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Localisation of Dirac modes in gauge theories and Goldstone's theorem at finite temperature

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Abstract: I discuss the possible effects of a finite density of localised near-zero Dirac modes in the chiral limit of gauge theories with N_f degenerate fermions. I focus in particular on the fate of the massless quasi-particle excitations predicted by the finite-temperature version of Goldstone's theorem, for which I provide an alternative and generalised proof based on a Euclidean $\mathrm{SU}(N_f)_A$ Ward-Takahashi identity. I show that localised near-zero modes can lead to a divergent pseudoscalar-pseudoscalar correlator that modifies this identity in the chiral limit. As a consequence, massless quasi-particle excitations can disappear from the spectrum of the theory in spite of a non-zero chiral condensate. Three different scenarios are possible, depending on the detailed behaviour in the chiral limit of the ratio of the mobility edge and the fermion mass, which I prove to be a renormalisation-group invariant quantity.

Keywords: Nonperturbative Effects, Random Systems, Spontaneous Symmetry Breaking

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1 Introduction

The low-energy physics of QCD at zero temperature is largely determined by two non-perturbative phenomena, namely spontaneous chiral symmetry breaking and confinement. The spontaneous breaking of chiral symmetry in the limit of massless quarks and the associated appearance of massless Goldstone bosons explain the lightness of pions for physical quark masses and their low-energy dynamics. The origin of confinement can be seen instead in the opposite infinite-mass, "quenched" (pure gauge) limit, where a linearly rising quark-antiquark potential forces quarks to be confined inside hadrons.

At finite temperature, confinement in SU(3) pure gauge theory is signalled by a divergent quark free energy, inferred from the vanishing of the Polyakov loop expectation value, which is in turn a consequence of the centre symmetry of the theory being unbroken. For finite quark masses this symmetry is explicitly broken but only mildly, and confinement persists at low temperatures, signalled by a disordered Polyakov loop in typical gauge configurations, resulting in a small expectation value of the Polyakov loop. At low temperatures also the effects of the spontaneous breaking of chiral symmetry in the massless limit are still clearly present.

At higher temperatures QCD undergoes a rapid but analytic crossover to a phase where chiral symmetry is approximately restored, and centre symmetry is spontaneously broken (on top of the explicit breaking mentioned above) with the Polyakov loop getting ordered, and quarks and gluons are liberated in the quark-gluon plasma [1, 2]. Despite the very different origin of chiral symmetry breaking and confinement, both chiral and confining properties change dramatically at the transition. A similar situation is found quite generally in gauge theories at finite temperature [3–10], with deconfinement always leading to a "more chirally symmetric" system, but the reasons for this behaviour are still not fully understood.

The QCD crossover is characterised also by another drastic change, that of the localisation properties of the low-lying modes of the Euclidean Dirac operator. In fact, while delocalised in the low-temperature phase, these modes become spatially localised on the scale of the inverse temperature in the high-temperature phase [11–20]. More precisely, above the transition temperature the low modes are localised up to a critical point, λ_c, in the spectrum, known as "mobility edge", beyond which they are again delocalised.[1] For a recent comprehensive review of localisation in gauge theories, the interested reader can consult ref. [21].

The relation between localisation and deconfinement has been extensively studied in QCD and QCD-like gauge theories in recent years. These studies have shown that these two phenomena are indeed intimately connected, with localised modes appearing exactly at the critical point when the phase transition is a genuine (i.e., non-analytic) transition [22–31]. A qualitative understanding of this connection is provided by the "sea/islands" picture of localisation [17, 21, 26, 32, 33], according to which the "islands" of fluctuations in the "sea" of ordered Polyakov loops in the deconfined phase provide "energetically" favourable regions where the low Dirac eigenmodes can localise. This is possible since the ordering of

[1] Localised modes are most likely found also in the deep ultraviolet region of the spectrum (see footnote 22), which, however, has no physical relevance.

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the Polyakov loop around 1 opens a pseudogap in the spectrum, which is then populated by a low but finite density of localised modes. Numerical evidence for the sea/islands picture has been provided in refs. [17, 19, 20, 30]. Remarkably, the existence of an ordered phase is all that is required for this mechanism to be at work, so that one expects localisation of low Dirac modes to appear in the deconfined phase of a generic gauge theory. This has been so far confirmed in a variety of different gauge theories [22–31], supporting the existence of a very close relation between deconfinement and the localisation properties of low Dirac modes.

The connection with chiral symmetry breaking has received instead less attention in recent years, despite being the original motivation for the study of localisation in gauge theories at finite temperature [13, 14]. As is well known, chiral symmetry breaking is related to the accumulation of near-zero Dirac modes [34]. In the "disordered medium scenario" [35] this is explained in terms of the mixing of topological zero modes of the Dirac operator caused by the overlapping of instantons and anti-instantons. While the "unperturbed" zero modes are localised at finite temperature, they become delocalised after mixing, with their eigenvalues broadening into a finite band around zero. It has been observed, however, that in the high-temperature phase the low modes of topological origin are not sufficient to quantitatively explain the amount of localised modes [17, 36]. The disordered medium scenario cannot therefore fully account for localised low modes, as is made clear by the fact that these are found also in theories without instantons [24, 30]. Nonetheless, topological modes are very likely to play an important role, complementing the sea/islands picture discussed above (see ref. [36]).

It is clear, in any case, that the low-lying eigenvalues of the Dirac operator and the corresponding eigenvectors are sensitive both to confinement and chiral symmetry breaking. The study of localisation could then lead not only to a better understanding of these two phenomena individually, but also to clarify their connection, with localisation possibly providing the mechanism through which deconfinement improves the chiral symmetry properties of gauge systems with fermions.

The appearance of localised modes in the spectrum of the Dirac operator becomes somewhat less surprising if one notices its formal analogy with the Hamiltonian of a disordered system. Such systems are known to display eigenmode localisation since the seminal work of Anderson [37], and have been intensely studied in the condensed-matter community for more than sixty years (see refs. [38–43] for a review). In fact, $-i\not{D}$ can be interpreted exactly as the Hamiltonian of a quantum system in the background of random gauge fields, fluctuating according to the distribution determined by the path-integral integration measure. With this insight, the critical features of localisation observed at the mobility edge [44–47] are understood as a consequence of universality, and of suitable symmetry considerations. On the other hand, the appearance of localised modes at the band centre (i.e., near zero) only in the high-temperature phase is surprising, calling for an explanation that is qualitatively provided by the sea/islands picture mentioned above.

The formal analogy with disordered systems, however, does not translate directly into an analogy between physical phenomena observed on its gauge theory side and on its condensed matter side. The reason for this is that, while energy levels and eigenvectors of a

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Hamiltonian have a direct physical meaning in condensed matter systems, it is not so for the Dirac eigenmodes in gauge theories, where observables are obtained only after summing over all the modes. The properties of individual points, and perhaps even of regions of the Dirac spectrum, are then difficult to connect to phenomenology. A notable exception to this state of affairs is the chiral limit of massless quarks, where only near-zero modes have physical relevance. This is famously exemplified by the Banks-Casher relation [34], stating that the chiral condensate is proportional to the density of near-zero Dirac modes in the chiral limit. It is then possible that in this limit the localisation properties of near-zero modes have clear observable consequences.

The investigation of this issue is made more pressing by the known consequences of localisation for the Goldstone excitations associated with the spontaneous breaking of a continuous symmetry [48] also at finite temperature [49–53]. It has been known for a long time that localisation can lead to the disappearance of Goldstone modes in non-relativistic disordered systems [54]. This phenomenon has been rediscovered more recently in the context of relativistic lattice gauge theories at zero temperature [55], although for an unphysical system, namely quenched SU(3) gauge theory with Wilson fermions, where it explains the disappearance of Goldstone bosons in the supercritical region outside the Aoki phase [56]. Localisation and the fate of quasi-particle Goldstone excitations in finite-temperature gauge theories have been studied only recently in refs. [57, 58], reaching similar conclusions: the presence of a finite density of localised near-zero modes possibly leads to the disappearance of Goldstone excitations from the spectrum of the theory.

In the absence of concrete models where localised near-zero modes are explicitly shown to be sufficiently dense, the results of refs. [57, 58] would probably be only of academic interest, closing a loophole in the proofs of Goldstone's theorem at zero and finite temperature. There are, however, interesting results that could make them more than simply a curiosity. The most intriguing one is the peak of localised near-zero modes in QCD right above the pseudocritical temperature, observed on the lattice studying overlap spectra in the background of HISQ configurations for near-physical quark masses [59]. While localisation properties have not been studied further, this peak has been observed to persist also at lower-than-physical light-quark masses [60–63]. This peak is usually ascribed to the topological modes mentioned above, and it is argued [59–61, 64] that it should shrink in the chiral limit as topological excitations become a non-interacting gas, so that its effects disappear except for what concerns the anomalous $\mathrm{U}(1)_A$ symmetry. However, it is not clear what mechanism should lead to instantons and anti-instantons forming a free gas in the chiral limit, and the evidence for the suggested behaviour of the peak is not conclusive. While there is no conclusive evidence for the peak surviving as a finite peak either, the possibility should not be excluded at this stage that a finite density of localised near-zero modes is found in the chiral limit.

Another interesting finding is the presence of two separate phase transitions in SU(3) gauge theory with $N_f = 2$ flavours of massless adjoint quarks [5, 6]. In this theory a deconfining transition at temperature T_{dec} separates a low-temperature, confining phase where chiral symmetry is broken by a non-zero chiral condensate, and an intermediate, deconfined phase where chiral symmetry is still broken but the chiral condensate is reduced.

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A second transition at a higher temperature $T_\chi > T_{\rm dec}$ separates this phase from the high-temperature, deconfined and chirally-restored phase. In the intermediate phase one surely finds a finite density of near-zero modes via the Banks-Casher relation; since this phase is deconfined, the sea/islands picture suggests that these modes will be localised, but no more or less direct evidence of this exists. Moreover, the results of refs. [5, 6] are consistent with the presence of massless Goldstone excitations. It is possible that, as a consequence of the relatively small volumes employed in those studies, localisation and its consequences could not manifest yet, and so larger volumes would be needed to make conclusive statements. However, if confirmed in larger volumes, their results do not contradict the analysis of refs. [57, 58], and a detailed study of correlation functions would be needed to test quantitatively the scenarios proposed there.

Although the basic approach is very similar to that of refs. [54, 55], the argument of refs. [57, 58] requires to deal with a certain number of technical complications, mostly originating in the loss of full O(4) invariance at finite temperature. While these complications can be overcome, they were discussed only briefly in refs. [54, 55], not to obscure the main points. In this paper I present the argument in full depth, providing details on the various aspects of the calculation. Before being able to discuss the fate of Goldstone quasi-particles, a few intermediate results need to be derived, that I believe are of interest in their own right. These include:

- a "Euclidean" derivation of Goldstone's theorem at finite temperature, and a generalisation thereof, in the case of broken non-singlet axial flavour symmetry, based on the corresponding Ward-Takahashi identities;
- the possible appearance in the chiral limit of a $1/m$ divergence in the pseudoscalar-pseudoscalar correlator, where m is the quark mass, in the presence of a finite density of localised near-zero modes;
- the renormalisation of the spectral correlators appearing in the mode decomposition of the pseudoscalar-pseudoscalar correlator, and the related proof that the ratio λ_c/m between the mobility edge and the quark mass is a renormalisation-group invariant quantity.

After a brief review of finite-temperature quantum field theory in section 2, and of gauge theories in section 3, the alternative proof of Goldstone's theorem at finite temperature is discussed in section 4. The study of the pseudoscalar-pseudoscalar correlator is reported in section 5, including the renormalisation of the spectral correlators and of the mobility edge, and the fate of Goldstone excitations in the presence of localised modes. Conclusions and discussion of future studies are found in section 6. To improve readability, most of the technical details are relegated to appendices A to H, which include: details on analyticity and reality properties of Euclidean correlation functions (appendix A); derivation of non-singlet axial Ward-Takahashi identities (appendix B) and their renormalisation (appendix C); another "Euclidean" proof of Goldstone's theorem at finite temperature in coordinate space (appendix D); details about the pseudoscalar-pseudoscalar correlator calculation, in particular estimates of the contribution of exponentially localised modes (appendix E),

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the study of the chiral limit (appendix F) and the renormalisation of the corresponding spectral correlators (appendix G), and the discussion of a bound on their large-distance behaviour (appendix H).

2 Finite-temperature quantum field theory

In this section I review a few relevant aspects of quantum field theory at finite temperature, mostly to set the notation. See, e.g., refs. [53, 65–71] for further details. The expectation value of an observable O for a system in thermal equilibrium at temperature $T = 1/\beta$ is obtained as follows from the density matrix of the canonical ensemble,

$$\langle\!\langle \hat{O} \rangle\!\rangle_\beta = \lim_{V\to\infty} \langle\!\langle \hat{O} \rangle\!\rangle_{\beta,V} \equiv \lim_{V\to\infty} \frac{\operatorname{Tr} e^{-\beta \hat{H}_V} \hat{O}}{\operatorname{Tr} e^{-\beta \hat{H}_V}} , \tag{2.1}$$

where $\hat{H}_V$ is the finite-volume Hamiltonian, and the volume V of the system is eventually sent to infinity in the thermodynamic limit. Here a caret denotes operators acting on the (zero-temperature) Hilbert space of the system, and Tr the trace over this space.

For (possibly composite) local bosonic field operators $\hat{\phi}_{1,2}(x)$, $x = (t, \vec{x})$, one defines the thermal (real-time) two-point correlation functions

$$\begin{aligned} G^{(+)}_{\phi_1\phi_2}(t,\vec{x}) &\equiv \langle\!\langle \hat{\phi}_1(t,\vec{x}) \hat{\phi}_2(0) \rangle\!\rangle_\beta \equiv \lim_{V\to\infty} \langle\!\langle e^{it\hat{H}_V} \hat{\phi}_1(0,\vec{x}) e^{-it\hat{H}_V} \hat{\phi}_2(0) \rangle\!\rangle_{\beta,V} , \\ G^{(-)}_{\phi_1\phi_2}(t,\vec{x}) &\equiv \langle\!\langle \hat{\phi}_2(0) \hat{\phi}_1(t,\vec{x}) \rangle\!\rangle_\beta \equiv \lim_{V\to\infty} \langle\!\langle \hat{\phi}_2(0) e^{it\hat{H}_V} \hat{\phi}_1(0,\vec{x}) e^{-it\hat{H}_V} \rangle\!\rangle_{\beta,V} . \end{aligned} \tag{2.2}$$

Throughout this paper, an arrow denotes collectively the three spatial components of a four-vector. Here and everywhere else in this paper, the temporal evolution of $\hat{\phi}_{1,2}(t,\vec{x})$ in the infinite-volume limit (also for complex time argument) is understood in the sense of the weak limit of H_V implied by eq. (2.2). Under suitable convergence conditions on the infinite-volume limit of the temporal evolution, the correlation functions eq. (2.2) satisfy [72] the KMS condition [73, 74]. Together with the relativistic locality condition, $[\hat{\phi}_i(x), \hat{\phi}_j(y)] = 0$ for spacelike Minkowski separation $(x-y)^2 < 0$, this implies that $G^{(\pm)}_{\phi_1\phi_2}(t,\vec{x})$ are the boundary values of an analytic function $G_{\phi_1\phi_2}(z,\vec{x})$, analytic in the cut complex plane $\mathcal{D} \equiv \mathbb{C}\backslash\{z \mid |\operatorname{Re} z| \geq |\vec{x}|, \operatorname{Im} z = n\beta, n \in \mathbb{Z}\}$, and furthermore periodic in the imaginary z direction with period β, $G_{\phi_1\phi_2}(z + in\beta, \vec{x}) = G_{\phi_1\phi_2}(z,\vec{x})$ (see refs. [53, 67–69]). The correlation functions $G^{(\pm)}_{\phi_1\phi_2}(t,\vec{x})$ for $t \in \mathbb{R}$ are then recovered as

$$\begin{aligned} G^{(+)}_{\phi_1\phi_2}(t,\vec{x}) &= G_{\phi_1\phi_2}(t - i\epsilon, \vec{x}) , \\ G^{(-)}_{\phi_1\phi_2}(t,\vec{x}) &= G_{\phi_1\phi_2}(t + i\epsilon, \vec{x}) = G_{\phi_1\phi_2}(t - i(\beta - \epsilon), \vec{x}) , \end{aligned} \tag{2.3}$$

where the limit $\epsilon \to 0^+$ at the end of the calculation is understood.

In particular, $G^{(\pm)}_{\phi_1\phi_2}$ can be obtained by analytic continuation from the Euclidean correlation function $\mathcal{G}_{\phi_1\phi_2}(t,\vec{x})$,

$$\begin{aligned} \mathcal{G}_{\phi_1\phi_2}(t,\vec{x}) &\equiv \langle\!\langle \theta(t) \hat{\phi}_1(-it,\vec{x}) \hat{\phi}_2(0) + \theta(-t) \hat{\phi}_2(0) \hat{\phi}_1(-it,\vec{x}) \rangle\!\rangle_\beta \\ &\equiv \langle\!\langle T\{\hat{\phi}_{E1}(t,\vec{x}) \hat{\phi}_{E2}(0)\} \rangle\!\rangle_\beta , \end{aligned} \tag{2.4}$$

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where $\hat{\phi}_{E1,2}(t,\vec{x}) \equiv \hat{\phi}_{1,2}(-it,\vec{x})$, and T stands for time-ordering in Euclidean (imaginary) time t. This function is the restriction to real z of $\mathcal{G}_{\phi_1\phi_2}(z,\vec{x}) = G_{\phi_1\phi_2}(-iz,\vec{x})$, analytic in the cut complex plane $i\mathcal{D}$, and periodic in the real direction, $\mathcal{G}_{\phi_1\phi_2}(z+n\beta,\vec{x}) = \mathcal{G}_{\phi_1\phi_2}(z,\vec{x})$. One then has

$$G^{(+)}_{\phi_1\phi_2}(t,\vec{x}) = \mathcal{G}_{\phi_1\phi_2}(\epsilon+it,\vec{x})\,, \qquad G^{(-)}_{\phi_1\phi_2}(t,\vec{x}) = \mathcal{G}_{\phi_1\phi_2}(-\epsilon+it,\vec{x})\,. \tag{2.5}$$

In particular, the thermal expectation value of the commutator $[\hat{\phi}_1(x),\hat{\phi}_2(0)]$ equals the discontinuity of $\mathcal{G}_{\phi_1\phi_2}$ along the imaginary axis (corresponding to real time),

$$\begin{aligned}\langle\!\langle[\hat{\phi}_1(t,\vec{x}),\hat{\phi}_2(0)]\rangle\!\rangle_\beta &= G^{(+)}_{\phi_1\phi_2}(t,\vec{x}) - G^{(-)}_{\phi_1\phi_2}(t,\vec{x}) = \mathcal{G}_{\phi_1\phi_2}(\epsilon+it,\vec{x}) - \mathcal{G}_{\phi_1\phi_2}(-\epsilon+it,\vec{x}) \\ &= \mathcal{G}_{\phi_1\phi_2}(\epsilon+it,\vec{x}) - \mathcal{G}_{\phi_1\phi_2}(\beta-\epsilon+it,\vec{x})\,,\end{aligned} \tag{2.6}$$

where $t \in \mathbb{R}$ and the second line follows from periodicity. As a further consequence of periodicity in Euclidean time, $\mathcal{G}_{\phi_1\phi_2}(t,\vec{x})$ with $t \in \mathbb{R}$ can be written as a mixed Fourier sum/Fourier transform as follows,

$$\mathcal{G}_{\phi_1\phi_2}(t,\vec{x}) = \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\, e^{-i(\omega_n t+\vec{k}\cdot\vec{x})}\tilde{\mathcal{G}}_{\phi_1\phi_2}(\omega_n,\vec{k})\,, \tag{2.7}$$

where ω_n are the bosonic Matsubara frequencies, $\omega_n = \frac{2\pi n}{\beta}$, $n \in \mathbb{Z}$, and

$$\tilde{\mathcal{G}}_{\phi_1\phi_2}(\omega_n,\vec{k}) \equiv \int_0^\beta dt \int d^3x\, e^{i(\omega_n t+\vec{k}\cdot\vec{x})}\mathcal{G}_{\phi_1\phi_2}(t,\vec{x}) \equiv \int_\beta d^4x\, e^{i(\omega_n t+\vec{k}\cdot\vec{x})}\mathcal{G}_{\phi_1\phi_2}(t,\vec{x})\,, \tag{2.8}$$

with the subscript β denoting compactification of the temporal direction.

