psi_{\Delta_1+\Delta_2}-\Psi_0+O(N). \tag{4.30}$$

The chemical potentials satisfy either $[\Delta_1+\Delta_2]_\omega=[\Delta_1]_\omega+[\Delta_2]_\omega$, or $[\Delta_1+\Delta_2]_\omega=[\Delta_1]_\omega+[\Delta_2]_\omega+1$. For the first case,

$$\begin{aligned}\mathcal{Z}=-\pi iN^2\frac{[\Delta_1]_\omega[\Delta_2]_\omega[\Delta_3]_\omega}{\tau\sigma}+\frac{N^2}{ab}\sum_{\Delta}\eta_\Delta\Bigg[&\sum_{\ell=0}^{\infty}\sum_{t=1}^{M_b-1}\frac{a}{1-e^{\frac{2\pi iat}{M_b}}}\frac{M_b^3\sin^2(\frac{\pi t}{M_b})}{\pi^2(M_b\ell+t)^3}\frac{e^{2\pi i(\ell+\frac{t}{M_b})(-\frac{1}{\omega}-\frac{\Delta}{\omega})}}{1-e^{-\frac{2\pi i}{\omega}(\ell+\frac{t}{M_b})}}+(a\leftrightarrow b)\\&+\sum_{\ell=0}^{\infty}\sum_{t=1}^{M_b-1}\frac{a}{1-e^{-\frac{2\pi iat}{M_b}}}\frac{M_b^3\sin^2(\frac{\pi t}{M_b})}{\pi^2(M_b\ell+t)^3}\frac{e^{2\pi i(\ell+\frac{t}{M_b})\frac{\Delta}{\omega}}}{1-e^{-\frac{2\pi i}{\omega}(\ell+\frac{t}{M_b})}}+(a\leftrightarrow b)\Bigg],\end{aligned} \tag{4.31}$$

where $[\Delta_3]_\omega\equiv\tau+\sigma-[\Delta_1]_\omega-[\Delta_2]_\omega-1$, the sum $\sum_\Delta$ is over $\Delta\in\{[\Delta_1]_\omega,[\Delta_2]_\omega,[\Delta_1+\Delta_2]_\omega,0\}$, and we define $\eta_\Delta=\{1,1,-1,-1\}$, respectively. For the second case a similar formula applies[11] using the function $[\Delta]'_\omega=[\Delta]_\omega+1$,

$$\begin{aligned}\mathcal{Z}=-\pi iN^2\frac{[\Delta_1]'_\omega[\Delta_2]'_\omega[\Delta_3]'_\omega}{\tau\sigma}+\frac{N^2}{ab}\sum_{\Delta}\eta_\Delta\Bigg[&\sum_{\ell=0}^{\infty}\sum_{t=1}^{M_b-1}\frac{a}{1-e^{\frac{2\pi iat}{M_b}}}\frac{M_b^3\sin^2(\frac{\pi t}{M_b})}{\pi^2(M_b\ell+t)^3}\frac{e^{2\pi i(\ell+\frac{t}{M_b})\frac{\Delta}{\omega}}}{1-e^{-\frac{2\pi i}{\omega}(\ell+\frac{t}{M_b})}}+(a\leftrightarrow b)\\&+\sum_{\ell=0}^{\infty}\sum_{t=1}^{M_b-1}\frac{a}{1-e^{-\frac{2\pi iat}{M_b}}}\frac{M_b^3\sin^2(\frac{\pi t}{M_b})}{\pi^2(M_b\ell+t)^3}\frac{e^{2\pi i(\ell+\frac{t}{M_b})(-\frac{1}{\omega}-\frac{\Delta}{\omega})}}{1-e^{-\frac{2\pi i}{\omega}(\ell+\frac{t}{M_b})}}+(a\leftrightarrow b)\Bigg],\end{aligned} \tag{4.32}$$

where now $[\Delta_3]'_\omega\equiv\tau+\sigma-[\Delta_1]'_\omega-[\Delta_2]'_\omega+1$, the sum $\sum_\Delta$ is over $\Delta\in\{[\Delta_1]'_\omega,[\Delta_2]'_\omega,[\Delta_1+\Delta_2]'_\omega,0\}$, and we define $\eta_\Delta=\{1,1,-1,-1\}$ as before.

[10]When computing $\lim_{\Delta\to 0}\Psi_\Delta$ some of the terms in Ψ_Δ diverge. When $M=1$, these divergences directly cancel with the second term in (4.5). In any case, the $O(N^2)$ terms we focus on here do not diverge, while presumably divergences in subleading orders cancel with the subleading terms we neglected.

[11]We neglected an imaginary part that determines the phase of the contribution to the index, which in any case is sensitive to $O(1)$ terms.

E. Dominance of new contributions

Are these new contributions to the index always subleading with respect to the $M = 1$ contribution analyzed in [10] (and the other contribution analyzed in [60] which is equal to it in the large N limit)? The answer turns out to be negative. We can verify numerically that there are choices of parameters, say $\omega = 0.45 + 0.84i$, $\Delta_1 = 0.07 + 0.13i$, $\Delta_2 = 0.02 + 0.11i$, $a = 2$, $b = 3$, $N = 300$, where the $M = 1$ contribution is smaller than, say, the $M = 6$ contribution. This is true both for our approximate large N value in (4.31) and (4.32) and for the exact evaluation (4.6).

Thus, in some cases this term may be more dominant. Note however, that there may exist even more dominant terms for other types of $\{m_j\}$ shifts, that we have not yet computed. Moreover, this ω is not necessarily in the regime where this Bethe ansatz solution is the most dominant one, and so these contributions might be canceled by contributions from other solutions to the Bethe ansatz equations. Analyzing these cancellations requires knowledge both about the phase of $\mathcal{I}_u$, which is sensitive to $O(1)$ terms in the expansion, and about other $\{m_j\}$ shifts, whose analysis is beyond the scope of this work.

One might wonder what happens in the Cardy limit, $\omega \to 0$, where the contribution from this solution is usually the dominant one to the index. In that limit one can show that the new, M-dependent, terms at large N in (4.31) become exponentially suppressed in $1/\omega$, and so the new contributions are similar to the $M = 1$ case.

V. THE BETHE ANSATZ FOR $SU(2)$

In this section we discuss the special case of $SU(2)$ gauge group, for which we analyzed in more detail the behavior of the index for large values of a and b, and in particular the case of $b = a + 1$ (which in the limit of large a, b should converge to the results for equal chemical potentials). In this case the solutions to the Bethe ansatz equation are fully classified ($u_{12} = \frac{1}{2}, \frac{\omega}{2}, \frac{1}{2} + \frac{\omega}{2}$), and there is a single variable $u = u_{12}$ (with $u_1 = -u_2$), and a single integer m labeling the different contributions (for each solution to the Bethe ansatz). One thing we will show is that the contribution to the index from some values of m is exponentially large in a for large a, such that large cancellations between different contributions must occur in the large a limit.

