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The dynamics of three-forms in thick branes

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Abstract: In this work, we investigate thick brane models with a single three-form field. We find novel solutions for thick braneworlds where only three-forms exist and interact gravitationally in the bulk, both with and without matter fields. We use an additional scalar field as proxy for the matter fields. As an initial study, we consider the results here in contrast to the single scalar field thick braneworld case. The properties of the specific three-form parameterisation limits the freedom we have to choose the form of the warp factor, leading to a closed system of equations with nontrivial yet unstable solutions. The stability of the gravitational sector for thick brane three-forms is investigated and the models are shown to be unstable against small perturbations of the metric, further indicating that three-forms cannot exist stably in thick braneworld settings.

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1 Introduction

An important problem in cosmology is the question "what provided the initial seeds for structure formation?". The theory of cosmological perturbations (for a thorough reference, see [1]) aims to describe the evolution of the universe *given* some collection of structures, but it does not account for the cause of the primordial density fluctuations. Inflation, coupled with quantum fluctuations (see [2]), is currently the most promising candidate. Inflation is attractive for its simplicity — we only need to add in a scalar field — and its ability to directly solve open theoretical issues with the big bang model. A scalar field is not the only possibility as the dynamical object driving inflation. Given that the empirical status of inflation is at present inconclusive (see [3] for an overview of the observational status and prospects), several possible models are discussed in the literature and have not yet been ruled out. Scalars are favoured for their simplicity, but there are no experimental reasons to dismiss alternatives.

In recent years, one such alternative is three-forms (for a subset of work on three-form inflation, see [4–11]). Moreover, three-forms have been considered as dark energy candidates, similarly to scalar fields [6, 12]. Three-forms have also been studied in other contexts, namely spherical objects (black holes, stars, and wormholes [13–17]), singularities in general relativity [18, 19], for use in screened cosmologies [20], and for studies on alternatives to FLRW spacetimes [21]. More infomation about the general three-form formalism can be found in [22–24].

A three-form is a three-indexed tensor field, A_{ABC}. Their inclusion as an alternative field is not without precedent, as they occur naturally in supersymmetry and string theory models (an overview can be found in [25]). They produce distinct signatures for all the aforementioned effects (inflation, dark energy, etc.). Although various inflationary mechanisms have been considered (for example, vector inflation [26]), we choose to pay three-forms particular attention because vectors and other two-form fields (e.g. Kalb-Ramond field) have been more considerably studied than three-forms (although not as thoroughly as scalars). Additionally, forms of four or higher can be shown to be equivalent to scalar fields [27]. Given their origin in string theoretic models, it is natural to study their dynamics in dimensions higher than 4. We could perform the analysis in the full string theoretic 11 dimensions, but it is useful to have some means of comparison to scalars. For that reason, we will study three-forms in branes (i.e. 5D).

Several studies have been done in branes, typically involving additional degrees of freedom in the gravitational theory (like scalar fields) [28–30] or some modification of the theory itself ($f(R)$, conformal gravity, see [31] and references in their introduction). An overview of scalars and thick brane solutions can be found in [32]. Few studies, however, have been done on three-forms in branes. There is an analysis of three-form cosmological solutions in the Randall-Sundrum II braneworld scenario [4]. This is an example of a thin-brane model [32]. To the authors' knowledge, there is no study of three-forms in thick braneworlds. This paper aims to address this knowledge gap.

We construct a model of a three-form in a warped flat thick brane, find the three-form analogue of the Klein-Gordon (3KG) equation of motion, the Einstein field equations (EFEs), analyse their solutions and their stability against metric perturbations, and do so for the cases with a matter source and without. Our analysis reveals crucial differences between three-forms and scalars in thick branes: we discuss extra terms in the equations of motion, degeneracy in the EFEs, and most saliently, instabilities. The classical thick brane background is not stable against linear perturbations when three-forms inhabit the bulk — this is in contrast to most scalar field models. Unlike the scalar field case, the overall system of equations governing three-form dynamics are not under-determined. They must be solved subject only to a choice of three-form dual parameterisation — there is no freedom to fix the warp factor. We find that the solutions reflect the instabilities found in the perturbative analysis. It appears as though, without some other modification or high degree of fine-tuning, three-forms are not stable in thick braneworlds. One caveat to the question of three-form stability is the choice of dual parameterisation.

We elaborate on the above and the possible implications thereof in the remainder of the paper. In section 2 we construct the model, section 3 contains the analysis of the background stability via perturbations and we solve the dynamical system in section 4. Results are discussed in section 5. Supplemental discussion of stability is included in the appendix, via an application of the formalism developed in [30] for a general q-form.

2 Thick brane model for three-forms

Throughout this work we use notation where (capital) Latin indices run through 5D, i.e. $B \in \{0, 1, 2, 3, 4\}$ and Greek indices run the standard 4D coordinates, i.e. $\mu \in \{0, 1, 2, 3\}$. The square of a tensor denotes contraction of all the indices, i.e. $A^2 = A_{ABC}A^{ABC}$, and a circle denotes contraction of all but the first index, i.e. $(A \circ A)_{AB} = A_A{}^{CD}A_{BCD}$. We work in units such that $c = 1$. Additionally, we use $\mathcal{W}$ to denote the warp function, and the three-form potential is denoted by $\mathbf{A}$. Boldface quantities refer to 5D vectors.

2.1 Three-form action

We consider the following action for a three-form field A_{ABC} minimally coupled to Einstein gravity in a 5D spacetime

$$S = \int d^4x dy \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{48} F^2 - V(A^2) \right], \tag{2.1}$$

where y identifies the coordinate of the fifth dimension (or 'bulk'), $g = \det g_{AB}$ is the determinant of the metric, R is the standard curvature scalar, $\kappa^2 = 8\pi G_5$ with G_5 being the 5D Newton's constant, $V(A^2)$ is the three-form self interacting potential and $\mathbf{F} = \mathrm{d}\mathbf{A}$ is the strength tensor of the three-form, with components,

$$F_{ABCD} = 4\nabla_{[A}A_{BCD]} \,. \tag{2.2}$$

In a torsion-free spacetime the covariant derivatives in the above equation can be replaced with ordinary partial ones, due to the symmetric nature of the connection [33]. In this case, the strength tensor plays the same role for the three-form theory as the kinetic field strength $\partial_\mu \phi$ does for standard scalar field theory or as $F_{\mu\nu}$ for classical Maxwell's electromagnetism, corresponding to a zero (ϕ) and one-form (A_μ) respectively. This three-form will naturally equip the field A_{ABC} with dynamics, depending on the choice of metric g_{AB}.

2.2 Metric and equations of motion

In this paper we will consider a conformally flat [34] 5D brane spacetime, with line element,

$$ds^2 = e^{2\mathcal{W}(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \,, \tag{2.3}$$

where $\eta_{\mu\nu}$ is the Minkowski 4D metric, with signature $(-,+,+,+)$, and the prefactor $e^{2\mathcal{W}(y)}$ is the so-called "warp factor" with $\mathcal{W}(y)$ being the "warp function".

In 5D the three-form dual, $\star A_{ABC} = B_{AB}$, is a two-form, B_{AB}, with components

$$B^{AB} = (\star A)^{AB} = \frac{1}{3!} \frac{1}{\sqrt{-g}} \epsilon^{ABCDE} A_{CDE} \,, \tag{2.4}$$

where here ϵ denotes the 5D Levi-Civita symbol [4]. We may now introduce a scalar function $\chi(y)$ that parametrizes B_{AB} and depends only on the fifth dimension, y. Using eq. (2.3), the dual (2.4) has the following non-vanishing components:

$$B_{0y} = -B_{y0} = e^{\mathcal{W}(y)} \chi(y) \,. \tag{2.5}$$

This is a fixed, free choice. This antisymmetric ansatz for the dual vector greatly simplifies the equations and allows the three-form components to be completely determined by the field $\chi(y)$. Inverting (2.4) gives

$$A_{ABC} = \sqrt{-g}\, \epsilon_{ABCDE} B^{DE} \,, \tag{2.6}$$

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which has the non-zero components:

$$A_{123} = A_{231} = A_{312} = -A_{132} = -A_{321} = -A_{213} = 2e^{3\mathcal{W}}\chi\,. \tag{2.7}$$

Finally, we can calculate the invariants:

$$A^2 = 24\chi^2\,, \tag{2.8}$$
$$F^2 = 96\left(\chi' + 3\mathcal{W}'\chi\right)^2\,, \tag{2.9}$$

where a prime denotes a derivative with respect to y, i.e. $\chi' = d\chi/dy$.

With the three-form invariants (2.8) and the brane metric (2.3) we are now ready to calculate the equations of motion using eq. (2.1). Varying the total action (2.1) with respect to the three-form yields the following equations of motion,

$$\nabla_A F^A{}_{BCD} = 12\frac{dV}{dA^2}A_{BCD}\,. \tag{2.10}$$

Due to the antisymmetric nature of the $\mathbf{A}$ field, $\mathbf{F}$ is a closed differential form, i.e. $\mathbf{dF} = 0$.