A central role in this paper is played by the spectral function,

$$\begin{aligned}\tilde{\rho}_{\phi_1\phi_2}(\omega,\vec{k}) &\equiv \int d^4x\, e^{i(\omega t-\vec{k}\cdot\vec{x})}\langle\!\langle\left[\hat{\phi}_1(t,\vec{x}),\hat{\phi}_2(0)\right]\rangle\!\rangle_\beta \\ &= \int d^4x\, e^{i(\omega t-\vec{k}\cdot\vec{x})}\left(\mathcal{G}_{\phi_1\phi_2}(\epsilon+it,\vec{x}) - \mathcal{G}_{\phi_1\phi_2}(-\epsilon+it,\vec{x})\right)\,,\end{aligned} \tag{2.9}$$

and by the closely related retarded and advanced propagators,

$$\begin{aligned}\tilde{r}_{\phi_1\phi_2}(\omega,\vec{k}) &\equiv i\int d^4x\, e^{i(\omega t-\vec{k}\cdot\vec{x})}\theta(t)\langle\!\langle\left[\hat{\phi}_1(t,\vec{x}),\hat{\phi}_2(0)\right]\rangle\!\rangle_\beta\,, \\ \tilde{a}_{\phi_1\phi_2}\left(\omega,\vec{k}\right) &\equiv -i\int d^4x\, e^{i\left(\omega t-\vec{k}\cdot\vec{x}\right)}\theta\left(-t\right)\langle\!\langle\left[\hat{\phi}_1(t,\vec{x}),\hat{\phi}_2(0)\right]\rangle\!\rangle_\beta\,.\end{aligned} \tag{2.10}$$

These are the boundary values of analytic functions, analytic respectively for $\mathrm{Im}\,\omega > 0$ and $\mathrm{Im}\,\omega < 0$, from which one obtains the spectral function as follows,

$$\tilde{\rho}_{\phi_1\phi_2}(\omega,\vec{k}) = -i\left(\tilde{r}_{\phi_1\phi_2}\left(\omega+i\epsilon,\vec{k}\right) - \tilde{a}_{\phi_1\phi_2}\left(\omega-i\epsilon,\vec{k}\right)\right)\,. \tag{2.11}$$

More directly, one can use the analytic continuation relations between the retarded and advanced propagators and the Fourier coefficients of the Euclidean correlator, $\tilde{\mathcal{G}}_{\phi_1\phi_2}(\omega_n,\vec{k})$, to reconstruct $\tilde{r}_{\phi_1\phi_2}$ and $\tilde{a}_{\phi_1\phi_2}$, and so $\tilde{\rho}_{\phi_1\phi_2}$, by analytic interpolation in the sense of

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Carlson's theorem, under reasonable hypotheses of moderate asymptotic growth in the complex time variable [68, 69]. For $n \neq 0$ one has [68–71]

$$\tilde{\mathcal{G}}_{\phi_1\phi_2}\left(\omega_n, \vec{k}\right) = \begin{cases} \tilde{r}_{\phi_1\phi_2}\left(i\omega_n, -\vec{k}\right) , & n > 0 , \\ \tilde{a}_{\phi_1\phi_2}\left(i\omega_n, -\vec{k}\right) , & n < 0 . \end{cases} \tag{2.12}$$

These results apply also to the case $n = 0$ if the spectral function is regular at the origin. If instead there is a transport peak, i.e., if

$$\tilde{\rho}_{\phi_1\phi_2}\left(\omega, \vec{k}\right) = 2\pi A_{\phi_1\phi_2}\left(\vec{k}\right)\omega\delta\left(\omega\right) + B_{\phi_1\phi_2}\left(\omega, \vec{k}\right) , \tag{2.13}$$

with $B_{\phi_1\phi_2}$ regular at $\omega = 0$, one has [70, 71]

$$\tilde{\mathcal{G}}_{\phi_1\phi_2}\left(0, \vec{k}\right) - \tilde{r}_{\phi_1\phi_2}\left(i\epsilon, -\vec{k}\right) = \tilde{\mathcal{G}}_{\phi_1\phi_2}\left(0, \vec{k}\right) - \tilde{a}_{\phi_1\phi_2}\left(-i\epsilon, -\vec{k}\right) = A_{\phi_1\phi_2}\left(-\vec{k}\right) . \tag{2.14}$$

Further details are given in appendix A.

Time-ordered Euclidean field correlators at finite temperature can be expressed in terms of a path integral (see, e.g., refs. [65, 66]),

$$\langle\langle T\left\{\hat{\phi}_{E1}\left(x_1\right)\ldots\hat{\phi}_{En}\left(x_n\right)\right\}\rangle\rangle_\beta = \langle\phi_1\left(x_1\right)\ldots\phi_n\left(x_n\right)\rangle_\beta \equiv \frac{\int_\beta\left[\mathscr{D}\chi\right]e^{-S_E[\chi]}\phi_1\left[\chi\left(x_1\right)\right]\ldots\phi_n\left[\chi\left(x_n\right)\right]}{\int_\beta\left[\mathscr{D}\chi\right]e^{-S_E[\chi]}} , \tag{2.15}$$

where S_E is a suitable Euclidean action, and the path integration $\int_\beta[\mathscr{D}\chi]$ is over sets of bosonic (c-number) and fermionic (Grassmann) field variables that are respectively periodic or antiperiodic in the time direction. Thermal correlation functions are then reconstructed by means of analytic continuation in the time coordinate according to eq. (2.5).

In the axiomatic setting, the analyticity properties of the real-time correlation functions follow from the general properties expected of quantum fields, and so the properties of the imaginary-time correlators are a consequence of the basic assumptions (see, e.g., ref. [68]). If one instead takes the Euclidean theory defined by eq. (2.15) as the starting point, these properties become necessary conditions that the theory must satisfy in order to be able to ultimately obtain a local relativistic quantum field theory. An example is discussed below in section 4.2.

3 Gauge theories with Dirac fermions

In this section I briefly describe gauge theories with Dirac fermions quantised in the path-integral approach. The discussion is quite general, and applies to four-dimensional gauge theories of compact Lie groups, minimally coupled to N_f degenerate "flavours" of Dirac fermions of mass m, transforming in some representation of the group, at finite temperature and in the imaginary-time formalism. The fermionic part of the Euclidean action is $S_{\mathrm{F}} = \int_\beta d^4x\,\mathscr{L}_{\mathrm{F}}$, with $\mathscr{L}_{\mathrm{F}}$ the following Euclidean Lagrangean density,

$$\mathscr{L}_{\mathrm{F}} = \bar{\psi}(\not{D} + m)\psi , \qquad \not{D} = \gamma_\mu D_\mu , \qquad D_\mu = \partial_\mu + igB_\mu . \tag{3.1}$$

Here ψ and $\bar{\psi}$ denote collectively two independent sets of Grassmann variables $\psi_{f\eta c}(x)$ and $\bar{\psi}_{f\eta c}(x)$, with flavour index $f = 1, \ldots, N_f$, "Dirac" index $\eta = 1, \ldots, 4$, and "colour"

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index $c = 1, \ldots, N_c$ corresponding to the gauge group representation. In eq. (3.1) and in the following, summation over the suppressed discrete indices, as well as over repeated explicit indices, is understood, unless explicitly stated otherwise. The (generally non-Abelian) Hermitean gauge fields $B_\mu(x) = B_\mu^\dagger(x)$ read $B_\mu = B_\mu^a T^a$, with B_μ^a real fields and $T^a = T^{a\dagger}$ a set of $N_c \times N_c$ matrices providing a representation of the gauge group generators. Both B_μ and the Dirac operator $\not{D}$ in the background of the gauge field act trivially on flavour space. The Euclidean Hermitean gamma matrices $\gamma_\mu = \gamma_\mu^\dagger$, $\mu = 1, \ldots, 4$, satisfy the anticommutation relations $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$; the fifth gamma matrix $\gamma_5 \equiv -\gamma_1\gamma_2\gamma_3\gamma_4$ satisfies $\gamma_5 = \gamma_5^\dagger$, $\gamma_5^2 = \mathbf{1}$, and $\{\gamma_5, \gamma_\mu\} = 0$.

Expectation values are formally defined in terms of a path integral as

$$\begin{aligned} \langle \mathcal{O} \rangle_\beta &\equiv Z_\beta^{-1} \int_\beta [\mathscr{D}B]\, e^{-S_{\mathrm{G}}[B]} \int_\beta [\mathscr{D}\psi]\,[\mathscr{D}\bar{\psi}]\, e^{-S_{\mathrm{F}}[\psi,\bar{\psi},B]}\, \mathcal{O}[\psi,\bar{\psi},B]\,, \\ Z_\beta &\equiv \int_\beta [\mathscr{D}B]\, e^{-S_{\mathrm{G}}[B]} \int_\beta [\mathscr{D}\psi]\,[\mathscr{D}\bar{\psi}]\, e^{-S_{\mathrm{F}}[\psi,\bar{\psi},B]}\,. \end{aligned} \tag{3.2}$$

Following for definiteness the Faddeev-Popov-De Witt gauge-invariant approach as in ref. [75], S_{G} is the gauge part of the action, including the usual Yang-Mills action and a gauge-fixing term, while the Faddeev-Popov determinant necessary to restore gauge invariance is included in the integration measure $\int_\beta[\mathscr{D}B]$.[2] Here the subscript β denotes the periodicity condition $B_\mu(\beta, \vec{x}) = B_\mu(0, \vec{x})$ to be imposed on the gauge fields. Integration over the fermion fields is done in the sense of Berezin integration of Grassmann variables, over field configurations satisfying the antiperiodicity conditions $\psi(\beta, \vec{x}) = -\psi(0, \vec{x})$ and $\bar{\psi}(\beta, \vec{x}) = -\bar{\psi}(0, \vec{x})$, denoted again by the subscript β. Since the fermionic action is quadratic, one formally has

$$\int_\beta [\mathscr{D}\psi]\,[\mathscr{D}\bar{\psi}]\, e^{-S_{\mathrm{F}}[\psi,\bar{\psi},B]} = \mathrm{Det}(\not{D}[B] + m)\,, \tag{3.3}$$

where Det denotes the functional determinant, and the dependence on the gauge fields has been made explicit. In general, after integrating out the fermion fields one is left with

$$\begin{aligned} \langle \mathcal{O} \rangle_\beta &= Z_\beta^{-1} \int_\beta [\mathscr{D}B]\, e^{-S_{\mathrm{G}}[B]} \mathrm{Det}(\not{D}[B] + m) \mathcal{O}_{\mathrm{G}}[B] \equiv \langle \mathcal{O}_{\mathrm{G}} \rangle_\beta\,, \\ \mathcal{O}_{\mathrm{G}}[B] &\equiv \left(\mathrm{Det}(\not{D}[B] + m)\right)^{-1} \int_\beta [\mathscr{D}\psi]\,[\mathscr{D}\bar{\psi}]\, e^{-S_{\mathrm{F}}[\psi,\bar{\psi},B]}\, \mathcal{O}[\psi,\bar{\psi},B]\,, \end{aligned} \tag{3.4}$$

with $\mathcal{O}_G$ built out of gauge fields and of fermionic propagators in a fixed gauge-field background, $(\not{D}[B] + m)^{-1}$.

3.1 Dirac eigenmodes and localisation

The eigenmodes of the Euclidean Dirac operator play an essential role in this paper. The Dirac operator is anti-Hermitean, with purely imaginary eigenvalues. At finite temperature

[2] Strictly speaking, the gauge-invariant continuum integration measure is not well defined beyond perturbation theory due to the existence of Gribov copies [76, 77]. This issue is dealt with by first formulating the theory in a gauge-invariant way on a lattice and taking eventually the continuum limit. The symbol $\int_\beta[\mathscr{D}B]$ should then be understood as a shorthand for this procedure. Here I prefer to avoid the technical complications of the lattice formulation for the sake of clarity, as they pose no obstacle to the development of the main argument (see section 3.5 for further comments).

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and in a finite spatial box of volume V, imposing periodic spatial boundary conditions on the gauge fields to preserve translation invariance, these eigenvalues form a discrete set $\{i\lambda_n\}$, $\lambda_n \in \mathbb{R}$, with corresponding eigenvectors ψ_n, $\not{D}\psi_n = i\lambda_n\psi_n$, obeying antiperiodic boundary conditions in the temporal direction and periodic boundary conditions in the spatial directions. Since the Dirac operator is trivial in flavour space, one treats the eigenmodes as carrying only Dirac and colour indices on top of the spacetime coordinate, $\psi_n = \psi_{n\,\eta c}(x)$. For future utility I introduce the following notation,

$$(\psi_{n'}(x), \Gamma\psi_n(x)) \equiv \sum_{\eta',c',\eta,c} \psi_{n'\,\eta' c'}(x)^* \Gamma_{\eta' c'\,\eta c} \psi_{n\,\eta c}(x)\,, \qquad \|\psi_n(x)\|^2 \equiv (\psi_n(x), \psi_n(x))\,, \tag{3.5}$$

i.e., the scalar product $(\cdot,\cdot)$ is restricted to Dirac and colour space, while the coordinate x is kept fixed. Throughout this paper I will always assume that Dirac modes have been orthonormalised, i.e.,

$$\int_\beta d^4x\, (\psi_{n'}(x), \psi_n(x)) = \delta_{n'n}\,. \tag{3.6}$$

Due to the chiral property $\{\gamma_5, \not{D}\} = 0$, nonzero eigenvalues appear in pairs $\pm i\lambda_n$, with corresponding eigenvectors ψ_n and $\gamma_5\psi_n$.[3] Moreover, one can choose the zero modes ψ_{n_0}, $\not{D}\psi_{n_0} = 0$, to have definite chirality, i.e., to obey $\gamma_5\psi_{n_0} = \xi_{n_0}\psi_{n_0}$ with $\xi_{n_0} = \pm 1$. For the fermionic determinant, eq. (3.3), one formally has in terms of the Dirac eigenvalues $\mathrm{Det}(\not{D} + m) = \prod_n (i\lambda_n + m)^{N_f} = m^{N_0 N_f} \prod_{n,\lambda_n>0} (\lambda_n^2 + m^2)^{N_f}$, with N_0 the number of exact zero modes.

Up to an unimportant factor of i, $\not{D}[B]$ for a fixed background field can be seen as the Hamiltonian of a four-dimensional quantum-mechanical system evolving in an additional, fictitious time. This system is effectively three-dimensional due to the compactification of the temporal direction. Moreover, for purely fermionic observables $\mathcal{O}$ the corresponding $\mathcal{O}_{\mathrm{G}}$ in eq. (3.4) can be expressed in terms of the eigenvalues and eigenvectors of this Hamiltonian, with the remaining integration being in practice a (gauge-invariant) average over the background gauge fields. This is formally identical to the ensemble average of a disordered system with (Hermitean) Hamiltonian $-i\not{D}[B]$ with energy levels $\lambda_n[B]$ and eigenvectors $\psi_n[B]$, with gauge field configurations providing different realisations of disorder, distributed according to the probability distribution determined by the path-integral measure after fermions have been integrated out, eq. (3.4).

It is well known that for disordered systems the eigenmodes can become localised, as was first realised by Anderson in his seminal paper [37]. Typically, localised and delocalised modes are found in disjoint spectral regions, separated by so-called *mobility edges* where a second-order phase transition takes place, known as *Anderson transition*. The subject of Anderson localisation and Anderson transitions has been and still is a very active area of research in condensed matter physics (see refs. [38–43] for a review), and has recently become of interest also in high-energy physics after the observation of localised modes in the high-temperature phase of gauge theories on the lattice [11–33, 36, 44–47].

[3] In case of degenerate eigenvalues one can always choose suitable eigenvectors that satisfy this relation.

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For the purposes of this paper, very little information is needed about localisation, besides the generic characterisation of localised and delocalised modes. Qualitatively, localised modes are supported essentially only in a finite spatial region whose size remains basically unchanged as the system size is increased. Delocalised modes, instead, extend over the whole system and keep spreading out as the system size grows, although not necessarily at the same rate. More precisely, the localisation properties of modes in a given spectral region are determined quantitatively by the scaling with the volume of the *inverse participation ratio*,

$$\mathrm{IPR}_n = \int_\beta d^4x \, \|\psi_n(x)\|^4 \,, \tag{3.7}$$

averaged over gauge configurations (i.e., realisations of disorder). Notice that since $\gamma_5^2 = \mathbf{1}$, one has that $\|\psi_n(x)\|^2 = \|\gamma_5\psi_n(x)\|^2$, and so ψ_n and $\gamma_5\psi_n$ have the same IPR. Working in a box of fixed temporal size β and varying spatial volume V, if modes in the given spectral region are typically non-negligible only in a spatial region of size $O(V^\alpha)$ one finds

$$\mathrm{IPR} \sim V^\alpha \left(V^{-\alpha}\right)^2 = V^{-\alpha} \tag{3.8}$$

where the exponent α is the *fractal dimension* of the modes.[4] For localised modes $\alpha = 0$, while delocalised modes have $0 < \alpha \leq 1$.[5]

3.2 Flavour symmetries and Ward-Takahashi identities

Besides local gauge symmetry, the fermionic action is manifestly invariant under a set of spacetime symmetries, namely translations, spatial rotations, and reflections through the hyperplanes orthogonal to the temporal and spatial directions. In the context of the Faddeev-Popov-De Witt approach, these are manifest also in the gauge action if one uses a covariant gauge, such as Lorenz gauge. The fermionic action also enjoys flavour symmetries related to transformations in the internal flavour space. In particular, the fermionic Lagrangean is invariant under a group of $\mathrm{U}(N_f) = \mathrm{U}(1)_B \times \mathrm{SU}(N_f)_V$ transformations, where $\mathrm{U}(1)_B$ corresponds to a common change of phase of all the different flavours, and $\mathrm{SU}(N_f)_V$ corresponds to special unitary rotations of the flavour components. For massless fermions, $m = 0$, the two chiral components of the fermionic fields are decoupled, and the symmetry is further enhanced to the chiral symmetry $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R = \mathrm{U}(1)_B \times \mathrm{U}(1)_A \times \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$. The $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ factor contains the subgroups of $\mathrm{SU}(N_f)_V$ (vector) and $\mathrm{SU}(N_f)_A$ (axial) transformations, given respectively by

$$\psi \to e^{i\alpha_a t^a}\psi \,, \qquad \bar\psi \to \bar\psi e^{-i\alpha_a t^a} \,, \qquad \alpha_a \in \mathbb{R} \,, \tag{3.9}$$

$$\psi \to e^{i\beta_a t^a \gamma_5}\psi \,, \qquad \bar\psi \to \bar\psi e^{i\beta_a t^a \gamma_5} \,, \qquad \beta_a \in \mathbb{R} \,, \tag{3.10}$$

[4] While other definitions of fractal dimension can be adopted (e.g., the "infrared dimension" of refs. [78–80]), it is the one obtained from the IPR that turns out to be important for our purposes (see section 5.2).

[5] Since

$$1 = \left| \int_\beta d^4x \, \|\psi_n(x)\|^2 \right|^2 \leq \int_\beta d^4x \, \|\psi_n(x)\|^4 \int_\beta d^4x \, 1 = \beta V \cdot \mathrm{IPR}_n \,,$$

one necessarily has $1 - \alpha \geq 0$. In the condensed matter literature the term "delocalised" is usually reserved to the case $\alpha = 1$, while modes with $0 < \alpha < 1$ are called "critical". For our purposes there is no need to distinguish these two cases (see section 5.2).

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where t^a, $a = 1, \dots N_f^2 - 1$ are the Hermitean and traceless generators of SU(N_f), obeying $[t^a, t^b] = if^{abc}t^c$ with totally antisymmetric and real structure constants f^{abc}. Normalisation is chosen so that $f^{abc}f^{abd} = N_f\delta^{cd}$ and $\mathrm{tr}\, t^a t^b = \frac{1}{2}\delta^{ab}$. The U(1)$_B$ and SU(N_f)$_V$ symmetries are expected not to break spontaneously, also in the massless limit [81]. Spontaneous breaking of the SU(N_f)$_A$ symmetry is instead possible. The U(1)$_A$ symmetry is known to be anomalous [82, 83] due to non-invariance of the functional integration measure [84, 85] and will not be considered in this paper.