A. Moving Z_m^u by $n\tau$

We assume without loss of generality that $b > a$. For $SU(2)$ we have (2.4)

$$\mathcal{Z}(u;\Delta,\sigma,\tau) = \frac{\tilde{\Gamma}(u+\Delta_1;\tau,\sigma)\tilde{\Gamma}(u+\Delta_2;\tau,\sigma)}{\tilde{\Gamma}(u;\tau,\sigma)\tilde{\Gamma}(u+\Delta_1+\Delta_2;\tau,\sigma)} \frac{\tilde{\Gamma}(-u+\Delta_1;\tau,\sigma)\tilde{\Gamma}(-u+\Delta_2;\tau,\sigma)}{\tilde{\Gamma}(-u;\tau,\sigma)\tilde{\Gamma}(-u+\Delta_1+\Delta_2;\tau,\sigma)}. \tag{5.1}$$

We will denote $Z_m^u = \mathcal{Z}(u - m\omega; \Delta, \sigma, \tau)$, such that the contribution of each solution u of the BA equation to the index is proportional to $\sum_{m=1}^{ab} Z_m^u$. This includes both the m shifts and the β shifts discussed above, since we showed in Sec. III that together they give $\sum_{m=1}^{2ab} Z_m^u = 2\sum_{m=1}^{ab} Z_m^u$.

Before beginning the analysis we derive a formula for how the contribution Z_m^u changes under shifts of m. Using (A13) we have

$$\begin{aligned} Z_{m+na}^u &= \mathcal{Z}(u - m\omega - na\omega; \Delta, \sigma, \tau) = \mathcal{Z}(u - m\omega - n\tau; \Delta, \sigma, \tau), \\ &= \mathcal{Z}(u - m\omega; \Delta, \sigma, \tau) \prod_{l=1}^{n} \frac{\theta_0(u - m\omega - (al \bmod b)\omega, \sigma)\theta_0(\Delta_1 + \Delta_2 + u - m\omega - (al \bmod b)\omega, \sigma)}{\theta_0(\Delta_1 + u - m\omega - (al \bmod b)\omega, \sigma)\theta_0(\Delta_2 + u - m\omega - (al \bmod b)\omega, \sigma)} \\ &\times \prod_{l=0}^{n-1} \frac{\theta_0(\Delta_1 - u + m\omega + (al \bmod b)\omega, \sigma)\theta_0(\Delta_2 - u + m\omega + (al \bmod b)\omega, \sigma)}{\theta_0(-u + m\omega + (al \bmod b)\omega, \sigma)\theta_0(\Delta_1 + \Delta_2 - u + m\omega + (al \bmod b)\omega, \sigma)}. \end{aligned} \tag{5.2}$$

Since a and b are mutually prime, for $n \le b$ we will get n distinct values of $(al \bmod b)$ in each product.

Let us look at the following a evaluations of $\mathcal{Z}$:

$$Z_{m-a}^u, \quad m = 1, \ldots, a, \tag{5.3}$$

and call these points our "basic window." Even though these values do not enter the sum in the index, we can construct from them all the terms that do enter the sum. To get Z_m^u using (5.2), we need to take

$$
\begin{aligned}
Z^u_m &= Z^u_{(m-a)+a} = \mathcal{Z}(u-(m-a)\omega - a\omega;\Delta,\sigma,\tau),\\
&= Z^u_{m-a}\frac{\theta_0(u-m\omega,\sigma)\theta_0(\Delta_1+\Delta_2+u-m\omega,\sigma)}{\theta_0(\Delta_1+u-m\omega,\sigma)\theta_0(\Delta_2+u-m\omega,\sigma)}\frac{\theta_0(\Delta_1-u+(m-a)\omega,\sigma)\theta_0(\Delta_2-u+(m-a)\omega,\sigma)}{\theta_0(-u+(m-a)\omega,\sigma)\theta_0(\Delta_1+\Delta_2-u+(m-a)\omega,\sigma)}.
\end{aligned}
\tag{5.4}
$$

In general, following (5.2), to construct $Z^u_{(m-a)+na}$ we take

$$
\begin{aligned}
&Z^u_{(m-a)+(n-1)a}\frac{\theta_0(u-(m-a)\omega-(an \bmod b)\omega,\sigma)}{\theta_0(\Delta_1+u-(m-a)\omega-(an \bmod b)\omega,\sigma)}\frac{\theta_0(\Delta_1+\Delta_2+u-(m-a)\omega-(an \bmod b)\omega,\sigma)}{\theta_0(\Delta_2+u-(m-a)\omega-(an \bmod b)\omega,\sigma)}\\
&\times\frac{\theta_0(\Delta_1-u+(m-a)\omega+(a(n-1) \bmod b)\omega,\sigma)}{\theta_0(-u+(m-a)\omega+(a(n-1) \bmod b)\omega,\sigma)}\frac{\theta_0(\Delta_2-u+(m-a)\omega+(a(n-1) \bmod b)\omega,\sigma)}{\theta_0(\Delta_1+\Delta_2-u+(m-a)\omega+(a(n-1) \bmod b)\omega,\sigma)}.
\end{aligned}
\tag{5.5}
$$

We get a recursive expression for $Z^u_{(m-a)+na}$ in terms of $Z^u_{(m-a)+(n-1)a}$ and one of b possible factors, labeled by the distinct label $(an \bmod b)$. Using it multiple times will give us an expression for $Z^u_{(m-a)+na}$ in terms of Z^u_{m-a} from the basic window, multiplied by n factors all labeled with distinct labels $(al \bmod b)$, $l=1,\ldots,n$ (if $n\le b$). This is enough motivation to define the factors

$$
\begin{aligned}
\Theta^u_r \equiv\;& \frac{\theta_0(u-(1-a)\omega-r\omega,\sigma)\theta_0(\Delta_1+\Delta_2+u-(1-a)\omega-r\omega,\sigma)}{\theta_0(\Delta_1+u-(1-a)\omega-r\omega,\sigma)\theta_0(\Delta_2+u-(1-a)\omega-r\omega,\sigma)}\frac{\theta_0(\Delta_1-u+(1-a)\omega+((r-a) \bmod b)\omega,\sigma)}{\theta_0(-u+(1-a)\omega+((r-a) \bmod b)\omega,\sigma)}\\
&\times\frac{\theta_0(\Delta_2-u+(1-a)\omega+((r-a) \bmod b)\omega,\sigma)}{\theta_0(\Delta_1+\Delta_2-u+(1-a)\omega+((r-a) \bmod b)\omega,\sigma)},\\
=\;&\frac{\theta_0(u-(1+r-a)\omega,\sigma)\theta_0(\Delta_1+\Delta_2+u-(1+r-a)\omega,\sigma)}{\theta_0(\Delta_1+u-(1+r-a)\omega,\sigma)\theta_0(\Delta_2+u-(1+r-a)\omega,\sigma)}\\
&\times\frac{\theta_0(\Delta_1-u+(1+r-2a)\omega,\sigma)\theta_0(\Delta_2-u+(1+r-2a)\omega,\sigma)}{\theta_0(-u+(1+r-2a)\omega,\sigma)\theta_0(\Delta_1+\Delta_2-u+(1+r-2a)\omega,\sigma)}.
\end{aligned}
\tag{5.6}
$$

It is easy to check that Θ^u_r are invariant under $r\to r+b \Leftrightarrow u\to u+\sigma$, such that the last equality is valid, and indeed we have b different factors and not more. Note that one can also shift $Z^u_{(m+1-a)}$ to $Z^u_{(m+1-a)+na}$ using only the same factors Θ^u_r, and so on. So if we calculate all of the b factors of Θ^u_r ($r=0,\ldots,b-1$), we get all the factors needed to produce all the terms in the sum out of our basic window values (5.3). If we arrange the Θ^u_r in the specific order

$$
(\Theta^u_{na \bmod b})^b_{n=1},
\tag{5.7}
$$

then we get the ordered factors needed to move Z^u_{1-a} to Z^u_{ab+1-a}, through all Z^u_{na+1-a} values. Using

$$
(\Theta^u_{(na+1) \bmod b})^b_{n=1}
\tag{5.8}
$$

we get the ordered factors needed to move Z^u_{2-a} to Z^u_{ab+2-a}, passing through all Z^u_{na+2-a} values, and so on.

