Chromonatural warm inflation

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Chromonatural inflation is a model where non-Abelian gauge fields are sustained by the coupling of the axion with the gauge field through the Chern-Simons term, while minimal warm inflation is a model where the axion produces a thermal bath of non-Abelian gauge particles through the Chern-Simons term. Since both axion inflation models are based on the same action, a natural question is if they are compatible or not. We study axion inflation with the Chern-Simons term and find that chromonatural inflation can accommodate radiation with a temperature much larger than the Hubble parameter during inflation, which is a characteristic feature of warm inflation. Thus, we conclude that chromonatural warm inflation exists, which must have phenomenologically interesting consequences.

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I. INTRODUCTION

The inflationary scenario can be realized by a single scalar field, the so-called inflaton. In the slow-roll limit, the inflaton acts as a cosmological constant. The cosmic no-hair theorem [1] tells us that the exponential expansion of the Universe driven by a cosmological constant erases all of the initial information of the Universe. More precisely, in the presence of the cosmological constant and matter satisfying the energy condition, the following occurs:

(1) The energy density of ordinary matter vanishes.
(2) The anisotropy of spacetime vanishes.
(3) The spatial curvature (except for the Bianchi IX spacetime, with a very large curvature scale compared to the Hubble scale) goes to zero.

Consequently, the Universe becomes a vacuum with homogeneous and isotropic flat spacetime where only quantum vacuum fluctuations remain. This is the reason why inflation predicts the scale-invariant spectrum with a statistically isotropic Gaussian distribution [2]. The belief that any inflation model leads to this prediction is named the cosmic no-hair conjecture [3–5].

In the presence of additional fields during inflation, however, conventional inflation could be unstable and a novel inflationary phase with cosmic hairs could appear [6]. In fact, there exist two clear counterexamples to the cosmic no-hair conjecture: namely, warm inflation [7], which violates (1); and anisotropic inflation [8] assisted by gauge fields, which violates (1) and (2). Generically, those novel inflation models give rise to qualitatively new predictions, such as the statistically anisotropic non-Gaussianity [9].

From the particle physics point of view, it is important to explore qualitatively novel inflation models with gauge fields [10–12]. As to the case of Abelian gauge fields, it has been shown that anisotropic inflation and warm inflation yield anisotropic warm inflation [13]. In the case of non-Abelian gauge fields, chromonatural inflation [14–16] and minimal warm inflation [17,18] are the counterexamples to the cosmic no-hair conjecture. In chromonatural inflation, the background gauge field exists and interacts with the axion through the Chern-Simons term to achieve slow-roll. In minimal warm inflation, on the other hand, the axion interacting with the gauge field through the Chern-Simons term produces a thermal bath, which induces a friction term in the axion dynamics realizing slow-roll inflation. Since both inflation models utilize a similar setup, it is intriguing to study if chromonatural inflation and warm inflation can coexist or not.

Curiously, although both models have the same action including the Chern-Simons interaction term, their predictions are apparently different. In fact, chromonatural inflation predicts the excess of primordial gravitational waves due to the instability of gauge field perturbations [12,19–22]. Meanwhile, in minimal warm inflation, it is a challenge to have detectable primordial gravitational waves [17,23]. It is natural to ask which happens in reality. Logically, both inflation models may occur simultaneously: namely, chromonatural warm inflation may exist. In that case, it is interesting to reveal the prediction of such a model on primordial gravitational waves. Thus, it is worth investigating the phase-space structure of axion inflation with the non-Abelian Chern-Simons term (see [24] for an earlier work on this issue).

The purpose of this paper is to clarify whether chromonatural warm inflation occurs or not. We study the effect of the dissipation of inflaton energy on chromonatural inflation by solving the slow-roll equation analytically and numerically. When the effect of dissipation is negligible, chromonatural inflation occurs. If we increase the dissipation rate, chromonatural inflation becomes warm. Thus, it turns out that chromonatural inflation could accommodate a thermal bath with a temperature T much larger than the Hubble parameter H, which is a characteristic feature of warm inflation. This can be nothing other than chromonatural warm inflation. If we further increase the dissipation rate, chromonatural inflation disappears, and only warm inflation occurs. Eventually, for very large dissipation, only cold inflation remains.

The paper is organized as follows: In Sec. II, we review chromonatural inflation. In Sec. III, we review minimal warm inflation. In Sec. IV, we present the slow-roll equations of natural inflation with the Chern-Simons term and clarify the slow-roll conditions. First, we analytically investigate the slow-roll equations and verify the existence of chromonatural warm inflation. Next, we study the slow-roll equations semianalytically and reveal the phase-space structure of axion inflation. As a consequence, we prove the compatibility of chromonatural inflation and warm inflation. The final section is devoted to our conclusion.

II. A REVIEW OF CHROMONATURAL INFLATION

In this section, we review chromonatural inflation, where the axion is coupled with non-Abelian gauge fields through the Chern-Simons term [14,15]. The interaction between the gauge field and the axion leads to an extra friction, which realizes the slow-roll inflation even for the steep potential. Hereafter, we shall work in natural units $(c = \hbar = M_{\rm pl} = 1)$.

The chromonatural inflation is described by the action

$$S = \int d^4x\sqrt{-G}\left[\frac{1}{2}R - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V(\chi) - \frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} - \frac{g^2\lambda}{8f}\chi\epsilon^{\mu\nu\lambda\rho}F^a_{\mu\nu}F^a_{\lambda\rho}\right], \tag{2.1}$$

where G is the determinant of the metric $G_{\mu\nu}$, R is the scalar curvature, χ is an axion field, the field strength of the non-Abelian gauge field is $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu$, and g is a coupling constant of the non-Abelian gauge field. Greek letters denote spacetime indices, while roman letters denote gauge indices. The last term is the Chern-Simons term with a coupling constant λ. We took the convention $\sqrt{-G}\epsilon^{0123} = 1$. The axion potential $V(\chi)$ reads

$$V(\chi) = \mu^4\left(1 + \cos\frac{\chi}{f}\right), \tag{2.2}$$

where the energy scale is given by μ, and f represents the decay constant.

Let us consider the cosmological background solution with the metric

$$ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j, \tag{2.3}$$

where $a(t) = e^{\alpha(t)}$ is a scale factor and $H = \dot\alpha$ is the Hubble parameter. The gauge field configuration is taken to be homogeneous and isotropic: $A^a_i = \psi(t)a(t)\delta^a_i$ and $A^a_0 = 0$. Then, the Einstein equations are given by

$$\dot\alpha^2 = \frac{1}{3}\left[\frac{1}{2}\dot\chi^2 + \mu^4\left(1 + \cos\frac{\chi}{f}\right) + \frac{3}{2}\dot\psi^2 + 3\dot\alpha\dot\psi\psi + \frac{3}{2}\dot\alpha^2\psi^2 + \frac{3}{2}g^2\psi^4\right], \tag{2.4}$$

$$\ddot\alpha = -\frac{1}{2}\dot\chi^2 - \dot\psi^2 - 2\dot\alpha\dot\psi\psi - \dot\alpha^2\psi^2 - g^2\psi^4. \tag{2.5}$$