The symmetry under the transformations eq. (3.10) implies an infinite set of Ward-Takahashi identities [86, 87]. Their derivation is rather standard, and is briefly reviewed for completeness in appendix B for the case at hand; here I only report the results, which are also not new (see, e.g., ref. [88] for $N_f = 2$). Defining the infinitesimal, x-dependent transformation

$$\delta_A\psi(x) = i\epsilon_a(x)t^a\gamma_5\psi(x)\,, \qquad \delta_A\bar\psi(x) = i\epsilon_a(x)\bar\psi(x)t^a\gamma_5\,, \tag{3.11}$$

one obtains for any observable $\mathcal{O}$ the identity

$$\left\langle \left(-\partial_\mu A^a_\mu(x) + 2mP^a(x)\right)\mathcal{O}\right\rangle_\beta = \left\langle -i\frac{\delta_A\mathcal{O}}{\delta\epsilon_a(x)}\right\rangle_\beta\,, \tag{3.12}$$

where A^a_μ are the flavour non-singlet axial-vector currents, and P^a are the flavour non-singlet pseudoscalar densities,

$$A^a_\mu(x) \equiv \bar\psi(x)\gamma_\mu\gamma_5 t^a\psi(x)\,, \qquad P^a(x) \equiv \bar\psi(x)\gamma_5 t^a\psi(x)\,. \tag{3.13}$$

Of particular interest here is the case $\mathcal{O} = P^b(y)$. A straightforward calculation leads to

$$-\partial_\mu\langle A^a_\mu(x)P^b(0)\rangle_\beta + 2m\langle P^a(x)P^b(0)\rangle_\beta = \delta^{(4)}(x)\delta^{ab}\Sigma\,, \tag{3.14}$$

where Σ is the chiral condensate, defined by $\langle\bar\psi_f\psi_g\rangle_\beta \equiv \delta_{fg}\Sigma$, which follows from vector flavour symmetry. The four-dimensional Dirac delta in eq. (3.14) is understood to be periodic in time,

$$\delta^{(4)}(x) = \delta_P(t)\delta^{(3)}(\vec{x})\,, \qquad \delta_P(t) = \sum_{n=-\infty}^{\infty}\delta(t - n\beta)\,. \tag{3.15}$$

Exploiting vector flavour symmetry further, one finds

$$\langle A^a_\mu(x)P^b(0)\rangle_\beta \equiv \delta^{ab}\mathcal{G}_{AP\,\mu}(x)\,, \qquad \langle P^a(x)P^b(0)\rangle_\beta \equiv \delta^{ab}\mathcal{G}_{PP}(x)\,, \tag{3.16}$$

and eq. (3.14) can be recast as

$$-\partial_\mu\mathcal{G}_{AP\,\mu}(x) + 2m\mathcal{G}_{PP}(x) = \delta^{(4)}(x)\Sigma\,. \tag{3.17}$$

The momentum-space version of this identity, obtained through a Fourier transform [see eq. (2.8)], reads

$$i\omega_n\tilde{\mathcal{G}}_{AP\,4}\left(\omega_n,\vec{k}\right) + i\vec{k}\cdot\vec{\tilde{\mathcal{G}}}_{AP}\left(\omega_n,\vec{k}\right) + 2m\tilde{\mathcal{G}}_{PP}\left(\omega_n,\vec{k}\right) = \Sigma\,. \tag{3.18}$$

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3.3 Time-reflection symmetry

The fermionic action is invariant under the following "time reflection" transformation,

$$\begin{aligned} &\psi\,(t,\vec{x}) \to \gamma_4\gamma_5\psi\,(\beta-t,\vec{x})\;, \qquad \bar{\psi}\,(t,\vec{x}) \to \bar{\psi}\,(\beta-t,\vec{x})\,\gamma_5\gamma_4\;, \\ &B_\mu\,(t,\vec{x}) \to \zeta_\mu B_\mu\,(\beta-t,\vec{x})\;, \end{aligned} \tag{3.19}$$

with $\zeta_4 = -1$ and $\zeta_{1,2,3} = 1$. No summation over μ is implied here and in the following equations. Using the Faddeev-Popov-De Witt procedure in a covariant gauge, the gauge action is also invariant under the transformation eq. (3.19), so this leaves the full action invariant. Under time reflection one has for the pseudoscalar densities and axial-vector currents

$$P^a(t,\vec{x}) \to -P^a(\beta-t,\vec{x})\,, \qquad A^a_\mu(t,\vec{x}) \to -\zeta_\mu A^a_\mu(\beta-t,\vec{x})\,. \tag{3.20}$$

For the correlators $\mathcal{G}_{AP\,\mu}$ and $\mathcal{G}_{PP}$ [see eq. (3.16)] one then finds

$$\mathcal{G}_{AP\,\mu}(t,\vec{x}) = \zeta_\mu\langle A^a_\mu(\beta-t,\vec{x})P^a(\beta,\vec{0})\rangle_\beta = \zeta_\mu\langle A^a_\mu(\beta-t,\vec{x})P^a(0,\vec{0})\rangle_\beta = \zeta_\mu\mathcal{G}_{AP\,\mu}(\beta-t,\vec{x})\,, \tag{3.21}$$

$$\mathcal{G}_{PP}(t,\vec{x}) = \langle P^a(\beta-t,\vec{x})P^a(\beta,\vec{0})\rangle_\beta = \langle P^a(\beta-t,\vec{x})P^a(0,\vec{0})\rangle_\beta = \mathcal{G}_{PP}(\beta-t,\vec{x})\,, \tag{3.22}$$

where antiperiodicity of ψ and $\bar{\psi}$ has been used to replace $P^a(\beta,\vec{0}) \to P^a(0,\vec{0})$, and no summation over a is implied. The symmetry properties of the coordinate-space correlators translate into the following relations for the momentum-space correlators,

$$\tilde{\mathcal{G}}_{AP\,\mu}(\omega_n,\vec{k}) = \zeta_\mu\tilde{\mathcal{G}}_{AP\,\mu}(-\omega_n,\vec{k})\,, \qquad \tilde{\mathcal{G}}_{PP}(\omega_n,\vec{k}) = \tilde{\mathcal{G}}_{PP}(-\omega_n,\vec{k})\,. \tag{3.23}$$

3.4 Analytic continuation

Since they will be used repeatedly, it is convenient to summarise the relevant analytic continuation relations needed to reconstruct the real-time, Minkowskian thermal expectation values from the imaginary-time, Euclidean correlation functions. The relevant Minkowskian operators are the axial-vector current and pseudoscalar density operators,

$$\hat{A}^a_\mu(x) \equiv \bar{\hat{\psi}}(x)\tilde{\gamma}_\mu\tilde{\gamma}_5 t^a\hat{\psi}(x)\,, \qquad \hat{P}^a(x) \equiv \bar{\hat{\psi}}(x)\tilde{\gamma}_5 t^a\hat{\psi}(x)\,. \tag{3.24}$$

Here $\tilde{\gamma}^\mu$, $\mu = 0,\ldots,3$ are the Minkowskian gamma matrices, obeying $\{\tilde{\gamma}^\mu,\tilde{\gamma}^\nu\} = 2\eta^{\mu\nu}$ with $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, and $\tilde{\gamma}^5 = i\tilde{\gamma}^0\tilde{\gamma}^1\tilde{\gamma}^2\tilde{\gamma}^3$. These are related with the Euclidean gamma matrices γ_μ and γ_5 as $\gamma_4 = \tilde{\gamma}^0$, $\gamma_j = -i\tilde{\gamma}^j$, and $\gamma_5 = \tilde{\gamma}^5$. In eq. (3.24), $\bar{\hat{\psi}} = \hat{\psi}^\dagger\tilde{\gamma}^0$, as usual. Using the general analytic continuation relation eq. (2.5), one finds that Euclidean and Minkowskian two-point correlation functions are related as follows,

$$\begin{aligned} \langle\!\langle\hat{A}^{a0}(t,\vec{x})\hat{P}^a(0)\rangle\!\rangle_\beta &= \mathcal{G}_{AP\,4}(\epsilon+it,\vec{x})\,, & \langle\!\langle\hat{P}^a(0)\hat{A}^{a0}(t,\vec{x})\rangle\!\rangle_\beta &= \mathcal{G}_{AP\,4}(-\epsilon+it,\vec{x})\,, \\ \langle\!\langle\hat{A}^{aj}(t,\vec{x})\hat{P}^a(0)\rangle\!\rangle_\beta &= i\mathcal{G}_{AP\,j}(\epsilon+it,\vec{x})\,, & \langle\!\langle\hat{P}^a(0)\hat{A}^{aj}(t,\vec{x})\rangle\!\rangle_\beta &= i\mathcal{G}_{AP\,j}(-\epsilon+it,\vec{x})\,, \\ \langle\!\langle\hat{P}^a(t,\vec{x})\hat{P}^a(0)\rangle\!\rangle_\beta &= \mathcal{G}_{PP}(\epsilon+it,\vec{x})\,, & \langle\!\langle\hat{P}^a(0)\hat{P}^a(t,\vec{x})\rangle\!\rangle_\beta &= \mathcal{G}_{PP}(-\epsilon+it,\vec{x})\,. \end{aligned} \tag{3.25}$$

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Here and in the rest of this subsection no summation over the flavour index a is implied. Of particular interest is the reconstruction of the spectral function, eq. (2.9), through that of the retarded and advanced propagators, eq. (2.10). In this work I need the spectral functions $\tilde{\rho}_{A^{0a}P^a}$ and $\tilde{\rho}_{P^aP^a}$, that are independent of a thanks to vector flavour invariance. For brevity I will denote them as follows,

$$\tilde{c}\left(\omega,\vec{k}\right) \equiv \tilde{\rho}_{A^{0a}P^a}\left(\omega,\vec{k}\right) , \qquad \tilde{c}^P\left(\omega,\vec{k}\right) \equiv \tilde{\rho}_{P^aP^a}\left(\omega,\vec{k}\right) . \tag{3.26}$$

These spectral functions can be obtained by analytic continuation using directly eq. (3.25) [see eq. (2.9)], or from the retarded and advanced propagators, which in turn can be reconstructed by analytic interpolation from the Fourier coefficients of the Euclidean correlator [68, 69]. Setting for brevity ($\mu = 0, \dots, 3$)

$$\begin{aligned} \tilde{r}^\mu\left(\omega,\vec{k}\right) &\equiv \tilde{r}_{A^{\mu a}P^a}\left(\omega,\vec{k}\right) , & \tilde{a}^\mu\left(\omega,\vec{k}\right) &\equiv \tilde{a}_{A^{\mu a}P^a}\left(\omega,\vec{k}\right) , \\ \tilde{r}^P\left(\omega,\vec{k}\right) &\equiv \tilde{r}_{P^aP^a}\left(\omega,\vec{k}\right) , & \tilde{a}^P\left(\omega,\vec{k}\right) &\equiv \tilde{a}_{P^aP^a}\left(\omega,\vec{k}\right) , \end{aligned} \tag{3.27}$$

one has from eq. (2.12) that for $n \neq 0$

$$\begin{aligned} \tilde{\mathcal{G}}_{AP\,4}\left(\omega_n,\vec{k}\right) &= \tilde{r}^0\left(i\omega_n,-\vec{k}\right) , & n>0\,, \qquad \tilde{\mathcal{G}}_{AP\,4}\left(\omega_n,\vec{k}\right) &= \tilde{a}^0\left(i\omega_n,-\vec{k}\right) , & n<0\,, \\ i\tilde{\mathcal{G}}_{AP\,j}\left(\omega_n,\vec{k}\right) &= \tilde{r}^j\left(i\omega_n,-\vec{k}\right) , & n>0\,, \qquad i\tilde{\mathcal{G}}_{AP\,j}\left(\omega_n,\vec{k}\right) &= \tilde{a}^j\left(i\omega_n,-\vec{k}\right) , & n<0\,, \\ \tilde{\mathcal{G}}_{PP}\left(\omega_n,\vec{k}\right) &= \tilde{r}^P\left(i\omega_n,-\vec{k}\right) , & n>0\,, \qquad \tilde{\mathcal{G}}_{PP}\left(\omega_n,\vec{k}\right) &= \tilde{a}^P\left(i\omega_n,-\vec{k}\right) , & n<0\,. \end{aligned} \tag{3.28}$$

These relations hold also for $n = 0$ if a transport peak is absent.

3.5 Regularisation and renormalisation

The discussion so far has been entirely formal, ignoring the ill-defined nature of path integrals. As is well known, these require a suitable regularisation to become mathematically well defined, and an appropriate renormalisation procedure to remove the divergences appearing when the regularisation is removed. Since regularisation usually breaks some of the symmetries, their recovery after renormalisation is carried out is not guaranteed in the general case, and this can spoil the formal results discussed above. In particular, Ward-Takahashi identities will be violated in the regulated theory if the regularisation breaks the corresponding symmetry, and it is not obvious that they can be recovered in the same form after renormalisation.

Before drawing any conclusion from the symmetry properties discussed in the previous subsections, it is important to make sure that they can be enforced in the finite, renormalised theory. The best way to enforce a symmetry is obviously to choose a regularisation that does not break it, in which case renormalisation will not spoil it. The best known non-perturbative regularisation of path integrals is the lattice regularisation (see, e.g., refs. [89–91]), which is especially convenient when dealing with gauge theories, since it allows one to mantain manifest gauge invariance. In the lattice approach, the formal functional integral is replaced with a well-defined finite-dimensional integral over fields defined only on the discrete elements of a finite lattice, eventually taking the limits of infinite volume and zero lattice spacing.

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This approach clearly breaks most of the spacetime symmetry; using a hypercubic lattice with periodic boundary conditions one can still retain symmetry under translations by multiples of the lattice spacing, a discrete subgroup of the SO(4) group, and reflections [so that eqs. (3.21) and (3.22) hold also in the regulated theory]. Nonetheless, for asymptotically free theories it is widely believed, and supported by a vast amount of numerical evidence, that a continuum limit exists (after a suitable renormalisation procedure) where spacetime symmetries are fully restored.

While vector flavour symmetry can be implemented exactly on the lattice, the axial flavour symmetry is problematic due to the known difficulties of implementing exact chiral symmetry for Dirac operators discretised on the lattice [92–94]. Nonetheless, for lattice Dirac operators satisfying the Ginsparg-Wilson relation [95], such as the fixed-point action [96, 97], domain-wall fermions [98, 99], and overlap fermions [100–103], one has an exact chiral-type symmetry that holds on any finite lattice [104], and that reduces to the usual chiral symmetry in the formal continuum limit. This implies exact Ward-Takahashi identities for suitably defined lattice currents and densities, that hold for any lattice spacing and tend to the continuum identities as the spacing goes to zero [104–107], and guarantee that the desired symmetry can be enforced in the renormalised theory. In particular, renormalised continuum correlation functions will satisfy the continuum Ward-Takahashi identities — of course, assuming that such a limit exists.

Taking the continuum limit and the associated restoration of spacetime symmetries for granted, one can use the continuum Ward-Takahashi identities as fully meaningful relations between renormalised quantities, and ignore where they came from. This will suffice for the discussion of the finite-temperature version of Goldstone's theorem in section 4. On the other hand, when studying the pseudoscalar-pseudoscalar correlator in detail in section 5 one has to keep track of the effects of renormalisation. Instead of dealing with the technical complications of the lattice approach, it is simpler to discuss the issue of renormalisation directly in the continuum limit. In fact, if this limit exists, then a renormalised theory with the desired symmetries can also be obtained directly in the continuum, starting from a regularised theory where the representation of fermionic observables in terms of sums over the eigenmodes of the continuum Dirac operator is cut off symmetrically at some ultraviolet scale Λ.[6] Ward-Takahashi identities are not exact anymore in this case, but their violations should disappear after appropriate renormalisation and removal of the cutoff.

A detailed discussion of the renormalisation issues related to the Ward-Takahashi identity eq. (3.14), in continuum language, is provided in appendix C. As mentioned above, using Ginsparg-Wilson fermions [95] in the lattice regularisation of the theory one can use the lattice Ward-Takahashi identities implied by the exact lattice chiral symmetry [104] to show nonperturbatively [105] that m renormalises only multiplicatively; that the composite operators A^a_μ require no renormalisation after the usual mass and coupling renormalisations have been carried out; and that the multiplicative renormalisation constants Z_P and Z_S of the non-singlet pseudoscalar and singlet scalar densities satisfy $Z_P = Z_S = Z_m^{-1}$ with Z_m the mass renormalisation constant. Combined with the properties of bilinear correlators

[6] A suitable regularisation of the integration over gauge fields is also required. However, this does not affect the argument.

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under the "$\mathcal{R}_5$-parity" transformation [108], the lattice chiral symmetry implies that all additive divergent contact terms drop from the lattice analogue of eq. (3.14) in the chiral limit. Based on the argument given above, it is then safe to use the continuum identity eq. (3.14) in its *regularised* version to discuss renormalisation issues in a simpler continuum language. After renormalisation, eq. (3.14) in its *renormalised* version can be used as the starting point for an alternative derivation of the finite temperature version of Goldstone's theorem [49–53]. While renormalisation is an essential part of the construction of the theory itself, and deserves a careful discussion as such, it will become clear that it plays a limited role in the arguments of this paper concerning the chiral limit, which is in fact dominated by the low-end, infrared part of the Dirac spectrum.

4 Goldstone's theorem at finite temperature

In this section I discuss an alternative derivation of the finite temperature analogue of Goldstone's theorem [49–53], including a slight but useful generalisation, based on the Ward-Takahashi identity eq. (3.14). In this section all Euclidean quantities are understood to be renormalised.

4.1 Review of the standard derivation

For quantum field theories at finite temperature, the analogue of Goldstone's theorem [48] proved in refs. [49–52] states that the spontaneous breaking of a continuous symmetry in a theory invariant under spatial translations leads to a gapless spectrum of "quasi-particle" excitations (see ref. [53] for a detailed discussion and a full list of references). A heuristic proof of this "Goldstone's theorem at finite temperature" is based on the following observations. Let $\hat{J}^\mu$ be the conserved Noether current associated with the symmetry, $\partial_\mu \hat{J}^\mu = 0$, and let

$$\hat{Q}_V(t) \equiv \int_V d^3x\, \hat{J}^0(t,\vec{x})\,, \tag{4.1}$$

be the corresponding charge, regularised by restricting spatial integration to a finite volume V. If a nonzero expectation value is found for the commutator

$$\lim_{V\to\infty} \langle\!\langle \left[i\hat{Q}_V(0), \hat{\mathcal{O}} \right] \rangle\!\rangle_\beta = b \neq 0\,, \tag{4.2}$$

for some local observable $\mathcal{O}$, then current conservation and relativistic locality imply

$$\lim_{V\to\infty} \langle\!\langle \left[i\hat{Q}_V(t), \hat{\mathcal{O}} \right] \rangle\!\rangle_\beta = b\,, \quad \forall t\,. \tag{4.3}$$

Taking the Fourier transform (in the sense of distributions) of eq. (4.3) one then finds

$$\lim_{\vec{k}\to 0} i\tilde{\rho}_{J^0\mathcal{O}}\left(\omega,\vec{k}\right) = \lim_{\vec{k}\to 0} \int d^4x\, e^{i\left(\omega t - \vec{k}\cdot\vec{x}\right)} \langle\!\langle \left[i\hat{J}^0\left(t,\vec{x}\right), \hat{\mathcal{O}} \right] \rangle\!\rangle_\beta = 2\pi b\delta(\omega)\,, \tag{4.4}$$

from which one infers the existence of massless quasi-particle excitations, i.e., such that their energy vanishes and their lifetime becomes infinite in the zero-momentum limit.[7] I will refer to these as Goldstone excitations or quasi-particles throughout this paper.

[7] The same conclusions hold more generally for non-relativistic systems, replacing the requirement of relativistic locality with Swieca's condition [51] for the commutator of the spatial part of the current with the relevant observable (see ref. [53]).

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In the case at hand, the relevant symmetry is the non-singlet axial part of chiral symmetry. After analytic continuation to Minkowski spacetime, eq. (3.12) expresses conservation of the non-singlet axial currents $\hat{A}^a_\mu$ in the chiral limit $m \to 0$, under the usual assumption that the second term on the left-hand side can be dropped. Equations (3.14) and (3.17) further show that axial flavour symmetry is spontaneously broken, in the sense of eq. (4.2), if $\Sigma \neq 0$. In fact, integrating eq. (3.17) in the $m \to 0$ limit over space and over the infinitesimal time interval $[-\epsilon, \epsilon]$ one gets as $\epsilon \to 0$,

$$\Sigma_* = -\int d^3x \, [\mathcal{G}_{AP\,4}(\epsilon, \vec{x}) - \mathcal{G}_{AP\,4}(-\epsilon, \vec{x})] = \int d^3x \, \langle\!\langle \Big[i\hat{A}^{a0}(0, \vec{x}), i\hat{P}^a(0) \Big] \rangle\!\rangle_\beta \,, \tag{4.5}$$

where no summation over a is implied and Σ_* is the chiral condensate in the chiral limit. Here I used continuity of $\int d^3x \, \vec{\nabla} \cdot \vec{\mathcal{G}}_{AP}(t, \vec{x})$ at $t = 0$,[8] and the analytic continuation relations eq. (3.25). Clearly, eq. (4.5) is nothing but eq. (4.2) at $t = 0$ with $\hat{Q}_V(0) = \int_V d^3x \, \hat{A}^{a0}(0, \vec{x})$ the finite-volume axial charge and $\hat{\mathcal{O}} = i\hat{P}^a(0)$. One can now use current conservation and relativistic locality to complete the argument, obtaining eq. (4.4) with $-\tilde{c}$ [see eq. (3.26)] on the left-hand side, and $b = \Sigma_*$ on the right-hand side, and infer the existence of pseudoscalar Goldstone quasi-particles, i.e., "quasi-pions" [see eq. (4.20) below].

