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Research Article

Relativistic Energies and Scattering Phase Shifts for the Fermionic Particles Scattered by Hyperbolical Potential with the Pseudo(spin) Symmetry

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In this paper, we studied the approximate scattering state solutions of the Dirac equation with the hyperbolical potential with pseudospin and spin symmetries. By applying an improved Greene-Aldrich approximation scheme within the formalism of functional analytical method, we obtained the spin-orbit quantum numbers dependent scattering phase shifts for the spin and pseudospin symmetries. The normalization constants, lower and upper radial spinor for the two symmetries, and the relativistic energy spectra were presented. Our results reveal that both the symmetry constants (C_{ps} and C_s) and the spin-orbit quantum number κ affect scattering phase shifts significantly.

1. Introduction

Scattering theory is very central to the study of several fields such as atomic, nuclear, high energy or condensed matter physics. It allows for descriptions and interpretations of many collisions processes such as excitation and ionization by particle or radiation impact [1–7]. Complete information about the quantum systems can only be obtained by investigating scattering state solutions of relativistic and nonrelativistic equations with quantum mechanical potential model.

As a result, several authors in quantum mechanics have strictly followed different approaches to study the scattering state solutions of the relativistic and nonrelativistic wave equations for central and noncentral potential models [8–15]. In their works, they have reported the calculations on phase shifts, transmission and reflection coefficients, resonances, normalized radial wave functions and properties of S-matrix for potential models of their interest. All these are sufficient enough to predict, correlate, and describe the behaviour of particles.

However, literature revealed that the investigations on the spin and pseudosymmetries [16, 17] of Dirac equation, which was previously on the bound state problems [18–21], have now been extended to scattering state problems [22, 23]. Just of recent, under the spin and pseudospin symmetry, the analysis of scattering state solutions of Dirac equation with certain potential function models of interest were considered by some authors. These include Yukawa potential [22] and Hellmann potential [23].

This is owing to the fact that the symmetries in Hadron and nuclear spectroscopy [24] as well as deformation and superdeformation in nuclei [25] can be best understood by studying scattering state problems under pseudo(spin) symmetry. Among the various relativistic potential function models, we consider hyperbolical potential (Schiöberg potential), which was proposed by Schiöberg in 1986 [26] in order to find a more suitable empirical potential energy function for diatomic molecules. In 2012, Wang et al. [27] constructed an improved version of hyperbolical potential (called new Schiöberg potential) by employing the dissociation energy and the equilibrium bond length for a diatomic molecule as explicit parameters. As Jia et al. [28, 29] also reported that the new Schiöberg potential, Deng-Fan potential, and the Manning-Rosen potential are the same empirical potential energy functions.

In view of the above works, we are motivated to investigate the scattering state solutions of Dirac equation with hyperbolical potential suggested by Schiöberg in 1986 and apply an improved Greene-Aldrich approximation scheme within the formalism of functional analytical method. In the present study the new Schiöberg potential is ignored and we focus on the effects of the symmetry constants and the positive potential parameters on the relativistic energy and the scattering phase shifts of the hyperbolical potential (Schiöberg potential).

This paper is organized as follows: Section 2 contains the basic equation. In Section 3, we studied the approximate scattering state solutions for the hyperbolical potential in detail. The discussions and conclusions are given in Section 4.

2. The Basic Equations

By considering the Dirac wave equation and its corresponding spinors, the two-coupled first-order differential equations for the upper and lower components of the spinor may be obtained as [30–33]

$$\left[\frac{d}{dr}+\frac{\kappa}{r}\right]F_{n\kappa}(r)=\left[M+E_{n\kappa}-\Delta(r)\right]G_{n\kappa}(r), \quad (1)$$

$$\left[\frac{d}{dr}-\frac{\kappa}{r}\right]G_{n\kappa}(r)=\left[M-E_{n\kappa}+\Sigma(r)\right]F_{n\kappa}(r), \quad (2)$$

where $\Delta(r)=V(r)-S(r)$ and $\Sigma(r)=V(r)+S(r)$. Solving for $G_{n\kappa}(r)$ in (1) and $F_{n\kappa}(r)$ in (2), we obtain the Schrödinger-like equations satisfying for upper radial spinor $F_{n\kappa}(r)$ and lower radial spinor $G_{n\kappa}(r)$, respectively, as

$$\left\{\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}+\left[-\left(M+E_{n\kappa}-\Delta(r)\right)\left(M-E_{n\kappa}+\Sigma(r)\right)+\frac{(d\Delta(r)/dr)(d/dr+\kappa/r)}{M+E_{n\kappa}-\Delta(r)}\right]\right\}F_{n\kappa}(r)=0, \quad (3)$$

$$\left\{\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}+\left[-\left(M+E_{n\kappa}-\Delta(r)\right)\left(M-E_{n\kappa}+\Sigma(r)\right)-\frac{(d\Sigma(r)/dr)(d/dr-\kappa/r)}{M-E_{n\kappa}+\Sigma(r)}\right]\right\}G_{n\kappa}(r)=0, \quad (4)$$

where $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1)$ and $\kappa(\kappa+1)=l(l+1)$.

2.1. Pseudospin Symmetry Limit for the Hyperbolical Potential. By following the pseudospin symmetry conditions and considering the hyperbolical potential $\Delta(r)$ satisfying relativistic model [34–37],

$$\Delta(r)=V(r)=D\left[1-\sigma_0\coth(\alpha r)\right]^2, \quad (5)$$

where D, α, and σ_0 are the three positive potential parameters that significantly affect the relativistic energy spectra and the relativistic scattering phase shifts. Schiöberg [26] reported that this potential is closely related to the Morse, the Kratzer, the Coulomb, and the harmonic oscillators and other potential functions in a particular limit. The properties and applications of this potential have been given by Lu et al. (2005) and Schiöberg (1986).

