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Manifestly gauge invariant exact renormalization group for quantum electrodynamics

Yuki Miyakawa[1], Hidenori Sonoda[2], and Hiroshi Suzuki[1,*]

[1] *Department of Physics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan*
[2] *Physics Department, Kobe University, Kobe 657-8501, Japan*
*E-mail: hsuzuki@phys.kyushu-u.ac.jp

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We formulate quantum electrodynamics on the basis of gauge (or BRST) covariant diffusion equations of fields. This is a particular example of the gradient flow exact renormalization group (GFERG). The resulting Wilson action fulfills a simple gauge Ward–Takahashi identity. We solve the GFERG equation around the Gaussian fixed point to the second order in gauge coupling and obtain the 1-loop beta function and anomalous dimensions. The anomalous dimension of the electron field coincides with that of the fermion field diffused by a gauge covariant flow equation of Lüscher.

Subject Index B05, B32

1. Introduction

A Wilson action is a functional of field variables with a finite momentum cutoff, say Λ [1]. If the underlying theory is a continuum limit, the theory is defined on all momentum scales. We obtain the interaction vertices of the Wilson action S_Λ by integrating out the fields with momenta larger than Λ. It is then natural to expect that only the correlations of the fields with momenta smaller than Λ are kept, but those with momenta larger than Λ are lost from S_Λ. In the exact renormalization group (ERG) formalism [1], in which a sharp momentum cutoff is replaced by a smooth function of momentum, this is not the case: we can still reconstruct the full correlation functions using the Wilson action. This makes gauge invariance compatible with a momentum cutoff. This viewpoint was first adopted for QED in Ref. [2]. A general framework for constructing non-Abelian gauge theories along this line was given in Ref. [3].

The realization of gauge invariance with a Wilson action has a long history starting in the 1980s. Early works in the 1990s such as Refs. [4–9] established the possibility of constructing gauge theories in the ERG formalism. (Ref. [5] gives references to the earlier works from the 1980s.) What is common in the realization of gauge invariance in the ERG formalism is that the gauge invariance is not what one expects naturally. For $\Lambda > 0$, the gauge transformation is modified so that the Jacobian is non-vanishing, and the resulting expression of gauge invariance is by no means manifest. This has been an obstacle for any calculation of the Wilson action beyond perturbation theory, since it is difficult to truncate the action keeping the non-manifest gauge invariance.

The original formulation of ERG is based on the diffusion of the fields [1]. Recently a proposal was made that we may be able to construct a manifestly gauge invariant Wilson action by replacing the diffusion equation by a gauge invariant diffusion equation [10]. This was inspired

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by the gauge invariant diffusion that generates a gradient flow of gauge fields, first discussed for lattice gauge theory in Refs. [11–13] and then by Lüscher and Weisz [14] for perturbative non-Abelian gauge theory. We call this new type of ERG the gradient flow exact renormalization group (GFERG). The present paper is a sequel to Ref. [15] where GFERG for fermions is discussed.

The paper is organized as follows. In Sect. 2 we first review the relation between a diffusion equation and the exact renormalization group transformation using a generic real scalar theory. We follow the discussions given in Ref. [16]; see also Ref. [17]. We then introduce a particular set of diffusion equations for QED that is consistent with the BRST invariance of the theory. We base our construction of GFERG on these diffusion equations. In Sect. 3 we construct a Wilson action S_Λ of QED with momentum cutoff Λ that keeps its BRST invariance as we lower Λ. We derive the cutoff dependence of S_Λ as a differential equation, and also derive an expression for the BRST invariance. The BRST transformation acts linearly on the action, and it is far simpler than the BRST invariance of the Wilson action in the ERG formulation, which is briefly reviewed in Appendix C. In Sect. 4 we introduce a dimensionless framework by measuring dimensionful fields and parameters in units of appropriate powers of the cutoff. We then construct the BRST invariant Wilson action perturbatively in Sect. 5. We only consider the Wilson action for the continuum limit parametrized by the gauge coupling, gauge-fixing parameter, and the electron mass parameter. Since the ghost fields are free, we can reduce the BRST invariance to the Ward–Takahashi (WT) identity. This WT identity can be interpreted as manifest gauge invariance even though the transformation of the gauge field is somewhat modified.[1] We construct the Wilson action satisfying the WT identity to second order in the gauge coupling. We conclude the paper in Sect. 6.

We work in the D-dimensional Euclidean space, where $D = 4 - \epsilon$. We use shorthand notation for the momentum integrals:

$$\int_p \equiv \int \frac{d^D p}{(2\pi)^D}. \tag{1}$$

We also use the convention that the momentum cutoff decreases along the flow of the renormalization group, and the beta functions and anomalous dimensions may have opposite signs to what the reader is familiar with.

2. Preparation

2.1 *ERG*

We would like to review the essence of the exact renormalization group (ERG for short). In one formulation of ERG we construct the Wilson action of a theory in terms of a field satisfying a simple diffusion equation. The flow of the Wilson action is generated by the diffusion of the field. For gauge theories, we can replace the simple diffusion equation by a covariant diffusion equation that is consistent with BRST invariance. The replacement results in the gradient flow exact renormalization group (GFERG for short). We will introduce the BRST covariant diffusion for QED in the next subsection.

Let $\phi(x)$ be a real scalar field renormalized at momentum scale μ, and let $S[\phi]$ be its action. We introduce a diffused field $\phi(t; x)$ as the solution of a simple diffusion equation

$$\partial_t \phi(t; x) = \partial^2 \phi(t; x) \tag{2}$$

[1] We ask the reader to bear with the overblown title of the paper.

satisfying the initial condition

$$\phi(0; x) = \phi(x). \tag{3}$$

We would like to construct a Wilson action equivalent to $S[\phi]$ in terms of the diffused field $\phi(t; x)$ instead of $\phi(x)$. Let Λ be a momentum scale smaller than μ given by

$$t = \frac{1}{\Lambda^2} - \frac{1}{\mu^2} > 0 \tag{4}$$

so that

$$\partial_t = \frac{\Lambda^2}{2}\left(-\Lambda\partial_\Lambda\right). \tag{5}$$

We introduce the Wilson action $S_\Lambda[\phi]$ for momentum cutoff Λ by

$$\exp\left(S_\Lambda[\phi]\right) \equiv \int [d\phi'] \exp\left\{-\frac{\Lambda^2}{2}\int_p \left[\phi(p) - z_\Lambda\phi'_\Lambda(p)\right]\left[\phi(-p) - z_\Lambda\phi'_\Lambda(-p)\right] + S[\phi']\right\}, \tag{6}$$

where the argument of S is the field $\phi'(x)$ (or its Fourier transform $\phi'(p)$), and $\phi'_\Lambda(p)$ is the Fourier transform of the diffused field

$$\phi'(t; x) = \int_p e^{ipx}\phi'_\Lambda(p) \tag{7}$$

satisfying the initial condition

$$\phi'(0; x) = \phi'(x). \tag{8}$$

In momentum space it is trivial to solve the diffusion equation to obtain

$$\phi'_\Lambda(p) = e^{-p^2\left(\frac{1}{\Lambda^2} - \frac{1}{\mu^2}\right)}\phi'(p). \tag{9}$$

By construction, $\phi(p)$ equals $z_\Lambda\phi'_\Lambda(p)$ with a squared fluctuation of order $1/\Lambda^2$. It is not exactly the same as $z_\Lambda\phi'_\Lambda(p)$, but it corresponds to it. The choice of z_Λ is not unique. For example, we can determine z_Λ to normalize the kinetic term of the Wilson action $S_\Lambda[\phi]$.

The Λ-dependence of the Wilson action $S_\Lambda[\phi]$ is given by the ERG differential equation:

$$-\Lambda\frac{\partial}{\partial\Lambda}e^{S_\Lambda[\phi]} = \int_p \left[\left(\frac{2p^2}{\Lambda^2} - \gamma_\Lambda\right)\phi(p)\frac{\delta}{\delta\phi(p)} + \frac{1}{\Lambda^2}\left(\frac{2p^2}{\Lambda^2} - \gamma_\Lambda + 1\right)\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right]e^{S_\Lambda[\phi]}, \tag{10}$$

where γ_Λ is defined by

$$\gamma_\Lambda \equiv -\Lambda\frac{d}{d\Lambda}\ln z_\Lambda. \tag{11}$$

For the correlation functions, we can give a precise relation between S_Λ and S:

$$\begin{aligned}&\left\langle \exp\left[-\frac{1}{2\Lambda^2}\int_p \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right]\phi(p_1)\cdots\phi(p_n)\right\rangle_{S_\Lambda} \\ &= z_\Lambda^n\left\langle\phi_\Lambda(p_1)\cdots\phi_\Lambda(p_n)\right\rangle_S = z_\Lambda^n\prod_{i=1}^n e^{-p_i^2\left(\frac{1}{\Lambda^2} - \frac{1}{\mu^2}\right)}\cdot\left\langle\phi(p_1)\cdots\phi(p_n)\right\rangle_S.\end{aligned} \tag{12}$$

In constructing S_Λ, we have "scrambled" the field $\phi(p)$ around $z_\Lambda\phi_\Lambda(p)$. We need to unscramble the field to get back the same correlation functions. This is the role played by the exponentiated differential operator.

An alternative definition of the Wilson action is given by [10]

$$\exp\left(S_\Lambda[\phi]\right) = \hat{s}_\Lambda\int [d\phi']\prod_p \delta\left(\phi(p) - z_\Lambda\phi'_\Lambda(p)\right)\cdot\exp\left(S[\phi']\right), \tag{13}$$

where

$$\hat{s}_\Lambda \equiv \exp\left[\frac{1}{2\Lambda^2}\int_p \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right] \tag{14}$$

is what we call the scrambler. The scrambler is necessary to maintain the locality of the Wilson action $S_\Lambda[\phi]$. The equality of the correlation functions can be written as

$$\left\langle \hat{s}_\Lambda^{-1}\left[\phi(p_1)\cdots\phi(p_n)\right]\right\rangle_{S_\Lambda} = z_\Lambda^n \left\langle \phi_\Lambda(p_1)\cdots\phi_\Lambda(p_n)\right\rangle_S \tag{15}$$

using the unscrambler, i.e., the inverse of the scrambler.

2.2 *Diffusion in QED*

In constructing the Wilson action of a gauge theory, we can use the simple diffusion equation for gauge fields and matter as explained above for the scalar theory. The Wilson action, thus constructed, retains gauge invariance (BRST invariance to be more precise), but its realization is not as straightforward as we wish [3]. This is partially due to the use of a simple diffusion of fields that does not respect the gauge invariance. In Ref. [10] we have introduced an alternative Wilson action based upon a covariant diffusion of fields, consistent with the gauge invariance of the theory.[2] In this subsection we would like to introduce such diffusion explicitly for the simple case of QED.

Let us consider QED renormalized at momentum scale μ in $D = 4 - \epsilon$-dimensional Euclidean space. We denote the gauge field by $A_\mu(x)$, and the electron field by $\psi(x)$ and $\bar{\psi}(x)$, and the free Faddeev–Popov ghost fields by $c(x)$ and $\bar{c}(x)$. The dimensionless gauge coupling renormalized at μ is e. The action S is invariant under the following BRST transformation of the renormalized fields:

$$\delta A_\mu(x) = \eta\partial_\mu c(x), \tag{16a}$$

$$\delta c(x) = 0, \tag{16b}$$

$$\delta\bar{c}(x) = \eta\frac{1}{\xi}\partial\cdot A(x), \tag{16c}$$

$$\delta\psi(x) = ie\mu^{\epsilon/2}\eta c(x)\psi(x), \tag{16d}$$

$$\delta\bar{\psi}(x) = -ie\mu^{\epsilon/2}\eta c(x)\bar{\psi}(x), \tag{16e}$$

where η is an arbitrary anticommuting number that keeps the statistics of the fields under the transformation.[3]

We introduce the following diffusion equations:

$$\partial_t A_\mu(t;x) = \partial^2 A_\mu(t;x), \tag{17a}$$

$$\partial_t c(t;x) = \partial^2 c(t;x), \tag{17b}$$

$$\partial_t \bar{c}(t;x) = \partial^2 \bar{c}(t;x), \tag{17c}$$

$$\partial_t \psi(t;x) = \left[\partial^2 - 2ie\mu^{\epsilon/2}A_\mu(t;x)\partial_\mu - e^2\mu^\epsilon A_\mu(t;x)A_\mu(t;x)\right]\psi(t;x), \tag{17d}$$

[2] Preceding Ref. [10], other gauge invariant ERG formulations had been introduced in Refs. [18–24].