Computing these factors numerically for generic parameters, their absolute values and phases generally look something like Fig. 2 (this is for the solution $u=\frac{\omega}{2}$, we will discuss the other solutions later). It should be noted

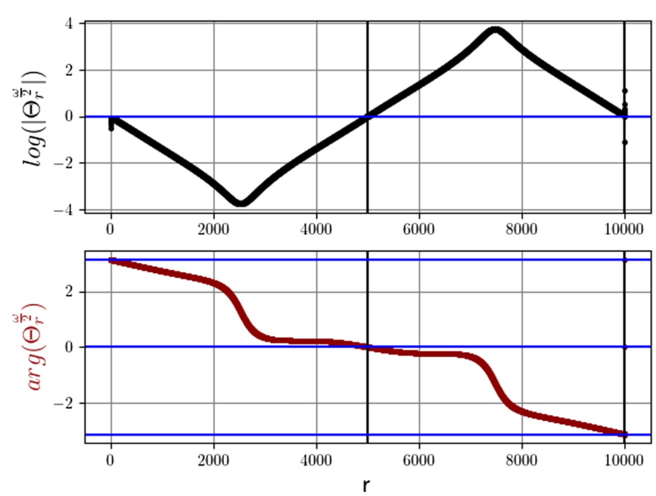


FIG. 2. We plot $\Theta_r^{u=\frac{\omega}{2}}$ for $a=9999$, $b=10000$, $\tau=-0.67+2.3i$, $\Delta_1=0.3+1.1i$, and $\Delta_2=0.11–0.6i$. The black vertical lines show two points in which $|\Theta^u_r|=1$, that represent the beginning and end of a streak of $|\Theta^u_r|$ values that are larger than 1. The blue horizontal lines mark the special values of 0 for $\log(|\Theta_r^{\frac{\omega}{2}}|)$, and $-\pi, 0, \pi$ for $\arg(\Theta_r^{\frac{\omega}{2}})$.

that for some parameters, the $|\Theta_r^u|$ graph looks a bit different. It can cross the $|\Theta_r^u| = 0$ line more often, and it can be relatively flat. With minor changes, the arguments below will hold also in these cases.

When we take the large a limit, Θ_r^u approaches a continuous function of (r/b) (as in Fig. 2). Naively one may think that since when we take $a, b \to \infty$ with fixed τ, ω goes to zero, one may be able to approximate the sum over m as a continuous integral over $x = m/a$ (in the range $[0, b]$). However, using the fact that most Θ_r^u are very different from one (even for large a), one can show that even if two adjacent values in our "basic window" approach each other in the large a limit, this is not true once they are shifted by a large amount (so that x is of order a). So Z_m^u does not really approach a continuous function of x in the large a limit.

1. The $b = a + 1$ case

Consider now the special case $b = a + 1$, for which $(na \bmod b) = ((-n) \bmod b)$. Then, the order of the factors that we multiply by will be just as plotted in Fig. 2 but reversed, with the specific starting point determined by m. This case is particularly interesting because for fixed τ, the limit $a \to \infty$ is equivalent to $\tau \to \sigma$, where we expect to converge to the results for equal chemical potentials $a = b = 1$ [which are given by (2.13) with no sum over m].

Looking at Fig. 2, we see that (5.6) is roughly divided to a continuous half of Θ_r^us with absolute value greater than 1, and half which are smaller than 1. It can also easily be shown that the factors in each half are just the inverses of the values in the other half, with a mirrored order (this is precisely true when $u = \frac{\omega}{2}$ and $\frac{1}{2} + \frac{\omega}{2}$, and true for large a for $u = \frac{1}{2}$). Note that the shape of the figure does not change when taking $a \to \infty, \tau = \text{const}$, it just becomes denser (except at the edges of the graph, in a region that becomes negligible as $a \to \infty$). The same is true for the basic window values of Z_{m-a}^u.

Moving the basic window of points by a to $Z_m^u, m = 1, \ldots, a$, each point will get multiplied by an appropriate factor from $\Theta_{b-1}^u, \Theta_0^u, \Theta_1^u, \ldots, \Theta_{b-3}^u$, in the order they are written (Θ_{b-2}^u does not take part). Moving the new (moved) window again will result in multiplying it by the same factors in the same order, but moved by one slot (so this time Θ_{b-3}^u does not take part). Most of the points that were multiplied by factors with absolute value larger than 1 before, will get this treatment again because of the topography of Θ_r^u. Moving a few windows forward from the basic window (when b is large we have plenty), less and less points have only been multiplied by factors with absolute value larger than 1, and more and more get multiplied by mixed factors. So after a while, a peak in the absolute value of Z_m^u arises in the region of the window that can be traced to the basic window only through factors with absolute value greater than 1. This is demonstrated

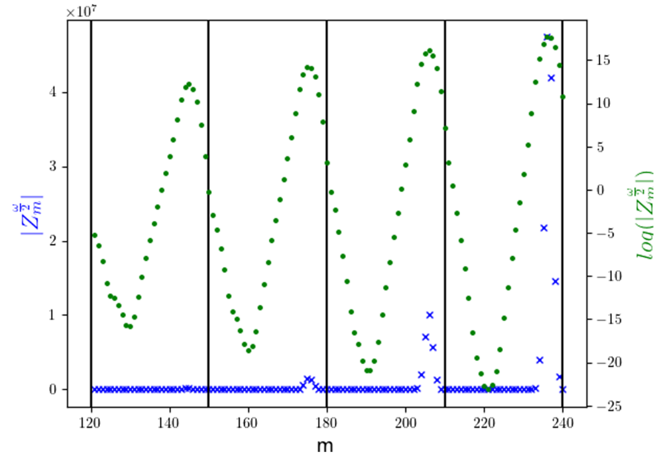


FIG. 3. We plot $|Z_m^{\frac{\omega}{2}}|$ (blue) and $\log(|Z_m^{\frac{\omega}{2}}|)$ (green) as a function of m. The horizontal lines mark the beginning and end of a-point windows (here $a = 30$, $b = 31$, and the rest of the parameters are as in Fig. 2). The values of each window can be calculated from the values of the former window by multiplying its values by $\Theta_r^{\frac{\omega}{2}}$ with the correct shift.

