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Constraining über gravity with recent observations and studying the H_0 problem

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Abstract This paper studies both ΛCDM and CDM models under the über gravity theory, named üΛCDM and üCDM respectively. We report bounds over their parameter phase-space using several cosmological data, in particular, the recent Pantheon+ sample. Based on the joint analysis, the best fit value of the über characteristic parameter is $z_\oplus = 0.046^{+0.047}_{-0.032}$ and $z_\oplus = 1.382^{+0.020}_{-0.021}$ at 68% confidence level for üΛCDM and üCDM respectively. Although über gravity can successfully mimics the cosmological constant, we find that the $\mathbb{H}0(z)$ diagnostic suggests the H_0 tension is not alleviated. Finally, both models are statistically compared with ΛCDM through the Akaike and Bayesian information criteria. Both über gravity models and ΛCDM are equally preferred for most of the single samples, in particular, üΛCDM is not rejected by the CMB data. However, there is strong evidence against them for the joint analysis.

1 Introduction

The Λ Cold Dark Matter (ΛCDM) model is one of the most successful theories that explain with great precision the beginning and the evolution of our Universe at its current stages. However, among its many conundrums, the dark matter and energy components are the most intriguing due to the extremely weak interactions (or possible negligible) with the standard fields, being only the gravitational field the sole probe of its existence. The dark energy (DE) is related to the current Universe acceleration first observed by the supernova teams lead by Riess, Perlmutter among others [1,2] and later sustained by the Planck Satellite [3]. Clearly an acceleration of the space-time is not expected, thus it requires an unnatural dynamics: the addition of an extra component into the Friedmann equations which is able to create an accelerated expansion of the Universe and produce a late de Sitter scale factor. Without doubts, the cosmological constant is the cheapest way to tackle this problem, being an essential part of the ΛCDM paradigm, having also an exquisite capability to simulate the late Universe [4]. However, it is important to emphasize that the cosmological constant is afflicted with several problems like those related to the vacuum energy density and the coincidence problem [5,6]. Is in this vein that the community is exploring diverse alternatives to the cosmological constant, being the problem not yet settled (see [7] for a compilation).

Additionally, cosmologists concentrate most of their efforts in tackling the case of the current expansion rate of the Universe, the so-called H_0 *tension*. The reduction of uncertainties has led to significant discrepancies between the Planck [3] and SH0ES [8] collaborations, whose disagreements are approximately at $\sim 5.0\sigma$. Whether these discrepancies are due to systematic errors or mistaken theoretical assumptions, the H_0 tension could represent a window into new physics. Although modifying the standard cosmological model, in order to address the tension, without compromis-

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ing its success in other areas has proven to be a very difficult task [9], the model-dependence of early H_0 measurements may suggest that it is worth exploring models beyond ΛCDM. The proposals include early dark energy [10–12], extra-relativistic degrees of freedom [13–15], new early dark energy [16,17], diffusion models [18,19], holographic dark energy [20–22], to mention some of them.

In the present study we focus on two proposals: the üΛCDM and üCDM models, based on the über-gravitational model [23], which understands the H_0 tension as a transition in the context of a gravitational theory [24]. über-gravity sets the idea of taking an ensemble average over all possible $f(R)$ theories [25], a process inspired by statistical mechanics. Although the criteria for assigning probabilities to each theory are not clear, the general properties of üΛCDM and üCDM can be studied and it shows promise in alleviating the H_0 tension. Indeed, the model contains two branches where the Hubble parameter evolves in different ways, with a threshold region determined by the free parameter $z_\oplus$, which is the redshift of transition to the über gravity. In this sense, it is also possible to determine the deceleration and jerk parameters in order to elucidate the characteristics of the component responsible for the acceleration. As stated, üΛCDM contains an extra parameter in comparison with üCDM, which contains the same number as ΛCDM. Therefore, in the case of üΛCDM, besides introducing a statistical penalization (e.g. Akaike (AIC) or Bayesian information criteria (BIC) respectively) due to the extra parameter, it also keep the unsolved problem associated to the cosmological constant.

Additionally, we constrain the free parameters through a Bayesian analysis, using recent observations of Observational Hubble Parameters (CMB), Type Ia Supernovae (SnIa), Baryon Acoustic Oscillations (BAO), HII Galaxies (HIIG), Cosmic Microwave Background radiation (CMB) and a joint analysis. Moreover, we implement a H_0 diagnostic technique to study a possible alleviation to the H_0 tension using the über-gravity paradigm.

The paper is organized as follows: in Sect. 2 we review the mathematical background of üΛCDM and üCDM models, emphasizing its main properties and defining the parameters to be analyzed. In Sect. 3 the data and methodology are described, and the results are shown in Sect. 4. Finally, in Sect. 5 we summarize the conclusions and give some outlooks on the subject. We henceforth use units in which $\hbar = k_B = c = 1$.

2 Mathematical background

The idea of taking an ensemble average over gravity theories is condensed in the following Lagrangian

$$\mathcal{L} = \left(\sum_{i=1}^{N} \mathcal{L}_i e^{-\beta \mathcal{L}_i}\right)\left(\sum_{i=1}^{N} e^{-\beta \mathcal{L}_i}\right)^{-1},\tag{1}$$

where the index i corresponds to the ith theory considered through the Lagrangian associated $\mathcal{L}$, being N the dimension of the gravitational models space over the manifold $\mathbb{M}$. The free parameter β assigns probabilities to each model. For greater rigor, the ensemble average is taken over all analytic models of gravity, described by the $f(R)$ family, where the R stands for the Ricci scalar. However, in [24] is shown that the Lagrangian, after a change of basis, can be written as

$$\mathcal{L}_{\text{uber}} = \left(\sum_{n=1}^{\infty} (\bar{R}^n - 2\Lambda) e^{-\beta(\bar{R}^n - 2\Lambda)}\right)\left(\sum_{n=1}^{\infty} e^{-\beta(\bar{R}^n - 2\Lambda)}\right)^{-1},\tag{2}$$

where $\bar{R} \equiv R/R_0$ contains a new cosmological parameter R_0, n is an integer, and Λ is the cosmological constant. Next, we will show two models associated with üΛCDM and üCDM.