[3] We have chosen the mass dimension of $c(x)$ as $(D-4)/2 = -\epsilon/2$ and that of $\bar{c}(x)$ as $D/2$ so that the mass dimension of η is zero.

$$\partial_t \bar{\psi}(t;x) = \left[\partial^2 + 2ie\mu^{\epsilon/2} A_\mu(t;x)\partial_\mu - e^2\mu^\epsilon A_\mu(t;x)A_\mu(t;x)\right]\bar{\psi}(t;x), \tag{17e}$$

where the fields match the renormalized fields at $t = 0$:

$$A_\mu(0;x) = A_\mu(x), \ldots, \bar{\psi}(0;x) = \bar{\psi}(x). \tag{18}$$

It is straightforward to check that the above diffusion equations are consistent with the BRST transformation. Namely, the BRST transformation of the fields at $t = 0$ implies the same BRST transformation of the diffused fields:

$$\delta A_\mu(t;x) = \eta\partial_\mu c(t;x), \tag{19a}$$

$$\delta c(t;x) = 0, \tag{19b}$$

$$\delta\bar{c}(t;x) = \eta\frac{1}{\xi}\partial\cdot A(t;x), \tag{19c}$$

$$\delta\psi(t;x) = ie\mu^{\epsilon/2}\eta c(t;x)\psi(t;x), \tag{19d}$$

$$\delta\bar{\psi}(t;x) = -ie\mu^{\epsilon/2}\eta c(t;x)\bar{\psi}(t;x). \tag{19e}$$

To show this, we need to check that δ commutes with ∂_t. Let us check only two here:

$$\delta\partial_t A_\mu(t;x) = \delta\partial^2 A_\mu(t;x) = \partial^2\delta A_\mu(t;x) = \eta\partial^2\partial_\mu c(t;x) \tag{20}$$

is consistent with

$$\partial_t\delta A_\mu(t;x) = \partial_t\eta\partial_\mu c(t;x) = \eta\partial_\mu\partial_t c(t;x) = \eta\partial_\mu\partial^2 c(t;x). \tag{21}$$

We also find that

$$\begin{aligned}
\delta\partial_t\psi(t;x) &= \delta\left[\partial^2 - 2ie\mu^{\epsilon/2}A_\mu(t;x)\partial_\mu - e^2\mu^\epsilon A_\mu(t;x)A_\mu(t;x)\right]\psi(t;x)\\
&= \left[\partial^2 - 2ie\mu^{\epsilon/2}A_\mu(t;x)\partial_\mu - e^2\mu^\epsilon A_\mu(t;x)A_\mu(t;x)\right]\left[\eta ie\mu^{\epsilon/2}c(t;x)\psi(t;x)\right]\\
&\quad + \left[-2ie\mu^{\epsilon/2}\eta\partial_\mu c(t;x)\partial_\mu - 2e^2\mu^\epsilon A_\mu(t;x)\eta\partial_\mu c(t;x)\right]\psi(t;x)\\
&= \eta ie\mu^{\epsilon/2}\partial^2 c(t;x)\cdot\psi(t;x) + \eta ie\mu^{\epsilon/2}c(t;x)\\
&\quad\times\left[\partial^2 - 2ie\mu^{\epsilon/2}A_\mu(t;x)\partial_\mu - e^2\mu^\epsilon A_\mu(t;x)A_\mu(t;x)\right]\psi(t;x)
\end{aligned} \tag{22}$$

is consistent with

$$\begin{aligned}
\partial_t\delta\psi(t;x) &= \partial_t\left[\eta ie\mu^{\epsilon/2}c(t;x)\psi(t;x)\right]\\
&= \eta ie\mu^{\epsilon/2}\partial^2 c(t;x)\psi(t;x) + \eta ie\mu^{\epsilon/2}c(t;x)\\
&\quad\times\left[\partial^2 - 2ie\mu^{\epsilon/2}A_\mu(t;x)\partial_\mu - e^2\mu^\epsilon A_\mu(t;x)A_\mu(t;x)\right]\psi(t;x).
\end{aligned} \tag{23}$$

We have thus introduced diffusion of fields preserving the form of the BRST transformation. Our aim is to construct a Wilson action of QED using the diffused fields as the elementary fields.

3. GFERG for QED

We introduce the Wilson action of QED as

$$
\begin{aligned}
e^{S_\Lambda[A_\mu,c,\bar{c},\psi,\bar{\psi}]} \equiv \int \left[dA'_\mu dc' d\bar{c}' d\psi' d\bar{\psi}'\right] \\
\times \exp\Bigg[-\frac{\Lambda^2}{2}\int d^D x \left(A_\mu - z_\Lambda A'_{\Lambda\mu}\right)^2 - \Lambda^2 \int d^D x \left(\bar{c} - \bar{c}'_\Lambda\right)\left(c - c'_\Lambda\right) \\
+ i\Lambda \int d^D x \left(\bar{\psi} - z_{F\Lambda}\bar{\psi}'_\Lambda\right)\left(\psi - z_{F\Lambda}\psi'_\Lambda\right) + S\left[A'_\mu, c', \bar{c}', \psi', \bar{\psi}'\right]\Bigg],
\end{aligned}
\tag{24}
$$

where z_Λ and $z_{F\Lambda}$ satisfy

$$
-\Lambda\frac{\partial}{\partial\Lambda}\ln z_\Lambda = \gamma_\Lambda, \tag{25a}
$$

$$
-\Lambda\frac{\partial}{\partial\Lambda}\ln z_{F\Lambda} = \gamma_{F\Lambda}. \tag{25b}
$$

We do not introduce any factor of wave function renormalization for the ghosts since they remain free fields (see below). The diffused fields satisfy

$$
-\Lambda\partial_\Lambda A_{\Lambda\mu} = \frac{2}{\Lambda^2}\partial^2 A_{\Lambda\mu}, \tag{26a}
$$

$$
-\Lambda\partial_\Lambda c_\Lambda = \frac{2}{\Lambda^2}\partial^2 c_\Lambda, \tag{26b}
$$

$$
-\Lambda\partial_\Lambda \bar{c}_\Lambda = \frac{2}{\Lambda^2}\partial^2 \bar{c}_\Lambda, \tag{26c}
$$

$$
-\Lambda\partial_\Lambda\psi_\Lambda = \frac{2}{\Lambda^2}\left(\partial^2 - 2ie\mu^{\epsilon/2}A_{\Lambda\mu}\partial_\mu - e^2\mu^\epsilon A_{\Lambda\mu}A_{\Lambda\mu}\right)\psi_\Lambda, \tag{26d}
$$

$$
-\Lambda\partial_\Lambda\bar{\psi}_\Lambda = \frac{2}{\Lambda^2}\left(\partial^2 + 2ie\mu^{\epsilon/2}A_{\Lambda\mu}\partial_\mu - e^2\mu^\epsilon A_{\Lambda\mu}A_{\Lambda\mu}\right)\bar{\psi}_\Lambda. \tag{26e}
$$

3.1 *GFERG differential equation*

We wish to obtain

$$
-\Lambda\partial_\Lambda e^{S_\Lambda[A_\mu,c,\bar{c},\psi,\bar{\psi}]} \tag{27}
$$

in terms of the functional derivatives with respect to the field variables. As a preparation, we note the following correspondence:

$$
-\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu(x)} \longleftrightarrow A_\mu(x) - z_\Lambda A'_{\Lambda\mu}(x), \tag{28a}
$$

$$
-\frac{1}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(x)} \longleftrightarrow c(x) - c'_\Lambda(x), \tag{28b}
$$

$$
-\frac{1}{\Lambda^2}\frac{\overleftarrow{\delta}}{\delta c(x)} \longleftrightarrow \bar{c}(x) - \bar{c}'_\Lambda(x), \tag{28c}
$$

$$
-\frac{i}{\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x)} \longleftrightarrow \bar{\psi}(x) - z_{F\Lambda}\bar{\psi}'_\Lambda(x), \tag{28d}
$$

$$
-\frac{i}{\Lambda}\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)} \longleftrightarrow \psi(x) - z_{F\Lambda}\psi'_\Lambda(x). \tag{28e}
$$

We need to be clear about what we mean by the above correspondence. Take Eq. (28a). The correspondence means

$$-\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu(x)}e^{S_\Lambda} = \int \left[dA'_\mu dc' d\bar{c}' d\psi' d\bar{\psi}'\right] \left[A_\mu(x) - z_\Lambda A'_{\Lambda\mu}(x)\right] \times \exp\left(\text{quadratic terms} + S\left[A'_\mu, c', \bar{c}', \psi', \bar{\psi}'\right]\right). \tag{29}$$

Differentiating this once more we obtain

$$\begin{aligned}(-)\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu(x_2)}(-)\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu(x_1)}e^{S_\Lambda} &= \int \left[dA'_\mu dc' d\bar{c}' d\psi' d\bar{\psi}'\right] \left[A_\mu(x_1) - z_\Lambda A'_{\Lambda\mu}(x_1)\right]\left[A_\mu(x_2) - z_\Lambda A'_{\Lambda\mu}(x_2)\right] \\ &\times \exp\left(\text{quadratic terms} + S\left[A'_\mu, c', \bar{c}', \psi', \bar{\psi}'\right]\right),\end{aligned} \tag{30}$$

where we have chosen $x_2 \neq x_1$ so that the second differentiation with respect to $A_\mu(x_2)$ acts only on the exponential, but not on $A_\mu(x_1)$ in the integrand. Taking the limit $x_2 \to x_1$, we obtain

$$\begin{aligned}\lim_{x_2\to x_1}(-)\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu(x_2)}(-)\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu(x_1)}e^{S_\Lambda} &= \int \left[dA'_\mu dc' d\bar{c}' d\psi' d\bar{\psi}'\right] \left[A_\mu(x_1) - z_\Lambda A'_{\Lambda\mu}(x_1)\right]\left[A_\mu(x_1) - z_\Lambda A'_{\Lambda\mu}(x_1)\right] \\ &\times \exp\left(\text{quadratic terms} + S\left[A'_\mu, c', \bar{c}', \psi', \bar{\psi}'\right]\right).\end{aligned} \tag{31}$$

Similarly, we obtain

$$\lim_{x'\to x}(-)\frac{1}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(x)}e^{S_\Lambda}(-)\frac{1}{\Lambda^2}\frac{\overleftarrow{\delta}}{\delta c(x')} = \int \left[dA'_\mu dc' d\bar{c}' d\psi' d\bar{\psi}'\right] \left[c(x) - c'_\Lambda(x)\right]\left[\bar{c}(x) - \bar{c}'_\Lambda(x)\right] \times \exp\left(\text{quadratic terms} + S\left[A'_\mu, c', \bar{c}', \psi', \bar{\psi}'\right]\right), \tag{32}$$

where we take the limit $x' \to x$ after the differentiation.

We now calculate

$$\begin{aligned}-\Lambda\frac{\partial}{\partial\Lambda}e^{S_\Lambda} = \int \left[dA'_\mu dc' d\bar{c}' d\psi' d\bar{\psi}'\right] e^{\text{quadratic terms}+S\left[A'_\mu, c', \bar{c}', \psi', \bar{\psi}'\right]} \\ \times \int d^D x \Bigg\{ \Lambda^2\left(A_\mu - z_\Lambda A'_{\Lambda\mu}\right)^2 + 2\Lambda^2\left(\bar{c} - \bar{c}'_\Lambda\right)\left(c - c'_\Lambda\right) - i\Lambda\left(\bar{\psi} - z_{F\Lambda}\bar{\psi}'_\Lambda\right) \\ \times\left(\psi - z_{F\Lambda}\psi'_\Lambda\right) + \Lambda^2\left(A_\mu - z_\Lambda A'_{\Lambda\mu}\right) z_\Lambda\left(\gamma_\Lambda A'_{\Lambda\mu} + \frac{2}{\Lambda^2}\partial^2 A'_{\Lambda\mu}\right) \\ + 2\partial^2\bar{c}'_\Lambda\left(c - c'_\Lambda\right) + 2\left(\bar{c} - \bar{c}'_\Lambda\right)\partial^2 c'_\Lambda \\ - i\Lambda z_{F\Lambda}\left[\gamma_{F\Lambda} + \frac{2}{\Lambda^2}\left(\partial^2 + 2ie\mu^{\epsilon/2}A'_{\Lambda\mu}\partial_\mu - e^2\mu^\epsilon A'_{\Lambda\mu}A'_{\Lambda\mu}\right)\right]\bar{\psi}' \\ \times\left(\psi - z_{F\Lambda}\psi'\right) - i\Lambda z_{F\Lambda}\left(\bar{\psi} - z_{F\Lambda}\bar{\psi}'\right) \\ \times\left[\gamma_{F\Lambda} + \frac{2}{\Lambda^2}\left(\partial^2 - 2ie\mu^{\epsilon/2}A'_{\Lambda\mu}\partial_\mu - e^2\mu^\epsilon A'_{\Lambda\mu}A'_{\Lambda\mu}\right)\right]\psi' \Bigg\}.\end{aligned} \tag{33}$$

