



Conformal higher-spin symmetries in twistor string theory

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Abstract

It is shown that similarly to massless superparticle, classical global symmetry of the Berkovits twistor string action is infinite-dimensional. We identify its superalgebra, whose finite-dimensional subalgebra contains $psl(4|4, \mathbb{R})$ superalgebra. In quantum theory this infinite-dimensional symmetry breaks down to $SL(4|4, \mathbb{R})$ one.

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1. Introduction

Twistor string theory [1,2] inspired remarkable progress in understanding spinor and twistor structures underlying scattering amplitudes in gauge theories and gravity. Unlike conventional superstrings the twistor string spectrum presumably includes only a finite number of oscillation modes, in particular those of the open string sector are exhausted by 4-dimensional $N = 4$ super-Yang–Mills theory and conformal supergravity [3]. Since the latter theory is non-unitary and one is unable beyond the tree level to disentangle its modes from those of super Yang–Mills, there were made over time other propositions of twistor string models [4–6]. However, for tree-level gluon amplitudes there was proved [7] the equivalence of the expressions obtained within the Berkovits model [2] and using the field-theoretic approach. To gain further insights into the properties of twistor strings it is helpful to identify their symmetries both classical and

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quantum. In Refs. [1,8] it was shown that except for an obvious $PSL(4|4, \mathbb{R})$ global symmetry twistor strings are also invariant under its Yangian extension that is closely related to infinite-dimensional symmetry of integrable $N = 4$ super-Yang–Mills theory [9,10].

In this paper we argue that the world-sheet action of Berkovits twistor string is invariant under infinite-dimensional global symmetry, whose superalgebra contains as a finite-dimensional subalgebra the generators of $PSL(4|4, \mathbb{R})$, ‘twisted’ $GL_t(1, \mathbb{R})$ symmetries and constant shifts of supertwistor components. For the twistor string model with ungauged $gl(1, \mathbb{R})$ current global symmetry algebra is isomorphic to the Dirac brackets (D.B.) algebra of the collection of all monomials constructed from an arbitrary number $l \geq 0$ of $PSL(4|4, \mathbb{R})$ supertwistors and the dual supertwistor. We identify this infinite-dimensional superalgebra as a twistor string algebra (TSA). Its finite-dimensional subalgebra is spanned by $gl(4|4, \mathbb{R})$ generators and dual supertwistor corresponding to $l = 1$ and $l = 0$ monomials respectively. The relations of the global symmetry algebra of the Berkovits model are obtained from those of TSA by setting to zero $gl(1, \mathbb{R})$ current. In the quantum theory we show that classical infinite-dimensional symmetry breaks down to $SL(4|4, \mathbb{R})$ one, whose consistency was proved in [8] using the world-sheet CFT techniques.

Infinite-dimensional nature of the symmetries of massless superparticles was revealed already in [11]. So in Section 2 we consider the (higher-spin) symmetries of $N = 4$ supersymmetric models of massless particles in the supertwistor formulation [12,13]. We included in this section also some of the known material, in particular on the finite-dimensional symmetries and spectrum identification, to make it self-contained and to prepare the ground for subsequent discussion of the twistor string symmetries in Section 3.

2. Higher-spin symmetries of $D = 4$ $N = 4$ massless superparticles

Kinetic term of the Shirafuji superparticle model [12] specialized to the case of $N = 4$ supersymmetry [13] has the form

$$S = \int d\tau \mathcal{L}, \quad \mathcal{L} = \frac{i}{2} (\bar{\mathcal{Z}}_A \dot{\mathcal{Z}}^A - \dot{\bar{\mathcal{Z}}}_A \mathcal{Z}^A). \tag{2.1}$$

$PSU(2, 2|4)$ supertwistor \mathcal{Z}^A has 4 bosonic components Z^α transforming in the fundamental representation of $SU(2, 2) \sim SO(2, 4)$ and 4 fermionic components ξ^i – in the fundamental representation of $SU(4)$ [14]. Components of the dual supertwistor

$$\bar{\mathcal{Z}}_A = (\bar{\mathcal{Z}}_\alpha, \bar{\xi}_i) = (\mathcal{Z})^\dagger \mathcal{H}, \quad \mathcal{H} = \begin{pmatrix} 0 & I_{2 \times 2} & \\ I_{2 \times 2} & 0 & \\ & & I_{4 \times 4} \end{pmatrix} \tag{2.2}$$

transform in the antifundamental representation of $SU(2, 2) \times SU(4)$.

In the canonical formulation non-trivial D.B. of the supertwistor components are

$$\{\mathcal{Z}^A, \bar{\mathcal{Z}}_B\}_{D.B.} = i\delta_B^A, \quad \{\bar{\mathcal{Z}}_B, \mathcal{Z}^A\}_{D.B.} = -i(-)^a \delta_B^A, \tag{2.3}$$

where a is the Grassmann parity equal 0 for the supertwistor bosonic components and 1 for the fermionic components.

2.1. Classical symmetries of $D = 4$ $N = 4$ massless superparticle

2.1.1. $U(2, 2|4)$ global symmetry

Action (2.1) is manifestly invariant under $U(2, 2|4)$ global symmetry generated by $G_{(1,1)} = \bar{Z}_B \Lambda^B{}_A Z^A$:

$$\delta Z^A = \{Z^A, G_{(1,1)}\}_{D.B.} = i \Lambda^A{}_B Z^B, \quad \delta \bar{Z}_A = \{\bar{Z}_A, G_{(1,1)}\}_{D.B.} = -i \bar{Z}_B \Lambda^B{}_A. \quad (2.4)$$

Supertwistor Z^A and its dual \bar{Z}_A are thus transform linearly under $U(2, 2|4)$. Associated Noether current up to a numerical factor coincides with the generator of $U(2, 2|4)$ transformations and is given by valence (1, 1) composite supertwistor

$$T_A{}^B = \bar{Z}_A Z^B \quad (2.5)$$

that on D.B. satisfies the relations of $u(2, 2|4)$ superalgebra

$$\{T_A{}^B, T_C{}^D\}_{D.B.} = i(\delta_C^B T_A{}^D - (-)^{\varepsilon_a^b \varepsilon_c^d} \delta_A^D T_C{}^B), \quad \varepsilon_a^b = (-)^{a+b}. \quad (2.6)$$

Irreducible components of (2.5)

$$T_A{}^B = \{\tilde{T}_\alpha{}^\beta, \tilde{T}_i{}^j; Q_\alpha{}^j, Q_i{}^\beta; T, U\} \quad (2.7)$$

include the generators of $SU(2, 2) \times SU(4)$ transformations, $D = 4$ $N = 4$ Poincaré and conformal supersymmetries, $U(1)$ phase rotation and ‘twisted’ $U_t(1)$ rotation respectively

$$\begin{aligned} \tilde{T}_\alpha{}^\beta &= \bar{Z}_\alpha Z^\beta - \frac{1}{4} \delta_\alpha^\beta (\bar{Z} Z), & \tilde{T}_i{}^j &= \bar{\xi}_i \xi^j - \frac{1}{4} \delta_i^j (\bar{\xi} \xi), \\ Q_\alpha{}^j &= \bar{Z}_\alpha \xi^j, & Q_i{}^\beta &= \bar{\xi}_i Z^\beta, \\ T &= \bar{Z} Z + \bar{\xi} \xi, & U &= \bar{Z} Z - \bar{\xi} \xi. \end{aligned} \quad (2.8)$$

Throughout this paper tilde over a tensor indicates that it is traceless under contraction of its upper and lower indices of the same sort. Component form of $u(2, 2|4)$ superalgebra relations (2.6) reads

$$\begin{aligned} \{\tilde{T}_\alpha{}^\beta, \tilde{T}_\gamma{}^\delta\}_{D.B.} &= i(\delta_\gamma^\beta \tilde{T}_\alpha{}^\delta - \delta_\alpha^\delta \tilde{T}_\gamma{}^\beta), \\ \{\tilde{T}_i{}^j, \tilde{T}_k{}^l\}_{D.B.} &= i(\delta_k^j \tilde{T}_i{}^l - \delta_i^l \tilde{T}_k{}^j), \\ \{Q_\alpha{}^j, Q_k{}^\delta\}_{D.B.} &= i\left(\delta_k^j \tilde{T}_\alpha{}^\delta + \delta_\alpha^\delta \tilde{T}_k{}^j + \frac{1}{4} \delta_\alpha^\delta \delta_k^j T\right), \\ \{\tilde{T}_\alpha{}^\beta, Q_\gamma{}^l\}_{D.B.} &= i\left(\delta_\gamma^\beta Q_\alpha{}^l - \frac{1}{4} \delta_\alpha^\beta Q_\gamma{}^l\right), \\ \{\tilde{T}_\alpha{}^\beta, Q_k{}^\delta\}_{D.B.} &= -i\left(\delta_\alpha^\delta Q_k{}^\beta - \frac{1}{4} \delta_\alpha^\beta Q_k{}^\delta\right), \\ \{\tilde{T}_i{}^j, Q_\gamma{}^l\}_{D.B.} &= -i\left(\delta_i^l Q_\gamma{}^j - \frac{1}{4} \delta_i^j Q_\gamma{}^l\right), \\ \{\tilde{T}_i{}^j, Q_k{}^\delta\}_{D.B.} &= i\left(\delta_k^j Q_i{}^\delta - \frac{1}{4} \delta_i^j Q_k{}^\delta\right), \\ \{U, Q_\alpha{}^j\}_{D.B.} &= 2i Q_\alpha{}^j, & \{U, Q_i{}^\beta\}_{D.B.} &= -2i Q_i{}^\beta. \end{aligned} \quad (2.9)$$

Few remarks regarding above relations are in order. $su(2, 2) \oplus su(4)$ and supersymmetry generators span $psu(2, 2|4)$ – the minimal superalgebra that includes conformal and R -symmetries. To obtain in closed form corresponding (anti)commutation relations it is common to set $T = 0$. The generators of $psu(2, 2|4)$ and T span $su(2, 2|4)$ superalgebra. Unlike the case $N \neq 4$ this superalgebra is not simple since T forms an Abelian ideal. Because U does not appear on the r.h.s. of (2.9) $u(2, 2|4)$ superalgebra has the structure of semidirect sum of $su(2, 2|4)$ and $u_t(1)$.