2.1 üΛCDM model

In background cosmology it is possible to assume homogenity and isotropy in the line element through the Friedmann–Lemaitre–Robertson–Walker (FLRW) equations as $ds^2 = -dt^2 + a(t)^2[dr^2 + r^2 d\Omega^2]$, where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and $a(t)$ is the scale factor. Thus, the Friedmann equation in this scenario can be written as [23]

$$E^2(z)_{\text{ü}\Lambda CDM} = \begin{cases} \Omega_{0m}(z+1)^3 + \Omega_{0r}(z+1)^4 + \Omega_{0\Lambda}, & z > z_\oplus \\ \frac{1}{2}\bar{R}_0 + \left(1 - \frac{1}{2}\bar{R}_0\right)(z+1)^4, & z < z_\oplus \end{cases}\tag{3}$$

where $E(z) \equiv H(z)/H_0$, $H(z)$ is the Hubble parameter, z is the redshift (i.e. $H_0 = H(z=0)$), Ω_{0m} is the matter density parameter, Ω_{0r} is the radiation density parameter, both at $z = 0$ and $\bar{R}_0 \equiv R_0/6H_0^2$. In this case, $\Omega_{0\Lambda}$ plays a role at $z > z_\oplus$, being $z_\oplus$ the region of transition to über gravity.

Using the continuity for $E(z)$ and $E'(z)$, where the prime denotes a derivative with respect to z, $\bar{R}_0$ takes the form

$$\frac{\bar{R}_0}{2} = \frac{(1 - \Omega_{0r})(z_\oplus + 1)^4 + 3\Omega_{0\Lambda}}{3 + (z_\oplus + 1)^4}.\tag{4}$$

One of the magnitudes of interest for the present study is the deceleration parameter, written in terms of the redshift,

which is given by

$$
q(z)_{ü\Lambda CDM} =
\begin{cases}
\frac{3\Omega_{0m}(z+1)^3 + 4\Omega_{0r}(z+1)^4}{2[\Omega_{0m}(z+1)^3 + \Omega_{0r}(z+1)^4 + \Omega_{0\Lambda}]} - 1, & z > z_\oplus \\[3mm]
\frac{2\left(1 - \frac{1}{2}\bar{R}_0\right)(z+1)^4}{\frac{1}{2}\bar{R}_0 + \left(1 - \frac{1}{2}\bar{R}_0\right)(z+1)^4} - 1, & z < z_\oplus
\end{cases}
\tag{5}
$$

Moreover, the jerk parameter is given by the expression

$$
j(z)_{ü\Lambda CDM} =
\begin{cases}
\frac{\Omega_{0m}(z+1)^3 + 3\Omega_{0r}(z+1)^4 + \Omega_{0\Lambda}}{\Omega_{0m}(z+1)^3 + \Omega_{0r}(z+1)^4 + \Omega_{0\Lambda}}, & z > z_\oplus \\[3mm]
\frac{2\left(1 - \frac{1}{2}\bar{R}_0\right)(z+1)^4}{\frac{1}{2}\bar{R}_0 + \left(1 - \frac{1}{2}\bar{R}_0\right)(z+1)^4} + 1, & z < z_\oplus
\end{cases}
\tag{6}
$$

which behaves like the standard ΛCDM model in the region $z > z_\oplus$, while for $z < z_\oplus$ the über gravity dominates its evolution.

Finally, the transition redshift z_T, which determines the beginning of the accelerated expansion of the universe ($q(z_T) = 0$), was calculated for both models. For $z_T < z_\oplus$ the transition redshift takes the form

$$
z_T = \sqrt[4]{\frac{3\Omega_{0\Lambda} + (1 - \Omega_{0r})(z_\oplus + 1)^4}{3(1 - \Omega_{0\Lambda}) + \Omega_{0r}(z_\oplus + 1)^4}} - 1,
\tag{7}
$$

whereas for $z_T > z_\oplus$ the value comes from solving

$$
\Omega_{0m}(z_T + 1)^3 + 2\Omega_{0r}(z_T + 1)^4 = 2\Omega_{0\Lambda},
\tag{8}
$$

which is obtained numerically. The transition in this model occurs for $z_T > z_\oplus$, so in this study only this last expression is taken into account.

2.2 üCDM model

In this case, we propose that the Universe acceleration is only driven by über contributions, thus we consider $\Omega_\Lambda = 0$, reducing the free parameters to the same number as in the standard model and assuming FLRW cosmology. Therefore, we have

$$
E^2(z)_{üCDM} =
\begin{cases}
\Omega_{0m}(z+1)^3 + \Omega_{0r}(z+1)^4, & z > z_\oplus \\[2mm]
\frac{1}{2}\bar{R}_0 + \left(1 - \frac{1}{2}\bar{R}_0\right)(z+1)^4, & z < z_\oplus
\end{cases}
\tag{9}
$$

being

$$
\frac{\bar{R}_0}{2} = \frac{(1 - \Omega_{0r})(z_\oplus + 1)^4}{3 + (z_\oplus + 1)^4}, \quad \Omega_{0m} = \frac{4(1 - \Omega_{0r})(z_\oplus + 1)}{3 + (z_\oplus + 1)^4}.
\tag{10}
$$