Using the correspondence given above, we can rewrite the rhs as

$$
\begin{aligned}
-\Lambda\frac{\partial}{\partial\Lambda}e^{S_\Lambda} = \int d^Dx \Bigg(& \frac{1}{\Lambda^2}\frac{\delta^2}{\delta A_\mu(x)\delta A_\mu(x')}e^{S_\Lambda} - \frac{2}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(x)}e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta c(x')} - \frac{i}{\Lambda}\,\mathrm{Tr}\,\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x')} \\
& - \frac{2}{\Lambda^2}\partial^2\left[A_\mu(x)+\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu(x)}\right]\frac{\delta}{\delta A_\mu(x')}e^{S_\Lambda} \\
& + \frac{2}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(x')}e^{S_\Lambda}\partial^2\left[\bar{c}(x)+\frac{1}{\Lambda^2}\frac{\overleftarrow{\delta}}{\delta c(x)}\right] + \frac{2}{\Lambda^2}\partial^2\left[c(x)+\frac{1}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(x)}\right]e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta c(x')} \\
& + \mathrm{Tr}\,\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x')}e^{S_\Lambda}\Bigg\{\frac{2}{\Lambda^2}\partial^2+\frac{4ie_\Lambda}{\Lambda^2}\left(A_\mu+\frac{1}{\Lambda^2}\frac{\overleftarrow{\delta}}{\delta A_\mu}\right)\partial_\mu \\
& - \frac{2e_\Lambda^2}{\Lambda^2}\left(A_\mu+\frac{1}{\Lambda^2}\frac{\overleftarrow{\delta}}{\delta A_\mu}\right)\left[A_\mu(x')+\frac{1}{\Lambda^2}\frac{\overleftarrow{\delta}}{\delta A_\mu(x')}\right]\Bigg\}\left[\bar{\psi}(x)+\frac{i}{\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right] \\
& + \mathrm{Tr}\Bigg\{\frac{2}{\Lambda^2}\partial^2-\frac{4ie_\Lambda}{\Lambda^2}\left(A_\mu+\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu}\right)\partial_\mu \\
& - \frac{2e_\Lambda^2}{\Lambda^2}\left(A_\mu+\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu}\right)\left[A_\mu(x')+\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu(x')}\right]\Bigg\} \\
& \times\left[\psi(x)+\frac{i}{\Lambda}\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\right]e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x')} - \gamma_\Lambda\left(A_\mu+\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu}\right)\frac{\delta}{\delta A_\mu}e^{S_\Lambda} \\
& + \gamma_{F\Lambda}\,\mathrm{Tr}\,\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x')}e^{S_\Lambda}\left[\bar{\psi}(x)+\frac{i}{\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right] \\
& + \gamma_{F\Lambda}\,\mathrm{Tr}\left[\psi(x)+\frac{i}{\Lambda}\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\right]e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x')}\Bigg) \\
= \int d^Dx \Bigg\{ & \Bigg[\left(-\frac{2}{\Lambda^2}\partial^2-\gamma_\Lambda\right)A_\mu(x)\cdot\frac{\delta}{\delta A_\mu(x)} \\
& + \left(-\frac{2}{\Lambda^2}\partial^2-\gamma_\Lambda+1\right)\frac{1}{\Lambda^2}\frac{\delta^2}{\delta A_\mu(x)\delta A_\mu(x')}\Bigg]e^{S_\Lambda} \\
& + \frac{2}{\Lambda^2}\left[-\partial^2\bar{c}(x)\frac{\overrightarrow{\delta}}{\delta\bar{c}(x)}e^{S_\Lambda} - e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta c(x)}\partial^2c(x)\right] \\
& + \left[\frac{1}{\Lambda^2}\left(\partial^2+\partial'^2\right)-1\right]\frac{2}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(x)}e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta c(x')} \\
& + \mathrm{Tr}\,\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x')}e^{S_\Lambda}\left(\frac{2}{\Lambda^2}\partial^2+\gamma_{F\Lambda}\right)\left[\bar{\psi}(x)+\frac{i}{\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right] \\
& + \mathrm{Tr}\left(\frac{2}{\Lambda^2}\partial^2+\gamma_{F\Lambda}\right)\left[\psi(x)+\frac{i}{\Lambda}\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\right]e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x')}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{i}{\Lambda}\,\mathrm{Tr}\,\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x')}\Bigg\} \\
&+ \int d^D x\;\mathrm{Tr}\Bigg(\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x')}e^{S_\Lambda}\Bigg\{\frac{4ie_\Lambda}{\Lambda^2}\left(A_\mu+\frac{1}{\Lambda^2}\frac{\overleftarrow{\delta}}{\delta A_\mu}\right)\partial_\mu \\
&- \frac{2e_\Lambda^2}{\Lambda^2}\left(A_\mu+\frac{1}{\Lambda^2}\frac{\overleftarrow{\delta}}{\delta A_\mu}\right)\left[A_\mu(x')+\frac{1}{\Lambda^2}\frac{\overleftarrow{\delta}}{\delta A_\mu(x')}\right]\Bigg\}\left[\bar{\psi}(x)+\frac{i}{\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right] \\
&+ \Bigg\{-\frac{4ie_\Lambda}{\Lambda^2}\left(A_\mu+\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu}\right)\partial_\mu \\
&- \frac{2e_\Lambda^2}{\Lambda^2}\left(A_\mu+\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu}\right)\left[A_\mu(x')+\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu(x')}\right]\Bigg\} \\
&\times\left[\psi(x)+\frac{i}{\Lambda}\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\right]e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x')}\Bigg),
\end{aligned}
\tag{34}
$$

where the limit $x' \to x$ is implied, and we have defined the gauge coupling of mass dimension $\epsilon/2$ by

$$
e_\Lambda \equiv \frac{e\mu^{\epsilon/2}}{z_\Lambda}. \tag{35}
$$

Note that we have given Eq. (34) in two parts. The first part reproduces the ERG differential equation. (See Appendix C for a quick review of ERG for QED.) The second part is unique to GFERG; it comes from the BRST covariance of the electron diffusion equations (26).

3.2 *BRST invariance*

We next derive the expression of BRST invariance of S_Λ, inherited from the invariance of $S[A'_\mu, c', \bar{c}', \psi', \bar{\psi}']$ under the BRST transformation:

$$
\delta A'_\mu = \eta\partial_\mu c', \tag{36a}
$$

$$
\delta c' = 0, \tag{36b}
$$

$$
\delta\bar{c}' = \eta\frac{1}{\xi}\partial_\mu A'_\mu, \tag{36c}
$$

$$
\delta\psi' = \eta ie\mu^{\epsilon/2}c'\psi', \tag{36d}
$$

$$
\delta\bar{\psi}' = \eta(-i)e\mu^{\epsilon/2}c'\bar{\psi}'. \tag{36e}
$$

As explained in Sect. 2.2, this induces the diffused fields to transform as

$$
\delta A'_{\Lambda\mu} = \eta\partial_\mu c'_\Lambda, \tag{37a}
$$

$$
\delta c'_\Lambda = 0, \tag{37b}
$$

$$
\delta\bar{c}'_\Lambda = \eta\frac{1}{\xi}\partial_\mu A'_{\Lambda\mu}, \tag{37c}
$$

$$\delta\psi'_\Lambda = \eta ie\mu^{\epsilon/2}c'_\Lambda\psi'_\Lambda, \tag{37d}$$

$$\delta\bar{\psi}'_\Lambda = \eta(-i)e\mu^{\epsilon/2}c'_\Lambda\bar{\psi}'_\Lambda. \tag{37e}$$

Hence, we obtain

$$\begin{aligned}
0 = &\int \left[dA'_\mu dc' d\bar{c}' d\psi' d\bar{\psi}'\right] e^{\text{quadratic terms}+S[A'_\mu,c',\bar{c}',\psi',\bar{\psi}']} \\
&\times \int d^Dx \Bigg[\Lambda^2 z_\Lambda \eta\partial_\mu c'_\Lambda \cdot \left(A_\mu - z_\Lambda A'_{\Lambda\mu}\right) + \Lambda^2\frac{1}{\xi}\eta\partial_\mu A'_{\Lambda\mu}\cdot\left(c - c'_\Lambda\right) \\
&+ i\Lambda z_{F\Lambda}\eta ie\mu^{\epsilon/2}c'_\Lambda\bar{\psi}'_\Lambda\left(\psi - z_{F\Lambda}\psi'_\Lambda\right) \\
&+ i\Lambda\left(\bar{\psi} - z_{F\Lambda}\bar{\psi}'_\Lambda\right)\left(-\eta z_{F\Lambda} ie\mu^{\epsilon/2}c'_\Lambda\psi'_\Lambda\right)\Bigg].
\end{aligned} \tag{38}$$

Dividing this by z_Λ, and defining

$$\xi_\Lambda \equiv \xi z_\Lambda^2, \tag{39}$$

we can rewrite this as

$$\begin{aligned}
0 = &\int \left[dA'_\mu dc' d\bar{c}' d\psi' d\bar{\psi}'\right] e^{\text{quadratic terms}+S[A'_\mu,c',\bar{c}',\psi',\bar{\psi}']} \\
&\times \int d^Dx \Bigg[\Lambda^2\partial_\mu c'_\Lambda \cdot \left(A_\mu - z_\Lambda A'_{\Lambda\mu}\right) + \Lambda^2\frac{1}{\xi_\Lambda}z_\Lambda\partial_\mu A'_{\Lambda\mu}\cdot\left(c - c'_\Lambda\right) \\
&+ i\Lambda z_{F\Lambda} ie_\Lambda c'_\Lambda\bar{\psi}'_\Lambda\left(\psi - z_{F\Lambda}\psi'_\Lambda\right) + i\Lambda\left(\bar{\psi} - z_{F\Lambda}\bar{\psi}'_\Lambda\right) z_{F\Lambda} ie_\Lambda c'_\Lambda\psi'_\Lambda\Bigg],
\end{aligned} \tag{40}$$

where we recall Eq. (35) defining e_Λ. Using the differentials, we can rewrite this further as

$$\begin{aligned}
&\int d^Dx \Bigg\{-\partial_\mu\left[c(x) + \frac{1}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(x)}\right]\frac{\delta}{\delta A_\mu(x')}e^{S_\Lambda} - \frac{1}{\xi_\Lambda}\partial_\mu\left[A_\mu(x) + \frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu}(x)\right]\frac{\overrightarrow{\delta}}{\delta\bar{c}(x')}e^{S_\Lambda} \\
&- ie_\Lambda\left[c(x) + \frac{1}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(x)}\right]\mathrm{Tr}\,\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x')}e^{S_\Lambda}\left[\bar{\psi}(x) + \frac{i}{\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right] \\
&+ ie_\Lambda\left[c(x) + \frac{1}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(x)}\right]\mathrm{Tr}\left[\psi(x) + \frac{i}{\Lambda}\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\right]e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x')}\Bigg\} = 0.
\end{aligned} \tag{41}$$

The terms with second-order differentials with respect to ψ and $\bar{\psi}$ cancel, and finally we obtain

$$\begin{aligned}
&\int d^Dx \Bigg\{-\partial_\mu\left[c(x) + \frac{1}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(x)}\right]\frac{\delta}{\delta A_\mu(x)}e^{S_\Lambda} - \frac{1}{\xi_\Lambda}\partial_\mu\left[A_\mu(x) + \frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu(x)}\right]\frac{\overrightarrow{\delta}}{\delta\bar{c}(x)}e^{S_\Lambda} \\
&+ ie_\Lambda\left[c(x) + \frac{1}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(x)}\right]\bar{\psi}(x)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}e^{S_\Lambda} \\
&- ie_\Lambda\left[c(x) + \frac{1}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(x)}\right]e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\psi(x)\Bigg\} = 0.
\end{aligned} \tag{42}$$

This comes pretty close to manifest BRST invariance. The simple expression is due to the BRST covariant diffusion of the fields, Eq. (26). In Appendix C we give the standard ERG formulation of QED for comparison.

4. GFERG and BRST in the dimensionless framework

To gain more insights into the scaling properties of the Wilson action, we adopt the dimensionless framework. Instead of the momentum cutoff Λ, we use the dimensionless logarithmic scale parameter τ defined by

$$\Lambda = \mu e^{-\tau}. \tag{43}$$

We write the Fourier transforms of the fields in terms of the dimensionless fields (with tildes) as

$$A_\mu(k) = \Lambda^{-(D+2)/2}\tilde{A}_\mu\left(k/\Lambda\right), \tag{44a}$$

$$c(k) = \Lambda^{-(D+4)/2}\tilde{c}\left(k/\Lambda\right), \tag{44b}$$

$$\bar{c}(-k) = \Lambda^{-D/2}\tilde{\bar{c}}\left(-k/\Lambda\right), \tag{44c}$$

$$\psi(p) = \Lambda^{-(D+1)/2}\tilde{\psi}\left(p/\Lambda\right), \tag{44d}$$

$$\bar{\psi}(-p) = \Lambda^{-(D+1)/2}\tilde{\bar{\psi}}\left(-p/\Lambda\right). \tag{44e}$$

The dimensionless gauge coupling is defined by

$$\tilde{e}_\tau \equiv \frac{e_\Lambda}{\Lambda^{\epsilon/2}} = \frac{e}{z_\Lambda}\left(\frac{\mu}{\Lambda}\right)^{\epsilon/2}. \tag{45}$$

The gauge-fixing parameter remains the same:

$$\tilde{\xi}_\tau = \xi_\Lambda = \xi z_\Lambda^2. \tag{46}$$

Denoting

$$\tilde{\gamma}_\tau = \gamma_\Lambda = -\Lambda\frac{\partial}{\partial\Lambda}\ln z_\Lambda, \tag{47}$$

we obtain

$$\frac{d}{d\tau}\tilde{e}_\tau = \left(\frac{\epsilon}{2} - \tilde{\gamma}_\tau\right)\tilde{e}_\tau, \tag{48}$$

$$\frac{d}{d\tau}\tilde{\xi}_\tau = 2\tilde{\gamma}_\tau\tilde{\xi}_\tau. \tag{49}$$

The Wilson action in the dimensionless framework is a functional of the dimensionless fields above:

$$\tilde{S}_\tau\left[\tilde{A}_\mu, \tilde{c}, \tilde{\bar{c}}, \tilde{\psi}, \tilde{\bar{\psi}}\right] \equiv S_\Lambda\left[A_\mu, c, \bar{c}, \psi, \bar{\psi}\right]. \tag{50}$$

In order to rewrite the GFERG differential equation (34) in the dimensionless framework, we need to use

$$\begin{aligned}
\frac{\partial}{\partial\tau}e^{\tilde{S}_\tau} &= -\Lambda\frac{\partial}{\partial\Lambda}e^{S_\Lambda} \\
&+ \int_k \left(\frac{D+2}{2} + k\cdot\partial_k\right)\tilde{A}_\mu(k)\cdot\frac{\delta}{\delta\tilde{A}_\mu(k)}e^{\tilde{S}_\tau} \\
&+ \int_k e^{\tilde{S}_\tau}\frac{\overleftarrow{\delta}}{\delta\tilde{c}(k)}\left(\frac{D+4}{2} + k\cdot\partial_k\right)\tilde{c}(k) + \int_k \left(\frac{D}{2} + k\cdot\partial_k\right)\tilde{\bar{c}}(-k)\cdot\frac{\overrightarrow{\delta}}{\delta\tilde{\bar{c}}(-k)}e^{\tilde{S}_\tau} \\
&+ \int_p e^{\tilde{S}_\tau}\frac{\overleftarrow{\delta}}{\delta\tilde{\psi}(p)}\left(\frac{D+1}{2} + p\cdot\partial_p\right)\tilde{\psi}(p) + \int_p \left(\frac{D+1}{2} + p\cdot\partial_p\right)\tilde{\bar{\psi}}(-p)\cdot\frac{\overrightarrow{\delta}}{\delta\tilde{\bar{\psi}}(-p)}e^{\tilde{S}_\tau}.
\end{aligned} \tag{51}$$