Substituting the metric (2.3) into eq. (2.10) one can express the equations of motion in terms of the χ field as

$$\chi'' + 4\mathcal{W}'\chi' + 3\chi\left(\mathcal{W}'' + \mathcal{W}'^2\right) - \frac{1}{4}\frac{dV}{d\chi} = 0\,, \tag{2.11}$$

where $V_\chi = dV/d\chi$. This is, in essence, the equation of motion for the three-form, via the parameterisation B_{AB} viz. its dual, A_{ABC}. The Einstein field equations are computed through the variation of eq. (2.1) with respect to the metric g^{AB}. They are

$$G_{AB} = \kappa^2 T_{AB}\,, \tag{2.12}$$

where G_{AB} is the standard Einstein tensor and the stress-energy tensor is sourced entirely by the three-form,

$$T_{AB} = \frac{1}{6}\left(F\circ F\right)_{AB} + 6\frac{dV}{dA^2}\left(A\circ A\right)_{AB} + g_{AB}\mathcal{L}_{3\mathrm{f}}\,, \tag{2.13}$$

where from eq. (2.1) we identify the three-form Lagrangian density as,

$$\mathcal{L}_{3\mathrm{f}} = -\frac{1}{48}F^2 - V(A^2)\,. \tag{2.14}$$

With our brane metric (2.3) the components of T_{AB} are:

$$T^0{}_0 = -V - \frac{1}{48}F^2\,, \tag{2.15}$$
$$T^1{}_1 = T^2{}_2 = T^3{}_3 = -T^0{}_0 - 2V + \chi V_\chi\,, \tag{2.16}$$
$$T^y{}_y = -T^0_0 - 2V\,, \tag{2.17}$$

with trace:

$$T = g^{AB}T_{AB} = 6\left(\chi' + 3\mathcal{W}'\chi\right)^2 + 3\chi V_\chi - 5V\,, \tag{2.18}$$

and the components of the Einstein tensor:

$$G^0{}_0 = 3\left(\mathcal{W}'' + 2\mathcal{W}'^2\right), \tag{2.19}$$

$$G^1{}_1 = G^2{}_2 = G^3{}_3 = G^0{}_0\,, \tag{2.20}$$

$$G^y{}_y = 6\mathcal{W}'^2. \tag{2.21}$$

We will now set $\kappa^2 = 1$ (cf. [31] or [29], who set it to 2). Our full system of equations, determining the dynamics of both the metric and three-form, is

$$\chi'' + 4\mathcal{W}'\chi' + 3\chi\left(\mathcal{W}'' + \mathcal{W}'^2\right) - \frac{1}{4}\frac{dV}{d\chi} = 0\,, \tag{2.22}$$

$$\begin{aligned} 3\left(\mathcal{W}'' + 2\mathcal{W}'^2\right) &= -V - \frac{1}{48}F^2\,, \\ 3\left(\mathcal{W}'' + 2\mathcal{W}'^2\right) &= \frac{1}{48}F^2 - V + \chi V_\chi\,, \\ 6\mathcal{W}'^2 &= \frac{1}{48}F^2 - V. \end{aligned} \tag{2.23}$$

2.3 Comparison with the scalar field case

It is interesting to compare this system of equations (2.22), (2.23) to the scalar field case. For a minimally coupled scalar field φ in our metric, the full system reads [29, 35]

$$\begin{aligned} 3\mathcal{W}'' + 6\mathcal{W}'^2 &= -\left[\frac{1}{2}\varphi'^2 + V(\varphi)\right], \\ 6\mathcal{W}'^2 &= \left[\frac{1}{2}\varphi'^2 - V(\varphi)\right], \\ \varphi'' + 4\mathcal{W}'\varphi' &= \frac{dV(\varphi)}{d\varphi}. \end{aligned} \tag{2.24}$$

The Einstein tensor is the same, but the stress-energy tensor of the scalar field has better degrees of symmetry: its 4D components are identical, so there is only one EFE. The 00 stress-energy component is the same as the three-form and is simply the Lagrangian of the field in question.

The differences between the three-form and scalar occur at the spatial (3D) level: the three-form's stress-energy tensor is different to the scalar field one, containing not only an opposite kinetic sign but a potential derivative. Finally, the yy component is the same as the three-form case. The true EFE difference manifests in the spatial 3D behaviour, not in the bulk. This is reflective of the nonzero components for the three-form in our parameterisation, eq. (2.7).

The equation of motion is also different: there's an additional $3\chi\left(\mathcal{W}'' + \mathcal{W}'^2\right)$ term and some scaling of the potential derivative for the three-form. This however is not only generically expected for a tensor field as compared to a scalar, but additionally the form that the three-form "Klein-Gordon" equation takes is highly dependant on the choice of parameterisation. This is not the case for the EFEs. This is crucial since the total system for the three-form, unlike the scalar field, is not overdetermined. Therefore, one cannot derive one equation in the system (2.22), (2.23) from the other three, one does have this liberty in

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eq. (2.24), however. This fact is exploited in typical scalar field studies to fix a known warp function and obtain closed solutions ([32]; see, for example, [31]). We can reduce our system to three independent equations in three unknown variables: χ, V, $\mathcal{W}$. Thus the system is not underdetermined and we are not free to impose any further choices.

3 Stability of the graviton through perturbations

In braneworld models, one must examine the stability of the zero-order modes against perturbations. That is to say, for a metric perturbation $h_{\mu\nu}$, obeying a Schrodinger-like equation of motion, does the Sturm-Liouville operator admit negative energy states? If so, the classical background is not stable [36]. We follow [36, 37]. The perturbed metric is only in 4D, the bulk remains unchanged. The metric is therefore

$$ds^2 = e^{2\mathcal{W}(y)}\left[\eta_{\mu\nu} + h_{\mu\nu}(x,y)\right]dx^\mu dx^\nu + dy^2. \tag{3.1}$$

The perturbed EFEs will therefore be

$$\delta R_{AB} = \delta T_{AB} - \frac{1}{3}\delta g_{AB}\delta T^C_C. \tag{3.2}$$

The perturbed Ricci tensor is the same for any minimally coupled braneworld model (here, we need only the 4D component, since it is this that enables us to analyse stability):

$$\begin{aligned}\delta R_{\mu\nu} = {}& e^{2\mathcal{W}}\left(\frac{1}{2}\partial_y^2 + 2\mathcal{W}'\partial_y + \mathcal{W}'' + 4\mathcal{W}'^2\right)h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}e^{2\mathcal{W}}\mathcal{W}'\partial_y\left(\eta^{\alpha\beta}h_{\alpha\beta}\right) - \frac{1}{2}\Box h_{\mu\nu}\\ &- \frac{1}{2}\eta^{\alpha\beta}\left(\partial_\mu\partial_\nu h_{\alpha\beta} - \partial_\mu\partial_\alpha h_{\nu\beta} - \partial_\nu\partial_\alpha h_{\mu\beta}\right)\end{aligned} \tag{3.3}$$

where $\Box$ is w.r.t. to $g_{\mu\nu}$, not g_{AB}. The transverse-traceless gauge [38] eliminates the second and fourth term, leaving

$$e^{2\mathcal{W}}\left(\frac{1}{2}\partial_y^2 + 2\mathcal{W}'\partial_y + \mathcal{W}'' + 4\mathcal{W}'^2\right)h_{\mu\nu} - \frac{1}{2}\Box h_{\mu\nu} = \delta T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}\delta T^C_C\,, \tag{3.4}$$

where the trace is with respect to all indices. The key calculation is the r.h.s.

For the scalar field case, we use the following change of variables

$$dz = e^{-\mathcal{W}(y)}dy; \qquad h_{\mu\nu} = e^{-ip\cdot x}e^{-3\mathcal{W}(z)/2}\bar{h}_{\mu\nu}(z). \tag{3.5}$$

we are able to eliminate $\mathcal{W}'' + 4\mathcal{W}'$, giving us the following wave equation for $h_{\mu\nu}$ [31]

$$\left[-\frac{d^2}{dz^2} + u(z)\right]\bar{h}_{\mu\nu}(z) = p^2\bar{h}_{\mu\nu}(z) \tag{3.6}$$

where the potential is

$$u(z) = \frac{9}{4}W'(z)^2 + \frac{3}{2}W''(z) \tag{3.7}$$

where W is a function of z. This operator $-\partial_z^2 + u(z)$ is bounded from below, and the lowest mode $p^2 = 0$ is normalisable. The crucial step is being able to eliminate $\mathcal{W}'' + 4\mathcal{W}'$ in eq. (3.3). If this is not possible, the differential operator changes.