For the deceleration parameter we have

$$
q(z)_{üCDM} =
\begin{cases}
\frac{3\Omega_{0m}(z+1)^3 + 4\Omega_{0r}(z+1)^4}{2[\Omega_{0m}(z+1)^3 + \Omega_{0r}(z+1)^4]} - 1, & z > z_\oplus \\[3mm]
\frac{2\left(1 - \frac{1}{2}\bar{R}_0\right)(z+1)^4}{\frac{1}{2}\bar{R}_0 + \left(1 - \frac{1}{2}\bar{R}_0\right)(z+1)^4} - 1, & z < z_\oplus
\end{cases}
\tag{11}
$$

and the jerk parameter reads

$$
j(z)_{üCDM} =
\begin{cases}
\frac{\Omega_{0m} + 3\Omega_{0r}(z+1)}{\Omega_{0m} + \Omega_{0r}(z+1)}, & z > z_\oplus \\[3mm]
\frac{2\left(1 - \frac{1}{2}\bar{R}_0\right)(z+1)^4}{\frac{1}{2}\bar{R}_0 + \left(1 - \frac{1}{2}\bar{R}_0\right)(z+1)^4} + 1, & z < z_\oplus
\end{cases}
\tag{12}
$$

being the first region $z > z_\oplus$ (for q and j) dominated by matter while the later one is controlled by the über effects that drive the acceleration.

Finally, in üCDM the transition is not allowed in the region $z_T > z_\oplus$, giving, as a result, the following analytic expression, valid for $z_T < z_\oplus$

$$
z_T = \sqrt[4]{\frac{(1 - \Omega_{0r})(z_\oplus + 1)^4}{3 + \Omega_{0r}(z_\oplus + 1)^4}} - 1,
\tag{13}
$$

Note that z_T shows a dependence only on $z_\oplus$ and Ω_{0r}.

3 Data and methodology

Both üΛCDM and üCDM cosmologies are confronted using CC, HIIG, SnIa, BAO, CMB and joint data through a Bayesian Markov Chain Monte Carlo (MCMC) analysis. To bound their free parameter phase-space, $(h, \Omega_{0\Lambda}, z_\oplus)$ for üΛCDM and $(h, z_\oplus)$ for üCDM, we use the emcee package [26] under Python language. We establish a configuration to achieve the convergence of the chains using the autocorrelation function, and generate a set of 3000 chains with 250 steps. Additionally, we use a Gaussian prior over h as $h = 0.7403 \pm 0.0142$ [27] and $h = 0.6766 \pm 0.0042$ [3] as a consistency probe, flat priors over $\Omega_{0\Lambda}$ and $z_\oplus$ in the region [0,1] and [0,2] respectively. Thus, the χ^2-function is given by

$$
\chi^2_{\text{Joint}} = \chi^2_{\text{CC}} + \chi^2_{\text{HIIG}} + \chi^2_{\text{SnIa}} + \chi^2_{\text{BAO}} + \chi^2_{\text{CMB}},
\tag{14}
$$

where each term corresponds to the χ^2 function per sample.

3.1 Cosmic chronometers

We use a sample of 31 measurements of the Hubble parameter using differential age method [28] (see also [29]). Due these points are cosmological model independent, they are useful

to test alternative cosmologies to ΛCDM. The χ^2 function can be built as

$$\chi^2_{\text{OHD}} = \sum_{i=1}^{31} \left(\frac{H_{th}(z_i) - H_{obs}(z_i)}{\sigma^i_{obs}} \right)^2, \tag{15}$$

where $H_{th}(z_i)$ is the theoretical Hubble parameter using Eq. (3), and $H_{obs}(z_i) \pm \sigma^i_{obs}$ is the observational counterpart with its uncertainty at the redshift z_i.

3.2 Type Ia Supernovae (Pantheon+)

Recently, a sample of 1701 measurements of the luminosity modulus coming from SNIa, namely Pantheon+ sample, is reported by [30,31]. This represents the largest sample and covers a region $0.001 < z < 2.26$. Considering that this sample is extracted from 1550 distinct SNIa, we build the χ^2 function as

$$\chi^2_{\text{SnIa}} = a + \log \left(\frac{e}{2\pi} \right) - \frac{b^2}{e}, \tag{16}$$

where

$$a = \Delta\tilde{\mu}^T \cdot \mathbf{Cov_P^{-1}} \cdot \Delta\tilde{\mu},$$
$$b = \Delta\tilde{\mu}^T \cdot \mathbf{Cov_P^{-1}} \cdot \Delta\mathbf{1},$$
$$e = \Delta\mathbf{1}^T \cdot \mathbf{Cov_P^{-1}} \cdot \Delta\mathbf{1}, \tag{17}$$

and $\Delta\tilde{\mu}$ is the vector of the difference between the theoretical distance modulus and the observed one, $\Delta\mathbf{1} = (1, 1, \ldots, 1)^T$, $\mathbf{Cov_P}$ is the covariance matrix formed by adding the systematic and statistic uncertainties. The transpose of the vectors are denoted with the super-index T on the previous expressions.

The theoretical counterpart of the distance modulus is estimated by

$$m_{th} = \mathcal{M} + 5 \log_{10} \left[\frac{d_L(z)}{10 \, \text{pc}} \right], \tag{18}$$

where $\mathcal{M}$ is a nuisance parameter which has been marginalized by Eq. (16). The luminosity distance, denoted as $d_L(z)$, is computed through

$$d_L(z) = (1+z)c \int_0^z \frac{dz'}{H(z')}, \tag{19}$$

where c is the speed of light.