Component form of $U(2, 2|4)$ transformations (2.4) is obtained by calculating D.B. of the supertwistor components with the individual generators in (2.8). In such a way we find transformation rules of the supertwistor components under $SU(2, 2) \times SU(4)$ rotations

$$\begin{aligned} \delta Z^\alpha &= i \Lambda^\alpha{}_\beta Z^\beta, & \delta \bar{Z}_\alpha &= -i \bar{Z}_\beta \Lambda^\beta{}_\alpha, & \Lambda^\alpha{}_\alpha &= 0; \\ \delta \xi^i &= i \Lambda^i{}_j \xi^j, & \delta \bar{\xi}_i &= -i \bar{\xi}_j \Lambda^j{}_i, & \Lambda^i{}_i &= 0, \end{aligned} \quad (2.10)$$

supersymmetry transformations

$$\begin{aligned} \delta Z^\alpha &= i \varepsilon^\alpha{}_i \xi^i, & \delta \bar{\xi}_i &= -i \bar{Z}_\alpha \varepsilon^\alpha{}_i; \\ \delta \bar{Z}_\alpha &= -i \bar{\xi}_i \varepsilon^i{}_\alpha, & \delta \xi^i &= i \varepsilon^i{}_\alpha Z^\alpha, & (\varepsilon^\alpha{}_i)^\dagger &= \bar{\varepsilon}^i{}_\alpha, \end{aligned} \quad (2.11)$$

as well as, $U(1)$ and $U_t(1)$ rotations

$$\delta Z^\alpha = i a Z^\alpha, \quad \delta \bar{Z}_\alpha = -i a \bar{Z}_\alpha, \quad \delta \xi^i = i a \xi^i, \quad \delta \bar{\xi}_i = -i a \bar{\xi}_i; \quad (2.12)$$

$$\delta Z^\alpha = i a_t Z^\alpha, \quad \delta \bar{Z}_\alpha = -i a_t \bar{Z}_\alpha, \quad \delta \xi^i = -i a_t \xi^i, \quad \delta \bar{\xi}_i = i a_t \bar{\xi}_i. \quad (2.13)$$

2.1.2. $OSp(8|8)$ global symmetry

Action (2.1) is also invariant under the symmetries generated by monomials composed of either supertwistors or dual supertwistors only. Linear functions $G_{(1,0)} = \bar{Z}_A \Lambda^A$ and $G_{(0,1)} = \bar{\Lambda}_A Z^A$ generate constant shifts of supertwistors

$$\delta Z^A = \{Z^A, G_{(1,0)}\}_{D.B.} = i \Lambda^A; \quad \delta \bar{Z}_A = \{\bar{Z}_A, G_{(0,1)}\}_{D.B.} = -i \bar{\Lambda}_A. \quad (2.14)$$

Consider also generating functions defined by valence (2, 0) and (0, 2) supertwistors

$$\begin{aligned} G_{(2,0)} &= \bar{Z}_{A_1} \bar{Z}_{A_2} \Lambda^{A_2 A_1} = \bar{Z}_{A(2)} \Lambda^{A(2)}, \\ G_{(0,2)} &= \bar{\Lambda}_{A_2 A_1} Z^{A_1} Z^{A_2} = \bar{\Lambda}_{A(2)} Z^{A(2)}, \end{aligned} \quad (2.15)$$

where convenient notation to be widely used below is $Z^{A(l)} = Z^{A_1} \dots Z^{A_l}$ ($Z^{A(0)} = 1$, $Z^{A(1)} = Z^A$) and $\bar{Z}_{A(l)} = \bar{Z}_{A_1} \dots \bar{Z}_{A_l}$ ($\bar{Z}_{A(0)} = 1$, $\bar{Z}_{A(1)} = \bar{Z}_A$).¹ Associated variations of supertwistors read

$$\begin{aligned} \delta Z^A &= \{Z^A, G_{(2,0)}\}_{D.B.} = 2i \bar{Z}_B \Lambda^{BA}; \\ \delta \bar{Z}_A &= \{\bar{Z}_A, G_{(0,2)}\}_{D.B.} = -2i \bar{\Lambda}_{AB} Z^B. \end{aligned} \quad (2.16)$$

Corresponding Noether currents can be identified with valence (2, 0) and (0, 2) supertwistors

$$T_{A(2)} = \bar{Z}_{A_1} \bar{Z}_{A_2}, \quad T^{A(2)} = Z^{A_1} Z^{A_2}. \quad (2.17)$$

¹ Composite objects like $Z^{A(l)}$ and $\bar{Z}_{A(l)}$ are graded symmetric in their indices. In general it is assumed graded symmetry in supertwistor indices denoted by the same letters. Similarly one defines the products of supertwistor bosonic and fermionic components as $Z^{\alpha(m)} = Z^{\alpha_1} \dots Z^{\alpha_m}$, $\bar{Z}_{\alpha(m)} = \bar{Z}_{\alpha_1} \dots \bar{Z}_{\alpha_m}$ and $\xi^{i[n]} = \xi^{i_1} \dots \xi^{i_n}$, $\bar{\xi}_{i[n]} = \bar{\xi}_{i_1} \dots \bar{\xi}_{i_n}$ ($n \leq N = 4$) that are (anti)symmetric. Antisymmetry in a set of n indices is indicated by placing n in square brackets. Both symmetrization and antisymmetrization are performed with unit weight.

Together with $u(2, 2|4)$ currents (2.5) they generate $OSp(8|8)$ global symmetry of the Shirafuji model (2.1). (Anti)commutation relations of $osp(8|8)$ superalgebra are given by (2.6) and

$$\begin{aligned} \{T_{A(2)}, T^{B(2)}\}_{D.B.} &= -i \left((-)^{a_2} \delta_{A_2}^{B_1} T_{A_1}{}^{B_2} + (-)^{a_2(b_1+b_2)} \delta_{A_2}^{B_2} T_{A_1}{}^{B_1} \right. \\ &\quad \left. + (-)^{b_1(a_1+a_2)} \delta_{A_1}^{B_1} T_{A_2}{}^{B_2} + (-)^{a_2 b_2 + a_1(b_1+b_2)} \delta_{A_1}^{B_2} T_{A_2}{}^{B_1} \right), \\ \{T_{A(2)}, T_B{}^C\}_{D.B.} &= -i \left((-)^{a_2 \varepsilon_b^c} \delta_{A_2}^C T_{A_1 B} + (-)^{a_1 \varepsilon_b^c + a_2 c} \delta_{A_1}^C T_{A_2 B} \right), \\ \{T^{A(2)}, T_B{}^C\}_{D.B.} &= i \left(\delta_B^{A_2} T^{A_1 C} + (-)^{a_2 b} \delta_B^{A_1} T^{A_2 C} \right). \end{aligned} \tag{2.18}$$

Decomposition of the generators (2.17)

$$\begin{aligned} T_{AB} = \{T_{\alpha\beta}, T_{ij}; Q_{\alpha j}\} : \quad T_{\alpha\beta} &= \bar{Z}_\alpha \bar{Z}_\beta, \quad T_{ij} = \bar{\xi}_i \bar{\xi}_j, \quad Q_{\alpha j} = \bar{Z}_\alpha \bar{\xi}_j; \\ T^{AB} = \{T^{\alpha\beta}, T^{ij}; Q^{\alpha j}\} : \quad T^{\alpha\beta} &= Z^\alpha Z^\beta, \quad T^{ij} = \xi^i \xi^j, \quad Q^{\alpha j} = Z^\alpha \xi^j \end{aligned} \tag{2.19}$$

allows to find component form of the transformations (2.16)

$$\begin{aligned} \delta Z^\alpha &= 2i \Lambda^{\alpha\beta} \bar{Z}_\beta; \\ \delta Z^\alpha &= -2i \Lambda^{\alpha i} \bar{\xi}_i, \quad \delta \xi^i = 2i \Lambda^{\alpha i} \bar{Z}_\alpha; \\ \delta \xi^i &= -2i \Lambda^{ij} \bar{\xi}_j \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} \delta \bar{Z}_\alpha &= -2i \bar{\Lambda}_{\alpha\beta} Z^\beta; \\ \delta \bar{Z}_\alpha &= -2i \bar{\Lambda}_{\alpha i} \xi^i, \quad \delta \bar{\xi}_i = -2i \bar{\Lambda}_{\alpha i} Z^\alpha; \\ \delta \bar{\xi}_i &= -2i \bar{\Lambda}_{ij} \xi^j. \end{aligned} \tag{2.21}$$

Component form of the relations (2.18) that involve $osp(8|8) \setminus u(2, 2|4)$ currents reads

$$\begin{aligned} \{T_{\alpha\beta}, T^{\gamma\delta}\}_{D.B.} &= -i \left(\delta_\beta^\gamma \tilde{T}_\alpha{}^\delta + \delta_\beta^\delta \tilde{T}_\alpha{}^\gamma + \delta_\alpha^\gamma \tilde{T}_\beta{}^\delta + \delta_\alpha^\delta \tilde{T}_\beta{}^\gamma \right) - \frac{i}{4} \left(\delta_\alpha^\gamma \delta_\beta^\delta + \delta_\alpha^\delta \delta_\beta^\gamma \right) (T + U), \\ \{T_{\alpha\beta}, Q^{\gamma l}\}_{D.B.} &= -i \left(\delta_\beta^\gamma Q_\alpha{}^l + \delta_\alpha^\gamma Q_\beta{}^l \right), \\ \{T_{ij}, T^{kl}\}_{D.B.} &= i \left(\delta_j^k \tilde{T}_i{}^l - \delta_j^l \tilde{T}_i{}^k - \delta_i^k \tilde{T}_j{}^l + \delta_i^l \tilde{T}_j{}^k \right) + \frac{i}{4} \left(\delta_i^k \delta_j^l - \delta_i^l \delta_j^k \right) (T - U), \\ \{T_{ij}, Q^{\gamma l}\}_{D.B.} &= i \left(\delta_j^l Q_i{}^\gamma - \delta_i^l Q_j{}^\gamma \right), \\ \{Q_{\alpha j}, T^{\gamma\delta}\}_{D.B.} &= -i \left(\delta_\alpha^\gamma Q_j{}^\delta + \delta_\alpha^\delta Q_j{}^\gamma \right), \quad \{Q_{\alpha j}, T^{kl}\}_{D.B.} = i \left(\delta_j^k Q_\alpha{}^l - \delta_j^l Q_\alpha{}^k \right), \\ \{Q_{\alpha j}, Q^{\gamma l}\}_{D.B.} &= i \left(\delta_j^l \tilde{T}_\alpha{}^\gamma - \delta_\alpha^\gamma \tilde{T}_j{}^l + \frac{1}{4} \delta_j^l \delta_\alpha^\gamma U \right). \end{aligned} \tag{2.22}$$