The field equations for the axion and the gauge field can be deduced as

$$\ddot\chi + 3\dot\alpha\dot\chi - \frac{\mu^4}{f}\sin\frac{\chi}{f} = -3\frac{g^3\lambda}{f}\psi^2(\dot\psi + \dot\alpha\psi), \tag{2.6}$$

$$\ddot\psi + 3\dot\alpha\dot\psi + \left[-\frac{1}{6}\dot\chi^2 + \frac{2}{3}\mu^4\left(1 + \cos\frac{\chi}{f}\right)\right]\psi + 2g^2\psi^3 - \frac{g^3\lambda}{f}\psi^2\dot\chi = 0. \tag{2.7}$$

In the slow-roll approximation, where we impose the condition $|\ddot\chi| \ll \dot\alpha|\dot\chi|$, $|\ddot\psi| \ll \dot\alpha|\dot\psi|$ and the inflaton potential is dominant in Eq. (2.4), we need to solve the equations

$$\dot\alpha^2 = \frac{\mu^4}{3}\left(1 + \cos\frac{\chi}{f}\right), \tag{2.8}$$

$$3\dot\alpha\dot\chi - \frac{\mu^4}{f}\sin\frac{\chi}{f} = -3\frac{g^3\lambda}{f}\psi^2(\dot\psi + \dot\alpha\psi), \tag{2.9}$$

$$3\dot\alpha\dot\psi + \frac{2}{3}\mu^4\left(1 + \cos\frac{\chi}{f}\right)\psi + 2g^2\psi^3 - \frac{g^3\lambda}{f}\psi^2\dot\chi = 0. \tag{2.10}$$

For $\lambda \neq 0$, the consistency conditions for the slow-roll approximation read

$$\epsilon_{\rm v} \equiv \frac{1}{2(1 + \frac{\sigma^2}{3})}\left(\frac{V'(\chi)}{V(\chi)}\right)^2 \ll 1, \tag{2.11}$$

$$\eta_{\rm v} \equiv \frac{1}{\left(1+\frac{\sigma^2}{3}\right)} \frac{V''(\chi)}{V(\chi)} \ll 1, \tag{2.12}$$

and

$$\delta \equiv \frac{1}{\left(1+\frac{\sigma^2}{3}\right)} \frac{\sigma^2}{\gamma} \ll 1, \tag{2.13}$$

where we define $\sigma \equiv \frac{g^3\lambda\psi^2}{f\dot{\alpha}}$ and $\gamma \equiv \frac{g^4\lambda^2\psi^2}{f^2}$. Notice that the slow-roll parameters have the extra factor $1+\sigma^2/3$. For $\lambda = 0$, we simply obtain the conventional slow-roll conditions of natural inflation [25].

$$\epsilon_{\rm v} = \frac{1}{2}\left(\frac{V'(\chi)}{V(\chi)}\right)^2 \sim \frac{1}{2f^2}\tan^2\frac{\chi}{2f} \ll 1, \tag{2.14}$$

$$\eta_{\rm v} = \frac{V''(\chi)}{V(\chi)} \sim \frac{1}{2f^2}\left(1-\tan^2\frac{\chi}{2f}\right) \ll 1. \tag{2.15}$$

We see that inflation can occur only for $f \gg 1$ in Planck units, which is unnatural from a particle physics perspective. For $\lambda \neq 0$, however, f can take a natural value, if $\sigma \gg 1$ and $\gamma \gg 1$. The conditions $\sigma \gg 1$ and $\gamma \gg 1$ require $\frac{g^2\lambda}{f} \gg 1$, because we need $g^2\psi^4 \ll \mu^4 \sim \dot{\alpha}^2$ and $\psi \ll 1$ for inflaton potential to be dominant in (2.4).

We rewrite Eqs. (2.9) and (2.10) using σ and γ as

$$3\dot{\alpha}\dot{\chi} = \frac{\mu^4}{f}\sin\frac{\chi}{f} - 3\sigma\dot{\alpha}(\dot{\psi}+\dot{\alpha}\psi), \tag{2.16}$$

$$3\dot{\alpha}\dot{\psi} = -2\left(1+\frac{\sigma^2}{\gamma}\right)\dot{\alpha}^2\psi + \sigma\dot{\alpha}\dot{\chi}. \tag{2.17}$$

From Eqs. (2.16) and (2.17), we can deduce the equations for $\dot{\psi}$ and $\dot{\chi}$. By imposing the slow-roll conditions, $\sigma \gg 1$ and $\gamma \gg 1$, we obtain the equation for the gauge field:

$$\dot{\alpha}\dot{\psi} \simeq \frac{\frac{\mu^4}{f}\sin\frac{\chi}{f}}{3\sigma} - \dot{\alpha}^2\psi. \tag{2.18}$$

Defining an effective potential for the gauge field,

$$V_{\rm eff}(\psi) = \dot{\alpha}^2\frac{\psi^2}{2} + \frac{\mu^4\sin\frac{\chi}{f}}{3g^3\lambda}\frac{\dot{\alpha}}{\psi}, \tag{2.19}$$

we can rewrite Eq. (2.18) as

$$\dot{\alpha}\dot{\psi} \simeq -V'_{\rm eff}. \tag{2.20}$$

Thus, we see the minimum of the effective potential

$$\psi_{\rm min} \simeq \left(\frac{\mu^4\sin\frac{\chi}{f}}{3g^3\lambda\dot{\alpha}}\right)^{1/3} \tag{2.21}$$

is an attractor. Indeed, from $V_{\rm eff}$, the effective mass of the gauge field ψ at $\psi_{\rm min}$ can be calculated as $m_\psi^2 \simeq 3\dot{\alpha}^2$, which is positive and large. Hence, the minimum is an attractor. Now, the slow-roll conditions can be expressed as

$$\sigma \simeq \frac{g^3\lambda\psi_{\rm min}^2}{f\dot{\alpha}} = \left(\frac{g^3\lambda\mu^8\sin^2\frac{\chi}{f}}{3^2f^3\dot{\alpha}^5}\right)^{\frac{1}{3}} \gg 1, \tag{2.22}$$

$$\gamma \simeq \frac{g^4\lambda^2\psi_{\rm min}^2}{f^2} = \left(\frac{g^6\lambda^4\mu^8\sin^2\frac{\chi}{f}}{3^2f^6\dot{\alpha}^2}\right)^{\frac{1}{3}} \gg 1. \tag{2.23}$$

Now, the axion equation reads from Eq. (2.17) by taking $\dot{\psi} \simeq 0$:

$$\dot{\alpha}\dot{\chi} \simeq \frac{2}{\sigma}\left(1+\frac{\sigma^2}{\gamma}\right)\dot{\alpha}^2\psi. \tag{2.24}$$

Using Eqs. (2.8), (2.21), and (2.24), we can calculate the number of e-folding as

$$\begin{aligned} N &= \int\frac{\dot{\alpha}}{\dot{\chi}}d\chi \simeq \int\frac{\sigma}{2\left(1+\frac{\sigma^2}{\gamma}\right)\psi}d\chi \\ &= g^2\lambda\int_{\frac{\chi_i}{f}}^{\pi}\frac{\beta(1+\cos X)^{\frac{2}{3}}\sin^{\frac{1}{3}}X}{3^{-\frac{1}{3}}\beta^2(1+\cos X)^{\frac{4}{3}}+\sin X^{\frac{2}{3}}}dX, \end{aligned} \tag{2.25}$$

where we define $X = \chi/f$ and $\beta = \lambda^{1/3}\mu^{4/3}$. For the current parameters, $\sigma \gg 1$ and $\gamma \gg 1$ (for example, $g \sim 10^{-6}$, $g^2\lambda \sim 10^2$, and $\mu \sim 10^{-4}$), we have $\beta \sim \mathcal{O}(1)$. If we take $X = 10^{-2}$ as an initial value, the number of e-folding is given by $N \sim g^2\lambda\mathcal{O}(1)$. Therefore, $g^2\lambda \sim \mathcal{O}(100)$ is enough for sufficient inflation to occur.