Under the pseudospin symmetry condition (4) yields

$$\left\{\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}-\gamma+\beta D\left[1-\frac{\sigma_0\left(1+e^{-2\alpha r}\right)}{1-e^{-2\alpha r}}\right]^2\right\}\cdot G_{\mathrm{ps},n\kappa}(r)=0, \quad (6)$$

where $\gamma=(M+E_{n\kappa})(M-E_{n\kappa}+C_{\mathrm{ps}})$ and $\beta=M-E_{n\kappa}+C_{\mathrm{ps}}$ are the pseudospin symmetry energy parameters.

2.2. Spin Symmetry Limit for the Hyperbolical Potential. In a similar way, we consider the spin symmetry conditions and take $\Sigma(r)$ as hyperbolical potential [34–37]. That is,

$$\Sigma(r)=V(r)=D\left[1-\sigma_0\coth(\alpha r)\right]^2, \quad (7)$$

and using the spin symmetry conditions, (3) becomes

$$\left\{\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}-\tilde{\gamma}-\tilde{\beta}D\left[1-\frac{\sigma_0\left(1+e^{-2\alpha r}\right)}{1-e^{-2\alpha r}}\right]^2\right\}\cdot F_{\mathrm{s},n\kappa}(r)=0, \quad (8)$$

where $\tilde{\gamma}=(M-E_{\mathrm{s},n\kappa})(M+E_{\mathrm{s},n\kappa}-C_{\mathrm{s}})$ and $\tilde{\beta}=(M+E_{\mathrm{s},n\kappa}-C_{\mathrm{s}})$ denote spin symmetry energy parameters.

2.3. Pekeris-Type Approximation. To obtain the approximate solutions in the presence of the spin symmetry and pseudospin symmetry, we use the Greene-Aldrich approximation [38] and apply the widely used scheme for the centrifugal term; namely,

$$\frac{1}{r^2}\approx 4\alpha^2\left[c_0+\frac{e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}\right], \quad (9)$$

where $c_0=1/12$ is a dimensionless constant [39]. This approximation has been used to obtain the approximate analytical solutions of Dirac equation with the hyperbolic potential in the presence of spin symmetry and pseudospin symmetry [21]. When $c_0=0$, this approximation reduces to the well-known Greene-Aldrich approximation [38]. Note that another approximation scheme has been proposed to overcome the effect of centrifugal term in [40]. Our interest here is to apply this improved Greene-Aldrich approximation and see whether the symmetry constants (C_{ps} and C_{s}) and the spin-orbit quantum number κ will have any influence on the scattering phase shifts.

3. Relativistic Scattering State Solutions

3.1. Pseudospin Symmetry Limit for Hyperbolical Potential. Defining a variable $z = 1 - e^{-2\alpha r}$ and applying approximation in (9), then (6) reduces to

$$\left\{\frac{d^2}{dz^2} - \frac{1}{(1-z)}\frac{d}{dz} + \left[\frac{Pz^2 + Qz + R}{z^2(1-z)^2}\right]\right\} G_{ps,nk}(r) = 0, \quad (10)$$

with the following useful definitions:

$$P = \frac{k_1^2}{4\alpha^2} + \frac{\beta D\sigma_o}{\alpha^2},$$
$$Q = \frac{4\alpha^2\kappa(\kappa-1)c_0 - 4\beta D\sigma_0(1+\sigma_0)}{4\alpha^2}, \quad (11)$$
$$R = \frac{-4\alpha^2\kappa(\kappa-1)c_0 + 4\beta D\sigma_0^2}{4\alpha^2},$$

where $k_1 = \sqrt{\beta D(1-\sigma_0)^2 - \gamma - 4\alpha^2\kappa(\kappa-1)c_0}$ is the asymptotic wave number for the pseudospin symmetry limit.

In order to solve (10) via the functional analytical method, we need to assume a wave function

$$G_{ps,n\kappa}(z) = z^\lambda (1-z)^{-ik_1/2\alpha} f(z), \quad (12)$$

with the pseudospin wave function parameter $\lambda = 1/2 + (1/2)\sqrt{1 + 4\kappa(\kappa-1) - 4\beta D\sigma_0^2/\alpha^2}$. Inserting (12) into (10) leads to the formation of hypergeometric equation [41]

$$z(1-z)f''(z) + \left[2\lambda - \left(2\lambda - \frac{ik_1}{\alpha} + 1\right)z\right]f'(z) + \left[\left(\lambda - \frac{ik_1}{2\alpha}\right)^2 + P\right]f(z) = 0. \quad (13)$$

By considering the boundary condition that $f(z)$ tends to finite when $z \to 0$, the lower component of radial wave functions for any arbitrary κ-wave scattering states for the hyperbolical potential is obtained as [42]

$$G_{ps,n\kappa}(z) = N_{n\kappa}\left(1 - e^{-2\alpha r}\right)^\lambda e^{ik_1 r} {}_2F_1\left(a, b, c; 1 - e^{-2\alpha r}\right), \quad (14)$$

where

$$a = \lambda - \frac{ik_1}{2\alpha} - \sqrt{-\frac{\beta D\sigma_o}{\alpha^2} - \frac{k_1^2}{4\alpha^2}},$$
$$b = \lambda - \frac{ik_1}{2\alpha} + \sqrt{-\frac{\beta D\sigma_o}{\alpha^2} - \frac{k_1^2}{4\alpha^2}}, \quad (15)$$
$$c = 2\lambda.$$

It is required that we consider the following conjugate relations which define the asymptotic phases:

$$c - a - b = (a + b - c)^* = \frac{ik_1}{\alpha}, \quad (16a)$$

$$c - b = \lambda + \frac{ik_1}{2\alpha} - \sqrt{-\frac{\beta D\sigma_o}{\alpha^2} - \frac{k_1^2}{4\alpha^2}} = a^*, \quad (16b)$$

$$c - a = \lambda + \frac{ik_1}{2\alpha} + \sqrt{-\frac{\beta D\sigma_o}{\alpha^2} - \frac{k_1^2}{4\alpha^2}} = b^* \quad (16c)$$

and $N_{n\kappa}$ is the normalization.