Accordingly D.B. relations involving both $u(2, 2|4)$ and $osp(8|8) \setminus u(2, 2|4)$ generators acquire the form

$$\begin{aligned} \{T_{\alpha\beta}, \tilde{T}_\gamma{}^\delta\}_{D.B.} &= -i \left(\delta_\beta^\delta T_{\alpha\gamma} + \delta_\alpha^\delta T_{\beta\gamma} - \frac{1}{2} \delta_\gamma^\delta T_{\alpha\beta} \right), \\ \{T_{\alpha\beta}, T\}_{D.B.} &= \{T_{\alpha\beta}, U\}_{D.B.} = -2i T_{\alpha\beta}, \\ \{T_{\alpha\beta}, Q_k{}^\delta\}_{D.B.} &= -i \left(\delta_\beta^\delta Q_{\alpha k} + \delta_\alpha^\delta Q_{\beta k} \right), \\ \{T_{ij}, \tilde{T}_k{}^l\}_{D.B.} &= -i \left(\delta_j^l T_{ik} + \delta_i^l T_{kj} - \frac{1}{2} \delta_k^l T_{ij} \right), \end{aligned}$$

$$\begin{aligned}
\{\mathbf{T}_{ij}, -\mathbf{T}\}_{D.B.} &= \{\mathbf{T}_{ij}, \mathbf{U}\}_{D.B.} = 2i\mathbf{T}_{ij}, \\
\{\mathbf{T}_{ij}, \mathbf{Q}_\gamma^l\}_{D.B.} &= i(\delta_j^l \mathbf{Q}_{\gamma i} - \delta_i^l \mathbf{Q}_{\gamma j}), \\
\{\mathbf{Q}_{\alpha j}, \tilde{\mathbf{T}}_\gamma^\delta\}_{D.B.} &= -i\left(\delta_\alpha^\delta \mathbf{Q}_{\gamma j} - \frac{1}{4}\delta_\gamma^\delta \mathbf{Q}_{\alpha j}\right), \quad \{\mathbf{Q}_{\alpha j}, \mathbf{T}\}_{D.B.} = -2i\mathbf{Q}_{\alpha j}, \\
\{\mathbf{Q}_{\alpha j}, \mathbf{Q}_\gamma^l\}_{D.B.} &= i\delta_j^l \mathbf{T}_{\alpha\gamma}, \quad \{\mathbf{Q}_{\alpha j}, \mathbf{Q}_k^\delta\}_{D.B.} = -i\delta_\alpha^\delta \mathbf{T}_{jk}, \\
\{\mathbf{Q}_{\alpha j}, \tilde{\mathbf{T}}_k^l\}_{D.B.} &= -i\left(\delta_j^l \mathbf{Q}_{\alpha k} - \frac{1}{4}\delta_k^l \mathbf{Q}_{\alpha j}\right),
\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
\{\mathbf{T}^{\alpha\beta}, \tilde{\mathbf{T}}_\gamma^\delta\}_{D.B.} &= i\left(\delta_\gamma^\beta \mathbf{T}^{\alpha\delta} + \delta_\gamma^\alpha \mathbf{T}^{\beta\delta} - \frac{1}{2}\delta_\gamma^\delta \mathbf{T}^{\alpha\beta}\right), \\
\{\mathbf{T}^{\alpha\beta}, \mathbf{T}\}_{D.B.} &= \{\mathbf{T}^{\alpha\beta}, \mathbf{U}\}_{D.B.} = 2i\mathbf{T}^{\alpha\beta}, \\
\{\mathbf{T}^{\alpha\beta}, \mathbf{Q}_\gamma^l\}_{D.B.} &= i(\delta_\gamma^\beta \mathbf{Q}^{\alpha l} + \delta_\gamma^\alpha \mathbf{Q}^{\beta l}), \\
\{\mathbf{T}^{ij}, \tilde{\mathbf{T}}_k^l\}_{D.B.} &= i\left(\delta_k^j \mathbf{T}^{il} + \delta_k^i \mathbf{T}^{lj} - \frac{1}{2}\delta_k^l \mathbf{T}^{ij}\right), \\
\{\mathbf{T}^{ij}, \mathbf{T}\}_{D.B.} &= \{\mathbf{T}^{ij}, -\mathbf{U}\}_{D.B.} = 2i\mathbf{T}^{ij}, \\
\{\mathbf{T}^{ij}, \mathbf{Q}_k^\delta\}_{D.B.} &= i(\delta_k^j \mathbf{Q}^{\delta i} - \delta_k^i \mathbf{Q}^{\delta j}), \\
\{\mathbf{Q}^{\alpha j}, \tilde{\mathbf{T}}_\gamma^\delta\}_{D.B.} &= i\left(\delta_\gamma^\alpha \mathbf{Q}^{\delta j} - \frac{1}{4}\delta_\gamma^\delta \mathbf{Q}^{\alpha j}\right), \quad \{\mathbf{Q}^{\alpha j}, \mathbf{T}\}_{D.B.} = 2i\mathbf{Q}^{\alpha j}, \\
\{\mathbf{Q}^{\alpha j}, \mathbf{Q}_\gamma^l\}_{D.B.} &= i\delta_\gamma^\alpha \mathbf{T}^{jl}, \quad \{\mathbf{Q}^{\alpha j}, \mathbf{Q}_k^\delta\}_{D.B.} = i\delta_k^j \mathbf{T}^{\alpha\delta}, \\
\{\mathbf{Q}^{\alpha j}, \tilde{\mathbf{T}}_k^l\}_{D.B.} &= i\left(\delta_k^j \mathbf{Q}^{\alpha l} - \frac{1}{4}\delta_k^l \mathbf{Q}^{\alpha j}\right).
\end{aligned} \tag{2.24}$$

One observes that the $u_t(1)$ generator U appears on the r.h.s. of (2.22) in addition to all the $su(2, 2|4)$ generators. As a digression let us note that $OSp(2N|8) \supset U(2, 2|N)$ symmetry is manifest in the superparticle models [15,16] formulated in generalized superspace with the bosonic coordinates described by symmetric 4×4 matrix that in addition to Minkowski 4-coordinates includes 6 coordinates described by the second rank antisymmetric tensor. Lagrangians of these models can be naturally written in terms of orthosymplectic supertwistors transforming linearly under $OSp(2N|8)$.²

2.1.3. Higher-spin symmetries

The complete global symmetry of the model (2.1) is infinite-dimensional. Generic form of the generating function for both finite-dimensional and higher-spin symmetries is

$$\mathbf{G}_{(k,l)} = \bar{Z}_{B_1} \dots \bar{Z}_{B_k} \Lambda^{B_k \dots B_1}{}_{A_l \dots A_1} Z^{A_1} \dots Z^{A_l}, \quad k, l \geq 0. \tag{2.25}$$

Parameters $\Lambda^{B(k)}{}_{A(l)}$ (anti)commute with themselves and with the supertwistor components depending on their parities defined by the sum $\varepsilon(\Lambda) = \sum_k b_k + \sum_l a_l$ of parities b_k and a_l that take

² String models formulated in terms of $OSp(N|8)$ supertwistors were considered in [17–19]. In Ref. [20] was discussed their relation to the Berkovits twistor string model [2].

values 0(1) for the indices corresponding to bosonic (fermionic) components of supertwistors. Using associated variation of supertwistors

$$\begin{aligned} \delta \mathcal{Z}^A &= \{ \mathcal{Z}^A, \mathbf{G}_{(k,l)} \}_{D.B.} = ik \bar{\mathcal{Z}}_{B_2} \dots \bar{\mathcal{Z}}_{B_k} \Lambda^{B_k \dots B_2 A}{}_{C(l)} \mathcal{Z}^{C(l)}, \\ \delta \bar{\mathcal{Z}}_A &= \{ \bar{\mathcal{Z}}_A, \mathbf{G}_{(k,l)} \}_{D.B.} = -il \bar{\mathcal{Z}}_{C(k)} \Lambda^{C(k)}{}_{AB_{l-1} \dots B_1} \mathcal{Z}^{B_1} \dots \mathcal{Z}^{B_{l-1}} \end{aligned} \tag{2.26}$$

one derives the variation of the superparticle’s Lagrangian

$$\delta \mathcal{L} = (k + l - 2) \frac{i}{2} \frac{d}{d\tau} (\bar{\mathcal{Z}}_{A(k)} \Lambda^{A(k)}{}_{B(l)} \mathcal{Z}^{B(l)}). \tag{2.27}$$

From this expression it becomes clear that $OSp(8|8)$ symmetry for which $(k, l) = (1, 1), (2, 0)$ or $(0, 2)$ is special since the action is invariant under corresponding variation. For other values of (k, l) the invariance is only up to a total divergence. So the complete infinite-dimensional symmetry of the superparticle model (2.1) is generated by the sum

$$\mathbf{G} = \sum_{k,l \geq 0} \mathbf{G}_{(k,l)} = \sum_{k,l \geq 0} \bar{\mathcal{Z}}_{B(k)} \Lambda^{B(k)}{}_{A(l)} \mathcal{Z}^{A(l)}. \tag{2.28}$$

Associated Noether currents are given by a collection of all possible monomials of the form

$$\mathbf{T}^{(k,l)}{}_{A(k)}{}^{B(l)} = \bar{\mathcal{Z}}_{A(k)} \mathcal{Z}^{B(l)}, \quad k, l \geq 0. \tag{2.29}$$

Such monomials span an infinite-dimensional superalgebra, whose (anti)commutation relations in schematic form read

$$\begin{aligned} \{ \mathbf{T}^{(k,l)}{}_{B(k)}{}^{A(l)}, \mathbf{T}^{(p,q)}{}_{D(p)}{}^{C(q)} \}_{D.B.} &= i (\delta_D^A \mathbf{T}^{(k+p-1,l+q-1)}{}_{B(k)D(p-1)}{}^{A(l-1)C(q)} \\ &\quad - \delta_B^C \mathbf{T}^{(k+p-1,l+q-1)}{}_{B(k-1)D(p)}{}^{A(l)C(q-1)}). \end{aligned} \tag{2.30}$$

2.2. Quantum symmetries of $D = 4$ $N = 4$ massless superparticle

At the quantum level D.B. relations (2.3) are replaced by (anti)commutators³

$$[\hat{\mathcal{Z}}^A, \hat{\mathcal{Z}}_B] = \delta_B^A, \quad [\hat{\mathcal{Z}}_B, \hat{\mathcal{Z}}^A] = -(-)^a \delta_B^A \tag{2.31}$$

and components of supertwistors and their duals become Hermitian conjugate operators $(\hat{\mathcal{Z}}^A)^\dagger = \hat{\mathcal{Z}}_A$.⁴ Thus global symmetry generators are promoted to Hermitian operators. Quantized $u(2, 2|4)$ generators $\hat{\mathbf{T}}_A{}^B$ are defined by the graded symmetrized (Weyl ordered) expression

$$\hat{\mathbf{T}}_A{}^B = \frac{1}{2} (\hat{\mathcal{Z}}_A \hat{\mathcal{Z}}^B + (-)^{ab} \hat{\mathcal{Z}}^B \hat{\mathcal{Z}}_A). \tag{2.32}$$

As far as component generators (2.7), (2.8) are concerned there are no ambiguities in the definition of $su(2, 2)$ and $su(4)$ generators, because of their tracelessness, and supersymmetry generators, while $u(1)$ and $u_t(1)$ generators can be presented in various forms

$$\hat{\mathbf{T}} = \frac{1}{2} (\hat{\mathcal{Z}} \hat{\mathcal{Z}} + \hat{\mathcal{Z}} \hat{\mathcal{Z}}) + \frac{1}{2} (\hat{\xi} \hat{\xi} - \hat{\xi} \hat{\xi}) = \hat{\mathcal{Z}} \hat{\mathcal{Z}} + \hat{\xi} \hat{\xi} = \hat{\mathcal{Z}} \hat{\mathcal{Z}} - \hat{\xi} \hat{\xi}, \tag{2.33}$$

$$\hat{\mathbf{U}} = \frac{1}{2} (\hat{\mathcal{Z}} \hat{\mathcal{Z}} + \hat{\mathcal{Z}} \hat{\mathcal{Z}}) - \frac{1}{2} (\hat{\xi} \hat{\xi} - \hat{\xi} \hat{\xi}) = \hat{\mathcal{Z}} \hat{\mathcal{Z}} - \hat{\xi} \hat{\xi} + 4 = \hat{\mathcal{Z}} \hat{\mathcal{Z}} + \hat{\xi} \hat{\xi} - 4. \tag{2.34}$$

³ Planck constant is omitted on the r.h.s.