in Fig. 3. This shows us that the points in which $|\Theta_r^u| = 1$ are important. We can find these points by noting that

$$\Theta_r^u = \Theta_0^{u - r\omega} \equiv \Theta_0^{\tilde{u}} \underset{a\to\infty}{\longrightarrow} -Q(\tilde{u}; \Delta, \tau), \tag{5.9}$$

so finding these points means solving

$$|Q(\tilde{u}; \Delta, \tau)| = 1. \tag{5.10}$$

This is similar to the BAE, except that Q can have a phase different from 0 or π, and we only search for solutions of the form $\alpha\omega$ or $\frac{1}{2} + \alpha\omega$ for $\alpha \in \mathbb{R}$. The known BAE solutions immediately provide us with the solutions $\tilde{u} = 0, \frac{\tau}{2}, \tau$ for $u = \frac{\omega}{2}$, and $\tilde{u} = \frac{1}{2}, \frac{1}{2} + \frac{\tau}{2}, \frac{1}{2} + \tau$ for $u = \frac{1}{2}$ and $u = \frac{1}{2} + \frac{\omega}{2}$. It is important to remember that in most of these points $-Q(\tilde{u}) = 1$, except for $\tilde{u} = 0, \tau$, in which $-Q(\tilde{u}) = -1$.

We can interpret Fig. 2 in light of this observation. We indeed have $|\Theta_r^u| = 1$ at $r\omega \approx 0, \frac{\tau}{2}, \tau$, with the phases we expect.

B. There are large cancellations between different m-shifted terms

The fact that many Θ_r^us are larger than 1 implies that for large a some Zs, arising from multiplying many of these Θs, will be exponentially large. One may wonder if perhaps the sum over m could be dominated by some specific large values, such that we can neglect the rest of the contributions. Clearly this is impossible, given that the large a limit of the sum should be a constant (namely the index for $\sigma = \tau$). It turns out that indeed the largest contributions partially cancel each other, with many terms taking part in this process. What is left from this partial cancellation can still be very large, but comparable to other terms in the sum.

The picture of Fig. 3, in which there is a peak in each window, with an increasing absolute value for the peak in each window, which arises because each window is multiplied by Θ_r^u factors with absolute value larger than 1, ends at $m \approx \frac{ab}{2}$. After that point, Θ_r^u becomes smaller than 1. The maximal point in the sum over shifts by na (for large enough a) will be the one that comes from the basic window by getting multiplied by all Θ_r^u factors in the range of values of r that obeys $|\Theta_r^u| > 1$, starting and ending with $|\Theta_r^u|$ values near 1. Note that the values near the beginning and end of this range are very close to 1, in fact infinitesimally close for $a \to \infty$. So around the maximum we have many other points that are very close to it in absolute value. But if we look at the phase of Θ_r^u near the beginning and end of the streak in Fig. 2, it is approximately π. So we have many large terms summed up with almost opposite phases, that cancel each others' contributions in a noisy manner. As we look further from the actual maximum, the change in the absolute value between adjacent terms grows, and the relative phase also slowly changes from π. But these terms still partially cancel each other, until we move enough to continuously shift to a different regime. So what is left after summing all these large terms is unclear, and can still be very large.

The important feature for these cancellations is that either at the beginning or at the end of the large $|\Theta_r^u|$ streak, the phase of Θ_r^u is not 0. Thus, the only cases that will not have this kind of cancellations are those in which $\Theta_r^u = 1$ in the beginning and end of the region, meaning when it begins and ends with $\tilde{u}$ that solves the reduced BAE. As mentioned above, this is the case with the other two solutions $u = \frac{1}{2}, \frac{\omega}{2} + \frac{1}{2}$, where the cancellations happen differently.

In these cases, it turns out that the cancellations happen between the two different solutions, rather than between different m movements of the same BA solution. In the large a limit, in the vicinity of the largest $|Z_m^u|$, the values of Z_m^u for many different nearby m terms are very similar; this is because they begin from nearby values in the "basic window," which are all then multiplied by almost all values of Θ_r^u that obey $|\Theta_r^u| > 1$. But if $Z_m^{\frac{1}{2}}$ is very close to $Z_{m+1}^{\frac{1}{2}}$, then they are both close to $Z_m^{\frac{\omega}{2}+\frac{1}{2}} = Z_{m+\frac{1}{2}}^{\frac{1}{2}}$, because $\mathcal{Z}$ is continuous. So the two solutions, $u = \frac{1}{2}$ and $u = \frac{\omega}{2} + \frac{1}{2}$, produce very similar values of Z_m^u for the largest $|Z_m^u|$ terms. Numerically we find that $H^{\frac{1}{2}} \approx -H^{\frac{\omega}{2}+\frac{1}{2}}$ at large a and (at least at) small q, so these large contributions mostly cancel each other. Note that if there are some extra solutions to (5.10) that break the streak with nonzero phase, it will just mean that the large contributions will cancel for the original reason, and not that they do not cancel.

Following the last note, the Θ_r^u picture is not always simple, depending on the precise parameters. Sometimes its absolute value crosses 1 multiple times as discussed above, and we get several large $|\Theta_r^u|$ streaks, and several small $|\Theta_r^u|$ streaks. This does not change the overall picture. When taking large a these regions become denser, and so some points in the sum become exponentially large in a. In this case it is not so easy to know what the global maximum is, since it depends on the sizes of the different large $|\Theta_r^u|$ streaks and on the value of the factors in these streaks. But everything we argued before will still be true for the global maximum in these cases, including the partial cancellation.

The discussion of this subsection is not a feature just of $b = a + 1$. It is true for any $b = a + \mathrm{const}$ when taking large a, except that these cases will have more $|\Theta_r^u| = 1$ crossings.

Numerical computations of (2.13) are consistent with the above discussion. The contribution of the sum over m-shifted solutions originating from $u = \frac{\omega}{2}$ partially cancels within itself. This can be seen from the fact that the whole sum is smaller in absolute value than the maximal term in the sum, sometimes by orders of magnitude.

The sums originating from $u = \frac{1}{2}, \frac{1}{2} + \frac{\omega}{2}$ are sometimes larger than their maximal value, but they partially cancel each other. The rest of the cancellation comes from adding the $\frac{\omega}{2}$ contribution, to get a value that is not exponentially large in a.

One could have thought that in the $a \to \infty$ limit, each HL solution to $Q(u; \Delta, \omega) = -1$ (including its m and β shifts) will contribute exactly the value that the solution to $Q(u; \Delta, \tau) = -1$ contributes at $\sigma = \tau$, matching the solutions via $\omega \leftrightarrow \tau$. However, we see that this is not true, as the contributions of some solutions to $Q(u; \Delta, \omega) = -1$ to the index grow exponentially for large a. Thus, the mapping between BA solutions and gravitational solutions described in [5] needs to be modified for this case of unequal angular momentum potentials.