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We now return to the r.h.s. for our three-form $S_{\mu\nu} = \delta T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}\delta T^C_C$,

$$\begin{aligned} S_{\mu\nu} = &\frac{1}{6}\delta\left(F\circ F\right)_{\mu\nu} + 6\delta\frac{dV}{dA^2}\left(A\circ A\right)_{\mu\nu} + 6\frac{dV}{dA^2}\delta\left(A\circ A\right)_{\mu\nu} + \eta_{\mu\nu}\delta\mathcal{L}_{3f} \\ &+ e^{2\mathcal{W}}h_{\mu\nu}\delta\mathcal{L}^0_{3f} - \frac{1}{3} + e^{2\mathcal{W}}h_{\mu\nu}T. \end{aligned} \tag{3.8}$$

Explicitly, the nonzero terms are given by

$$\left(A\circ A\right)_{\mu\nu} = \left(A\circ A\right)_{ii} = 8e^{2\mathcal{W}}\chi^2. \tag{3.9}$$

$$\left(F\circ F\right)_{\mu\nu} = \left(F\circ F\right)_{ii} = \left(F\circ F\right)_{yy} = 24e^{2\mathcal{W}}\left(3\chi\mathcal{W}' + \chi'\right)^2 \tag{3.10}$$

where $i = \{1,2,3\}$. The three-form field has vector and scalar perturbations. But in our case we only have (cf. eq. (2.7)) perturbations [12]

$$A_{ijk} = \epsilon_{ijk}\left(\chi_0 + \delta\chi\right). \tag{3.11}$$

As such, the terms $\delta\left(F\circ F\right)_{\mu\nu}$ and $\delta\left(A\circ A\right)_{\mu\nu}$ will contain several perturbed terms, but none proportional to $h_{\mu\nu}$. To linear order then,

$$\begin{aligned} \delta\left(A\circ A\right)_{\mu\nu} &= 8e^{2\mathcal{W}}(2\chi_0\delta\chi) \\ \delta\left(F\circ F\right)_{\mu\nu} &= 48e^{2\mathcal{W}}\left(3\delta\chi\mathcal{W}' + \delta\chi'\right)^2. \end{aligned} \tag{3.12}$$

This is similarly true for $\eta_{\mu\nu}\delta\mathcal{L}_{3f}$, since $\mathcal{L}_{3f}$ contains only F^2 and V terms. The terms of concern are therefore the last two in the source term (3.8),

$$S_{\mu\nu} \propto e^{2\mathcal{W}}h_{\mu\nu}\left(-\frac{1}{48}F^2 - V(A^2)\right)^{(0)} - \frac{1}{3}e^{2\mathcal{W}}h_{\mu\nu}T^{(0)}. \tag{3.13}$$

The term we require, in order to cancel out the term we want, is from eq. (2.23)

$$\mathcal{W}'' + 4\mathcal{W}' = \frac{1}{144}F^2 - \frac{V}{3}. \tag{3.14}$$

Thus we have,

$$S_{\mu\nu} \propto \frac{2}{3}e^{2\mathcal{W}}h_{\mu\nu}V. \tag{3.15}$$

This does not equal the ideal term, and as such does not cancel. The equation of motion for $h_{\mu\nu}$ is therefore

$$e^{2\mathcal{W}}\left(\frac{1}{2}\partial_y^2 + 2\mathcal{W}'\partial_y + \mathcal{W}'' + 4\mathcal{W}'^2 - \frac{2}{3}V\right)h_{\mu\nu} - \frac{1}{2}\Box h_{\mu\nu} = \delta\tilde{T}_{\mu\nu}, \tag{3.16}$$

where

$$\delta\tilde{T}_{\mu\nu} = \frac{1}{6}\delta\left(F\circ F\right)_{\mu\nu} + 6\delta\frac{dV}{dA^2}\left(A\circ A\right)_{\mu\nu} + 6\frac{dV}{dA^2}\delta\left(A\circ A\right)_{\mu\nu} + \eta_{\mu\nu}\delta\mathcal{L}_{3f} \tag{3.17}$$

rearrangement yields

$$\left(\partial_y^2 + 4\mathcal{W}'\partial_y + 2\mathcal{W}'' + 8\mathcal{W}'^2 - \frac{4}{3}V - e^{-2\mathcal{W}}\Box\right)h_{\mu\nu} = 2e^{-2\mathcal{W}}\delta\tilde{T}_{\mu\nu}. \tag{3.18}$$

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This a sourced wave equation. Pertinent to our analysis is the form of the differential operator after the change of variable, $dz = e^{-\mathcal{W}(y)}dy$ and

$$h_{\mu\nu} = e^{-ip\cdot x}e^{-3\mathcal{W}(z)/2}\bar{h}_{\mu\nu}(z). \tag{3.19}$$

We can perform this variable transformation step-by-step. Firstly, the d'Alembertian only acts on $e^{-ip\cdot x}$ and, with our metric signature, gives $-p^2$. So,

$$\Box h_{\mu\nu} = -p^2 e^{-ip\cdot x}e^{-3\mathcal{W}(z)/2}\bar{h}_{\mu\nu}(z)\,. \tag{3.20}$$

Next, our derivatives become [36]

$$\frac{d}{dy} = e^{-2W(z)}\frac{3}{4}\frac{d}{dz}, \qquad \frac{d^2}{dy^2} = e^{-2W(z)}\frac{d^2}{dz^2} \tag{3.21}$$

where now W is a function of z, and a prime is a now a derivative w.r.t. to z (the change from $\mathcal{W}(y) \to W(z)$ is given by integrating the above equation). The exponential can be moved from the r.h.s., leaving

$$\left(\partial_z^2 + 3W'\partial_y + e^{2W}\left[2W'' + 8W'^2 - \frac{4}{3}V\right] + p^2\right)h_{\mu\nu} = 2\delta\tilde{T}_{\mu\nu}. \tag{3.22}$$

Applying the l.h.s. to eq. (3.19)

$$3W'\partial_z\left(e^{-ip\cdot x}e^{-3W(z)/2}\bar{h}_{\mu\nu}(z)\right) = e^{-ip\cdot x}3W'e^{-3W(z)/2}\bar{h}'_{\mu\nu} - e^{-ip\cdot x}\frac{9}{2}W'^2e^{-3W(z)/2}\bar{h}_{\mu\nu}; \tag{3.23}$$

$$\begin{aligned}\partial_z^2\left(e^{-ip\cdot x}e^{-3W(z)/2}\bar{h}_{\mu\nu}(z)\right) &= e^{-ip\cdot x}e^{-3W(z)/2}\bar{h}''_{\mu\nu} - e^{-ip\cdot x}3W'e^{-3W(z)/2}\bar{h}'_{\mu\nu}\\ &\quad + e^{-ip\cdot x}\frac{9}{4}W'^2e^{-3W(z)/2}\bar{h}_{\mu\nu} - e^{-ip\cdot x}\frac{3}{2}W''e^{-3W(z)/2}\bar{h}_{\mu\nu}.\end{aligned} \tag{3.24}$$

We see the $h'_{\mu\nu}$ terms cancel, leaving us to divide through by $e^{-ip\cdot x}e^{-3W(z)/2}$:

$$\begin{aligned}&\left(\frac{d^2}{dz^2} + u(z) + p^2\right)\bar{h}_{\mu\nu} = \bar{S}_{\mu\nu};\\ &\qquad u(z) = \left[8e^{2W} - \frac{9}{4}\right]W'^2 + \left[2e^{2W} - \frac{3}{2}\right]W'' - e^{2W}\frac{4}{3}V;\\ &\qquad \bar{S}_{\mu\nu} = e^{ip\cdot x}e^{3W(z)/2}2\delta\tilde{T}_{\mu\nu}.\end{aligned} \tag{3.25}$$

This is now a Schrodinger-type equation. As a sanity-check, when there is no three-form the r.h.s. vanishes, and the effective potential $u(z)$ would match the scalar field case since the extra e^{2W} terms would not appear in earlier steps in the derivation. The three-form has changed the differential operator to, for $p > 0$,

$$D^2 = -\frac{d^2}{dz^2} - u(z). \tag{3.26}$$

This operator is not factorisable, so the gravity sector on the brane is not linearly stable [31, 34, 36, 37]. Even with a source term this conclusion holds: the term p^2 corresponds

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to the energy eigenvalue, and this term is not necessarily bounded from below. There may be regions where this is the case, but the background metric is unstable against perturbations, and so the energy can become unbounded at certain points. With the introduction of the three-form field, the gravity sector of the braneworld is linearly unstable. We expect high sensitivity in the dynamics of the three-form away from stable fixed points, and study this in the following section.

4 Three-form solutions

We now turn to obtaining solutions for the three-form and explicit analysis. The complex nature of our system of equations requires the use of numerical methods. A dynamicla system is constructed to reduce the derivative order from second to first. We will do so for both the model we have considered thus far as well as with an additional scalar field acting as a matter source.

4.1 Solutions without matter

Using the dynamical variables:

$$x = \chi \quad z = \chi' + 3\mathcal{W}'\chi \quad f = \mathcal{W}' \tag{4.1}$$

our system of equations (2.23) and (2.22) can be expressed as the following dynamical system:

$$x' = z - 3fx\,, \tag{4.2}$$

$$z' = -z\left(\frac{z}{x} + f\right)\,, \tag{4.3}$$

$$f' = -\frac{4}{3}z^2\,. \tag{4.4}$$

It should be noted that this choice of variables does not impact the form of the dynamical system and thus does not affect the results.