Since we work only in the dimensionless framework from now, we omit the tildes altogether, and we obtain the GFERG equation as

$$\begin{aligned}
\partial_\tau e^{S_\tau} &= \int_k \Bigg[\left(2k^2 + \frac{D+2}{2} - \gamma_\tau + k\cdot\partial_k\right)A_\mu(k)\cdot\frac{\delta}{\delta A_\mu(k)}e^{S_\tau} \\
&+ \left(2k^2 + 1 - \gamma_\tau\right)\frac{\delta^2}{\delta A_\mu(k)\delta A_\mu(-k)}e^{S_\tau}\Bigg] \\
&+ \int_k \Bigg[\left(2k^2 + \frac{D}{2} + k\cdot\partial_k\right)\bar{c}(-k)\cdot\frac{\overrightarrow{\delta}}{\delta\bar{c}(-k)}e^{S_\tau} \\
&+ e^{S_\tau}\frac{\overleftarrow{\delta}}{\delta c(k)}\left(2k^2 + \frac{D+4}{2} + k\cdot\partial_k\right)c(k) - 2(2k^2+1)\frac{\overrightarrow{\delta}}{\delta\bar{c}(-k)}e^{S_\tau}\frac{\overleftarrow{\delta}}{\delta c(k)}\Bigg] \\
&+ \int_p \Bigg[e^{S_\tau}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\left(2p^2 + \frac{D+1}{2} - \gamma_{F\tau} + p\cdot\partial_p\right)\psi(p) \\
&+ \left(2p^2 + \frac{D+1}{2} - \gamma_{F\tau} + p\cdot\partial_p\right)\bar{\psi}(-p)\cdot\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}e^{S_\tau} \\
&- i\left(4p^2 + 1 - 2\gamma_{F\tau}\right)\mathrm{Tr}\,\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}e^{S_\tau}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\Bigg] \\
&+ \int d^Dx\ \mathrm{Tr}\,\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x')}e^{S_\tau}\Bigg\{4ie_\tau\left[A_\mu(x) + \frac{\overleftarrow{\delta}}{\delta A_\mu(x)}\right]\partial_\mu \\
&- 2e_\tau^2\left[A_\mu(x) + \frac{\overleftarrow{\delta}}{\delta A_\mu(x)}\right]\left[A_\mu(x') + \frac{\overleftarrow{\delta}}{\delta A_\mu(x')}\right]\Bigg\}\left[\bar{\psi}(x) + i\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right] \\
&+ \int d^Dx\ \mathrm{Tr}\Bigg\{-4ie_\tau\left[A_\mu(x) + \frac{\delta}{\delta A_\mu(x)}\right]\partial_\mu \\
&- 2e_\tau^2\left[A_\mu(x) + \frac{\delta}{\delta A_\mu(x)}\right]\left[A_\mu(x') + \frac{\delta}{\delta A_\mu(x')}\right]\Bigg\}\left[\psi(x) + i\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\right]e^{S_\tau}\frac{\overleftarrow{\delta}}{\delta\psi(x')}.
\end{aligned} \tag{52}$$

Except for the last two integrals, the rhs coincides with the ERG equation. For the last two integrals we have kept the coordinate space notation to take the limit $x' \to x$ carefully.[4]

Similarly, we can rewrite the BRST invariance in the dimensionless framework. In coordinate space it is given by

$$\int d^D x \left\{ -\partial_\mu \left[c(x) + \frac{\overrightarrow{\delta}}{\delta \bar{c}(x)} \right] \frac{\delta}{\delta A_\mu(x)} e^{S_\tau} - \frac{1}{\xi_\tau} \partial_\mu \left[A_\mu(x) + \frac{\delta}{\delta A_\mu(x)} \right] \frac{\overrightarrow{\delta}}{\delta \bar{c}(x)} e^{S_\tau} \right.$$
$$\left. + ie_\tau \left[c(x) + \frac{\overrightarrow{\delta}}{\delta \bar{c}(x)} \right] \bar{\psi}(x) \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(x)} e^{S_\tau} - ie_\tau \left[c(x) + \frac{\overrightarrow{\delta}}{\delta \bar{c}(x)} \right] e^{S_\tau} \frac{\overleftarrow{\delta}}{\delta \psi(x)} \psi(x) \right\} = 0. \quad (53)$$

5. Perturbative solution

In constructing the Wilson action S_τ, we assume that it does not depend on τ explicitly: we assume that its τ-dependence comes only through the τ-dependence of three dimensionless parameters, i.e., the gauge coupling e_τ, gauge-fixing parameter ξ_τ, and the electron mass parameter m_τ. This assumption is not valid, however, for S_τ, the dimensionless version of S_Λ given by Eq. (24). As long as the renormalization scale μ is finite, S_Λ depends on $\Lambda/\mu = e^{-\tau}$ explicitly. To remove this, we must take the "continuum limit"; i.e., we must take $\mu \to +\infty$.

Both the differential equation (52) and the BRST invariance (53) have been derived based on the integral formula (24), but neither has explicit dependence on τ. In practice we can construct the continuum limit of S_τ by solving Eqs. (52) and (53) simultaneously under the above assumption. The gauge coupling e_τ and the gauge-fixing parameter ξ_τ are introduced through the BRST invariance (53). We normalize the kinetic terms of the gauge and electron fields; this fixes the anomalous dimensions γ_τ and $\gamma_{F\tau}$ as functions of e_τ^2 although they may also depend on ξ_τ. (We believe that neither depends on ξ_τ.) We introduce a normalization condition of the electron mass term that determines the τ-dependence of m_τ in the form

$$\frac{d}{d\tau} m_\tau = \left[1 + \beta_m \left(e_\tau^2 \right) \right] m_\tau. \quad (54)$$

We believe that β_m is also independent of ξ_τ. We thus obtain

$$\partial_\tau e^{S_\tau} = \partial_\tau e^{S(e_\tau, \xi_\tau, m_\tau)}$$
$$= \left\{ \left[\frac{\epsilon}{2} - \gamma \left(e_\tau^2 \right) \right] e_\tau \frac{\partial}{\partial e_\tau} + 2\gamma \left(e_\tau^2 \right) \xi_\tau \frac{\partial}{\partial \xi_\tau} + \left[1 + \beta_m \left(e_\tau^2 \right) \right] m_\tau \frac{\partial}{\partial m_\tau} \right\} e^{S(e_\tau, \xi_\tau, m_\tau)}. \quad (55)$$

We will drop the suffix τ from the parameters e_τ, ξ_τ, and m_τ.[5] Thus, our GFERG differential equation becomes

$$\left\{ \left[\frac{\epsilon}{2} - \gamma(e^2) \right] e \frac{\partial}{\partial e} + 2\gamma(e^2) \xi \frac{\partial}{\partial \xi} + \left[1 + \beta_m(e^2) \right] m \frac{\partial}{\partial m} \right\} e^{S(e, \xi, m)}$$
$$= \text{rhs of Eq. (4.10)}. \quad (56)$$

If we define the beta function of e^2 by

$$\beta(e^2) \equiv -2\gamma(e^2) e^2, \quad (57)$$

we can rewrite

$$\left[\frac{\epsilon}{2} - \gamma(e^2) \right] e \frac{\partial}{\partial e} = \left[\frac{\epsilon}{2} + \frac{\beta(e^2)}{2e^2} \right] e \frac{\partial}{\partial e}. \quad (58)$$

[4] The dimensionless coordinate is given by $\tilde{x}_\mu \equiv \Lambda x_\mu$. We have omitted the tilde in the GFERG equation.

[5] We write e for e_τ. It should not be confused with the renormalized coupling e of the original action S.

Our purpose is to solve the GFERG equation (56) and the BRST invariance (53) together perturbatively in powers of e. For this purpose we expand the Wilson action in powers of e:

$$S(e,\xi,m) = S^{(0)}(\xi,m) + eS^{(1)}(\xi,m) + e^2 S^{(2)}(\xi,m) + \cdots . \tag{59}$$

5.1 *Tree level*

$S^{(0)}$ satisfies the GFERG equation

$$\begin{aligned} m\partial_m S^{(0)} = \int_k \Bigg[& \left(2k^2 + \frac{D+2}{2} + k\cdot\partial_k\right) A_\mu(k)\cdot\frac{\delta S^{(0)}}{\delta A_\mu(k)} + 2(2k^2+1)\frac{1}{2}\frac{\delta S^{(0)}}{\delta A_\mu(k)}\frac{\delta S^{(0)}}{\delta A_\mu(-k)}\Bigg] \\ & + \int_k \Bigg[S^{(0)}\frac{\overleftarrow{\delta}}{\delta c(k)}\left(2k^2 + \frac{D+4}{2} + k\cdot\partial_k\right) c(k) \\ & + \left(2k^2 + \frac{D}{2} + k\cdot\partial_k\right)\bar{c}(-k)\cdot\frac{\overrightarrow{\delta}}{\delta\bar{c}(-k)}S^{(0)} \\ & + 2(2k^2+1)S^{(0)}\frac{\overleftarrow{\delta}}{\delta c(k)}\cdot\frac{\overrightarrow{\delta}}{\delta\bar{c}(-k)}S^{(0)}\Bigg] \\ & + \int_p \Bigg[S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\left(2p^2 + \frac{D+1}{2} + p\cdot\partial_p\right)\psi(p) \\ & + \left(2p^2 + \frac{D+1}{2} + p\cdot\partial_p\right)\bar{\psi}(-p)\cdot\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S^{(0)} \\ & + i(4p^2+1)S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S^{(0)}\Bigg]. \end{aligned} \tag{60}$$

This is solved by

$$\begin{aligned} S^{(0)} = & -\frac{1}{2}\int_k A_\mu(k)A_\nu(-k)\left[\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{k^2}{e^{-2k^2}+k^2} + \frac{k_\mu k_\nu}{k^2}\frac{k^2}{\xi e^{-2k^2}+k^2}\right] \\ & - \int_k \bar{c}(-k)c(k)\frac{k^2}{e^{-2k^2}+k^2} - \int_p \bar{\psi}(-p)\frac{\not{p}+im}{e^{-2p^2}+i(\not{p}+im)}\psi(p). \end{aligned} \tag{61}$$

We have normalized the kinetic terms appropriately.

The ξ-dependence of the longitudinal part is determined by the BRST invariance (53); at tree level it gives

$$\int_k \left\{ k_\mu\left[c(k) + \frac{\overrightarrow{\delta}}{\delta\bar{c}(-k)}S^{(0)}\right]\frac{\delta S^{(0)}}{\delta A_\mu(k)} - \frac{1}{\xi}k_\mu\left[A_\mu(-k) + \frac{\delta S^{(0)}}{\delta A_\mu(k)}\right]\frac{\overrightarrow{\delta}}{\delta\bar{c}(-k)}S^{(0)}\right\} = 0. \tag{62}$$

Let us check that Eq. (61) satisfies this:

$$\begin{aligned} & \int_k k_\mu\left[c(k) + \frac{\overrightarrow{\delta}}{\delta\bar{c}(-k)}S^{(0)}\right]\frac{\delta S^{(0)}}{\delta A_\mu(k)} \\ & = \int_k c(k)\left(1 - \frac{k^2}{e^{-2k^2}+k^2}\right)k_\mu A_\mu(-k)\frac{-k^2}{\xi e^{-2k^2}+k^2} \\ & = \int_k c(k)\frac{e^{-2k^2}}{e^{-2k^2}+k^2}k_\mu A_\mu(-k)\frac{-k^2}{\xi e^{-2k^2}+k^2}, \end{aligned} \tag{63a}$$

$$\int_k \frac{1}{\xi} k_\mu \left[A_\mu(-k) + \frac{\delta S^{(0)}}{\delta A_\mu(k)} \right] \frac{\overrightarrow{\delta}}{\delta \bar{c}(-k)} S^{(0)}$$
$$= \int_k \frac{1}{\xi} k_\mu A_\mu(-k) \left(1 - \frac{k^2}{\xi e^{-2k^2} + k^2} \right) \frac{-k^2}{e^{-2k^2} + k^2} c(k)$$
$$= \int_k k_\mu A_\mu(-k) \frac{e^{-2k^2}}{\xi e^{-2k^2} + k^2} \frac{-k^2}{e^{-2k^2} + k^2} c(k). \tag{63b}$$

Hence, $S^{(0)}$ is BRST invariant.

5.2 *BRST invariance simplified to manifest gauge invariance*

Before proceeding to calculate $S^{(1)}$, we stop to simplify our expression for the BRST invariance given in Eq. (53).

The GFERG equation (56) implies the absence of higher-order corrections to the ghost-dependent part of the action S.[6] Hence, the ghost part of the action is exactly as given in Eq. (61):

$$S_{\text{ghost}} = -\int_k \bar{c}(-k) \frac{k^2}{e^{-2k^2} + k^2} c(k). \tag{64}$$

Hence, we can rewrite the BRST invariance as

$$\int_k \left\{ (-ik_\mu) \frac{e^{-2k^2}}{e^{-2k^2} + k^2} c(k) \frac{\delta S}{\delta A_\mu(k)} + \frac{1}{\xi} ik_\mu \left[A_\mu(-k) + \frac{\delta S}{\delta A_\mu(k)} \right] \frac{-k^2}{e^{-2k^2} + k^2} c(k) \right.$$
$$\left. + ie \frac{e^{-2k^2}}{e^{-2k^2} + k^2} c(k) \int_p \bar{\psi}(-p-k) \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(-p)} S - ie \frac{e^{-2k^2}}{e^{-2k^2} + k^2} c(k) \int_p S \frac{\overleftarrow{\delta}}{\delta \psi(p+k)} \psi(p) \right\} = 0. \tag{65}$$

The integrand is proportional to $c(k)$, and its coefficient must vanish. This results in the Ward–Takahashi (WT) identity given by

$$\frac{\xi e^{-2k^2} + k^2}{\xi e^{-2k^2}} k_\mu \frac{\delta S_I}{\delta A_\mu(k)} = e \int_p \left[\bar{\psi}(-p-k) \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(-p)} S - S \frac{\overleftarrow{\delta}}{\delta \psi(p+k)} \psi(p) \right], \tag{66}$$

where

$$S_I \equiv S - S^{(0)} \tag{67}$$

is the interaction part, and we have used the BRST invariance of $S^{(0)}$. Equation (66) differs from the classical gauge invariance

$$k_\mu \frac{\delta S_{\text{classical}}}{\delta A_\mu(k)} = e \int_p \left[\bar{\psi}(-p-k) \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(-p)} S_{\text{classical}} - S_{\text{classical}} \frac{\overleftarrow{\delta}}{\delta \psi(p+k)} \psi(p) \right] \tag{68}$$

merely by the k-dependent factor on the lhs.