III. A REVIEW OF MINIMAL WARM INFLATION

Warm inflation is an attractive model, because it might provide a mechanism for slow-roll inflation without a shallow potential and the thermalization of the Universe without reheating [7,26–28]. The point is that the decay rate of the inflaton satisfies $\Gamma_\chi \gtrsim H$ during inflation. The inflaton produces other light fields and generates a thermal bath. However, since the thermal bath gives a temperature typically much larger than the Hubble parameter, thermal backreaction to the inflaton potential may destroy the warm inflation scenario [29]. To resolve the thermal backreaction issue, minimal warm inflation utilizing the shift symmetry of an axion has been proposed [17]. Although there are other models protecting quantum and thermal corrections to the inflaton potential [30–33], we focus on the minimal warm inflation model in this paper.

The Lagrangian of minimal warm inflation driven by the axion field χ is given by

$$S = \int d^4x\sqrt{-G}\left[\frac{1}{2}R - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V(\chi) - \frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} - \frac{g^2\lambda}{8f}\chi\epsilon^{\mu\nu\lambda\rho}F^a_{\mu\nu}F^a_{\lambda\rho}\right]. \tag{3.1}$$

The action is exactly the same as that of chromonatural inflation. In the scenario of warm inflation, it is crucial to take into account the dissipation rate of the inflaton Γ_χ, calculated from the sphaleron transition rate [34] as

$$\Gamma_\chi(T) = \kappa g^{10}T^3\frac{\lambda^2}{f^2}, \tag{3.2}$$

where the parameter κ depends on g, the number of colors N_c and flavors N_f of the gauge group [35]. The natural value is $\kappa \sim \mathcal{O}(100)$. It is convenient to define the dimensionless quantity Q by the relation $\Gamma_\chi = 3HQ$—namely,

$$Q = \kappa g^{10}T^3\frac{\lambda^2}{3Hf^2}. \tag{3.3}$$

In warm inflation, the dissipation of the energy of the axion field χ due to the coupling to gauge fields leads to an effective friction in the equation of the axion

$$\ddot{\chi} + 3H(1+Q)\dot{\chi} + V'(\chi) = 0. \tag{3.4}$$

Moreover, the Friedman equation reads

$$H^2 = \frac{1}{3M_{\rm pl}^2}\left(V(\chi) + \frac{1}{2}\dot{\chi}^2 + \rho_{\rm R}\right), \tag{3.5}$$

where we take into account the energy density of radiation with the internal degrees of freedom g_i:

$$\rho_R = \frac{\pi^2}{30}g_iT^4. \tag{3.6}$$

The conservation of energy is described by

$$\dot{\rho}_{\rm R} + 4H\rho_{\rm R} = 3HQ\dot{\chi}^2. \tag{3.7}$$

Using the slow-roll approximation $|\ddot{\chi}| \ll H|\dot{\chi}|$ in Eq. (3.4); $\dot{\chi}^2 \ll V(\chi)$, $\rho_{\rm R} \ll V(\chi)$ in Eq. (3.5); and $\dot{\rho}_{\rm R} \ll H\rho_{\rm R}$ in Eq. (3.7), we obtain the slow-roll equations

$$3H(1+Q)\dot{\chi} + V'(\chi) = 0, \tag{3.8}$$

$$H^2 = \frac{V(\chi)}{3M_{\rm pl}^2}, \tag{3.9}$$

$$\rho_{\rm R} = \frac{3}{4}Q\dot{\chi}^2. \tag{3.10}$$

The consistency conditions for the slow-roll approximation are given by

$$\epsilon_{\rm v} \equiv \frac{1}{2(1+Q)}\left(\frac{V'(\chi)}{V(\chi)}\right)^2 \ll 1, \tag{3.11}$$

$$\eta_{\rm v} \equiv \frac{1}{1+Q}\frac{V''(\chi)}{V(\chi)} \ll 1, \tag{3.12}$$

$$\beta \equiv \frac{1}{1+Q}\frac{Q'V'(\chi)}{QV(\chi)} \ll 1. \tag{3.13}$$

We can see that the factor $1+Q$ alleviates the condition on the inflaton potential.

Since we know the axion potential (2.2), we can calculate the number of e-folding as

$$N = \int\frac{\dot{\alpha}}{\dot{\chi}}d\chi \simeq \int\frac{3(1+Q)\dot{\alpha}^2f^2}{\mu^4\sin\frac{\chi}{f}}d\chi = f^2\int_{\frac{\chi_i}{f}}^{\pi}\frac{(1+Q)(1+\cos X)}{\sin X}dX, \tag{3.14}$$

where we define $X = \chi/f$. If we take $X = 10^{-2}$ as an initial value, the number of e-folding is given by $N \sim 10f^2(1+Q)$. For example, $1+Q \sim f^{-2}\mathcal{O}(10)$ gives $N \sim \mathcal{O}(100)$. Hence, Q should be large, $Q \gg 1$, for a natural value of the decay constant $f < 1$.

IV. CHROMONATURAL WARM INFLATION

As shown in the previous sections, both chromonatural inflation and minimal warm inflation assume interaction between the inflaton and non-Abelian gauge fields through the Chern-Simons term. Therefore, it is natural to ask if chromonatural inflation and minimal warm inflation could be compatible. The purpose of this section is to show the existence of chromonatural warm inflation.

A. Slow-roll equations of axion inflation with Chern-Simons term

We consider the same Lagrangian with chromonatural inflation,

$$S = \int d^4x\sqrt{-G}\left[\frac{1}{2}R - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V(\chi) - \frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} - \frac{g^2\lambda}{8f}\chi\epsilon^{\mu\nu\lambda\rho}F^a_{\mu\nu}F^a_{\lambda\rho}\right], \tag{4.1}$$

where the axion potential $V(\chi)$ is given in Eq. (2.2). Here, we note the dual role of the SU(2) gauge field. In chromonatural inflation, it is crucial to assume that the SU(2) gauge field configuration in the background is rotation invariant: $A^a_i = \psi(t)a(t)\delta^a_i$. In minimal warm inflation, it is crucial to notice that the axion field can decay into a thermal gas of a