⁴ Hermitian conjugation is assumed to reverse the order for both bosonic and fermionic operators.

There are no numerical constants in ‘asymmetric’ representations of \widehat{T} since equal in number bosonic and fermionic components of supertwistor and its dual give contributions that cancel each other. Also no ambiguity arises in the definition of quantized $osp(8|8) \setminus u(2, 2|4)$ generators (2.17). In general higher-spin generators $\widehat{T}_{A(k)}^{(k,l) B(l)}$ are defined by the sum of all graded permutations of constituent supertwistors

$$\widehat{T}_{B(k)}^{(k,l) A(l)} = \frac{1}{(k+l)!} (\widehat{Z}_{B(k)} \widehat{Z}^{A(l)} + \dots). \tag{2.35}$$

They satisfy (anti)commutation relations [21] that in schematic form read

$$\begin{aligned} & \left\{ \widehat{T}_{B(k)}^{(k,l) A(l)}, \widehat{T}_{D(p)}^{(p,q) C(q)} \right\} \\ & \sim \sum_{\substack{m \leq \min(l,p) \\ n \leq \min(k,q) \\ m,n \geq 0 \\ m+n \text{ odd}}} \delta_{D(m)}^{A(m)} \delta_{B(n)}^{C(n)} \widehat{T}_{B(k-n)D(p-m)}^{(k+p-m-n, l+q-m-n) A(l-m)C(q-n)}. \end{aligned} \tag{2.36}$$

2.3. Higher-spin symmetries of massless superparticle and higher-spin superalgebras

It is worthwhile to compare considered classical and quantum relations of the higher-spin currents with those of the higher-spin superalgebras based on orthosymplectic symmetries [22]. Generating function (2.28) of the infinite-dimensional global symmetry of the superparticle action (2.1) can be considered as a symbol of the operator \widehat{G} that is defined by the same expression (2.28) in which (Weyl ordered products of) quantized supertwistors (2.31) should be substituted. Associative algebra $aq(8|8)$ (aq = ‘associative quantum’) of such operators is isomorphic to the $*$ -product algebra of their symbols. The $*$ -product can be brought to the following form in terms of $PSU(2, 2|4)$ supertwistors [21]

$$A(\mathcal{Z}, \bar{\mathcal{Z}}) * B(\mathcal{Z}, \bar{\mathcal{Z}}) = Ae^\Delta B, \quad \Delta = \left(\frac{\overleftarrow{\partial}}{\partial \bar{\mathcal{Z}}} \frac{\overrightarrow{\partial}}{\partial \bar{\mathcal{Z}}} - \frac{\overrightarrow{\partial}}{\partial \mathcal{Z}} \frac{\overleftarrow{\partial}}{\partial \mathcal{Z}} \right), \tag{2.37}$$

where $A(\mathcal{Z}, \bar{\mathcal{Z}})$ and $B(\mathcal{Z}, \bar{\mathcal{Z}})$ are symbols of the operators $\widehat{A}(\widehat{\mathcal{Z}}, \widehat{\bar{\mathcal{Z}}})$ and $\widehat{B}(\widehat{\mathcal{Z}}, \widehat{\bar{\mathcal{Z}}})$. Introduction of the Lie superalgebra structure in $aq(8|8)$ requires assignment of parities to the monomials $\widehat{T}_{B(k)}^{(k,l) A(l)}$ (and associated expansion coefficients) in G . The prescription [22] appropriate for the construction of higher-spin gauge theories consists in ascribing parity 1(0) to $SU(2, 2)$ ($SU(4)$) indices. Such a choice agrees with the spin-statistics relation for the expansion coefficients that are identified with the potentials (field strengths) of higher-spin gauge fields but, for instance, the generators of such a Lie superalgebra defined by the product of an odd number of supertwistor bosonic components should satisfy anticommutation relations, while those equal the product of supertwistor fermionic components – commutation relations. We adhere to alternative prescription motivated by the symmetries of the superparticle action that, however, results in wrong spin–statistics relation for some of the parameters in (2.28). Both prescriptions match for the superalgebras spanned by the generators composed of an even number of supertwistors, in particular for the finite-dimensional symmetries generated by quadratic monomials.

The r.h.s. of (anti)commutators $A * B - (-)^{\varepsilon(A)\varepsilon(B)} B * A$ of $aq(8|8)$ elements coincides with that of quantized generators (2.35) so that relations (2.36) can be identified as corresponding to infinite-dimensional Lie superalgebra $Lie[aq](8|8)$. Then D.B. relations (2.30) can be identified with the classical limit $\hbar \rightarrow 0$ of $Lie[aq](8|8)$ that can be named $Lie[ac](8|8)$. In general

the classical limit of Lie superalgebras based on $*$ -product associative algebras is obtained by introducing explicit dependence on \hbar

$$(A * B - (-)^{\varepsilon(A)\varepsilon(B)} B * A)(\mathcal{Z}, \bar{\mathcal{Z}}) = \frac{1}{\hbar} (Ae^{\Delta_h} B - (-)^{\varepsilon(A)\varepsilon(B)} B e^{\Delta_h} A),$$

$$\Delta_h = \hbar \Delta \tag{2.38}$$

and taking $\hbar \rightarrow 0$. \hbar plays the role of the contraction parameter as was explained in [23,22].

2.4. $D = 4$ $N = 4$ massless superparticle with gauged $U(1)$ symmetry

Modification of the model (2.1) considered by Shirafuji [12] consists in gauging $U(1)$ symmetry (2.12) by adding T with the Lagrange multiplier to the action

$$S_{U(1)} = \int d\tau \lambda T. \tag{2.39}$$

Gauging phase symmetry means that superparticle propagates on projective supertwistor space. Global symmetry of the action (2.1), (2.39) is described by a subalgebra of $Lie[ac\ell](8|8)$ spanned by the generators commuting with T . We identify this superalgebra as $iu_{cl}(2, 2|4)$. It is the classical limit of $iu(2, 2|4)$ ($iu =$ ‘infinite-dimensional unitary’) superalgebra introduced in [21]. Both $iu(2, 2|4)$ and $iu_{cl}(2, 2|4)$ share the same finite-dimensional subalgebra $u(2, 2|4)$. $iu_{cl}(2, 2|4)$ is the centralizer of T in $Lie[ac\ell](8|8)$ analogously to the case of corresponding Lie superalgebras based on associative $*$ -product algebras (see, e.g., relevant discussion in [24]). Generators of $iu_{cl}(2, 2|4)$ are the same as those of $iu(2, 2|4)$ and are given by the monomials in (2.29) with equal number of supertwistors and dual supertwistors

$$T^{(L,L)}_{A(L)}{}^{B(L)} \equiv T^{(L)}_{A(L)}{}^{B(L)} = \bar{\mathcal{Z}}_{A(L)} \mathcal{Z}^{B(L)}, \quad L \geq 1. \tag{2.40}$$

Number L we shall call the level of the generator following [25]. D.B. relations of the generators (2.40) can be derived from (2.30)

$$\{T^{(L_1)}_{B(L_1)}{}^{A(L_1)}, T^{(L_2)}_{D(L_2)}{}^{C(L_2)}\}_{D.B.} = i \left(\delta_D^A T^{(L_1+L_2-1)}_{B(L_1)D(L_2-1)}{}^{A(L_1-1)C(L_2)} - \delta_B^C T^{(L_1+L_2-1)}_{B(L_1-1)D(L_2)}{}^{A(L_1)C(L_2-1)} \right). \tag{2.41}$$

Going on the constraint shell $T \approx 0$ implies setting to zero those generators of $iu_{cl}(2, 2|4)$ that are multiples of T . This narrows down higher-spin symmetry to the subalgebra of $iu_{cl}(2, 2|4)$. For $N \neq 4$ such a symmetry is generated by the classical limit of $isl(2, 2|N)$ superalgebra [21]. The construction of this superalgebra is based on the direct sum representation of $u(2, 2|N)$ as $su(2, 2|N) \oplus T_N^5$ and its higher-spin generalization. The $N = 4$ case requires special treatment since $su(2, 2|4) = psu(2, 2|4) \oplus T_4$.

2.5. Spectrum identification

Important realization [26,12] of quantized (super)twistors related to the definition of twistor wave functions is to treat components of $\hat{\mathcal{Z}}^A$ as classical quantities, while components of the dual supertwistor $\hat{\mathcal{Z}}_A$ are considered as differential operators

⁵ In this and some of the subsequent formulas to avoid confusion T is endowed with the subscript explicitly indicating the number of odd components of associated supertwistors.

$$\hat{Z}_\alpha \rightarrow -\frac{\partial}{\partial Z^\alpha}, \quad \hat{\xi}_i \rightarrow \frac{\bar{\partial}}{\partial \xi^i} \tag{2.42}$$

acting in the space of (homogeneous) functions of (Z, ξ) . Alternatively, components of $\hat{\mathcal{Z}}_A$ may be treated as c -numbers, while components of $\hat{\mathcal{Z}}^A$ are replaced by differential operators.

