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Universality in anomaly flow

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Universality in anomaly flow by an Aharonov–Bohm phase θ_H is shown in the flat $M^4 \times (S^1/Z_2)$ spacetime and in the Randall–Sundrum (RS) warped space. We analyze the $SU(2)$ gauge theory with doublet fermions. With orbifold boundary conditions the $U(1)$ part of the gauge symmetry remains unbroken at $\theta_H = 0$ and π. Chiral anomalies smoothly vary with θ_H in the RS space. It is shown that the anomaly coefficients associated with this anomaly flow are expressed in terms of the values of the wave functions of the gauge fields at the UV and IR branes in the RS space. The anomaly coefficients depend on θ_H, the warp factor of the RS space, and the orbifold boundary conditions for fermions, but not on the bulk mass parameters of fermions.

Subject Index B00, B06, B31, B40, B43

1. Introduction

In gauge–Higgs unification (GHU), gauge symmetry is dynamically broken by an Aharonov–Bohm (AB) phase, θ_H, in the fifth dimension [1–7]. It has been shown recently that chiral anomalies [8–11] in GHU flow with θ_H, i.e. anomaly coefficients smoothly change with θ_H in the Randall–Sundrum (RS) warped space [12]. In the GUT-inspired $SO(5) \times U(1)_X \times SU(3)_C$ GHU models in the RS space, chiral quarks and leptons at $\theta_H = 0$ are transformed to vector-like fermions at $\theta_H = \pi$ [13]. As θ_H varies from 0 to π, $SU(2)_L \times U(1)_Y \times SU(3)_C$ gauge symmetry is converted to $SU(2)_R \times U(1)_{Y'} \times SU(3)_C$ gauge symmetry. Chiral fermions appearing as zero modes of fermion multiplets in the spinor representation of $SO(5)$ at $\theta_H = 0$ become massive fermions having vector-like gauge couplings at $\theta_H = \pi$. The chiral anomaly induced by each quark or lepton at $\theta_H = 0$ smoothly changes and vanishes at $\theta_H = \pi$.

In the RS space, each fermion multiplet is characterized by its own dimensionless bulk mass parameter c which controls the mass and wave function of the fermion. In Ref. [12] it was recognized by numerical evaluation that the anomaly coefficients depend on θ_H, but not on the bulk mass parameter c. This fact leads to a puzzle. How can the θ_H dependence of the anomaly coefficients be determined and expressed independently of the details of the fermion field? This is the main theme addressed in this paper. We show that the anomaly coefficients at general θ_H are expressed in terms of the values of the wave functions of gauge fields at the UV and IR branes in the RS space. The anomaly coefficients depend on θ_H, the warp factor z_L of the RS space, and the boundary conditions of the fermion field, but not on the bulk mass parameter c. The universality of the anomaly flow is observed.

We stress that the universal behavior is highly nontrivial. In GHU in the RS space the gauge couplings of each fermion mode depend on θ_H, z_L, and c. To find the total anomaly coefficients one needs to sum all contributions coming from triangle loop diagrams in which all possible

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Kaluza–Klein (KK) excited modes of fermions are running. The universality of the anomaly flow is established only when all the contributions are taken into account.

The phenomenon of anomaly flow is different from that of anomaly inflow in which anomalies and fermion zero modes on defects such as strings and domain walls or on the boundary of spacetime are intertwined and related to each other [14–16]. In orbifold gauge theory the gauge couplings of fermion modes vary with the AB phase θ_H in the fifth dimension, and anomalies also vary with θ_H. We are going to show that the θ_H dependence of the anomalies is expressed by a holographic formula involving the values of the wave functions of gauge fields.

In this paper we analyze $SU(2)$ GHU models in the flat $M^4 \times (S^1/Z_2)$ spacetime and in the RS warped space with orbifold boundary conditions which break $SU(2)$ to $U(1)$. The $U(1)$ gauge symmetry survives at $\theta_H = 0$ and π. Fermion doublet multiplets have zero modes at $\theta_H = 0$ or π, depending on their boundary conditions. Chiral anomalies appear in various combinations of KK modes of the gauge fields. In the flat $M^4 \times (S^1/Z_2)$ spacetime all four-dimensional (4D) gauge couplings are determined analytically, but the KK mass spectrum of the gauge and fermion fields exhibits level crossings as θ_H varies. In the RS space no level crossing occurs in the spectrum, and all gauge couplings vary smoothly with θ_H. The flat-spacetime limit of the RS space gives rise to singular behavior of the anomalies as functions of θ_H, reproducing the known result in the flat spacetime.

In Sect. 2, $SU(2)$ GHU models are introduced in both the flat $M^4 \times (S^1/Z_2)$ spacetime and in the RS space. In Sect. 3, chiral anomalies are evaluated and expressed in a simple form which involves the values of the wave functions of gauge fields at the UV and IR branes and the boundary conditions of fermion fields. In Sect. 4, conditions for anomaly cancellation are derived. Section 5 is devoted to a summary and discussions.

2. *SU*(2) GHU

We consider $SU(2)$ GHU in the flat $M^4 \times (S^1/Z_2)$ spacetime with coordinate x^M ($M = 0, 1, 2, 3, 5$, $x^5 = y$) whose action is given by

$$
\begin{aligned}
I_{\rm flat} &= \int d^4x \int_0^L dy\, \mathcal{L}_{\rm flat}, \\
\mathcal{L}_{\rm flat} &= -\frac{1}{2} \mathrm{Tr}\, F_{MN} F^{MN} + \overline{\Psi} \gamma^M D_M \Psi,
\end{aligned} \tag{1}
$$

where $\mathcal{L}_{\rm flat}(x^\mu, y) = \mathcal{L}_{\rm flat}(x^\mu, y + 2L) = \mathcal{L}_{\rm flat}(x^\mu, -y)$. Here, $A_M = \frac{1}{2} \sum_{a=1}^3 A_M^a \tau^a$, where the τ^a are Pauli matrices, and $F_{MN} = \partial_M A_N - \partial_N A_N - i g_A [A_M, A_N]$. We adopt the metric $\eta_{MN} = \mathrm{diag}\,(-1, 1, 1, 1, 1)$. Ψ is an $SU(2)$ doublet and $D_M = \partial_M - i g_A A_M$; $\overline{\Psi} = i \Psi^\dagger \gamma^0$. Orbifold boundary conditions are given, with $(y_0, y_1) = (0, L)$, by

$$
\begin{aligned}
\begin{pmatrix} A_\mu \\ A_y \end{pmatrix} (x, y_j - y) &= P_j \begin{pmatrix} A_\mu \\ -A_y \end{pmatrix} (x, y_j + y) P_j^{-1}, \\
\Psi(x, y_j - y) &= \begin{cases} +P_j \gamma^5 \Psi(x, y_j + y) & \text{(type 1A)}, \\ -P_j \gamma^5 \Psi(x, y_j + y) & \text{(type 1B)}, \\ (-1)^j P_j \gamma^5 \Psi(x, y_j + y) & \text{(type 2A)}, \\ (-1)^{j+1} P_j \gamma^5 \Psi(x, y_j + y) & \text{(type 2B)}, \end{cases} \\
P_0 = P_1 &= \tau^3.
\end{aligned} \tag{2}
$$

The $SU(2)$ symmetry is broken to $U(1)$ by the boundary conditions in Eq. (2). A^3_μ, $A^{1,2}_y$ are parity even at both y_0 and y_1, and have constant zero modes. The zero mode of A^3_μ is the 4D $U(1)$ gauge field, and the 4D gauge coupling is given by

$$g_4 = \frac{g_A}{\sqrt{L}}. \tag{3}$$

We denote the doublet field as $\Psi = (u, d)^t$. In type 1A (1B), u_R and d_L (u_L and d_R) are parity even at both y_0 and y_1, and have zero modes, leading to chiral structure.

The zero modes of $A^{1,2}_y$ may develop nonvanishing expectation values. Without loss of generality one may assume that $\langle A^1_y \rangle = 0$. An AB phase θ_H along the fifth dimension is given by

$$P \exp\left\{ i g_A \int_0^{2L} dy \langle A_y \rangle \right\} = e^{i\theta_H \tau^2} = \begin{pmatrix} \cos\theta_H & \sin\theta_H \\ -\sin\theta_H & \cos\theta_H \end{pmatrix},$$
$$\theta_H = g_4 L \langle A^2_y \rangle. \tag{4}$$

The AB phase θ_H is a physical quantity. It couples to fields, affecting their mass spectrum. One can change the value of θ_H by a gauge transformation, which also alters the boundary conditions. Under a large gauge transformation given by

$$\tilde{A}_M = \Omega A_M \Omega^{-1} + \frac{i}{g_A} \Omega \partial_M \Omega^{-1}, \qquad \tilde{\Psi} = \Omega \Psi,$$
$$\Omega = \exp\left(\frac{i}{2} \theta(y) \tau^2 \right), \qquad \theta(y) = \theta_H \left(1 - \frac{y}{L} \right), \tag{5}$$

$\tilde{\theta}_H = 0$, and the boundary condition matrices become

$$\tilde{P}_j = \Omega(y_j - y) P_j \Omega^{-1}(y_j + y),$$
$$\tilde{P}_0 = \begin{pmatrix} \cos\theta_H & -\sin\theta_H \\ -\sin\theta_H & -\cos\theta_H \end{pmatrix}, \qquad \tilde{P}_1 = \tau^3. \tag{6}$$

Although the AB phase $\tilde{\theta}_H$ vanishes, the boundary conditions become nontrivial; the physics remains the same. This gauge is called the twisted gauge [17,18].

Fields in the twisted gauge satisfy free equations. KK expansions for $\tilde{A}^1_\mu$ and $\tilde{A}^3_\mu$ are given by

$$\begin{pmatrix} \tilde{A}^1_\mu(x, y) \\ \tilde{A}^3_\mu(x, y) \end{pmatrix} = \sum_{n=-\infty}^{\infty} B^{(n)}_\mu(x) \frac{1}{\sqrt{\pi R}} \begin{pmatrix} \sin\left[\frac{ny}{R} - \theta(y) \right] \\ \cos\left[\frac{ny}{R} - \theta(y) \right] \end{pmatrix}, \tag{7}$$

where $L = \pi R$. In the original gauge they become

$$\begin{pmatrix} A^1_\mu(x, y) \\ A^3_\mu(x, y) \end{pmatrix} = \sum_{n=-\infty}^{\infty} B^{(n)}_\mu(x) \frac{1}{\sqrt{\pi R}} \begin{pmatrix} \sin \frac{ny}{R} \\ \cos \frac{ny}{R} \end{pmatrix}. \tag{8}$$

The mass of the $B^{(n)}_\mu(x)$ mode is $m_n(\theta_H) = R^{-1} \left| n + \frac{\theta_H}{\pi} \right|$. The spectrum is periodic in θ_H with period π.