SU(2)-gauge field through the Chern-Simons coupling. It should be emphasized that these two components are different. We take into account the energy density of the thermal gas ρ_R in Eqs. (2.4) and (2.5) as

$$\dot{\alpha}^2 = \frac{1}{3}\left[\frac{1}{2}\dot{\chi}^2 + \mu^4\left(1+\cos\frac{\chi}{f}\right) + \frac{3}{2}\dot{\psi}^2 + 3\dot{\alpha}\dot{\psi}\psi + \frac{3}{2}\dot{\alpha}^2\psi^2 + \frac{3}{2}g^2\psi^4 + \rho_R\right], \tag{4.2}$$

$$\ddot{\alpha} = -\frac{1}{2}\dot{\chi}^2 - \dot{\psi}^2 - 2\dot{\alpha}\dot{\psi}\psi - \dot{\alpha}^2\psi^2 - g^2\psi^4 - \frac{2}{3}\rho_R, \tag{4.3}$$

and Eqs. (2.6) and (2.7) read

$$\ddot{\chi} + 3\dot{\alpha}(1+Q)\dot{\chi} - \frac{\mu^4}{f}\sin\frac{\chi}{f} = -3\frac{g^3\lambda}{f}\psi^2(\dot{\psi}+\dot{\alpha}\psi), \tag{4.4}$$

$$\ddot{\psi} + 3\dot{\alpha}\dot{\psi} + \left[-\frac{1}{6}\dot{\chi}^2 + \frac{2}{3}\mu^4\left(1+\cos\frac{\chi}{f}\right)\right]\psi + 2g^2\psi^3 - \frac{g^3\lambda}{f}\psi^2\dot{\chi} = 0, \tag{4.5}$$

where we have added the dissipation term only for the axion equation of motion. It is convenient to define ρ_χ and ρ_ψ as

$$\rho_\chi = \mu^4\left(1+\cos\frac{\chi}{f}\right), \tag{4.6}$$

$$\rho_\psi = \frac{3}{2}\dot{\alpha}^2\psi^2 + \frac{3}{2}g^2\psi^4. \tag{4.7}$$

The energy conservation implies

$$\dot{\rho}_{\mathrm{R}} + 4\dot{\alpha}\rho_{\mathrm{R}} = 3\dot{\alpha}Q\dot{\chi}^2. \tag{4.8}$$

In the slow-roll approximation, we impose the condition $|\ddot{\chi}| \ll \dot{\alpha}(1+Q)|\dot{\chi}|$, $|\ddot{\psi}| \ll \dot{\alpha}|\dot{\psi}|$, $\dot{\rho}_{\mathrm{R}} \ll \dot{\alpha}\rho_{\mathrm{R}}$, and the inflaton potential is dominant in Eqs. (4.2), (4.4), (4.5), and (4.8). The equations are simplified as follows:

$$\dot{\alpha}^2 = \frac{\mu^4}{3}\left(1+\cos\frac{\chi}{f}\right), \tag{4.9}$$

$$3\dot{\alpha}(1+Q)\dot{\chi} = \frac{\mu^4}{f}\sin\frac{\chi}{f} - 3\sigma\dot{\alpha}(\dot{\psi}+\dot{\alpha}\psi), \tag{4.10}$$

$$3\dot{\alpha}\dot{\psi} = -2\left(1+\frac{\sigma^2}{\gamma}\right)\dot{\alpha}^2\psi + \sigma\dot{\alpha}\dot{\chi}, \tag{4.11}$$

$$\rho_{\mathrm{R}} = \frac{3}{4}Q\dot{\chi}^2. \tag{4.12}$$

Note that we have defined $\sigma \equiv \frac{g^3\lambda\psi^2}{f\dot{\alpha}}$ and $\gamma \equiv \frac{g^4\lambda^2\psi^2}{f^2}$. The consistency conditions for the slow-roll approximation can be derived as follows. From the condition $\dot{\rho}_{\mathrm{R}} \ll \dot{\alpha}\rho_{\mathrm{R}}$, we obtain

$$\epsilon_{\mathrm{v}} \equiv \frac{1}{2(1+Q)\left(1+\frac{\sigma^2}{3(1+Q)}\right)}\left(\frac{V'(\chi)}{V(\chi)}\right)^2 \ll 1, \tag{4.13}$$

$$\eta_{\mathrm{v}} \equiv \frac{1}{(1+Q)\left(1+\frac{\sigma^2}{3(1+Q)}\right)}\frac{V''(\chi)}{V(\chi)} \ll 1, \tag{4.14}$$

$$\beta \equiv \frac{1}{1+Q}\frac{Q'V'(\chi)}{QV(\chi)} \ll 1. \tag{4.15}$$

We can see that the slow-roll conditions (2.11), (2.12), and (3.11)–(3.13) are included as special cases of Eqs. (4.13)–(4.15). The requirement $|\ddot{\psi}| \ll \dot{\alpha}|\dot{\psi}|$ does not imply any additional condition. From the condition $|\ddot{\chi}| \ll \dot{\alpha}(1+Q)|\dot{\chi}|$, we can see that the slow-roll condition (2.13) is updated to

$$\delta \equiv \frac{1}{\left(1+\frac{\sigma^2}{3(1+Q)}\right)}\frac{\sigma^2}{\gamma}\frac{1}{(1+Q)} \ll 1. \tag{4.16}$$

Note that these slow-roll conditions can be satisfied when all the parameters σ, γ, and Q are larger than 1.

After diagonalizing Eqs. (4.10) and (4.11), the equations of the axion and the gauge field are given by

$$\dot{\alpha}\dot{\chi} = \frac{1}{9\left(1+\frac{\sigma^2}{3(1+Q)}\right)}\left[\frac{3\frac{\mu^4}{f}\sin\frac{\chi}{f}}{1+Q} - \left(1-\frac{2\sigma^2}{\gamma}\right) \times \frac{\mu^4\left(1+\cos\frac{\chi}{f}\right)\sigma\psi}{1+Q}\right], \tag{4.17}$$

$$\dot{\alpha}\dot{\psi} = \frac{1}{9\left(1+\frac{\sigma^2}{3(1+Q)}\right)}\left[\frac{\sigma\frac{\mu^4}{f}\sin\frac{\chi}{f}}{1+Q} - \left(\frac{\sigma^2}{1+Q}+2+\frac{2\sigma^2}{\gamma}\right)\mu^4 \times \left(1+\cos\frac{\chi}{f}\right)\psi\right]. \tag{4.18}$$

B. Effect of dissipation on chromonatural inflation

It is possible to cast Eq. (4.18) into the form $\dot{\alpha}\dot{\psi} \simeq -V'(\psi)_{\mathrm{eff}}$, where we define the effective potential

$$V'(\psi)_{\mathrm{eff}} = \frac{\mu^4}{3\left(1+\frac{3(1+Q)}{\sigma^2}\right)}\left[-\frac{\sin\frac{\chi}{f}}{f\sigma} + \left(1+\frac{2(1+Q)}{\sigma^2} + \frac{2(1+Q)}{\gamma}\right)\left(1+\cos\frac{\chi}{f}\right)\psi\right]. \tag{4.19}$$

From Eq. (4.19), we obtain the fixed point

$$\psi_{\text{fixed}} = \left(\frac{\mu^2 \sin\frac{\chi}{f}}{\sqrt{3} g^3 \lambda \sqrt{1+\cos\frac{\chi}{f}} \left(1 + \frac{2(1+Q)}{\sigma^2} + \frac{2(1+Q)}{\gamma}\right)} \right)^{1/3}. \tag{4.20}$$

The second and third terms in the denominator represent corrections to the attractor solution of chromonatural inflation due to the effect of dissipation Q. This formal solution shows that chromonatural inflation can be realized when Q is much smaller than σ^2 and γ. However, once Q becomes comparable to σ^2 and γ, we would expect that warm inflation commences. What we would like to clarify is whether chromonatural and warm inflation coexist or not.