3.1.1. Pseudospin Symmetry Phase Shifts and Normalization Constant. To obtain the phase shifts δ_l and normalization constant, we apply the following recurrence relation of hypergeometric function or analytic-continuation formula [41]:

$$\begin{aligned}{}_2F_1(a,b,c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \cdot {}_2F_1(a;b;1+a+b-c;1-z) + (1-z)^{c-a-b} \\ &\cdot \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma a \Gamma(b)} \cdot {}_2F_1(c-a;c-b;c-a-b+1;1-z).\end{aligned} \quad (17)$$

Considering (17) and the property ${}_2F_1(a,b;c;0) = 1$, as $r \to \infty$, we have

$${}_2F_1\left(a,b;c;1-e^{-2\alpha r}\right) = \Gamma(c) \cdot \left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} + \frac{\Gamma(c-a-b)}{\Gamma(a^*)\Gamma(b^*)}e^{-\alpha(c-a-b)r}\right|; \quad (18)$$

using (16a), (16b), and (16c) we may transform (18) as

$${}_2F_1\left(a,b;c;1-e^{-2\alpha r}\right) = \Gamma(c) \cdot \left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} + \left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|^* e^{-ik_1 r}\right|. \quad (19)$$

By taking $\Gamma(c-a-b)/\Gamma(c-a)\Gamma(c-b) = |\Gamma(c-a-b)/\Gamma(c-a)\Gamma(c-b)|e^{i\delta}$ and inserting in (19), we have

$${}_2F_1\left(a,b;c;1-e^{-2\alpha r}\right) = \Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right| \cdot e^{-ik_1 r}\left[e^{i(k_1 r+\delta)} + e^{-i(k_1 r+\delta)}\right]. \quad (20)$$

Therefore, we obtain the asymptotic form of the lower spinor for $r \to \infty$ as

$$G_{ps,n\kappa}(r) = 2N_{n,\kappa}\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right| \times \sin\left(k_1 r + \frac{\pi}{2} + \delta\right). \quad (21)$$

On comparison of (20) with the boundary condition $r \to \infty \Rightarrow G_{\mathrm{ps},n\kappa}(\infty) \to 2\sin(k_1 r + \delta_{l,n\kappa} - l\pi/2)$ [42]. Thus, we finally obtain the explicit pseudospin symmetry phase shifts and the normalization constant, respectively, as

$$\begin{aligned}\delta_{l\mathrm{ps},n\kappa} &= \frac{\pi}{2}(l+1) + \arg\Gamma\left(\frac{ik_1}{\alpha}\right) \\ &- \arg\Gamma\left(\lambda + \frac{ik_1}{2\alpha} + \sqrt{-\frac{\beta D\sigma_o}{\alpha^2} - \frac{k_1^2}{4\alpha^2}}\right) \\ &- \arg\Gamma\left(\lambda + \frac{ik_1}{2\alpha} - \sqrt{-\frac{\beta D\sigma_o}{\alpha^2} - \frac{k_1^2}{4\alpha^2}}\right),\end{aligned} \tag{22}$$

$$\begin{aligned}N_{\mathrm{ps},n\kappa} &= \frac{\left|\Gamma\left(\lambda + ik_1/2\alpha + \sqrt{-\beta D\sigma_o/\alpha^2 - k_1^2/4\alpha^2}\right)\right|}{\sqrt{2\lambda}} \\ &\times \left|\frac{\Gamma\left(\lambda + ik_1/2\alpha - \sqrt{-\beta D\sigma_o/\alpha^2 - k_1^2/4\alpha^2}\right)}{\Gamma(ik_1/\alpha)}\right|.\end{aligned} \tag{23}$$

3.1.2. Analytical Properties of S-Matrix for the Pseudospin Symmetry Limit. Here, the analytical properties of partial-wave s-matrix are investigated to verify the fact that the poles of the s-matrix in the complex energy plane correspond to bound states for real poles [42]; thus, we consider $\Gamma(\lambda + ik_1/2\alpha + \sqrt{-\beta D\sigma_o/\alpha^2 - k_1^2/4\alpha^2})$, where its first-order poles are at the point [42]

$$\begin{aligned}&\left(\lambda + \frac{ik_1}{2\alpha} + \sqrt{-\frac{\beta D\sigma_o}{\alpha^2} - \frac{k_1^2}{4\alpha^2}}\right) = 0, -1, -2, -3, \ldots \\ &= -n \quad (n = 0, 1, 2, \ldots).\end{aligned} \tag{24}$$

Consequently, the bound state energy levels for the pseudospin symmetry limit are obtained as

$$\begin{aligned}\frac{\gamma}{4\alpha^2} &= \frac{\beta D}{4\alpha^2}(1-\sigma_0)^2 - \kappa(\kappa-1)c_0 \\ &+ \left[\frac{(n+\lambda)^2 + \beta D\sigma_0/\alpha^2}{2(n+\lambda)}\right]^2.\end{aligned} \tag{25}$$

3.2. Spin Symmetry Limit for the Hyperbolical Potential. Using the previously defined transformation variable and approximation, (8) becomes

$$\begin{aligned}&\left\{\frac{d^2}{dz^2} - \frac{1}{(1-z)}\frac{d}{dz} + \left[\frac{\widetilde{P}z^2 + \widetilde{Q}z + \widetilde{R}}{z^2(1-z)^2}\right]\right\} F_{\mathrm{s},nk}(r) \\ &= 0,\end{aligned} \tag{26}$$

with the following spin symmetry phase parameters:

$$\begin{aligned}\widetilde{P} &= \frac{k_2^2}{4\alpha^2} - \frac{\widetilde{\beta} D\sigma_0}{\alpha^2}, \\ \widetilde{Q} &= \frac{4\alpha^2\kappa(\kappa+1)c_0 + 4\widetilde{\beta}D\sigma_0(1+\sigma_0)}{4\alpha^2}, \\ \widetilde{R} &= \frac{-4\alpha^2\kappa(\kappa+1)c_0 - 4\widetilde{\beta}D\sigma_0^2}{4\alpha^2},\end{aligned} \tag{27}$$

where $k_2 = \sqrt{-\widetilde{\beta}D(1-\sigma_0)^2 - \widetilde{\gamma} - 4\alpha^2\kappa(\kappa+1)c_0}$ is the asymptotic wave number for the spin symmetry limit.