Quantum generator (2.33) of $U(1)$ phase symmetry in the realization (2.42) can be brought to the form

$$\hat{T} = 2\hat{s} + 2 - \xi \frac{\partial}{\partial \xi}, \tag{2.43}$$

where $\hat{s} = -1 - \frac{1}{2}Z \frac{\partial}{\partial Z}$ is the helicity operator [27]. Any homogeneous function on the supertwistor space is its eigenfunction including the wave function $F(Z, \xi)$ of the superparticle propagating on the projective supertwistor space that satisfies

$$\left(\hat{s} + 1 - \frac{1}{2}\xi \frac{\partial}{\partial \xi} \right) F(Z, \xi) = 0. \tag{2.44}$$

The solution to this equation is

$$F(Z, \xi) = f(Z) + \xi^i \varphi_i(Z) + \frac{1}{2!} \xi^{i[2]} f_{i[2]}(Z) + \frac{1}{3!} \xi^{i[3]} \varphi_{i[3]}(Z) + \frac{1}{4!} \xi^{i[4]} f_{i[4]}(Z). \tag{2.45}$$

Even components in the expansion f , $f_{i[2]}$ and $f_{i[4]}$ upon the twistor transform [27] describe bosonic particles of helicities $-1, 0$ and $+1$ on (complexified conformally-compactified) Minkowski space–time, whereas odd components φ_i and $\varphi_{i[3]}$ – fermions of helicities $-1/2$ and $+1/2$, altogether forming $D = 4, N = 4$ super-Yang–Mills multiplet that is CPT self-conjugate, i.e. includes particles of opposite helicities.

The spectrum of the superparticle model (2.1) is described by an infinite series of even $F_{2k}(Z, \xi)$ and odd $\Phi_{2k+1}(Z, \xi)$ ($k \in \mathbb{Z}$) eigenfunctions of the operator (2.43)

$$\left(\hat{s} + 1 - \frac{1}{2}\xi \frac{\partial}{\partial \xi} + \frac{a}{2} \right) \mathcal{F}_a(Z, \xi) = 0: \tag{2.46}$$

$$\begin{cases} (\hat{s} + 1 - \frac{1}{2}\xi \frac{\partial}{\partial \xi} + k) F_{2k} = 0, & a = 2k, \\ (\hat{s} + 1 - \frac{1}{2}\xi \frac{\partial}{\partial \xi} + k + \frac{1}{2}) \Phi_{2k+1} = 0, & a = 2k + 1, \end{cases}$$

where the subscript indicates homogeneity degree. Pairs of functions \mathcal{F}_a and \mathcal{F}_{-a} ($a > 0$) describe (upon the twistor transform) CPT conjugate doubleton supermultiplets [28,29] with helicities ranging from $-a/2 - 1$ to $a/2 + 1$ that are even⁶ for even values of a and odd for odd values of a . This explains the notation introduced in Eq. (2.46). The low-spin supermultiplets in this series are $\Phi_{\pm 1}$ with particles of helicities $0, \pm 1/2, \pm 1, \pm 3/2$ and $F_{\pm 2}$ describing $D = 4, N = 4$ Einstein supergravity. The only exception is $F_0 \equiv F$ twistor function corresponding to the self-conjugate super-Yang–Mills multiplet considered in the previous paragraph.

Another way to derive the spectrum [13] of the superparticle model (2.1) is to transform [30,31] supertwistor components into two pairs of bosonic and two pairs of fermionic $SU(2)$ oscillators realizing $[SU(2)]^4$ – the maximal compact subgroup of $SU(2, 2|4)$. They provide

⁶ The supermultiplet is even/odd if the particle with the highest value of the modulus of helicity is boson/fermion.

minimal (single generation) oscillator realization of the $su(2, 2|4)$ superalgebra and are used to construct doubleton supermultiplets [32,28,29]. All these supermultiplets assemble into two singleton supermultiplets of $osp(8|8)$ that arise upon quantization of the massless superparticle on superspace with bosonic 4×4 matrix coordinates [16].

3. Higher-spin supersymmetries in twistor string models

For Lorentzian signature world sheet the simplest twistor string action can be presented as

$$S = \int d\tau d\sigma (\mathcal{L}_L + \mathcal{L}_R):$$

$$\mathcal{L}_L = -2(Y_\alpha \partial_- Z^\alpha + \eta_i \partial_- \xi^i) + \mathcal{L}_{L\text{-mat}}, \quad \mathcal{L}_R = -2(\bar{Y}_\alpha \partial_+ \bar{Z}^\alpha + \bar{\eta}_i \partial_+ \bar{\xi}^i) + \mathcal{L}_{R\text{-mat}},$$
(3.1)

where $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$, $\sigma^\pm = \tau \pm \sigma$, $Y_{+\alpha} \equiv Y_\alpha$, $\bar{Y}_{-\alpha} \equiv \bar{Y}_\alpha$, $\eta_{+i} \equiv \eta_i$, $\bar{\eta}_{-i} \equiv \bar{\eta}_i$ and $\mathcal{L}_{L(R)\text{-mat}}$ are Lagrangians for left- and right-moving non-twistor matter variables, whose contribution to the world-sheet conformal anomaly equals $c = \bar{c} = 26$ to cancel that of (b, c) -ghosts. Such variables may contain a current algebra for some Lie group (see, e.g., [3]). In Berkovits twistor-string model [2] global scale symmetry for both left- and right-movers

$$\delta Z^\alpha = \Lambda Z^\alpha, \quad \delta Y_\alpha = -\Lambda Y_\alpha, \quad \delta \xi^i = \Lambda \xi^i, \quad \delta \eta_i = -\Lambda \eta_i;$$

$$\delta \bar{Z}^\alpha = \bar{\Lambda} \bar{Z}^\alpha, \quad \delta \bar{Y}_\alpha = -\bar{\Lambda} \bar{Y}_\alpha, \quad \delta \bar{\xi}^i = \bar{\Lambda} \bar{\xi}^i, \quad \delta \bar{\eta}_i = -\bar{\Lambda} \bar{\eta}_i$$
(3.2)

is gauged by adding to the action (3.1) appropriate constraints $T = Y_\alpha Z^\alpha + \eta_i \xi^i \approx 0$ and $\bar{T} = \bar{Y}_\alpha \bar{Z}^\alpha + \bar{\eta}_i \bar{\xi}^i \approx 0$ with the Lagrange multipliers

$$S_{GL(1, \mathbb{R})} = \int d\tau d\sigma (\lambda T + \bar{\lambda} \bar{T}).$$
(3.3)

This necessitates add two units to the central charges of the matter variables to compensate that of (b, c) -ghosts and ghosts for the gauged $GL(1, \mathbb{R})$ symmetry.

Definition of the open string sector, that to date is the only one well-understood, is based on the conditions $Z^A = \bar{Z}^A$, $\mathcal{Y}_B = \bar{\mathcal{Y}}_B$ imposed on the world-sheet boundary on the supertwistors $Z^A = (Z^\alpha, \xi^i)$, $\bar{Z}^A = (\bar{Z}^\alpha, \bar{\xi}^i)$ and their duals $\mathcal{Y}_B = (Y_\beta, \eta_j)$, $\bar{\mathcal{Y}}_B = (\bar{Y}_\beta, \bar{\eta}_j)$. So taking into account reality condition of the Lagrangian one is led to consider left(right)-moving supertwistors Z^A (\bar{Z}^A) and dual supertwistors \mathcal{Y}_B ($\bar{\mathcal{Y}}_B$) as independent variables with real components. Such supertwistors are adapted for the description of fields on $D = 4$ $N = 4$ superspace for the space–time of signature $(+ - - -)$.⁷ Conformal group of Minkowski space–time of such a signature is $SO(3, 3) \sim SL(4, \mathbb{R})$ and its minimal $N = 4$ supersymmetric extension is $PSL(4|4, \mathbb{R})$ with the bosonic subgroup $SL(4, \mathbb{R}) \times SL(4, \mathbb{R})$ implying that bosonic and odd components of Z^A belong to the fundamental representation of $SL(4, \mathbb{R})_L \times SL(4, \mathbb{R})_L$, whereas bosonic and odd components of \mathcal{Y}_A belong to the antifundamental representation. Correspondingly components of \bar{Z}^A and $\bar{\mathcal{Y}}_A$ transform according to the (anti)fundamental representation of $SL(4, \mathbb{R})_R \times SL(4, \mathbb{R})_R$.⁸

⁷ Detailed discussion of the reality conditions of the twistor string Lagrangian for both Lorentzian and Euclidean world sheets, and different real structures in the complex supertwistor space associated with $D = 4$ space–times of various signatures can be found, e.g., in [4].

⁸ In this section the same letters are utilized to label supertwistors, their components and indices as in the previous one, although here they are strictly speaking different mathematical objects related to another real structure in the complex

Focusing on the sector of left-movers of the model (3.1) and applying the Dirac approach yields equal-time D.B. relations

$$\{Z^\alpha(\sigma), Y_\beta(\sigma')\}_{D.B.} = \delta_\beta^\alpha \delta(\sigma - \sigma'), \quad \{\xi^i(\sigma), \eta_j(\sigma')\}_{D.B.} = \delta_j^i \delta(\sigma - \sigma') \quad (3.4)$$

that in terms of the $PSL(4|4, \mathbb{R})$ supertwistors can be written as

$$\begin{aligned} \{Z^A(\sigma), \mathcal{Y}_B(\sigma')\}_{D.B.} &= \delta_B^A \delta(\sigma - \sigma'), \\ \{\mathcal{Y}_B(\sigma), Z^A(\sigma')\}_{D.B.} &= -(-)^a \delta_B^A \delta(\sigma - \sigma'). \end{aligned} \quad (3.5)$$

Similar relations hold for the right-movers.

3.1. Classical symmetries of twistor strings

Global symmetry of the left-moving part of the action (3.1) is generated on D.B. by the function

$$G = \int d\sigma \sum_{L \geq 0} G_{(L)}(\sigma), \quad G_{(L)}(\sigma) = \mathcal{Y}_B(\sigma) \Lambda^B{}_{A_L \dots A_1} Z^{A_1}(\sigma) \dots Z^{A_L}(\sigma). \quad (3.6)$$

For arbitrary value of the order L transformation rules for the supertwistors read

$$\begin{aligned} \delta Z^A(\sigma) &= \Lambda^A{}_{B(L)} Z^{B(L)}(\sigma), \\ \delta \mathcal{Y}_A(\sigma) &= -L \mathcal{Y}_C(\sigma) \Lambda^C{}_{AB_{L-1} \dots B_1} Z^{B_1}(\sigma) \dots Z^{B_{L-1}}(\sigma). \end{aligned} \quad (3.7)$$

Associated Noether current densities up to irrelevant numerical factor are given by the monomials

$$T^{(L)}{}_B{}^{A(L)}(\sigma) = \mathcal{Y}_B Z^{A(L)}, \quad L \geq 0 \quad (3.8)$$

that enter generating functions $G_{(L)}$. On D.B. they generate the TSA⁹

$$\begin{aligned} \{T^{(L)}{}_B{}^{A(L)}(\sigma), T^{(M)}{}_D{}^{C(M)}(\sigma')\}_{D.B.} &= (\delta_D^A T^{(L+M-1)}{}_B{}^{A(L-1)C(M)} \\ &\quad - \delta_B^C T^{(L+M-1)}{}_D{}^{A(L)C(M-1)})(\sigma) \delta(\sigma - \sigma'). \end{aligned} \quad (3.9)$$

The finite-dimensional subalgebra of TSA is spanned, apart from the order 0 generator $\mathcal{Y}_A(\sigma)$ that is responsible for constant shift of the supertwistor components, by quadratic monomial in supertwistors

$$T_A{}^B(\sigma) = \mathcal{Y}_A Z^B, \quad (3.10)$$

generating $gl(4|4, \mathbb{R})$ superalgebra

supertwistor space. For the dual supertwistor and its components we use independent notation since quantities with bars are reserved to label variables of the right-moving sector of the twistor string. We hope this will not cause a confusion since in each section only one kind of supertwistors is considered.