VI. ALLOWED SHIFTS FROM THE GRAVITY SIDE

While most of this paper concentrated on some specific contributions to the index at $O(N^2)$, in this section we consider the mapping in the opposite direction, and we argue that when $\tau \neq \sigma$ there may be some additional gravitational contributions whose origin within the Bethe ansatz formalism is unknown at the moment. In order to analyze the different backgrounds, remember that the chemical potentials on the gravity side are the same as those on the conformal field theory up to an integer shift [5]

$$\Delta_{g,1} = \Delta_1 + n_1, \qquad \Delta_{g,2} = \Delta_2 + n_2,$$
$$\tau_g = \tau + k_2, \qquad \sigma_g = \sigma + k_1. \tag{6.1}$$

On the gravity side the on-shell action (on the first branch) takes the form

$$I = \pi i N^2 \frac{\Delta_{g,1}\Delta_{g,2}\Delta_{g,3}}{\tau_g \sigma_g}, \tag{6.2}$$

where $\Delta_{g,3} = \tau_g + \sigma_g - \Delta_{g,1} - \Delta_{g,2} - 1$. The entire partition function is periodic under integer shifts of any of the four chemical potentials, while tuning $\Delta_{g,3}$ to preserve this linear relation. However, the on-shell action is not periodic. In [5] the periodicity was understood as coming from the contribution of different bulk geometries. The case where $\tau = \sigma$ was considered, and some of the different bulk geometries were associated with the shift $(\tau_g, \sigma_g) \to (\tau_g + 1, \sigma_g + 1)$. These were then argued to be matched with the contribution of the Hong-Liu solutions (2.15) with different rs.

Ostensibly, there could be also shifts in the Δ_gs, which do not seem to match to any Hong-Liu solution. However, it turned out [5] that the resulting bulks were all unstable to brane nucleation, such that the stable bulks matched in a one-to-one fashion with the contributions coming from the Bethe ansatz solutions. The brane involved is a Euclidean D3-brane that wraps an $S^1 \subset \mathrm{AdS}_5$ and an $S^3 \subset S^5$, and has an action which is one of

$$I_{D_3} = \begin{cases} 2\pi N \frac{\Delta_{g,i}}{\tau_g} \\ 2\pi N \frac{\Delta_{g,i}}{\sigma_g} \end{cases} \tag{6.3}$$

depending on the exact cycles that the brane wraps around, see details in [5]. Since the branes do not wrap the thermal cycle, their contribution to the Euclidean partition function is $e^{iI_{D_3}}$, and so the geometry is stable only if

$$\mathbb{Im}(I_{D_3}) \geq 0. \tag{6.4}$$

We will now repeat this analysis for the case where $\tau = a\omega$ and $\sigma = b\omega$, so that we have $\tau_g = a\omega + k_1$ and $\sigma_g = b\omega + k_2$. We will consider only shifts of τ_g and σ_g, so we consider stability bounds that are independent of the Δ_gs. The stability conditions coming from combinations of three D3-branes that wrap the same $S^1 \subset \mathrm{AdS}_5$ and three different choices of $S^3 \subset S^5$ are

$$\mathbb{Im}\left(\frac{\tau_g - 1}{\sigma_g}\right) \geq 0, \qquad \mathbb{Im}\left(\frac{\sigma_g - 1}{\tau_g}\right) \geq 0. \tag{6.5}$$

This implies[12]

$$\begin{aligned} &\mathbb{Im}(\tau_g) > \mathbb{Im}(\tau_g)\mathbb{R}e(\sigma_g) - \mathbb{R}e(\tau_g)\mathbb{Im}(\sigma_g) > -\mathbb{Im}(\sigma_g), \\ &\quad a > a\mathbb{R}e(\sigma_g) - b\mathbb{R}e(\tau_g) > -b, \\ &\quad a > ak_2 - bk_1 > -b. \end{aligned} \tag{6.6}$$

[12]Note that the inequality is independent of $\mathbb{R}e(\omega)$, as it cancels between the two terms of the second inequality.

Starting from a stable bulk solution, shifting $\tau_g \to \tau + a$ and $\sigma_g \to \sigma_g + b$ is always allowed. These shifts keep $\frac{\tau_g}{\sigma_g} = \frac{a}{b}$, and correspond to $\omega \to \omega + 1$. They reproduce the Hong-Liu solutions with different r, (2.15). However, (6.6) can have other solutions. For example, when $(a, b) = (2, 3)$ we can also choose $k_2 = 2$, $k_1 = 1$, and this shift is not reproduced by merely considering the different Hong-Liu solutions. Conceivably, these other shifts might be reproduced from the sum over the different $\{m_j\}$ shifts considered in the rest of this paper, but this is beyond the scope of this paper.

ACKNOWLEDGMENTS

We would like to thank Francesco Benini for useful discussions. This work was supported in part by an Israel Science Foundation (ISF) center for excellence grant (Grant No. 2289/18), by ISF Grant No. 2159/22, by Simons Foundation Grant No. 994296 (Simons Collaboration on Confinement and QCD Strings), by Grant No. 2018068 from the United States-Israel Binational Science Foundation (BSF), by the Minerva Foundation with funding from the Federal German Ministry for Education and Research, by the German Research Foundation through a German-Israeli Project Cooperation (DIP) grant “Holography and the Swampland,” and by a research grant from Martin Eisenstein. O. A. is the Samuel Sebba Professorial Chair of Pure and Applied Physics. O. M. is supported by the ERC-COG Grant No. NP-QFT 864583 “Nonperturbative dynamics of quantum fields: from new deconfined phases of matter to quantum black holes,” by the MUR-FARE2020 Grant No. R20E8NR3HX “The Emergence of Quantum Gravity from Strong Coupling Dynamics.” O. M. is also partially supported by the INFN “Iniziativa Specifica GAST.”

APPENDIX: SPECIAL FUNCTIONS

We will use the notations[13]

$$q = e^{2i\pi\tau}, \qquad p = e^{2\pi i\sigma}, \qquad z = e^{2\pi i u}. \tag{A1}$$

1. q-Pochhammer symbol

The q-Pochhamemer symbol is

$$(z;q)_n \equiv \prod_{k=0}^{n-1}(1 - zq^k), \quad (z;q)_\infty \equiv \prod_{k=0}^{\infty}(1 - zq^k), \ \text{ for } |q| < 1. \tag{A2}$$

[13]Note that this convention is more common in the literature concerning modular functions and transformations. In some of the literature concerning elliptic functions one uses $q' = e^{\pi i\tau} = \sqrt{q}$, even though it is denoted by q there.

There are also a series expansion and a Plethystic representation for $(z;q)_\infty$,

$$(z;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n-1)}}{(q;q)_n} z^n = \exp\left[-\sum_{k=1}^{\infty}\frac{1}{k}\frac{z^k}{1-q^k}\right], \tag{A3}$$

where the first converges for $|q|<1$ while the second converges for $|z|,|q|<1$.

