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Thermodynamic properties of novel dilatonic BTZ black holes under the influence of rainbow gravity

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ABSTRACT

Through exact solution of the coupled scalar and gravitational field equations, in an energy dependent spacetime, two classes of novel dilatonic BTZ black holes have been found. The black hole solutions have only one horizon and their asymptotic behaviors are non-flat and non-AdS. The black hole mass, entropy and temperature have been calculated, as the conserved and thermodynamic quantities, and it has been shown that, although these quantities get modified in the presence of rainbow functions, they satisfy the first law of black hole thermodynamics in its standard form. The black hole heat capacity and Gibbs free energy have been calculated and the local and global stabilities of the black holes have been analyzed making use of the canonical and grand canonical ensembles, respectively. Then, by considering the black hole thermal fluctuations, the quantum gravitational effects on the local and global stabilities have been studied.

1. Introduction

One of the outstanding achievements in perturbative string theory and loop quantum gravity is the prediction of a fundamental measurable length which is of the order of Planck length. Based on the existence of such observer independent fundamental length there are some interest, in almost all of the various quantum gravity approaches, to promote the usual energy momentum relation to the well-known modified dispersion relation [1–3]. The modified dispersion relation violates the Lorentz invariance. Deformed (modified) special relativity, as the Planck-scale version of the usual special relativity, has been pout forwarded initially based on the nonlinear Lorentz transformations to make the modified dispersion relation Lorentz invariant. In the deformed special relativity theory in addition to the speed of light Planck energy is an invariant quantity too. The light speed and Planck energy are the upper limit of the amount of speed and energy that a particle can attain. It is evident that the modified dispersion relation reduces to its usual form when the infrared limit is taken [4–8].

Gravity's rainbow is considered as a simple extension of the deformed special relativity to include gravity. Indeed, gravity's rainbow is a deformed general theory of relativity in which the impacts of string theory and loop quantum gravity are taken into account by considering the minimal measurable length. Thus, according to the correspondence principle, this theory is expected to recover the standard general relativity at low energy regime. In this regard, it is believed that this theory can be successful in explaining the well-known problems of the standard theory of gravity [9–11]. Now the gravity's rainbow has been the subject of many interesting works and a lot of papers have appeared in which the thermodynamic properties of the black holes have been studied at the framework of high energy physics by considering the impacts of rainbow functions [12–15].

On the other hand, Hawking et al. showed that black holes are thermodynamic systems having well-defined thermodynamic quantities such as temperature and entropy. Nowadays, study of the black hole thermodynamic properties, and especially thermodynamic stability of the black holes, have attracted an increasing interest and it is an important subject area in the context of black hole physics. In the context of canonical ensemble, one is able to analyze the local stability of the black holes by use of the black hole heat capacity, with the black hole charge as a constant, or noting the signature of the Hessian determinant. Geometric thermodynamics is the other approach for studying the black hole local stability [16–18]. Global stability of the black holes can be investigated regarding the signature of the Gibbs free energy. Thermodynamic local stability or phase transition of black holes in the three- and four-dimensional gravity's rainbow have been studied in refs. [4,7,19,20]. Local and global stabilities as well as the Hawking-Page phase transition of dilatonic black holes with power-law electrodynamics have been studied in our previous work [21]. Here,

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we tend to obtain novel exact three-dimensional dilatonic black hole solutions in rainbow gravity and to study their thermodynamic properties. Also, we investigate thermodynamic local and global stabilities or type one, type two and Hawking-Page phase transitions of the novel dilatonic black holes in gravity's rainbow. Next we examine the impacts of black hoe thermal fluctuations on the thermodynamic quantities and thermodynamic local and global stabilities.

The paper is outlined based on the following order: In Sec. 2, starting from the suitable action of the three-dimensional Einstein-dilaton gravity theory, the related field equations have been solved in a static, circularly symmetric and energy dependent spacetime. Two classes of novel dilatonic BTZ black holes have been introduced in the presence of the rainbow functions. The black holes have one horizon only and their asymptotic behavior is non-flat and non-AdS. In Sec. 3, the black hole entropy, temperature and mass have been calculated and it has been shown that, even in the presence of the rainbow functions, they satisfy the standard form of the first law of black hole thermodynamics. Section 4 is devoted to the study of the black hole local and global stabilities in the canonical and grand canonical ensembles, respectively. By calculating the black hole heat capacity and Gibbs free energy, the ranges at which the novel dilatonic black holes are locally or globally stable have been determined. Also, the points of type one, type two and Hawking-Page phase transitions have been characterized. In Sec. 5, by consideration of the thermal fluctuations, the quantum gravitational corrections on the thermodynamic properties of the black holes have been studied. It has been shown that the black hole entropy gets logarithmic correction and other thermodynamic quantities remain unchanged when the first order corrections are taken into account. It is found that black hole local and global stability conditions are affected when the logarithmic corrected entropy is utilized. The results and discussions are presented in Sec. 6.