3.3 Baryon acoustic oscillations

The Baryon Acoustic Oscillations signature is an standard ruler useful to constrain cosmological model parameters and is the result of the interactions between baryons and photons

in the recombination era. We use 6 correlated points obtained by [32–34] and collected by [35]. We confront them with the cosmological models by building the χ^2 function as

$$\chi^2_{\text{BAO}} = \vec{X}^T C_{BAO}^{-1} \vec{X} \tag{20}$$

where $\vec{X}^T$ is the residual vector between observational measurements and theoretical values of the ratio $d_A(z_*)/D_V(z)$, where $d_A(z_d)$ is the comoving angular diameter distance at the the photon decoupling epoch (z_*) and the dilation scale is given by [36]

$$D_V(z) = \left[d_A^2(z) cz/H(z) \right]^{1/3} \tag{21}$$

where c is again the speed of light. For this work we use $z_* = 1089.80 \pm 0.21$ [3].

3.4 HII galaxies

A sample of 181 measurements coming from Hydrogen II galaxies (HIIG), with their luminosity dominated by young massive burst of star formation, is reported by [37,38]. This sample which covers a region $0.01 < z < 2.6$ and is useful to establish bounds over cosmological parameters due the correlation between the measured luminosity L of the galaxies and the inferred velocity dispersion σ of their ionized gas [39–42]. The χ^2-function is built as

$$\chi^2_{\text{HIIG}} = \sum_i^{181} \frac{[\mu_{th}(z_i, \Theta) - \mu_{obs}(z_i)]^2}{\epsilon_i^2}, \tag{22}$$

where ϵ_i is the observational uncertainty measured at z_i having 68% of confidence level. Additionally, the observational distance modulus (μ_{obs}) is expressed

$$\mu_{obs} = 2.5(\alpha + \beta \log \sigma - \log f - 40.08). \tag{23}$$

Here, α and β are the intercept and slope of the L-σ relation and f is the measured flux. The theoretical estimate is given as

$$\mu_{th}(z, \Theta) = 5 \log_{10} \left[\frac{d_L(z, \Theta)}{1 \, \text{Mpc}} \right] + 25, \tag{24}$$

where d_L is the luminosity distance measured in Mpc (see Eq. (19)).

3.5 Cosmic microwave background radiation

The CMB temperature anisotropies are useful measurements to establish constraints over cosmological parameters. A way to use them without performing a full perturbative analysis is to compress the full information into some parameters.

Fig. 1 2D contours at 1σ (inner region) and 3σ (outermost region) CL for the üΛCDM
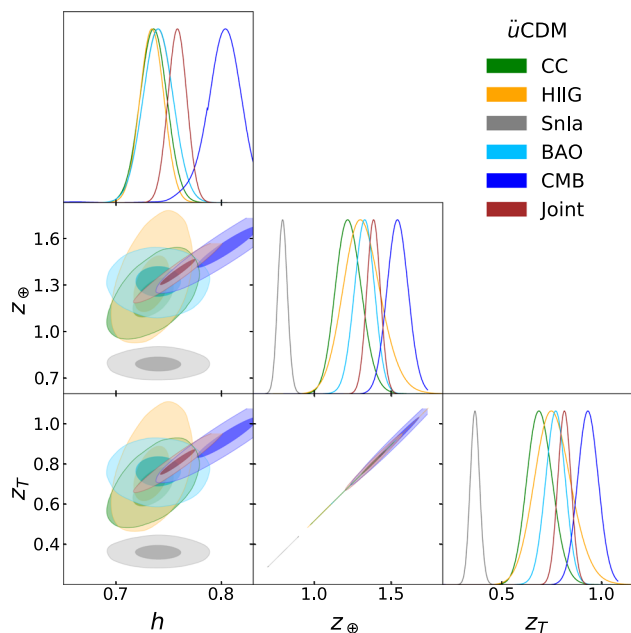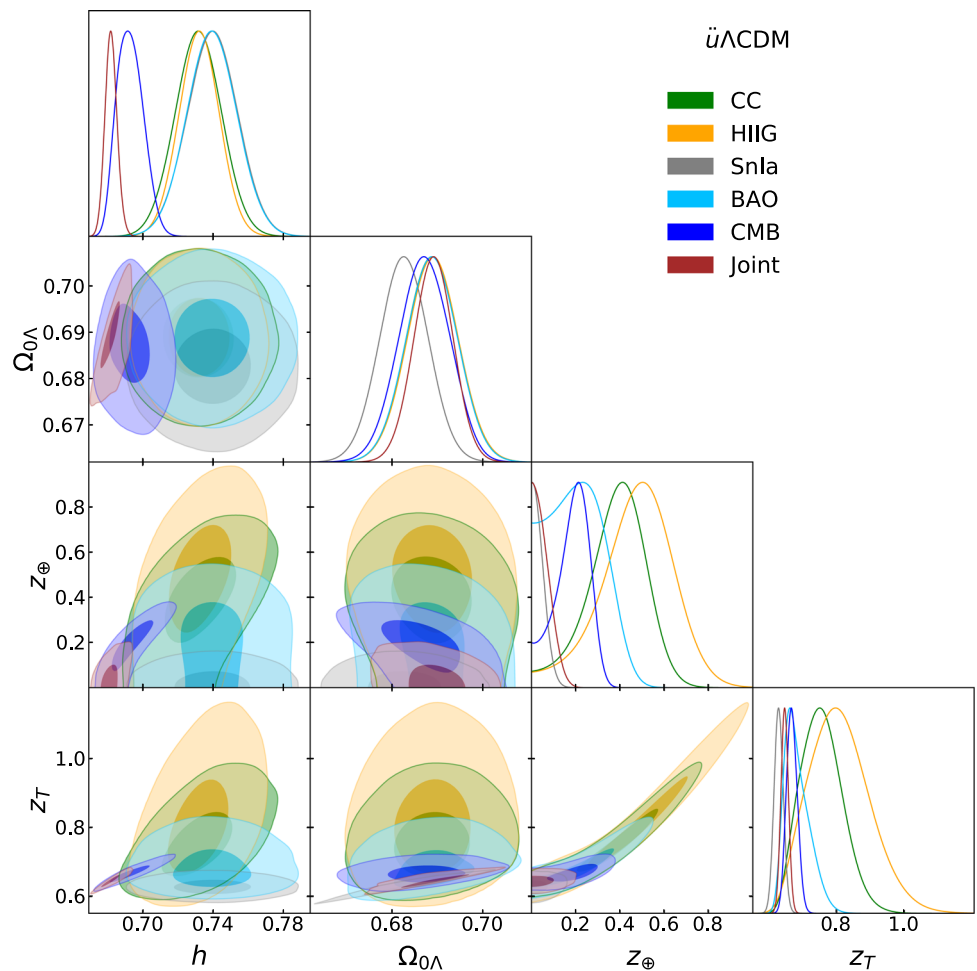
Fig. 2 2D contours at 1σ (inner region) and 3σ (outermost region) CL for the üCDM