In fact Eq. (66) implies that the action

$$S_{\text{inv}} \equiv S + \frac{1}{2} \int_k A_\mu(k) A_\nu(-k) \frac{k_\mu k_\nu}{\xi e^{-2k^2} + k^2} + \int_k \bar{c}(-k) \frac{k^2}{e^{-2k^2} + k^2} c(k) \tag{69}$$

without the gauge-fixing and ghost terms is invariant under the infinitesimal "gauge" transformation

[6] Recall Eq. (24). Since the original action S is quadratic in ghost fields, S_Λ produced by the Gaussian functional integration of S remains quadratic in ghost fields.

$$\delta A_\mu(k) = -\frac{\xi e^{-2k^2} + k^2}{\xi e^{-2k^2}} k_\mu \chi(k), \tag{70a}$$

$$\delta\psi(p) = -e\int_k \chi(k)\psi(p-k), \tag{70b}$$

$$\delta\bar{\psi}(-p) = e\int_k \chi(k)\bar{\psi}(-p-k), \tag{70c}$$

where $\chi(k)$ is an arbitrary infinitesimal function. On this account we may call $S_{\rm inv}$ manifestly gauge invariant. The meaning of this gauge invariance is left for future studies. Please note that Eq. (66) is valid in the presence of additional interaction parameters in the action. In deriving Eq. (66) we have only assumed that the ghost part is given by Eq. (64); as long as the action satisfies the BRST invariance (53), we can derive Eq. (66).

5.3 *First order*

Since the gauge coupling e in Eq. (56) accompanies the gauge field, the first-order term $S^{(1)}$ must have the structure:

$$S^{(1)} = \int_{p,k} \bar{\psi}(-p-k)V_\mu(p,k)\psi(p)A_\mu(k). \tag{71}$$

Because of the charge conjugation symmetry of our formulation, we can exclude the term cubic in the gauge potential [15].

Equation (56) gives

$$\begin{aligned}
&\left(\frac{4-D}{2} + m\frac{\partial}{\partial m}\right) S^{(1)} \\
&= \int_k \left[\left(2k^2 + \frac{D+2}{2} + k\cdot\frac{\partial}{\partial k}\right) A_\mu(k) + 2(2k^2+1)\frac{\delta S^{(0)}}{\delta A_\mu(-k)}\right]\frac{\delta S^{(1)}}{\delta A_\mu(k)} \\
&\quad + \int_p S^{(1)}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\left[\left(2p^2 + \frac{D+1}{2} + p\cdot\frac{\partial}{\partial p}\right)\psi(p) + i(4p^2+1)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S^{(0)}\right] \\
&\quad + \int_p \left[\left(2p^2 + \frac{D+1}{2} + p\cdot\frac{\partial}{\partial p}\right)\bar{\psi}(-p) + i(4p^2+1)S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\right]\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S^{(1)} \\
&\quad + 4\int_{p,k} \mathrm{Tr}\left\{p_\mu\left[\psi(p) + i\frac{\delta}{\delta\bar{\psi}(-p)}S^{(0)}\right]S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(p+k)}\right\}\left[A_\mu(k) + \frac{\delta S^{(0)}}{\delta A_\mu(-k)}\right] \\
&\quad + 4\int_{p,k} \mathrm{Tr}\left\{\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S^{(0)}\left[\bar{\psi}(-p-k) + iS^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(p+k)}\right](p+k)_\mu\right\} \\
&\quad \times\left[A_\mu(k) + \frac{\delta S^{(0)}}{\delta A_\mu(-k)}\right].
\end{aligned} \tag{72}$$

At this stage, it is very helpful to introduce new variables by[7]

$$\mathcal{A}_\mu(k) \equiv A_\mu(k) + \frac{\delta S^{(0)}}{\delta A_\mu(-k)} = e^{-2k^2}h_{\mu\nu}(k)A_\nu(k), \tag{73a}$$

[7]Note that $e^{k^2}\mathcal{A}_\mu(k)$, $e^{p^2}\Psi(p)$, and $e^{p^2}\bar{\Psi}(p)$ are the variables of the 1PI action in the lowest order in perturbation theory (see, e.g., Eq. (23) of Ref. [25]). We thus expect that interaction vertices simplify if expressed in terms of these variables.

$$\Psi(p) \equiv \psi(p) + i\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S^{(0)} = e^{-2p^2}\frac{1}{i}h_F(p)\psi(p), \tag{73b}$$

$$\bar{\Psi}(-p) \equiv \bar{\psi}(-p) + iS^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(p)} = \bar{\psi}(-p)e^{-2p^2}\frac{1}{i}h_F(p), \tag{73c}$$

where

$$h_{\mu\nu}(k) \equiv \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{e^{-2k^2}+k^2} + \frac{k_\mu k_\nu}{k^2}\frac{\xi}{\xi e^{-2k^2}+k^2}, \tag{74a}$$

$$h_F(p) \equiv \frac{i}{e^{-2p^2} + i(\not{p} + im)}, \tag{74b}$$

from Eq. (61). These h-functions are the high-momentum propagators satisfying

$$\left(k\cdot\frac{\partial}{\partial k} + 2\right)h_{\mu\nu}(k) = 2(2k^2+1)e^{-2k^2}h_{\mu\rho}(k)h_{\rho\nu}(k), \tag{75a}$$

$$\left(p\cdot\frac{\partial}{\partial p} + m\frac{\partial}{\partial m} + 1\right)h_F(p) = (4p^2+1)e^{-2p^2}\frac{1}{i}h_F(p)^2, \tag{75b}$$

and

$$(\not{p} + im)h_F(p) = ie^{-2p^2}h_F(p) + 1. \tag{76}$$

Using Eq. (73), it is straightforward to show

$$\begin{aligned}&\left(k\cdot\frac{\partial}{\partial k} + \frac{D+2}{2}\right)\left[e^{k^2}\mathcal{A}_\mu(k)\right]\cdot\frac{\delta}{\delta\left[e^{k^2}\mathcal{A}_\mu(k)\right]}\\ &\quad= \left[\left(2k^2 + \frac{D+2}{2} + k\cdot\frac{\partial}{\partial k}\right)A_\mu(k) + 2(2k^2+1)\frac{\delta S^{(0)}}{\delta A_\mu(-k)}\right]\cdot\frac{\delta}{\delta A_\mu(k)},\end{aligned} \tag{77}$$

$$\begin{aligned}&\frac{\overleftarrow{\delta}}{\delta\left[e^{p^2}\Psi(p)\right]}\left(p\cdot\frac{\partial}{\partial p} + m\frac{\partial}{\partial m} + \frac{D+1}{2}\right)\left[e^{p^2}\Psi(p)\right]\\ &\quad= \frac{\overleftarrow{\delta}}{\delta\psi(p)}\left[\left(2p^2 + \frac{D+1}{2} + p\cdot\frac{\partial}{\partial p}\right)\psi(p) + i(4p^2+1)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S^{(0)}\right],\end{aligned} \tag{78}$$

and

$$\begin{aligned}&\left(p\cdot\frac{\partial}{\partial p} + m\frac{\partial}{\partial m} + \frac{D+1}{2}\right)\left[\bar{\Psi}(-p)e^{p^2}\right]\cdot\frac{\overrightarrow{\delta}}{\delta\left[\bar{\Psi}(-p)e^{p^2}\right]}\\ &\quad= \left[\left(2p^2 + \frac{D+1}{2} + p\cdot\frac{\partial}{\partial p}\right)\bar{\psi}(-p) + i(4p^2+1)S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\right]\cdot\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}.\end{aligned} \tag{79}$$

In terms of these new variables, Eq. (72) becomes quite simple:

$$\begin{aligned}
&\left(\frac{4-D}{2}+m\frac{\partial}{\partial m}\right)S^{(1)}\\
&=\int_k\left(k\cdot\frac{\partial}{\partial k}+\frac{D+2}{2}\right)\left[e^{k^2}\mathcal{A}_\mu(k)\right]\cdot\frac{\delta S^{(1)}}{\delta\left[e^{k^2}\mathcal{A}_\mu(k)\right]}\\
&\quad+\int_p S^{(1)}\frac{\overleftarrow{\delta}}{\delta\left[e^{p^2}\Psi(p)\right]}\left(p\cdot\frac{\partial}{\partial p}+m\frac{\partial}{\partial m}+\frac{D+1}{2}\right)\left[e^{p^2}\Psi(p)\right]\\
&\quad+\int_p\left(p\cdot\frac{\partial}{\partial p}+m\frac{\partial}{\partial m}+\frac{D+1}{2}\right)\left[\bar\Psi(-p)e^{p^2}\right]\cdot\frac{\overrightarrow{\delta}}{\delta\left[\bar\Psi(-p)e^{p^2}\right]}S^{(1)}\\
&\quad+4\int_{p,k}p_\mu\bar\Psi(-p-k)e^{2(p+k)^2}(\not p+\not k+im)\Psi(p)\mathcal{A}_\mu(k)\\
&\quad+4\int_{p,k}\bar\Psi(-p-k)(\not p+im)e^{2p^2}\Psi(p)(p+k)_\mu\mathcal{A}_\mu(k).
\end{aligned}\tag{80}$$

We now write

$$S^{(1)}=\int_{p,k}\bar\Psi(-p-k)e^{(p+k)^2}\widetilde{V}_\mu(p,k)e^{p^2}\Psi(p)e^{k^2}\mathcal{A}_\mu(k),\tag{81}$$

so that the vertex part $\widetilde{V}_\mu$ satisfies the inhomogeneous scaling equation

$$\begin{aligned}
&\left(p\cdot\frac{\partial}{\partial p}+k\cdot\frac{\partial}{\partial k}+m\frac{\partial}{\partial m}\right)\widetilde{V}_\mu(p,k)\\
&=4e^{(p+k)^2-p^2-k^2}(\not p+\not k+im)p_\mu+4e^{p^2-(p+k)^2-k^2}(\not p+im)(p+k)_\mu.
\end{aligned}\tag{82}$$

We wish to find a local solution that can be expanded in powers of p and k at zero momenta. Equation (82) determines $\widetilde{V}_\mu(p,k)$ up to a constant vector. A particular solution is obtained by the formula in Appendix A. The general solution is

$$\begin{aligned}
\widetilde{V}_\mu(p,k)=\widetilde{V}_\mu&+2(\not p+\not k+im)p_\mu F((p+k)^2-p^2-k^2)\\
&+2(\not p+im)(p+k)_\mu F(p^2-(p+k)^2-k^2),
\end{aligned}\tag{83}$$

where $\tilde V_\mu$ is a constant vector, and

$$F(x)\equiv\frac{e^x-1}{x}.\tag{84}$$

$\widetilde{V}_\mu$ is determined by imposing the WT identity (66), which requires

$$k_\mu\widetilde{V}_\mu(p,k)=e^{(p+k)^2-p^2-k^2}(\not p+\not k+im)-e^{p^2-(p+k)^2-k^2}(\not p+im).\tag{85}$$

This gives $\widetilde{V}_\mu=\gamma_\mu$, and we obtain

$$\begin{aligned}
\widetilde{V}_\mu(p,k)=\gamma_\mu&+2(\not p+\not k+im)p_\mu F((p+k)^2-p^2-k^2)\\
&+2(\not p+im)(p+k)_\mu F(p^2-(p+k)^2-k^2).
\end{aligned}\tag{86}$$

It follows from this that

$$\widetilde{V}_\mu(p+k,-k)=\widetilde{V}_\mu(p,k),\tag{87}$$

which will be used frequently below.

Our result for $S^{(1)}$ coincides with the first-order term of the gauge invariant local Wilson action obtained in Ref. [15].

5.4 *Second order*

We expect that the anomalous dimensions are second order in e:

$$\gamma = O(e^2), \qquad \gamma_F = O(e^2), \qquad \beta_m = O(e^2). \tag{88}$$

In what follows we denote

$$\gamma \equiv \gamma_1 e^2 + \cdots, \qquad \gamma_F \equiv \gamma_{F1} e^2 + \cdots, \qquad \beta_m \equiv \beta_{m1} e^2 + \cdots. \tag{89}$$

Extracting the second-order terms in Eq. (56) is already a laborious task. We obtain

$$\begin{aligned}
&(4-D)S^{(2)} + m\frac{\partial}{\partial m}S^{(2)} + 2\gamma_1\xi\frac{\partial}{\partial\xi}S^{(0)} + \beta_{m1}m\frac{\partial}{\partial m}S^{(0)} \\
&\quad + \gamma_1\int_k \mathcal{A}_\mu(k)\frac{\delta S^{(0)}}{\delta A_\mu(k)} + \gamma_{F1}\int_p\left[S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\Psi(p) + \bar\Psi(-p)\frac{\overrightarrow{\delta}}{\delta\bar\psi(-p)}S^{(0)}\right] \\
&= \int_k\left(k\cdot\frac{\partial}{\partial k} + \frac{D+2}{2}\right)\left[e^{k^2}\mathcal{A}_\mu(k)\right]\cdot\frac{\delta S^{(2)}}{\delta\left[e^{k^2}\mathcal{A}_\mu(k)\right]} \\
&\quad + \int_k (2k^2+1)\frac{\delta^2 S^{(2)}}{\delta A_\mu(k)\delta A_\mu(-k)} \\
&\quad + \int_p S^{(2)}\frac{\overleftarrow{\delta}}{\delta\left[e^{p^2}\Psi(p)\right]}\left(p\cdot\frac{\partial}{\partial p} + m\frac{\partial}{\partial m} + \frac{D+1}{2}\right)\left[e^{p^2}\Psi(p)\right] \\
&\quad + \int_p \left(p\cdot\frac{\partial}{\partial p} + m\frac{\partial}{\partial m} + \frac{D+1}{2}\right)\left[e^{p^2}\bar\Psi(-p)\right]\cdot\frac{\overrightarrow{\delta}}{\delta\left[e^{p^2}\bar\Psi(-p)\right]}S^{(2)} \\
&\quad + \int_p (-i)(4p^2+1)\,\mathrm{Tr}\left[\frac{\overrightarrow{\delta}}{\delta\bar\psi(-p)}S^{(2)}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\right] \\
&\quad + \int_p i(4p^2+1)S^{(1)}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\frac{\overrightarrow{\delta}}{\delta\bar\psi(-p)}S^{(1)} \\
&\quad + \int d^Dx\, 4i\mathcal{A}_\mu(x)S^{(1)}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\partial_\mu\Psi(x) \\
&\quad + \int d^Dx\,(-4i)\mathcal{A}_\mu(x)\partial_\mu\bar\Psi(x)\cdot\frac{\overrightarrow{\delta}}{\delta\bar\psi(x)}S^{(1)} \\
&\quad + \int d^Dx\,(-4)\mathcal{A}_\mu(x)S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\partial_\mu\frac{\overrightarrow{\delta}}{\delta\bar\psi(x)}S^{(1)} \\
&\quad + \int d^Dx\, 4\mathcal{A}_\mu(x)S^{(1)}\partial_\mu\frac{\overleftarrow{\delta}}{\delta\psi(x)}\cdot\frac{\overrightarrow{\delta}}{\delta\bar\psi(x)}S^{(0)} \\
&\quad + \int d^Dx\, 2\mathcal{A}_\mu(x)\mathcal{A}_\mu(x)S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\Psi(x) \\
&\quad + \int d^Dx\, 2\mathcal{A}_\mu(x)\mathcal{A}_\mu(x)\bar\Psi(x)\frac{\overrightarrow{\delta}}{\delta\bar\psi(x)}S^{(0)}
\end{aligned}$$