2. The basic equations and solutions

We start with the proper action of the three-dimensional Einstein gravity theory nonminimally coupled to a scalar dilatonic field. It can be written in the following general form [22,23]

$$I=-\frac{1}{16\pi}\int\sqrt{-g}\left[\mathcal{R}-V(\phi)-2g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]d^3x. \tag{2.1}$$

Here, $\mathcal{R}$ is the Ricci scalar, ϕ is a scalar field which is assumed to coupled to itself via the dilatonic potential $V(\phi)$. By use of the variational principle the action (2.1) gives the related field equations. Varying it respects to various fields, one obtains the following field equations

$$\mathcal{R}_{\mu\nu}=V(\phi)g_{\mu\nu}+2\partial_\mu\phi\partial_\nu\phi, \tag{2.2}$$

$$4\Box\phi=\frac{dV(\phi)}{d\phi},\qquad \phi=\phi(r). \tag{2.3}$$

We consider the following ansutz, as the three-dimensional static and circularly symmetric solution to the gravitational field equations (2.2), in an energy dependent spacetime [24,25]

$$ds^2=-\frac{U(r)}{f^2(\varepsilon)}dt^2+\frac{1}{g^2(\varepsilon)}\left[\frac{1}{U(r)}dr^2+r^2R^2(r)d\theta^2\right], \tag{2.4}$$

note that $U(r)$ and $R(r)$ are two unknown functions to be determined. $U(r)$ is named as metric function and $R(r)$ denotes the impacts of dilaton field on the spacetime geometry. The structure of the line element (2.4) shows that it is a dimensionless function. The functions $f(\varepsilon)$ and $g(\varepsilon)$ are the temporal and spacial rainbow functions, respectively. There are three alternative proposed models of rainbow functions with the following explicit forms:

Model I: In this model, which is motivated by the results of loop quantum gravity and noncommutative geometry, the explicit form of the temporal and spacial rainbow functions are written the following form [5,6]

$$f(\varepsilon)=1,\qquad g(\varepsilon)=\sqrt{1-\eta\varepsilon^n}. \tag{2.5}$$

Model II: The hard spectra from gamma-ray burster's is the motivation by which the rainbow functions are constructed out Amelino-Camelia et al. [8]. That is

$$g(\varepsilon)=1,\qquad f(\varepsilon)=\frac{e^{\beta\varepsilon}-1}{\beta\varepsilon}. \tag{2.6}$$

Model III: This model is proposed based on the constancy of the light speed with the following explicit form [7]

$$f(\varepsilon)=g(\varepsilon)=\frac{1}{1-\zeta\varepsilon}. \tag{2.7}$$

It must be noted that the coefficients η, β and ζ, known as the rainbow parameters, are of the order of unity, $\varepsilon\le 1$ and the power n is a positive integer [13,15,26].

Now, making use of the line element (2.4) in the gravitational field equations (2.2), one is able to obtain the following differential equations

$$e_{tt}\equiv U''(r)+\left(\frac{1}{r}+\frac{R'(r)}{R(r)}\right)U'(r)+2\frac{V(\phi)}{g^2(\varepsilon)}=0, \tag{2.8}$$

$$e_{rr}\equiv e_{tt}+2U(r)\left(\frac{R''(r)}{R(r)}+\frac{2R'(r)}{rR(r)}+2\phi'^2(r)\right)=0, \tag{2.9}$$

$$e_{\theta\theta}\equiv\frac{U'(r)}{r}+\frac{R''(r)}{R(r)}U(r)+\frac{R'(r)}{R(r)}\left(U'(r)+\frac{2}{r}U(r)\right)+\frac{V(\phi)}{g^2(\varepsilon)}=0. \tag{2.10}$$

It can be shown that Eqs. (2.8) and (2.10) are not independent. Thus we can solve the first order differential equation (2.10) and ensure that its solution satisfies the Eq. (2.8) [21,27]. Indeed, we have four unknown functions $U(r)$, $R(r)$, $\phi(r)$ and $V(\phi)$, while there are only three independent equations [Eqs. (2.3), (2.9) and (2.10)]. In order to overcome this problem, we must to use an ansutz. Noting the structure of the line element presented in Eq. (2.4), one can argue that $R(r)$ is a dimensionless function of r, which indicates the effects of dilaton field on the spacetime geometry. As the result, there are the following alternatives:

- $R(r)$ can be considered as an exponential function in the form of $R(r)=e^{2\beta\phi(r)}$ in which the dilaton field is appeared explicitly. Similar exponential function has been used by many authors for the charged dilaton black holes [19,22,23,28–30].
- It also can be written in the form of a power-law function as $R(r)=\left(\frac{r}{r_0}\right)^\nu$. Here, the power ν is the dilaton parameter, and r_0 is a dimensional constant. A similar power-law function was initially used by Chan and Mann [31,32] and then in refs. [33–35].