Similarly, the fermion field Ψ in the twisted gauge,

$$\tilde{\Psi} = \begin{pmatrix} \tilde{u} \\ \tilde{d} \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2}\theta(y) & \sin\frac{1}{2}\theta(y) \\ -\sin\frac{1}{2}\theta(y) & \cos\frac{1}{2}\theta(y) \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}, \tag{9}$$

satisfies free equations in the bulk region $0 < y < L$. The KK expansion of $\tilde{\Psi}$ in type 1A is given by

$$\begin{pmatrix} \tilde{u}_R(x,y) \\ \tilde{d}_R(x,y) \end{pmatrix} = \sum_{n=-\infty}^{\infty} \psi_R^{(n)}(x) \frac{1}{\sqrt{\pi R}} \begin{pmatrix} \cos\left[\frac{ny}{R} - \frac{1}{2}\theta(y)\right] \\ \sin\left[\frac{ny}{R} - \frac{1}{2}\theta(y)\right] \end{pmatrix},$$

$$\begin{pmatrix} \tilde{u}_L(x,y) \\ \tilde{d}_L(x,y) \end{pmatrix} = \sum_{n=-\infty}^{\infty} \psi_L^{(n)}(x) \frac{1}{\sqrt{\pi R}} \begin{pmatrix} -\sin\left[\frac{ny}{R} - \frac{1}{2}\theta(y)\right] \\ \cos\left[\frac{ny}{R} - \frac{1}{2}\theta(y)\right] \end{pmatrix}. \tag{10}$$

In the original gauge it becomes

$$\text{type 1A}: \quad \begin{pmatrix} u_R(x,y) \\ d_R(x,y) \end{pmatrix} = \sum_{n=-\infty}^{\infty} \psi_R^{(n)}(x) \frac{1}{\sqrt{\pi R}} \begin{pmatrix} \cos\frac{ny}{R} \\ \sin\frac{ny}{R} \end{pmatrix},$$

$$\begin{pmatrix} u_L(x,y) \\ d_L(x,y) \end{pmatrix} = \sum_{n=-\infty}^{\infty} \psi_L^{(n)}(x) \frac{1}{\sqrt{\pi R}} \begin{pmatrix} -\sin\frac{ny}{R} \\ \cos\frac{ny}{R} \end{pmatrix}. \tag{11}$$

$\psi_R^{(n)}$ and $\psi_L^{(n)}$ combine to form the $\psi^{(n)}(x)$ mode, whose mass is given by $m_n(\theta_H) = R^{-1}\left|n + \frac{\theta_H}{2\pi}\right|$. The spectrum is periodic in θ_H with period 2π. The KK expansion for type 1B is obtained by interchanging the left-handed and right-handed components in Eq. (11).

For Ψ in type 2A the KK expansion is

$$\text{type 2A}: \quad \begin{pmatrix} u_R(x,y) \\ d_R(x,y) \end{pmatrix} = \sum_{n=-\infty}^{\infty} \psi_R^{(n+\frac{1}{2})}(x) \frac{1}{\sqrt{\pi R}} \begin{pmatrix} \cos\frac{(n+\frac{1}{2})y}{R} \\ \sin\frac{(n+\frac{1}{2})y}{R} \end{pmatrix},$$

$$\begin{pmatrix} u_L(x,y) \\ d_L(x,y) \end{pmatrix} = \sum_{n=-\infty}^{\infty} \psi_L^{(n+\frac{1}{2})}(x) \frac{1}{\sqrt{\pi R}} \begin{pmatrix} -\sin\frac{(n+\frac{1}{2})y}{R} \\ \cos\frac{(n+\frac{1}{2})y}{R} \end{pmatrix}. \tag{12}$$

$\psi_R^{(n+\frac{1}{2})}$ and $\psi_L^{(n+\frac{1}{2})}$ combine to form the $\psi^{(n+\frac{1}{2})}(x)$ mode, whose mass is given by $m_{n+\frac{1}{2}}(\theta_H) = R^{-1}\left|n + \frac{1}{2} + \frac{\theta_H}{2\pi}\right|$. The KK expansion for type 2B is obtained by interchanging the left-handed and right-handed components in Eq. (12).

Next, we examine $SU(2)$ GHU in the RS space whose metric is given by [19]

$$ds^2 = e^{-2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{13}$$

where $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$, $\sigma(y) = \sigma(y + 2L) = \sigma(-y)$, and $\sigma(y) = ky$ for $0 \le y \le L$. It has the same topology as $M^4 \times (S^1/Z_2)$. In the fundamental region $0 \le y \le L$ the metric can be written, in terms of the conformal coordinate $z = e^{ky}$, as

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + \frac{dz^2}{k^2}\right) \qquad (1 \le z \le z_L = e^{kL}). \tag{14}$$

z_L is called the warp factor of the RS space. The action in RS is

$$\begin{aligned} I_{\rm RS} &= \int d^5x \sqrt{-\det G}\,\mathcal{L}_{\rm RS}, \\ \mathcal{L}_{\rm RS} &= -\frac{1}{2}{\rm Tr}F_{MN}F^{MN} + \overline{\Psi}\mathcal{D}(c)\Psi, \\ \mathcal{D}(c) &= \gamma^A e_A{}^M\left(D_M + \frac{1}{8}\omega_{MBC}[\gamma^B, \gamma^C]\right) - c\sigma', \end{aligned} \tag{15}$$

where $\sigma'(y) = k$ for $0 \le y \le L$. Note that $\mathcal{L}_{\rm RS}(x^\mu, y) = \mathcal{L}_{\rm RS}(x^\mu, -y) = \mathcal{L}_{\rm RS}(x^\mu, y + 2L)$. The fields A_M and Ψ satisfy the same boundary conditions, Eq. (2), as in the flat spacetime. The dimensionless bulk mass parameter c in $\mathcal{D}(c)$ controls the mass and wave function of the fermion field. The KK mass scale is given by

$$m_{\rm KK} = \frac{\pi k}{z_L - 1}, \tag{16}$$

which becomes $1/R$ in the flat-spacetime limit $k \to 0$.

In the KK expansion in the z coordinate, $A_z^a(x, z) = k^{-1/2}\sum A_z^{a(n)}(x)h_n(z)$, the zero mode $A_z^{2(0)}$ has a wave function $h_0(z) = \sqrt{2/(z_L^2 - 1)}\,z$. In the y-coordinate, $A_y^{2(0)}$ has a wave function $v_0(y) = ke^{ky}h_0(z)$ for $0 \le y \le L$, and $v_0(-y) = v_0(y) = v_0(y + 2L)$. The AB phase θ_H in Eq. (4) becomes

$$\theta_H = \frac{\langle A_z^{2(0)}\rangle}{f_H}, \qquad f_H = \frac{1}{g_4}\sqrt{\frac{2k}{L(z_L^2 - 1)}}. \tag{17}$$

The twisted gauge [17,18], in which $\tilde{\theta}_H = 0$, is related to the original gauge by a large gauge transformation,

$$\Omega(z) = e^{i\theta(z)\tau^2/2}, \qquad \theta(z) = \theta_H\frac{z_L^2 - z^2}{z_L^2 - 1}. \tag{18}$$

In the y-coordinate it becomes

$$\Omega(y) = \exp\left\{i\theta_H\sqrt{\frac{2}{z_L^2 - 1}}\int_y^L dy v_0(y)\cdot\frac{\tau^2}{2}\right\}. \tag{19}$$

In the twisted gauge $\tilde{A}_\mu^{1,3}(x, z)$ satisfy free equations in $1 \le z \le z_L$ and the boundary conditions in Eq. (6). The mass spectrum $\{m_n(\theta_H) = k\lambda_n(\theta_H)\}$ $(\lambda_0 < \lambda_1 < \lambda_2 < \cdots)$ is given by

$$Z_\mu^{(n)}: \quad SC'(1; \lambda_n) + \lambda_n\sin^2\theta_H = 0, \tag{20}$$

where $S(z; \lambda)$ and $C(z; \lambda)$ are expressed in terms of Bessel functions and are given by Eq. (A1). The KK expansions in the twisted gauge in the region $1 \le z \le z_L$ are written as[1]

$$\begin{pmatrix}\tilde{A}_\mu^1(x, z)\\ \tilde{A}_\mu^3(x, z)\end{pmatrix} = \frac{1}{\sqrt{L}}\sum_{n=0}^\infty Z_\mu^{(n)}(x)\tilde{\mathbf{h}}_n(z), \qquad \tilde{\mathbf{h}}_n(z) = \begin{pmatrix}\tilde{h}_n(z)\\ \tilde{k}_n(z)\end{pmatrix}, \tag{21}$$

[1]Note the change in the normalization of mode functions: $\tilde{\mathbf{h}}_n(z)$ in the present paper corresponds to $\sqrt{kL}\tilde{\mathbf{h}}_n(z)$ in Ref. [12].

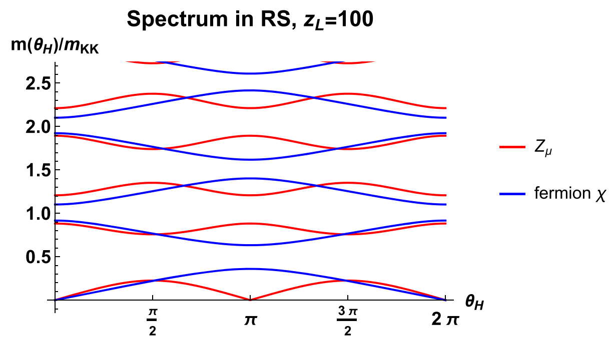

Fig. 1. The mass spectrum of the gauge fields $Z_\mu^{(n)}$ and fermion fields $\chi^{(n)}$ (type 1A) in the RS warped space, The warp factor is $z_L = 100$ and the bulk mass parameter of Ψ is $c = 0.25$. There is no level crossing in the spectrum.

where the mode functions $\tilde{\mathbf{h}}_n(z)$ are given in Eq. (B1). In the original gauge the KK expansions of $A_\mu^{1,3}(x, y)$ become

$$\begin{aligned}
\begin{pmatrix} A_\mu^1(x,y) \\ A_\mu^3(x,y) \end{pmatrix} &= \frac{1}{\sqrt{L}} \sum_{n=0}^{\infty} Z_\mu^{(n)}(x) \begin{pmatrix} h_n(y) \\ k_n(y) \end{pmatrix}, \\
\begin{pmatrix} h_n(y) \\ k_n(y) \end{pmatrix} &= \begin{pmatrix} -h_n(-y) \\ k_n(-y) \end{pmatrix} = \begin{pmatrix} h_n(y+2L) \\ k_n(y+2L) \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta(z) & \sin\theta(z) \\ -\sin\theta(z) & \cos\theta(z) \end{pmatrix} \begin{pmatrix} \tilde{h}_n(z) \\ \tilde{k}_n(z) \end{pmatrix} \qquad \text{for } 0 \le y \le L.
\end{aligned} \tag{22}$$