Similarly, we also assume the following upper wave function for the spin symmetry

$$F_{\mathrm{s},n\kappa}(z) = z^{\widetilde{\lambda}}(1-z)^{-ik_2/2\alpha} f(z), \tag{28}$$

with the spin symmetry wave function parameter $\widetilde{\lambda} = 1/2 + (1/2)\sqrt{1 + 4\kappa(\kappa+1) - 4\widetilde{\beta}D\sigma_0^2/\alpha^2}$.

To avoid repetition, we follow the same procedures in previous subsection and write the upper component of spin symmetry radial wave functions for any arbitrary κ-wave scattering states as

$$\begin{aligned}&F_{\mathrm{s},n\kappa}(z) \\ &= N_{n\kappa}\left(1 - e^{-2\alpha r}\right)^{\widetilde{\lambda}} e^{ik_2 r}{}_2F_1\left(a, b, c; 1 - e^{-2\alpha r}\right),\end{aligned} \tag{29}$$

where we have used the following wave function parameters:

$$\begin{aligned}a &= \widetilde{\lambda} - \frac{ik_2}{2\alpha} - \sqrt{\frac{\widetilde{\beta}D\sigma_0}{\alpha^2} - \frac{k_2^2}{4\alpha^2}}, \\ b &= \widetilde{\lambda} - \frac{ik_2}{2\alpha} + \sqrt{\frac{\widetilde{\beta}D\sigma_0}{\alpha^2} - \frac{k_2^2}{4\alpha^2}}, \\ c &= 2\widetilde{\lambda},\end{aligned} \tag{30}$$

where $N_{n\kappa}$ is the normalization constant depending on n and κ.

3.2.1. Spin Symmetry Phase Shifts and Normalization Constant. Following the same steps in Section 3.1.1, we write the explicit spin symmetry phase shifts and the corresponding normalization constant, respectively, as

$$\begin{aligned}\widetilde{\delta}_{l\mathrm{s},n\kappa} &= \frac{\pi}{2}(l+1) + \arg\Gamma\left(\frac{ik_2}{\alpha}\right) \\ &- \arg\Gamma\left(\widetilde{\lambda} + \frac{ik_2}{2\alpha} + \sqrt{\frac{\widetilde{\beta}D\sigma_0}{\alpha^2} - \frac{k_2^2}{4\alpha^2}}\right)\end{aligned}$$

$$-\arg\Gamma\left(\tilde{\lambda}+\frac{ik_2}{2\alpha}-\sqrt{\frac{\tilde{\beta}D\sigma_0}{\alpha^2}-\frac{k_2^2}{4\alpha^2}}\right), \tag{31}$$

$$N_{s,n\kappa}=\frac{\left|\Gamma\left(\tilde{\lambda}+ik_2/2\alpha+\sqrt{\tilde{\beta}D\sigma_0/\alpha^2-k_2^2/4\alpha^2}\right)\right|}{\sqrt{2\tilde{\lambda}}}\times\left|\frac{\Gamma\left(\tilde{\lambda}+ik_2/2\alpha-\sqrt{\tilde{\beta}D\sigma_0/\alpha^2-k_2^2/4\alpha^2}\right)}{\Gamma\left(ik_2/\alpha\right)}\right|, \tag{32}$$

where we have employed the following phase shifts parameters for simplicity:

$$c-a-b=(a+b-c)^*=\frac{ik_2}{\alpha},$$
$$c-b=\tilde{\lambda}+\frac{ik_2}{2\alpha}-\sqrt{\frac{\tilde{\beta}D\sigma_0}{\alpha^2}-\frac{k_2^2}{4\alpha^2}}=a^*, \tag{33}$$
$$c-a=\tilde{\lambda}+\frac{ik_2}{2\alpha}+\sqrt{\frac{\tilde{\beta}D\sigma_0}{\alpha^2}-\frac{k_2^2}{4\alpha^2}}=b^*.$$

3.2.2. Analytical Properties of S-Matrix for the Spin Symmetry Limit. Following the same fashion in Section 3.1.2, the corresponding bound state energy levels for the spin symmetry are determined by the following energy equation:

$$\kappa(\kappa+1)c_0+\frac{\tilde{\gamma}}{4\alpha^2}+\frac{\tilde{\beta}D}{4\alpha^2}(1-\sigma_0)^2-\left[\frac{\left(n+\tilde{\lambda}\right)^2-\tilde{\beta}D\sigma_0/\alpha^2}{2\left(n+\tilde{\lambda}\right)}\right]^2=0. \tag{34}$$

3.3. Nonrelativistic Limit for the Scattering State Solution. To study the nonrelativistic limit, we apply the following appropriate mapping to (34):

$$\kappa(\kappa+1)=l(l+1),$$
$$E_{s,nk}-M\longrightarrow E_{nl},$$
$$M+E_{s,nk}\longrightarrow\frac{2\mu}{\hbar^2}, \tag{35}$$
$$C_s=0.$$

Consequently, we obtain the nonrelativistic bound state energy levels for any arbitrary l as

$$E_{nl}=\frac{2\alpha^2\hbar^2 l(l+1)c_0}{\mu}+D(1-\sigma_0)^2-\frac{\alpha^2\hbar^2}{8\mu}\left[\frac{(\Lambda+2n)^2-8\mu D\sigma_0/\alpha^2\hbar^2}{(\Lambda+2n)}\right]^2, \tag{36}$$

where $\Lambda=1+\sqrt{1+4l(l+1)+8\mu D\sigma_0^2/\alpha^2\hbar^2}$.