Let us take a look at the following expressions:

$$\sigma = \frac{g^3 \lambda \psi^2}{f \dot{\alpha}}, \qquad \gamma = \frac{g^4 \lambda^2 \psi^2}{f^2}, \qquad Q = \kappa g^{10} T^3 \frac{\lambda^2}{3 \dot{\alpha} f^2}. \tag{4.21}$$

Once the gauge field is captured at the minimum of the effective potential, σ^2, γ, and Q are determined by the parameters g, f, λ, μ, and κ and the field χ. Now, we change the parameter κ, which controls the effect of dissipation, and fix the other parameters and the field χ. For $\kappa = 0$, the solution reduces to that of chromonatural inflation in Sec. II. As we increase κ, the effect of dissipation becomes larger. Eventually, we see that chromonatural warm inflation can be realized. We will find that chromonatural inflation ceases to occur for larger κ.

From now on, we fix the parameters $\{g, f, \lambda, \mu\} = \{0.2, 0.01, 9\times10^5, 3\times10^{-4}\}$. For $\kappa = 0$, we have an attractor solution

$$\psi_{\min} = \left(\frac{\mu^2 \sin\frac{\chi}{f}}{\sqrt{3} g^3 \lambda \sqrt{1+\cos\frac{\chi}{f}}} \right)^{1/3} \simeq \left(\frac{\mu^2}{\sqrt{3} g^3 \lambda} \right)^{1/3}$$
$$\simeq 0.0002 \left(\frac{0.2}{g}\right) \left(\frac{9\times10^5}{\lambda}\right)^{\frac{1}{3}} \left(\frac{\mu}{3\times10^{-4}}\right)^{\frac{2}{3}}, \tag{4.22}$$

where we assume that the χ-dependent part gives an $\mathcal{O}(1)$ contribution. Then, σ and γ would be given by

$$\sigma \simeq 10^5 \left(\frac{g}{0.2}\right) \left(\frac{10^{-2}}{f}\right) \left(\frac{\lambda}{9\times10^5}\right)^{\frac{1}{3}} \left(\frac{3\times10^{-4}}{\mu}\right)^{\frac{2}{3}} \tag{4.23}$$

and

$$\gamma \simeq 10^6 \left(\frac{g}{0.2}\right)^2 \left(\frac{10^{-2}}{f}\right)^2 \left(\frac{\lambda}{9\times10^5}\right)^{\frac{4}{3}} \left(\frac{\mu}{3\times10^{-4}}\right)^{\frac{4}{3}}. \tag{4.24}$$

Since $\sigma \gg 1$ and $\gamma \gg 1$, the slow-roll conditions of chromonatural inflation are satisfied. Notice that, for these parameters, $\sigma^2 \gg \gamma$ is satisfied.

Let us start to increase κ gradually. As long as the effect of dissipation can be negligible, $\sigma^2 \gg Q$ and $\gamma \gg Q$, the fixed point is approximated by that of chromonatural inflation $\psi_{\min}$. The slow-roll conditions, from (4.13) to (4.16), are reduced into those of chromonatural inflation, $\sigma \gg 1$ and $\gamma \gg 1$. Using Eqs. (3.3), (3.6), and (4.12), we obtain

$$T = \frac{3}{40\pi^2} \kappa g^{10} \frac{\lambda^2}{\dot{\alpha} f^2} \dot{\chi}^2, \tag{4.25}$$

where we choose $g_i = 100$ in Eq. (3.6). Since the inequality $\sigma^2 \gg \gamma$ holds and $\dot{\psi} \simeq 0$ is satisfied, Eq. (4.11) yields

$$\dot{\chi} = \frac{2f}{g\lambda} \psi_{\min}. \tag{4.26}$$

Substituting Eqs. (4.22) and (4.26) into Eq. (4.25), we can analytically obtain the ratio of temperature to the Hubble parameter as

$$\frac{T}{\dot{\alpha}} = \frac{1}{30} \kappa g^6 \left(\frac{g^2 \psi_{\min}^2}{\dot{\alpha}^2}\right) \simeq \kappa \left(\frac{g}{0.2}\right)^6 \left(\frac{3\times10^{-4}}{\mu}\right)^{\frac{8}{3}} \left(\frac{9\times10^5}{\lambda}\right)^{\frac{2}{3}}. \tag{4.27}$$

Thus, for $\kappa > 1$, we see that warm inflation is realized. Using Eq. (4.27), we can evaluate the dissipation parameter (3.3) as

$$Q = \kappa g^{10} \frac{\lambda^2}{3\dot{\alpha} f^2} T^3 \simeq 10^{-7} \kappa^4 \left(\frac{g}{0.2}\right)^{28} \left(\frac{3\times10^{-4}}{\mu}\right)^4 \left(\frac{10^{-2}}{f}\right)^2. \tag{4.28}$$

We can also deduce the following expressions:

$$\frac{Q}{\sigma^2} \simeq 10^{-18} \kappa^4 \left(\frac{g}{0.2}\right)^{26} \left(\frac{3\times10^{-4}}{\mu}\right)^{\frac{8}{3}} \left(\frac{9\times10^5}{\lambda}\right)^{\frac{2}{3}}, \tag{4.29}$$

and

$$\frac{Q}{\gamma} \simeq 10^{-12} \kappa^4 \left(\frac{g}{0.2}\right)^{26} \left(\frac{3\times10^{-4}}{\mu}\right)^{\frac{16}{3}} \left(\frac{9\times10^5}{\lambda}\right)^{\frac{4}{3}}. \tag{4.30}$$

From the above results, we see that the current assumptions $\sigma^2 \gg Q$ and $\gamma \gg Q$ hold as long as κ satisfies the condition $\kappa \ll 10^3$. For $1 < \kappa$, we see that the temperature could exceed the Hubble parameter, which is a characteristic feature of warm inflation. In particular, for $10^{1.75} < \kappa \ll 10^3$, we obtain strong warm inflation $Q > 1$. Thus, chromonatural warm inflation occurs for the parameter region $1 < \kappa \ll 10^3$. Note that these conditions change depending on the choice of other parameters. For example, when we consider $g \sim \mathcal{O}(10^{-2})$, the condition for warm inflation to occur becomes $10^6 < \kappa \ll 10^9$.

From Eqs. (4.26) and (4.28), we obtain the radiation energy density as

$$\rho_R = \frac{3}{4} Q\dot{\chi}^2 \simeq 10^{-5} \frac{\kappa^4 g^{32}}{\dot{\alpha}^4} \psi_{\rm min}^8 \simeq 10^{-28}\kappa^4 \left(\frac{g}{0.2}\right)^{24} \left(\frac{3\times 10^{-4}}{\mu}\right)^{\frac{8}{3}} \left(\frac{9\times 10^5}{\lambda}\right)^{\frac{8}{3}}, \tag{4.31}$$

where we use the relation $\sigma^2 \gg \gamma$ in the last equality. The energy density of the gauge field can be obtained by substituting the fixed point (4.22) into Eq. (4.7) as

$$\rho_\psi = \frac{3}{2} g^2 \psi_{\rm min}^4 \simeq 10^{-16} \left(\frac{\mu}{3\times 10^{-4}}\right)^{\frac{8}{3}} \left(\frac{0.2}{g}\right)^2 \left(\frac{9\times 10^5}{\lambda}\right)^{\frac{4}{3}}. \tag{4.32}$$

We can also obtain the number of e-folding using Eq. (2.25). For the current parameters, we obtain $\beta = \lambda^{1/3}\mu^{4/3}$ is $\mathcal{O}(10^{-2})$, and $N \simeq g^2\lambda\mathcal{O}(10^{-2}) \sim \mathcal{O}(100)$.