Authors in [43] compress the CMB information from Planck 2018 Temperature Power Spectrum (TT), for high TE multipoles, polarization spectra EE modes + lowE data [3] in the acoustic scale l_A which characterizes the CMB temperature in the transverse direction, the shift parameter R which influences the CMB temperature along the line-of-sight direction, and the quantity $\Omega_{b0}h^2$ where Ω_{b0} is the density of baryons today ($z = 0$). Thus, the figure-of-merit is built as

$$\chi^2_{\text{CMB}} = V_{\text{CMB}} \text{Cov}^{-1}_{\text{CMB}} V^T_{\text{CMB}}, \tag{25}$$

where V_{CMB} is

$$V_{\text{CMB}} = \begin{pmatrix} R^{th} - 1.7493 \\ l_A^{th} - 301.462 \\ \Omega_b h^{2th} - 0.02239 \end{pmatrix}, \tag{26}$$

the superscripts th refers to the theoretical estimates, and $\text{Cov}^{-1}_{\text{CMB}}$ represents the inverse of

$$\text{Cov}_{\text{CMB}} = 10^{-8} \begin{pmatrix} 2162.25 & 19560.23 & -46.04 \\ 19560.23 & 801025.00 & -456.45 \\ -46.04 & -456.45 & 2.25 \end{pmatrix}, \tag{27}$$

Table 1 Bestfit values and their uncertainties at 1σ of the free parameters for both üΛCDM and üCDM models using a Gaussian prior on H_0 on the SH0ES value and Planck value respectively. Additionally, we show values for the AICc, BIC, ΔAICc $\equiv$ AICc $-$ AICc$^{\Lambda CDM}$ and ΔBIC $\equiv$ BIC $-$ BIC$^{\Lambda CDM}$ for both üΛCDM and üCDM models