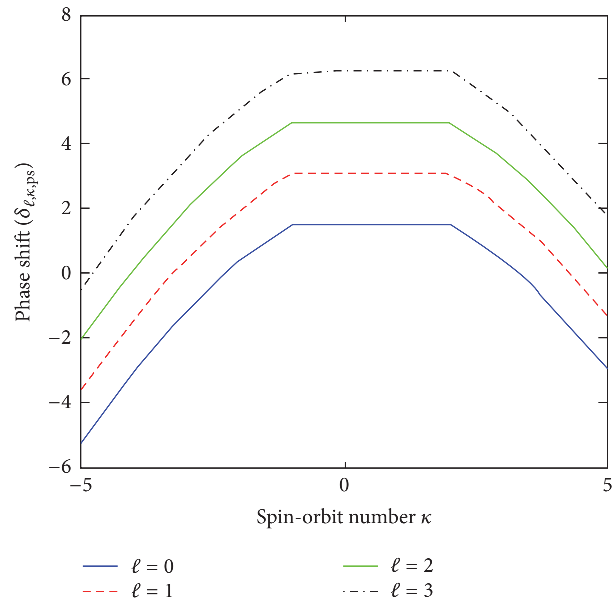

FIGURE 1: A plot of pseudospin scattering phase shifts for the hyperbolical potential as a function of spin-orbit number κ for $l = 0, 1, 2, 3$ with positive potential parameter $\sigma_0 = 0.10$, $\alpha = 0.10$, $c_o = 1/12$, $C_{ps} = 0$, $D = 10$, and $E = M = 1$. The relativistic mass and energy are equal in all the calculations.

4. Discussion and Conclusion

The pseudospin symmetry bound states energy spectra displayed in Table 1 are obtained from (25) and the corresponding spin symmetry bound states energy spectra displayed in Table 2 are obtained from (34) while the nonrelativistic bound state energy spectra are obtained from (36). The pseudospin symmetry and spin symmetry phase shifts are obtained from (22) and (31), respectively.

In Tables 1 and 2, for fixed values principal quantum number n, the relativistic bound state energies increase with decreasing values of spin-orbit quantum numbers κ whether the symmetry constants are present or not. The relativistic bound state energy increases with increase in principal quantum number n for all $\kappa < 0$. An increase positive parameter σ_0 decreases the relativistic bound state energy for all n and all $\kappa < 0$. The results reasonably showed that the presence symmetry constants contribute significantly to the relativistic bound state energies. Table 3 displayed improved nonrelativistic energies for the states 2p, 3p, 3d, 4p, 4d, and 4f for positive potential parameter $\sigma_0 = 0.10$.

The pseudospin symmetry and spin symmetry phase shifts are displayed in Tables 4 and 5, respectively. To see the clearer behaviour of the phase shifts in the spin and pseudospin symmetry limits we plot the numerical results of phase shift in both pseudospin and spin symmetries against the spin-orbit quantum numbers κ in Figures 1–4. In Figure 1, phase shifts are slightly decreasing exponentially to the left and to the right for negative and positive values of spin-orbit quantum numbers, respectively, for zero pseudospin constant. A steady phase shifts is observed at $-1 \le \kappa \le 1$. This suggests that the spin-orbit number affects phase shifts

Table 1: Pseudospin symmetry bound state energies at the poles of s-matrix (in units fm^{-1}) for the hyperbolical potential as a function of positive potential parameter σ_0 for different values n and $\kappa < 0$, $\alpha = 0.1$ in atomic units ($\mu = \hbar = 1$), and $c_o = 1/12$.

n	κ	σ_0	$E_{ps,n\kappa}$, $D = 5$, $C_{ps} = 0$	$E_{ps,n,\kappa}$, $D = 10$, $C_{ps} = -5$
1	−1	0.10	2.279163696, 1.029316001	2.1233814120, −3.993501276
1	−1	0.15	1.861759552, 1.031095657	1.3219168610, −3.993392561
1	−1	0.20	1.595805327, 1.034022382	0.8412753547, −3.993236371
1	−1	0.25	1.410036018, 1.038851771	0.5132318062, −3.993028490
1	−2	0.10	2.499986930, 1.049190512	2.3968826230, −3.989115288
1	−2	0.15	2.038697268, 1.051647743	1.4930060260, −3.988979900
1	−2	0.20	1.734781325, 1.055647948	0.9587751244, −3.988786147
1	−2	0.25	1.518635403, 1.062147486	0.5989218610, −3.988529604
1	−3	0.10	2.699088639, 1.074153615	2.7421356920, −3.983691020
1	−3	0.15	2.214785842, 1.077522035	1.7216780460, −3.983519794
1	−3	0.20	1.879801953, 1.082984677	1.1206711830, −3.983275390
1	−3	0.25	1.634694558, 1.091816104	0.7192390968, −3.982952907
1	−4	0.10	2.860475911, 1.104471112	3.1199566410, −3.977227902
1	−4	0.15	2.372200392, 1.109000893	1.9871570450, −3.977014236
1	−4	0.20	2.015688603, 1.116352376	1.3149710440, −3.976709786
1	−4	0.25	1.745900039, 1.128273463	0.8667745016, −3.976309013
2	−1	0.10	2.615061529, 1.055125495	3.1191738270, −3.987973646
2	−1	0.15	2.170952096, 1.059520099	2.1743591190, −3.987691971
2	−1	0.20	1.854820744, 1.066898930	1.5763871650, −3.987286660
2	−1	0.25	1.610697844, 1.079577280	1.1529573900, −3.986746084
2	−2	0.10	2.751633202, 1.082483207	3.3080559170, −3.982048026
2	−2	0.15	2.286444614, 1.087766383	2.2967933160, −3.981747832
2	−2	0.20	1.948294269, 1.096614037	1.6626435590, −3.981316476
2	−2	0.25	1.684969993, 1.111784132	1.2171612360, −3.980742235
2	−3	0.10	2.880415615, 1.115474816	3.5565298640, −3.975069259
2	−3	0.15	2.407308925, 1.122116505	2.4653804670, −3.974727589
2	−3	0.20	2.050458310, 1.133236516	1.7841645900, −3.974237859
2	−3	0.25	1.766983989, 1.152366772	1.3088621540, −3.973588060
2	−4	0.10	2.985001626, 1.154437579	3.8391364550, −3.967043330
2	−4	0.15	2.516623028, 1.162869448	2.6672425580, −3.966648795
2	−4	0.20	2.146938664, 1.177051056	1.9336438780, −3.966084415
2	−4	0.25	1.844677707, 1.201768936	1.4235514910, −3.965337543