The standard argument outlined above makes essential use of current conservation as an operator equation to infer eq. (4.3) from eq. (4.2). In the following I discuss a more direct argument that requires only knowledge of the Ward-Takahashi identity eq. (3.14) in its energy-momentum-space form, eq. (3.18), works directly with Euclidean quantities, and allows for a simple but useful generalisation. This argument has been presented briefly in ref. [57]; here I provide a more detailed discussion. In appendix D I discuss the coordinate-space version of the argument, which is new. Of course, appropriate analyticity conditions must be satisfied in order to be able to reconstruct the physical, Minkowskian correlation functions. Moreover, a suitable regularity condition must also be satisfied to guarantee relativistic locality of the reconstructed theory. This regularity condition plays an important role and will be discussed next.

4.2 Regularity condition

In this subsection I "reverse-engineer" a condition that has to be imposed on $\vec{\tilde{\mathcal{G}}}_{AP}$ in order to obtain the desired locality properties of quantum field theory in Minkowski space. Starting from the relativistic locality condition, $[\hat{\vec{A}}^a(x), \hat{P}^b(0)] = 0$ for $x^2 < 0$, one finds (in the sense of distributions) that

$$\begin{aligned}
&\lim_{k\to 0} \vec{k} \cdot \int d^3x \, e^{-i\vec{k}\cdot\vec{x}} \langle\!\langle \Big[\vec{\hat{A}}^a(t, \vec{x}), \hat{P}^b(0) \Big] \rangle\!\rangle_\beta = \lim_{k\to 0} i \int d^3x \, \Big\{ \vec{\nabla} e^{-i\vec{k}\cdot\vec{x}} \Big\} \cdot \langle\!\langle \Big[\vec{\hat{A}}^a(t, \vec{x}), \hat{P}^b(0) \Big] \rangle\!\rangle_\beta \\
&= \lim_{k\to 0} i \int d^3x \, \vec{\nabla} \cdot \Big\{ e^{-i\vec{k}\cdot\vec{x}} \langle\!\langle \Big[\vec{\hat{A}}^a(t, \vec{x}), \hat{P}^b(0) \Big] \rangle\!\rangle_\beta \Big\} - i \int d^3x \, e^{-i\vec{k}\cdot\vec{x}} \vec{\nabla} \cdot \langle\!\langle \Big[\vec{\hat{A}}^a(t, \vec{x}), \hat{P}^b(0) \Big] \rangle\!\rangle_\beta \\
&= -i \int d^3x \, \vec{\nabla} \cdot \langle\!\langle \Big[\vec{\hat{A}}^a(t, \vec{x}), \hat{P}^b(0) \Big] \rangle\!\rangle_\beta = 0 \,,
\end{aligned} \tag{4.6}$$

[8] This follows from $\int d^3x \, \vec{\nabla} \cdot \vec{\mathcal{G}}_{AP}(t, \vec{x}) = \lim_{R\to\infty} \int_{\partial B_R} d^2\vec{\Sigma} \cdot \vec{\mathcal{G}}_{AP}(t, \vec{x})$, with B_R the ball of radius R and $d^2\vec{\Sigma}$ the corresponding infinitesimal surface element oriented outwards, and from continuity in t of $\vec{\mathcal{G}}_{AP}(t, \vec{x})$ for $t < |\vec{x}|$ (see section 2). It is assumed that $\vec{\mathcal{G}}_{AP}(t, \vec{x})$ vanishes sufficiently fast at spatial infinity (at least like $|\vec{x}|^{-2}$) so that the surface integral is convergent (but not necessarily zero).

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since the first term on the second line and the term on the third line vanish due to finiteness of the support of the commutator at fixed t.[9] This holds independently of the quark mass m, and should be true in particular in the chiral limit if one is to obtain a decent quantum field theory. In this limit, the final equality implies the time-independence of the regularised charge commutator if the Noether current is conserved, so that eq. (4.3) follows from eq. (4.2) (see the discussion in ref. [53], section 15.2.II).

To work out the implications of eq. (4.6) in the Euclidean setting, one uses the relations eq. (3.28) between the retarded and advanced propagators $\tilde{r}^j$ and $\tilde{a}^j$ and the Euclidean correlator $\tilde{\mathcal{G}}_{AP\,j}$. Plugging them into eq. (4.6), one finds ($n > 0$, no summation over a)

$$\begin{aligned}\lim_{\vec{k}\to 0} \vec{k}\cdot\vec{\tilde{\mathcal{G}}}_{AP}\left(\omega_{\pm n},\vec{k}\right) &= \pm \lim_{\vec{k}\to 0} \vec{k}\cdot \int_{-\infty}^{\infty} dt\,\theta\left(\pm t\right) e^{-\omega_{\pm n} t} \int d^3x\, e^{i\vec{k}\cdot\vec{x}} \langle\!\langle \left[\vec{\hat{A}}^a\left(t,\vec{x}\right), \hat{P}^a\left(0\right)\right] \rangle\!\rangle_\beta \\ &= \pm \int_{-\infty}^{\infty} dt\,\theta\left(\pm t\right) e^{-\omega_n |t|} \lim_{\vec{k}\to 0} \vec{k}\cdot \int d^3x\, e^{i\vec{k}\cdot\vec{x}} \langle\!\langle \left[\vec{\hat{A}}^a\left(t,\vec{x}\right), \hat{P}^a(0)\right] \rangle\!\rangle_\beta = 0\,,\end{aligned} \tag{4.7}$$

where the exchange of the limit $\vec{k}\to 0$ and integration over t is justified by the exponential damping factor. From the Euclidean perspective, this is a necessary condition that the Euclidean correlators must obey in order to reconstruct a decent Minkowskian theory. I then assume that $\vec{\tilde{\mathcal{G}}}_{AP}$ obeys the *regularity condition* $\vec{k}\cdot\vec{\tilde{\mathcal{G}}}_{AP}(\omega_n,\vec{k}) \to 0$ as $\vec{k}\to 0$ for $n \neq 0$.

4.3 Euclidean proof of Goldstone's theorem in momentum space

To obtain a Euclidean proof of Goldstone's theorem at finite temperature, one starts from the Ward-Takahashi identity in energy-momentum-space eq. (3.18). Setting

$$\mathcal{R}(x) \equiv 2m\mathcal{G}_{PP}(x)\,, \qquad \widetilde{\mathcal{R}}\left(\omega_n,\vec{k}\right) \equiv 2m\widetilde{\mathcal{G}}_{PP}\left(\omega_n,\vec{k}\right)\,, \tag{4.8}$$

this identity becomes

$$i\omega_n \widetilde{\mathcal{G}}_{AP\,4}\left(\omega_n,\vec{k}\right) + i\vec{k}\cdot\vec{\tilde{\mathcal{G}}}_{AP}\left(\omega_n,\vec{k}\right) + \widetilde{\mathcal{R}}\left(\omega_n,\vec{k}\right) = \Sigma\,. \tag{4.9}$$

The symmetry properties eq. (3.23) imply that $\widetilde{\mathcal{G}}_{AP\,4}(\omega_n,\vec{k}) = -\widetilde{\mathcal{G}}_{AP\,4}(-\omega_n,\vec{k})$, so in particular $\widetilde{\mathcal{G}}_{AP\,4}(0,\vec{k}) = 0$, while the regularity condition eq. (4.7) requires $\vec{k}\cdot\vec{\tilde{\mathcal{G}}}_{AP}(\omega_n,\vec{k}) \to 0$ as $\vec{k}\to 0$ for $n \neq 0$ (also in the chiral limit). Notice also $\widetilde{\mathcal{R}}(-\omega_n,\vec{k}) = \widetilde{\mathcal{R}}(\omega_n,\vec{k})$. One then obtains from eq. (4.9)

$$i\vec{k}\cdot\vec{\tilde{\mathcal{G}}}_{AP}\left(0,\vec{k}\right) + \widetilde{\mathcal{R}}\left(0,\vec{k}\right) = \Sigma\,, \tag{4.10}$$

$$\lim_{\vec{k}\to 0}\left\{ i\omega_n \widetilde{\mathcal{G}}_{AP\,4}\left(\omega_n,\vec{k}\right) + \widetilde{\mathcal{R}}\left(\omega_n,\vec{k}\right)\right\} = \Sigma\,, \quad n \neq 0\,. \tag{4.11}$$

It is usually (although perhaps implicitly) assumed that the pseudoscalar-pseudoscalar correlator is sufficiently regular as a function of m in the chiral limit, so that $\widetilde{\mathcal{R}}(\omega_n,\vec{k}) \to 0$

[9] More formally, after smearing over t with a function of compact support $h(t)$, the quantity $\vec{C}(\vec{x}) = \int dt\, h(t) \langle\!\langle [\vec{\hat{A}}^a(t,\vec{x}), \hat{P}^a(0)] \rangle\!\rangle_\beta$ is a distribution in $\vec{x}$ of compact support, and so by the Paley-Wiener theorem for distributions its Fourier transform $\vec{\tilde{C}}(\vec{k}) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} \vec{C}(\vec{x})$ is an entire function, so that $\vec{k}\cdot\vec{\tilde{C}}(\vec{k}) \to 0$ as $\vec{k}\to 0$ follows.

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as $m \to 0$. I will refer to this as the standard scenario. As I show below in section 5, this may not be the case if a finite density of localised near-zero Dirac modes is present. More precisely, in the presence of localised near-zero modes the pseudoscalar-pseudoscalar correlator can develop a $1/m$ divergence, that exactly cancels out the factor of m and leaves behind a finite $\widetilde{\mathcal{R}}$ in the chiral limit. I will refer to this as the non-standard scenario.

Denoting by a subscript $*$ the quantities obtained in the chiral limit, taking now $m \to 0$ in eq. (4.9) followed by $\vec{k} \to 0$ one obtains in full generality

$$i\vec{k} \cdot \vec{\widetilde{\mathcal{G}}}_{AP\,*}\left(0,\vec{k}\right) + \widetilde{\mathcal{R}}_*\left(0,\vec{k}\right) = \Sigma_*\,, \tag{4.12}$$

$$\lim_{\vec{k}\to 0}\left\{i\omega_n \widetilde{\mathcal{G}}_{AP\,4\,*}\left(\omega_n,\vec{k}\right) + \widetilde{\mathcal{R}}_*\left(\omega_n,\vec{k}\right)\right\} = \Sigma_*\,, \quad n \neq 0\,. \tag{4.13}$$

Exploiting rotation invariance, one finds from the equation for zero Matsubara frequency, eq. (4.12), that

$$\vec{\widetilde{\mathcal{G}}}_{AP\,*}\left(0,\vec{k}\right) = -i\vec{k}\left(\frac{\Sigma_* - \widetilde{\mathcal{R}}_*\left(0,\vec{k}\right)}{\vec{k}^2}\right) \underset{\vec{k}\to 0}{\to} -i\left(\Sigma_* - \widetilde{\mathcal{R}}_*\left(0,\vec{0}\right)\right)\frac{\vec{k}}{\vec{k}^2}\,, \tag{4.14}$$

and so it follows that $\vec{\widetilde{\mathcal{G}}}_{AP\,*}(0,\vec{k})$ has a pole at $\vec{k}=0$ if $\Sigma_* - \widetilde{\mathcal{R}}_*(0,\vec{0}) \neq 0$, so in particular in the standard scenario, where $\widetilde{\mathcal{R}}_*(0,\vec{0}) = 0$, if $\Sigma_* \neq 0$. However, the existence of this pole does not imply *a priori* that massless Goldstone quasi-particles are present. This is different from the zero-temperature case, where one would find $\widetilde{\mathcal{G}}_{AP\,\mu\,*} \propto p_\mu/p^2$ due to O(4) invariance. In that case, after analytic continuation $(p_4,\vec{p}) \to (-ip^0,\vec{p})$ to Minkowski space one finds a pole at $(p^0)^2 - \vec{p}^2 = 0$ in the axial-vector-pseudoscalar correlator, which directly implies the presence of massless particles in the spectrum. At finite temperature full O(4) invariance is lost, and the connection with the spectrum is encoded in the axial-vector-pseudoscalar spectral function, whose reconstruction requires the analytic interpolation of the discrete Fourier components of the Euclidean correlator. The presence of a pole in $\vec{\widetilde{\mathcal{G}}}_{AP\,*}$ at zero frequency is therefore by itself not sufficient to infer the relevant properties of the spectral function at zero frequency.

To make progress one needs to exploit eq. (4.13). To this end, one sets

$$\mathrm{G}_*\left(\omega_n\right) \equiv \lim_{\vec{k}\to 0}\widetilde{\mathcal{G}}_{AP\,4\,*}\left(\omega_n,\vec{k}\right)\,, \qquad \mathrm{R}_*\left(\omega_n\right) \equiv \lim_{\vec{k}\to 0}\widetilde{\mathcal{R}}_*\left(\omega_n,\vec{k}\right)\,, \tag{4.15}$$

and using eq. (4.13) one finds

$$\mathrm{G}_*(\omega_n) = \frac{\Sigma_* - \mathrm{R}_*(\omega_n)}{i\omega_n}\,, \quad n \neq 0\,, \tag{4.16}$$

while $\mathrm{G}_*(0) = 0$. It is instructive to discuss first the standard scenario in which $\widetilde{\mathcal{R}}_*$ vanishes and so $\mathrm{R}_* = 0$. In this case eq. (4.16) entirely determines $\mathrm{G}_*(\omega_n)$, up to the value of Σ_*. It is then easy to obtain its unique analytic interpolation (in the sense of Carlson's theorem) to a function $\bar{\mathrm{G}}_*(\Omega)$ of a complex variable $\Omega \in \mathbb{C}$, and reconstruct the relevant Minkowskian

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quantities by analytic continuation [68, 69]. One finds[10]

$$\bar{\mathrm{G}}_*(\Omega) = \frac{\Sigma_*}{i\Omega}\,, \tag{4.17}$$

which is analytic in the whole complex plane except for a pole at $\Omega = 0$. This is enough to reconstruct the spectral function in the zero-momentum limit. In fact, using eq. (3.28), one sees that the interpolation eq. (4.17) for $\mathrm{Im}\,\Omega > 0$ and $\mathrm{Im}\,\Omega < 0$ corresponds respectively to the retarded and advance propagators $\tilde{r}^0$ and $\tilde{a}^0$ in the chiral and zero-momentum limit,

$$\lim_{\vec{k}\to 0} \tilde{r}^0_*\left(\Omega, \vec{k}\right) = \frac{\Sigma_*}{\Omega} \text{ for } \mathrm{Im}\,\Omega > 0\,, \qquad \lim_{\vec{k}\to 0} \tilde{a}^0_*\left(\Omega, \vec{k}\right) = \frac{\Sigma_*}{\Omega} \text{ for } \mathrm{Im}\,\Omega < 0\,. \tag{4.18}$$

Since the spectral density is given by

$$i\tilde{c}\left(\omega, \vec{k}\right) = \tilde{r}^0\left(\omega + i\epsilon, \vec{k}\right) - \tilde{a}^0\left(\omega - i\epsilon, \vec{k}\right)\,, \qquad \omega \in \mathbb{R}\,, \tag{4.19}$$

one finds in the chiral limit

$$\lim_{\vec{k}\to 0} i\tilde{c}_*(\omega, \vec{k}) = \frac{\Sigma_*}{\omega + i\epsilon} - \frac{\Sigma_*}{\omega - i\epsilon} = -2\pi i \Sigma_* \delta(\omega)\,, \tag{4.20}$$

and so if $\Sigma_* \neq 0$ one finds massless Goldstone excitations in the spectrum. This completes this alternative proof of Goldstone's theorem at finite temperature under the usual assumptions on the symmetry breaking term.

In the non-standard scenario where $\widetilde{\mathcal{R}}_* \neq 0$, $\bar{\mathrm{G}}_*(\Omega)$ is not fully determined, but one can still use eq. (4.16) to relate it with the analytic interpolation of $\mathrm{R}_*(\omega_n)$, which will be denoted with $\bar{\mathrm{R}}_*(\Omega)$. One finds

$$\bar{\mathrm{G}}_*(\Omega) = \frac{\Sigma_* - \bar{\mathrm{R}}_*(\Omega)}{i\Omega}\,. \tag{4.21}$$

Using eq. (4.19) and the well known result $\frac{1}{\omega \pm i\epsilon} = \mathrm{PV}\frac{1}{\omega} \mp i\pi\delta(\omega)$, together with the symmetry property $\bar{\mathrm{R}}_*(-\Omega) = \bar{\mathrm{R}}_*(\Omega)$ that follows from eq. (3.23) by analytic continuation, one finds for the spectral function in the chiral limit

$$\begin{aligned}
\lim_{\vec{k}\to 0} i\tilde{c}_*(\omega, \vec{k}) &= \bar{\mathrm{G}}_*(\epsilon - i\omega) - \bar{\mathrm{G}}_*(-\epsilon - i\omega) \\
&= -i\pi\delta(\omega)\left[2\Sigma_* - \bar{\mathrm{R}}_*(\epsilon) - \bar{\mathrm{R}}_*(-\epsilon)\right] - \mathrm{PV}\frac{1}{\omega}\left[\bar{\mathrm{R}}_*(\epsilon - i\omega) - \bar{\mathrm{R}}_*(-\epsilon - i\omega)\right] \\
&= -i2\pi\delta(\omega)\left[\Sigma_* - \bar{\mathrm{R}}_*(\epsilon)\right] - \mathrm{PV}\frac{1}{\omega}\left[\bar{\mathrm{R}}_*(\epsilon - i\omega) - \bar{\mathrm{R}}_*(\epsilon + i\omega)\right]\,.
\end{aligned} \tag{4.22}$$

The quantity in the second square bracket on the right-hand side of eq. (4.22) is manifestly antisymmetric, so that one can drop the principal-value prescription. To discuss its

[10] Formally, one defines $F(\Omega) = \frac{\Sigma}{i\Omega} - \bar{\mathrm{G}}_*(\Omega)$, $\Omega \neq 0$, and $F(0) = 0$, and looks for analytic interpolations $\bar{\mathrm{G}}_*(\Omega)$ obeying eq. (4.16) $\forall n \neq 0$. This function obeys $F(\omega_n) = 0$ $\forall n$, and Carlson's theorem implies that its unique interpolation analytic in the upper half of the complex plane is $F(\Omega) \equiv 0$, which further extends uniquely to $F(\Omega) \equiv 0$ on the whole complex plane. This leads to eq. (4.17). Notice that $\lim_{\Omega\to 0} \bar{\mathrm{G}}_*(\Omega) \neq \mathrm{G}_*(0) = 0$.

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regularity properties it is convenient to express it in terms of the pseudoscalar spectral function. Recalling eq. (3.28), one sees that

$$\begin{aligned}\bar{\mathrm{R}}_*\left(\epsilon - i\omega\right) &= \lim_{\vec{k}\to 0}\lim_{m\to 0} 2m\tilde{r}^P\left(i\epsilon+\omega,-\vec{k}\right),\\ \bar{\mathrm{R}}_*\left(-\epsilon - i\omega\right) &= \lim_{\vec{k}\to 0}\lim_{m\to 0} 2m\tilde{a}^P\left(-i\epsilon+\omega,-\vec{k}\right),\end{aligned} \tag{4.23}$$

and so

$$\begin{aligned}\bar{\mathrm{R}}_*\left(\epsilon - i\omega\right) - \bar{\mathrm{R}}_*\left(-\epsilon - i\omega\right) &= \lim_{\vec{k}\to 0} i\int d^4x\, e^{i\left(\omega t-\vec{k}\cdot\vec{x}\right)}\lim_{m\to 0} 2m\langle\!\langle\left[\hat{P}^a\left(t,\vec{x}\right),\hat{P}^a\left(0\right)\right]\rangle\!\rangle_\beta\\ &= i\lim_{\vec{k}\to 0}\lim_{m\to 0} 2m\tilde{c}^P\left(\omega,\vec{k}\right).\end{aligned} \tag{4.24}$$

In the last passage it is assumed that the chiral limit can be exchanged with the Fourier transform at finite $\vec{k}$.

The pseudoscalar spectral function at $\vec{k}=0$ is an antisymmetric function of ω [see eq. (A.18) in appendix A], expected to be regular at $\omega=0$ and so vanishing at least like ω (see ref. [71]). Moreover, no transport peak $\propto\omega\delta(\omega)$ is expected in the pseudoscalar channel. These expectations are supported both by analytic perturbative results (also in the chiral limit) and by numerical lattice calculations [109–111]. It is then reasonable to assume that the transport peak is absent, and that $\tilde{c}^P$ is regular at $\omega=0$, and remains so as $m\to 0$.[11] Under these assumptions one finds from eq. (2.14)

$$\bar{\mathrm{R}}_*\left(\epsilon\right) = \lim_{\vec{k}\to 0}\lim_{m\to 0} 2m\tilde{r}^P\left(i\epsilon,-\vec{k}\right) = \lim_{\vec{k}\to 0}\lim_{m\to 0} 2m\widetilde{\mathcal{G}}_{PP}\left(0,\vec{k}\right) = \lim_{\vec{k}\to 0}\widetilde{\mathcal{R}}_*\left(0,\vec{k}\right) = \mathrm{R}_*(0)\,, \tag{4.25}$$

so that eq. (4.22) reduces to

$$\lim_{\vec{k}\to 0} i\tilde{c}_*\left(\omega,\vec{k}\right) = -2\pi i\delta\left(\omega\right)\left[\Sigma_* - \mathrm{R}_*\left(0\right)\right] - \frac{1}{\omega}\left[\bar{\mathrm{R}}_*\left(\epsilon-i\omega\right) - \bar{\mathrm{R}}_*\left(\epsilon+i\omega\right)\right], \tag{4.26}$$

with the second term regular as $\omega\to 0$.[12] Only the first term affects the presence of massless Goldstone excitations in the spectrum, which exist if $\Sigma_* - \mathrm{R}_*(0)\neq 0$. This completes the proof of the generalised Goldstone's theorem at finite temperature in the presence of a nonzero $\mathrm{R}_*(0)$.