By relating the symbol to the Dedekind eta function,

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}}(q;q)_\infty, \tag{A4}$$

one obtains the properties of the q-Pochhammer symbol under modular transformations

$$(\tilde{q};\tilde{q})_\infty = \sqrt{-i\tau}e^{\frac{\pi i}{12}(\tau+1/\tau)}(q;q)_\infty, \tag{A5}$$

where $\tilde{q}=e^{-2\pi i/\tau}$. Finally, we have the asymptotic behavior

$$\begin{aligned} (z;q)_\infty &\sim 1-z \quad \text{for } q\to 0, \\ \log(z;q)_\infty &\sim -\frac{z}{1-q} \quad \text{for } z\to 0. \end{aligned} \tag{A6}$$

2. Elliptic theta function

The elliptic theta function is

$$\theta_0(u;\tau) \equiv (z;q)_\infty(q/z;q)_\infty = \prod_{k=0}^{\infty}(1-zq^k)(1-z^{-1}q^{k+1}), \tag{A7}$$

which gives an analytic function on $|q|<1$ with simple zeros at $z=q^k$ for $k\in\mathbb{Z}$ and no singularities. The infinite product is convergent in the whole domain. We can also give a plethystic expansion

$$\theta_0(u;\tau) = \exp\left[-\sum_{k=1}^{\infty}\frac{1}{k}\frac{z^k+(qz^{-1})^k}{1-q^k}\right], \tag{A8}$$

which converges for $|q|<|z|<1$. The periodicity relations are

$$\begin{aligned} \theta_0(u+n+m\tau;\tau) &= (-1)^m e^{-\pi i m(2u+(m-1)\tau)}\theta_0(u;\tau), \quad m,n\in\mathbb{Z} \\ \theta_0(u;\tau) &= \theta_0(\tau-u;\tau) = -e^{2\pi i u}\theta_0(-u;\tau), \end{aligned} \tag{A9}$$

and under modular transformations

$$\begin{aligned} \theta_0(u;\tau+1) &= \theta_0(u;\tau), \\ \theta_0\left(\frac{u}{\tau};-\frac{1}{\tau}\right) &= -ie^{\frac{\pi i}{\tau}(u^2+u+\frac{1}{6})-\pi i u+\frac{\pi i \tau}{6}}\theta_0(u;\tau). \end{aligned} \tag{A10}$$

3. The elliptic gamma function

The elliptic gamma function is defined by

$$\tilde{\Gamma}(u;\sigma,\tau) \equiv \prod_{m,n=0}^{\infty}\frac{1-p^{m+1}q^{n+1}z^{-1}}{1-p^m q^n z}. \tag{A11}$$

This definition gives a meromorphic single valued function on $|p|,|q|<1$ with simple zeros at $z=p^{m+1}q^{n+1}$ and simple poles at $z=p^{-m}q^{-n}$ for $m,\ n\geq 0$. The infinite product is convergent on the whole domain. We can also give a plethystic definition

$$\tilde{\Gamma}(u;\sigma,\tau) = \exp\left[\sum_{k=1}^{\infty}\frac{1}{k}\frac{z^k-(pqz^{-1})^k}{(1-p^k)(1-q^k)}\right], \tag{A12}$$

which converges for $|pq|<|z|<1$. The function has the following periodicity relations:

$$\begin{aligned} \tilde{\Gamma}(u;\sigma,\tau) &= \tilde{\Gamma}(u;\tau,\sigma), \\ \tilde{\Gamma}(u;\sigma,\tau) &= \tilde{\Gamma}(u+1;\sigma,\tau) = \tilde{\Gamma}(u;\sigma+1,\tau) \\ &= \tilde{\Gamma}(u;\sigma,\tau+1), \\ \tilde{\Gamma}(u+\sigma;\sigma,\tau) &= \theta_0(u;\tau)\tilde{\Gamma}(u;\sigma,\tau), \\ \tilde{\Gamma}(u+\tau;\sigma,\tau) &= \theta_0(u;\sigma)\tilde{\Gamma}(u;\sigma,\tau). \end{aligned} \tag{A13}$$

Moreover,

$$\tilde{\Gamma}(u;\sigma,\tau)\tilde{\Gamma}(\sigma+\tau-u;\sigma,\tau) = 1. \tag{A14}$$

The elliptic gamma function has $SL(3,\mathbb{Z})$ modular properties. For $\sigma,\tau,\sigma/\tau,\sigma+\tau\in\mathbb{C}\backslash\mathcal{R}$ there is a "modular formula" [66]:

$$\begin{aligned} \tilde{\Gamma}(u;\sigma,\tau) &= e^{-\pi i \mathcal{Q}(u;\sigma,\tau)}\frac{\tilde{\Gamma}(\frac{u}{\tau};\frac{\sigma}{\tau},-\frac{1}{\tau})}{\tilde{\Gamma}(\frac{u-\tau}{\sigma};-\frac{1}{\sigma},-\frac{\tau}{\sigma})} \\ &= e^{-\pi i \mathcal{Q}(u;\sigma,\tau)}\frac{\tilde{\Gamma}(\frac{u}{\sigma};-\frac{1}{\sigma},\frac{\tau}{\sigma})}{\tilde{\Gamma}(\frac{u-\sigma}{\tau};-\frac{\sigma}{\tau},-\frac{1}{\tau})}, \end{aligned} \tag{A15}$$

where $\mathcal{Q}(u;\sigma,\tau)$ is the cubic polynomial

$$\mathcal{Q}(u;\sigma,\tau) = \frac{u^3}{3\sigma\tau} - \frac{\sigma+\tau-1}{2\sigma\tau}u^2 + \frac{\sigma^2+\tau^2+3\sigma\tau-3\sigma-3\tau+1}{6\sigma\tau}u + \frac{(\sigma+\tau-1)(\sigma+\tau-\sigma\tau)}{12\sigma\tau}. \tag{A16}$$

In the degenerate case $\sigma = \tau$ the formula above is not valid. For $u \in \mathbb{C}\backslash(\mathbb{Z}+\tau\mathbb{Z})$, however, there is a degenerate relation

$$\tilde{\Gamma}(u;\tau,\tau) = \frac{e^{-\pi i \mathcal{Q}(u;\tau,\tau)}}{\theta_0(\frac{u}{\tau};-\frac{1}{\tau})}\prod_{k=0}^{\infty}\frac{\psi(\frac{k+1+u}{\tau})}{\psi(\frac{k-u}{\tau})}, \tag{A17}$$

where the function ψ is the elliptic digamma function defined below and the polynomial $\mathcal{Q}$ reduces to

$$\mathcal{Q}(u;\tau,\tau) = \frac{(2u-2\tau+1)(2u(u+1)-2\tau(2u+1)+\tau^2)}{12\tau^2}. \tag{A18}$$

Using

$$\mathcal{Q}(u+1;\tau,\tau) - \mathcal{Q}(u;\tau,\tau) = \frac{(u+1)(u+1-2\tau)}{\tau^2} + \frac{5}{6}, \tag{A19}$$

one can check that $\tilde{\Gamma}(u;\tau,\tau)$ is invariant under $u \to u+1$.