For a fermion field $\Psi(x, z)$ it is most convenient to express its KK expansion for $\check{\Psi}(x, z) = z^{-2}\Psi(x, z)$. The equations of motion in the region $1 \le z \le z_L$ become

$$\begin{aligned}
&-kD_-(c)\check{\Psi}_R + \sigma^\mu \partial_\mu \check{\Psi}_L = 0, \qquad -kD_+(c)\check{\Psi}_L + \bar{\sigma}^\mu \partial_\mu \check{\Psi}_R = 0, \\
&\sigma^\mu = (I_2, \vec{\sigma}), \qquad \bar{\sigma}^\mu = (-I_2, \vec{\sigma}), \qquad D_\pm(c) = \pm\frac{\partial}{\partial z} + \frac{c}{z}.
\end{aligned} \tag{23}$$

In the presence of gauge fields, ∂_M is replaced by $\partial_M - ig_A A_M$. The Neumann boundary conditions at $z = (z_0, z_1) = (1, z_L)$, corresponding to even parity, for left- and right-handed components are given by $D_+(c)\check{\Psi}_L\big|_{z_j} = 0$ and $D_-(c)\check{\Psi}_R\big|_{z_j} = 0$.

The spectrum of the KK modes of the fermion field Ψ is determined by

$$\chi^{(n)} : \begin{cases} S_L S_R(1; \lambda_n, c) + \sin^2 \frac{1}{2}\theta_H = 0, & \text{for type 1A/B}, \\ S_L S_R(1; \lambda_n, c) + \cos^2 \frac{1}{2}\theta_H = 0, & \text{for type 2A/B}, \end{cases} \tag{24}$$

where the functions $S_{L/R}(z; \lambda, c)$ are given in Eq. (A4). The spectrum is periodic in θ_H with period 2π. A massless mode appears at $\theta_H = 0$ for type 1A and 1B, whereas it appears at $\theta_H = \pi$ for type 2A and 2B. There is no level crossing in the spectrum except for the case $c = 0$. The spectra of the gauge fields in Eq. (20) and fermion fields in Eq. (24) are displayed in Fig. 1.

The KK expansion of the fermion field Ψ in the twisted gauge in the region $1 \le z \le z_L$ is expressed as

$$\begin{pmatrix} \tilde{\check{u}}_R(x,z) \\ \tilde{\check{d}}_R(x,z) \end{pmatrix} = \sqrt{k} \sum_{n=0}^{\infty} \chi_R^{(n)}(x) \tilde{\mathbf{f}}_{Rn}(z), \qquad \tilde{\mathbf{f}}_{Rn}(z) = \begin{pmatrix} \tilde{f}_{Rn}(z) \\ \tilde{g}_{Rn}(z) \end{pmatrix},$$
$$\begin{pmatrix} \tilde{\check{u}}_L(x,z) \\ \tilde{\check{d}}_L(x,z) \end{pmatrix} = \sqrt{k} \sum_{n=0}^{\infty} \chi_L^{(n)}(x) \tilde{\mathbf{f}}_{Ln}(z), \qquad \tilde{\mathbf{f}}_{Ln}(z) = \begin{pmatrix} \tilde{f}_{Ln}(z) \\ \tilde{g}_{Ln}(z) \end{pmatrix}. \tag{25}$$

The mode functions $\tilde{\mathbf{f}}_{Rn}(z)$ and $\tilde{\mathbf{f}}_{Ln}(z)$ for type 1A are given in Eq. (B2). In the original gauge the expansions of $\check{u}(x,y)$ and $\check{d}(x,y)$ become

$$\begin{pmatrix} \check{u}_R(x,y) \\ \check{d}_R(x,y) \end{pmatrix} = \sqrt{k} \sum_{n=0}^{\infty} \chi_R^{(n)}(x) \begin{pmatrix} f_{Rn}(y) \\ g_{Rn}(y) \end{pmatrix},$$
$$\begin{pmatrix} \check{u}_L(x,y) \\ \check{d}_L(x,y) \end{pmatrix} = \sqrt{k} \sum_{n=0}^{\infty} \chi_L^{(n)}(x) \begin{pmatrix} f_{Ln}(y) \\ g_{Ln}(y) \end{pmatrix}, \tag{26}$$

where

$$\begin{aligned}
\text{type 1A}: \begin{pmatrix} f_{Rn}(y) \\ g_{Rn}(y) \end{pmatrix} &= \begin{pmatrix} f_{Rn}(-y) \\ -g_{Rn}(-y) \end{pmatrix} = \begin{pmatrix} f_{Rn}(y+2L) \\ g_{Rn}(y+2L) \end{pmatrix} \\
&= \begin{pmatrix} \cos\frac{1}{2}\theta(z) & -\sin\frac{1}{2}\theta(z) \\ \sin\frac{1}{2}\theta(z) & \cos\frac{1}{2}\theta(z) \end{pmatrix} \begin{pmatrix} \tilde{f}_{Rn}(z) \\ \tilde{g}_{Rn}(z) \end{pmatrix} \qquad \text{for } 0 \le y \le L, \\
\begin{pmatrix} f_{Ln}(y) \\ g_{Ln}(y) \end{pmatrix} &= \begin{pmatrix} -f_{Ln}(-y) \\ g_{Ln}(-y) \end{pmatrix} = \begin{pmatrix} f_{Ln}(y+2L) \\ g_{Ln}(y+2L) \end{pmatrix} \\
&= \begin{pmatrix} \cos\frac{1}{2}\theta(z) & -\sin\frac{1}{2}\theta(z) \\ \sin\frac{1}{2}\theta(z) & \cos\frac{1}{2}\theta(z) \end{pmatrix} \begin{pmatrix} \tilde{f}_{Ln}(z) \\ \tilde{g}_{Ln}(z) \end{pmatrix} \qquad \text{for } 0 \le y \le L; \\
\text{type 2A}: \begin{pmatrix} f_{Rn}(y) \\ g_{Rn}(y) \end{pmatrix} &= \begin{pmatrix} f_{Rn}(-y) \\ -g_{Rn}(-y) \end{pmatrix} = \begin{pmatrix} -f_{Rn}(y+2L) \\ -g_{Rn}(y+2L) \end{pmatrix}, \\
\begin{pmatrix} f_{Ln}(y) \\ g_{Ln}(y) \end{pmatrix} &= \begin{pmatrix} -f_{Ln}(-y) \\ g_{Ln}(-y) \end{pmatrix} = \begin{pmatrix} -f_{Ln}(y+2L) \\ -g_{Ln}(y+2L) \end{pmatrix}.
\end{aligned} \tag{27}$$

For type 1B (2B), the parity of $f_{R/Ln}, g_{R/Ln}$ is reversed compared to type 1A (2A).

3. Anomalies

Doublet fermions in type 1A or 1B are chiral at $\theta_H = 0$. Massless modes appear for right-handed u and left-handed d (left-handed u and right-handed d) for type 1A (1B). They become massive as θ_H varies, and their gauge couplings become purely vector-like at $\theta_H = \pi$. Chiral anomalies exist at $\theta_H = 0$, smoothly vary as θ_H in the RS space, and vanish at $\theta_H = \pi$. This phenomenon is called the anomaly flow by an AB phase [12].

Chiral anomalies arise from triangular loop diagrams. Gauge couplings of fermions have been obtained in Ref. [12]. Substituting the KK expansions in Eqs. (22) and (26) into

$$g_A \int_1^{z_L} \frac{dz}{k} \Big\{ \check{\Psi}_R^\dagger \bar{\sigma}^\mu A_\mu \check{\Psi}_R - \check{\Psi}_L^\dagger \sigma^\mu A_\mu \check{\Psi}_L \Big\}, \tag{28}$$

one finds that the couplings in

$$\frac{g_4}{2}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} Z_\mu^{(n)}(x)\Big\{t^R_{n\ell m}\chi_R^{(\ell)}(x)^\dagger\bar\sigma^\mu\chi_R^{(m)}(x)+t^L_{n\ell m}\chi_L^{(\ell)}(x)^\dagger\sigma^\mu\chi_L^{(m)}(x)\Big\} \tag{29}$$

are given by

$$\begin{aligned}
t^R_{n\ell m} &= \int_1^{z_L} dz\Big\{\tilde h_n(z)\big(\tilde f^*_{R\ell}(z)\tilde g_{Rm}(z)+\tilde g^*_{R\ell}(z)\tilde f_{Rm}(z)\big)+\tilde k_n(z)\big(\tilde f^*_{R\ell}(z)\tilde f_{Rm}(z)-\tilde g^*_{R\ell}(z)\tilde g_{Rm}(z)\big)\Big\}\\
&= \frac{k}{2}\int_{-a}^{2L-a} dy e^{\sigma(y)}\Big\{h_n(y)\big(f^*_{R\ell}(y)g_{Rm}(y)+g^*_{R\ell}(y)f_{Rm}(y)\big)+k_n(y)\big(f^*_{R\ell}(y)f_{Rm}(y)\\
&\quad -g^*_{R\ell}(y)g_{Rm}(y)\big)\Big\},\\
t^L_{n\ell m} &= -\int_1^{z_L} dz\Big\{h_n(z)\big(f^*_{L\ell}(z)g_{Lm}(z)+g^*_{L\ell}(z)f_{Lm}(z)\big)+k_n(z)\big(f^*_{L\ell}(z)f_{Lm}(z)-g^*_{L\ell}(z)g_{Lm}(z)\big)\Big\}\\
&= -\frac{k}{2}\int_{-a}^{2L-a} dy e^{\sigma(y)}\Big\{h_n(y)\big(f^*_{L\ell}(y)g_{Lm}(y)+g^*_{L\ell}(y)f_{Lm}(y)\big)+k_n(y)\big(f^*_{L\ell}(y)f_{Lm}(y)\\
&\quad -g^*_{L\ell}(y)g_{Lm}(y)\big)\Big\}.
\end{aligned} \tag{30}$$

The couplings $t^R_{n\ell m}$ and $t^L_{n\ell m}$ are gauge invariant. In the integral formulas in the y-coordinate the constant a is arbitrary as the integrands are periodic functions with period $2L$. It is convenient to take $0 < a < L$ in the following discussions. We note that the couplings $t^{R/L}_{n\ell m}$ depend not only on θ_H and z_L, but also on the bulk mass parameter c of the fermion field Ψ.