In this work, we proceed with a power-law solution of the form $R(r)=\left(\frac{r}{r_0}\right)^\nu$, and set $r_0=1$ without loss of generality. Noting Eqs. (2.8) and (2.9) we obtain

$$\frac{R''(r)}{R(r)}+\frac{2}{r}\frac{R'(r)}{R(r)}+2\phi'^2(r)=0. \tag{2.11}$$

Substituting in Eq. (2.11), one can show that

$$\phi(r)=\gamma\ln\left(\frac{b}{r}\right),\quad \text{with } \gamma=\sqrt{-\nu(\nu+1)/2}. \tag{2.12}$$

It is valid for positive b and the ν-values in the range $-1<\nu<0$. A similar power-law scalar field has been used previously for studying the three- four- and five-dimensional black hole solutions [33,35,36].

Making use of these solutions in Eq. (2.10) we have

$$U'(r)+\frac{\nu}{r}U(r)+\frac{r}{\nu+1}\frac{V(\phi)}{g^2(\varepsilon)}=0. \tag{2.13}$$

For solving this equation for the metric function $f(r)$, we need to calculate the functional form of $V(\phi(r))$ as a function of the radial coordinate. To do so, we proceed to solve the scalar field equation (2.3). It can be written as

$$\frac{4\gamma}{r}\left(U'(r)+\frac{\nu}{r}U(r)\right)+\frac{1}{g^2(\varepsilon)}\frac{dV(\phi)}{d\phi}=0. \tag{2.14}$$

By combining the Eqs. (2.13) and (2.14), after some simplifications, we obtained the dilatonic potential $V(\phi)$, in terms of the cosmological constant $\Lambda=-\ell^{-2}$ as

$$V(\phi)=2\Lambda e^{2a\phi}, \quad \text{with } a=\frac{2\gamma}{\nu+1}. \tag{2.15}$$

Now, by combining Eqs. (2.15) and (2.13) the metric function $U(r)$ can be calculated as

$$U(r)=\begin{cases} -mr^{2/3}-\frac{6\Lambda b^2}{g^2(\varepsilon)}\left(\frac{b}{r}\right)^{-\frac{2}{3}}\ln\left(\frac{r}{\ell}\right), & \nu=-\frac{2}{3}, \text{ (a)},\\ -\frac{m}{r^{\nu}}-\frac{2\Lambda r^2}{(1+\nu)(2+3\nu)g^2(\varepsilon)}\left(\frac{b}{r}\right)^{-2\nu}, & \\ & \nu\neq-\frac{2}{3},\ -1<\nu<0, \text{ (b)}, \end{cases} \tag{2.16}$$

where, m is an integration constant related to the black holes total mass. The roots of the metric functions (2.16), known as the horizon radiuses, are located at

$$r_+=\begin{cases} \ell\exp\left(\frac{m\ell^2g^2(\varepsilon)}{6b^{\frac{4}{3}}}\right), & \text{for } \nu=-\frac{2}{3},\\ \left[\frac{m(1+\nu)(2+3\nu)g^2(\varepsilon)}{2(\ell b^{\nu})^{-2}}\right]^{\frac{1}{2+3\nu}}, & \text{for } \nu\neq-\frac{2}{3}. \end{cases} \tag{2.17}$$

Thus, in order to have real and positive horizon radius with positive mass the parameter ν must be restricted to the range $-2/3<\nu<0$. The position of the horizons to be exactly determined, it is better to show the plots of $U(r)$ versus r. Thus the numerical values of $f(\varepsilon)$ and $g(\varepsilon)$ are needed. Noting Eqs. (2.5)–(2.7) and the related explanations, one can argue that the numerical values of the temporal and spatial rainbow functions can be approximated as equal to or slightly different from unity. We prefer to use the numerical values similar to those of refs. [7,24]. These values are suitable approximations of almost all of the proposed functional forms of the rainbow functions. The plots of the metric functions (2.16) have been shown in Figs. 1 and 2.

Now, we check the spacetime curvature singularities. The most important curvature scalars which can produce significant information about the spacetime singularities are the Ricci and Kretschmann scalars. As a matter of calculation one is able to show that they take the following forms

$$\mathcal{R}=\begin{cases} -\frac{2}{3}\left(\frac{b}{r}\right)^{4/3}\left\{3\Lambda\left[1+\frac{2}{3}\ln\left(\frac{r}{\ell}\right)\right]+\frac{m}{3}g^2(\varepsilon)\right\}, & \text{for } \nu=-\frac{2}{3},\\ \frac{4\Lambda(3+5\nu)}{2+3\nu}\left(\frac{b}{r}\right)^{-2\nu}+\frac{\nu(\nu+1)m}{r^{2+\nu}}g^2(\varepsilon), & \text{for } \nu\neq-\frac{2}{3}, \end{cases} \tag{2.18}$$