4. Function ψ

Define, for $\mathbb{Im}(t) < 0$, the function

$$\psi(t) = \exp\left[t\log(1-e^{-2\pi i t}) - \frac{1}{2\pi i}\mathrm{Li}_2(e^{-2\pi i t})\right] = \exp\left[-\sum_{\ell=1}^{\infty}\left(\frac{t}{\ell}+\frac{1}{2\pi i \ell^2}\right)e^{-2\pi i t\ell}\right]. \tag{A20}$$

The branch of the logarithm is determined by its series expansion $\log(1-z) = -\sum_{\ell=1}^{\infty} z^\ell/\ell$, whereas $\mathrm{Li}_2(z) = \sum_{\ell=1}^{\infty} z^\ell/\ell^2$ is the dilogarithm. One can show that the branch cut discontinuities of the logarithm and the dilogarithm cancel in the definition of $\psi(t)$, such that the latter extends to a meromorphic function on the whole complex plane. Some useful properties of $\psi(t)$ are

$$\psi(t)\psi(-t) = e^{-\pi i(t^2-1/6)}, \qquad \psi(t+n) = (1-e^{-2\pi i t})^n\psi(t) \quad \text{for } n\in\mathbb{Z}. \tag{A21}$$

In particular, from (A20), $\psi(0) = e^{\pi i/12}$.

[1] Francesco Benini, Kiril Hristov, and Alberto Zaffaroni, Black hole microstates in AdS_4 from supersymmetric localization, J. High Energy Phys. 05 (2016) 054.
[2] Alejandro Cabo-Bizet, Davide Cassani, Dario Martelli, and Sameer Murthy, Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS_5 black holes, J. High Energy Phys. 10 (2019) 062.
[3] Sunjin Choi, Joonho Kim, Seok Kim, and June Nahmgoong, Large AdS black holes from QFT, arXiv:1810.12067.
[4] Francesco Benini and Elisa Milan, Black holes in 4D $\mathcal{N}=4$ Super-Yang-Mills field theory, Phys. Rev. X **10**, 021037 (2020).
[5] Ofer Aharony, Francesco Benini, Ohad Mamroud, and Elisa Milan, A gravity interpretation for the Bethe Ansatz expansion of the $\mathcal{N}=4$ SYM index, Phys. Rev. D **104**, 086026 (2021).
[6] Arash Arabi Ardehali, Junho Hong, and James T. Liu, Asymptotic growth of the 4d $\mathcal{N}=4$ index and partially deconfined phases, J. High Energy Phys. 07 (2020) 073.
[7] Alejandro Cabo-Bizet, Davide Cassani, Dario Martelli, and Sameer Murthy, The large-N limit of the 4d $\mathcal{N}=1$ superconformal index, J. High Energy Phys. 11 (2020) 150.
[8] Sameer Murthy, The growth of the $\frac{1}{16}$-BPS index in 4d $\mathcal{N}=4$ SYM, arXiv:2005.10843.
[9] Prarit Agarwal, Sunjin Choi, Joonho Kim, Seok Kim, and June Nahmgoong, AdS black holes and finite N indices, Phys. Rev. D **103**, 126006 (2021).
[10] Francesco Benini, Edoardo Colombo, Saman Soltani, Alberto Zaffaroni, and Ziruo Zhang, Superconformal indices at large N and the entropy of $\mathrm{AdS}_5 \times \mathrm{SE}_5$ black holes, Classical Quantum Gravity **37**, 215021 (2020).
[11] Alfredo González Lezcano, Junho Hong, James T. Liu, and Leopoldo A. Pando Zayas, Sub-leading structures in superconformal indices: Subdominant saddles and logarithmic contributions, J. High Energy Phys. 01 (2021) 001.
[12] Christian Copetti, Alba Grassi, Zohar Komargodski, and Luigi Tizzano, Delayed deconfinement and the Hawking-Page transition, J. High Energy Phys. 04 (2022) 132.

[13] Alejandro Cabo-Bizet, From multi-gravitons to black holes: The role of complex saddles, arXiv:2012.04815.

[14] Antonio Amariti, Marco Fazzi, and Alessia Segati, The SCI of $\mathcal{N}=4$ $USp(2N_c)$ and $SO(N_c)$ SYM as a matrix integral, J. High Energy Phys. 06 (2021) 132.

[15] Seyed Morteza Hosseini and Alberto Zaffaroni, Large N matrix models for 3d $\mathcal{N}=2$ theories: Twisted index, free energy and black holes, J. High Energy Phys. 08 (2016) 064.

[16] Masazumi Honda, Quantum black hole entropy from 4d supersymmetric Cardy formula, Phys. Rev. D **100**, 026008 (2019).

[17] Francesco Benini and Alberto Zaffaroni, Supersymmetric partition functions on Riemann surfaces, Proc. Symp. Pure Math. **96**, 13 (2017).

[18] Francesco Benini, Kiril Hristov, and Alberto Zaffaroni, Exact microstate counting for dyonic black holes in AdS_4, Phys. Lett. B **771**, 462 (2017).

[19] Seyed Morteza Hosseini, Anton Nedelin, and Alberto Zaffaroni, The Cardy limit of the topologically twisted index and black strings in AdS_5, J. High Energy Phys. 04 (2017) 014.

[20] Alejandro Cabo-Bizet, Victor I. Giraldo-Rivera, and Leopoldo A. Pando Zayas, Microstate counting of AdS_4 hyperbolic black hole entropy via the topologically twisted index, J. High Energy Phys. 08 (2017) 023.

[21] Francesco Azzurli, Nikolay Bobev, P. Marcos Crichigno, Vincent S. Min, and Alberto Zaffaroni, A universal counting of black hole microstates in AdS_4, J. High Energy Phys. 02 (2018) 054.

[22] Seyed Morteza Hosseini, Kiril Hristov, and Achilleas Passias, Holographic microstate counting for AdS_4 black holes in massive IIA supergravity, J. High Energy Phys. 10 (2017) 190.

[23] Francesco Benini, Hrachya Khachatryan, and Paolo Milan, Black hole entropy in massive type IIA, Classical Quantum Gravity **35**, 035004 (2018).

[24] Seyed Morteza Hosseini, Itamar Yaakov, and Alberto Zaffaroni, Topologically twisted indices in five dimensions and holography, J. High Energy Phys. 11 (2018) 119.

[25] P. Marcos Crichigno, Dharmesh Jain, and Brian Willett, 5d partition functions with a twist, J. High Energy Phys. 11 (2018) 058.

[26] Seyed Morteza Hosseini, Kiril Hristov, Achilleas Passias, and Alberto Zaffaroni, 6D attractors and black hole microstates, J. High Energy Phys. 12 (2018) 001.

[27] Arash Arabi Ardehali, Cardy-like asymptotics of the 4d $\mathcal{N}=4$ index and AdS_5 blackholes, J. High Energy Phys. 06 (2019) 134.

[28] Minwoo Suh, Supersymmetric AdS_6 black holes from matter coupled $F(4)$ gauged supergravity, J. High Energy Phys. 02 (2019) 108.

[29] Martin Fluder, Seyed Morteza Hosseini, and Christoph F. Uhlemann, Black hole microstate counting in type IIB from 5d SCFTs, J. High Energy Phys. 05 (2019) 134.

[30] Dongmin Gang, Nakwoo Kim, and Leopoldo A. Pando Zayas, Precision microstate counting for the entropy of wrapped M5-branes, J. High Energy Phys. 03 (2020) 164.