Sample	χ^2_{min}	h	$\Omega_{0\Lambda}$	$z_\oplus$	z_T	AICc	ΔAICc	BIC	ΔBIC
üΛCDM + H_0 SH0ES									
CC	16.24	$0.732^{+0.013}_{-0.013}$	$0.689^{+0.006}_{-0.006}$	$0.395^{+0.114}_{-0.137}$	$0.752^{+0.064}_{-0.061}$	23.13	1.06	26.54	2.03
HIIG	436.64	$0.732^{+0.011}_{-0.011}$	$0.689^{+0.006}_{-0.006}$	$0.486^{+0.145}_{-0.167}$	$0.803^{+0.094}_{-0.086}$	442.78	2.06	452.24	5.19
SnIa	2011.90	$0.740^{+0.014}_{-0.014}$	$0.683^{+0.005}_{-0.005}$	$0.033^{+0.034}_{-0.023}$	$0.628^{+0.014}_{-0.013}$	2017.91	15.50	2034.22	20.94
BAO	2.81	$0.739^{+0.014}_{-0.014}$	$0.689^{+0.006}_{-0.006}$	$0.203^{+0.127}_{-0.133}$	$0.675^{+0.045}_{-0.029}$	32.81	19.99	7.64	1.60
CMB	15.37	$0.693^{+0.008}_{-0.007}$	$0.687^{+0.006}_{-0.006}$	$0.194^{+0.064}_{-0.090}$	$0.667^{+0.016}_{-0.015}$	27.37	3.57	21.61	0.05
Joint	2494.24	$0.682^{+0.003}_{-0.003}$	$0.689^{+0.004}_{-0.004}$	$0.046^{+0.047}_{-0.032}$	$0.646^{+0.011}_{-0.011}$	2500.25	7.00	2516.93	12.56
üΛCDM + H_0 Planck									
CC	15.09	$0.678^{+0.004}_{-0.004}$	$0.688^{+0.005}_{-0.006}$	$0.111^{+0.106}_{-0.077}$	$0.654^{+0.026}_{-0.018}$	21.98	3.02	25.39	3.99
HIIG	441.63	$0.679^{+0.004}_{-0.004}$	$0.689^{+0.006}_{-0.006}$	$0.160^{+0.146}_{-0.111}$	$0.665^{+0.046}_{-0.024}$	447.77	2.42	457.23	5.55
SnIa	2011.87	$0.676^{+0.004}_{-0.004}$	$0.683^{+0.005}_{-0.005}$	$0.032^{+0.034}_{-0.022}$	$0.628^{+0.013}_{-0.013}$	2017.88	15.48	2034.19	20.92
BAO	2.79	$0.677^{+0.004}_{-0.004}$	$0.689^{+0.006}_{-0.005}$	$0.211^{+0.124}_{-0.135}$	$0.677^{+0.046}_{-0.030}$	32.79	19.99	7.62	1.60
CMB	5.40	$0.679^{+0.003}_{-0.003}$	$0.685^{+0.004}_{-0.004}$	$0.055^{+0.051}_{-0.038}$	$0.636^{+0.009}_{-0.009}$	17.40	10.50	11.64	6.98
Joint	2473.66	$0.678^{+0.003}_{-0.003}$	$0.685^{+0.004}_{-0.004}$	$0.033^{+0.035}_{-0.023}$	$0.633^{+0.009}_{-0.009}$	2479.67	5.11	2496.35	10.67
üCDM + H_0 SH0ES									
CC	15.67	$0.735^{+0.013}_{-0.013}$	–	$1.220^{+0.087}_{-0.081}$	$0.686^{+0.066}_{-0.061}$	20.10	-1.97	22.54	-1.97
HIIG	436.10	$0.734^{+0.011}_{-0.011}$	–	$1.306^{+0.125}_{-0.114}$	$0.751^{+0.095}_{-0.087}$	440.17	-0.55	446.50	-0.55
SnIa	2017.91	$0.740^{+0.014}_{-0.014}$	–	$0.792^{+0.031}_{-0.030}$	$0.361^{+0.023}_{-0.023}$	2021.92	19.51	2032.79	19.51
BAO	2.59	$0.740^{+0.014}_{-0.014}$	–	$1.323^{+0.064}_{-0.066}$	$0.765^{+0.049}_{-0.050}$	12.59	-0.23	5.81	-0.23
CMB	26.05	$0.803^{+0.007}_{-0.007}$	–	$1.535^{+0.023}_{-0.024}$	$0.925^{+0.018}_{-0.019}$	32.45	8.65	30.21	8.65
Joint	2748.84	$0.758^{+0.006}_{-0.006}$	–	$1.382^{+0.020}_{-0.021}$	$0.809^{+0.015}_{-0.016}$	2752.85	259.60	2763.97	259.60
üCDM + H_0 Planck									
CC	16.81	$0.677^{+0.004}_{-0.004}$	–	$1.022^{+0.067}_{-0.063}$	$0.536^{+0.051}_{-0.048}$	21.24	2.28	23.68	2.28
HIIG	442.25	$0.679^{+0.004}_{-0.004}$	–	$1.062^{+0.100}_{-0.094}$	$0.566^{+0.076}_{-0.072}$	446.32	0.97	452.65	0.97
SnIa	2017.90	$0.677^{+0.004}_{-0.004}$	–	$0.791^{+0.031}_{-0.031}$	$0.361^{+0.024}_{-0.023}$	2021.91	19.51	2032.78	19.51
BAO	2.59	$0.677^{+0.004}_{-0.004}$	–	$1.324^{+0.064}_{-0.065}$	$0.766^{+0.049}_{-0.050}$	12.59	-0.21	5.81	-0.21
CMB	271.73	$0.709^{+0.004}_{-0.004}$	–	$1.219^{+0.013}_{-0.014}$	$0.685^{+0.010}_{-0.010}$	278.13	271.23	275.89	271.23
Joint	2866.96	$0.701^{+0.004}_{-0.004}$	–	$1.191^{+0.012}_{-0.013}$	$0.665^{+0.009}_{-0.010}$	2870.97	396.41	2882.09	396.41

which is the covariance matrix for V_{CMB}. The theoretical counterparts are estimated by

$$l_A^{th} = (1 + z_*)\pi \frac{D_A(z_*)}{r_s(z_*)}, \tag{28}$$

and

$$R^{th}(z_*) = \frac{(1 + z_*)D_A(z_*)\sqrt{\Omega_{0m}H_0^2}}{c}, \tag{29}$$

where z_* is the redshift at the photon decoupling phase that takes the form

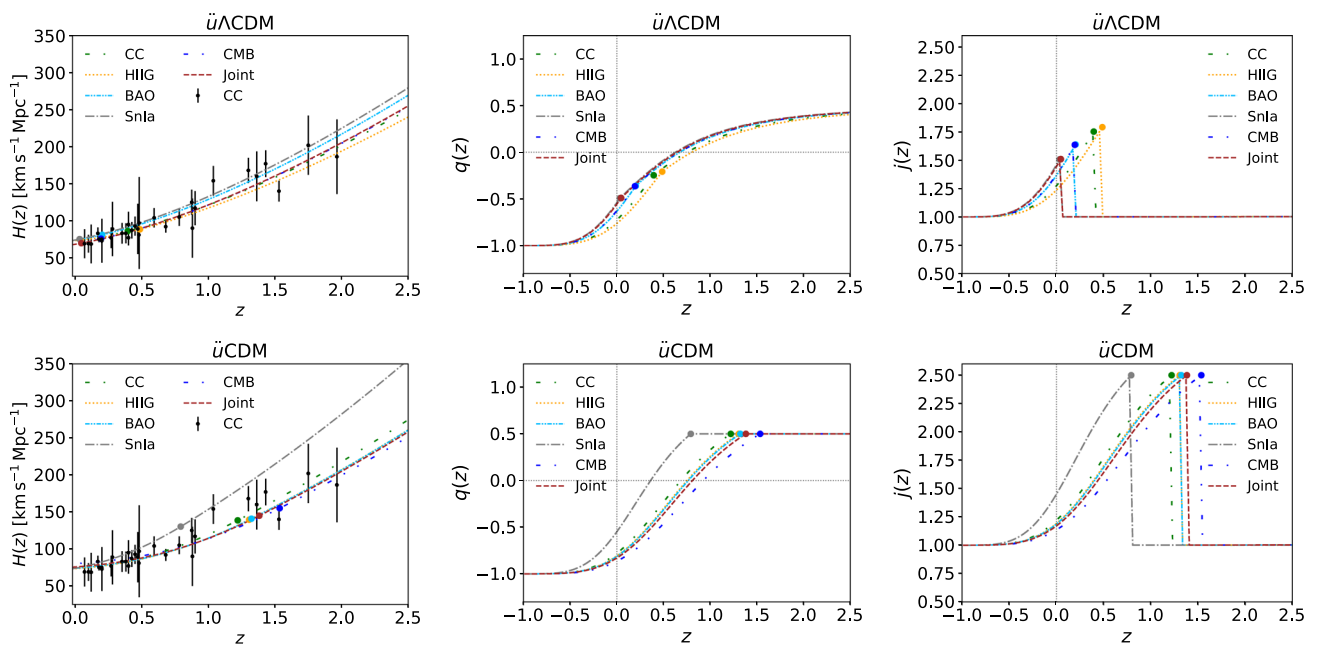$$z_* = 10488[1 + 0.00124(\Omega_b h^2)^{-0.738}][1 + g_1(\Omega_m h^2)^{g_2}], \tag{30}$$