$$\mathcal{R}^{\mu\nu\rho\lambda}\mathcal{R}_{\mu\nu\rho\lambda}=\begin{cases} 4\left(\frac{b}{r}\right)^{8/3}\left\{11\Lambda^2-\frac{2}{27}m^2g^4(\varepsilon)-\Lambda\left[\Lambda+\frac{28}{27}mg^2(\varepsilon)b^{-4/3}\right]\right. & \\ \left.\quad\times\ln\left(\frac{r}{\ell}\right)+\frac{\Lambda m}{b^{4/3}}g^2(\varepsilon)-\frac{8}{3}\Lambda^2\left[\ln\left(\frac{r}{\ell}\right)\right]^2\right\}, & \text{for } \nu=-\frac{2}{3},\\ A_1\left(\frac{b}{r}\right)^{-4\nu}+A_2\left(\frac{b}{r}\right)^{2-\nu}+\frac{2\nu^2(\nu+1)^2m^2g^4(\varepsilon)}{r^{4+2\nu}}, & \text{for } \nu\neq-\frac{2}{3}, \end{cases} \tag{2.19}$$

where,

$$A_1=\frac{8\Lambda^2(16\nu^2+17\nu+6)}{(2+3\nu)^2}, \quad \text{and}$$

$$A_2=-\frac{2\nu(\nu+1)(2-9\nu)m\Lambda g^2(\varepsilon)}{(2+3\nu)b^{2+\nu}}.$$

It is understood from Eqs. (2.18) and (2.19) that the Ricci and Kretschmann scalars are finite for finite values of r. There is an essential (not coordinate) singularity located at $r=0$, which can be recovered with an event horizon. As the result, with the help of Eq. (2.17) or Figs. 1 and 2, our exact solutions can be interpreted as black holes with one event horizon.

3. Thermodynamic properties

In this section, we seek for satisfaction of the first law of black hole thermodynamics for both of the new three-dimensional dilatonic black hole solutions in the presence of the rainbow functions. For this purpose we proceed to calculate the conserved and thermodynamic quantities related to the new black hole solutions.

The entropy of the black holes, as an important thermodynamic quantity, can be calculated by use of the Hawking-Bekenstein entropy-area law. According to the entropy-area law, as a geometrical method, the entropy of the black hole is equal to one-quarter of the surface are of the black hole horizon. It leads to the following relation for the entropy of the black holes

$$S=\frac{\pi r_+^{\nu+1}}{2g(\varepsilon)}, \tag{3.1}$$

where, r_+ is the black hole horizon radius which is the real root of the relation $U(r_+)=0$. Note that Eq. (3.1) reduces to its standard form for the three-dimensional Einstein black holes in the absence of dilatonic potential when its infrared limit is taken.

The other conserved quantity to be calculated is the black hole mass, M. It can be calculated in terms of the mass parameter, m. The total mass of the three-dimensional dilatonic black holes introduced here can be obtained as [31,32] (see also [33,34])

$$M=\frac{\nu+1}{8f(\varepsilon)}m. \tag{3.2}$$

The black hole mass given in Eq. (3.2) reduces to that of BTZ black holes in the absence of dilaton potential if one set $f(\varepsilon)=1$.

Next, the Hawking temperature associated to the black hole horizon, $r=r_+$, can be calculated by use of the concept of surface gravity κ, as

$$T=\frac{\kappa}{2\pi}=\frac{g(\varepsilon)}{4\pi f(\varepsilon)}\frac{d}{dr}U(r)|_{r=r_+}$$
$$=\begin{cases} -\frac{3\Lambda b}{2\pi f(\varepsilon)g(\varepsilon)}\left(\frac{b}{r_+}\right)^{\frac{1}{3}}, & \text{for } \nu=-\frac{2}{3} \text{ (a)},\\ -\frac{\Lambda r_+}{2\pi(\nu+1)f(\varepsilon)g(\varepsilon)}\left(\frac{b}{r_+}\right)^{-2\nu}, & \text{for } \nu\neq-\frac{2}{3} \text{ (b)}. \end{cases} \tag{3.3}$$