The anomaly coefficient associated with the three legs of $Z^{(n_1)}_{\mu_1}Z^{(n_2)}_{\mu_2}Z^{(n_3)}_{\mu_3}$ is given by

$$\begin{aligned}
a_{n_1n_2n_3} &= a^R_{n_1n_2n_3}+a^L_{n_1n_2n_3},\\
a^R_{n_1n_2n_3} &= \mathrm{Tr}\,T^R_{n_1}T^R_{n_2}T^R_{n_3}, \qquad (T^R_n)_{m\ell}=t^R_{nm\ell}\\
a^L_{n_1n_2n_3} &= \mathrm{Tr}\,T^L_{n_1}T^L_{n_2}T^L_{n_3}, \qquad (T^L_n)_{m\ell}=t^L_{nm\ell}.
\end{aligned} \tag{31}$$

The anomaly coefficient $a_{n_1n_2n_3}$ depends on θ_H, exhibiting the anomaly flow. It was observed by numerical evaluation in Ref. [12] that $a_{n_1n_2n_3}$ does not depend on the bulk mass parameter c, though $a^R_{n_1n_2n_3}$ and $a^L_{n_1n_2n_3}$ do depend on c. We show here that $a_{n_1n_2n_3}(\theta_H, z_L)$ is expressed in terms of the values of the wave functions $k_{n_j}(y)$ at $y=0$ and $y=L$.

To see this, we insert the formulas for $t^{R/L}_{n\ell m}$ in Eq. (30) into Eq. (31), and rearrange the traces:

$$\begin{aligned}
a_{n_1n_2n_3} &= \Big(\frac{k}{2}\Big)^3\int\int\int_{-a}^{2L-a} dy_1dy_2dy_3 e^{\sigma(y_1)+\sigma(y_2)+\sigma(y_3)}\\
&\quad\times\Big[k_1k_2k_3\big\{A_R(1,2)A_R(2,3)A_R(3,1)-B_R(1,2)B_R(2,3)B_R(3,1)\\
&\quad +B_L(1,2)B_L(2,3)B_L(3,1)-A_L(1,2)A_L(2,3)A_L(3,1)\big\}\\
&\quad +k_1h_2h_3\big\{A_R(1,2)B_R(2,3)A_R(3,1)-B_R(1,2)A_R(2,3)B_R(3,1)\\
&\quad +B_L(1,2)A_L(2,3)B_L(3,1)-A_L(1,2)B_L(2,3)A_L(3,1)\big\}\\
&\quad +h_1k_2h_3\big\{A_R(1,2)A_R(2,3)B_R(3,1)-B_R(1,2)B_R(2,3)A_R(3,1)\\
&\quad +B_L(1,2)B_L(2,3)A_L(3,1)-A_L(1,2)A_L(2,3)B_L(3,1)\big\}\\
&\quad +h_1h_2k_3\big\{B_R(1,2)A_R(2,3)A_R(3,1)-A_R(1,2)B_R(2,3)B_R(3,1)\\
&\quad +A_L(1,2)B_L(2,3)B_L(3,1)-B_L(1,2)A_L(2,3)A_L(3,1)\big\}\Big],
\end{aligned} \tag{32}$$

where

$$
\begin{aligned}
&k_j = k_{n_j}(y_j), \qquad h_j = h_{n_j}(y_j), \\
&\begin{pmatrix} A_{R/L}(j,k) \\ B_{R/L}(j,k) \end{pmatrix} = \begin{pmatrix} A_{R/L} \\ B_{R/L} \end{pmatrix}(y_j, y_k) = \sum_{n=0}^{\infty} \begin{pmatrix} f_{R/Ln}(y_j) f^*_{R/Ln}(y_k) \\ g_{R/Ln}(y_j) g^*_{R/Ln}(y_k) \end{pmatrix}.
\end{aligned} \tag{33}
$$

Equations (25) and (26) along with the orthonormality relations of the mode functions imply that

$$
\begin{aligned}
&\begin{pmatrix} \check{u}_{R/L}(x,y) \\ \check{d}_{R/L}(x,y) \end{pmatrix} = \frac{k}{2} \int_{-a}^{2L-a} dy' e^{\sigma(y')} \begin{pmatrix} A_{R/L} & C_{R/L} \\ D_{R/L} & B_{R/L} \end{pmatrix}(y,y') \begin{pmatrix} \check{u}_{R/L}(x,y') \\ \check{d}_{R/L}(x,y') \end{pmatrix}, \\
&\begin{pmatrix} C_{R/L} \\ D_{R/L} \end{pmatrix}(y,y') = \sum_{n=0}^{\infty} \begin{pmatrix} f_{R/Ln}(y) g^*_{R/Ln}(y') \\ g_{R/Ln}(y) f^*_{R/Ln}(y') \end{pmatrix}.
\end{aligned} \tag{34}
$$

We made use of the relation $C_{R/L} = D_{R/L} = 0$ in deriving Eq. (32). With the choice of the AB phase θ_H in Eq. (17), all mode functions $\{f_{Rn}(y)\}$ etc. can be taken to be real so that $A_{R/L}(y,y') = A_{R/L}(y',y)$ and $B_{R/L}(y,y') = B_{R/L}(y',y)$.

In addition to the relation in Eq. (34), $A_{R/L}$ and $B_{R/L}$ must satisfy the parity relations and boundary conditions of the mode functions. With $(y_0, y_1) = (0, L)$,

$$
\begin{aligned}
&\text{type 1A}: \quad \begin{pmatrix} A_R \\ B_R \\ A_L \\ B_L \end{pmatrix}(y_j - y, y') = \begin{pmatrix} A_R \\ -B_R \\ -A_L \\ B_L \end{pmatrix}(y_j + y, y'), \\
&\begin{pmatrix} \hat{D}_-(c) A_R(y,y') \\ \hat{D}_+(c) B_L(y,y') \end{pmatrix}_{y=\epsilon, L-\epsilon} = 0, \qquad \hat{D}_\pm(c) = \pm \frac{\partial}{\partial y} + ck, \\
&B_R(y_j, y') = A_L(y_j, y') = 0; \\
&\text{type 2A}: \quad \begin{pmatrix} A_R \\ B_R \\ A_L \\ B_L \end{pmatrix}(y_j - y, y') = \begin{pmatrix} (-1)^j A_R \\ (-1)^{j+1} B_R \\ (-1)^{j+1} A_L \\ (-1)^j B_L \end{pmatrix}(y_j + y, y'), \\
&\begin{pmatrix} \hat{D}_-(c) A_R(y,y') \\ \hat{D}_+(c) B_L(y,y') \end{pmatrix}_{y=\epsilon} = \begin{pmatrix} \hat{D}_-(c) B_R(y,y') \\ \hat{D}_+(c) A_L(y,y') \end{pmatrix}_{y=L-\epsilon} = 0, \\
&B_R(0,y') = A_R(L,y') = A_L(0,y') = B_L(L,y') = 0.
\end{aligned} \tag{35}
$$

The conditions for type 1B (2B) are obtained by interchanging R (right-handed) and L (left-handed) in those for type 1A (2A). For $c \neq 0$, parity-even components of $A_{R/L}$ and $B_{R/L}$ functions exhibit the cusp behavior at $y, y' = 0, \pm L, \cdots$.

It is not easy to explicitly write down the $A_{R/L}(y,y')$ and $B_{R/L}(y,y')$ functions for $c \neq 0$ which satisfy the relations in both Eqs. (34) and (35). In Ref. [12] it was recognized that the anomaly coefficient $a_{n_1 n_2 n_3}$ in Eq. (32) is independent of c. With this observation we now derive an analytical expression for $a_{n_1 n_2 n_3}$ by evaluating it in the case $c = 0$. We confirm later that the numerically evaluated $a_{n_1 n_2 n_3}$ for $c \neq 0$ agrees with the analytical formula.

Fermion wave functions for $c = 0$ are expressed in terms of trigonometric functions; they are summarized in Appendix B.3. Inserting the wave functions in Eq. (B5) into

$A_R(z, z') = \sum f_{Rn}(z) f^*_{Rn}(z')$, for instance, one finds for type 1A that, for $1 \le z, z' \le z_L$,

$$\begin{aligned}
A_R(z, z')^{c=0} &= \frac{1}{z_L - 1} \sum_{n=-\infty}^{\infty} \cos\left(n\pi \frac{z - z_L}{z_L - 1} + \alpha(z)\right) \cos\left(n\pi \frac{z' - z_L}{z_L - 1} + \alpha(z')\right) \\
&= \delta_{2(z_L-1)}(z - z') \cos\left\{\alpha(z) - \alpha(z')\right\} + \delta_{2(z_L-1)}(z + z' - 2) \cos\left\{\alpha(z) + \alpha(z')\right\} \\
&= \delta_{2(z_L-1)}(z - z') + \delta_{2(z_L-1)}(z + z' - 2), \\
\alpha(z) &= \frac{1}{2}\left\{\theta_H \frac{z - z_L}{z_L - 1} + \theta(z)\right\}, \qquad \alpha(1) = \alpha(z_L) = 0.
\end{aligned} \tag{36}$$

Here, $\delta_L(x) = \sum_n \delta(x - nL)$. With the extension in Eq. (27) in the y-coordinate and similar manipulation, one finds that

$$\begin{aligned}
\text{type 1A},\ c = 0: \quad A_R(y, y') = B_L(y, y') &= \frac{e^{-\sigma(y)}}{k}\left\{\delta_{2L}(y - y') + \delta_{2L}(y + y')\right\}, \\
B_R(y, y') = A_L(y, y') &= \frac{e^{-\sigma(y)}}{k}\left\{\delta_{2L}(y - y') - \delta_{2L}(y + y')\right\},
\end{aligned} \tag{37}$$

The formulas for type 1B are obtained by interchanging R and L.