$$
\begin{aligned}
&+ \int_k (2k^2+1)\frac{\delta S^{(1)}}{\delta A_\mu(k)}\frac{\delta S^{(1)}}{\delta A_\mu(-k)} \\
&+ \int d^D x\, 4iS^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\partial_\mu\Psi(x)\cdot\frac{\delta S^{(1)}}{\delta A_\mu(x)} \\
&+ \int d^D x\,(-4i)\partial_\mu\bar{\Psi}(x)\cdot\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S^{(0)}\cdot\frac{\delta S^{(1)}}{\delta A_\mu(x)} \\
&+ \int d^D x\, 4\mathcal{A}_\mu(x)\,\mathrm{Tr}\left[\partial_\mu\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S^{(1)}\cdot\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right] \\
&+ \int d^D x\,(-4)\mathcal{A}_\mu(x)\,\mathrm{Tr}\left[\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S^{(1)}\partial_\mu\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right] \\
&+ \int d^D x\,(-4i)\mathcal{A}_\mu(x)\mathcal{A}_\mu(x)\,\mathrm{Tr}\left[\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(x')}\right] \\
&+ \int d^D x\, 4i\frac{\delta S^{(1)}}{\delta A_\mu(x)}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\partial_\mu\Psi(x) \\
&+ \int d^D x\,(-4i)\partial_\mu\bar{\Psi}(x)\cdot\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\frac{\delta S^{(1)}}{\delta A_\mu(x)} \\
&+ \int d^D x\,(-4)S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\partial_\mu\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\cdot\frac{\delta S^{(1)}}{\delta A_\mu(x)} \\
&+ \int d^D x\, 4\frac{\delta S^{(1)}}{\delta A_\mu(x)}\partial_\mu\frac{\overleftarrow{\delta}}{\delta\psi(x)}\cdot\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S^{(0)} \\
&+ \int d^D x\, 2\frac{\delta^2 S^{(0)}}{\delta A_\mu(x)\delta A_\mu(x')}S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\Psi(x) \\
&+ \int d^D x\, 2\frac{\delta^2 S^{(0)}}{\delta A_\mu(x)\delta A_\mu(x')}\bar{\Psi}(x)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S^{(0)} \\
&+ 4\int d^D x\,\frac{\delta S^{(1)}}{\delta A_\mu(x)}\,\mathrm{Tr}\left[\partial_\mu\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S^{(0)}\cdot\frac{\overleftarrow{\delta}}{\delta\psi(x')}-\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x')}S^{(0)}\partial_\mu\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right], \qquad (90)
\end{aligned}
$$

where we have used variables defined in Eq. (73). Note that we take the limit $x' \to x$ only after taking differentials as has been explained in Sect. 3. In Appendix B we elaborate on how this limit actually works in this case.

We have four types of terms:

$$
S^{(2)} = S^{(2)}\big|_{\bar{\psi}AA\psi} + S^{(2)}\big|_{\bar{\psi}\psi\bar{\psi}\psi} + S^{(2)}\big|_{AA} + S^{(2)}\big|_{\bar{\psi}\psi}. \qquad (91)
$$

We compute them one by one.

5.5 $\bar{\psi}AA\psi$ term

Let us first consider the term proportional to $\bar{\psi}AA\psi$:

$$
S^{(2)}|_{\bar{\psi}AA\psi} \equiv \int_{p,k,l}\bar{\Psi}(-p-k-l)e^{(p+k+l)^2}\widetilde{V}_{\mu\nu}(p,k,l)e^{p^2}\Psi(p)e^{k^2}\mathcal{A}_\mu(k)e^{l^2}\mathcal{A}_\nu(l), \qquad (92)
$$

where $\widetilde{V}_{\mu\nu}(p,k,l)=\widetilde{V}_{\nu\mu}(p,l,k)$. Equation (90) gives

$$\begin{aligned}
&\left(p\cdot\frac{\partial}{\partial p}+k\cdot\frac{\partial}{\partial k}+l\cdot\frac{\partial}{\partial l}+m\frac{\partial}{\partial m}+1\right)\widetilde{V}_{\mu\nu}(p,k,l)\\
&=\frac{1}{2}\widetilde{V}_{\mu}(p+l,k)(-1)[4(p+l)^2+1]e^{-2(p+l)^2}h_F(p+l)^2\widetilde{V}_{\nu}(p,l)\\
&\quad+2\widetilde{V}_{\mu}(p+l,k)h_F(p+l)\Big[e^{(p+l)^2-p^2-l^2}(\not{p}+\not{l}+im)p_{\nu}\\
&\quad+e^{p^2-(p+l)^2-l^2}(\not{p}+im)(p+l)_{\nu}\Big]\\
&\quad+2\Big[e^{(p+k+l)^2-(p+l)^2-k^2}(\not{p}+\not{k}+\not{l}+im)(p+l)_{\mu}\\
&\quad+e^{(p+l)^2-(p+k+l)^2-k^2}(\not{p}+\not{l}+im)(p+k+l)_{\mu}\Big]h_F(p+l)\widetilde{V}_{\nu}(p,l)\\
&\quad-2\left[e^{(p+l)^2-p^2-l^2}\widetilde{V}_{\mu}(p+l,k)p_{\nu}+e^{(p+l)^2-(p+k+l)^2-k^2}(p+k+l)_{\mu}\widetilde{V}_{\nu}(p,l)\right]\\
&\quad+(\mu\leftrightarrow\nu,k\leftrightarrow l)\\
&\quad-2\delta_{\mu\nu}e^{-k^2-l^2}\left[e^{(p+k+l)^2-p^2}(\not{p}+\not{k}+\not{l}+im)+e^{p^2-(p+k+l)^2}(\not{p}+im)\right],
\end{aligned}\tag{93}$$

where we have used the relation (76). Noting further the properties (75), we can simplify the above to

$$\begin{aligned}
&\left(p\cdot\frac{\partial}{\partial p}+k\cdot\frac{\partial}{\partial k}+l\cdot\frac{\partial}{\partial l}+m\frac{\partial}{\partial m}+1\right)\widetilde{V}_{\mu\nu}(p,k,l)\\
&=\left(p\cdot\frac{\partial}{\partial p}+k\cdot\frac{\partial}{\partial k}+l\cdot\frac{\partial}{\partial l}+m\frac{\partial}{\partial m}+1\right)\\
&\quad\times\frac{1}{2}\left[\widetilde{V}_{\mu}(p+l,k)h_F(p+l)\widetilde{V}_{\nu}(p,l)+\widetilde{V}_{\nu}(p+k,l)h_F(p+k)\widetilde{V}_{\mu}(p,k)\right]\\
&\quad-2\left[e^{(p+l)^2-p^2-l^2}\widetilde{V}_{\mu}(p+l,k)p_{\nu}+e^{(p+l)^2-(p+k+l)^2-k^2}(p+k+l)_{\mu}\widetilde{V}_{\nu}(p,l)\right]\\
&\quad+(\mu\leftrightarrow\nu,k\leftrightarrow l)\\
&\quad-2\delta_{\mu\nu}e^{-k^2-l^2}\left[e^{(p+k+l)^2-p^2}(\not{p}+\not{k}+\not{l}+im)+e^{p^2-(p+k+l)^2}(\not{p}+im)\right].
\end{aligned}\tag{94}$$

The last line can be integrated by the formula in Appendix A as

$$\begin{aligned}
&-2\delta_{\mu\nu}e^{-k^2-l^2}\left[e^{(p+k+l)^2-p^2}(\not{p}+\not{k}+\not{l}+im)+e^{p^2-(p+k+l)^2}(\not{p}+im)\right]\\
&=-\delta_{\mu\nu}\left(p\cdot\frac{\partial}{\partial p}+k\cdot\frac{\partial}{\partial k}+l\cdot\frac{\partial}{\partial l}+m\frac{\partial}{\partial m}+1\right)\\
&\quad\times\left[(\not{p}+\not{k}+\not{l}+im)F((p+k+l)^2-p^2-k^2-l^2)\right.\\
&\quad\left.+(\not{p}+im)F(p^2-(p+k+l)^2-k^2-l^2)\right],
\end{aligned}\tag{95}$$

where the function $F(x)$ is defined by Eq. (84). Therefore, the solution to Eq. (94) is given by

$$\begin{aligned}
\widetilde{V}_{\mu\nu}(p,k,l)&=\frac{1}{2}\left[\widetilde{V}_{\mu}(p+l,k)h_F(p+l)\widetilde{V}_{\nu}(p,l)+\widetilde{V}_{\nu}(p+k,l)h_F(p+k)\widetilde{V}_{\mu}(p,k)\right]\\
&\quad-\delta_{\mu\nu}\left[(\not{p}+\not{k}+\not{l}+im)F((p+k+l)^2-p^2-k^2-l^2)\right.\\
&\quad\left.+(\not{p}+im)F(p^2-(p+k+l)^2-k^2-l^2)\right]\\
&\quad-4X_{\mu\nu}(p,k,l),
\end{aligned}\tag{96}$$

where $X_{\mu\nu}(p,k,l)$ satisfies

$$\begin{aligned}
&\left(p\cdot\frac{\partial}{\partial p}+k\cdot\frac{\partial}{\partial k}+l\cdot\frac{\partial}{\partial l}+m\frac{\partial}{\partial m}+1\right)X_{\mu\nu}(p,k,l)\\
&=\frac{1}{2}\left[e^{(p+l)^2-p^2-l^2}\widetilde{V}_\mu(p+l,k)p_\nu+e^{(p+l)^2-(p+k+l)^2-k^2}(p+k+l)_\mu\widetilde{V}_\nu(p,l)\right]\\
&\quad+(\mu\leftrightarrow\nu,k\leftrightarrow l)\\
&=\frac{1}{2}\Big\{e^{(p+l)^2-p^2-l^2}\\
&\quad\times\left[\gamma_\mu+2(\not{p}+\not{k}+\not{l}+im)(p+l)_\mu F((p+k+l)^2-(p+l)^2-k^2)\right.\\
&\quad\left.+2(\not{p}+\not{l}+im)(p+k+l)_\mu F((p+l)^2-(p+k+l)^2-k^2)\right]p_\nu\\
&\quad+e^{(p+l)^2-(p+k+l)^2-k^2}(p+k+l)_\mu\\
&\quad\times\left[\gamma_\nu+2(\not{p}+\not{l}+im)p_\nu F((p+l)^2-p^2-l^2)\right.\\
&\quad\left.+2(\not{p}+im)(p+l)_\nu F(p^2-(p+l)^2-l^2)\right]\Big\}\\
&\quad+(\mu\leftrightarrow\nu,k\leftrightarrow l).
\end{aligned}\tag{97}$$

This can be solved again by the formula in Appendix A to yield

$$\begin{aligned}
X_{\mu\nu}(p,k,l)&=\frac{1}{4}\gamma_\mu p_\nu F((p+l)^2-p^2-l^2)+\frac{1}{4}(p+k+l)_\mu\gamma_\nu F((p+l)^2-(p+k+l)^2-k^2)\\
&\quad+\frac{1}{2}(\not{p}+\not{l}+im)(p+k+l)_\mu p_\nu F((p+l)^2-(p+k+l)^2-k^2)\\
&\quad\times F((p+l)^2-p^2-l^2)+\frac{1}{2}\frac{(\not{p}+\not{k}+\not{l}+im)(p+l)_\mu p_\nu}{(p+k+l)^2-(p+l)^2-k^2}\\
&\quad\times\left[F((p+k+l)^2-p^2-k^2-l^2)-F((p+l)^2-p^2-l^2)\right]\\
&\quad+\frac{1}{2}\frac{(\not{p}+im)(p+k+l)_\mu(p+l)_\nu}{p^2-(p+l)^2-l^2}\\
&\quad\times\left[F(p^2-(p+k+l)^2-k^2-l^2)-F((p+l)^2-(p+k+l)^2-k^2)\right]\\
&\quad+(\mu\leftrightarrow\nu,k\leftrightarrow l),
\end{aligned}\tag{98}$$

where we have used the identity

$$F(x)F(y)=\left(\frac{1}{x}+\frac{1}{y}\right)F(x+y)-\frac{F(x)}{y}-\frac{F(y)}{x}.\tag{99}$$

Equation (96) with $X_{\mu\nu}$ given by Eq. (98) gives a local solution to Eq. (93). The solution is unique because the homogeneous equation

$$\left(p\cdot\frac{\partial}{\partial p}+k\cdot\frac{\partial}{\partial k}+l\cdot\frac{\partial}{\partial l}+m\frac{\partial}{\partial m}+1\right)\widetilde{V}_{\mu\nu}(p,k,l)=0\tag{100}$$

has no solution analytic in momenta and m.