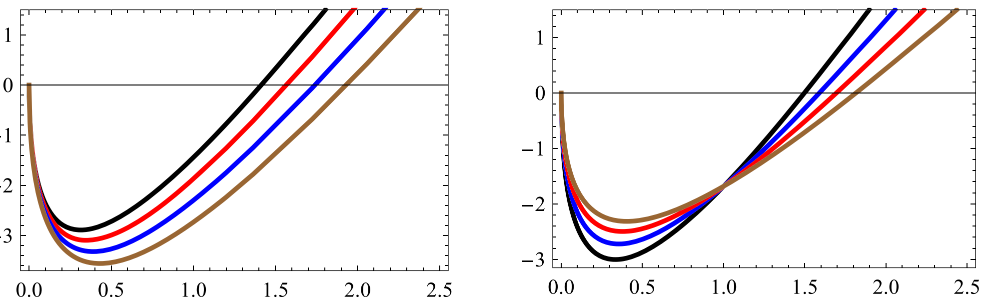

Fig. 1. $U(r)$ versus r for $\Lambda=-1$, $M=1$, $b=2.5$, and $\nu=-\frac{2}{3}$, Eq. (2.16-a). Left: $g(\varepsilon)=0.7$ and $f(\varepsilon)=0.6, 0.78, 0.96, 1.14$ for black, blue, red and brown curves, respectively. Right: $f(\varepsilon)=0.7$ and $g(\varepsilon)=0.7, 0.75, 0.8, 0.85$ for black, blue, red and brown curves, respectively.

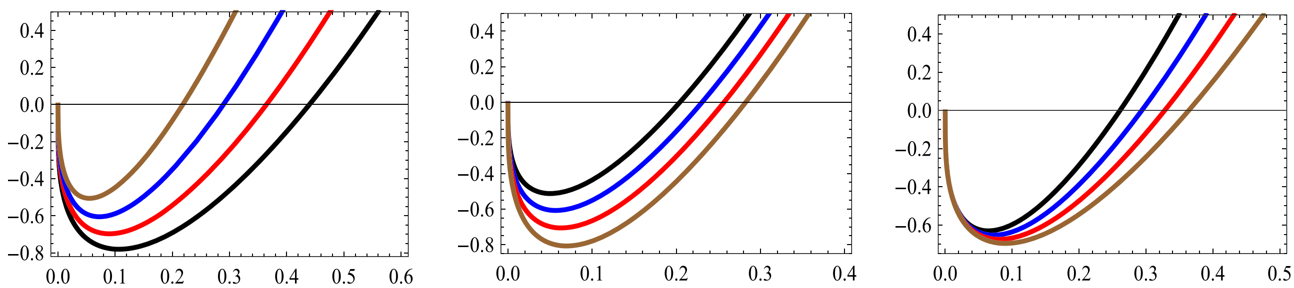

Fig. 2. $U(r)$ versus r for $b=2.5$, $M=0.2$, $\Lambda=-1$ and $\nu\neq-\frac{2}{3}$, Eq. (2.16-b). Left: $f(\varepsilon)=0.8$, $g(\varepsilon)=0.9$ and $-\nu=0.2, 0.3, 0.33, 0.36$ from down to top, respectively. Middle: $\nu=-0.3$, $g(\varepsilon)=0.7$ and $f(\varepsilon)=0.7, 0.8, 0.9, 1$ from top to down, respectively. Right: $\nu=-0.3$, $f(\varepsilon)=0.8$ and $g(\varepsilon)=0.75, 0.8, 0.85, 0.9$ from top to down, respectively.

Note that we have used the condition $U(r_+)=0$ for eliminating the mass parameter m from the above given relations. Since, the cosmological constant Λ is negative, from Eq. (3.3) one can argue that the black hole temperature is positive valued for both of the new dilatonic black hole solutions with the dilaton parameter ν in its acceptable range. The black holes with positive temperature are physically reasonable. The other point to be mentioned is that the extreme black holes (i.e. the black holes having zero temperature) can not occur.

In order to calculate the thermodynamic and conserved quantities, related to our new black holes from the thermodynamical methods, we need to have a Smarr-type mass formula. It can be derived from Eq. (2.16) by imposing the condition $U(r_+)=0$ and substituting the mass parameter m within Eq. (3.2). The Smarr mass formula which gives the black hole mass as the function of thermodynamical extensive parameter S is obtained as

$$M(S)=\begin{cases}-\frac{\Lambda[r_+(S)]^{\frac{4}{3}}}{4f(\varepsilon)g^2(\varepsilon)}\ln\left(\frac{r_+(S)}{l}\right), & \text{for } \nu=-\frac{2}{3},\\ -\frac{\Lambda[r_+(S)]^{2+\nu}}{4(2+3\nu)f(\varepsilon)g^2(\varepsilon)}\left(\frac{b}{r_+(S)}\right)^{-2\nu}, & \text{for } \nu\neq-\frac{2}{3}.\end{cases} \tag{3.4}$$

From the thermodynamical point of wive, the black hole mass can be interpreted as the internal energy that has to be positive for a thermodynamical system. Noting the fact that $\Lambda=-\ell^{-2}<0$ the black hole mass is positive valued for the case $\nu=-\frac{2}{3}$. Also, the mass of the black holes with $\nu\neq-\frac{2}{3}$ is positive if the dilaton parameter ν be chosen in the range $-\frac{2}{3}<\nu<0$.