We begin the analysis by first considering cases with analytical solutions. This system has one line of finite fixed points $\mathcal{P}_L$ at $(x, z, f) = (x, 0, 0)$ corresponding to a 5D Minkowski solution (with no brane). The three-form field is constant in this case, $\chi = \chi_0$.[1] Additionally, just $z = 0$ is an invariant submanifold. On this submanifold, assuming $\mathcal{W}' \neq 0$ and $\chi' \neq 0$, we find solutions which are analytic:

$$\mathcal{W}(y) = by + \mathcal{W}_0, \tag{4.5}$$

$$\chi(y) = \chi_0 e^{-3by}, \tag{4.6}$$

$$V = -6b^2, \tag{4.7}$$

where b is a constant. These are plotted in figure 1. At the level of the action, this submanifold corresponds to $F^2 = 0$ and a (negative) constant three-form potential in 5D. The means our potential plays the role of an effective cosmological constant and our action is no more than a de Sitter-like action in 5D.

[1] Here we are not looking at perturbations, χ_0 is not the zeroth order field. Hereafter, subscript 0 denotes the value of a variable at $y = 0$, i.e. $\chi(0) = \chi_0$.

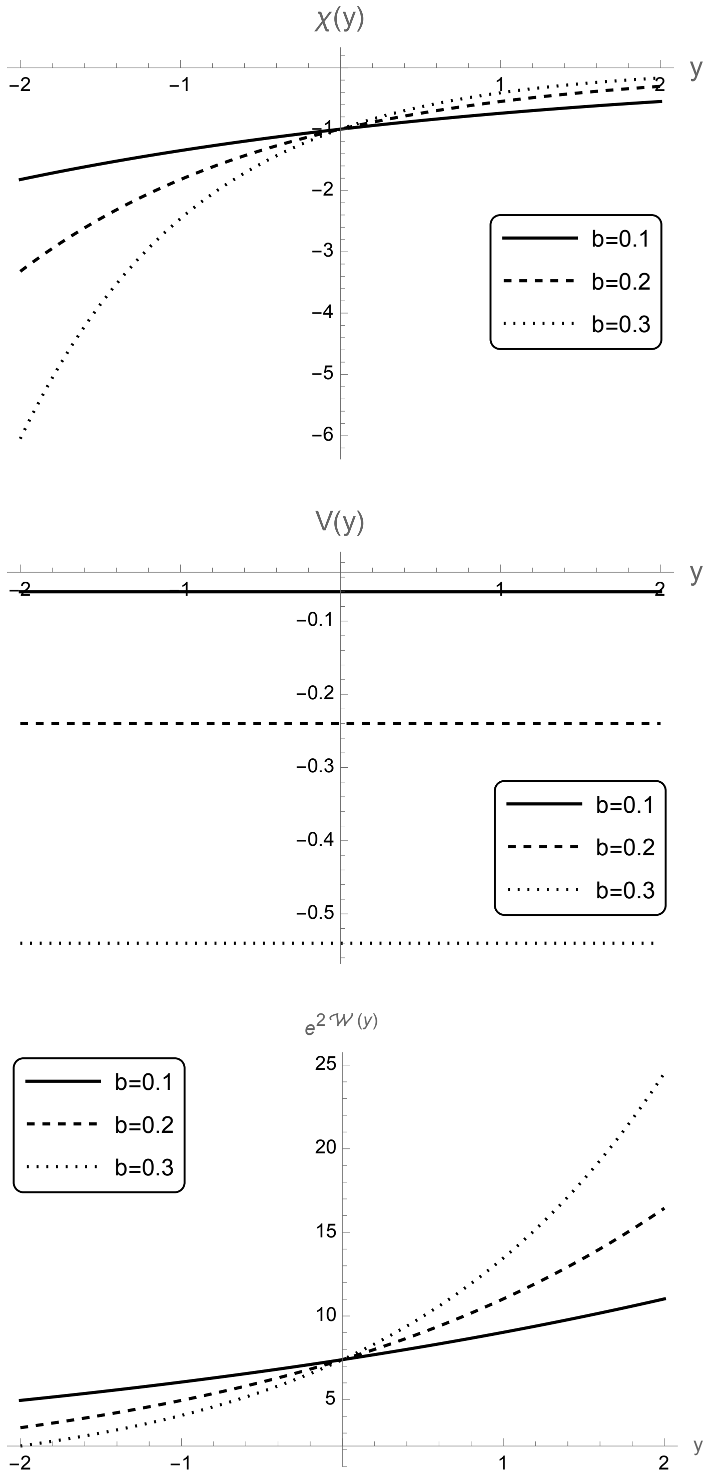

Figure 1. Analytic solutions on the invariant submanifold $z = 0$ for the warp factor $e^{2\mathcal{W}(y)}$, the three-form field $\chi(y)$ and its associated potential V for different values of the constant b. For all solutions, the values $\mathcal{W}_0 = 1$ and $\chi_0 = -1$ are fixed. All the parameters we have fixed are free to fix, and their actual choice has no bearing on the general behaviour.

For $z \neq 0$, the system must be solved numerically. To do this, we need to impose boundary conditions. In the case of $f(R,T)$ scalar fields [31] boundary conditions at the origin are set to guarantee symmetric solutions, i.e. $\mathcal{W}'(0) = 0$, $\chi'(0) = 0$. Z_2 symmetry is not required but it is assumed unless there is an explicit reason this assumption cannot hold [32]. In our case, these conditions correspond directly to the fixed points $\mathcal{P}_L$, admitting only the trivial solution $\chi = \chi_0$, $\mathcal{W} = \mathcal{W}_0$, $V = 0$.

Therefore the only case admitting symmetric solutions for both χ and $\mathcal{W}$ is the Minkowski solution with no brane. Nontrivial solutions exist which satisfy either condition $\chi'(0) = 0$ or $\mathcal{W}'(0) = 0$ separately. This alone, however, does not produce symmetry and instead both conditions to be satisfied. As such, we cannot assume Z_2 symmetry for our initial conditions and must solve the system with a different set of conditions.

There is no reason to favour one set of initial conditions over another, so for now we choose those initial conditions generating solutions as close to symmetric as possible. We impose only the condition $\mathcal{W}'(0) \equiv \mathcal{W}_0' = 0$ and examine solutions for small values of χ_0' and varying values of χ_0. Solutions are shown in figures 2 and 3. Figure 2 shows the solutions for a range of initial three-form values and figure 3 shows the solutions for a range of initial three-form derivative values.

The solution for χ is asymmetric. As mentioned earlier, despite imposing $\mathcal{W}'(0) = 0$, this condition is not sufficient to ensure symmetry. The solution for the warp factor will be influenced by the asymmetric nature of χ. This would imply differential behaviour on different sides of the brane, depending on position in the bulk.

It is also noted that χ grows rapidly further from the centre of the brane, regardless of the value of either χ_0' or χ_0. The three-form field exhibits pathological behaviour at some point in its evolution. The precise value of y is initial condition dependent, but the overall result is not. This implies the three-form is unstable off the brane. The instability corresponds to a runaway three-form potential: there are finite minima on the brane, in which the three-form is itself finite, but as the potential grows for large (positive and negative) values of y in the bulk it becomes infinite. The warp factor solution obtained is a thick brane solution in the sense that the maxima is reached on the brane itself, $y = 0$, but it is not a regular function.

This corroborates the perturbation analysis. The energy spectrum is not bounded from below and the background spacetime is unstable against perturbations. This would explain why the field grows rapidly for y values too far from the initial configuration, and why the other behavioural properties are so sensitive to initial values. We can further see this through the plot for $u(y)$. The graviton potential has a minimum, and despite the apparent trend to flatten out away from the brane, it also diverges (we have kept the y range smaller to keep it consistent with the other plots). This is expected, as we have shown analytically that the metric perturbation does not have a stable energy spectrum.

4.2 Introducing a matter source

We now introduce an additional matter source to our model,

$$S = \int d^4x dy \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \mathcal{L}_{3\mathrm{f}} \right] + S_m(g_{AB}, \psi) \,, \tag{4.8}$$