For fermions in type 2A, one finds, for $1 \le z, z' \le z_L$, that

$$\begin{aligned}
A_R(z, z')^{c=0} &= \frac{1}{z_L - 1} \sum_{n=-\infty}^{\infty} \sin\left(n\pi \frac{z - z_L}{z_L - 1} + \beta(z)\right) \sin\left(n\pi \frac{z' - z_L}{z_L - 1} + \beta(z')\right) \\
&= \delta_{2(z_L-1)}(z - z') \cos\left\{\beta(z) - \beta(z')\right\} - \delta_{2(z_L-1)}(z + z' - 2) \cos\left\{\beta(z) + \beta(z')\right\}, \\
\beta(z) &= \frac{1}{2}\left\{(\theta_H + \pi) \frac{z - z_L}{z_L - 1} + \theta(z)\right\}, \qquad \beta(1) = -\frac{1}{2}\pi\ ,\ \beta(z_L) = 0.
\end{aligned} \tag{38}$$

Noting the relations in Eq. (27), one finds in the y-coordinate that

$$\begin{aligned}
\text{type 2A},\ c = 0: \quad A_R(y, y') = B_L(y, y') &= \frac{e^{-\sigma(y)}}{k}\left\{\hat{\delta}_{2L}(y - y') + \hat{\delta}_{2L}(y + y')\right\}, \\
B_R(y, y') = A_L(y, y') &= \frac{e^{-\sigma(y)}}{k}\left\{\hat{\delta}_{2L}(y - y') - \hat{\delta}_{2L}(y + y')\right\}, \\
\hat{\delta}_{2L}(y) &= \delta_{4L}(y) - \delta_{4L}(y - 2L).
\end{aligned} \tag{39}$$

The formulas for type 2B are obtained by interchanging R and L.

We insert the expressions in Eqs. (37) or (39) into Eq. (32). The products of three delta functions appear in the integrand. Take $0 < a < L$. Then, in the integration range $-a \le y_1, y_2, y_3 \le 2L - a$, the products of the delta functions reduce to

$$\begin{aligned}
&\left.\begin{matrix} \delta_{2L}(y_1 - y_2)\delta_{2L}(y_2 - y_3)\delta_{2L}(y_3 + y_1) \\ \delta_{2L}(y_1 + y_2)\delta_{2L}(y_2 + y_3)\delta_{2L}(y_3 + y_1) \end{matrix}\right\} \Rightarrow \frac{1}{2}\left\{\delta(y_1)\delta(y_2)\delta(y_3) + \delta(y_1 - L)\delta(y_2 - L)\delta(y_3 - L)\right\}, \\
&\left.\begin{matrix} \hat{\delta}_{2L}(y_1 - y_2)\hat{\delta}_{2L}(y_2 - y_3)\hat{\delta}_{2L}(y_3 + y_1) \\ \hat{\delta}_{2L}(y_1 + y_2)\hat{\delta}_{2L}(y_2 + y_3)\hat{\delta}_{2L}(y_3 + y_1) \end{matrix}\right\} \Rightarrow \frac{1}{2}\left\{\delta(y_1)\delta(y_2)\delta(y_3) - \delta(y_1 - L)\delta(y_2 - L)\delta(y_3 - L)\right\}.
\end{aligned} \tag{40}$$

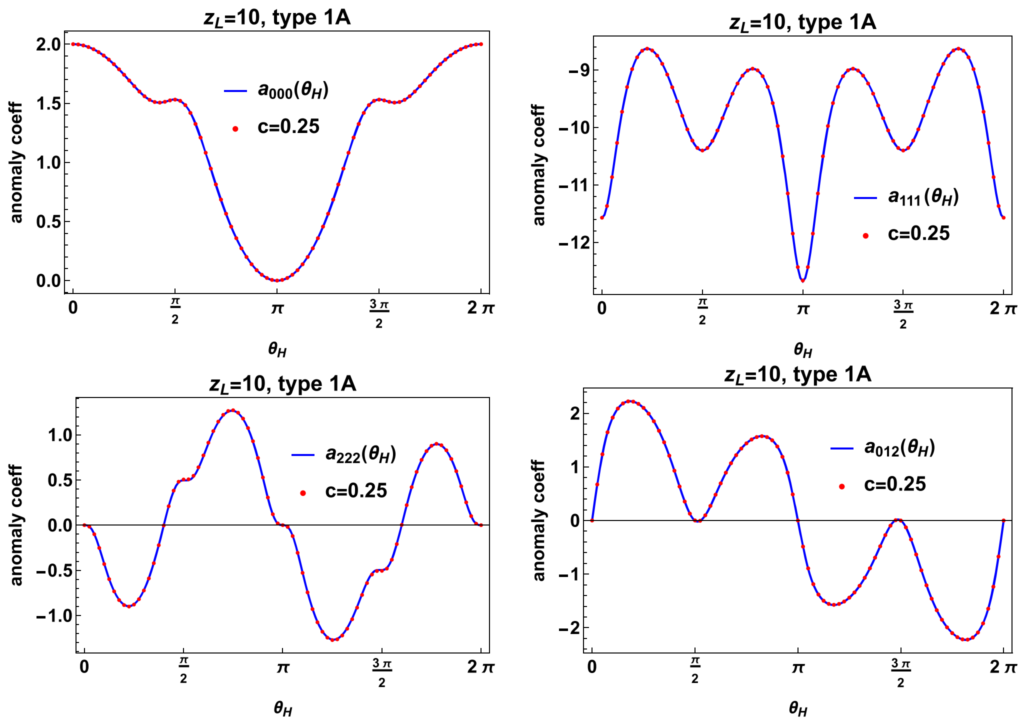

Fig. 2. The anomaly coefficients a_{000}, a_{111}, a_{222}, and a_{012} as functions of θ_H for type 1A fermions for $z_L = 10$. The blue curves represent the universal curves given by Eq. (41). The red dots represent the values determined from the gauge couplings $t^{R/L}_{n\ell m}$ $(0 \le \ell, m \le \ell_0)$ in Eq. (30) and then taking the traces of the $(\ell_0 + 1)$-dimensional matrices in Eq. (31) for fermions with $c = 0.25$ and $\ell_0 = 10$.

As $h_n(0) = h_n(L) = 0$, only the terms proportional to $k_1k_2k_3$ in Eq. (32) survive. We find the formula for the anomaly coefficients:

$$a_{n\ell m}(\theta_H, z_L) = Q_0 k_n(0)k_\ell(0)k_m(0) + Q_1 k_n(L)k_\ell(L)k_m(L),$$
$$(Q_0, Q_1) = \begin{cases} (+1, +1) & \text{(type 1A)}, \\ (-1, -1) & \text{(type 1B)}, \\ (+1, -1) & \text{(type 2A)}, \\ (-1, +1) & \text{(type 2B)}. \end{cases} \tag{41}$$

The anomaly coefficients are determined by the values of the wave functions of the gauge fields at the UV and IR branes and the parity conditions of the fermion fields.

The formula in Eq. (41) is strikingly simple. The wave function $k_n(y)$ depends on θ_H and z_L. The sum of the chiral anomalies arising from all possible fermion KK modes are summarized in terms of $k_n(0)$ and $k_n(L)$. The c-independence of those anomalies is confirmed numerically. The anomaly coefficients $a_{n\ell m}$ given by Eq. (41) are compared with those determined by first evaluating the gauge couplings $t^{R/L}_{n\ell m}$ $(0 \le \ell, m \le \ell_0)$ in Eq. (30) and then taking the traces of the $(\ell_0 + 1)$-dimensional matrices in Eq. (31). In Fig. 2 the results for a_{000}, a_{111}, a_{222}, and a_{012} are shown for type 1A fermions with $c = 0.25$, $\ell_0 = 10$, and $z_L = 10$. One sees that the numerically evaluated values for $c = 0.25$ fall on the universal curves given by Eq. (41). We have checked that the numerically evaluated values for other values of c fall on the universal curves as well.

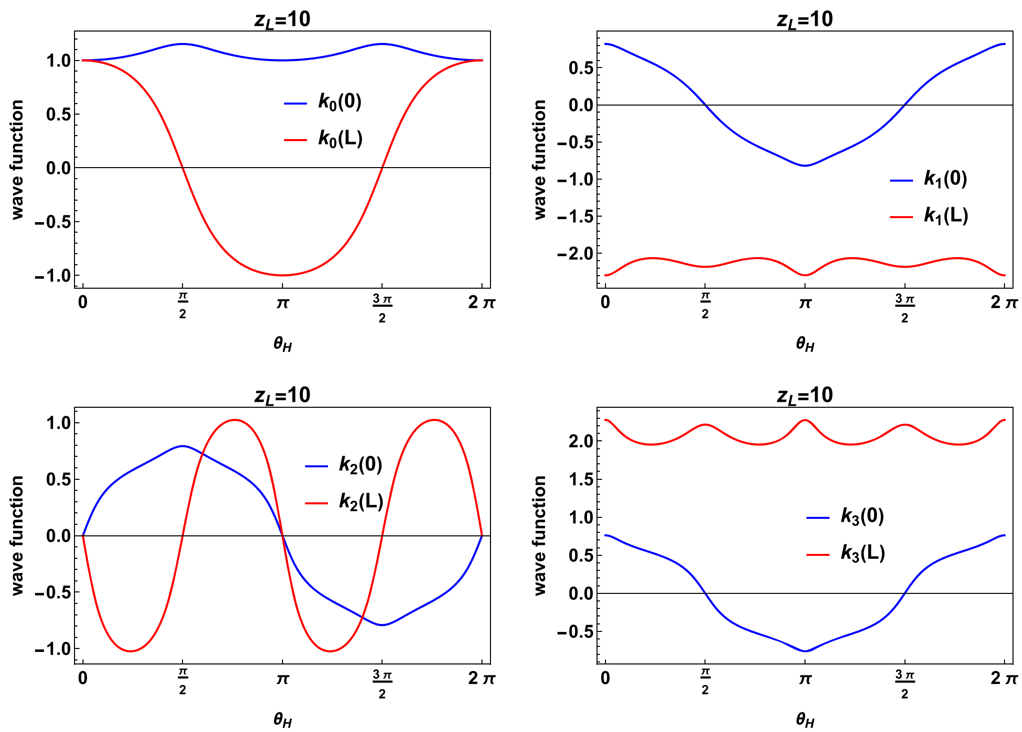

Fig. 3. The values of the gauge wave functions $k_n(y; \theta_H)$ ($n = 0, 1, 2, 3$) at $y = 0$ (blue curves) and $y = L$ (red curves) for $z_L = 10$.

Some $k_n(0; \theta_H)$ and $k_n(L; \theta_H)$ are plotted in Fig. 3. Note that for $n = 1, 3, 5, \ldots$, $|k_n(L; \theta_H)|$ is much larger than $|k_n(0; \theta_H)|$ for $z_L \geq 10$. Massless gauge bosons ($Z_\mu^{(0)}$) exist at $\theta_H = 0$ and π. $k_0(0; 0) = k_0(L; 0) = 1$ and $k_0(0; \pi) = -k_0(L; \pi) = 1$, so that $a_{000}(\theta_H = 0) = 2$ and $a_{000}(\theta_H = \pi) = 0$ for type 1A fermions and $a_{000}(\theta_H = 0) = 0$ and $a_{000}(\theta_H = \pi) = 2$ for type 2A fermions. The anomaly flow is reflected in the behavior of the wave functions of the gauge fields at $y = 0$ and L.