Making use of Eq. (3.4) and treating the black hole entropy as the thermodynamical extensive parameter, one can obtain the parameter T as the intensive parameter conjugate to entropy. It can lead to confirmation of the first law of black hole thermodynamics. For this purpose it is easily shown that

$$\frac{dM}{dS}=\left(\frac{\partial M}{\partial r_+}\right)\left(\frac{\partial S}{\partial r_+}\right)^{-1}=T,\quad \text{for both } \nu=-\frac{2}{3} \text{ and } \nu\neq-\frac{2}{3}. \tag{3.5}$$

Therefore, even if the thermodynamic quantities are affected by the rainbow functions, the first law of black hole thermodynamics still remains valid for both classes of the new three-dimensional dilatonic black holes in the following form

$$dM=TdS. \tag{3.6}$$

4. Thermal stability analysis

In this section, we investigate thermal stability of the new black hole solutions obtained in section 2. The local and global stabilities of the black holes can be analyzed by use of the canonical and grand canonical ensembles, respectively. In the canonical ensemble one is able to perform a local stability analysis regarding the black hole heat capacity [37,38]. Also, global stability of the black holes can be studied noting the Gibbs free energy of the black holes [39]. Now, we proceed to explore the local and global stabilities of our new dilatonic black holes by use of the canonical and grand canonical ensembles, separately.

4.1. Black hole local stability

In order to study the local stability or thermodynamic phase transition of the black holes, we calculate the black hole heat capacity for both of the new black hole solutions, found here. It can be calculated by considering the following relation

$$\mathcal{H}=T\left(\frac{\partial S}{\partial T}\right). \tag{4.1}$$

It is well-known that the positivity of heat capacity $\mathcal{H}$, for the physical black holes, guaranties the local stability. The unstable black holes experience thermodynamic phase transitions to be stabilized. The real root(s) of the black hole heat capacity are the locations of the type one phase transition. Also, the divergent points of black hole heat capacity indicate the type two phase transition points. Equivalently, the real root(s) of the denominator of the black hole heat capacity are the points of type two phase transition [37,38] (see also [40–42]). Keeping these facts in mind, we are able to analyze the thermal stability or phase transition of both of our new black hole solutions.

Making use of Eqs. (3.1) and (3.3) in Eq. (4.1), one is able to show that

$$\mathcal{H}=\begin{cases} -\frac{\pi}{2g(\varepsilon)}r_+^{1/3}, & \text{for } \nu=-\frac{2}{3},\\ \frac{\pi(1+\nu)}{2(1+2\nu)g(\varepsilon)}r_+^{\nu+1}, & \text{for } \nu\neq-\frac{2}{3},\end{cases} \tag{4.2}$$

from which we can say that the heat capacity of the black holes with $\nu=-\frac{2}{3}$ is negative everywhere. Thus, no type one or type two phase transition can take place because it neither vanishes nor diverges. They are locally unstable. The heat capacity of the black holes correspond to $\nu\neq-\frac{2}{3}$ is positive for $\nu>-\frac{1}{2}$. As the result they are locally stable with the dilaton parameter ν in the range $-\frac{1}{2}<\nu<0$. No type one phase transition can take place. If we set $\nu=-\frac{1}{2}$ the heat capacity of this kind of black holes diverges and they undergo type two phase transition. There is no point of type one phase transition. It is notable that in the absence of dilaton field (i.e. $\nu=0$) the black hole heat capacity is positive valued and the BTZ black holes are locally stable even in the presence of rainbow functions.

4.2. Black hole global stability

In order to investigate the global stability of the novel dilatonic BTZ black holes we need to calculate the Gibbs free energy. Regarding the signature and vanishing points of the Gibbs free energy the global stability or Hawking-Page phase transition of the black holes can be studied. In the case under consideration, it can be defined via the following relation [43,44]

$$G=M-ST. \tag{4.3}$$

As a matter of calculation, one is able to show that

$$G=\begin{cases} \frac{\Lambda r_+^{\frac{4}{3}}}{4f(\varepsilon)g^2(\varepsilon)}\left[3\left(\frac{b}{r_+}\right)^{\frac{4}{3}}-\ln\left(\frac{r_+}{\ell}\right)\right], & \text{for } \nu=-\frac{2}{3},\\ \frac{\Lambda(2\nu+1)r_+^{2+\nu}}{4f(\varepsilon)g^2(\varepsilon)(\nu+1)(3\nu+2)}\left(\frac{r_+}{b}\right)^{2\nu}, & \text{for } \nu\neq-\frac{2}{3}.\end{cases} \tag{4.4}$$

With the purpose of analyzing the black hole global stability, it must be noted that the black hole Gibbs free energy has a real root located at

$$r_+\equiv r_1=2\sqrt{2}b\,[Lw(\eta)]^{-\frac{3}{4}},\quad \text{for } \nu=-\frac{2}{3}, \tag{4.5}$$

where $Lw(\eta)$ is the Lambert function and $\eta=4\left(\frac{b}{\ell}\right)^{4/3}$. For more details on the Lambert function see [45] and references therein. It is positive for $r_+>r_1$ and negative for $r_+<r_1$. Therefore, the black holes with horizon radius equal to r_1 experience Hawking-Page phase transition. Those have horizon radii in the range $r_+>r_1$ are globally stable. Otherwise they prefer to be in the radiation phase.