[31] Gergely Kántor, Constantinos Papageorgakis, and Paul Richmond, AdS_7 black-hole entropy and 5D $\mathcal{N}=2$ Yang-Mills, J. High Energy Phys. 01 (2020) 017.

[32] Sunjin Choi, Chiung Hwang, and Seok Kim, Quantum vortices, M2-branes and black holes, arXiv:1908.02470.

[33] Sunjin Choi, Saebyeok Jeong, Seok Kim, and Eunwoo Lee, Exact QFT duals of AdS black holes, J. High Energy Phys. 09 (2023) 138.

[34] Nikolay Bobev and P. Marcos Crichigno, Universal spinning black holes and theories of class $\mathcal{R}$, J. High Energy Phys. 12 (2019) 054.

[35] Jun Nian and Leopoldo A. Pando Zayas, Microscopic entropy of rotating electrically charged AdS_4 black holes from field theory localization, J. High Energy Phys. 03 (2020) 081.

[36] Francesco Benini, Dongmin Gang, and Leopoldo A. Pando Zayas, Rotating black hole entropy from M5-branes, J. High Energy Phys. 03 (2020) 057.

[37] Arash Arabi Ardehali and Sameer Murthy, The 4d superconformal index near roots of unity and 3d Chern-Simons theory, J. High Energy Phys. 10 (2021) 207.

[38] Joonho Kim, Seok Kim, and Jaewon Song, A 4d $\mathcal{N}=1$ Cardy formula, J. High Energy Phys. 01 (2021) 025.

[39] Marina David, Alfredo Lezcano González, Jun Nian, and Leopoldo A. Pando Zayas, Logarithmic corrections to the entropy of rotating black holes and black strings in AdS_5, J. High Energy Phys. 04 (2022) 160.

[40] Marina David, Jun Nian, and Leopoldo A. Pando Zayas, Gravitational Cardy limit and AdS black hole entropy, J. High Energy Phys. 11 (2020) 041.

[41] Antonio Amariti and Alessia Segati, Kerr-Newman black holes from $\mathcal{N}=1^*$, J. High Energy Phys. 06 (2023) 216.

[42] Seyed Morteza Hosseini, Itamar Yaakov, and Alberto Zaffaroni, The joy of factorization at large N: Five-dimensional indices and AdS black holes, J. High Energy Phys. 02 (2022) 097.

[43] Junho Hong, The topologically twisted index of $\mathcal{N}=4$ SU(N) Super-Yang-Mills theory and a black hole Farey tail, J. High Energy Phys. 10 (2021) 145.

[44] Kevin Goldstein, Vishnu Jejjala, Yang Lei, Sam van Leuven, and Wei Li, Residues, modularity, and the Cardy limit of the 4d $\mathcal{N}=4$ superconformal index, J. High Energy Phys. 04 (2021) 216.

[45] Vishnu Jejjala, Yang Lei, Sam Van Leuven, and Wei Li, $SL(3,\mathbb{Z})$ modularity and new Cardy limits of the $\mathcal{N}=4$ superconformal index, J. High Energy Phys. 11 (2021) 047.

[46] Vishnu Jejjala, Yang Lei, Sam van Leuven, and Wei Li, Modular factorization of superconformal indices, J. High Energy Phys. 10 (2023) 105.

[47] Ohad Mamroud, The SUSY index beyond the Cardy limit, J. High Energy Phys. 01 (2024) 111.

[48] Pietro Benetti Genolini, Alejandro Cabo-Bizet, and Sameer Murthy, Supersymmetric phases of $\mathrm{AdS}_4/\mathrm{CFT}_3$, J. High Energy Phys. 06 (2023) 125.

[49] Alejandro Cabo-Bizet, Davide Cassani, Dario Martelli, and Sameer Murthy, The asymptotic growth of states of the 4d $\mathcal{N}=1$ superconformal index, J. High Energy Phys. 08 (2019) 120.

[50] Sunjin Choi, Seungkyu Kim, and Jaewon Song, Large N universality of 4d $\mathcal{N}=1$ superconformal index and AdS black holes, arXiv:2309.07614.
[51] Nikolay Bobev, Marina David, Junho Hong, Valentin Reys, and Xuao Zhang, A compendium of logarithmic corrections in AdS/CFT, arXiv:2312.08909.
[52] Alejandro Cabo-Bizet, Marina David, and Alfredo González Lezcano, Thermodynamics of black holes with probe D-branes, arXiv:2312.12533.
[53] Yiming Chen, Matthew Heydeman, Yifan Wang, and Mengyang Zhang, Probing supersymmetric black holes with surface defects, J. High Energy Phys. 10 (2023) 136.
[54] Antonio Amariti, Ivan Garozzo, and Gabriele Lo Monaco, Entropy function from toric geometry, Nucl. Phys. **B973**, 115571 (2021).
[55] Alfredo González Lezcano and Leopoldo A. Pando Zayas, Microstate counting via Bethe Ansätze in the 4d $\mathcal{N}=1$ superconformal index, J. High Energy Phys. 03 (2020) 088.
[56] Assaf Lanir, Anton Nedelin, and Orr Sela, Supersymmetric phases of 4d $\mathcal{N}=4$ SYM at large N, J. High Energy Phys. 04 (2020) 091.
[57] Alejandro Cabo-Bizet and Sameer Murthy, Supersymmetric phases of 4d $N=4$ SYM at large N, J. High Energy Phys. 09 (2020) 184.
[58] Francesco Benini and Elisa Milan, A Bethe Ansatz type formula for the superconformal index, Commun. Math. Phys. **376**, 1413 (2020).
[59] Cyril Closset, Heeyeon Kim, and Brian Willett, $\mathcal{N}=1$ supersymmetric indices and the four-dimensional A-model, J. High Energy Phys. 08 (2017) 090.
[60] Edoardo Colombo, The large-N limit of 4d superconformal indices for general BPS charges, J. High Energy Phys. 12 (2022) 013.
[61] Francesco Benini and Giovanni Rizi, Superconformal index of low-rank gauge theories via the Bethe Ansatz, J. High Energy Phys. 05 (2021) 061. arXiv:
[62] Justin Kinney, Juan Martin Maldacena, Shiraz Minwalla, and Suvrat Raju, An index for 4 dimensional super conformal theories, Commun. Math. Phys. **275**, 209 (2007).
[63] Christian Romelsberger, Counting chiral primaries in $\mathcal{N}=1$, $d=4$ superconformal field theories, Nucl. Phys. **B747**, 329 (2006).
[64] Ofer Aharony, Francesco Benini, and Ohad Mamroud (to be published).
[65] Junho Hong and James T. Liu, The topologically twisted index of $\mathcal{N}=4$ super-Yang-Mills on $\mathrm{T}^2\times S^2$ and the elliptic genus, J. High Energy Phys. 07 (2018) 018.
[66] Giovanni Felder and Alexander Varchenko, The elliptic gamma function and $\mathrm{SL}(3,\mathbb{Z})\ltimes Z_3$, Adv. Math. **156**, 44 (2000).