significantly for any arbitrary angular momentum quantum number.

Figure 2 illustrates the behaviour of phase shifts in the presence of pseudospin constants; the graphs follow the same pattern with more negative phase shifts. The negativity is an indication that pseudospin constant strongly influence scattering phase shifts. However, Figures 3 and 4 illustrate the behaviour of spin symmetry phase shifts. An exponential rise in phase shifts to the left and to the right for negative and positive value of the spin-orbit quantum numbers κ, respectively, is observed for all angular momentum quantum number l.

In conclusion, we have studied the approximate scattering state solution of Dirac equation with the hyperbolical potential using a short-range approximation within the framework of functional analytical method. We have obtained the spin and pseudospin symmetry bound state energies and their corresponding nonrelativistic energies, spin and pseudospin symmetry phase shifts, normalization constants, pseudospin symmetry lower component, and spin symmetry upper com-

Table 2: Spin symmetry energies at the poles of s-matrix (in units fm^{-1}) for the hyperbolical potential as a function of positive potential parameter σ_0 for different values n and $\kappa < 0$, $\alpha = 0.1$ and $D = 10$ in atomic units ($\mu = \hbar = 1$), and $c_o = 1/12$.

n	κ	σ_0	$E_{n,\kappa<0}$, $D = 10$, $C_s = 0$	$E_{n,\kappa<0}$, $D = 10$, $C_s = 5$
0	−2	0.10	2.852765813, −0.9971975973	4.894486374, 4.004871783
0	−2	0.15	2.286257855, −0.9971671352	4.484669284, 4.005013210
0	−2	0.20	1.976243190, −0.9971235606	4.291486196, 4.005228383
0	−2	0.25	1.777926670, −0.9970658947	4.186281947, 4.005540044
0	−3	0.10	3.349240971, −0.9941222745	5.374173995, 4.010207082
0	−3	0.15	2.597878123, −0.9940696246	4.789639030, 4.010461236
0	−3	0.20	2.190050696, −0.9939945667	4.496311909, 4.010844649
0	−3	0.25	1.933251119, −0.9938956907	4.328836785, 4.011393011
0	−4	0.10	3.899001828, −0.9899271098	5.855422017, 4.017491955
0	−4	0.15	2.970664174, −0.9898454341	5.119514795, 4.017895505
0	−4	0.20	2.458599114, −0.9897291933	4.729861208, 4.018501874
0	−4	0.25	2.135123157, −0.9895764138	4.498048414, 4.019363991
0	−5	0.10	4.445008402, −0.9846098450	6.309268779, 4.026738666
0	−5	0.15	3.365034712, −0.9844924012	5.451065869, 4.027328768
0	−5	0.20	2.754614425, −0.9843254210	4.975122387, 4.028213547
0	−5	0.25	2.364503739, −0.9841062477	4.681992554, 4.029467529
1	−2	0.10	4.307228207, −0.9929477534	5.855267990, 4.012335498
1	−2	0.15	3.494799666, −0.9928150205	5.188967231, 4.012888014
1	−2	0.20	3.001609225, −0.9926233942	4.804823066, 4.013746766
1	−2	0.25	2.661245323, −0.9923666479	4.556984308, 4.015030469
1	−3	0.10	4.595853112, −0.9881911614	6.178641705, 4.020607401
1	−3	0.15	3.680442629, −0.9880244067	5.414291047, 4.021342479
1	−3	0.20	3.132109286, −0.9877847131	4.970695195, 4.022473524
1	−3	0.25	2.758390431, −0.9874654265	4.683014161, 4.024139623
1	−4	0.10	4.953432668, −0.9823076582	6.534290370, 4.030858520
1	−4	0.15	3.924112012, −0.9820955447	5.675159767, 4.031828444
1	−4	0.20	3.308851693, −0.9817915310	5.167984043, 4.033311444
1	−4	0.25	2.892600410, −0.9813881332	4.835489514, 4.035475765
1	−5	0.10	5.338358623, −0.9752962991	6.885521151, 4.043099865
1	−5	0.15	4.202075842, −0.9750304881	5.945991145, 4.044348875
1	−5	0.20	3.517208069, −0.9746502453	5.378418925, 4.046250921
1	−5	0.25	3.054278778, −0.9741470070	5.001075188, 4.049010392

ponent of radial spinor wave functions for any arbitrary κ-wave scattering states.