where

$$g_1 = \frac{0.0738(\Omega_b h^2)^{-0.238}}{1 + 39.5(\Omega_b h^2)^{0.763}}, \tag{31}$$

$$g_2 = \frac{0.560}{1 + 21.1(\Omega_b h^2)^{1.81}}, \tag{32}$$

and D_A is the angular diameter distance for a flat geometry that reads as

$$D_A(z) = \frac{c}{H_0(z + 1)} \int_0^z \frac{dz'}{E(z')}, \tag{33}$$

Fig. 3 Reconstruction of $H(z)$ (left panel), $q(z)$ (middle panel) and $j(z)$ (right panel) for üΛCDM and üCDM, for each dataset. Circle marker represents the best value of $z_\oplus$ for each sample

while r_s is the comoving sound horizon given by

$$r_s(z) = \frac{c}{H_0} \int_0^{1/(z+1)} \frac{da}{a^2 E(a)\sqrt{3\left(1 + \frac{3\Omega_b h^2}{4\Omega_\gamma h^2}a\right)}}, \quad (34)$$

being Ω_b and Ω_γ the baryons and photons density parameters respectively [44].

4 Results

We use the über gravity in the context of üΛCDM and üCDM models to understand the nature of the current Universe acceleration and help us to interpret the origin of the H_0 tension. Due to the consistency between the results obtained by using H_0 Gaussian priors from Planck and Shoes, hereafter we centered our discussion in the case of H_0 with priors coming from low-redshift (see Table 1). In Figs. 1 and 2 we present the 2D parameter likelihood contours at 68% (1σ) and 99.7% (3σ) confidence level (CL) respectively for both models. Moreover Table 1 shows the mean values of the parameters and their uncertainties at 1σ.

Additionally to this, we statistically compare these results with ΛCDM model, applying the corrected Akaike information criterion (AICc) [45] and the Bayesian information criterion (BIC) [46]. Both criteria penalize according to the size of the data sample (N) and the number of degrees of freedom (k) defined as $\text{AICc} = \chi^2_{\min} + 2k + 2(k^2 + k)/(N - k - 1)$ and $\text{BIC} = \chi^2_{\min} + k \log(N)$, respectively, where $\chi^2_{\min}$ is the

minimum of χ^2. Therefore, a model with a lower value of AICc (BIC) is preferred by the data.

For the ΔAICc we have the following conditions: if $|\Delta\text{AICc}| < 4$ both models are statistically equivalent, if $4 < \text{AICc} < 10$ the data still support the given model (über gravity) but less than the preferred one (concordance model) and finally if $\text{AICc} > 10$ indicates that the data do not support the given model. In addition, for ΔBIC the interpretation is as follows: if $\Delta\text{BIC} < 2$ there is no evidence against the new model which in this case is über gravity, if $2 < \Delta\text{BIC} < 6$ there is modest evidence against the new model, and finally if $\Delta\text{BIC} > 10$, gives the strongest evidence against it.

We start by showing the AICc and BIC for the ΛCDM model: for AICc values we have 22.07 (CC), 440.72 (HIIG), 2002.41 (SnIa), 12.82 (BAO), 23.80 (CMB) and 2493.25 (Joint); as for BIC it has 24.51 (CC), 447.05 (HIIG), 2013.28 (SnIa), 6.04 (BAO), 21.56 (CMB) and 2504.37 (Joint). Table 1 shows both ΔAICc and ΔBIC by considering ΛCDM as the reference scenario. According to AICc, we find that both üΛCDM and ΛCDM are preferred equally by CC, HIIG and CMB while the üΛCDM is not supported by SnIa and BAO, but there is a still support for the über gravity when the combined data are considered. Regarding BIC, we find the strongest evidence against üΛCDM for SnIa and the combined data and a modest evidence against the uber gravity for HIIG.

As for üCDM, we observe that üCDM and ΛCDM are equally preferred by CC, HIIG and BAO while the strongest evidence against üCDM is provided by SnIa and Joint analysis, and a strong evidence against über gravity is given by

Fig. 4 The $\mathbb{H}0(z)$ diagnostic for über gravity, showing üCDM and üΛCDM models. Here we present the behavior of $\mathbb{H}0(z)$ under the constraints of the four data-samples and the joint analysis. The continuous red line represents the results for ΛCDM cosmology assuming $h = 0.6766$ and $\Omega_{0m} = 0.3111$ according to [3]
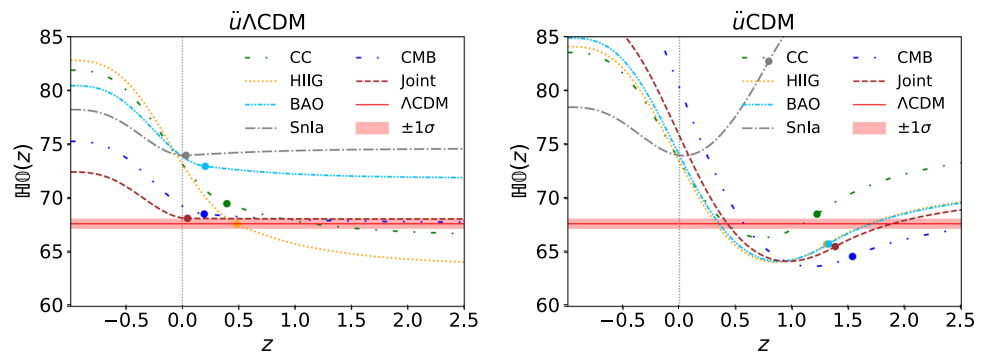
CMB. It is worth to mention that ΔAICc and ΔBIC are reduced to $\Delta\chi^2$ because both models have same number of free parameters.