To check the analytical results, we numerically solved the basic Eqs. (4.4), (4.5), and (4.8) with the parameter set $\{\kappa, g, f, \lambda, \mu\} = \{10^2, 2\times 10^{-1}, 10^{-2}, 9\times 10^5, 3\times 10^{-4}\}$. In Fig. 1, we can see that the gauge field value has settled down into the potential minimum $\psi_{\rm min}$, indicating that the slow-roll conditions are satisfied during inflation. We also show T/H for various κ in Fig. 2. As we increase κ, we see that the temperature gradually increases.

In this subsection, we have analytically investigated chromonatural warm inflation. Although we have fixed the parameters other than κ, it is easy to change other parameters and to see what kind of inflation is realized. In particular, we should note that warm inflation is highly sensitive to the gauge coupling constant g, as one can see from Eqs. (4.27), (4.28), and (4.31).

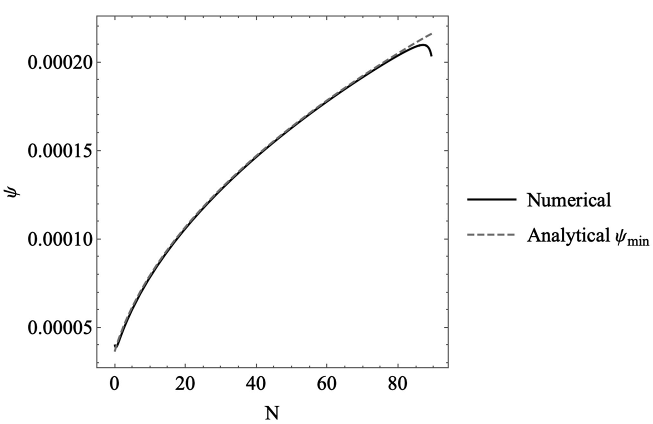

FIG. 1. The solid curve represents the time evolution of the gauge field numerically obtained with $\kappa = 10^2$. The dashed curve represents the analytical solution $\psi_{\rm min}$.

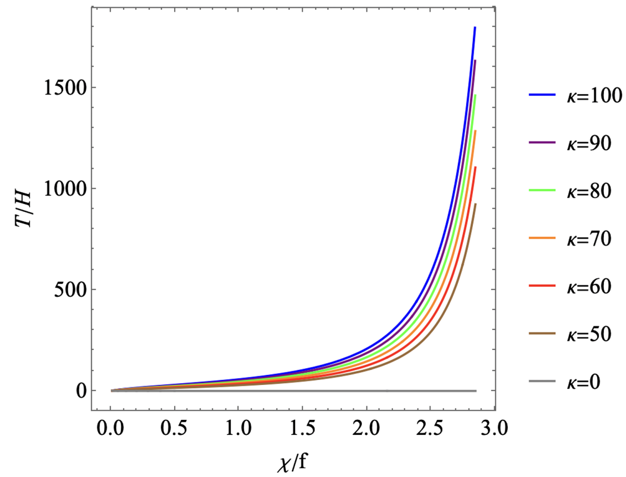

FIG. 2. The horizontal axis represents χ/f. Inflation ends when χ/f reaches π. The graph shows that larger κ gives a larger temperature. We use the parameters $g = 0.2$, $f = 0.01$, $\lambda = 9\times 10^5$, $\mu = 3\times 10^{-4}$ and the initial conditions $T_0 = 10^{-7}$, $\chi_0 = 10^{-4}$, $v_{\chi_0} = 0$, $\psi_0 = 4\times 10^{-5}$, $v_{\psi_0} = 0$.

C. The existence of chromonatural warm inflation

If we increase κ, the condition $\gamma \gg Q$ will eventually be violated: namely, the effect of dissipation Q cannot be negligible. According to Eq. (4.20), the gauge field would become smaller as we increase κ. Hence, the slow-roll conditions of chromonatural inflation, $\sigma \gg 1$ and $\gamma \gg 1$, will be violated in the end. In order to see the behavior, we will analyze the κ dependence of the gauge field using the slow-roll Eqs. (4.17) and (4.18).

The temperature can be determined by the time derivative of the axion $\dot{\chi}$ as in Eq. (4.25). The slow-roll equation of the axion field gives

$$\dot{\chi} = \frac{\sqrt{3}}{9(1+Q+\frac{\sigma^2}{3})} \left\{ \frac{3\mu^2}{f} \frac{\sin\frac{\chi}{f}}{\sqrt{1+\cos\frac{\chi}{f}}} - \left(1 - \frac{2\sigma^2}{\gamma}\right)\mu^2 \sqrt{1+\cos\frac{\chi}{f}}\,\sigma\psi \right\}. \tag{4.33}$$

By substituting $\dot{\chi}$ into the temperature (4.25), we obtain

$$T = \frac{3\kappa g^{10}}{40\pi^2} \frac{\lambda^2}{\dot{\alpha} f^2} \left[\frac{\sqrt{3}}{9(1+Q+\frac{\sigma^2}{3})} \left\{ \frac{3\mu^2}{f} \frac{\sin\frac{\chi}{f}}{\sqrt{1+\cos\frac{\chi}{f}}} - \left(1 - \frac{2\sigma^2}{\gamma}\right)\mu^2 \sqrt{1+\cos\frac{\chi}{f}}\,\sigma\psi \right\} \right]^2. \tag{4.34}$$

Moreover, since the gauge field is settled down into the minimum of the effective potential, $\dot{\psi} \simeq 0$ will be realized. Thus, Eq. (4.18) implies

$$1+Q=\frac{\sigma^2}{2}\left(1+\frac{\sigma^2}{\gamma}\right)^{-1}\left\{\frac{1}{f\sigma\psi}\frac{\sin\frac{\chi}{f}}{1+\cos\frac{\chi}{f}}-1\right\}. \tag{4.35}$$

By using Eq. (4.35) in Eq. (4.34), the temperature T can be written as a function of ψ. On the other hand, using the expression of Q (3.3) in Eq. (4.35), the temperature T can also be written by ψ as

$$T=\left(\frac{\sqrt{3}\mu^2 f^2\sigma^2}{2\kappa g^{10}\lambda^2}\right)^{\frac{1}{3}}\left[\left(1+\frac{\sigma^2}{\gamma}\right)^{-1}\times\left\{\frac{1}{f\sigma\psi}\frac{\sin\frac{\chi}{f}}{\left(1+\cos\frac{\chi}{f}\right)}-1\right\}-\frac{2}{\sigma^2}\right]^{\frac{1}{3}}. \tag{4.36}$$

The attractor solutions of slow-roll inflation could be obtained as the intersection of (4.34) and (4.36). To make the analysis precise, we fix the parameters g, f, λ, μ. Thus, the attractor of the gauge field and the temperature can be determined by the axion field χ once κ is determined. During inflation, χ is almost constant. Hence, we can take a representative value.

Now, we will take a look at the κ dependence of ψ, σ, γ, T/H, Q, and $\dot{\chi}$. In the numerical calculations, we use the parameters $g=0.2$, $f=0.01$, $\lambda=9\times10^5$, $\mu=3\times10^{-4}$. We also choose a reference point $\chi=0.01$ for the axion field.