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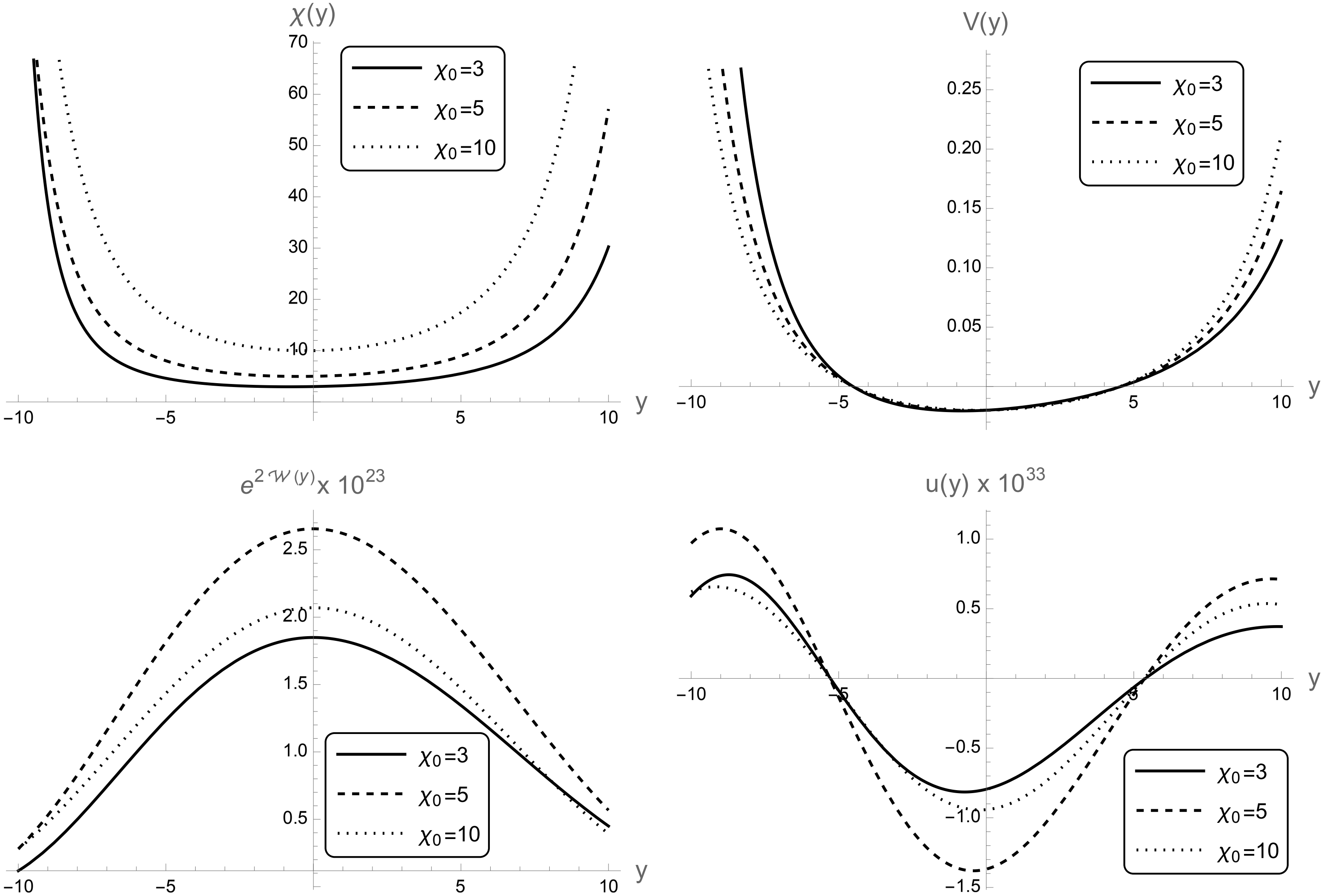

Figure 2. Solutions for different values of χ_0 for the three-form field $\chi(y)$, its associated potential $V(y)$, the warp factor $e^{2\mathcal{W}(y)}$. The bottom panel is the graviton potential eq. (3.25) but in terms of y, $u(y)$ (see appendix C), with fixed initial conditions corresponding to $\chi'(0) = 0.1$ and $\mathcal{W}'(0) = 0$.

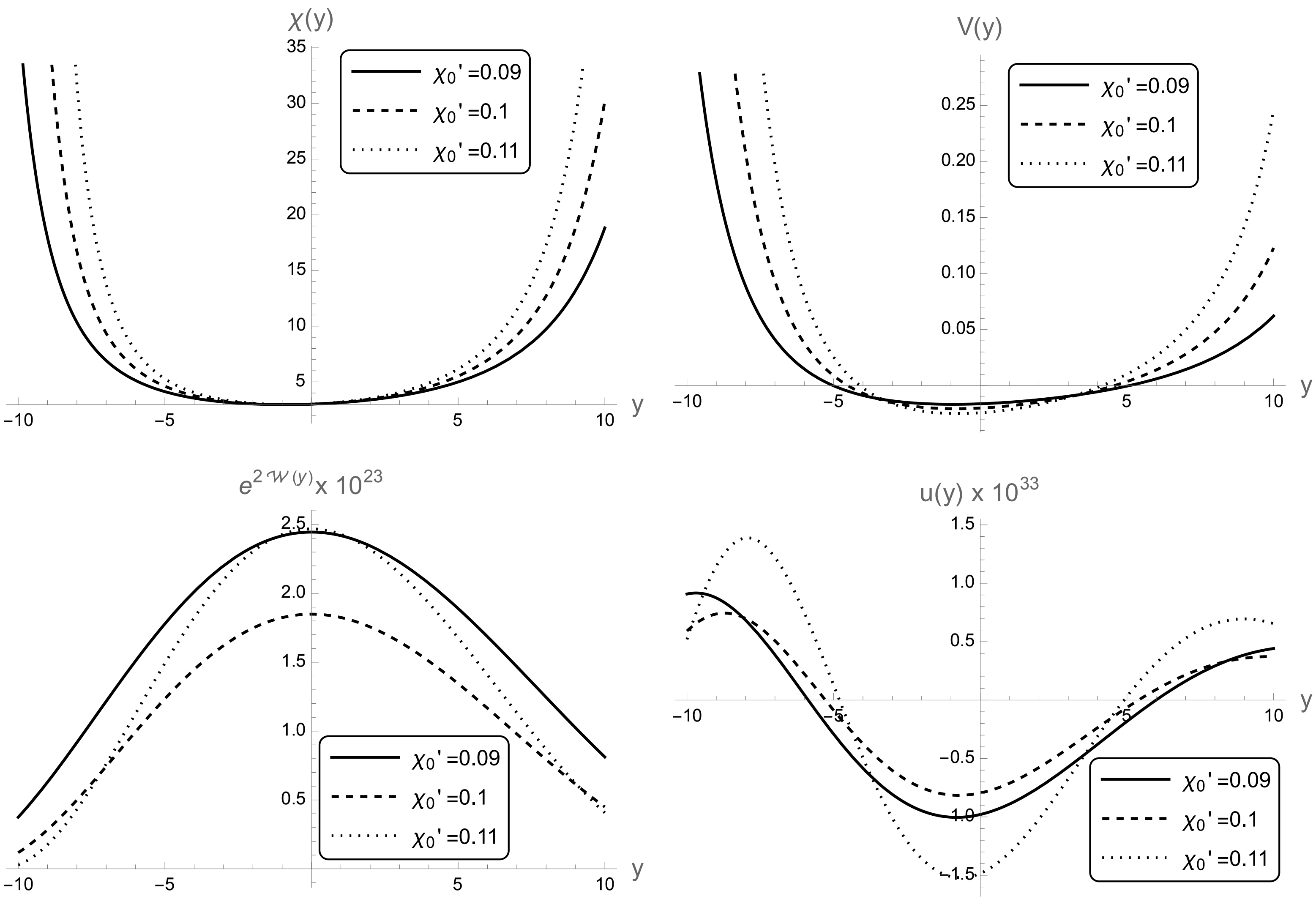

Figure 3. Solutions for different values of χ_0' for the three-form field $\chi(y)$, its associated potential $V(y)$, and the warp factor $e^{2\mathcal{W}(y)}$. The bottom panel is the graviton potential eq. (3.25) but in terms of y, $u(y)$ (see appendix C), with fixed initial conditions corresponding to $\chi(0) = 3$ and $\mathcal{W}'(0) = 0$.

where S_m is the matter action and ψ denotes the matter fields. We will consider matter described by a single dynamical scalar field $\psi(y)$, dependent only on the extra dimension, with an interaction potential $U(\psi)$. Hence the matter action is

$$S_m = -\int d^4x dy \sqrt{-g} \left[\frac{1}{2} \partial^C \psi \partial_C \psi + U(\psi) \right]. \tag{4.9}$$

Taking the variation of eq. (4.9) with respect to the scalar field ψ, we obtain the stress-energy tensor for the matter field

$$T^{(\mathrm{m})}{}_{AB} = -g_{AB} \left[\frac{1}{2} \partial^C \psi \partial_C \psi + U(\psi) \right] + \partial_A \psi \partial_B \psi. \tag{4.10}$$

The total stress-energy tensor is the sum of the stress-energy tensor for the three-form $T^{(3\mathrm{f})}{}_{AB}$ defined in (2.13) and that of the scalar field ψ,

$$T^{(\mathrm{tot})}{}_{AB} = T^{(3\mathrm{f})}{}_{AB} + T^{(\mathrm{m})}{}_{AB}. \tag{4.11}$$

Varying eq. (4.8) with respect to ψ, we obtain the standard Klein-Gordon equation for the matter field,

$$\psi'' + 4\mathcal{W}'\psi' - U_\psi = 0, \tag{4.12}$$

where the prime denotes differentiation with respect to y and the subscript $_\psi$ denotes differentiation with respect to ψ.