The dependence of the anomaly coefficients $a_{n\ell m}$ on fermion types has a simple pattern: $a_{n\ell m}(\theta_H)^{\text{type1A}} = -a_{n\ell m}(\theta_H)^{\text{type1B}}$ and $a_{n\ell m}(\theta_H)^{\text{type2A}} = -a_{n\ell m}(\theta_H)^{\text{type2B}}$. Further, $a_{n\ell m}(\theta_H + \pi)^{\text{type1A}} = a_{n\ell m}(\theta_H)^{\text{type2A}}$ or $a_{n\ell m}(\theta_H)^{\text{type2B}}$ (see Fig. 4). This follows from the property that $[k_n(0), k_n(L)]_{\theta_H+\pi} = [k_n(0), -k_n(L)]_{\theta_H}$ or $[-k_n(0), k_n(L)]_{\theta_H}$.

Formulas in the flat $M^4 \times (S^1/Z_2)$ spacetime simplify. With the KK expansions in Eqs. (8), (11), and (12), the gauge couplings are written as

$$\frac{g_4}{2} \sum_{n=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} B_\mu^{(n)}(x) \Big\{ s_{n\ell m}^R \psi_R^{(\ell)}(x)^\dagger \bar\sigma^\mu \psi_R^{(m)}(x) + s_{n\ell m}^L \psi_L^{(\ell)}(x)^\dagger \sigma^\mu \psi_L^{(m)}(x) \Big\} \tag{42}$$

for type 1A and 1B fermions. For type 2A and 2B fermions, $\psi_{R/L}^{(m)}(x)$ should be replaced by $\psi_{R/L}^{(m+\frac{1}{2})}(x)$. The anomaly coefficient associated with the three legs of $B_{\mu_1}^{(n_1)} B_{\mu_2}^{(n_2)} B_{\mu_3}^{(n_3)}$ is given by

$$\begin{aligned} b_{n_1 n_2 n_3} &= b_{n_1 n_2 n_3}^R + b_{n_1 n_2 n_3}^L, \\ b_{n_1 n_2 n_3}^R &= \mathrm{Tr} S_{n_1}^R S_{n_2}^R S_{n_3}^R, \qquad (S_n^R)_{m\ell} = s_{nm\ell}^R, \\ b_{n_1 n_2 n_3}^L &= \mathrm{Tr} S_{n_1}^L S_{n_2}^L S_{n_3}^L, \qquad (S_n^L)_{m\ell} = s_{nm\ell}^L. \end{aligned} \tag{43}$$

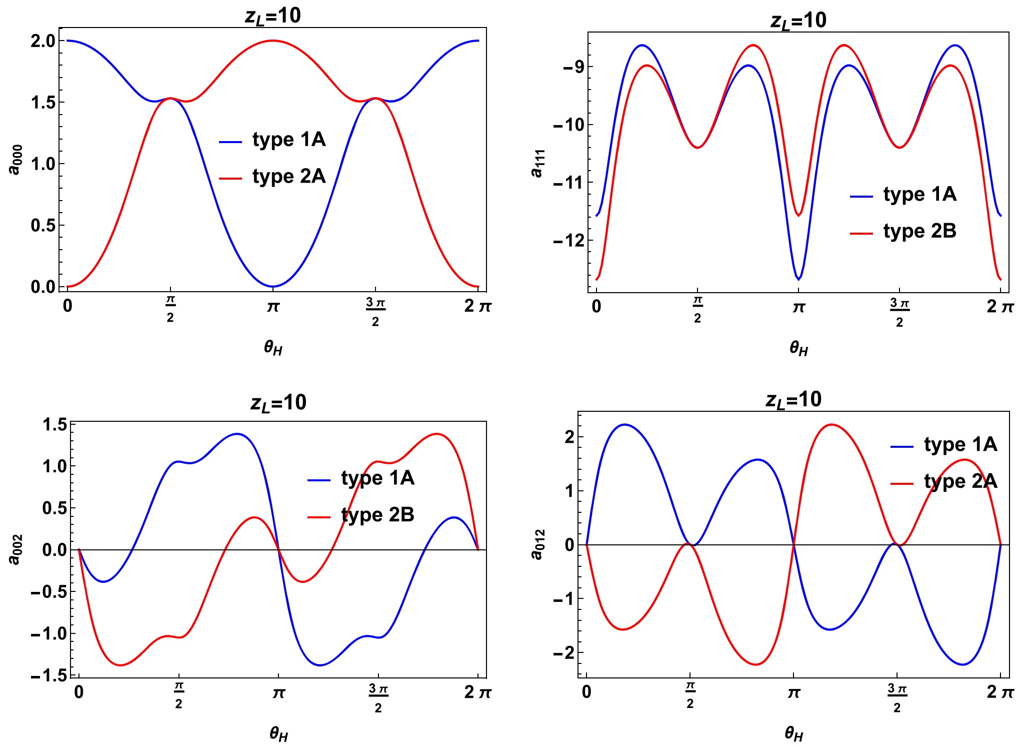

Fig. 4. The dependence of the anomaly coefficients a_{000}, a_{111}, a_{002}, and a_{012} on fermion types is shown for $z_L = 10$. One sees that $a_{n\ell m}(\theta_H + \pi)^{\text{type1A}} = a_{n\ell m}(\theta_H)^{\text{type2A}}$ or $a_{n\ell m}(\theta_H)^{\text{type2B}}$.

Applying the same argument as in the case of the RS space, one finds that

$$b_{n\ell m} = Q_0 k_n^{\text{flat}}(0) k_\ell^{\text{flat}}(0) k_m^{\text{flat}}(0) + Q_1 k_n^{\text{flat}}(L) k_\ell^{\text{flat}}(L) k_m^{\text{flat}}(L), \tag{44}$$

where Q_0 and Q_1 are given in Eq. (41). Since $k_n^{\text{flat}}(y) = \cos(n\pi y/L)$ from Eq. (8), one finds that

$$b_{n\ell m} = Q_0 + (-1)^{n+\ell+m} Q_1, \tag{45}$$

which agrees with the result in Ref. [12]. The formula in Eq. (45) also results in the flat-spacetime limit of Eq. (41). In the flat spacetime the level-crossing in the mass spectrum of gauge fields occurs at $\theta_H = 0, \pm\frac{1}{2}\pi, \pm\pi, \ldots$. For this reason the flat-spacetime limit of Eq. (41) becomes singular, as shown in Ref. [12].

4. Anomaly cancellation

The universality of the anomaly flow, expressed in Eq. (41), has a profound implication in model building, particularly in the GHU scenario. Chiral anomalies associated with gauge currents must be cancelled for the consistency of the theory in four dimensions [20,21]. The fact that the anomaly coefficients are independent of the bulk mass parameters of fermions implies that anomaly cancellation can be achieved among various distinct fermions in the theory. In this section we examine this problem in the $SU(2)$ model.

Let us first recall that the equations following from the action $I_{\rm RS}$ in Eq. (15) are, at the classical level,

$$\begin{aligned}\frac{1}{\sqrt{-\det G}}\partial_M\big(\sqrt{-\det G}F^{MN}\big) - ig_A[A_M, F^{MN}] + J^N = 0,\\ \mathcal{D}(c)\Psi = 0,\\ J^N = J^{Na}\frac{\tau^a}{2}, \qquad J^{Na} = -ig_A\overline{\Psi}\gamma^A {e_A}^N\frac{\tau^a}{2}\Psi.\end{aligned} \tag{46}$$

The current in five dimensions is covariantly conserved:

$$\frac{1}{\sqrt{-\det G}}\partial_N\big(\sqrt{-\det G}J^N\big) - ig_A[A_N, J^N] = 0. \tag{47}$$

Note that the derivative term in the fifth coordinate generates mass terms in four dimensions when expanded in the KK modes. At the quantum level an anomaly term arises on the right-hand side of Eq. (47). The four-dimensional current $j^\mu_{(n)}(x)$ which couples with $Z^{(n)}_\mu(x)$ is

$$\begin{aligned} j^\mu_{(n)}(x) &= \int_0^L dy\sqrt{-\det G}\big\{h_n(y)J^{\mu 1} + k_n(y)J^{\mu 3}\big\}\\ &= \frac{g_4}{2}\sum_{\ell=0}^\infty\sum_{m=0}^\infty\Big\{t^R_{n\ell m}\chi_R^{(\ell)}(x)^\dagger\bar\sigma^\mu\chi_R^{(m)}(x) + t^L_{n\ell m}\chi_L^{(\ell)}(x)^\dagger\sigma^\mu\chi_L^{(m)}(x)\Big\}.\end{aligned} \tag{48}$$

The divergence $\partial_\mu j^\mu_{(n)}$ picks up an anomalous term $j^{\rm anomaly}_{(n)}$ given by

$$j^{\rm anomaly}_{(n)} = -\Big(\frac{g_4}{2}\Big)^3\sum_{\ell=0}^\infty\sum_{m=0}^\infty\frac{a_{n\ell m}}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}Z^{(\ell)}_{\mu\nu}Z^{(m)}_{\rho\sigma}, \tag{49}$$

where $Z^{(\ell)}_{\mu\nu} = \partial_\mu Z^{(\ell)}_\nu - \partial_\nu Z^{(\ell)}_\mu$.

The conditions for the cancellation of the gauge anomalies are simple. Let the numbers of doublet fermions of types 1A, 1B, 2A and 2B be n_{1A}, n_{1B}, n_{2A}, and n_{2B}, respectively. It follows from Eq. (41) that the anomalies are cancelled if

$$n_{1A} = n_{1B}, \qquad n_{2A} = n_{2B}. \tag{50}$$

In the presence of brane fermions, namely fermions living only on the UV or IR brane, the conditions are generalized. Suppose that there are $\hat n_R$ right-handed and $\hat n_L$ left-handed doublet brane fermions on the UV brane at $y = 0$. As the $Z^{(n)}_\mu$ coupling of each brane fermion is given by $(g_4/2)k_n(0)$, the anomaly cancellation conditions become

$$\begin{aligned} n_{1A} - n_{1B} + n_{2A} - n_{2B} + \hat n_R - \hat n_L = 0,\\ n_{1A} - n_{1B} - n_{2A} + n_{2B} = 0.\end{aligned} \tag{51}$$

We stress that the conditions in Eqs. (50) and (51) do not depend on θ_H and z_L. Furthermore, the conditions guarantee that not only the zero mode anomaly a_{000} but also all other anomalies $a_{n\ell m}$ are cancelled at once.