It is worth commenting on the result above, especially in relation with the usual Goldstone's theorem at finite temperature. As already discussed above, a nonzero $\mathrm{R}_*(0)$ can only appear if the pseudoscalar-pseudoscalar correlator develops a $1/m$ divergence as $m\to 0$, cancelling out the factor of m appearing in the Ward-Takahashi identity and leaving a finite contribution in the chiral limit. This mechanism is reminiscent of the formation of

[11] These assumptions can be weakened. The effects of a transport peak can be taken into account, see footnote 12. Concerning the behaviour at $\omega=0$, it is sufficient to assume that $\tilde{c}^P$ has an integrable singularity, which can further be demanded only in the relevant chiral and zero-momentum limits.

[12] In the presence of a transport peak $\tilde{c}^P(\omega,\vec{k})|_{\mathrm{tp}} = 2\pi A_{PP}(\vec{k})\omega\delta(\omega)$, the principal value prescription must be kept in eq. (4.26), and since $\mathrm{PV}\frac{1}{\omega}\omega\delta(\omega) = \frac{\omega^2}{\omega^2+\epsilon^2}\delta(\omega) = 0$ one finds no contribution to $\tilde{c}_*(\omega,\vec{k}\to 0)$ from the second term. Setting $A_* = \lim_{\vec{k}\to 0}\lim_{m\to 0} 2mA_{PP}(\vec{k})$, from eq. (2.14) it follows $\lim_{\epsilon\to 0}\bar{\mathrm{R}}_*(\epsilon) = \mathrm{R}_*(0) - A_*$, and so the delta term is changed to $-2\pi i\delta(\omega)[\Sigma_* - \mathrm{R}_*(0) + A_*]$.

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anomalies, although here one is sensitive to the infrared rather than the ultraviolet regime of the theory. For this reason, I will refer to a nonzero $\mathrm{R}_*(0)$ as an *anomalous remnant*.

It is clear from eq. (4.26) that in principle $\mathrm{R}_*(0)$ could cancel Σ_*, so that in spite of the apparent spontaneous breaking of a continuous symmetry by a nonzero expectation value of an order parameter one would find no Goldstone excitations, as one would expect from the usual Goldstone's theorem at finite temperature. However, an anomalous remnant radically modifies the usual Ward-Takahashi identity in the chiral limit, signalling that the non-singlet axial currents are *not* conserved in this limit. More precisely, the anomalous remnant makes the non-singlet axial flavour symmetry *explicitly* broken even in the chiral limit. As a consequence, one evades Goldstone's theorem at finite temperature, since this has current conservation as one of its main hypotheses. The presence of massless quasi-particle excitations is not guaranteed by a nonzero condensate alone, and it is rather the difference between the amount of spontaneous breaking, measured by Σ_*, and of explicit breaking, measured by $\mathrm{R}_*(0)$, that determines the fate of Goldstone excitations.[13]

It is interesting to notice that the presence of a pole at zero spatial momentum in the correlator $\vec{\tilde{\mathcal{G}}}_{AP}(0,\vec{k})$ is after all sufficient to infer the existence of massless Goldstone excitations. In fact, the residue at this pole equals $\Sigma_* - \widetilde{\mathcal{R}}_*(0,\vec{0}) = \Sigma_* - \mathrm{R}_*(0)$ up to a constant factor [see eq. (4.14)], so that the pole is present if and only if a Dirac-delta term is present in the axial-vector-pseudoscalar spectral function.

As a final remark, I mention that one can extend the calculation done above to the case of finite fermion mass without any difficulty. For the singular part of the spectral function $\tilde{c}(\omega,\vec{k})$ one finds

$$\lim_{\vec{k}\to 0} i\tilde{c}\left(\omega,\vec{k}\right)|_{\mathrm{sing}} = -2\pi i\delta(\omega)[\Sigma - \mathrm{R}(0)]\,, \tag{4.27}$$

analogously to eq. (4.26). At finite quark mass one generally finds a nonzero $\mathrm{R}(0) = \lim_{\vec{k}\to 0}\widetilde{\mathcal{R}}(0,\vec{k}) = \int d^4x\, 2m\mathcal{G}_{PP}(x) = 2m\chi_\pi$, where χ_π is the pseudoscalar susceptibility, and a nonvanishing condensate Σ. However, one can show that at finite m

$$\Sigma = \mathrm{R}(0) = 2m\chi_\pi\,, \tag{4.28}$$

and so the singular part of $\tilde{c}$ vanishes and no massless quasi-particle excitation appears, as one expects when $m \neq 0$. This requires one to show that there is no pole in $\vec{\tilde{\mathcal{G}}}_{AP}(0,\vec{k})$ at $\vec{k} = 0$ [see eq. (4.10)]. Equivalently, one notices that eq. (4.28) is just the usual integrated Ward-Takahashi identity relating the chiral condensate and the pseudoscalar susceptibility, which holds if one can drop the boundary term when integrating eq. (3.17) over spacetime. Both the absence of a pole and the vanishing of the boundary term follow if $\vec{\mathcal{G}}_{AP}(x)$ falls off sufficiently fast at large distances. This can be shown to be the case. An argument by Vafa and Witten [81, 112] establishes an exponential bound on the two-point correlation function of any flavour non-singlet gauge-invariant local operator, at any nonzero m in theories with a positive path-integral measure. This applies in particular to the axial-vector currents A^a_μ, and to the finite-temperature gauge theories considered here. Using the

[13] I show below in section 5.7 that $|\Sigma_*| - |\mathrm{R}_*(0)| = -(\Sigma_* - \mathrm{R}_*(0)) \geq 0$.

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Ward-Takahashi identity

$$\partial_\mu \langle A^a_\mu(x) A^b_\nu(0)\rangle_\beta = 2m\langle P^a(x) A^b_\nu(0)\rangle_\beta = 2m\delta^{ab}\mathcal{G}_{AP\,\nu}(x)\,, \tag{4.29}$$

obtained by setting $\mathcal{O} = A^b_\nu(0)$ in eq. (3.12), this implies an exponential bound on $\vec{\mathcal{G}}_{AP}$, and so the desired result follows.[14] Notice that if $\Sigma_* \neq 0$, eq. (4.28) implies a divergent pseudoscalar susceptibility in the chiral limit, and a finite limit for $\mathrm{R}(0) \to \Sigma_*$ as $m \to 0$. This limit in general does *not* coincide with $\mathrm{R}_*(0)$, which is obtained by taking first the chiral limit followed by the zero-momentum limit [see comment before eq. (4.12)], since the two limits generally do not commute. While the localisation properties of Dirac modes play no role in establishing the integrated Ward-Takahashi identity eq. (4.28), I will show below in section 5 that they are crucial in the determination of $\mathrm{R}_*(0)$.

5 Pseudoscalar correlator in the chiral limit

As discussed in the previous section, it is usually assumed that the contribution $\mathcal{R}$ [see eq. (4.8)] of the pseudoscalar-pseudoscalar correlator to the non-singlet axial Ward-Takahashi identity eq. (3.14) vanishes in the chiral limit. As anticipated, I argue now that this is not the case if a finite density of localised modes is found near the origin of the Dirac spectrum: under certain technical conditions, such modes lead to the development of a $1/m$ infrared divergence, that compensates the factor of m and thus gives a finite contribution to the Ward-Takahashi identity also in the chiral limit. As discussed above at the end of section 4.3, this leads to an anomalous remnant $\mathrm{R}_*(0)$ [see eq. (4.15)] that competes with the chiral condensate to determine the fate of Goldstone excitations.

In this section I initially work with "bare", unrenormalised quantities (denoted by a subscript B), appearing in a suitably regularised version of the path integral, eq. (3.2). This could be, e.g., a lattice regularisation with Ginsparg-Wilson fermions [95–103], that guarantees control over the chiral properties of the theory [104–107]. However, as explained in section 3.5, it is justified to work directly in the continuum, which allows for a simpler and clearer treatment of the main issues, without having to deal with the technicalities of the lattice approach.

The starting point is the decomposition of the bare pseudoscalar correlator $\langle P^a_B(x) P^b_B(0)\rangle_\beta$ in terms of the eigenmodes of the Dirac operator. For infrared (IR) regularisation purposes one works in a finite spatial volume V, imposing periodic boundary conditions in the spatial directions. Antiperiodic boundary conditions are imposed instead in the (compact) time direction due to the antiperiodicity condition on fermion fields, see section 3. In this setting the spectrum of $\not{D}$ becomes discrete, and so the eigenvalues $i\lambda_n$ and the corresponding eigenvectors ψ_n will be labeled by an index n taking integer values. Moreover, as discussed above in section 3.5, for ultraviolet (UV) regularisation purposes one cuts off the spectrum at some scale Λ, including in the mode decompositions only modes with $|\lambda_n| \leq \Lambda$. Cutting off modes in this way makes the chiral condensate and the relevant correlation functions

[14] In fact, the bound of ref. [81] is easily extended to the two-point function of any pair of flavour non-singlet local bilinear operators like A^a_μ and P^a, so it applies directly to $\vec{\mathcal{G}}_{AP}(x)$.

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finite, introducing violations in the Ward-Takahashi identity that, however, disappear as $\Lambda \to \infty$. These violations, as well as UV divergences, are of no concern here, since they originate in the UV part of the Dirac spectrum, while in the chiral limit only the low-end, IR part of the spectrum plays a role, as will become clear below. Nonetheless, in spite of the fact that they have no physical effect, UV modes should be carefully handled to obtain physically meaningful, renormalised quantities.

5.1 Mode decomposition

A straightforward calculation gives the following result for the pseudoscalar-pseudoscalar correlator expressed in terms of Dirac modes,

$$-\langle P_B^a(x) P_B^b(0)\rangle_\beta = \lim_{V\to\infty} \frac{\delta^{ab}}{2} \left\langle \sum_{n,n'} \frac{\mathcal{O}^{\gamma_5}_{n'n}(x)\mathcal{O}^{\gamma_5}_{nn'}(0)}{(i\lambda_n + m_B)(i\lambda_{n'} + m_B)} \right\rangle_\beta \equiv -\delta^{ab}\mathcal{G}_{PP\,B}(x)\,. \tag{5.1}$$

Here and in the following equations, only expectation values of functionals $\mathcal{O}_{\mathrm{G}}[B_B]$ of the (bare) gauge fields appear. In eq. (5.1) m_B denotes the bare fermion mass, and tr denotes the trace over Dirac and colour indices. The dependence of $\mathcal{G}_{PP\,B}$ on m_B and on the temperature $T = 1/\beta$ is left implicit. Moreover [see eq. (3.5)],

$$\mathcal{O}^\Gamma_{n'n}(x) \equiv (\psi_{n'}(x), \Gamma\psi_n(x))\,. \tag{5.2}$$

In this paper I will be concerned only with $\Gamma = \mathbf{1}, \gamma_5$, in which case the following properties hold,

$$\mathcal{O}^\Gamma_{-n'-n}(x) = \mathcal{O}^\Gamma_{n'n}(x)\,, \qquad \mathcal{O}^\Gamma_{n'n}(x)^* = \mathcal{O}^\Gamma_{nn'}(x)\,, \qquad \mathcal{O}^\Gamma_{-n'n}(x) = \mathcal{O}^{\gamma_5\Gamma}_{n'n}(x)\,, \tag{5.3}$$

where the notation $-n$ indicates that the mode $\psi_{-n} \equiv \gamma_5\psi_n$ is involved. As mentioned above, the sums over modes in eq. (5.1), as well as the product in the determinant $\mathrm{Det}(\not{D} + m) = \prod_n (i\lambda_n + m)^{N_f}$ appearing in eq. (3.4), are restricted to $|\lambda_n| \le \Lambda$.

Taking into account the symmetry of the spectrum, the relation between eigenvectors implied by the chiral property discussed in section 3.1, and the properties eq. (5.3), one can recast eq. (5.1) as

$$-\mathcal{G}_{PP\,B}(x) = \lim_{V\to\infty} \frac{1}{2}\int_{-\Lambda}^{\Lambda} d\lambda \int_{-\Lambda}^{\Lambda} d\lambda' \, \frac{(m_B^2 - \lambda\lambda')C^{\gamma_5}_{V,\Lambda}(\lambda,\lambda';x;m_B)}{(\lambda^2 + m_B^2)(\lambda'^2 + m_B^2)}\,, \tag{5.4}$$

where I introduced the *spectral correlators*

$$C^\Gamma_{V,\Lambda}(\lambda,\lambda';x;m_B) \equiv \left\langle \sum_{n,n'} \delta(\lambda - \lambda_n)\delta(\lambda' - \lambda_{n'}) \mathrm{Re}\left\{\mathcal{O}^\Gamma_{n'n}(x)\mathcal{O}^\Gamma_{nn'}(0)\right\} \right\rangle_\beta\,, \tag{5.5}$$

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and made explicit their dependence on the bare fermion mass, the volume, and the UV regulator. It is convenient to separate the cases $\lambda_n = \pm\lambda_{n'}$ from the rest, and set[15]

$$\begin{aligned} C^{\Gamma}_{s\,V,\Lambda}(\lambda;x;m_B) &\equiv \Big\langle \sum_n \delta(\lambda-\lambda_n)\mathcal{O}^{\Gamma}_{nn}(x)\mathcal{O}^{\Gamma}_{nn}(0)\Big\rangle_\beta\,, \\ \bar{C}^{\Gamma}_{V,\Lambda}(\lambda,\lambda';x;m_B) &\equiv \Big\langle \sum_{\substack{n,n'\\ n\neq\pm n'}} \delta(\lambda-\lambda_n)\delta(\lambda'-\lambda_{n'})\mathrm{Re}\left\{\mathcal{O}^{\Gamma}_{n'n}(x)\mathcal{O}^{\Gamma}_{nn'}(0)\right\}\Big\rangle_\beta\,, \end{aligned} \tag{5.6}$$

in terms of which one has

$$\begin{aligned} C^{\gamma_5}_{V,\Lambda}(\lambda,\lambda';x;m_B) = \delta(\lambda+\lambda')C^{\mathbf{1}}_{s\,V,\Lambda}(\lambda;x;m_B) + \delta(\lambda-\lambda')C^{\gamma_5}_{s\,V,\Lambda}(\lambda;x;m_B) \\ + \bar{C}^{\gamma_5}_{V,\Lambda}(\lambda,\lambda';x;m_B)\,. \end{aligned} \tag{5.7}$$

In the language of random Hamiltonians (see section 3.1), the quantities defined in eqs. (5.5) and (5.6) are a type of Green's functions measuring the correlation between eigenmodes. Using eq. (5.3) one finds the following symmetry relations,

$$\begin{aligned} \bar{C}^{\Gamma}_{V,\Lambda}(-\lambda,-\lambda';x;m_B) &= \bar{C}^{\Gamma}_{V,\Lambda}(\lambda,\lambda';x;m_B)\,, & \bar{C}^{\Gamma}_{V,\Lambda}(\lambda',\lambda;x;m_B) &= \bar{C}^{\Gamma}_{V,\Lambda}(\lambda,\lambda';x;m_B)\,, \\ \bar{C}^{\Gamma}_{V,\Lambda}(-\lambda,\lambda';x;m_B) &= \bar{C}^{\gamma_5\Gamma}_{V,\Lambda}(\lambda,\lambda';x;m_B)\,, & C^{\Gamma}_{s\,V,\Lambda}(-\lambda;x;m_B) &= C^{\Gamma}_{s\,V,\Lambda}(\lambda;x;m_B)\,. \end{aligned} \tag{5.8}$$

As they hold in any volume, these relations will hold also in the thermodynamic limit $V\to\infty$.

5.2 Large-volume limit

In this subsection I discuss the large-volume limit of the spectral correlators, and show that here the localisation properties of the Dirac eigenmodes play a crucial role: while they do not affect the large-volume behaviour of $\bar{C}^{\Gamma}_{V,\Lambda}$, they strongly affect whether $C^{\Gamma}_{s\,V,\Lambda}$ survives the infinite-volume limit or not.

In order to see how the localisation properties of the modes affect their contribution to the spectral correlators, eq. (5.6), notice first the exact bounds

$$|\mathcal{O}^{\Gamma}_{n'n}(x)|^2 = |(\psi_{n'}(x),\Gamma\psi_n(x))|^2 \leq \|\Gamma\|^2\|\psi_{n'}(x)\|^2\,\|\psi_n(x)\|^2\,, \tag{5.9}$$

where the matrix norm $\|\Gamma\|=1$ for $\Gamma=\mathbf{1},\gamma_5$, and

$$\begin{aligned} \Big|\langle\mathcal{O}^{\Gamma}_{n'n}(x)\mathcal{O}^{\Gamma}_{nn'}(0)\rangle_\beta\Big| &\leq \langle\Big|\mathcal{O}^{\Gamma}_{n'n}(x)\Big|\,\Big|\mathcal{O}^{\Gamma}_{nn'}(0)\Big|\rangle_\beta \\ &\leq \|\Gamma\|^2\langle\|\psi_{n'}(x)\|\,\|\psi_n(x)\|\,\|\psi_{n'}(0)\|\,\|\psi_n(0)\|\rangle_\beta \\ &\leq \|\Gamma\|^2\sqrt{\langle\|\psi_{n'}(x)\|^2\,\|\psi_n(x)\|^2\rangle_\beta}\sqrt{\langle\|\psi_{n'}(0)\|^2\,\|\psi_n(0)\|^2\rangle_\beta} \\ &= \|\Gamma\|^2\langle\|\psi_{n'}(x)\|^2\,\|\psi_n(x)\|^2\rangle_\beta\,. \end{aligned} \tag{5.10}$$

[15] Strictly speaking, in the definition of $\bar{C}^{\Gamma}_{V,\Lambda}$ the condition $n\neq\pm n'$ should read $\lambda_n\neq\pm\lambda_{n'}$, while contributions from distinct but exactly degenerate modes, $\lambda_n=\lambda_{n'}$ with $n\neq n'$, should be included in $C^{\Gamma}_{s\,V,\Lambda}$. However, zero modes do not contribute in the thermodynamic limit (see below), and degenerate nonzero modes are expected to appear only on a set of configurations of zero measure, and can be ignored. See footnote 20 for further comments.

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These follow from the Cauchy-Schwarz inequality, on the third line applied to the expectation value $\langle \ldots \rangle_\beta$ which is defined by a path-integral with positive-definite integration measure. Translation invariance is used in the last passage. In a first approximation, one can treat these bounds more loosely as estimates of the magnitude of $\mathcal{O}^\Gamma_{n'n}$ and their correlation functions, in order to obtain the volume dependence of the individual contributions to eq. (5.6). While these estimates should be supplemented by suitable factors in order to take into account the dependence on Γ and, more importantly, on the correlation between modes n and n' and on the distance between 0 and x, they should suffice if all one is interested in is their volume dependence.