⁹ To be more precise one has to introduce TSA as an infinite-dimensional Lie superalgebra and then consider its loop version pertinent to twistor-string global symmetry. Let us also note that the subscript L in the notation of symmetry groups and algebras will be omitted as the discussion is concentrated on the sector of left-movers only. On the boundary left- and right-moving variables are identified and thus also no subscripts are needed.

$$\{T_A{}^B(\sigma), T_C{}^D(\sigma')\}_{D.B.} = (\delta_C{}^B T_A{}^D - (-)^{\varepsilon_a \varepsilon_c} \delta_A{}^D T_C{}^B)(\sigma) \delta(\sigma - \sigma'),$$

$$\varepsilon_a{}^b = (-)^{a+b}. \tag{3.11}$$

Irreducible components of $gl(4|4, \mathbb{R})$ current densities (3.10) are

$$T_A{}^B(\sigma) = \{\tilde{T}_\alpha{}^\beta, \tilde{T}_i{}^j; Q_\alpha{}^j, Q_i{}^\beta; T, U\};$$

$$\tilde{T}_\alpha{}^\beta = Y_\alpha Z^\beta - \frac{1}{4} \delta_\alpha^\beta (YZ), \quad \tilde{T}_i{}^j = \eta_i \xi^j - \frac{1}{4} \delta_i^j (\eta \xi);$$

$$Q_\alpha{}^j = Y_\alpha \xi^j, \quad Q_i{}^\beta = \eta_i Z^\beta; \quad T = Y_\alpha Z^\alpha + \eta_i \xi^i, \quad U = Y_\alpha Z^\alpha - \eta_i \xi^i. \tag{3.12}$$

The densities of $sl(4, \mathbb{R}) \times sl(4, \mathbb{R})$ currents $\tilde{T}_\alpha{}^\beta(\sigma)$, $\tilde{T}_i{}^j(\sigma)$ and those of the supersymmetry currents $Q_\alpha{}^j(\sigma)$, $Q_i{}^\beta(\sigma)$ span $psl(4|4, \mathbb{R})$ superalgebra, while these generators and $T(\sigma)$ span $sl(4|4, \mathbb{R})$. On D.B. they generate infinitesimal $SL(4, \mathbb{R}) \times SL(4, \mathbb{R})$ rotations of the supertwistor components

$$\delta Z^\alpha(\sigma) = \Lambda^\alpha{}_\beta Z^\beta(\sigma), \quad \delta Y_\alpha(\sigma) = -Y_\beta(\sigma) \Lambda^\beta{}_\alpha, \quad \Lambda^\alpha{}_\alpha = 0;$$

$$\delta \xi^i(\sigma) = \Lambda^i{}_j \xi^j(\sigma), \quad \delta \eta_i(\sigma) = -\eta_j(\sigma) \Lambda^j{}_i, \quad \Lambda^i{}_i = 0 \tag{3.13}$$

and supersymmetry transformations

$$\delta Z^\alpha(\sigma) = \varepsilon^\alpha{}_i \xi^i(\sigma), \quad \delta \eta_i(\sigma) = -Y_\alpha(\sigma) \varepsilon^\alpha{}_i;$$

$$\delta Y_\alpha(\sigma) = -\eta_i(\sigma) \varepsilon^i{}_\alpha, \quad \delta \xi^i(\sigma) = \varepsilon^i{}_\alpha Z^\alpha(\sigma), \tag{3.14}$$

where $\varepsilon^\alpha{}_i$ and $\varepsilon^i{}_\alpha$ are independent odd parameters with 16 real components each. $T(\sigma)$ generates $GL(1, \mathbb{R})$ transformations (3.2) and $U(\sigma)$ – ‘twisted’ $GL_t(1, \mathbb{R})$ transformations

$$\delta Z^\alpha(\sigma) = \Lambda_t Z^\alpha(\sigma), \quad \delta Y_\alpha(\sigma) = -\Lambda_t Y_\alpha(\sigma),$$

$$\delta \xi^i(\sigma) = -\Lambda_t \xi^i(\sigma), \quad \delta \eta_i(\sigma) = \Lambda_t \eta_i(\sigma). \tag{3.15}$$

$gl(4|4, \mathbb{R})$ relations (3.11) are spelt out in terms of irreducible components (3.12) as

$$\{\tilde{T}_\alpha{}^\beta(\sigma), \tilde{T}_\gamma{}^\delta(\sigma')\}_{D.B.} = (\delta_\gamma^\beta \tilde{T}_\alpha{}^\delta - \delta_\alpha^\delta \tilde{T}_\gamma{}^\beta)(\sigma) \delta(\sigma - \sigma'),$$

$$\{\tilde{T}_i{}^j(\sigma), \tilde{T}_k{}^l(\sigma')\}_{D.B.} = (\delta_k^j \tilde{T}_i{}^l - \delta_i^l \tilde{T}_k{}^j)(\sigma) \delta(\sigma - \sigma'),$$

$$\{Q_\alpha{}^j(\sigma), Q_k{}^\delta(\sigma')\}_{D.B.} = \left(\delta_k^j \tilde{T}_\alpha{}^\delta + \delta_\alpha^\delta \tilde{T}_k{}^j + \frac{1}{4} \delta_\alpha^\delta \delta_k^j T \right)(\sigma) \delta(\sigma - \sigma'),$$

$$\{\tilde{T}_\alpha{}^\beta(\sigma), Q_\gamma{}^l(\sigma')\}_{D.B.} = \left(\delta_\gamma^\beta Q_\alpha{}^l - \frac{1}{4} \delta_\alpha^\beta Q_\gamma{}^l \right)(\sigma) \delta(\sigma - \sigma'),$$

$$\{\tilde{T}_\alpha{}^\beta(\sigma), Q_k{}^\delta(\sigma')\}_{D.B.} = -\left(\delta_\alpha^\delta Q_k{}^\beta - \frac{1}{4} \delta_\alpha^\beta Q_k{}^\delta \right)(\sigma) \delta(\sigma - \sigma'),$$

$$\{\tilde{T}_i{}^j(\sigma), Q_\gamma{}^l(\sigma')\}_{D.B.} = -\left(\delta_i^l Q_\gamma{}^j - \frac{1}{4} \delta_i^j Q_\gamma{}^l \right)(\sigma) \delta(\sigma - \sigma'),$$

$$\{\tilde{T}_i{}^j(\sigma), Q_k{}^\delta(\sigma')\}_{D.B.} = \left(\delta_k^j Q_i{}^\delta - \frac{1}{4} \delta_i^j Q_k{}^\delta \right)(\sigma) \delta(\sigma - \sigma'),$$

$$\{U(\sigma), Q_\alpha{}^j(\sigma')\}_{D.B.} = 2Q_\alpha{}^j(\sigma) \delta(\sigma - \sigma'),$$

$$\{U(\sigma), Q_i{}^\beta(\sigma')\}_{D.B.} = -2Q_i{}^\beta(\sigma) \delta(\sigma - \sigma'). \tag{3.16}$$

$T(\sigma)$ commutes on D.B. with all other $gl(4|4, \mathbb{R})$ current densities thus forming an Abelian ideal. The density $U(\sigma)$ of ‘twisted’ $gl_t(1, \mathbb{R})$ current does not appear on the r.h.s. of (3.16) that allows to consider $gl(4|4, \mathbb{R})$ as the semidirect sum of $sl(4|4, \mathbb{R})$ and $gl_t(1, \mathbb{R})$.

3.2. Quantum symmetries of twistor strings

It was shown in [8] that $SL(4|4, \mathbb{R})$ symmetry is preserved at the quantum level, whereas the generator U of ‘twisted’ $GL_t(1, \mathbb{R})$ symmetry has anomalous OPE with the world-sheet stress–energy tensor implying that corresponding symmetry is broken in twistor string theory. Thus possible type of infinite-dimensional symmetry that could survive in the quantum theory is restricted to that based on $sl(4|4, \mathbb{R})$ as finite-dimensional subalgebra. Since $gl(4|4, \mathbb{R})$ superalgebra belongs to the family of $gl(M|M, \mathbb{R})$ superalgebras, whose properties differ from those of $gl(M|N, \mathbb{R})$ superalgebras with $M \neq N$, one is forced to take components of supertwistors as building blocks of the generators for sl -type superalgebras.

3.2.1. Superalgebraic perspective on quantum higher-spin symmetries

In the bosonic limit TSA reduces to TSA_b – an infinite-dimensional Lie algebra, whose generators are obtained from (3.8) by setting to zero fermionic components of the supertwistors. Order 0 and 1 generators are given by the dual bosonic twistor Y_α and $gl(4, \mathbb{R})$ generators $Y_\alpha Z^\beta$. The latter divide into $sl(4, \mathbb{R})$ \tilde{T}_α^β and $gl(1, \mathbb{R})$ $T_0 = Y_\alpha Z^\alpha$ ones. Higher-order generators $Y_\alpha Z^{\beta(L)}$ divide into

$$\tilde{T}_\alpha^{\beta(L)} = Y_\alpha Z^{\beta(L)} - \frac{1}{L+3} (YZ) \delta_\alpha^{\beta(1)} Z^{\beta(L-1)} \tag{3.17}$$

and $T_0 Z^{\beta(L-1)}$. Expression (3.17) is an obvious generalization of \tilde{T}_α^β from (3.12) to the case $L > 1$.

Proceeding to TSA superalgebra, from (3.9) one infers that the D.B. relations of order L and M generators close on order $L + M - 1$ generators. So that order 1 generators, i.e. $gl(4|4, \mathbb{R})$ ones (3.12), play a special role: D.B. relations of the generators of an arbitrary order L with those of order 1 yield again order L generators. This feature can be used to characterize irreducible higher-order generators.