In the case $\nu\neq-\frac{2}{3}$ it is easy to show that the Gibbs free energy vanishes at $\nu=-\frac{1}{2}$, the point at which Hawking-Page phase transition occurs. It is positive in the range $-\frac{2}{3}<\nu<-\frac{1}{2}$, thus the black holes with the ν-values in this range are globally stable. Also, the black holes with the ν-values in ranges $-1<\nu<-\frac{2}{3}$ and $-\frac{1}{2}<\nu<0$ have negative Gibbs free energy and they prefer the radiation phase.

5. Thermal fluctuations and corrected thermodynamics

Up to now, we have studied thermodynamic properties of our novel dilatonic black holes by ignoring the black hole thermal fluctuations. Now, we come back to the study of thermodynamics and thermal stability of the black holes in the presence of quantum gravitational effects by considering the black hole thermal fluctuations. It is well-known that, when the first order corrections are studied through consideration of the black hole thermal fluctuations, the black hole temperature remains unchanged while the black hole entropy gets logarithmic correction [46,47]. Thus the first order corrected entropy is written as [48,49]

$$S^{(C)}=S-\frac{\xi}{2}\ln\left(ST^2\right). \tag{5.1}$$

Note that S and T are the uncorrected black hole entropy and temperature presented in Eqs. (3.1) and (3.3), respectively. The coefficient ξ is the thermal fluctuations or correction parameter with the dimension of Length. $S^{(C)}$ denotes the first order corrected entropy with the thermal fluctuations are taken into account.

The Helmholtz free energy is an important thermodynamic quantity to be calculated. With the black hole temperature and corrected entropy in hand, one is able to calculate the Helmholtz free energy making use of the following relation [46]

$$\begin{aligned} F^{(C)}&=-\int S^{(C)}dT\\ &=\begin{cases} \frac{9\Upsilon b^2}{\pi}\left(\frac{b}{r_+}\right)^{\frac{2}{3}}\left\{\frac{\xi}{2}\ln\left[\frac{\Upsilon b^{1/3}}{g(\varepsilon)}\left(\frac{b}{r_+}\right)^{\frac{7}{3}}\right]-\frac{7\xi}{4}-\frac{\pi r_+^{1/3}}{g(\varepsilon)}\right\}, & \\ \qquad\qquad \text{for } \nu=-\frac{2}{3}, & \text{(a)}\\ \frac{\Upsilon r_+^2}{\pi(1+\nu)^2}\left(\frac{r_+}{b}\right)^{4\nu}\left\{\frac{\xi}{2}\ln\left[\frac{\Upsilon(1+2\nu)^2 r_+^{1+\nu}}{(1+\nu)^2 g(\varepsilon)}\left(\frac{r_+}{b}\right)^{4\nu}\right]\right. & \\ \qquad \left.-\frac{\xi(1+5\nu)}{4(1+2\nu)}-\frac{\pi(1+2\nu)r_+^{\nu+1}}{g(\varepsilon)(3+5\nu)}\right\}, \quad \text{for } \nu\neq-\frac{2}{3}, & \text{(b)}\end{cases}\end{aligned} \tag{5.2}$$

where,

$$\Upsilon=\frac{\Lambda^2}{8\pi f^2(\varepsilon)g^2(\varepsilon)}. \tag{5.3}$$

In terms of the Helmholtz free energy, the black hole mass $M^{(C)}=F^{(C)}+TS^{(C)}$, is obtained as

$$M^{(C)}=F^{(C)}+ST-\frac{\xi T}{2}\ln\left(ST^2\right), \tag{5.4}$$

where, $F^{(C)}$ is given by Eq. (5.2), S and T are the uncorrected entropy and temperature, respectively.

Starting from Eq. (5.4), after some algebraic calculations, one can show that the first law of black hole thermodynamics is valid in the following form

$$dM^{(C)}=T^{(C)}dS^{(C)},\quad \text{with } T^{(C)}=T, \tag{5.5}$$

even when the black hole thermal fluctuations are taken into account.