We also studied the behaviour of phase shifts with spin-orbit quantum numbers under spin and pseudospin symmetries and we have successfully showed that relativistic scattering phase shifts largely depend on the symmetry constants (C_{ps} and C_s) and spin-orbit quantum numbers κ

Disclosure

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Table 3: Nonrelativistic energies at the poles of s-matrix (in units fm^{-1}) for the hyperbolical potential as a function of positive potential parameter σ_0 for different states in atomic units ($\mu = \hbar = 1$), $c_o = 1/12$, and $D = 10$ in all the calculations.

n	l	σ_0	States	E_{nl} for $\alpha = 0.10$	E_{nl} for $\alpha = 0.15$	E_{nl} for $\alpha = 0.20$	E_{nl} for $\alpha = 0.25$
		0.10		2.61890	3.90580	5.00395	5.88694
0	1	0.15	2p	1.68043	2.57796	3.43332	4.21023
		0.20		1.20892	1.86672	2.52064	3.14766
		0.10		4.73556	6.04579	6.91727	7.48500
1	1	0.15	3p	3.46030	4.62316	5.50084	6.15070
		0.20		2.68324	3.67163	4.46580	5.09331
		0.10		3.62747	5.29513	6.47684	7.25824
0	2	0.15	3d	2.27024	3.56732	4.69715	5.59908
		0.20		1.57921	2.54881	3.48311	4.31406
		0.10		6.00303	7.11562	7.71968	8.02132
2	1	0.15	4p	4.66775	5.80666	6.52479	6.95757
		0.20		3.75708	4.81251	5.53175	6.00386
		0.10		5.33170	6.73691	7.54672	7.97921
1	2	0.15	4d	3.85825	5.19508	6.13602	6.75665
		0.20		2.95305	4.10518	5.00371	5.66610
		0.10		4.69061	6.43208	7.43782	7.98144
0	3	0.15	4f	3.00365	4.60199	5.79916	6.61027
		0.20		2.07438	3.35838	4.47793	5.35424

Table 4: Pseudospin scattering phase shifts for hyperbolical potential with positive potential parameter $\sigma_0 = 0.10$, $\alpha = 0.10$, $c_o = 1/12$, and $E = 1$. The relativistic mass is $m = 1$ in atomic units ($m = \hbar = 1$).

l	κ	$\delta_{l,\text{ps},\kappa}$ for $C_{\text{ps}} = 0, D = 10$	$\delta_{l,\text{ps},\kappa}$ for $C_{\text{ps}} = 0.05, D = 10$
	−1, 1	1.521873210037270, 1.570796326794897	−8.076217263995554, −6.037944299866905
	−2, 2	0.446819990038913, 1.521873210037270	−9.291029083349983, −8.076217263995554
0	−3, 3	−1.097154578790937, 0.446819990038913	−9.904139528549850, −9.291029083349983
	−4, 4	−2.972292099735542, −1.097154578790937	−10.013382196552033, −9.904139528549850
	−5, 5	−5.103985153357378, −2.972292099735542	−9.621579256682395, −10.013382196552033
	−1, 1	3.092669536832167, 3.141592653589793	−6.505420937200658, −4.467147973072009
	−2, 2	2.017616316833810, 3.092669536832167	−7.720232756555086, −6.505420937200658
1	−3, 3	0.473641748003960, 2.017616316833810	−8.333343201754955, −7.720232756555086
	−4, 4	−1.401495772940645, 0.473641748003960	−8.442585869757139, −8.333343201754955
	−5, 5	−3.533188826562482, −1.401495772940645	−8.050782929887498, −8.442585869757139
	−1, 1	4.663465863627064, 4.712388980384690	−4.934624610405762, −2.896351646277112
	−2, 2	3.588412643628707, 4.663465863627064	−6.149436429760190, −4.934624610405762
2	−3, 3	2.044438074798856, 3.588412643628707	−6.762546874960058, −6.149436429760190
	−4, 4	0.169300553854252, 2.044438074798856	−6.871789542962241, −6.762546874960058
	−5, 5	−1.962392499767585, 0.169300553854252	−6.479986603092602, −6.871789542962241
	−1, 1	6.234262190421960, 6.283185307179586	−3.363828283610865, −1.325555319482215
	−2, 2	5.159208970423603, 6.234262190421960	−4.578640102965293, −3.363828283610865
3	−3, 3	3.615234401593753, 5.159208970423603	−5.191750548165161, −4.578640102965293
	−4, 4	1.740096880649148, 3.615234401593753	−5.300993216167345, −5.191750548165161
	−5, 5	−0.391596172972688, 1.740096880649148	−4.909190276297705, −5.300993216167345

TABLE 5: Spin scattering phase shifts for hyperbolical potential with positive potential parameter $\sigma_0 = 0.50$, $\alpha = 0.50$, $c_o = 1/12$, and $E = 1$. The relativistic mass is $m = 1$ in atomic units ($m = \hbar = 1$).

l	κ	$\delta_{l,s,\kappa}$ for $C_s = 5$, $D = 10$	$\delta_{l,s,\kappa}$ for $C_s = 10$, $D = 10$
0	−1, 1	−15.357449458632775, −15.177770279380065	−34.356717558868027, −34.211165018163769
	−2, 2	−15.177770279380065, −14.803384517679635	−34.211165018163769, −33.916113793943268
	−3, 3	−14.803384517679635, −14.204750986373117	−33.916113793943268, −33.463715091492688
	−4, 4	−14.204750986373117, −13.337466036355023	−33.463715091492688, −32.842249720235664
	−5, 5	−13.337466036355023, −12.138509328495319	−32.842249720235664, −32.036048006960613
1	−1, 1	−13.786653131837877, −13.606973952585166	−32.785921232073129, −32.640368691368877
	−2, 2	−13.606973952585166, −13.232588190884737	−32.640368691368877, −32.345317467148369
	−3, 3	−13.232588190884737, −12.633954659578222	−32.345317467148369, −31.892918764697786
	−4, 4	−12.633954659578222, −11.766669709560125	−31.892918764697786, −31.271453393440765
	−5, 5	−11.766669709560125, −10.567713001700424	−31.271453393440765, −30.465251680165718
2	−1, 1	−12.215856805042982, −12.036177625790272	−31.215124905278238, −31.069572364573975
	−2, 2	−12.036177625790272, −11.661791864089839	−31.069572364573975, −30.774521140353471
	−3, 3	−11.661791864089839, −11.063158332783328	−30.774521140353471, −30.322122437902888
	−4, 4	−11.063158332783328, −10.195873382765226	−30.322122437902888, −29.700657066645867
	−5, 5	−10.195873382765226, −8.9969166749055290	−29.700657066645867, −28.894455353370819
3	−1, 1	−10.645060478248087, −10.465381298995377	−29.644328578483339, −29.498776037779084
	−2, 2	−10.465381298995377, −10.090995537294944	−29.498776037779084, −29.203724813558573
	−3, 3	−10.090995537294944, −9.4923620059884290	−29.203724813558573, −28.751326111107996
	−4, 4	−9.4923620059884290, −8.6250770559703320	−28.751326111107996, −28.129860739850976
	−5, 5	−8.6250770559703320, −7.4261203481106310	−28.129860739850976, −27.323659026575921