Since we know the behavior of solutions up to $\kappa=10$ from the analysis in Secs. IV and IV B, we will start with $\kappa=10$. In Fig. 3, we plot the gauge field for various κ. In Figs. 4 and 5, the parameters σ and γ which characterize the slow-roll chromonatural inflation are depicted. Up to $\kappa=10^2$, ψ, σ, and γ do not change, as shown in Figs. 3–5, which is consistent with the results in Secs. IV and IV B. In the parameter region $\kappa\gtrsim10^3$, we see the behavior $\psi\propto1/\sqrt{\kappa}$ due to the dissipation. As a consequence, σ and γ decrease linearly, $1/\kappa$. Around $\kappa=10^7$, we can see that σ and γ become $\mathcal{O}(1)$, which means that the chromonatural inflation ceases to occur. In fact, we numerically checked that the intersection point disappears when we take $\kappa>10^7$.

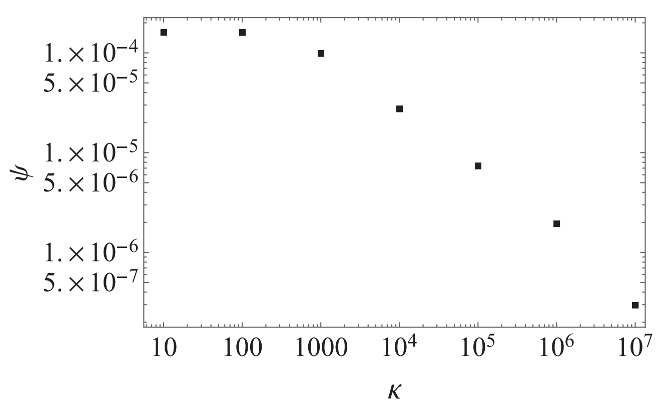

FIG. 3. The gauge field ψ is depicted for various κ. The gauge field does not depend on κ for $\kappa<10^2$. For $\kappa>10^2$, the gauge field gradually decreases as $1/\sqrt{\kappa}$.

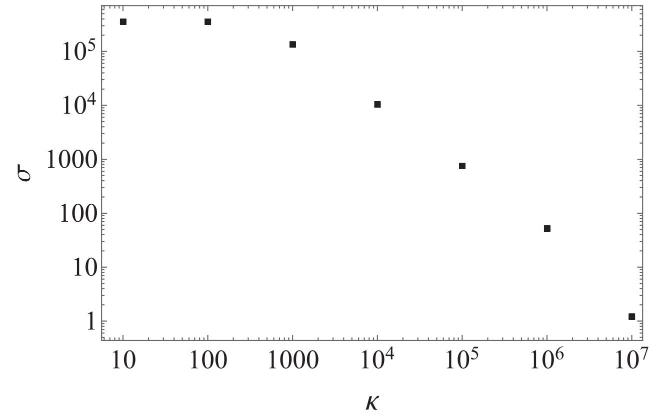

FIG. 4. The variable σ is depicted for various κ. Since $\sigma\gg1$ is the slow-roll condition for chromonatural inflation, we see that chromonatural inflation ceases to occur around $\kappa=10^7$.

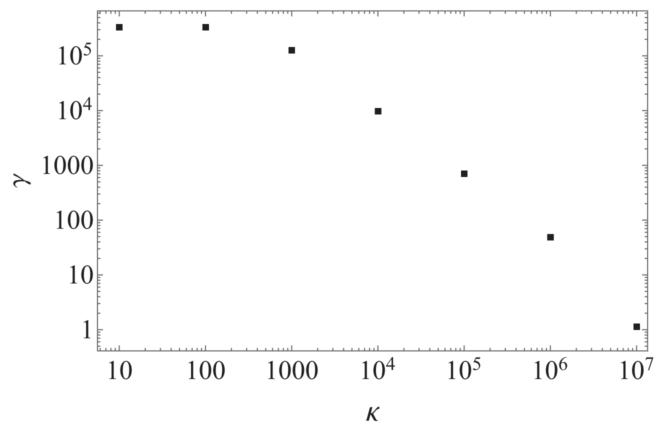

FIG. 5. We plot γ for various κ. Since $\gamma\gg1$ is the slow-roll condition for chromonatural inflation, again we see that chromonatural inflation ceases to occur around $\kappa=10^7$.

In Fig. 6, we plot the ratio of temperature to the Hubble parameter T/H for various κ. We find that the ratio T/H takes the maximum value $T/H\simeq600$ around $\kappa=10^3$. Beyond $\kappa=10^3$, the ratio T/H decreases.

We also plot the dissipation parameter Q in Fig. 7. From Fig. 7, we see that the dissipation parameter Q rapidly increases up to $\kappa=10^3$, beyond which Q increases slowly. Note that the dissipation parameter Q is always much larger than 1 for large κ.

Remarkably, for $\kappa>10^3$, the temperature T decreases in spite of increasing Q. This is because $\dot{\chi}$ decreases faster than the increasing rate of Q. Note that the larger κ enhances the dissipation, and as a consequence, the friction becomes large. That makes $\dot{\chi}$ small. To verify this, we plot $\dot{\chi}$ in Fig. 8. This is the reason why the ratio T/H starts to decrease at some point.

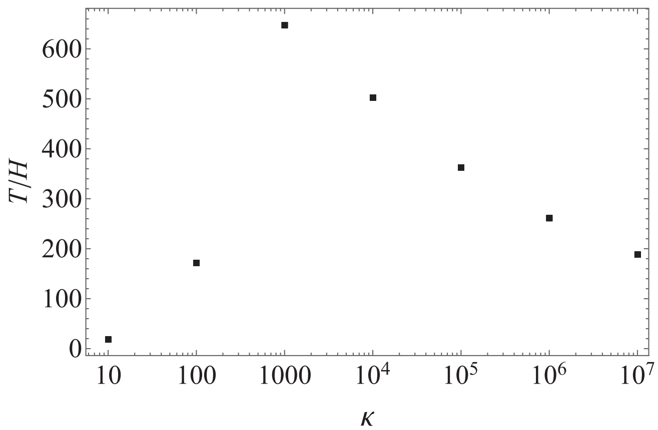

FIG. 6. The ratio T/H is depicted for various κ. Around $\kappa = 10^3$, the ratio becomes maximum.

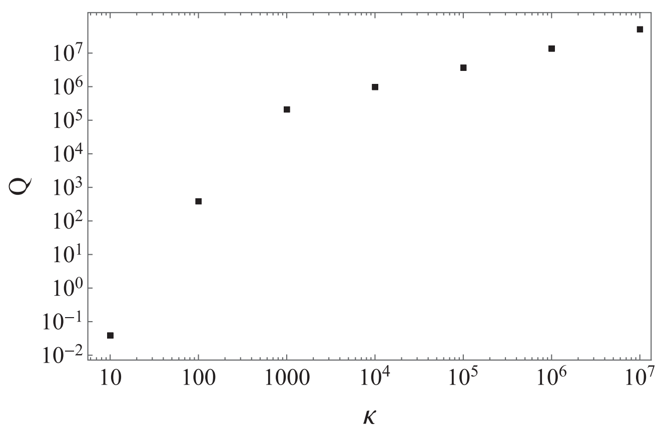

FIG. 7. The dissipation parameter Q is depicted for various κ. We see Q rapidly increase up to $\kappa = 10^3$, beyond which Q increases slowly.