Thus the form of irreducible order 2 generators can be found by D.B.-commuting corresponding bosonic generator (3.17) with order 1 supersymmetry generators Q_i^β and Q_α^j , dividing generators that appear on the r.h.s. into irreducible $SL(4, \mathbb{R}) \times SL(4, \mathbb{R})$ tensors, then D.B.-commuting them with Q_i^β and Q_α^j and so on. In such a way we obtain

$$\{Q_i^\beta(\sigma), \tilde{T}_\gamma^{\delta(2)}(\sigma')\}_{D.B.} = \left(\delta_\gamma^\beta Q_i^{\delta(2)} - \frac{1}{5} \delta_\gamma^{\delta(1)} Q_i^{\delta(2)\beta} \right) (\sigma) \delta(\sigma - \sigma'), \tag{3.18}$$

where

$$Q_i^{\delta(2)} = \eta_i Z^{\delta(2)} \tag{3.19}$$

D.B.-commutes with Q_i^β . Analogously calculation of D.B. relations of $\tilde{T}_\gamma^{\delta(2)}$ and Q_α^j yields

$$\{Q_\alpha^j(\sigma), \tilde{T}_\gamma^{\delta(2)}(\sigma')\}_{D.B.} = - \left(\delta_\alpha^{\delta(1)} \tilde{Q}_\gamma^{\delta(2)j} - \frac{1}{5} \delta_\gamma^{\delta(1)} \tilde{Q}_\alpha^{\delta(2)j} \right) (\sigma) \delta(\sigma - \sigma'), \tag{3.20}$$

where another order 2 supersymmetry generator

$$\tilde{Q}_\gamma^{\delta i} = \tilde{T}_\gamma^{\delta \xi i} \tag{3.21}$$

D.B.-commutes with Q_α^j . Applying Q_α^j to $Q_k^{\delta(2)}$ gives

$$\begin{aligned} \{Q_\alpha^j(\sigma), Q_k^{\delta(2)}(\sigma')\}_{D.B.} &= \delta_k^j \tilde{T}_\alpha^{\delta(2)}(\sigma) \delta(\sigma - \sigma') \\ &\quad + \delta_\alpha^{(\delta_1)} \left(\tilde{T}_k^{\delta_2 j} + \delta_k^j \left(\frac{9}{40} T - \frac{1}{40} U \right) Z^{\delta_2} \right) (\sigma) \delta(\sigma - \sigma') \end{aligned} \tag{3.22}$$

and similarly

$$\begin{aligned} \{Q_i^\beta(\sigma), \tilde{Q}_\gamma^{\delta l}(\sigma')\}_{D.B.} &= \left(\delta_\gamma^\beta \tilde{T}_i^{\delta l} - \frac{1}{4} \delta_\gamma^\delta \tilde{T}_i^{\beta l} \right) (\sigma) \delta(\sigma - \sigma') \\ &\quad + \delta_i^l \left[\tilde{T}_\gamma^{\beta \delta} + \left(\frac{9}{40} T - \frac{1}{40} U \right) \right. \\ &\quad \left. \times \left(\delta_\gamma^\beta Z^\delta - \frac{1}{4} \delta_\gamma^\delta Z^\beta \right) \right] (\sigma) \delta(\sigma - \sigma'), \end{aligned} \tag{3.23}$$

where

$$\tilde{T}_i^{\beta j} = \tilde{T}_i^j Z^\beta. \tag{3.24}$$

Continuing further one recovers the set of irreducible order 2 generators

$$\begin{aligned} \tilde{T}_\alpha^{\beta(2)}, \quad \tilde{T}_i^{\alpha j}, \quad T_\alpha^{j[2]} &= Y_\alpha \xi^{j[2]}, \\ Q_i^{\alpha(2)}, \quad \tilde{Q}_\alpha^{\beta j}, \quad \tilde{Q}_i^{j[2]} &= \eta_i \xi^{j[2]} - \frac{1}{3} (\eta \xi) \delta_i^{[j_1} \xi^{j_2]} \end{aligned} \tag{3.25}$$

and

$$TZ^\alpha, \quad UZ^\alpha, \quad T\xi^i, \quad U\xi^i. \tag{3.26}$$

The operators associated with the generators (3.26), as will be shown below, are not the primary fields in the world-sheet CFT and hence corresponding symmetries are broken at the quantum level. Since these generators appear on the r.h.s. of (3.22), (3.23) this implies breaking of the order 2 supersymmetries $Q_i^{\alpha(2)}$, $\tilde{Q}_\alpha^{\beta j}$ and, in view of (3.18), (3.20) breaking of the bosonic symmetry generated by $\tilde{T}_\gamma^{\delta(2)}$. So that classical order 2 symmetries break in the quantum theory.

For order $L > 2$ calculation of D.B. relations of the corresponding bosonic generator (3.17) and order 1 supersymmetry generators gives

$$\{Q_i^\beta(\sigma), \tilde{T}_\gamma^{\delta(L)}(\sigma')\}_{D.B.} = \left(\delta_\gamma^\beta Q_i^{\delta(L)} - \frac{1}{L+3} \delta_\gamma^{\delta(1)} Q_i^{\delta(L-1)\beta} \right) (\sigma) \delta(\sigma - \sigma'), \tag{3.27}$$

and

$$\{Q_\alpha^j(\sigma), \tilde{T}_\gamma^{\delta(L)}(\sigma')\}_{D.B.} = - \left(\delta_\alpha^{\delta(1)} \tilde{Q}_\gamma^{\delta(L-1)j} - \frac{1}{L+3} \delta_\gamma^{\delta(1)} \tilde{Q}_\alpha^{\delta(L-1)j} \right) (\sigma) \delta(\sigma - \sigma'), \tag{3.28}$$

where order L supersymmetry generators are defined by the expressions

$$Q_i^{\delta(L)} = \eta_i Z^{\delta(L)}, \quad \tilde{Q}_\gamma^{\delta(L-1)j} = \tilde{T}_\gamma^{\delta(L-1)} \xi^j. \tag{3.29}$$

Their D.B. relations with order 1 supersymmetry generators read

$$\{Q_\alpha^j(\sigma), Q_k^{\delta(L)}(\sigma')\}_{D.B.} = \delta_k^j \tilde{T}_\alpha^{\delta(L)}(\sigma) \delta(\sigma - \sigma') + \delta_\alpha^{\delta(1)} \left[\tilde{T}_k^{\delta(L-1)j} + \delta_k^j \left(\frac{L+7}{8(L+3)} T - \frac{L-1}{8(L+3)} U \right) Z^{\delta(L-1)} \right] (\sigma) \delta(\sigma - \sigma') \tag{3.30}$$

and

$$\{Q_i^\beta(\sigma), \tilde{Q}_\gamma^{\delta(L-1)l}(\sigma')\}_{D.B.} = \left(\delta_\gamma^\beta \tilde{T}_i^{\delta(L-1)l} - \frac{1}{L+2} \delta_\gamma^{\delta(1)} \tilde{T}_i^{\beta\delta(L-2)l} \right) (\sigma) \delta(\sigma - \sigma') + \delta_i^l \left[\tilde{T}_\gamma^{\beta\delta(L-1)} + \left(\frac{L+7}{8(L+3)} T - \frac{L-1}{8(L+3)} U \right) \times \left(\delta_\gamma^\beta Z^{\delta(L-1)} - \frac{1}{L+2} \delta_\gamma^{\delta(1)} Z^\beta Z^{\delta(L-2)} \right) \right] (\sigma) \times \delta(\sigma - \sigma'), \tag{3.31}$$

where

$$\tilde{T}_k^{\delta(L-1)j} = \tilde{T}_k^j Z^{\delta(L-1)}. \tag{3.32}$$

Continuing further calculation of D.B. relations of $gl(4|4, \mathbb{R})$ supersymmetry generators and order L generators allows to find complete set of irreducible order L bosonic

$$\begin{aligned} \tilde{T}_\alpha^{\beta(p)j[q]} &= \tilde{T}_\alpha^{\beta(p)} \xi^{j[q]}, & q = 0, 2, 4, & p + q = L; \\ \tilde{T}_i^{\beta(p)j[q]} &= \tilde{T}_i^{j[q]} Z^{\beta(p)}, & q = 1, 3, & p + q = L \end{aligned} \tag{3.33}$$

and fermionic generators

$$\begin{aligned} \tilde{Q}_\alpha^{\beta(p)j[q]} &= \tilde{T}_\alpha^{\beta(p)} \xi^{j[q]}, & q = 1, 3, & p + q = L; \\ \tilde{Q}_i^{\beta(p)j[q]} &= \tilde{Q}_i^{j[q]} Z^{\beta(p)}, & q = 0, 2, 4, & p + q = L. \end{aligned} \tag{3.34}$$

Relevant (traceless) products of bosonic components of supertwistors are defined in (3.17) and the definition of (traceless) products of fermionic components is given in (3.12), (3.25) and by the expressions

$$\begin{aligned} \tilde{T}_i^{j[3]} &= \eta_i \xi^{j[3]} - \frac{1}{2} (\eta \xi) \delta_i^{[j1} \xi^{j2} \xi^{j3]}, \\ Q_i &= \eta_i, & Q_i^{j[4]} &= \eta_i \xi^{j[4]}. \end{aligned} \tag{3.35}$$

There are also generators of the form

$$TZ^{\alpha(p)} \xi^{i[q]}, \quad UZ^{\alpha(p)} \xi^{i[q]}, \quad p \geq 0, \quad 0 \leq q \leq 4. \tag{3.36}$$

It is these generators that correspond to non-tensor operators in the world-sheet CFT. They are present on the r.h.s. of (3.30) and (3.31) implying breaking of order L symmetries in analogy with those of order 2.

In Berkovits twistor string theory $GL(1, \mathbb{R})$ symmetry is gauged so that generators carrying the factor of T are set to zero. However, $GL_t(1, \mathbb{R})$ symmetry, being anomalous, cannot be gauged thus the generators carrying the factor of U cannot be put to zero. So we conclude that for any order L it is not possible to find a set of generators with closed D.B. relations that would correspond to the primary fields. As a result the quantum symmetry of the twistor string reduces to $SL(4|4, \mathbb{R}) \times SL(4|4, \mathbb{R})$ for the sector of closed strings and its diagonal subgroup for the sector of open strings.