Consider first the case $n \neq \pm n'$. If modes n near λ have fractal dimension $\alpha(\lambda)$, i.e., are mostly supported in regions whose size scales like $V^{\alpha(\lambda)}$, then $\|\psi_n(x)\|^2 \sim V^{-\alpha(\lambda)}$ inside the supporting region while being negligible outside. Translation invariance implies that the probability of finding a given spacetime point inside the support of the mode is $V^{\alpha(\lambda)}/V$. Finally, the correlation between modes is expected to decrease as $|\lambda_n - \lambda_{n'}|$ increases, and so it will be small for most pairs of modes. One can then estimate

$$\langle \|\psi_{n'}(x)\|^2 \, \|\psi_n(x)\|^2 \rangle_\beta \sim \left(\frac{1}{V^{\alpha(\lambda')}} \frac{V^{\alpha(\lambda')}}{V} \right) \left(\frac{1}{V^{\alpha(\lambda)}} \frac{V^{\alpha(\lambda)}}{V} \right) = \frac{1}{V^2} \,, \tag{5.11}$$

irrespectively of the localisation properties of the modes. From the last line in eq. (5.10) one then estimates

$$\left| \langle \mathcal{O}^\Gamma_{n'n}(x) \mathcal{O}^\Gamma_{nn'}(0) \rangle_\beta \right| \sim \frac{1}{V^2} \,. \tag{5.12}$$

Since $\bar{C}^\Gamma_{V,\Lambda}$ involves a double sum over modes, and since the number of modes per unit spectral interval typically scales like the volume,[16] one has $O(V^2)$ contributions of order $O(1/V^2)$. One then expects $\bar{C}^\Gamma_\Lambda(\lambda, \lambda'; x; m_B) \equiv \lim_{V\to\infty} \bar{C}^\Gamma_{V,\Lambda}(\lambda, \lambda'; x; m_B)$ to be nonzero as long as both spectral densities $\rho_B(\lambda)$ and $\rho_B(\lambda')$ are nonzero, independently of the localisation properties of modes near λ and λ'. The *bare spectral density* is defined as

$$\rho_{B,V}(\lambda) \equiv \frac{1}{\beta V} \left\langle {\sum_n}' \delta(\lambda - \lambda_n) \right\rangle_\beta \,, \qquad \rho_B(\lambda) \equiv \lim_{V\to\infty} \rho_{B,V}(\lambda) \,, \tag{5.13}$$

with $\rho_{B,V}$ the bare spectral density in a finite volume. Here $\sum'_n$ denotes the sum over nonzero modes only. Exact zero modes are explicitly excluded even though they would not contribute in the thermodynamic limit, since their number scales only like $N_0 \sim \sqrt{V}$ at large volume. As a further consequence, since the estimate eq. (5.12) applies in particular to pairs of distinct zero modes, and pairs of a zero and a nonzero mode, one finds that the corresponding total contributions scale respectively like V/V^2 and $V\sqrt{V}/V^2$, and so contributions involving zero modes can be dropped from $\bar{C}^\Gamma_{V,\Lambda}$ in eq. (5.6).

The situation is different in the case $n = \pm n'$, where correlations cannot be neglected. Using the last line in eq. (5.10), one can estimate for modes near λ that

$$\left| \langle \mathcal{O}^\Gamma_{nn}(x) \mathcal{O}^\Gamma_{nn}(0) \rangle_\beta \right| \sim \langle \|\psi_n(x)\|^4 \rangle_\beta = \frac{1}{\beta V} \langle \mathrm{IPR}_n \rangle_\beta \sim \frac{1}{V V^{\alpha(\lambda)}} \,, \tag{5.14}$$

[16] The case in which the growth is faster than $O(V)$, leading to points in the spectrum where the spectral density has an integrable singularity in the infinite-volume limit (van Hove singularity), is not considered here.

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where translation invariance was used. Since a single sum over modes appears in $C^{\Gamma}_{sV,\Lambda}$, there are $O(V)$ such contributions in any spectral region with a finite density of modes, and so one expects $C^{\Gamma}_{sV,\Lambda}(\lambda;x) \to 0$ in the large-volume limit *except if modes near* λ *are localised*, in which case one expects to find a finite value. Equation (5.14) can be turned into a rigorous bound showing that $C^{\Gamma}_{sV,\Lambda}(\lambda;x)$ vanishes in the large-volume limit if the fractal dimension of modes near λ is nonzero, i.e., if they are delocalised. From the last line in eq. (5.10), in fact, one has[17]

$$\begin{aligned}|C^{\Gamma}_{sV,\Lambda}(\lambda;x;m_B)| &\le \Big\langle \sum_n \delta(\lambda-\lambda_n)|\mathcal{O}^{\Gamma}_{nn}(x)||\mathcal{O}^{\Gamma}_{nn}(0)|\Big\rangle_\beta \le \Big\langle \sum_n \delta(\lambda-\lambda_n)\|\psi_n(x)\|^4\Big\rangle_\beta \\ &= \frac{1}{\beta V}\Big\langle \sum_n \delta(\lambda-\lambda_n)\mathrm{IPR}_n\Big\rangle_\beta = \rho_{B,V}(\lambda)\overline{\mathrm{IPR}}_V(\lambda) + \delta(\lambda)\frac{N_0}{\beta V}\overline{\mathrm{IPR}}^0_V\,,\end{aligned} \tag{5.15}$$

where I used translation invariance, and where

$$\overline{\mathrm{IPR}}_V(\lambda) \equiv \frac{1}{\beta V \rho_{B,V}(\lambda)}\Big\langle {\sum_n}' \delta(\lambda-\lambda_n)\mathrm{IPR}_n\Big\rangle_\beta\,, \qquad \overline{\mathrm{IPR}}^0_V \equiv \frac{1}{N_0}\Big\langle \sum_{\substack{n_0\\ \lambda_{n_0}=0}} \mathrm{IPR}_{n_0}\Big\rangle_\beta\,, \tag{5.16}$$

is the average IPR computed locally in the spectrum.[18] Assuming that modes have fractal dimension $\alpha(\lambda)$ near λ, one has $\overline{\mathrm{IPR}}_V(\lambda) \sim V^{-\alpha(\lambda)}$. If modes are delocalised near $\lambda \neq 0$, i.e., $\alpha(\lambda) > 0$, one finds that $C^{\Gamma}_{sV,\Lambda}$ vanishes in the thermodynamic limit. On the other hand, if modes are localised then $\alpha(\lambda) = 0$ and $C^{\Gamma}_{sV,\Lambda}$ need not vanish, and as shown above one expects that it does not.[19] One then expects $C^{\Gamma}_{\mathrm{loc}\,\Lambda} \equiv \lim_{V\to\infty} C^{\Gamma}_{sV,\Lambda}$ to be finite where a finite density of localised modes is present and only there, as signalled by the subscript "loc". An explicit calculation shows that this is the case (see section 5.6 below). Notice that since the last term in eq. (5.15) scales like $\sqrt{V}/V$, exact zero modes give no contribution to $C^{\Gamma}_{sV,\Lambda}$ in the thermodynamic limit independently of their localisation properties.[20]

Summarising, in the thermodynamic limit one finds that $\bar{C}^{\Gamma}_{\Lambda}$ is nonzero where the spectral density is nonzero, while $C^{\Gamma}_{\mathrm{loc}\,\Lambda}$ is nonzero only in spectral regions where modes are

[17] The Dirac deltas in eq. (5.15) can be handled rigorously, without changing the result, by integrating first over small intervals around λ, λ', using then eq. (5.10), and finally taking the limit of infinitesimal intervals.

[18] The dependence of $\overline{\mathrm{IPR}}_V$ on Λ and m_B is irrelevant here and has been suppressed. In computing $\overline{\mathrm{IPR}}^0_V$ one should average over the degenerate zero-mode subspace using the procedure of ref. [30], treating separately the two chiralities, but this would not change the fact that zero modes do not contribute here (see below).

[19] While capturing correctly the overall volume dependence, the simple estimate eq. (5.14) entirely misses the x-dependence of the spectral correlators. Taking localisation literally, one would find $\big|\langle \mathcal{O}^{\Gamma}_{nn}(x)\mathcal{O}^{\Gamma}_{nn}(0)\rangle_\beta\big| \sim \frac{1}{\ell^3}$ for spatial separation smaller than the typical localisation length ℓ, and zero otherwise. A more precise analysis is carried out in appendix E using a more realistic exponential envelope of localised modes, that leads to expect an exponential suppression in $|\vec{x}|$ rather than strictly finite support.

[20] Exactly degenerate but distinct nonzero modes could also contribute to $C^{\Gamma}_{sV,\Lambda}$. For these modes the estimate eq. (5.12) applies so that their contribution is suppressed like $N_{\mathrm{deg}}V/V^2 = N_{\mathrm{deg}}V^{-1}$ in the thermodynamic limit, with N_{deg} their typical (possibly λ-dependent) degeneracy, unless they are strongly spatially correlated. In this case, the estimate eq. (5.14) applies instead, leading to a total contribution of order $N_{\mathrm{deg}}V^{-\alpha}$. These modes can then be relevant only if localised, strongly spatially correlated, and appearing on a set of configurations of finite measure. This seems very unlikely to happen.

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localised.[21] In the infinite-volume limit eq. (5.4) then reduces to

$$-\mathcal{G}_{PP\,B}\,(x) = \int_0^\Lambda d\lambda \left(\frac{C^{\mathbf{1}}_{\mathrm{loc}\,\Lambda}\,(\lambda;x;m_B)}{\lambda^2+m_B^2} + \frac{(m_B^2-\lambda^2)\,C^{\gamma_5}_{\mathrm{loc}\,\Lambda}\,(\lambda;x;m_B)}{(\lambda^2+m_B^2)^2} \right) + \frac{1}{2}\int_{-\Lambda}^{\Lambda} d\lambda \int_{-\Lambda}^{\Lambda} d\lambda' \, \frac{(m_B^2-\lambda\lambda')\,\bar{C}^{\gamma_5}_{\Lambda}\,(\lambda,\lambda';x;m_B)}{(\lambda^2+m_B^2)\,(\lambda'^2+m_B^2)}\,, \tag{5.17}$$

having used the symmetry relations eq. (5.8).

5.3 Renormalisation

The bare pseudoscalar-pseudoscalar correlator needs to be suitably renormalised before the UV cutoff Λ is removed. As discussed in detail in appendix C, both additive and multiplicative renormalisation are required, and the renormalised pseudoscalar-pseudoscalar correlator reads

$$\mathcal{G}_{PP}(x) = \lim_{\Lambda\to\infty} Z_m^2\,[\mathcal{G}_{PP\,B}(x) - \mathrm{CT}_{PP}(x)]\,, \tag{5.18}$$

where $\mathrm{CT}_{PP}(x)$ are divergent contact terms and Z_m is the mass renormalisation constant, with the renormalised mass defined as $m_B = Z_m m$. Divergent contact terms originate from large Dirac eigenvalues in eq. (5.17). However, they are polynomial in the fermion mass and so drop from $\mathcal{R}$ in the chiral limit, and therefore can essentially be ignored as far as the Ward-Takahashi identity, eqs. (3.14) and (3.17), is concerned. More generally, all the contributions to $\mathcal{R}$ coming from large eigenvalues vanish in the chiral limit, including any finite term remaining after the subtraction procedure, since they yield at most a constant term in the pseudoscalar-pseudoscalar correlator as $m \to 0$. Multiplicative renormalisation, on the other hand, is required also in the chiral limit. To identify the divergent contact terms, one splits the integrals in eq. (5.17) at a suitably chosen m-independent subtraction scale. To disentangle additive and multiplicative divergences, it is convenient to work with renormalised spectral correlators. This also allows one to see more clearly how the remaining finite terms behave in the chiral limit.

The procedure is most easily illustrated in the case of the chiral condensate. The bare chiral condensate Σ_B is obtained from the spectral density, eq. (5.13), via the Banks-Casher relation [34],

$$-\Sigma_B = \int_0^\Lambda d\lambda\,\rho_B(\lambda,m_B)\frac{2m_B}{\lambda^2+m_B^2}\,. \tag{5.19}$$

Here the dependence of ρ_B on the bare mass has been made explicit. The spectral density grows like λ^3 at large λ, thus leading to quadratic and logarithmic additive divergences, which originate from the mixing of the scalar density with the identity operator (see appendix C). A possible renormalisation scheme to take care of them is the following (see, e.g, ref. [113]). One splits the integral at some (mass-independent) subtraction scale μ_B,

[21] As pointed out in section 3, localised and delocalised modes usually do not coexist in the same spectral region. However, the qualitative estimate given above under eq. (5.16) does not depend on this non-coexistence assumption. If for some system one could separate localised and delocalised modes in a given spectral region and define the density $\rho_{\mathrm{loc}}(\lambda)$ of localised modes, then $C^{\Gamma}_{\mathrm{loc}\,\Lambda}$ would be nonzero where $\rho_{\mathrm{loc}} \neq 0$.

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and for $\lambda > \mu_B$ one expands the denominator in powers of m_B^2/λ^2, obtaining

$$\begin{aligned} -\Sigma_B = &\int_0^{\mu_B} d\lambda\, \rho_B(\lambda, m_B) \frac{2m_B}{\lambda^2 + m_B^2} + \int_{\mu_B}^{\Lambda} d\lambda\, \rho_B(\lambda, m_B) \frac{2m_B^5}{\lambda^4(\lambda^2 + m_B^2)} \\ &+ \int_{\mu_B}^{\Lambda} d\lambda\, \rho_B(\lambda, m_B) \left(\frac{2m_B}{\lambda^2} - \frac{2m_B^3}{\lambda^4} \right) = \Sigma_B^{(1)} + \Sigma_B^{(2)} \,. \end{aligned} \tag{5.20}$$

The integrals on the first line, that define $\Sigma_B^{(1)}$, are formally convergent as $\Lambda \to \infty$, i.e., if one ignores that the bare spectral density is also Λ-dependent, and furthermore are made finite by multiplicative renormalisation. In fact, multiplying by Z_m one finds after a change of variables

$$Z_m \Sigma_B^{(1)} = \int_0^{\frac{\mu_B}{Z_m}} d\lambda\, Z_m \rho_B\left(Z_m\lambda, Z_m m\right) \frac{2m}{\lambda^2 + m^2} + \int_{\frac{\mu_B}{Z_m}}^{\frac{\Lambda}{Z_m}} d\lambda\, Z_m \rho_B\left(Z_m\lambda, Z_m m\right) \frac{2m^5}{\lambda^4\left(\lambda^2 + m^2\right)} \,. \tag{5.21}$$

As it was shown in refs. [114, 115], the quantity

$$\rho(\lambda, m) \equiv \lim_{\Lambda\to\infty} Z_m \rho_B(Z_m\lambda, Z_m m) \tag{5.22}$$

is finite, and so eq. (5.21) has a finite limit as $\Lambda \to \infty$ if $\mu \equiv \mu_B/Z_m$ is kept fixed. Notice that since Z_m depends on Λ only logarithmically, the (bare) subtraction scale $\mu_B = Z_m\mu$ has to depend logarithmically on the cutoff, while Λ/Z_m still diverges as $\Lambda \to \infty$. The terms on the second line of eq. (5.20), that define $\Sigma_B^{(2)}$, remain divergent also after multiplication by Z_m, and need to be subtracted. One then defines the renormalised condensate as

$$-\Sigma \equiv \lim_{\Lambda\to\infty} -Z_m \Sigma_B^{(1)} = \int_0^{\mu} d\lambda\, \rho(\lambda, m) \frac{2m}{\lambda^2 + m^2} + \int_{\mu}^{\infty} d\lambda\, \rho(\lambda, m) \frac{2m^5}{\lambda^4(\lambda^2 + m^2)} \,. \tag{5.23}$$

The second term is at least of order $O(m^5)$ and vanishes in the chiral limit. For the first term one finds instead the well-known result [34]

$$-\Sigma_* = \lim_{m\to 0} \rho(0, m) \int_0^{\frac{\mu}{m}} dz\, \frac{2}{1 + z^2} + \lim_{m\to 0} \int_0^{\mu} d\lambda\, \frac{2m f(\lambda)}{\lambda^2 + m^2} = \pi\rho(0, 0) \,, \tag{5.24}$$

where $\rho(0, m) \equiv \lim_{\lambda\to 0} \rho(\lambda, m)$ and $\rho(0, 0) \equiv \lim_{m\to 0} \rho(0, m)$, and where $f(\lambda) = \rho(\lambda, m) - \rho(0, m)$ is assumed to vanish at least as fast as some power law as $\lambda \to 0$, i.e., $\lambda^{-\gamma} f(\lambda) \to 0$ as $\lambda \to 0$ for some $\gamma > 0$. In fact, in this case, the second term in eq. (5.24) is of order at most $m^{\frac{1}{2^N}}$ for some sufficiently large N such that $2^N\gamma > 1$, as shown in appendix F [see eq. (F.6)].

The same procedure can now be repeated for the pseudoscalar-pseudoscalar correlator. In this case, after splitting the integrals and multiplying by the renormalisation constant Z_m^2, one finds

$$\mathcal{G}_{PP\,B}(x) = \sum_{i,j=1}^{2} \mathcal{G}_{PP\,B}^{(ij)}(x) \,, \tag{5.25}$$

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with

$$
\begin{aligned}
-Z_m^2 \mathcal{G}^{(ij)}_{PP\,B}(x) = \delta_{ij} \int_{I_i} d\lambda\, \frac{Z_m C^{\mathbf{1}}_{\mathrm{loc}\,\Lambda}(Z_m\lambda; x; Z_m m)}{\lambda^2 + m^2} \\
+ \delta_{ij} \int_{I_i} d\lambda\, \frac{(m^2 - \lambda^2)\, Z_m C^{\gamma_5}_{\mathrm{loc}\,\Lambda}(Z_m\lambda; x; Z_m m)}{(\lambda^2 + m^2)^2} \\
+ \int_{I_i} d\lambda \int_{I_j} d\lambda'\, \frac{(m^2 + \lambda\lambda')\, Z_m^2 \bar{C}^{\mathbf{1}}_{\Lambda}(Z_m\lambda, Z_m\lambda'; x; Z_m m)}{(\lambda^2 + m^2)\,(\lambda'^2 + m^2)} \\
+ \int_{I_i} d\lambda \int_{I_j} d\lambda'\, \frac{(m^2 - \lambda\lambda')\, Z_m^2 \bar{C}^{\gamma_5}_{\Lambda}(Z_m\lambda, Z_m\lambda'; x; Z_m m)}{(\lambda^2 + m^2)\,(\lambda'^2 + m^2)}\,,
\end{aligned}
\tag{5.26}
$$

where $I_1 = [0, \mu]$ and $I_2 = [\mu, \Lambda/Z_m]$. Here the properties eq. (5.8) have been used to restrict the double integral on the second line of eq. (5.17) to the positive part of the spectrum. It is clear that additive divergences can originate only from $\mathcal{G}^{(12)}_{PP\,B}$, $\mathcal{G}^{(21)}_{PP\,B}$, and $\mathcal{G}^{(22)}_{PP\,B}$. The symmetry of the integrand under $\lambda \leftrightarrow \lambda'$ implies $\mathcal{G}^{(12)}_{PP\,B} = \mathcal{G}^{(21)}_{PP\,B}$. Using an argument similar to that of refs. [114, 115], I show in appendix G that the functions

$$
\begin{aligned}
C^{\Gamma}_{\mathrm{loc}}(\lambda; x; m) &\equiv \lim_{\Lambda\to\infty} Z_m C^{\Gamma}_{\mathrm{loc}\,\Lambda}(Z_m\lambda; x; Z_m m)\,, \\
\bar{C}^{\,\Gamma}(\lambda, \lambda'; x; m) &\equiv \lim_{\Lambda\to\infty} Z_m^2 \bar{C}^{\Gamma}_{\Lambda}(Z_m\lambda, Z_m\lambda'; x; Z_m m)\,,
\end{aligned}
\tag{5.27}
$$

are renormalised, finite quantities. This makes $\mathcal{G}^{(11)}_{PP} \equiv \lim_{\Lambda\to\infty} Z_m^2 \mathcal{G}^{(11)}_{PP\,B}$ finite in the large-Λ limit. It remains only to identify and subtract the additively divergent contributions to $\mathcal{G}^{(12)}_{PP\,B}$ and $\mathcal{G}^{(22)}_{PP\,B}$. For the purposes of this paper this need not be done explicitly: all that matters is that the remaining finite terms stay finite also in the chiral limit, which is easy to show. Details are provided in appendix F. One then concludes that

$$
Z_m^2 \left(\mathcal{G}^{(12)}_{PP\,B}(x) + \mathcal{G}^{(21)}_{PP\,B}(x) + \mathcal{G}^{(22)}_{PP\,B}(x) \right) = Z_m^2 \mathrm{CT}_{PP}(x) + F(x; m) + \dots\,,
\tag{5.28}
$$

with omitted terms vanishing as $\Lambda \to \infty$, and F a finite Λ-independent quantity that remains finite also as $m \to 0$. One has then $\mathcal{G}_{PP} = \mathcal{G}^{(11)}_{PP} + F$, and for the quantity of interest, namely $\mathcal{R}_* = \lim_{m\to 0} \mathcal{R}$ [see eq. (4.8)] one finds $\mathcal{R}_* = \lim_{m\to 0} 2m\mathcal{G}^{(11)}_{PP}$, and the following spectral representation,

$$
\begin{aligned}
-\mathcal{R}_*(x) = \lim_{m\to 0} 2m \int_0^{\mu} d\lambda \left(\frac{C^{\mathbf{1}}_{\mathrm{loc}}(\lambda; x; m)}{\lambda^2 + m^2} + \frac{(m^2 - \lambda^2) C^{\gamma_5}_{\mathrm{loc}}(\lambda; x; m)}{(\lambda^2 + m^2)^2} \right) \\
+ 2m \int_0^{\mu} d\lambda \int_0^{\mu} d\lambda'\, \frac{(m^2 + \lambda\lambda')\bar{C}^{\mathbf{1}}(\lambda, \lambda'; x; m) + (m^2 - \lambda\lambda')\bar{C}^{\gamma_5}(\lambda, \lambda'; x; m)}{(\lambda^2 + m^2)(\lambda'^2 + m^2)}\,.
\end{aligned}
\tag{5.29}
$$

5.4 Renormalisation of the mobility edge

The result eq. (5.27) has an important consequence for the renormalisation properties of the mobility edges. As discussed above in section 5.2, the unrenormalised spectral correlators $C^{\Gamma}_{\mathrm{loc}\,\Lambda}(\lambda; x; m_B)$ have support only in disjoint regions where modes are localised, separated by mobility edges $\lambda^{(i)}_{c\,B}$ from regions where modes are instead delocalised. Since

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renormalisation consists only in a multiplicative factor and a rescaling of λ and m_B, the support of $C^{\Gamma}_{\mathrm{loc}}(\lambda;x;m)$ is still made of disjoint regions delimited by the renormalised mobility edges $\lambda_c^{(i)} = \lim_{\Lambda\to\infty} Z_m^{-1}\lambda_{cB}^{(i)}$. In other words, the mobility edges renormalise like the fermion mass, so that the ratios $\lambda_{cB}^{(i)}/m_B$ are renormalisation-group-invariant quantities free from UV divergences, i.e., the equality

$$\frac{\lambda_{cB}^{(i)}}{m_B} = \frac{\lambda_c^{(i)}}{m} + o(\Lambda^0) \tag{5.30}$$

holds up to corrections that vanish as the UV regulator is removed. This had been suggested before [18], and was supported by numerical results on the lattice [18, 31], but had not been proved yet. Details are provided in appendix G.