The full set of field equations now read

$$3\left(\mathcal{W}'' + 2\mathcal{W}'^2\right) = -V - \frac{1}{48}F^2 - \frac{1}{2}\psi'^2 - U, \tag{4.13}$$

$$3\left(\mathcal{W}'' + 2\mathcal{W}'^2\right) = -V + \frac{1}{48}F^2 + \chi V_\chi - \frac{1}{2}\psi'^2 - U, \tag{4.14}$$

$$6\mathcal{W}'^2 = -V + \frac{1}{48}F^2 + \frac{1}{2}\psi'^2 - U. \tag{4.15}$$

There is freedom to fix the warp function, since between the three EFFs (4.13), (4.14), (4.15) and two KG equations (2.11), (4.12), we only have four independent equations and five unknowns:

$$\psi'' + 4\mathcal{W}'\psi' - U_\psi = 0, \tag{4.16}$$

$$\chi'' + 4\mathcal{W}'\chi' + 3\chi\left(\mathcal{W}'' + \mathcal{W}'^2\right) - \frac{1}{4}\frac{dV}{d\chi} = 0, \tag{4.17}$$

$$3\mathcal{W}'' = \chi V_\chi - \psi'^2, \tag{4.18}$$

$$3\left(\mathcal{W}'' + 2\mathcal{W}'^2\right) = -V - 2(\chi' + 3\mathcal{W}'\chi)^2 - \frac{1}{2}\psi'^2 - U. \tag{4.19}$$

With this new freedom, we choose the standard form of the warp factor,

$$\mathcal{W}(y) = \mathcal{W}_0 \log[\mathrm{sech}(ky)], \tag{4.20}$$

where k, $\mathcal{W}_0$ are constants. With this choice, the resulting set of equations are for χ, V, ψ and U are

$$\psi'^2 = 3k^2\mathcal{W}_0 \operatorname{sech}^2(ky) - 4(\chi' - 3k\mathcal{W}_0\chi \tanh(ky))^2 \tag{4.21}$$

$$\chi'' = 3k^2\mathcal{W}_0\chi \operatorname{sech}^2(ky)(-2\mathcal{W}_0 \cosh(2ky) + 2\mathcal{W}_0 + 1) + 10k\mathcal{W}_0\chi' \tanh(ky) - \frac{\chi'^2}{\chi} \tag{4.22}$$

$$V' = \frac{\chi'}{\chi}(-3k^2\mathcal{W}_0 \operatorname{sech}^2(ky) + \psi'^2) \tag{4.23}$$

$$U' = \psi'(\psi'' - 4k\mathcal{W}_0 \tanh(ky)\psi'). \tag{4.24}$$

As in the matter-free case, we use dynamical systems methods to solve this by introducing the dynamical variables

$$\begin{aligned} x &= \chi, \qquad & z &= \chi' - 3k\mathcal{W}_0 \tanh(ky)\chi, \\ \phi &= \psi, \qquad & \Phi &= \psi'. \end{aligned} \tag{4.25}$$

The resulting dynamical system is

$$x' = z + 3k\mathcal{W}_0 \tanh(ky)x \tag{4.26}$$

$$z' = k\mathcal{W}_0 \tanh(ky)z - \frac{z^2}{2x} \tag{4.27}$$

$$\phi' = \Phi \tag{4.28}$$

$$\begin{aligned} \Phi' = 4k\mathcal{W}_0 \tanh(ky)\Phi + \frac{1}{\Phi}\Bigg[&\frac{2z^3}{x} + 6k\mathcal{W}_0 \tanh(ky)z^2 \\ &- 3k^2\mathcal{W}_0 \operatorname{sech}^2(ky)\tanh(ky)(1+2\mathcal{W}_0)\Bigg]. \end{aligned} \tag{4.29}$$

This system has many features in common with the matter-free case. $z = 0$ is an invariant submanifold on which an analytical solution can be found:

$$\chi(y) = b \cosh^{3\mathcal{W}_0}(ky) \tag{4.30}$$

$$\psi(y) = b \pm \frac{2\sqrt{6}\sqrt{\mathcal{W}_0}\cosh(ky)\tan^{-1}\left(\tanh\left(\frac{ky}{2}\right)\right)}{\sqrt{\cosh(2ky)+1}} \tag{4.31}$$

$$V(y) = b - 3k^2\mathcal{W}_0^{3/2}\left(\pm\frac{\sqrt{3}}{k\sqrt{\cosh^2(ky)}} - \frac{3}{2}\sqrt{\mathcal{W}_0}\operatorname{sech}^2(ky)\right) \tag{4.32}$$

$$U(y) = b + 6k^2\mathcal{W}_0^2 \operatorname{sech}^2(ky) + \frac{3}{2}k^2\mathcal{W}_0 \operatorname{sech}^2(ky), \tag{4.33}$$

where $\mathcal{W}_0$, k and b are constants. These are plotted in figures 4 and 5. The inclusion of matter has fundamentally changed the dynamics. The potential now evolves, and we can see that the negative solution V_- should be discounted as "unphysical", since its behaviour does not correspond with the three-form's. The positive solution does correspond, however,

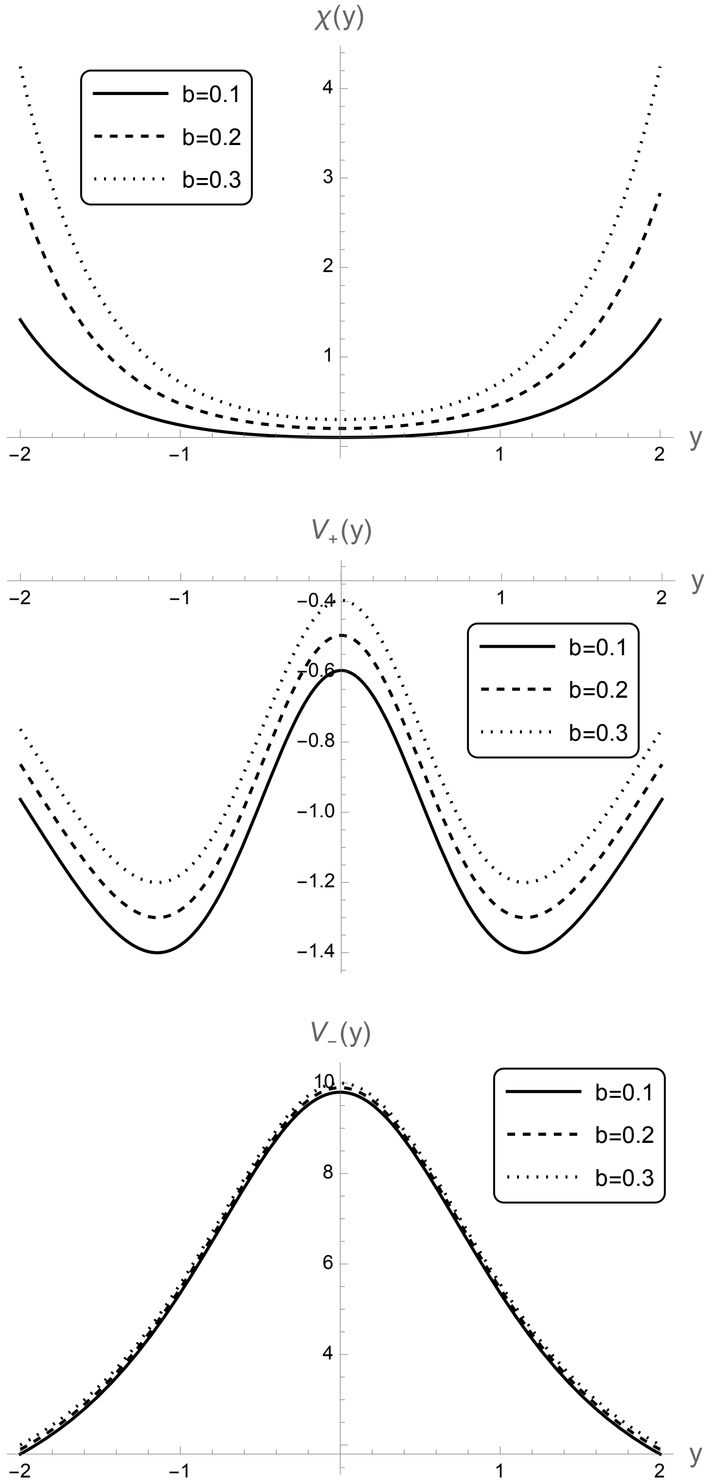

Figure 4. Analytic solutions for the matter case on the invariant submanifold $z = 0$ for the three-form field $\chi(y)$ and its associated potentials V_+ and V_- for different values of the constant b. For all solutions, the values $\mathcal{W}_0 = 1$ and $k = 1$ are fixed.