5.6 $\bar{\psi}\psi\bar{\psi}\psi$ term

We observe that the inhomogeneous terms of Eq. (90) that can contribute to the four-Fermi term always contain the factor $\delta S^{(1)}/\delta A_\mu(x)$, where $S^{(1)}$ is given by Eq. (71). This suggests the

structure

$$S^{(2)}|_{\bar\psi\psi\bar\psi\psi} = \frac{1}{2}\int_{p,q,k} \bar\Psi(-p-k)e^{(p+k)^2}\Gamma_\mu(p,k)e^{p^2}\Psi(p) \\ \times \bar\Psi(-q)e^{q^2}\Gamma_\nu(q+k,-k)e^{(q+k)^2}\Psi(q+k)D_{\mu\nu}(k). \tag{101}$$

Using Eq. (82) and the properties (75), we find that the ERG equation for Eq. (101) takes the following extremely simple form:

$$\left(p\cdot\frac{\partial}{\partial p}+q\cdot\frac{\partial}{\partial q}+k\cdot\frac{\partial}{\partial k}+m\frac{\partial}{\partial m}+2\right)\left[\Gamma_\mu(p,k)\cdot\Gamma_\nu(q+k,k)D_{\mu\nu}(k)\right] \\ =\left(p\cdot\frac{\partial}{\partial p}+q\cdot\frac{\partial}{\partial q}+k\cdot\frac{\partial}{\partial k}+m\frac{\partial}{\partial m}+2\right)\left[\widetilde V_\mu(p,k)\cdot\widetilde V_\nu(q+k,k)h_{\mu\nu}(k)\right]. \tag{102}$$

Since the corresponding homogeneous equation

$$\left(p\cdot\frac{\partial}{\partial p}+q\cdot\frac{\partial}{\partial q}+k\cdot\frac{\partial}{\partial k}+m\frac{\partial}{\partial m}+2\right)\left[\Gamma_\mu(p,k)\cdot\Gamma_\nu(q+k,k)D_{\mu\nu}(k)\right]=0 \tag{103}$$

has no solution analytic in momenta and m, we get the unique local solution

$$S^{(2)}|_{\bar\psi\psi\bar\psi\psi} = \frac{1}{2}\int_{p,q,k} \bar\Psi(-p-k)e^{(p+k)^2}\widetilde V_\mu(p,k)e^{p^2}\Psi(p) \\ \times \bar\Psi(-q)e^{q^2}\widetilde V_\nu(q+k,-k)e^{(q+k)^2}\Psi(q+k)h_{\mu\nu}(k). \tag{104}$$

5.7 *AA term and* γ_1

Now, we can study the second-order correction to the AA term. We will see that the analyticity of this term determines the first non-trivial order coefficient of the anomalous dimension, γ_1 in Eq. (89). We first note that the WT identity (66) requires that this correction be transverse:

$$S^{(2)}|_{AA} \equiv \frac{1}{2}\int_k e^{k^2}\mathcal{A}_\mu(k)e^{k^2}\mathcal{A}_\nu(-k)\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\widetilde V_T(k). \tag{105}$$

For this to be local, $\tilde V_T(k)$ must be of order k^2 at $k=0$. We may also normalize the kinetic term by demanding $\tilde V_T(k)$ to be of order $(k^2)^2$.

Now, the part of the ERG equation (90) relevant to the AA term gives

$$\frac{1}{2}\left\{\left[k\cdot\frac{\partial}{\partial k}+m\frac{\partial}{\partial m}-2+(4-D)\right]\widetilde V_T(k)-2\gamma_1 k^2\right\}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right) \\ =-\int_p \mathrm{Tr}\left[\left(p\cdot\frac{\partial}{\partial p}+m\frac{\partial}{\partial m}+1\right)h_F(p)\cdot\widetilde V_{\mu\nu}(p,-k,k)\right] \\ -4ie^{-k^2}\int_p e^{-(p+k)^2-p^2}\,\mathrm{Tr}\left[h_F(p+k)\widetilde V_\nu(p,k)h_F(p)\right](2p+k)_\mu \\ -4e^{-2k^2}\delta_{\mu\nu}\int_p e^{-2p^2}\,\mathrm{Tr}\left[\frac{1}{e^{-2p^2}+i(\not p+im)}\right] \\ \equiv I_{\mu\nu}(k), \tag{106}$$

where we have used Eq. (75). Since the lhs is symmetric under $k \leftrightarrow -k$ and $\mu \leftrightarrow \nu$, we make this symmetry manifest on the rhs too by rewriting

$$\begin{aligned} I_{\mu\nu}(k) = &-\frac{1}{2}\int_p \mathrm{Tr}\left[\left(p\cdot\frac{\partial}{\partial p} + m\frac{\partial}{\partial m} + 1\right) h_F(p)\cdot\widetilde{V}_{\mu\nu}(p,-k,k)\right] \\ &- \frac{1}{2}\int_p \mathrm{Tr}\left[\left(p\cdot\frac{\partial}{\partial p} + k\cdot\frac{\partial}{\partial k} + m\frac{\partial}{\partial m} + 1\right) h_F(p+k)\cdot\widetilde{V}_{\nu\mu}(p+k,k,-k)\right] \\ &- 2ie^{-k^2}\int_p e^{-(p+k)^2-p^2}\,\mathrm{Tr}\left[h_F(p+k)\widetilde{V}_\nu(p,k)h_F(p)\right](2p+k)_\mu \\ &- 2ie^{-k^2}\int_p e^{-(p+k)^2-p^2}\,\mathrm{Tr}\left[h_F(p)\widetilde{V}_\mu(p,k)h_F(p+k)\right](2p+k)_\nu \\ &- 4e^{-2k^2}\delta_{\mu\nu}\int_p e^{-2p^2}\,\mathrm{Tr}\left[\frac{1}{e^{-2p^2}+i(\not{p}+im)}\right], \end{aligned} \tag{107}$$

where we have used Eq. (87). By using Eqs. (82), (96), (97), and

$$\left(k\cdot\frac{\partial}{\partial k}+2\right)F(-2k^2) = 2e^{-2k^2}, \tag{108}$$

we can show

$$\begin{aligned} I_{\mu\nu}(k) = &-\frac{1}{2}\int_p \left(p\cdot\frac{\partial}{\partial p} + k\cdot\frac{\partial}{\partial k} + m\frac{\partial}{\partial m} + 2\right) \\ &\times \mathrm{Tr}\Big\{h_F(p)\widetilde{V}_\mu(p,k)h_F(p+k)\widetilde{V}_\nu(p,k) \\ &- 4\left[h_F(p)X_{\mu\nu}(p,-k,k) + h_F(p+k)X_{\nu\mu}(p+k,k,-k)\right] \\ &- 4i\delta_{\mu\nu}F(-2k^2)e^{-2p^2}h_F(p)\Big\}. \end{aligned} \tag{109}$$

Now, let us compute γ_1 for $D = 4$. The lhs of Eq. (106) gives

$$-2\gamma_1\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right) \tag{110}$$

to order k^2. We can determine γ_1 by calculating $I_{\mu\nu}(k)$ to the same order. Since $k\cdot\partial/\partial k = 2$ for the k^2 term, Eq. (109) gives

$$\begin{aligned} -2\gamma_1\left(\delta_{\mu\nu}k^2 - k_\mu k_\nu\right) = -\int_p \frac{\partial}{\partial p_\rho}\Big(&p_\rho\,\mathrm{Tr}\Big\{h_F(p)\widetilde{V}_\mu(p,k)h_F(p+k)\widetilde{V}_\nu(p,k) \\ &- 4\left[h_F(p)X_{\mu\nu}(p,-k,k) + h_F(p+k)X_{\nu\mu}(p+k,k,-k)\right] \\ &- 4i\delta_{\mu\nu}F(-2k^2)e^{-2p^2}h_F(p)\Big\}\Big)\Big|_{m=0,O(k^2)}. \end{aligned} \tag{111}$$

The 4-momentum integral on the rhs is thus given by a surface integral at $|p| \to \infty$. From the explicit form of the integrand, it is not difficult to find the surface term that contributes to the integral at $|p| = \infty$. For instance, the last term does not contribute because of the factor e^{-2p^2}. In this way, we obtain

$$-2\gamma_1 = -\frac{1}{(4\pi)^2}\frac{8}{3}. \tag{112}$$

Hence, from Eq. (57) the beta function of e^2 is

$$\beta(e^2) = -2\gamma(e^2)e^2 \simeq -\frac{1}{(4\pi)^2}\frac{8}{3}(e^2)^2. \tag{113}$$

This agrees with the 1-loop beta function of QED.

5.8 $\bar{\psi}\psi$ term and γ_{F1}, β_m

Finally, we consider the second-order correction to the fermion kinetic and mass terms:

$$S^{(2)}|_{\bar{\psi}\psi} \equiv \int_p \bar{\Psi}(-p) e^{p^2} \widetilde{V}_F(p) e^{p^2} \Psi(p). \tag{114}$$

The GFERG equation is given by

$$\begin{aligned}
&\left[p \cdot \frac{\partial}{\partial p} + m \frac{\partial}{\partial m} - 1 + (4-D) \right] \widetilde{V}_F(p) - \beta_{m1} i m - 2\gamma_{F1}(\not{p} + im) \\
&= \int_k \left(k \cdot \frac{\partial}{\partial k} + 2 \right) h_{\mu\nu}(k) \cdot \widetilde{V}_{\mu\nu}(p, -k, k) \\
&\quad + \int_k h_{\mu\nu}(k) \widetilde{V}_\mu(p,k) \left(p \cdot \frac{\partial}{\partial p} + k \cdot \frac{\partial}{\partial k} + m \frac{\partial}{\partial m} + 1 \right) h_F(p+k) \cdot \widetilde{V}_\nu(p,k) \\
&\quad + 4i \int_k h_{\mu\nu}(k) e^{-(p+k)^2 - p^2 - k^2} h_F(p+k) p_\mu \widetilde{V}_\nu(p,k) \\
&\quad + 4 \int_k h_{\mu\nu}(k) e^{p^2 - (p+k)^2 - k^2} (\not{p} + im)(p+k)_\mu h_F(p+k) \widetilde{V}_\nu(p,k) \\
&\quad + 4i \int_k h_{\mu\nu}(k) \widetilde{V}_\mu(p,k) e^{-(p+k)^2 - p^2 - k^2} h_F(p+k) p_\nu \\
&\quad + 4 \int_k h_{\mu\nu}(k) \widetilde{V}_\mu(p,k) e^{p^2 - (p+k)^2 - k^2} h_F(p+k)(\not{p} + im)(p+k)_\nu \\
&\quad - 4 \int_k h_{\mu\mu}(k)(\not{p} + im) e^{-2k^2} \\
&\equiv I_F(p),
\end{aligned} \tag{115}$$

where we have used Eq. (75) for the first two lines on the rhs. We have also used Eq. (87). We may normalize the kinetic and mass terms so that $\tilde{V}_F(p)$ has no term proportional to either $\not{p}$ or m. By using Eqs. (96), (82), (97), (76), and (108), we obtain

$$I_F(p) = \int_k \left(p \cdot \frac{\partial}{\partial p} + k \cdot \frac{\partial}{\partial k} + m \frac{\partial}{\partial m} + 3 \right) \left[h_{\mu\nu}(k) \widetilde{V}_{\mu\nu}(p, -k, k) \right]. \tag{116}$$

Now, let us compute β_{m1}, γ_{F1} for $D = 4$. The lhs of Eq. (115) gives

$$-\beta_{m1} i m - 2\gamma_{F1} (\not{p} + im) \tag{117}$$

to first order in $\not{p}$ and m. We can determine β_{m1} and γ_{F1} by calculating $I_F(p)$ to the same order. We can take $p \cdot \partial/\partial p + m\partial/\partial m = 1$, and Eqs. (115) and (116) give

$$-\beta_{m1} i m - 2\gamma_{F1}(\not{p} + im) = \int_k \frac{\partial}{\partial k_\rho} \left[k_\rho h_{\mu\nu}(k) \widetilde{V}_{\mu\nu}(p, -k, k) \right]\Big|_{O(m), O(p)}. \tag{118}$$

It is again straightforward to find the surface term at $|k| = \infty$, which contributes to this integral, and we obtain

$$\beta_{m1} = \frac{6}{(4\pi)^2}, \qquad \gamma_{F1} = \frac{3}{(4\pi)^2}. \tag{119}$$

The former is the usual mass anomalous dimension in QED. Interestingly, the latter coincides with the anomalous dimension resulting from the wave function renormalization of the *flowed* or *diffused* (i.e., not usual) fermion field; the 1-loop renormalization factor has been given in Eq. (2.16) of Ref. [26], where $C_F = 1$ for QED with the electron. Note that this anomalous dimension is independent of the gauge-fixing parameter ξ. This is expected because the

Wilson action (24), by construction, reproduces the correlation functions of flowed or diffused fields up to contact terms [10].

6. Conclusion

In this paper we have constructed the gradient flow exact renormalization group (GFERG) for QED, based on the BRST invariant diffusion equations (17). With the exclusion of the gauge-fixing term, the Wilson action (69) becomes manifestly invariant under the gauge transformation (70). We have computed the action perturbatively in powers of the coupling e to the order e^2, reproducing the 1-loop beta function and anomalous dimensions. It was especially pleasing to find the anomalous dimension of the electron field as gauge invariant.

Our perturbative calculations show that the Wilson action becomes complex despite the simplicity in gauge invariance. The complexity comes from that of the GFERG differential equations. But we believe that the manifest gauge invariance will turn out to be a big advantage when we attempt to solve the GFERG differential equations non-perturbatively (with some gauge invariant approximations).

Whether the GFERG differential equation (52) has a non-trivial fixed point satisfying the WT identity (66) is of great interest to be studied in the future; see, e.g., Ref. [27] and references cited therein for related studies. At present we do not even know what it means to have a fixed point in the GFERG formalism.

The GFERG formalism was originally introduced for non-Abelian gauge theories [10]. It would be interesting to extend the analysis of this paper to see how far we can simplify the realization of non-Abelian gauge invariance compared with the standard ERG formalism.