In order to perform a black hole stability analysis in the presence of the thermal fluctuations, we need to calculate the corrected black hole heat capacity. Starting from the relation $\mathcal{H}^{(C)}=T\frac{\partial S^{(C)}}{\partial T}$, one can calculate the quantum corrected black hole heat capacity. It is a matter of calculation to show that

$$\mathcal{H}^{(C)}=\begin{cases} -\frac{1}{2}\left(\frac{\pi}{g(\varepsilon)}r_+^{1/3}+\frac{\xi}{2}\right), & \text{for } \nu=-\frac{2}{3},\\ \frac{1}{2(1+2\nu)}\left[\frac{\pi(1+\nu)}{g(\varepsilon)}r_+^{\nu+1}-\frac{\xi}{2}(3+5\nu)\right], & \text{for } \nu\neq-\frac{2}{3}.\end{cases} \tag{5.6}$$

Eq. (5.6) indicates the corrected heat capacity of dilatonic BTZ black holes in rainbow gravity with consideration of the thermal fluctuations. The black hole heat capacity is negative everywhere and the black holes corresponding to the case of $\nu = -\frac{2}{3}$ are locally unstable even in the presence of the thermal fluctuations. For the black holes with $\nu \neq -\frac{2}{3}$ the corrected black hole heat capacity vanishes at

$$r_+ \equiv R_1 = \left[\frac{\xi(3+5\nu)}{2\pi(1+\nu)}\right]^{\frac{1}{1+\nu}}, \quad \text{for } -\frac{3}{5} < \nu < 0. \tag{5.7}$$

Eq. (5.7) indicates the horizon radius of the black holes that undergo type one phase transition. In the range $-1 < \nu < -\frac{3}{5}$ both of the terms in the brackets are positive and $\mathcal{H}^{(C)}$ becomes negative due to the presence of coefficient $1+2\nu$. Thus for the ν in the range $-1 < \nu < -\frac{3}{5}$ the black holes are locally unstable. The black holes with horizon radius in the range $r_+ < R_1$ are locally stable.

Now, we explore the global stability in the presence of black hole thermal fluctuations. The corrected black hole Gibbs free energy can be determined by use of the relation $G^{(C)} = M^{(C)} - S^{(C)}T$. It is easy to show that it is just equal to the Helmholtz free energy presented in Eq. (5.2). We need to have the real root(s) of $G^{(C)} = 0$, but it is too difficult to be solved analytically. Numerical calculations show that there is only one real root we label by $r_+ = r_2$, $G^{(C)} > 0$ for $r_+ < r_2$ and $G^{(C)} < 0$ for $r_+ > r_2$. As the result, the black holes with $r_+ = r_2$ undergo Hawking-Page phase transition. Those with $r_+ < r_2$ are globally stable and black holes with the horizon radius in the range $r_+ > r_2$ are in the phase of radiation.

6. Conclusion

We studied the exact solutions of the Einstein-dilaton gravity theory in a three-dimensional circularly symmetric and energy dependent geometry. By solving the coupled equations of the gravitational and scalar fields, we found that the scalar potential takes the form of Liouville potential and reduces to the cosmological constant when the dilaton field disappears. Also, two classes of novel one horizon dilatonic black holes have been introduced with the non-flat and non-AdS asymptotic behavior. Through consideration of curvature scalars we argued that based on existence of the event horizon and appearance of the curvature singularities our new exact solutions are really black holes. The solutions reduce to the metric function of the well-known BTZ black holes in the absence of dilaton field.

Next, in order to investigate the thermodynamic properties of the new black hole solutions, we just obtained, we calculated the black hole temperature, entropy and total mass and showed that, although these quantities get modified in the presence of rainbow functions, they satisfy the standard form of the first law of black hole thermodynamics.

Then, we analyzed thermal stability or thermodynamic phase transition of the black holes by use of the canonical and grand canonical ensembles. By calculating the black hole heat capacity we determined the points of type one and type two phase transitions. Also, we indicated that the black holes corresponding to $\nu = -\frac{2}{3}$ are locally unstable while the black holes corresponding to $\nu \neq -\frac{2}{3}$ are stable with the dilaton parameter in the range $-\frac{1}{2} < \nu < 0$. In addition, we obtained the Gibbs free energy of the black holes and by analyzing its signature we obtained the points of Hawking-Page phase transition and characterized the ranges where the black holes are globally stable. For the black holes with $\nu = -\frac{2}{3}$ there is a point of Hawking-Page phase transition r_1, indicated by Eq. (4.5). This class of black holes are globally stable provided that their horizon radii are greater than r_1. Also, the second class of dilatonic black holes experience Hawking-Page phase transition if $\nu = -\frac{1}{2}$ is chosen. They are globally stable in the range $-\frac{2}{3} < \nu < -\frac{1}{2}$.

Finally, with the aim of studying the quantum gravitational impacts on the thermodynamic behavior of the black holes, we considered black hole thermal fluctuations. In the presence of thermal fluctuations, when the first order corrections are taken into account, the black hole entropy gets logarithmic correction and the other thermodynamic quantities remain unchanged. By use of the logarithmic corrected black hole entropy we found that although the first law of black hole thermodynamics is still valid but the local and global stabilities as well as the points of type one, type two and Hawking-Page phase transitions get some modifications.

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