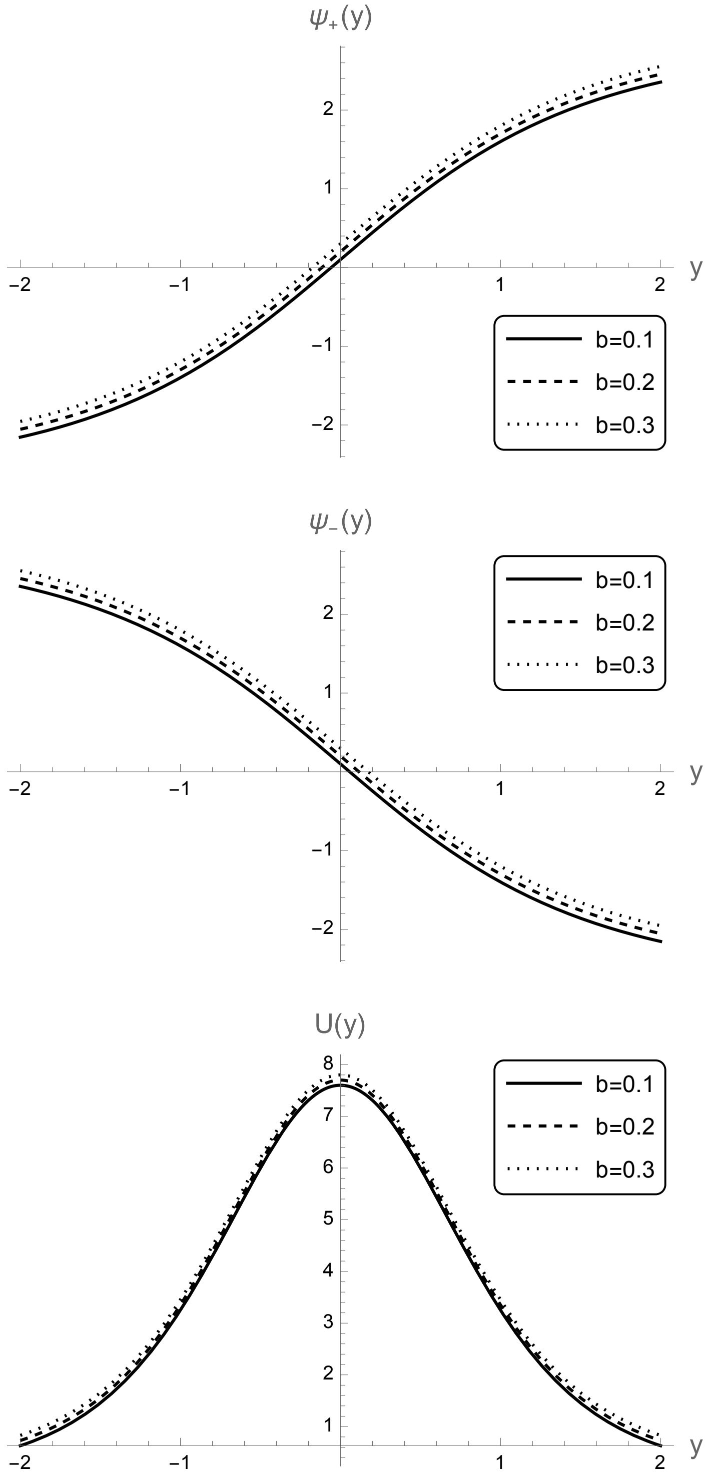

Figure 5. Analytic solutions for the matter case on the invariant submanifold $z = 0$ for the matter field $\psi_+(y)$ and $\psi_-(y)$ and the associated potential $U(y)$ for different values of the constant b. For all solutions, the values $\mathcal{W}_0 = 1$ and $k = 1$ are fixed.

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and similarly exhibits instability at larger values of y. The matter fields inherit the apparent instability due to their dependence on the three-form. Despite the matter potential being well-behaved, the matter field dynamics are not.

Assuming $z \neq 0$, solutions must be found numerically. Again, $z \neq 0$ implies that $\chi'(0) \neq 0$ so numerical, symmetric solutions for χ are not possible. Additionally, it is not possible to obtain symmetric solutions for ψ since eq. (4.29) is singular at $\Phi = 0$. Therefore, similar to the case for $\psi = 0$, initial conditions are imposed on $\chi(0) = \chi_0$ and $\chi'(0) = \chi'_0$ to achieve near-symmetric solutions which remain finite near the brane. As is the case for χ_0 and χ'_0, the system is highly sensitive to values of ϕ_0 and ϕ'_0. Solutions for a range of χ_0 and ϕ_0 values are shown in figure 6 and we plot the two potentials for a range of χ'_0 values; these are shown in figure 7.

Much like the analytical sub-case, the inclusion of matter and an additional degree of freedom does not change the fact that the three-form field increases unboundedly at sufficiently high values off the center of the brane. It does, however, keep the three-form stable for a larger range of y. The three-form potential is unable to maintain symmetry when χ'_0 is too high. For smaller values, as seen in figure 7, the three-form potential maintains some degree of symmetry and does not grow as quickly compared to the matter-free scenario. However, in general, for both, the initial 'kick' must not be too large. This sensitivity provides further evidence that three-forms are unstable. Our brane now is symmetric by construction, and yet the instabilities in the dynamics of the three-form continue to manifest themselves.

The scalar field, acting as matter, produces a more stable potential and dynamical behaviour (figure 7). But since the dynamics are connected to the three-form, they similarly rapidly grow.

5 Discussions and conclusions

In this work, we have studied thick braneworld solutions in general relativity, in a 5D bulk, with the inclusion of a three-form field. This has extended the work done by [4], who considered three-form inflation in Randall-Sundrum II braneworlds. We have constructed novel solutions for the three-form field in the extra dimension in a warped Minkowski spacetime. We first developed the general formalism required to investigate the problem, and found that we could not study specific solutions to the problem without matter, due to the exactly determined set of governing equations. We have also studied three-forms in branes with matter, expanding our model to include a scalar field as a matter source. This allowed us to fix the warp function specifically and study that particular configuration.

Section 3 contains novel results of interest. The three-form radically changes the study of the stability of the gravity sector of the model. Following procedure from [36], we have derived a differential equation describing the stability of the graviton. The potential, normally a stability potential for scalar-tensor theories, is not stable in the three-form case. The gravity zero mode is therefore not normalisable. This indicates the background is not stable against perturbations, and that the brane cannot support internal structure (i.e. thick branes specifically are unstable).

This finding is further supported by the new results in section 4. We have found new numerical solutions for the three-form field, its potential, and the warp function. We can see

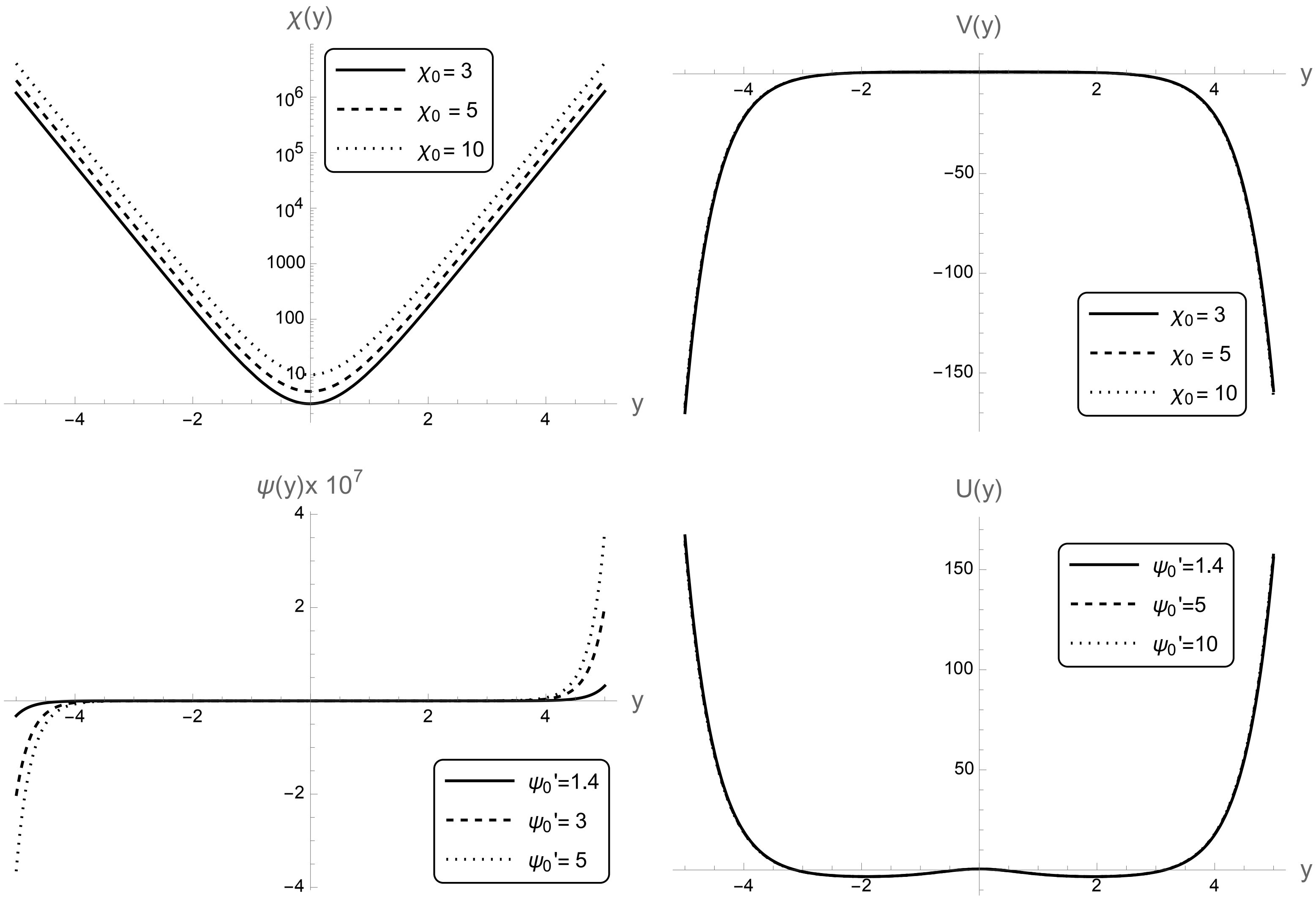

Figure 6. Solutions for different values of χ_0 and ϕ_0 for the three-form field $\chi(y)$, its associated potential $V(y)$, the matter field $\psi(y)$ and its associated potential $U(y)$ with fixed conditions corresponding to $\chi'(0) = 0.1$ and $\psi(0) = 1$.