3.2.2. Higher-spin symmetries from the world-sheet CFT perspective

This subsection we devote to consideration of the twistor part of the left-moving world-sheet CFT justifying the arguments above discussion relied on. To apply the 2d CFT technique to the model (3.1) it is helpful to carry out Wick rotation to Euclidean signature world-sheet

$$\tau \rightarrow i\sigma^2, \quad \sigma \rightarrow \sigma^1 \quad \Rightarrow \quad \sigma^+ \rightarrow z = \sigma^1 + i\sigma^2, \quad \sigma^- \rightarrow -\bar{z} = -(\sigma^1 - i\sigma^2). \tag{3.37}$$

The following changes of the world-sheet derivatives

$$\partial_+ \rightarrow \partial_z = \frac{1}{2}(\partial_1 - i\partial_2) \equiv \partial, \quad \partial_- \rightarrow -\partial_{\bar{z}} = -\frac{1}{2}(\partial_1 + i\partial_2) \equiv -\bar{\partial}, \tag{3.38}$$

2d volume element

$$d\tau d\sigma \rightarrow i d\sigma^1 d\sigma^2 = \frac{i}{2} d^2z, \tag{3.39}$$

and supertwistor components

$$\mathcal{Y}_A \rightarrow \mathcal{Y}_{A(z)}, \quad \bar{\mathcal{Y}}_A \rightarrow -\bar{\mathcal{Y}}_{A(\bar{z})}, \tag{3.40}$$

result in the Euclidean action

$$S_E = \int d^2z (\mathcal{Y}_A \bar{\partial} \mathcal{Z}^A + \bar{\mathcal{Y}}_A \partial \bar{\mathcal{Z}}^A). \tag{3.41}$$

Non-trivial OPE’s for the supertwistor components of the left-moving sector, on which we focus,

$$\mathcal{Z}^\alpha(z) Y_\beta(w) \sim \frac{\delta_\beta^\alpha}{z-w}, \quad \xi^i(z) \eta_j(w) \sim \frac{\delta_j^i}{z-w} \tag{3.42}$$

in terms of the supertwistors can be written as

$$\mathcal{Z}^A(z) \mathcal{Y}_B(w) \sim \frac{\delta_B^A}{z-w}, \quad \mathcal{Y}_B(z) \mathcal{Z}^A(w) \sim -\frac{(-)^a \delta_B^A}{z-w}. \tag{3.43}$$

By definition primary fields are characterized by the following general form of the OPE with the world-sheet stress-energy tensor

$$L(z)\mathcal{O}(w) \sim \frac{h}{(z-w)^2} \mathcal{O}(w) + \frac{1}{z-w} \partial\mathcal{O}(w), \tag{3.44}$$

where h is conformal weight of the primary field.¹⁰ The supertwistor part of the left-moving stress–energy tensor for the twistor string model (3.1) equals

$$L_{tw}(z) = -\mathcal{Y}_A \partial \mathcal{Z}^A \tag{3.45}$$

so that \mathcal{Y}_B and \mathcal{Z}^A are primary fields of conformal weight 1 and 0 respectively.

From the world-sheet CFT perspective the necessary condition for the considered global symmetries to survive in the quantum theory is that their generators become primary fields, i.e. their OPE’s with the stress–energy tensor are anomaly free, in other words, on the r.h.s. of (3.44) there

¹⁰ It is assumed that composite operators depending on a single argument are normal-ordered but normal ordering signs :: will be omitted.

should not appear terms with poles of order higher than two. As we find the generators containing the factor of T or U fail to comply with this requirement.

Using the relations

$$Y_\gamma \partial Z^\gamma(z) Y_\beta Z^\alpha(w) \sim \frac{\delta_\beta^\alpha}{(z-w)^3} - \frac{1}{(z-w)^2} Y_\beta Z^\alpha(w) - \frac{1}{(z-w)} \partial(Y_\beta Z^\alpha)(w) \tag{3.46}$$

and

$$\eta_k \partial \xi^k(z) \eta_j \xi^i(w) \sim -\frac{\delta_j^i}{(z-w)^3} - \frac{1}{(z-w)^2} \eta_j \xi^i(w) - \frac{1}{z-w} \partial(\eta_j \xi^i)(w), \tag{3.47}$$

it follows that $sl(4|4, \mathbb{R})$ generators $\tilde{T}_\alpha^\beta, \tilde{T}_i^j, Q_\alpha^j, Q_i^\beta$ and T are primary fields of unit weight, while U is not [8]

$$L_{\text{tw}}(z)U(w) \sim \frac{-8}{(z-w)^3} + \frac{1}{(z-w)^2}U(w) + \frac{1}{z-w}\partial U(w). \tag{3.48}$$

Higher-order generators (3.33), (3.34) also become primary fields of unit weight. While OPE’s of the generators (3.36) with the stress–energy tensor are anomalous

$$\begin{aligned} L_{\text{tw}}(z)TZ^{\alpha(p)}\xi^{i[q]}(w) &\sim -\frac{p+q}{(z-w)^3}Z^{\alpha(p)}\xi^{i[q]}(w) + \mathcal{O}((z-w)^{-2}) \\ L_{\text{tw}}(z)UZ^{\alpha(p)}\xi^{i[q]}(w) &\sim -\frac{8+p-q}{(z-w)^3}Z^{\alpha(p)}\xi^{i[q]}(w) + \mathcal{O}((z-w)^{-2}). \end{aligned} \tag{3.49}$$

In the case $p = q = 0$ one recovers discussed above OPE’s of $gl(1, \mathbb{R})$ and $gl_t(1, \mathbb{R})$ generators with the stress–energy tensor. For $p \neq 0, q \neq 0$ anomalous terms do not vanish so that associated symmetries are broken. Since generators (3.33), (3.34) are linked with other order L generators by order 1 supersymmetries (cf. Eqs. (3.27)–(3.31)) it appears that higher-spin symmetry is broken for arbitrary value of L except for $L = 1$, for which quantum-mechanically consistent global symmetry is isomorphic to $SL(4|4, \mathbb{R})$.

4. Conclusion and discussion

In this paper we performed the analysis of higher-spin global symmetries of $D = 4, N = 4$ massless superparticle models in supertwistor formulation extending the consideration of Ref. [11]. Discussed infinite-dimensional conformal superalgebras stemming from the $aq(8|8)$ algebra require further study as they could underly $N = 4$ supersymmetric extension of interacting higher-spin theories on AdS_5 [33,34] and conformal higher-spin theories on $D = 4$ Minkowski space–time [35]. We have also revealed infinite-dimensional classical symmetries in the Berkovits twistor string model and its extension with ungauged $GL(1, \mathbb{R})$ symmetry. Noether current densities associated with these symmetries have been constructed in terms of $PSL(4|4, \mathbb{R})$ supertwistors. In the generalized twistor string model the D.B. relations of the Noether current densities have been shown to form the TSA infinite-dimensional Lie superalgebra, whose finite-dimensional subalgebra is spanned by $gl(4|4, \mathbb{R})$ generators and the generator of constant shifts of the supertwistor components. The full classical symmetry of the twistor string action is generated by the direct sum of two copies of TSA superalgebra for the left- and right-movers that for the open string sector are identified on the boundary. Classical symmetry of the Berkovits model is described by the subalgebra of TSA obtained by going on the constraint shell $Y_\alpha Z^\alpha + \eta_i \xi^i \approx 0$.

Its finite-dimensional subalgebra is spanned by $psl(4|4, \mathbb{R})$, ‘twisted’ $gl_t(1, \mathbb{R})$ generators and that of shifts of the supertwistor components.

The fact that the symmetry of twistor string action is infinite-dimensional is anticipated due to the symmetry enhancement in $N = 4$ super-Yang–Mills theory at zero coupling [36,37]. One could similarly anticipate infinite-dimensional symmetry of free $N = 4$ conformal supergravity [38] that is present in the spectrum of Berkovits twistor string on equal footing with $N = 4$ super-Yang–Mills theory. Observed infinite-dimensional symmetry breaking down to $SL(4|4, \mathbb{R})$ at the quantum level also agrees with the higher-spin symmetry breaking in $N = 4$ super-Yang–Mills once the interactions are switched on [37]. Looking at the symmetry enhancement on the stringy side [36,39] in the weak coupling regime of gauge/gravity duality our results seem to support the evidence [20] for the tensionless nature of twistor strings or rather certain equivalence of the limits of zero and infinite tension [40]. Interesting question is whether other twistor string models [4–6] are invariant under higher-spin symmetries.

To conclude let us make a few comments on the twistor string spectrum. There are three kinds of states in the open string sector of the Berkovits model [3]. Twistor counterpart of $N = 4$ super-Yang–Mills multiplet is described by the vertex operator

$$V_{YM}(z) = j_R(z) F_0^R(\mathcal{Z}(z)), \tag{4.1}$$

where j_R ($R = 1, \dots, \dim G$) represent currents of unit conformal weight from the current algebra G that enters the Lagrangian $\mathcal{L}_{L(R)\text{-mat}}$ in (3.1) and $F_0^R(\mathcal{Z})$ is a scalar function on the supertwistor space of homogeneity degree zero. Other options to construct vertices of overall conformal weight one and homogeneity degree zero are

$$V_f(z) = \mathcal{Y}_A(z) f^A(\mathcal{Z}(z)), \quad V_g(z) = g_A(\mathcal{Z}(z)) \partial \mathcal{Z}^A \tag{4.2}$$

with the supertwistor functions $f^A(\mathcal{Z})$ and $g_A(\mathcal{Z})$ having homogeneity degrees ($GL(1, \mathbb{R})$ charges) $+1$ and -1 . They satisfy the constraints $\partial_A f^A = \mathcal{Z}^A g_A = 0$ and are defined modulo the gauge invariances $\delta f^A = \mathcal{Z}^A \mathfrak{f}$, $\delta g_A = \partial_A \mathfrak{g}$ to match upon the twistor transform the states of $N = 4$ conformal supergravity [3]. As far as the open string sector of the model (3.1) is concerned the vertex operators are formally remain the same as above but the condition of zero homogeneity degree in supertwistor components is relaxed so that $F_a^R(\mathcal{Z})$ describes not only $N = 4$ super-Yang–Mills states but also all the doubleton supermultiplets via the pairs of functions $F_{\pm a}(\mathcal{Z})$ having opposite homogeneity degrees $+a$ and $-a$. In particular, functions $F_{\pm 2}(\mathcal{Z})$ describe $N = 4$ Einstein supergravity multiplet. It is then natural to take j_R corresponding to some Abelian algebra. The states of $N = 4$ Einstein supergravity also reside in conformal supergravity vertices (4.2) with the supertwistor functions constrained by the ansatz $f^A(\mathcal{Z}) = I^{AB} \partial_B F'_{+2}(\mathcal{Z})$ and $g_A(\mathcal{Z}) = F'_{-2}(\mathcal{Z}) I_{AB} \mathcal{Z}^B$, where I^{AB} , I_{AB} are infinity supertwistors [41]. Supertwistor functions $f^A(\mathcal{Z})$ and $g_A(\mathcal{Z})$ with other values of $GL(1, \mathbb{R})$ charges correspond to higher-spin counterparts of $N = 4$ conformal supergravity multiplet and deserve further study. In the Berkovits twistor string model important role is played by the gauged $GL(1, \mathbb{R})$ symmetry that allows to shift conformal weights of the supertwistor fields and reproduce scattering amplitudes for various helicity configurations of external particles. In the ungauged case to be able to study scattering amplitudes of particles from, for instance, doubleton supermultiplets some additional variables should be introduced. This could impose further restrictions or lead to the determination of the structure of yet undetermined matter Lagrangians in (3.1).

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