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Appendix A. Integration formula

A particular solution to

$$\left(\sum_i p_i \cdot \frac{\partial}{\partial p_i} + m\frac{\partial}{\partial m} + \zeta\right) F(p,m) = f(p,m), \tag{A1}$$

where

$$\lim_{\alpha\to 0} \alpha^{\zeta} f(\alpha p, \alpha m) = 0, \tag{A2}$$

is given by

$$F(p,m) = \int_0^1 d\alpha\, \alpha^{\zeta-1} f(\alpha p, \alpha m). \tag{A3}$$

Note that this solution is analytic in the momenta and m.

The proof is straightforward. Noting

$$\frac{\partial}{\partial\alpha} f(\alpha p, \alpha m) = \frac{1}{\alpha}\left(\sum_i p_i \cdot \frac{\partial}{\partial p_i} + m\frac{\partial}{\partial m}\right) f(\alpha p, \alpha m) \tag{A4}$$

under the prerequisite (A2), we have

$$\begin{aligned}
&\left(\sum_i p_i \cdot \frac{\partial}{\partial p_i} + m\frac{\partial}{\partial m}\right) \int_0^1 d\alpha\, \alpha^{\zeta-1} f(\alpha p, \alpha m) \\
&= \int_0^1 d\alpha\, \alpha^{\zeta} \frac{\partial}{\partial \alpha} f(\alpha p, \alpha m) \\
&= f(p,m) - \zeta \int_0^1 d\alpha\, \alpha^{\zeta-1} f(\alpha p, \alpha m).
\end{aligned} \tag{A5}$$

This is Eq. (A1) for Eq. (A3).

Appendix B. The working of the limit $x' \to x$

In Sect. 3.1 we have explained how to take second- and higher-order functional differentials at the same point as a limit of functional differentials at different points. This careful treatment is necessary to avoid unphysical singularities. In deriving the GFERG differential equation for the second-order Wilson action $S^{(2)}$ we need to practice the treatment. There are three integrals to consider.

B.1 AA term

We compute

$$\begin{aligned}
&\int d^D x\, (-4i)\mathcal{A}_\mu(x)\mathcal{A}_\mu(x)\, \mathrm{Tr}\left[\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)} S^{(0)} \frac{\overleftarrow{\delta}}{\delta\psi(x')}\right] \\
&= \int_k (-4i)\mathcal{A}_\mu(k)\mathcal{A}_\mu(-k) \int_p \mathrm{Tr}(-) \frac{\not{p} + im}{e^{-2p^2} + i(\not{p} + im)} e^{-ip(x-x')} \\
&= \int_k (-4i)\mathcal{A}_\mu(k)\mathcal{A}_\mu(-k) \int_p i\,\mathrm{Tr}\, \frac{-e^{-2p^2} + e^{-2p^2} + i(\not{p} + im)}{e^{-2p^2} + i(\not{p} + im)} e^{-ip(x-x')} \\
&= -4 \int_k \mathcal{A}_\mu(k)\mathcal{A}_\mu(-k) \int_p e^{-ip(x-x')} e^{-2p^2}\, \mathrm{Tr}\, \frac{1}{e^{-2p^2} + i(\not{p} + im)} \\
&\quad + 16 \int_k \mathcal{A}_\mu(k)\mathcal{A}_\mu(-k) \int_p e^{-ip(x-x')}.
\end{aligned} \tag{B1}$$

For $x \neq x'$, we find

$$\int_p e^{-ip(x-x')} = \delta(x - x') = 0. \tag{B2}$$

Hence, in the limit $x' \to x$, we obtain

$$-4 \int_k \mathcal{A}_\mu(k)\mathcal{A}_\mu(-k) \int_p e^{-2p^2}\, \mathrm{Tr}\, \frac{1}{e^{-2p^2} + i(\not{p} + im)}, \tag{B3}$$

where the integral over p is absolutely convergent.

B.2 $\bar{\psi}\psi$ term

We compute

$$
\begin{aligned}
&2\int d^Dx\,\frac{\delta^2 S^{(0)}}{\delta A_\mu(x)\delta A_\mu(x')}\left[S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\Psi(x)+\bar{\Psi}(x)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S^{(0)}\right]\\
&\quad=2\int_k e^{ik(x-x')}\left[(D-1)\frac{k^2}{e^{-2k^2}+k^2}+\frac{k^2}{\xi e^{-2k^2}+k^2}\right]\\
&\qquad\times\int_p\left[\bar{\psi}(-p)\frac{\not{p}+im}{e^{-2p^2}+i(\not{p}+im)}\Psi(p)+\bar{\Psi}(-p)\frac{\not{p}+im}{e^{-2p^2}+i(\not{p}+im)}\psi(p)\right]\\
&\quad=2\int_k e^{ik(x-x')}\left[D-(D-1)\frac{e^{-2k^2}}{e^{-2k^2}+k^2}-\frac{\xi e^{-2k^2}}{\xi e^{-2k^2}+k^2}\right]\\
&\qquad\times\int_p\left[\bar{\psi}(-p)\frac{\not{p}+im}{e^{-2p^2}+i(\not{p}+im)}\Psi(p)+\bar{\Psi}(-p)\frac{\not{p}+im}{e^{-2p^2}+i(\not{p}+im)}\psi(p)\right].
\end{aligned}
\tag{B4}
$$

Ignoring the delta function again, we obtain

$$
\begin{aligned}
\xrightarrow{x'\to x}\ &-2\int_k e^{-2k^2}h_{\mu\mu}(k)\int_p\left[\bar{\psi}(-p)\frac{\not{p}+im}{e^{-2p^2}+i(\not{p}+im)}\Psi(p)\right.\\
&\left.+\bar{\Psi}(-p)\frac{\not{p}+im}{e^{-2p^2}+i(\not{p}+im)}\psi(p)\right],
\end{aligned}
\tag{B5}
$$

where $h_{\mu\nu}(k)$ is defined by Eq. (74a).

B.3 Vanishing terms

We examine the last integral of Eq. (90):

$$
\begin{aligned}
&4\int d^Dx\,\frac{\delta S^{(1)}}{\delta A_\mu(x)}\,\mathrm{Tr}\left[\partial_\mu\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(x')}-\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x')}S^{(0)}\partial_\mu\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right]\\
&\quad=4\left.\frac{\delta S^{(1)}}{\delta A_\mu(k)}\right|_{k=0}\int_p(-i)p_\mu\,\mathrm{Tr}\left\{\left[e^{ip(x-x')}+e^{ip(x'-x)}\right]\frac{\not{p}+im}{e^{-2p^2}+i(\not{p}+im)}\right\}\\
&\quad=4\left.\frac{\delta S^{(1)}}{\delta A_\mu(k)}\right|_{k=0}\int_p(-)p_\mu\,\mathrm{Tr}\left\{\left[e^{ip(x-x')}+e^{ip(x'-x)}\right]\frac{e^{-2p^2}+i(\not{p}+im)-e^{-2p^2}}{e^{-2p^2}+i(\not{p}+im)}\right\}\\
&\quad=4\left.\frac{\delta S^{(1)}}{\delta A_\mu(k)}\right|_{k=0}\int_p\left[e^{ip(x-x')}+e^{ip(x'-x)}\right]p_\mu\left\{-4+e^{-2p^2}\,\mathrm{Tr}\,\frac{1}{e^{-2p^2}+i(\not{p}+im)}\right\}.
\end{aligned}
\tag{B6}
$$

Ignoring the derivative of the delta function, we obtain a vanishing integral over p:

$$
\xrightarrow{x'\to x}\ 8\left.\frac{\delta S^{(1)}}{\delta A_\mu(k)}\right|_{k=0}\int_p p_\mu e^{-2p^2}\,\mathrm{Tr}\,\frac{1}{e^{-2p^2}+i(\not{p}+im)}=0.
\tag{B7}
$$

Appendix C. ERG for QED

In the ERG formalism, the Wilson action is constructed as

$$
\begin{aligned}
e^{S_\Lambda[A_\mu,c,\bar{c},\psi,\bar{\psi}]} \equiv \int \left[dA'_\mu dc' d\bar{c}' d\psi' d\bar{\psi}'\right] \exp\Bigg\{ & S\left[A'_\mu, c', \bar{c}', \psi', \bar{\psi}'\right] \\
& - \frac{\Lambda^2}{2}\int_k \left[A_\mu(k) - z_\Lambda e^{-k^2/\Lambda^2} A'_\mu(k)\right]\left[A_\mu(-k) - z_\Lambda e^{-k^2/\Lambda^2} A'_\mu(-k)\right] \\
& - \Lambda^2 \int_k \left[\bar{c}(-k) - e^{-k^2/\Lambda^2}\bar{c}'(-k)\right]\left[c(k) - e^{-k^2/\Lambda^2} c'(k)\right] \\
& + i\Lambda \int_p \left[\bar{\psi}(-p) - z_{F\Lambda} e^{-p^2/\Lambda^2}\bar{\psi}'(-p)\right]\left[\psi(p) - z_{F\Lambda} e^{-p^2/\Lambda^2}\psi'(p)\right]\Bigg\}, \qquad \text{(C1)}
\end{aligned}
$$

where the electron fields are diffused according to the standard diffusion equation. z_Λ and $z_{F\Lambda}$ here differ from those in Eq. (24).

The Wilson action satisfies the ERG differential equation

$$
\begin{aligned}
& -\Lambda\frac{\partial}{\partial\Lambda} e^{S_\Lambda} \\
& = \int_k \left[\left(2\frac{k^2}{\Lambda^2} - \gamma_\Lambda\right) A_\mu(k)\frac{\delta}{\delta A_\mu(k)} + \left(2\frac{k^2}{\Lambda^2} + 1 - \gamma_\Lambda\right)\frac{1}{\Lambda^2}\frac{\delta^2}{\delta A_\mu(k)\delta A_\mu(-k)}\right] e^{S_\Lambda} \\
& \quad + \int_k \left[2\frac{k^2}{\Lambda^2}\bar{c}(-k)\frac{\overrightarrow{\delta}}{\delta\bar{c}(-k)} e^{S_\Lambda} + e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta c(k)} 2\frac{k^2}{\Lambda^2} c(k) - 2\left(2\frac{k^2}{\Lambda^2} + 1\right)\frac{1}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(-k)} e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta c(k)}\right] \\
& \quad + \int_p \left[e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\left(2\frac{p^2}{\Lambda^2} - \gamma_{F\Lambda}\right)\psi(p) + \left(2\frac{p^2}{\Lambda^2} - \gamma_{F\Lambda}\right)\bar{\psi}(-p)\cdot\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)} e^{S_\Lambda}\right. \\
& \quad \left. - i\left(4\frac{p^2}{\Lambda^2} + 1 - 2\gamma_{F\Lambda}\right)\frac{1}{\Lambda}\,\mathrm{Tr}\,\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)} e^{S_\Lambda}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\right]. \qquad \text{(C2)}
\end{aligned}
$$

This is the same as the first part of Eq. (34); the second part proportional to $e_\Lambda \equiv e\mu^{\epsilon/2}/z_\Lambda$ and e_Λ^2 present in Eq. (34) are missing here. The difference is due to the simple diffusion equations that we have adopted for the electron fields in ERG.

The BRST invariance of the original S is inherited by the Wilson action as

$$
\begin{aligned}
& \frac{1}{\xi_\Lambda} e^{-S_\Lambda}\int_k k_\mu \left[A_\mu(-k) + \frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu(k)}\right]\frac{\overrightarrow{\delta}}{\delta\bar{c}(-k)} e^{S_\Lambda} \\
& \quad = \int_k e^{-k^2/\Lambda^2} k_\mu\,[c(k)]\,\frac{\delta}{\delta A_\mu(k)} S_\Lambda \\
& \quad\quad - e_\Lambda e^{-S_\Lambda}\int_p e^{-p^2/\Lambda^2}\,\mathrm{Tr}\left[\int_k [c(k)]\,[\psi(p-k)]\, e^{S_\Lambda}\right]\frac{\overleftarrow{\delta}}{\delta\psi(p)} \\
& \quad\quad - e_\Lambda e^{-S_\Lambda}\int_p e^{-p^2/\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}\left[e^{S_\Lambda}\int_k [c(k)]\,[\bar{\psi}(-p-k)]\right], \qquad \text{(C3)}
\end{aligned}
$$

where we have defined the composite operators as

$$
[c(k)] \equiv e^{k^2/\Lambda^2}\left[c(k) + \frac{1}{\Lambda^2}\frac{\overrightarrow{\delta}}{\delta\bar{c}(-k)} S_\Lambda\right], \qquad \text{(C4a)}
$$

$$[\psi(p)] \equiv e^{p^2/\Lambda^2}\left[\psi(p) + \frac{i}{\Lambda}\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_\Lambda\right], \tag{C4b}$$

$$\left[\bar{\psi}(-p)\right] \equiv e^{p^2/\Lambda^2}\left[\bar{\psi}(-p) + \frac{i}{\Lambda}S_\Lambda\frac{\overleftarrow{\delta}}{\delta\psi(p)}\right]. \tag{C4c}$$

Since the ghost part of the action is given by

$$S_{\Lambda,\text{ghost}} = -\int_k \bar{c}(-k)\frac{k^2}{k^2/\Lambda^2 + e^{-2k^2/\Lambda^2}}c(k), \tag{C5}$$

the BRST invariance reduces to the WT identity

$$\begin{aligned}
&\frac{\xi e^{-2k^2} + k^2}{\xi e^{-k^2}}k_\mu\frac{\delta S_I}{\delta A_\mu(k)} \\
&= ee^{-S}\int_p e^{-(p+k)^2+p^2}\,\mathrm{Tr}\left\{\left[\psi(p) + i\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}\right]e^S\right\}\frac{\overleftarrow{\delta}}{\delta\psi(p+k)} \\
&\quad - ee^{-S}\int_p e^{-p^2+(p+k)^2}\,\mathrm{Tr}\,\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}\left\{e^S\left[\bar{\psi}(-p-k) + i\frac{\overleftarrow{\delta}}{\delta\psi(p+k)}\right]\right\}
\end{aligned} \tag{C6}$$

in the dimensionless notation. Because of the mismatch of the exponential cutoff functions, the WT identity is non-linear in S, not as simple as our WT identity (66).

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