Fermion multiplets in the triplet representation do not contribute to anomalies in the $SU(2)$ gauge theory, as is easily confirmed. The anomaly cancellation is achieved by the condition in Eqs. (50) or (51), namely by the condition for the numbers of doublet fermions with four types of orbifold boundary conditions. It does not depend on the AB phase θ_H, namely the vacuum expectation value of A_y. The situation is very similar to the anomaly cancellation condition in the Standard Model (SM).

5. Summary and discussions

We have examined the anomaly flow by the AB phase θ_H in the $SU(2)$ gauge theory in the RS space and in the flat $M^4 \times (S^1/Z_2)$ spacetime. The anomaly coefficients $a_{n\ell m}(\theta_H, z_L)$ induced by a fermion field in the bulk smoothly change in θ_H in the RS space. Although the gauge couplings of the fermion, $t^{R/L}_{n\ell m}(\theta_H, z_L, c)$, nontrivially depend on the bulk mass parameter c of the fermion, the total anomaly coefficients $a_{n\ell m}$ are independent of c. We have shown that those anomaly coefficients $a_{n\ell m}$ are expressed in terms of the values of the wave functions of the gauge fields at the UV and IR branes. The holographic formula in Eq. (41) manifestly exhibits the c-independence. We have confirmed that the values of the anomaly coefficients numerically evaluated directly from $t^{R/L}_{n\ell m}(\theta_H, z_L, c)$ fall precisely on the curves given by Eq. (41). It has been left for future investigation to find an analytic proof of the c-independence of Eq. (32).

As has been mentioned in the previous section, universality in anomaly flow is critically important in the construction of realistic models of particle physics. GHU models have been proposed to unify the 4D Higgs boson with gauge fields in the framework of gauge theory on five-dimensional orbifolds in which the gauge hierarchy problem is naturally solved [5,7,22–34]. In particular, $SO(5) \times U(1)_X \times SU(3)_C$ GHU in the RS space with $\theta_H \sim 0.1$ and $z_L = 10^5 \sim 10^{10}$ has been shown to reproduce nearly the same phenomenology at low energies as the SM [31,33]. As in the case of the SM, all chiral anomalies associated with gauge currents must be cancelled. Generalization of the argument on universality to the group $SO(5) \times U(1)_X \times SU(3)_C$ is necessary. Further, the technology developed in the present paper can be applied to the evaluation of anomalies of global currents such as baryon and lepton numbers. The phenomenon of anomaly flow may possibly be related to Chern–Simons terms in five dimensions [35–37]. These issues will be clarified in separate papers.

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Appendix A. Basis functions

Wave functions of gauge fields and fermions are expressed in terms of the following basis functions. For gauge fields we introduce

$$\begin{aligned}
C(z;\lambda) &= \frac{\pi}{2}\lambda z z_L F_{1,0}(\lambda z, \lambda z_L),\\
S(z;\lambda) &= -\frac{\pi}{2}\lambda z F_{1,1}(\lambda z, \lambda z_L),\\
C'(z;\lambda) &= \frac{\pi}{2}\lambda^2 z z_L F_{0,0}(\lambda z, \lambda z_L),\\
S'(z;\lambda) &= -\frac{\pi}{2}\lambda^2 z F_{0,1}(\lambda z, \lambda z_L),\\
F_{\alpha,\beta}(u,v) &\equiv J_\alpha(u)Y_\beta(v) - Y_\alpha(u)J_\beta(v),
\end{aligned} \tag{A1}$$

where $J_\alpha(u)$ and $Y_\alpha(u)$ are Bessel functions of the first and second kind. They satisfy

$$\begin{aligned}
&-z\frac{d}{dz}\frac{1}{z}\frac{d}{dz}\begin{pmatrix} C \\ S \end{pmatrix} = \lambda^2 \begin{pmatrix} C \\ S \end{pmatrix},\\
&\qquad C(z_L;\lambda) = z_L, \qquad C'(z_L;\lambda) = 0,\\
&\qquad S(z_L;\lambda) = 0, \qquad S'(z_L;\lambda) = \lambda,\\
&CS' - SC' = \lambda z.
\end{aligned} \tag{A2}$$

To express wave functions of KK modes of gauge fields, we make use of

$$\begin{aligned}
&\hat{S}(z;\lambda) = N_0(\lambda)S(z;\lambda), \qquad \hat{C}(z;\lambda) = N_0(\lambda)^{-1}C(z;\lambda),\\
&\check{S}(z;\lambda) = N_1(\lambda)S(z;\lambda), \check{C}(z;\lambda) = N_1(\lambda)^{-1}C(z;\lambda),\\
&N_0(\lambda) = \frac{C(1;\lambda)}{S(1;\lambda)}, N_1(\lambda) = \frac{C'(1;\lambda)}{S'(1;\lambda)}.
\end{aligned} \tag{A3}$$

For fermion fields with a bulk mass parameter c, we define

$$\begin{aligned}
\begin{pmatrix} C_L \\ S_L \end{pmatrix}(z;\lambda,c) &= \pm\frac{\pi}{2}\lambda\sqrt{z z_L}F_{c+\frac{1}{2},c\mp\frac{1}{2}}(\lambda z, \lambda z_L),\\
\begin{pmatrix} C_R \\ S_R \end{pmatrix}(z;\lambda,c) &= \mp\frac{\pi}{2}\lambda\sqrt{z z_L}F_{c-\frac{1}{2},c\pm\frac{1}{2}}(\lambda z, \lambda z_L).
\end{aligned} \tag{A4}$$

These functions satisfy

$$\begin{aligned}
&D_+(c)\begin{pmatrix} C_L \\ S_L \end{pmatrix} = \lambda\begin{pmatrix} S_R \\ C_R \end{pmatrix},\\
&D_-(c)\begin{pmatrix} C_R \\ S_R \end{pmatrix} = \lambda\begin{pmatrix} S_L \\ C_L \end{pmatrix}, \qquad D_\pm(c) = \pm\frac{d}{dz} + \frac{c}{z},\\
&\qquad C_R = C_L = 1, \quad S_R = S_L = 0 \qquad \text{at } z = z_L,\\
&C_L C_R - S_L S_R = 1.
\end{aligned} \tag{A5}$$

Also, $C_L(z; \lambda, -c) = C_R(z; \lambda, c)$ and $S_L(z; \lambda, -c) = -S_R(z; \lambda, c)$. To express wave functions of KK modes of fermion fields, we make use of

$$\begin{aligned}
&\hat{S}_L(z; \lambda, c) = N_L(\lambda, c) S_L(z; \lambda, c), \quad \hat{C}_L(z; \lambda, c) = N_R(\lambda, c) C_L(z; \lambda, c),\\
&\hat{S}_R(z; \lambda, c) = N_R(\lambda, c) S_R(z; \lambda, c), \quad \hat{C}_R(z; \lambda, c) = N_L(\lambda, c) C_R(z; \lambda, c),\\
&\check{S}_L(z; \lambda, c) = N_R(\lambda, c)^{-1} S_L(z; \lambda, c), \quad \check{C}_L(z; \lambda, c) = N_L(\lambda, c)^{-1} C_L(z; \lambda, c),\\
&\check{S}_R(z; \lambda, c) = N_L(\lambda, c)^{-1} S_R(z; \lambda, c), \quad \check{C}_R(z; \lambda, c) = N_R(\lambda, c)^{-1} C_R(z; \lambda, c),\\
&N_L(\lambda, c) = \frac{C_L(1; \lambda, c)}{S_L(1; \lambda, c)}, \quad N_R(\lambda, c) = \frac{C_R(1; \lambda, c)}{S_R(1; \lambda, c)}.
\end{aligned} \tag{A6}$$

Appendix B. Wave functions in RS

B.1 Gauge fields $Z_\mu^{(n)}$

The mode functions of the gauge fields $Z_\mu^{(n)}(x)$ in Eq. (21) are given by

$$\begin{aligned}
&\tilde{\mathbf{h}}_0(z) = \bar{\mathbf{h}}_0^a(z),\\
&\tilde{\mathbf{h}}_{2\ell-1}(z) = (-1)^\ell \begin{cases} \bar{\mathbf{h}}_{2\ell-1}^a(z) & (-\frac{1}{2}\pi < \theta_H < \frac{1}{2}\pi),\\ \bar{\mathbf{h}}_{2\ell-1}^b(z) & (0 < \theta_H < \pi),\\ -\bar{\mathbf{h}}_{2\ell-1}^a(z) & (\frac{1}{2}\pi < \theta_H < \frac{3}{2}\pi),\\ -\bar{\mathbf{h}}_{2\ell-1}^b(z) & (\pi < \theta_H < 2\pi),\\ \bar{\mathbf{h}}_{2\ell-1}^a(z) & (\frac{3}{2}\pi < \theta_H < \frac{5}{2}\pi), \end{cases} \qquad (\ell = 1, 2, 3, \ldots),\\
&\tilde{\mathbf{h}}_{2\ell}(z) = (-1)^\ell \begin{cases} \bar{\mathbf{h}}_{2\ell}^b(z) & (-\frac{1}{2}\pi < \theta_H < \frac{1}{2}\pi),\\ -\bar{\mathbf{h}}_{2\ell}^a(z) & (0 < \theta_H < \pi),\\ -\bar{\mathbf{h}}_{2\ell}^b(z) & (\frac{1}{2}\pi < \theta_H < \frac{3}{2}\pi),\\ \bar{\mathbf{h}}_{2\ell}^a(z) & (\pi < \theta_H < 2\pi),\\ \bar{\mathbf{h}}_{2\ell}^b(z) & (\frac{3}{2}\pi < \theta_H < \frac{5}{2}\pi), \end{cases} \qquad (\ell = 1, 2, 3, \ldots),\\
&\bar{\mathbf{h}}_n^a(z) = \frac{1}{\sqrt{r_n^a}} \begin{pmatrix} -s_H \hat{S}(z; \lambda_n) \\ c_H C(z; \lambda_n) \end{pmatrix}, \qquad \bar{\mathbf{h}}_n^b(z) = \frac{1}{\sqrt{r_n^b}} \begin{pmatrix} c_H S(z; \lambda_n) \\ s_H \check{C}(z; \lambda_n) \end{pmatrix},\\
&s_H = \sin\theta_H, \qquad c_H = \cos\theta_H,\\
&r_n = \frac{1}{kL} \int_1^{z_L} \frac{dz}{z} \big\{ |\hat{h}_n(z)|^2 + |\hat{k}_n(z)|^2 \big\} \qquad \text{for } \begin{pmatrix} \hat{h}_n(z) \\ \hat{k}_n(z) \end{pmatrix}.
\end{aligned} \tag{B1}$$

$\hat{S}$ and $\check{C}$ are given in Eq. (A3). In the above formulas, the two expressions given in an overlapping θ_H region are the same. The connection formulas are necessary as one of them fails to make sense at the boundary in θ_H.