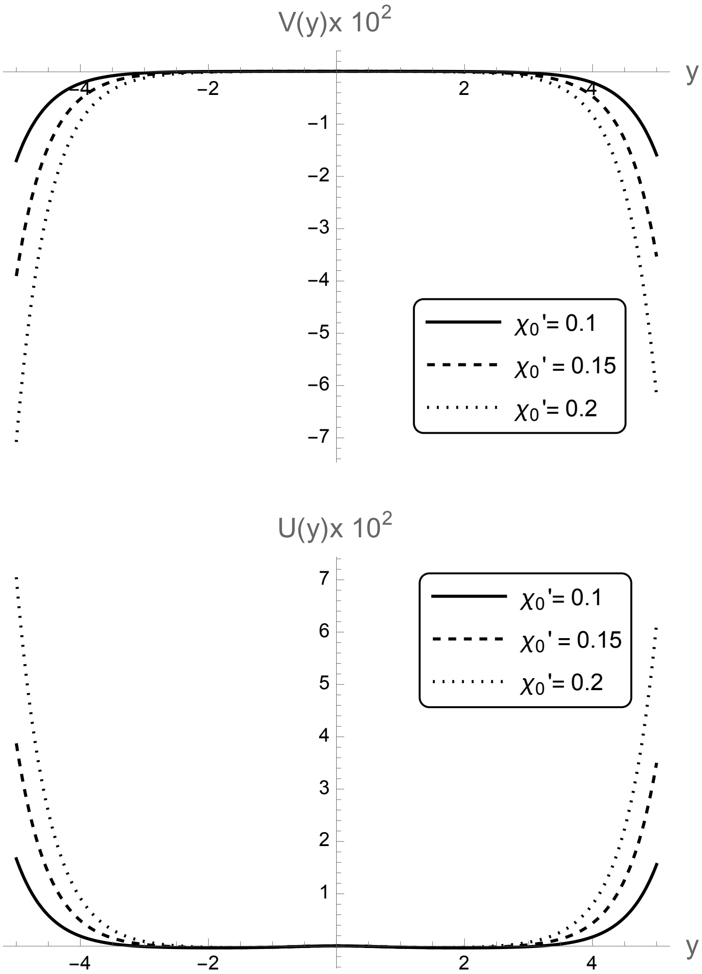

Figure 7. Solutions for different values of χ_0' for the three-form potential $V(y)$ and the matter potential $U(y)$ with fixed conditions corresponding to $\chi(0) = 3$, $\psi(0) = 1$ and $\psi'(0) = 3$.

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from the solutions that the three-form is unstable, diverging off the center of the brane (the center being the initial starting point for our evolution). As expected from section 3, the system is highly sensitive to initial conditions. The system also admits no degree of symmetry, displaying differential behaviour for the same point but on the other side of the brane, further enforcing the notion that the dynamics are unstable. The addition of matter does not remedy this situation. Despite affording the freedom to fix a symmetric and well-behaved warp function, the system's evolution still diverges and shows a varying degree of symmetry in the three-form and the scalar, depending on the conditions. In either case, however, there is no means to obtain regular behaviour past a certain point in the system's evolution.

There is a substantial caveat to the analysis performed in this paper. The one true free choice available is the freedom to choose eq. (2.5). This ansantz is arbitrary, and has great impact on both the three-form equation of motion and the stress-energy tensor in the EFEs. If the three-form were to be parameterised differently, the system of equations would be different and possibly well-behaved. The perturbations would potentially be stable too. This paper represents a first step in studying three-forms in thick branes, in conjunction with their scalar field counterparts. The indication from this work is that there exist properties of thick braneworlds that render three-forms unstable. A comprehensive proof of this — by showing this to be the case for a general three-form parameterisation — is still needed, as is a more thorough theoretical analysis of the properties of thick branes which render them incompatible with three-forms. This remains to be done as future work.

Acknowledgments

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A Stability analysis through localisation

There is an alternative way to study stability of tensor forms in branes that will further lend support to our conclusions. We employ the methods of [39]: in this work they use the conformal coupling of a scalar to localise a vector field on the brane. However, their method is applicable to any q-form and for any sort of matter coupling, including minimal coupling.

We can glean some insight from their condition for stability, which we quote here. The general condition is, for a minimally coupled q-form,

$$\int_{-\infty}^{+\infty} \left(e^{2\mathcal{W}(y)}\right)^{3-2q} < \infty. \tag{A.1}$$

In our case, we can't fix any warp factor; ours is determined solely by the three-form parameterisation choice. In any event, even without some analytical expression to use here, we can numerically use the result for $\mathcal{W}(y)$ in figure 2 and find that the integral diverges. Visually, our warp factor diverges after certain values of y, which means the integral above won't converge everywhere. Hence, we can conclude once more, via this method, that three-forms aren't stable in thick branes.

B Scalar "mimicing"?

Is the under-determined set of equations the missing ingredient to finding consistent and stable solutions? To answer this, a naive approach is to 'force' the three-form equations to look like their scalar field counterparts (cf. (2.24)). This means we want the components of the T^i_i to match T^0_0 in eq. (2.23), which means $2T^0_0 = -2V + \chi V_\chi$, and so

$$\begin{aligned} \chi'' + 4\mathcal{W}'\chi' + 3\chi\left(\mathcal{W}'' + \mathcal{W}'^2\right) - \frac{1}{4}\frac{dV}{d\chi} &= 0 \\ 3\left(\mathcal{W}'' + 2\mathcal{W}'^2\right) &= -V - \frac{1}{48}F^2 \\ 6\mathcal{W}'^2 &= \frac{1}{48}F^2 - V. \end{aligned} \tag{B.1}$$

Eliminating the V term, we have

$$\begin{aligned}\chi'' + 4\mathcal{W}'\chi' + 3\chi\left(\mathcal{W}'' + \mathcal{W}'^2\right) - \frac{1}{4}\frac{dV}{d\chi} = 0 \\ 3\left(\mathcal{W}'' + 4\mathcal{W}'^2\right) = -4\left(\chi' + 3\mathcal{W}'\chi\right)^2.\end{aligned} \tag{B.2}$$

We use the chain rule an to rewrite the potential derivative, $dV/d\chi = dV/dydy/d\chi = V'/\chi'$. We therefore have

$$\begin{aligned}\chi'' + 4\mathcal{W}'\chi' + 3\chi\left(\mathcal{W}'' + \mathcal{W}'^2\right) - \frac{1}{4}\frac{V'}{\chi'} = 0 \\ 3\left(\mathcal{W}'' + 4\mathcal{W}'^2\right) = -4\left(\chi' + 3\mathcal{W}'\chi\right)^2.\end{aligned} \tag{B.3}$$

In principle we should be able to pick one unknown; for example, we can pick $\mathcal{W} = \log(\operatorname{sech} y)$ like [31] and solve. However, due to the 'forcing' there is a constraint equation that can be written as

$$4\left(\chi' + 3\mathcal{W}'\chi\right)^2 + \frac{\chi}{\chi'}V' = 0. \tag{B.4}$$

This exactly determines our system again. Indeed, picking $\mathcal{W} = \log(\operatorname{sech} y)$ solves some of the equations here but not all. This alone is a sufficient counterexample to illustrate that there's no simple forcing the system to be under-determined.

C Graviton potential redefinition

Our graviton potential plots in figures 2 and 3 plot them as a function of y. We quote them as a function of z in eq. (3.25). The potential in terms of y is

$$u(y) = \frac{e^{2\mathcal{W}(y)}}{4}\left[\mathcal{W}'(y)^2\left(9e^{\mathcal{W}(y)} - \frac{21}{4}\right) + \mathcal{W}''(y)\left(3e^{\mathcal{W}(y)} - 3\right)\right] - e^{\mathcal{W}(y)}V. \tag{C.1}$$

For prosterity, we quote here the full backwards-transformation from $W(z) \to \mathcal{W}(y)$:

$$W(z) = \frac{1}{2}\left(-\ln(4/3) + \mathcal{W}(y)\right) \tag{C.2}$$

$$W'(z) = \frac{1}{2}e^{\mathcal{W}(y)}\mathcal{W}'(y) \tag{C.3}$$

$$W''(z) = \frac{1}{2}e^{2\mathcal{W}(y)}\left(\mathcal{W}'(y)^2 + \mathcal{W}''(y)\right) \tag{C.4}$$

where a prime denotes differentiation w.r.t. the function argument.

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