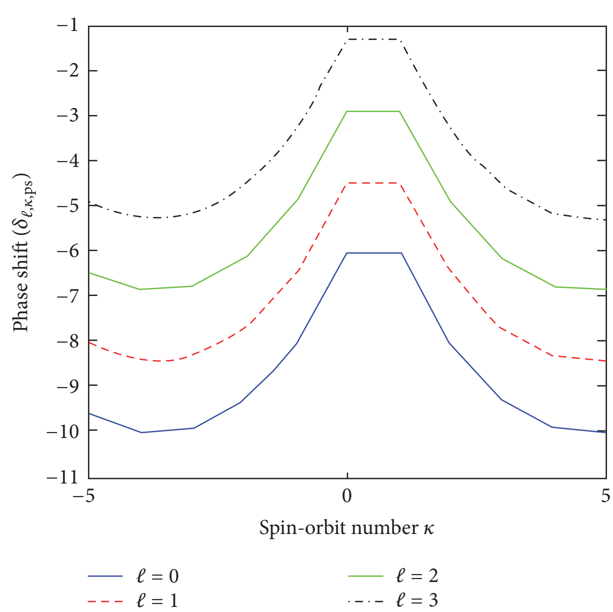

FIGURE 2: A plot of pseudospin scattering phase shifts for the hyperbolical potential as a function of spin-orbit number κ for $l = 0, 1, 2, 3$ with positive potential parameter $\sigma_0 = 0.10$, $\alpha = 0.10$, $c_o = 1/12$, $C_{ps} = 0.05$, $D = 10$, and $E = M = 1$. The relativistic mass and energy are equal in all the calculations.

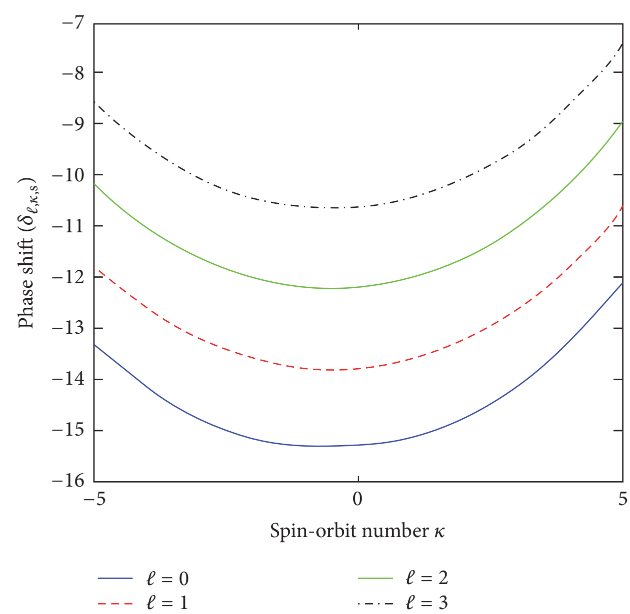

FIGURE 3: A plot of spin scattering phase shifts for the hyperbolical potential as a function of spin-orbit number κ for $l = 0, 1, 2, 3$ with positive potential parameter $\sigma_0 = 0.50$, $\alpha = 0.50$, $c_o = 1/12$, $C_s = 5$, $D = 10$, and $E = M = 1$. The relativistic mass and energy are equal in all the calculations.

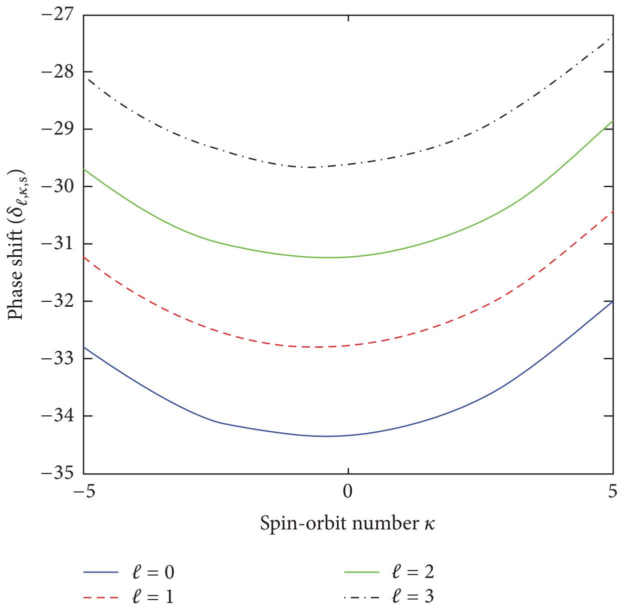

Figure 4: A plot of spin scattering phase shifts for the hyperbolical potential as a function of spin-orbit number κ for $l = 0, 1, 2, 3$ with positive potential parameter $\sigma_0 = 0.50$, $\alpha = 0.50$, $c_o = 1/12$, $C_s = 10$, $D = 10$, and $E = M = 1$. The relativistic mass and energy are equal in all the calculations.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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