5.5 Chiral limit

The final step in order to obtain $\mathcal{R}_*$ is to determine the behaviour of the various terms appearing in eq. (5.29) as $m \to 0$. To proceed it is necessary to make assumptions about the position of the mobility edge(s), if present. It is assumed from now on that modes are localised in the range $[0, Z_m\lambda_c]$ in the UV-regulated theory, and so in the range $[0, \lambda_c]$ of the "renormalised" spectrum, with $\lambda_c = \lambda_c(m)$ the renormalised mobility edge. It is also assumed that, if other regions with localised modes are present, then these are found above some renormalised lower mobility edge λ_c' and remain separated from $\lambda = 0$ in the chiral limit.[22] With these assumptions, eq. (5.29) becomes

$$\mathcal{R}_*(x) = \mathcal{R}_*^{(1)}(x) + \mathcal{R}_*^{(2)}(x) + \mathcal{R}_*^{(3)}(x)\,, \tag{5.31}$$

where

$$-\mathcal{R}_*^{(1)}(x) = \lim_{m\to 0} \quad 2m\int_0^{\lambda_c} d\lambda \left(\frac{C^{\mathbf{1}}_{\mathrm{loc}}(\lambda;x;m)}{\lambda^2+m^2} + \frac{(m^2-\lambda^2)C^{\gamma_5}_{\mathrm{loc}}(\lambda;x;m)}{(\lambda^2+m^2)^2}\right) \tag{5.32}$$

and[23]

$$-\mathcal{R}_*^{(2)}(x) = \lim_{m\to 0} 2m\int_{\lambda_c'}^{\mu} d\lambda \left(\frac{C^{\mathbf{1}}_{\mathrm{loc}}(\lambda;x;m)}{\lambda^2+m^2} + \frac{(m^2-\lambda^2)C^{\gamma_5}_{\mathrm{loc}}(\lambda;x;m)}{(\lambda^2+m^2)^2}\right) \tag{5.33}$$

receive contributions only from localised modes, while

$$-\mathcal{R}_*^{(3)}(x) = \lim_{m\to 0} 2m\int_0^{\mu} d\lambda \int_0^{\mu} d\lambda' \, \frac{(m^2+\lambda\lambda')\bar{C}^{\mathbf{1}}(\lambda,\lambda';x;m) + (m^2-\lambda\lambda')\bar{C}^{\gamma_5}(\lambda,\lambda';x;m)}{(\lambda^2+m^2)(\lambda'^2+m^2)} \tag{5.34}$$

[22] Localised modes have been observed at the high end ($|\lambda| > \lambda_c'$) of the spectrum of staggered fermions in pure gauge $\mathbb{Z}_2$ gauge theory in 2+1 dimensions on the lattice [30], and it is likely that this feature is found also in other theories. However, these are ultraviolet modes that should not affect the continuum physics. In particular, it is unlikely that λ_c' reaches down to the origin in the chiral limit.

[23] The upper limit of integration is set to μ for generality, and does not mean that modes are localised in the whole interval $[\lambda_c', \mu]$.

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receives contributions from both localised and delocalised modes. It is shown in appendix F that in the chiral limit the integrals appearing in $\mathcal{R}_*^{(1)}$, eq. (5.32), behave as follows as functions of m,

$$\begin{aligned}\int_0^{\lambda_c} d\lambda\, \frac{C^{\mathbf{1}}_{\mathrm{loc}}(\lambda;x;m)}{\lambda^2+m^2} &= \frac{1}{m}\left(\arctan\left(\frac{\lambda_c}{m}\right) C^{\mathbf{1}}_{\mathrm{loc}}(0;x;0) + o(m^0)\right),\\ \int_0^{\lambda_c} d\lambda\, \frac{(m^2-\lambda^2)C^{\gamma_5}_{\mathrm{loc}}(\lambda;x;m)}{(\lambda^2+m^2)^2} &= \frac{1}{m}\left(\frac{\frac{\lambda_c}{m}}{1+\frac{\lambda_c^2}{m^2}} C^{\gamma_5}_{\mathrm{loc}}(0;x;0) + o(m^0)\right),\end{aligned} \tag{5.35}$$

so possibly diverging like $1/m$ and leading to a finite $\mathcal{R}_*$, while the integrals appearing in $\mathcal{R}_*^{(2,3)}$ tend to constants in the chiral limit, and so give no contribution to $\mathcal{R}_*$. These results are valid under the rather mild technical assumptions that finite limits exist for the spectral correlators as $\lambda \to 0$ and/or $\lambda' \to 0$, and that such limits are approached at least as fast as some power law, and moreover that the resulting quantities have a finite limit as $m \to 0$. One then obtains the main result of this subsection,

$$-\mathcal{R}_*(x) = \pi\xi C^{\mathbf{1}}_{\mathrm{loc}}(0;x;0) + \eta C^{\gamma_5}_{\mathrm{loc}}(0;x;0)\,, \tag{5.36}$$

where

$$\xi \equiv \frac{2}{\pi}\arctan\kappa\,, \qquad \eta \equiv \frac{2\kappa}{1+\kappa^2}\,, \qquad \kappa \equiv \lim_{m\to 0}\frac{\lambda_c(m)}{m}\,. \tag{5.37}$$

Notice that κ is renormalisation-group invariant. Notice also that $C^{\Gamma}_{\mathrm{loc}}(0;x;0)$ are obtained using the following order of limits, starting from the bare, finite-volume spectral correlators,

$$C^{\Gamma}_{\mathrm{loc}}(0;x;0) = \lim_{m\to 0}\lim_{\lambda\to 0}\lim_{\Lambda\to\infty}\lim_{V\to\infty} Z_m C^{\Gamma}_{s\,V,\Lambda}(Z_m\lambda;x;Z_m m_B)\,. \tag{5.38}$$

Finally, notice that the subtraction scale μ does not affect the final result, as it should be.

Assuming that $C^{\Gamma}_{\mathrm{loc}}(0;x;0)$ is nonzero in the chiral limit, there are three possible scenarios depending on how the mobility edge scales with m in the chiral limit.

(i). If λ_c vanishes faster than m, then $\kappa \to 0$. In this case one finds $\mathcal{R}_* = 0$, so that the presence of localised near-zero modes does not affect the Ward-Takahashi identity eq. (3.14). If $\lambda_c \sim m^{\delta+1}$, then $\kappa \sim m^{\delta}$, and **(i-a)** if $\delta \geq 1$, the pseudoscalar-pseudoscalar correlator remains finite in the chiral limit, while **(i-b)** if $0 < \delta < 1$ it develops an infrared divergence $1/m^{1-\delta}$.[24]

(ii). If λ_c vanishes as fast as m, then $\kappa \to$ constant. In this case the pseudoscalar-pseudoscalar correlator develops an infrared divergence $1/m$, and one finds $\mathcal{R}_* \neq 0$. One finds for the coefficient of the first term $\pi\xi < \pi$, and the second term is finite.

(iii). If λ_c vanishes more slowly than m, including not vanishing at all,[25] then $\kappa \to \infty$. Also in this case the pseudoscalar-pseudoscalar correlator develops an infrared divergence $1/m$, and one finds $\mathcal{R}_* \neq 0$. The coefficient of the first term is maximal and equal to π in this case, while the second term vanishes.

[24] If $\delta \geq 1$, no divergence can appear in the terms omitted in eq. (5.35), which are subleading as long as $\delta > 0$, see eq. (F.17) and subsequent discussion in appendix F.

[25] The quite unlikely case of λ_c diverging in the chiral limit is contained in case (iii). In fact, below some value of m one would find $\lambda_c > \mu$, and so λ_c would be replaced by the cutoff μ in the calculations above.

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5.6 Anomalous remnant

The anomalous remnant $\mathrm{R}_*(0)$ is obtained by integrating $\mathcal{R}_*(x)$ over Euclidean spacetime. If one can exchange the order of integration and of the sequence of limits appearing in eq. (5.38), then the calculation is straightforward. Set

$$-\mathrm{R}_*(0) = -\int_\beta d^4x\, \mathcal{R}_*(x) = \pi\xi I_*^{\mathbf{1}} + \eta I_*^{\gamma_5}\,, \tag{5.39}$$

where [see eqs. (5.6) and (5.27)]

$$\begin{aligned} I_*^\Gamma &= \int_\beta d^4x \lim_{m\to 0}\lim_{\lambda\to 0} C^\Gamma_{\mathrm{loc}}(\lambda;x;m) \\ &= \int_\beta d^4x \lim_{m\to 0}\lim_{\lambda\to 0}\lim_{\Lambda\to\infty}\lim_{V\to\infty} Z_m \sum_n \Big\langle \delta(Z_m\lambda-\lambda_n)\mathcal{O}^\Gamma_{nn}(x)\mathcal{O}^\Gamma_{nn}(0)\Big\rangle_\beta\,, \end{aligned} \tag{5.40}$$

and denote $\mathrm{Lim} \equiv \lim_{m\to 0}\lim_{\lambda\to 0}\lim_{\Lambda\to\infty}\lim_{V\to\infty}$. Under the interchangeability assumption made above one finds

$$\begin{aligned} I_*^{\mathbf{1}} &= \mathrm{Lim}\, Z_m \sum_n \Big\langle \delta(Z_m\lambda-\lambda_n)\int_\beta d^4x\, \mathcal{O}^{\mathbf{1}}_{nn}(x)\mathcal{O}^{\mathbf{1}}_{nn}(0)\Big\rangle_\beta \\ &= \mathrm{Lim}\, Z_m \sum_n \Big\langle \delta(Z_m\lambda-\lambda_n)\mathcal{O}^{\mathbf{1}}_{nn}(0)\Big\rangle_\beta = \mathrm{Lim}\, Z_m \frac{1}{\beta V}\sum_n \langle \delta(Z_m\lambda-\lambda_n)\rangle_\beta \\ &= \mathrm{Lim}\, Z_m\rho_{B,V}(Z_m\lambda, Z_m m) = \lim_{m\to 0}\lim_{\lambda\to 0}\lim_{\Lambda\to\infty} Z_m\rho_B(Z_m\lambda, Z_m m) \\ &= \lim_{m\to 0}\lim_{\lambda\to 0}\rho(\lambda,m) = \lim_{m\to 0}\rho(0,m) = \rho(0,0)\,, \end{aligned} \tag{5.41}$$

where $\int_\beta d^4x\, \mathcal{O}^{\mathbf{1}}_{n'n}(x) = \delta_{n'n}$ was used [see eq. (3.6)] and exact zero modes were dropped, while

$$I_*^{\gamma_5} = \mathrm{Lim}\, Z_m \sum_n \Big\langle \delta(Z_m\lambda-\lambda_n)\int_\beta d^4x\, \mathcal{O}^{\gamma_5}_{nn}(x)\mathcal{O}^{\gamma_5}_{nn}(0)\Big\rangle_\beta = 0\,, \tag{5.42}$$

since $\int_\beta d^4x\, \mathcal{O}^{\gamma_5}_{n'n}(x) = \int_\beta d^4x\, \mathcal{O}^{\mathbf{1}}_{n'-n}(x) = \delta_{n'-n}$. In conclusion,

$$-\mathrm{R}_*(0) = \pi\xi\rho(0,0)\,. \tag{5.43}$$

This is the main result of this section: in the presence of a finite density of localised near-zero modes, and if $\xi \neq 0$, one finds a nonvanishing anomalous remnant $\mathrm{R}_*(0)$.

Of course, one can consider the spacetime integral in eq. (5.40) also away from the chiral and zero-eigenvalue limits. Under the assumption that the other limits can be suitably interchanged, the calculation above shows that

$$I^\Gamma(\lambda;m) = \int_\beta d^4x\, C^\Gamma_{\mathrm{loc}}(\lambda;x;m) = \delta_{\Gamma,\mathbf{1}}\rho(\lambda,m)\,, \tag{5.44}$$

if $\alpha(\lambda) = 0$, and zero otherwise, and so $C^{\mathbf{1}}_{\mathrm{loc}}$ must be nonzero in the presence of a finite density of localised modes, as anticipated in section 5.2.

The result eq. (5.43) crucially depends on the possibility of integrating over spacetime before taking the various limits. For localised modes this is justified as follows.

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- In the finite-volume and UV-regularised theory (e.g., on a finite lattice), the mode sum is over a finite number of modes, and can certainly be exchanged with integration.

- Averaging over gauge fields in the regularised setting should cause no problem. For example, in the lattice regularisation the discretised gauge field is represented in terms of link variables which are elements of the gauge group. Averaging then consists in a multiple compact Haar integral over the gauge group, that can certainly be exchanged with spacetime integration, i.e., with summation over lattice sites.

- The infinite-volume limit and the removal of the UV cutoff are difficult to control analytically. Nonetheless, it is expected that at finite fermion mass the pseudoscalar-pseudoscalar correlator is bounded exponentially as a function of $|\vec{x}|$ (or at least by some integrable function), independently of the localisation properties of the modes,[26] uniformly in V and Λ. This is confirmed by numerical experience on the lattice (see, e.g., ref. [116]). One can then use the dominated convergence theorem (see, e.g., ref. [117]) to justify exchanging integration with these limits.

- The crucial step is the chiral limit. In this limit, the exponential (or, more generally, integrable) bound mentioned above may be lost. This is what one expects if near-zero modes, which are the only ones of physical relevance, are delocalised: indeed, in the standard scenario ($\mathcal{R}_* = 0$) the finite-temperature Goldstone theorem leads to massless pseudoscalar excitations and a non-integrable algebraic decay $1/|\vec{x}|$ of the pseudoscalar-pseudoscalar correlator. On the other hand, for localised modes an integrable bound is expected to be inherited by the correlator from the modes, and one can use again dominated convergence to justify the exchange of integration and chiral limit. A more detailed argument showing that the $1/m$-divergent part of the pseudoscalar-pseudoscalar correlator should inherit the fast decay properties of the localised near-zero modes from the spectral correlators is provided in appendix H.

As a final remark, notice that integration over spacetime is equivalent to the zero-momentum limit of the Fourier transform. This is not expected to commute with the chiral limit in general: for example, it does not if massless Goldstone bosons are present. These limits should, however, commute if near-zero modes are localised.

5.7 Fate of Goldstone excitations

The main result of the previous subsection, eq. (5.43), can now be used to discuss the fate of Goldstone excitations in the chiral limit of the theory. The singular part in the spectral function, eq. (4.26), reads

$$\lim_{\vec{k}\to 0} \tilde{c}_*(\omega, \vec{k})|_{\text{singular}} = -2\pi\delta(\omega)[\Sigma_* - \mathrm{R}_*(0)] = 2\pi\delta(\omega)\rho(0,0)(1-\xi)\,, \tag{5.45}$$

[26] For localised modes this is shown below in appendix E. For delocalised modes, averaging over gauge fields leads to destructive interference effects among modes, and one expects an exponential decay of the correlator.

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where I have used the Banks-Casher relation [34] in the chiral limit, $\Sigma_* = -\pi\rho(0,0)$. Notice that the coefficient of the Dirac delta is larger than or equal to 0. There are three possible scenarios, corresponding to scenarios (i)–(iii) for the behaviour of the pseudoscalar correlator.

(i). If λ_c vanishes faster than m, then $\xi = 0$, and the presence of massless quasi-particles depends exclusively on $\rho(0,0)$ being non-zero. This is identical to the standard scenario encountered in the usual finite-temperature Goldstone's theorem.

(ii). If λ_c vanishes as fast as m, then $0 < \xi < 1$, and massless quasi-particles are present if $\rho(0,0) \neq 0$. This is similar to the standard scenario, although the coefficient of the singular term is reduced. This results in a reduction of the residue $-i[\Sigma_* - \mathrm{R}_*(0)] = 2\pi i\rho(0,0)(1-\xi)$ of the pole at zero momentum in $\vec{\tilde{\mathcal{G}}}_{AP}(0,\vec{k})$ [see eq. (4.14)].

(iii). If λ_c vanishes more slowly than m, including if it remains finite, then $\xi = 1$ and Goldstone excitations disappear from the spectrum, even if the spectral density is finite at the origin.[27]

6 Conclusions and outlook

Besides its connection with deconfinement, supported by an increasing amount of evidence, the physical consequences of the localisation of low Dirac modes in high-temperature gauge theories [13, 14, 16–33, 36, 44–47] have so far remained quite elusive. In this paper I have discussed the effects of a finite density of localised near-zero modes in the chiral limit on the pseudoscalar-pseudoscalar correlator and on the massless Goldstone excitations expected from Goldstone's theorem at finite temperature [49–53]. These effects were discussed first in refs. [57, 58], of which this work is the completion, including a number of technical details omitted there. In summary, the main result is that if a finite density of localised near-zero modes is present in the chiral limit, and if the corresponding mobility edge vanishes more slowly than the fermion mass, then no massless excitation are found in this limit, in contrast with one's expectations from Goldstone's theorem. If the mobility edge vanishes as fast as the fermion mass, massless excitations are present but the coefficient of the corresponding delta-function term in the axial-vector-pseudoscalar spectral function is reduced. To my knowledge, these are so far the only direct (although in the chiral limit) physical consequences of localisation of Dirac modes that have been found.

The argument presented here and, in shortened form, in refs. [57, 58], requires a number of intermediate results that I believe are valuable for their own sake:

- a proof and an extension of Goldstone's theorem at finite temperature, in the case of the non-singlet axial flavour symmetry of gauge theories with massless fermions, based on the corresponding Ward-Takahashi identity in Euclidean space;

[27] If localised and delocalised modes coexist in the same spectral region (see footnote 21), then $\rho(0,0)$ should be replaced by the density of localised near-zero modes, $\rho_{\mathrm{loc}}(0,0) < \rho(0,0)$, in eq. (5.43); and ξ should be replaced by $\xi\frac{\rho_{\mathrm{loc}}(0,0)}{\rho(0,0)} < 1$ in eq. (5.45) and in the discussion above. Notice that Goldstone excitations cannot disappear in this case.

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- a detailed calculation of the contribution of localised modes to the pseudoscalar-pseudoscalar correlator, showing that they can lead to a $1/m$ divergence in the fermion mass if the mobility edge, λ_c, vanishes as fast as or more slowly than m;
- a proof of the renormalisation-group-invariance of the mobility edge in units of the fermion mass, λ_c/m.

The extension of Goldstone's theorem at finite temperature closes a loophole in the usual proof (see refs. [49–53]), related to the possibility that the relevant symmetry remains effectively explicitly broken in the chiral limit. In this case the axial flavour symmetry remains not conserved even in the chiral limit, and the usual Goldstone's theorem at finite temperature is evaded. There is therefore no contradiction between this theorem and the possible disappearance of massless excitations mentioned above, in spite of the chiral condensate being nonzero due to the presence of a finite density of near-zero modes.

A term explicitly breaking the symmetry, referred to as the anomalous remnant in this paper, is present in the relevant Ward-Takahashi identity in the chiral limit if near-zero modes are localised and have a finite spectral density, and if their mobility edge does not vanish faster than the fermion mass m. This results, in a manner similar to the formation of anomalies, from the cancellation of the symmetry-breaking parameter m in the Ward-Takahashi identity against a $1/m$ divergence in the pseudoscalar-pseudoscalar correlator. This is the origin of the effects discussed above on the massless Goldstone modes that are usually expected in the chiral limit.

The renormalisation-group-invariance of the ratio of the mobility edge and the fermion mass was suggested in ref. [18], and is supported by numerical evidence in lattice QCD (also at finite imaginary chemical potential) showing that this ratio has limited sensitivity to the lattice spacing [18, 31]. The proof presented here puts this on firmer ground, and further supports the fact that localisation of the low Dirac modes is a genuine physical effect in Euclidean spacetime, and not a lattice artefact.