B.2 Fermion fields $\chi_{R/L}^{(n)}$

The mode functions of the fermion fields $\chi_{R/L}^{(n)}(x)$ in Eq. (25) are given, for type 1A and $c > 0$, by

$$
\text{type 1A}: \quad \tilde{\mathbf{f}}_{R,2\ell}(z) = \begin{cases} \bar{\mathbf{f}}^a_{R,2\ell}(z) & (-\pi < \theta_H < \pi), \\ \bar{\mathbf{f}}^b_{R,2\ell}(z) & (0 < \theta_H < 2\pi), \\ -\bar{\mathbf{f}}^a_{R,2\ell}(z) & (\pi < \theta_H < 3\pi), \\ -\bar{\mathbf{f}}^b_{R,2\ell}(z) & (2\pi < \theta_H < 4\pi), \\ \bar{\mathbf{f}}^a_{R,2\ell}(z) & (3\pi < \theta_H < 5\pi), \end{cases} \qquad (\ell = 0, 1, 2, \ldots),
$$

$$
\tilde{\mathbf{f}}_{R,2\ell-1}(z) = \begin{cases} \bar{\mathbf{f}}^c_{R,2\ell-1}(z) & (-\pi < \theta_H < \pi), \\ \bar{\mathbf{f}}^d_{R,2\ell-1}(z) & (0 < \theta_H < 2\pi), \\ -\bar{\mathbf{f}}^c_{R,2\ell-1}(z) & (\pi < \theta_H < 3\pi), \\ -\bar{\mathbf{f}}^d_{R,2\ell-1}(z) & (2\pi < \theta_H < 4\pi), \\ \bar{\mathbf{f}}^c_{R,2\ell-1}(z) & (3\pi < \theta_H < 5\pi), \end{cases} \qquad (\ell = 1, 2, 3, \ldots),
$$

$$
\tilde{\mathbf{f}}_{L0}(z) = \bar{\mathbf{f}}^a_{L0}(z),
$$

$$
\tilde{\mathbf{f}}_{L,2\ell-1}(z) = \begin{cases} \bar{\mathbf{f}}^a_{L,2\ell-1}(z) & (-\pi < \theta_H < \pi), \\ \bar{\mathbf{f}}^b_{L,2\ell-1}(z) & (0 < \theta_H < 2\pi), \\ -\bar{\mathbf{f}}^a_{L,2\ell-1}(z) & (\pi < \theta_H < 3\pi), \\ -\bar{\mathbf{f}}^b_{L,2\ell-1}(z) & (2\pi < \theta_H < 4\pi), \\ \bar{\mathbf{f}}^a_{L,2\ell-1}(z) & (3\pi < \theta_H < 5\pi), \end{cases} \qquad (\ell = 1, 2, 3, \ldots),
$$

$$
\tilde{\mathbf{f}}_{L,2\ell}(z) = \begin{cases} \bar{\mathbf{f}}^c_{L,2\ell}(z) & (-\pi < \theta_H < \pi), \\ \bar{\mathbf{f}}^d_{L,2\ell}(z) & (0 < \theta_H < 2\pi), \\ -\bar{\mathbf{f}}^c_{L,2\ell}(z) & (\pi < \theta_H < 3\pi), \\ -\bar{\mathbf{f}}^d_{L,2\ell}(z) & (2\pi < \theta_H < 4\pi), \\ \bar{\mathbf{f}}^c_{L,2\ell}(z) & (3\pi < \theta_H < 5\pi), \end{cases} \qquad (\ell = 1, 2, 3, \ldots). \tag{B2}
$$

Here,

$$
\begin{aligned}
&\bar{\mathbf{f}}^a_{Rn}(z) = \frac{1}{\sqrt{r^a_n}} \begin{pmatrix} \bar{c}_H C_R(z;\lambda_n,c) \\ -\bar{s}_H \hat{S}_R(z;\lambda_n,c) \end{pmatrix}, \qquad \bar{\mathbf{f}}^b_{Rn}(z) = \frac{1}{\sqrt{r^b_n}} \begin{pmatrix} \bar{s}_H C_R(z;\lambda_n,c) \\ \bar{c}_H \check{S}_R(z;\lambda_n,c) \end{pmatrix}, \\
&\bar{\mathbf{f}}^c_{Rn}(z) = \frac{1}{\sqrt{r^c_n}} \begin{pmatrix} \bar{s}_H \hat{C}_R(z;\lambda_n,c) \\ \bar{c}_H S_R(z;\lambda_n,c) \end{pmatrix}, \qquad \bar{\mathbf{f}}^d_{Rn}(z) = \frac{1}{\sqrt{r^d_n}} \begin{pmatrix} -\bar{c}_H \check{C}_R(z;\lambda_n,c) \\ \bar{s}_H S_R(z;\lambda_n,c) \end{pmatrix}, \\
&\bar{\mathbf{f}}^a_{Ln}(z) = \frac{1}{\sqrt{r^a_n}} \begin{pmatrix} \bar{s}_H \hat{S}_L(z;\lambda_n,c) \\ \bar{c}_H C_L(z;\lambda_n,c) \end{pmatrix}, \qquad \bar{\mathbf{f}}^b_{Ln}(z) = \frac{1}{\sqrt{r^b_n}} \begin{pmatrix} -\bar{c}_H \check{S}_L(z;\lambda_n,c) \\ \bar{s}_H C_L(z;\lambda_n,c) \end{pmatrix}, \\
&\bar{\mathbf{f}}^c_{Ln}(z) = \frac{1}{\sqrt{r^c_n}} \begin{pmatrix} \bar{c}_H S_L(z;\lambda_n,c) \\ -\bar{s}_H \hat{C}_L(z;\lambda_n,c) \end{pmatrix}, \qquad \bar{\mathbf{f}}^d_{Ln}(z) = \frac{1}{\sqrt{r^d_n}} \begin{pmatrix} \bar{s}_H S_L(z;\lambda_n,c) \\ \bar{c}_H \check{C}_L(z;\lambda_n,c) \end{pmatrix}, \\
&\bar{c}_H = \cos\tfrac{1}{2}\theta_H, \qquad \bar{s}_H = \sin\tfrac{1}{2}\theta_H, \\
&r_n = \int_1^{z_L} dz \big\{ |\hat{f}_n(z)|^2 + |\hat{g}_n(z)|^2 \big\} \qquad \text{for } \begin{pmatrix} \hat{f}_n(z) \\ \hat{g}_n(z) \end{pmatrix}.
\end{aligned} \tag{B3}
$$

The functions $\hat{S}_{R/L}$, $\check{S}_{R/L}$, etc. are defined in Eq. (A6). In Eq. (B2), two expressions in an overlapping region in θ_H are the same.

B.3 Fermion fields $\chi^{(n)}_{R/L}$ for c = 0

For $c = 0$, $C_{R/L}(z;\lambda,0)$ and $S_{R/L}(z;\lambda,0)$ reduce to trigonometric functions:

$$\begin{pmatrix} C_L \\ S_L \end{pmatrix}(z;\lambda,0) = \begin{pmatrix} \cos\lambda(z-z_L) \\ \sin\lambda(z-z_L) \end{pmatrix},$$
$$\begin{pmatrix} C_R \\ S_R \end{pmatrix}(z;\lambda,0) = \begin{pmatrix} \cos\lambda(z-z_L) \\ -\sin\lambda(z-z_L) \end{pmatrix}. \tag{B4}$$

The spectrum and wave functions in $1 \le z = e^{ky} \le z_L$ in the original gauge for type 1A are:

$$\text{type 1A}: \qquad \lambda_n = \frac{1}{z_L-1}\left|n\pi + \tfrac{1}{2}\theta_H\right| \qquad (-\infty < n < \infty),$$
$$\begin{pmatrix} f_{Rn}(y) \\ g_{Rn}(y) \end{pmatrix} = \frac{1}{\sqrt{z_L-1}} \begin{pmatrix} \cos\left\{(n\pi + \tfrac{1}{2}\theta_H)\frac{z-z_L}{z_L-1} + \tfrac{1}{2}\theta(z)\right\} \\ \sin\left\{(n\pi + \tfrac{1}{2}\theta_H)\frac{z-z_L}{z_L-1} + \tfrac{1}{2}\theta(z)\right\} \end{pmatrix},$$
$$\begin{pmatrix} f_{Ln}(y) \\ g_{Ln}(y) \end{pmatrix} = \frac{1}{\sqrt{z_L-1}} \begin{pmatrix} -\sin\left\{(n\pi + \tfrac{1}{2}\theta_H)\frac{z-z_L}{z_L-1} + \tfrac{1}{2}\theta(z)\right\} \\ \cos\left\{(n\pi + \tfrac{1}{2}\theta_H)\frac{z-z_L}{z_L-1} + \tfrac{1}{2}\theta(z)\right\} \end{pmatrix}, \tag{B5}$$

and for type 2A:

$$\text{type 2A}: \qquad \lambda_n = \frac{1}{z_L-1}\left|(n+\tfrac{1}{2})\pi + \tfrac{1}{2}\theta_H\right| \qquad (-\infty < n < \infty),$$
$$\begin{pmatrix} f_{Rn}(y) \\ g_{Rn}(y) \end{pmatrix} = \frac{1}{\sqrt{z_L-1}} \begin{pmatrix} -\sin\left\{(n\pi + \tfrac{1}{2}\pi + \tfrac{1}{2}\theta_H)\frac{z-z_L}{z_L-1} + \tfrac{1}{2}\theta(z)\right\} \\ \cos\left\{(n\pi + \tfrac{1}{2}\pi + \tfrac{1}{2}\theta_H)\frac{z-z_L}{z_L-1} + \tfrac{1}{2}\theta(z)\right\} \end{pmatrix},$$
$$\begin{pmatrix} f_{Ln}(y) \\ g_{Ln}(y) \end{pmatrix} = \frac{1}{\sqrt{z_L-1}} \begin{pmatrix} \cos\left\{(n\pi + \tfrac{1}{2}\pi + \tfrac{1}{2}\theta_H)\frac{z-z_L}{z_L-1} + \tfrac{1}{2}\theta(z)\right\} \\ \sin\left\{(n\pi + \tfrac{1}{2}\pi + \tfrac{1}{2}\theta_H)\frac{z-z_L}{z_L-1} + \tfrac{1}{2}\theta(z)\right\} \end{pmatrix}. \tag{B6}$$

Note that the expressions in Eqs. (B5) and (B6) reduce, up to normalization factors, to the expressions in Eqs. (11) and (12) in the flat-spacetime limit, respectively. For other regions in y, the wave functions are defined by Eq. (27).

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