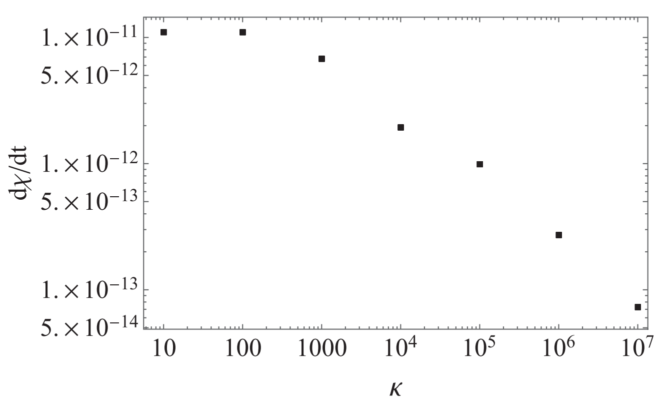

FIG. 8. The time derivative of the axion field $\dot{\chi}$ is depicted for various κ. From $\kappa = 10^3$, $\dot{\chi}$ decreases rapidly.

To understand the behavior of the Universe in the parameter region $\kappa > 10^7$, we checked analytically and numerically the maximum value of κ where warm inflation can occur. Beyond $\kappa = 10^7$, since the gauge field becomes trivial, the behavior of the axion field would return to that of warm inflation. The time derivative of the axion field $\dot{\chi}$ is given by taking $\psi = 0$ in Eq. (4.33) as

$$\dot{\chi} = \frac{1}{\sqrt{3}(1+Q)} \frac{\mu^2}{f} \frac{\sin\frac{\chi}{f}}{\sqrt{1+\cos\frac{\chi}{f}}}. \tag{4.37}$$

From Eqs. (3.6) and (4.12), the temperature is given by

$$T = \frac{1}{\kappa^{\frac{1}{7}}} \left\{ \frac{9}{40\pi^2} \frac{\dot{\alpha}\mu^4}{g^{10}\lambda^2} \frac{\sin^2\frac{\chi}{f}}{\left(1+\cos\frac{\chi}{f}\right)} \right\}^{\frac{1}{7}}. \tag{4.38}$$

Note that we have assumed $Q \gg 1$, as shown in Fig. 6. As κ is increased, we see that the temperature decreases. Thus, eventually we will reach the critical point $T = \dot{\alpha}$. Now, assuming the parameter set $g = 0.2$, $f = 0.01$, $\lambda = 9 \times 10^5$, $\mu = 3 \times 10^{-4}$ and the axion field $\chi = 0.01$, we can obtain κ at the critical point:

$$\kappa = \frac{3^5}{40\pi^2} \frac{1}{g^{10}\lambda^2\mu^8} \frac{\sin^2\frac{\chi}{f}}{\left(1+\cos\frac{\chi}{f}\right)^4} \simeq 10^{22}. \tag{4.39}$$

Thus, for $\kappa > 10^{22}$, the temperature is below the Hubble parameter—that is, we have cold inflation. In Sec. III, we obtained the number of e-folding (3.14), which can be approximately written as $N \sim 10 f^2 Q$. Substituting (4.38) into (3.3), we can derive the dissipation parameter Q as

$$Q = \kappa^{\frac{4}{7}} \left\{ \frac{3}{(40\pi^2)^3} \frac{g^{40}\lambda^8\mu^4}{f^{14}} \frac{\sin^6\frac{\chi}{f}}{\left(1+\cos\frac{\chi}{f}\right)^5} \right\}^{\frac{1}{7}}. \tag{4.40}$$

We see that the number of e-folding increases as $N \propto \kappa^{\frac{4}{7}}$. Therefore, the Universe looks like an eternal inflation for $\kappa > 10^{22}$.

In Sec. IV B, we have analytically shown that chromonatural warm inflation exists for $1 < \kappa < 10^3$. In this subsection, we have numerically shown that chromonatural warm inflation exists in the parameter region $10 < \kappa < 10^7$. For $\kappa > 10^7$, warm inflation is realized up to $\kappa < 10^{22}$. If we consider much larger $\kappa > 10^{22}$, almost eternal inflation is realized with the number of e-folding $N \propto \kappa^{\frac{4}{7}}$. Thus, we can conclude that chromonatural warm inflation exists for $1 < \kappa < 10^7$.

V. CONCLUSION

We studied axion inflation with the non-Abelian Chern-Simons interaction term. In particular, we examined if chromonatural inflation and minimal warm inflation can occur simultaneously. We showed that these inflation models can coexist. Indeed, chromonatural inflation could have a temperature $T > H$, which is a characteristic feature of warm inflation.

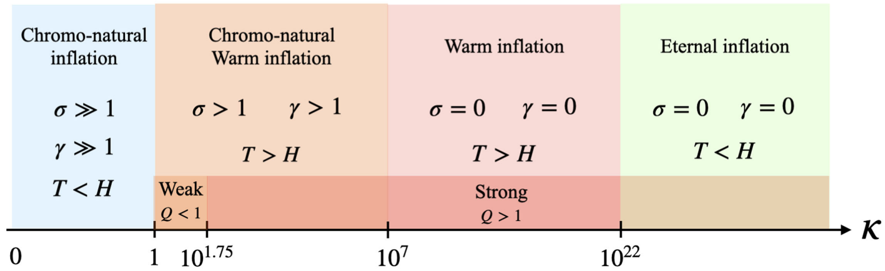

FIG. 9. For the same parameters as those in Fig. 3, we depict a one-dimensional phase diagram of inflation. This diagram clearly shows that chromonatural warm inflation exists for a wide range of parameters, including a natural one at $\kappa = 10^2$.

We analyzed the system in two ways. First, starting from the chromonatural inflation where $\kappa = 0$, we gradually increased the parameter κ. We analytically showed that chromonatural warm inflation occurs once κ exceeds 1. In particular, when κ is larger than $10^{1.75}$, strong warm inflation appears. Beyond $\kappa = 10^3$, the approximation became invalid. Hence, second, we numerically solved attractor equations. We found that gauge fields decrease as $\psi \propto 1/\sqrt{\kappa}$. Hence, σ and γ decrease linearly as $1/\kappa$. Around $\kappa = 10^7$, we see that σ and γ become $\mathcal{O}(1)$. Namely, chromonatural inflation ceases to occur there. We also found that the ratio of the temperature to the Hubble parameter T/H is always larger than 1 in the parameter region $1 < \kappa < 10^7$. Thus, we found that chromonatural warm inflation occurs in the parameter region $1 < \kappa < 10^7$. For $\kappa > 10^7$, we have warm inflation. If we further increase κ, since $\dot{\chi}$ decreases, warm inflation ceases to occur around $\kappa = 10^{22}$. For $\kappa > 10^{22}$, inflation becomes cold and looks like eternal inflation. We can summarize our findings in Fig. 9.

In our analysis, we have always fixed the parameters other than κ. However, our analytic expressions are useful for getting information about inflation with other parameter values. For example, we can see the strong dependence of warm inflation on the gauge coupling constant g. Moreover, our numerical analysis can be easily repeated for other parameter cases. We merely focused on the existence of chromonatural warm inflation in this paper.

As we mentioned in the Introduction, the counterexample of the cosmic no-hair conjecture can lead to interesting phenomenological consequences. In the present case, minimal warm inflation predicts the absence of primordial gravitational waves due to the low Hubble scale. On the other hand, in chromonatural inflation, the non-Abelian gauge field contains the tensor modes interacting with the metric tensor modes, leading to an excess of primordial gravitational waves due to the tachyonic instability. However, in chromonatural warm inflation, the growth of the gauge field would be suppressed. Hence, it might happen that tachyonic instability could produce an appropriate amount of gravitational waves to be observable. We will leave this issue for future work.

ACKNOWLEDGMENTS

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