On the other hand, notice that the üΛCDM scenario produces a late acceleration, constraining the free parameter of über gravity according to the joint analysis as $z_\oplus \simeq 0.046$, which also coincides with the epoch of transition to über gravity. The parameter $\Omega_{0\Lambda} \simeq 0.689$ marks a cosmological constant domination, however the unsolved questions about its characteristics remains. The redshift transition is compatible with ΛCDM model which is at 0.646, as estimated from the joint analysis. The transition from GR to über gravity is perfectly observed through the jerk factor shown in Fig. 3, where a discontinuity in the $j(z)$ function is observed. According to the $\mathbb{H}_0$ diagnostic presented in Fig. 4, the tension in H_0 is not alleviated because it keeps the trend towards values consistent with supernovae results.

Additionally, we explore the üCDM model which is not studied in the literature. In this case, the über component $z_\oplus$ acts like a cosmological constant, having the same free parameters as ΛCDM. For the joint analysis we conclude that the value of the über parameter us $z_\oplus \simeq 1.382$ and the transition to an accelerated Universe happens earlier in its evolution $z_T \simeq 0.809$. From Fig. 3, in particular with $q(z)$ and $j(z)$, we observe that the transition to über gravity domination is more extreme in comparison with the üΛCDM model. Additionally to this, in this figure we also observe a trend to values that are consistent with supernova results instead of those obtained from Planck, concluding that the H_0 tension persist despite the über gravity presence.

5 Conclusions and outlooks

This paper presents an exhaustive revision of the über gravity for the üΛCDM and üCDM models, obtaining stringent constrictions with different cosmological data samples, in particular the recent Pantheon+ sample. First, we focus on our results obtained assuming a prior on h provided by SHOES because these are consistent with those we obtain using a prior from Planck (see Table 1). We statistically compare both models with ΛCDM by using AICc and BIC. We find that the

über cosmologies and ΛCDM are equally preferred for CC, HIIG, and BAO. When CMB data are used, üΛCDM does not present evidence against but üCDM presents a strong evidence against. However, for both über cosmologies, üΛCDM and üCDM, the joint analysis shows the strongest evidence against them.

First, the üΛCDM contains one extra free parameter in comparison with the standard paradigm, maintaining the same open questions related to the understanding of the cosmological constant. Our results for the joint analysis point out that $\Omega_{0\Lambda} \simeq 0.689$, consistent with the standard cosmological model, but with a subdominant value for the über parameter $z_\oplus \simeq 0.046$. It is important to remind that the über parameter $z_\oplus$ is the point where the über gravity starts to dominate over the standard GR, thus, the presence of the ΛCDM delays the apparition of the über gravity as observed in Fig. 3 for the $H(z)$, $q(z)$ and $j(z)$ parameters. The transition redshift is also in consistency with the standard cosmological model (see Table 1). Regarding $\mathbb{H}0(z)$ diagnostic, Fig. 4 reveals that, according to the joint analysis, the H_0 value is more consistent with the supernova data than with the Planck ones at $z = 0$, unable to reduce the tension under this scenario. This could be the result of ΛCDM being behind the dynamics while über gravity start its domination at $z_\oplus \simeq 0.046$. Additionally to this, the new sample of Pantheon+ could generate a tendency to a value greater than the one predicted by Planck.

Furthermore, we explore the üCDM, where the über parameter plays the role of the cosmological constant, thus having the same parameters as in the standard cosmology. In this case, über parameter generates the late acceleration of the Universe and its presence start earlier, specifically at $z_\oplus \simeq 1.382$ according to the joint analysis presented in Table 1. The transition also happens earlier than in the standard model, at $z_T \simeq 0.809$ (see Table 1). The evolution of $H(z)$, $q(z)$ and $j(z)$ is presented in Fig. 3 where we can see that the behavior is more abrupt, mainly in the $q(z)$ and $j(z)$ behavior. However, über gravity eventually mimics the cosmological constant because $j = 1$ when $z = -1$. The $\mathbb{H}0(z)$ diagnostic plot presented in Fig. 4 shows that at $z = 0$ the value for H_0 does not coincide with the Planck result for ΛCDM, being in concordance with supernova results, which is also a sign

that the über parameter acting like a cosmological constant is inadequate to alleviate the tension in H_0.

Finally, it is worth to point out the difference in the values of $z_\oplus$ for üCDM and üΛCDM. The discrepancy is caused mainly because in the first one the über parameter acts like a cosmological constant and not only as a transition to über gravity, which is the case for üΛCDM.

In summary, we observe that the über gravity is an alternative to study the late Universe acceleration by mimicking the cosmological constant behavior through the über parameter $z_\oplus$. However, the mystery of the Hubble tension remains unsolvable and gives values that are compatibles with supernova results. Nevertheless, we consider that a deeper exploration of the über Lagrangian for other values of n is necessary in order to elucidate any extra dynamics that could help us understand the reason for the Hubble tension. Such exploration will be presented elsewhere.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: The data used in the manuscript are referenced in section 3, "Data and methodology".]

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