The main result of this paper would be of more limited interest if one could prove that localised modes must not be present in the chiral limit, lest one violates some general property expected of a decent quantum field theory at finite temperature. Nonetheless, in this case it could be reversed to show that no $1/m$ divergence should appear in the pseudoscalar-pseudoscalar correlator, and that the massless Goldstone modes cannot be removed if a nonzero condensate is present. However, I am not aware of any general property of finite-temperature quantum field theory in contrast with the assumption of localised near-zero modes in the chiral limit.

The most pessimistic scenario, from the theoretical point of view, is that while not violating any general principle, localised near-zero modes are in practice not found in any useful model. It is then important to follow up on the interesting clues about the possible realisation of such a scenario, found in $N_f = 2$ massless adjoint QCD [5, 6] and, most importantly, in QCD towards the chiral limit [59–61].

The connection between localisation and disappearance of Goldstone modes has been known for a long time in condensed matter physics [54], and for quite some time in zero-temperature lattice field theory as well, although in an unphysical setup [55] (see also

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ref. [58]). The mechanism behind this connection is the same in those cases and in finite-temperature quantum field theory, and boils down to how the susceptibility appearing in the integrated version of a suitable Ward-Takahashi identity blows up in the limit in which the symmetry-breaking parameter is sent to zero. In the case at hand this is the non-singlet pseudoscalar susceptibility, which diverges like $1/m$ in the chiral limit. However, in the usual case where low Dirac modes are delocalised this happens only *after* the spacetime integration that connects the susceptibility to the pseudoscalar-pseudoscalar correlator. Instead, if low Dirac modes are localised the $1/m$ divegence is already present in the correlator *before* spacetime integration. This mechanism is identical to the one already discussed in refs. [54, 55]. The major differences between refs. [54, 55] and this work and refs. [57, 58] are on the one hand technical, due to the reduced amount of symmetry in finite-temperature quantum field theory in the imaginary-time formulation stemming from the presence of a compactified direction; and on the other hand of practical relevance (in principle...), as this work is concerned with physically more realistic theories.

In this respect, it would be interesting to work out the consequences for the finite-mass theory of the realisation of a non-standard scenario in the chiral limit, with Goldstone modes removed or at least modified by the presence of localised modes. This would require to preliminarily understand what kind of finite-temperature transition one would find in the chiral limit, separating a low-temperature phase where low modes are delocalised from a phase at higher temperature where they are localised, but with a finite density on both sides of the transition.[28] What follows is largely speculative.

Although it may seem natural to expect that the mobility edge is vanishingly small at such a transition, it is not necessarily so, as displayed by SU(3) pure gauge theory where a finite mobility edge appears at the transition in the trivial Polyakov-loop sector (see ref. [118]). This may be a consequence of the first-order nature of the transition. Results at the first-order reconfinement transition in trace-deformed SU(3) gauge theory at high temperature are also compatible with a finite mobility edge at the transition [29]. Independently of this, the order of the chiral transition is determined by the behaviour of the chiral condensate rather than that of the mobility edge. In this respect, the radical change in the nature of the low modes would more naturally suggest a finite discontinuity in the condensate, and so a first-order transition, but a continuous behaviour cannot be excluded. In any case, the fact that the anomalous remnant is identically zero in one phase and nonzero in the other indicates the presence of a non-analyticity in the partition function (as a function of an extended set of variables including a suitable chemical potential), and so that the phase transition is genuine. This is the case as long as a nonzero anomalous remnant appears, irrespectively of whether it is able to remove the Goldstone excitations from the spectrum or not.

It would be interesting to study how localisation of finitely-dense low modes could be included in the theoretical analysis of phase transitions in gauge systems, extending the

[28] One should mention the claim of refs. [78, 79] that modes in the immediate vicinity of the origin are *critical*, i.e., with nontrivial localisation properties, rather than localised, in high-temperature SU(3) pure gauge theory probed with overlap fermions. However, this claim has still to be fully confirmed by a finite size scaling analysis, and it is unclear how it would affect the opposite, chiral limit of massless fermions.

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standard analysis of ref. [119]. This predicts a second-order chiral transition for $N_f = 2$ and a first-order one for $N_f \geq 3$ (see, however, the recent analysis of ref. [120] claiming the possibility of a continuous phase transition also in this case). According to the standard lore, isolated first-order points are not expected, and a line of first-order transitions should reach out from zero mass to a critical endpoint at some nonzero fermion mass where the transition is second order. However, no second-order endpoint has been observed so far for $N_f = 3$, and there are recent claims that no such critical endpoint is present up to $N_f = 6$ flavours of light fermions [121]. The inclusion of localisation effects may lead to revise one's expectations.

A transition from delocalised to localised finitely-dense near-zero modes could be followed by a second transition at a higher temperature, where the density of near-zero modes goes to zero and chiral symmetry gets fully restored. This kind of scenario would fit what is known about $N_f = 2$ massless adjoint QCD [5, 6]: one would find Goldstone modes being "weakened" at the deconfinement transition by the formation of an anomalous remnant [like in scenario (ii) discussed in section 5.7], and then gradually disappearing from the spectrum until full restoration at higher temperature. It is also possible that extending the study of refs. [5, 6] to larger volumes the effects of localised near-zero modes become fully visible, and Goldstone modes get entirely removed from the spectrum at deconfinement [scenario (iii) in section 5.7]. In either case, if something similar happened with massless fundamental fermions, then the fact that in QCD (approximate) deconfinement and chiral restoration take place together in the same relatively narrow temperature range would be a consequence of the "blurring" of the two separate sharp transitions into a single analytic crossover.

Whether any of the non-standard scenarios discussed in this paper applies to QCD depends on the behaviour of the finite peak of near-zero localised modes observed at near-physical quark masses [59]. Such a peak is known to be present in pure gauge theory [36], and so in the limit of large quark masses; and to survive at the lower-than-physical quark masses investigated so far (although localisation properties were not studied) [60–63]. If it survived the chiral limit, this peak would then be a common feature in the whole quark mass range, most likely appearing at the same temperature where near-zero modes first become localised. This is the case in pure gauge theory, where this temperature also coincides with the deconfinement temperature [22, 23]. Moreover, if the peak consisted of localised modes all the way to the chiral limit, then the temperature at which it appears there would mark a genuine phase transition, distinct from the chiral transition, even if the peak had a vanishing width in this limit (as long as this does not vanish too fast with m). This would naturally, albeit unexpectedly, suggest to identify the deconfinement transition in the presence of dynamical fermions as the transition to localised near-zero modes. Presumably, for any finite quark mass the near-zero peak disappears at some sufficiently high temperature, corresponding to a second pseudocritical temperature above the crossover one for physical masses. In the chiral limit, this should extrapolate to the critical temperature of a second transition, where the near-zero peak disappears and chiral symmetry is restored.

The crucial question is whether or not the two transitions coincide in the chiral limit. If they do, then localisation of near-zero modes and chiral restoration take place at the same

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temperature, and no peak of localised near-zero modes ever forms at $m = 0$. This would fit with the disordered medium scenario of ref. [35], according to which the accumulation of Dirac eigenvalues near the origin takes place together with the delocalisation of the corresponding eigenmodes. If the two transitions do not coincide, an intermediate phase would appear where chiral symmetry is broken and Goldstone excitations are weakened or absent altogether. Obviously, this is possible only if the temperature where near-zero modes localise is below the chiral transition temperature. A recent estimate of the latter is $T_c^0 \simeq 132\,\mathrm{MeV}$ [122], obtained by extrapolating observables to zero light-quark mass according to O(4) scaling. This suggests the presence of massless Goldstone modes below T_c^0, which can be reconciled with the presence of an intermediate phase in the chiral limit if scenario (ii) with weakened Goldstone excitations is realised. On the other hand, the localisation temperature is known only at the physical point, where it is obtained by extrapolation from higher temperatures [18]. While the result clearly falls within the crossover range, somewhere above the position of the peak of the chiral susceptibility, the accuracy of the extrapolation is not fully under control. To this end, it would be interesting to improve on the results of ref. [18] and determine the temperature at which localisation appears by a direct study near the crossover at physical and lower-than-physical quark masses.

It should be noted that the existence of an intermediate phase in QCD, between the crossover temperature and a (much) higher one, has been suggested several times in the literature, although with various different motivations (see, e.g., refs. [123, 124] and the discussion in ref. [125]). It would be interesting to investigate what happens to the near-zero peak at the higher temperatures where the conjectured intermediate phase should end.

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A Analyticity and reality properties of two-point functions

In this appendix I provide further details on the analytic continuation of two-point functions and on their reality properties.

Analytic continuation relations. The analytic continuation relations, eqs. (2.12) and (2.14), between the Fourier coefficients of the Euclidean correlator, eq. (2.8), and the retarded and advanced propagators, eq. (2.10), can be obtained following the approach of ref. [68] based on the deformation of the integration path of a suitable complex integral. One defines the integrals

$$I_{\gtrless}\left(\omega, \vec{k}\right) \equiv i \int_{\mathcal{C}_{\gtrless}} dz \int d^3x \, e^{i\left(\omega z - \vec{k}\cdot\vec{x}\right)} G_{\phi_1\phi_2}\left(z, \vec{x}\right) , \tag{A.1}$$

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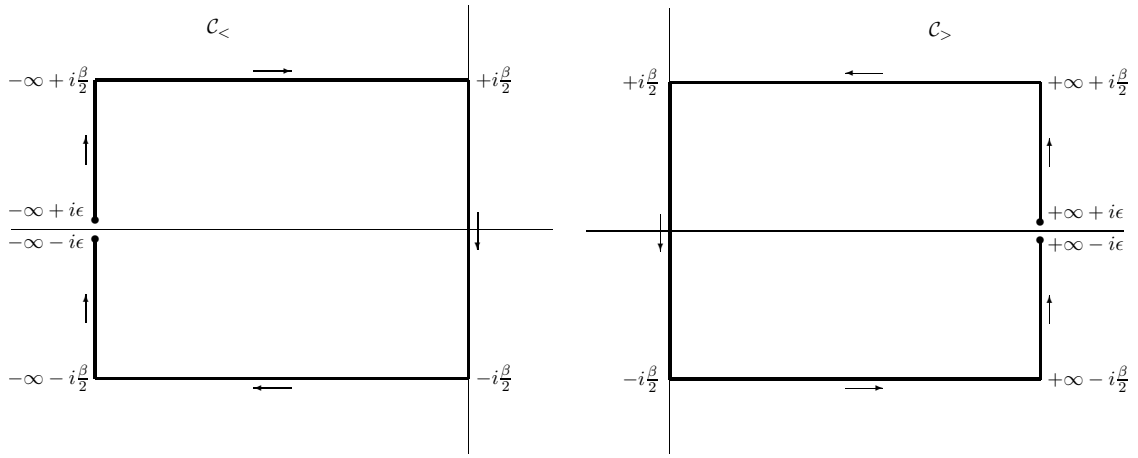

Figure 1. Integration paths in eq. (A.1).

for $\mathrm{Im}\,\omega \gtrless 0$, respectively, where the paths $\mathcal{C}_\gtrless$ are shown in figure 1. The contribution of the paths along the imaginary direction at large $|\mathrm{Re}\,z|$ are suppressed exponentially as the paths are pushed to infinity and so can be discarded. Independently of ω, the paths can be shrunk to run close to the real axis, leading to

$$I_> \left(\omega, \vec{k}\right) = \tilde{r}_{\phi_1\phi_2}\left(\omega, \vec{k}\right) , \quad \mathrm{Im}\,\omega > 0\,; \qquad I_< \left(\omega, \vec{k}\right) = \tilde{a}_{\phi_1\phi_2}\left(\omega, \vec{k}\right) , \quad \mathrm{Im}\,\omega < 0\,. \tag{A.2}$$

On the other hand, for the special values $\omega = i\omega_n$, $n \neq 0$, one finds that thanks to periodicity the contributions from the paths reaching to infinity cancel each other out, so that

$$I_> \left(i\omega_n, \vec{k}\right) = \tilde{\mathcal{G}}_{\phi_1\phi_2}\left(\omega_n, -\vec{k}\right) , \quad n > 0\,; \qquad I_< \left(i\omega_n, \vec{k}\right) = \tilde{\mathcal{G}}_{\phi_1\phi_2}\left(\omega_n, -\vec{k}\right) , \quad n < 0\,. \tag{A.3}$$

Comparing eqs. (A.2) and (A.3), one obtains eq. (2.12), as in ref. [68]. The case $n = 0$, instead, was not discussed in detail there. In this case one starts from $I_>(i\epsilon, \vec{k}\,)$ and $I_<(-i\epsilon, \vec{k}\,)$, eventually taking $\epsilon \to 0$. One finds

$$\begin{aligned} I_\gtrless \left(\pm i\epsilon, \vec{k}\right) = & \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} G_{\phi_1\phi_2}\left(-i\tau, \vec{x}\right) \\ & - \epsilon\beta \int d^4x\, \theta\left(\pm t\right) e^{\mp\epsilon t} e^{-i\vec{k}\cdot\vec{x}} G_{\phi_1\phi_2}\left(t - i\tfrac{\beta}{2}, \vec{x}\right) , \end{aligned} \tag{A.4}$$

up to terms that vanish in the limit $\epsilon \to 0$, and so

$$\lim_{\epsilon\to 0} I_\gtrless \left(\pm i\epsilon, \vec{k}\right) = \tilde{\mathcal{G}}_{\phi_1\phi_2}\left(0, -\vec{k}\right) - A^{(\pm)}_{\phi_1\phi_2}\left(\vec{k}\right) , \tag{A.5}$$

where

$$A^{(\pm)}_{\phi_1\phi_2}\left(\vec{k}\right) \equiv \lim_{t\to\pm\infty} \beta \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} G_{\phi_1\phi_2}\left(t - i\tfrac{\beta}{2}, \vec{x}\right) , \tag{A.6}$$

assuming that this quantity is finite.

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Transport peak. As discussed in refs. [70, 71], $A^{(\pm)}_{\phi_1\phi_2}$ in eq. (A.5) can be identified with the coefficient of the transport peak. If $\tilde{\rho}_{\phi_1\phi_2}/\omega$ has a Dirac delta singularity at the origin [see eq. (2.13)],

$$\tilde{\rho}_{\phi_1\phi_2}\left(\omega,\vec{k}\right) = 2\pi A_{\phi_1\phi_2}\left(\vec{k}\right)\omega\delta\left(\omega\right) + B_{\phi_1\phi_2}\left(\omega,\vec{k}\right)\,, \tag{A.7}$$

with $B_{\phi_1\phi_2}$ regular, one finds[29]

$$\tilde{\mathcal{G}}_{\phi_1\phi_2}\left(0,-\vec{k}\right) - \tilde{r}_{\phi_1\phi_2}\left(i\epsilon,\vec{k}\right) = \tilde{\mathcal{G}}_{\phi_1\phi_2}\left(0,-\vec{k}\right) - \tilde{a}_{\phi_1\phi_2}\left(-i\epsilon,\vec{k}\right) = A_{\phi_1\phi_2}\left(\vec{k}\right)\,, \tag{A.8}$$

where the limit $\epsilon \to 0$ is understood, so that $A_{\phi_1\phi_2} = A^{(+)}_{\phi_1\phi_2} = A^{(-)}_{\phi_1\phi_2}$. Notice that if the Euclidean fields $\phi_{E1,2}$ corresponding to $\phi_{1,2}$ have simple transformation properties under time-reflection (see section 3.3), $\phi_{E1,2}(t,\vec{x}) \to \varsigma_{1,2}\phi_{E1,2}(\beta - t,\vec{x})$, $\varsigma_{1,2} = \pm 1$, then $A^{(\pm)}_{\phi_1\phi_2}(\vec{k}) = \varsigma_1\varsigma_2 A^{(\mp)}_{\phi_1\phi_2}(\vec{k})$, and so a transport peak can only be present if $\varsigma_1\varsigma_2 = 1$.

Reality of Euclidean correlators. The correlators $\mathcal{G}_{AP\,\mu}(t,\vec{x})$ and $\mathcal{G}_{PP}(t,\vec{x})$, and more generally the Euclidean correlation function of an arbitrary number of non-singlet axial-vector and pseudoscalar densities, are real functions. To see this, consider the generating functional

$$Z_{A,P}[j] \equiv \int_\beta [\mathscr{D}\psi][\mathscr{D}\bar{\psi}] e^{-\int_\beta d^4x\,\bar{\psi}K[j]\psi}\,, \qquad K[j] \equiv \not{D} + m\mathbf{1} + j^a_{A\mu}\gamma_\mu\gamma_5 t^a + j^a_P\gamma_5 t^a\,, \tag{A.9}$$

where j denotes collectively the set of real sources $j^a_{A\mu}(x)$ and $j^a_P(x)$. Performing the Grassmann integral one obtains

$$Z_{A,P}[j] = \mathrm{Det}\, K[j]\,. \tag{A.10}$$

Since $K[j]^\dagger = \gamma_5 K[j]\gamma_5$, one finds that $Z_{A,P}[j]$ is real,

$$Z_{A,P}\,[j]^* = (\mathrm{Det}\, K\,[j])^* = \mathrm{Det}\, K\,[j]^\dagger = \mathrm{Det}\, K\,[j] = Z_{A,P}[j]\,. \tag{A.11}$$

Taking functional derivatives with respect to the currents, setting $j = 0$, and averaging over gauge fields as in eq. (3.2),[30] the claimed result follows. This formal derivation holds also in a regularised, lattice version of the theory as long as the discretised Dirac operator satisfies the γ_5-Hermiticity property $\gamma_5\not{D}\gamma_5 = \not{D}^\dagger$.

Reality properties of thermal expectation values. A consequence of the reality property discussed above is that the analytic extensions[31] of $\mathcal{G}_{AP\,\mu}$ and $\mathcal{G}_{PP}$ in the complex-t plane satisfy the Schwarz reflection principle, i.e.,

$$\mathcal{G}_{AP\,\mu}(z^*,\vec{x}) = \mathcal{G}_{AP\,\mu}(z,\vec{x})^*\,, \qquad \mathcal{G}_{PP}(z^*,\vec{x}) = \mathcal{G}_{PP}(z,\vec{x})^*\,. \tag{A.12}$$

[29] While only the result for the retarded propagator is discussed in refs. [70, 71], the one for the advanced propagator can be obtained by a simple extension of the calculation.

[30] In Lorenz gauge, the gauge action and integration measure are manifestly real.

[31] While contact terms may be present in the Euclidean correlation functions, it is understood that they are not involved in the process of analytic continuation.

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This can be combined with the symmetry properties and the periodicity of the correlators as follows. Setting $z = \epsilon - it$, one finds, using the analytic extension of the relation eq. (3.21), that

$$\mathcal{G}_{AP\,\mu}(\epsilon + it, \vec{x})^* = \mathcal{G}_{AP\,\mu}(\epsilon - it, \vec{x}) = \zeta_\mu \mathcal{G}_{AP\,\mu}(\beta - \epsilon + it, \vec{x}) = \zeta_\mu \mathcal{G}_{AP\,\mu}(-\epsilon + it, \vec{x})\,. \tag{A.13}$$

Using the analytic continuation relations eq. (3.25), one sees that these are relations between the real-time correlation functions of the axial-vector and pseudoscalar Minkowskian operators, that can be summarised as

$$\langle\!\langle \hat{A}^a_\mu(t, \vec{x}) i\hat{P}^b(0) \rangle\!\rangle^*_\beta = \langle\!\langle i\hat{P}^b(0) \hat{A}^a_\mu(t, \vec{x}) \rangle\!\rangle_\beta\,. \tag{A.14}$$

Similarly, using eq. (3.22), one finds

$$\mathcal{G}_{PP}(\epsilon + it, \vec{x})^* = \mathcal{G}_{PP}(\epsilon - it, \vec{x}) = \mathcal{G}_{PP}(\beta - \epsilon + it, \vec{x}) = \mathcal{G}_{PP}(-\epsilon + it, \vec{x})\,, \tag{A.15}$$

and, using again eq. (3.25),

$$\langle\!\langle \hat{P}^a(t, \vec{x}) \hat{P}^b(0) \rangle\!\rangle^*_\beta = \langle\!\langle \hat{P}^b(0) \hat{P}^a(t, \vec{x}) \rangle\!\rangle_\beta\,. \tag{A.16}$$

Equations (A.14) and (A.16) imply

$$\begin{aligned} \langle\!\langle \left[\hat{A}^a_\mu\left(t, \vec{x}\right), \hat{P}^b\left(0\right) \right] \rangle\!\rangle^*_\beta &= \langle\!\langle \left[\hat{A}^a_\mu\left(t, \vec{x}\right), \hat{P}^b\left(0\right) \right] \rangle\!\rangle_\beta\,, \\ \langle\!\langle \left[\hat{P}^a\left(t, \vec{x}\right), \hat{P}^b\left(0\right) \right] \rangle\!\rangle^*_\beta &= -\langle\!\langle \left[\hat{P}^a\left(t, \vec{x}\right), \hat{P}^b\left(0\right) \right] \rangle\!\rangle_\beta\,, \end{aligned} \tag{A.17}$$

for the thermal expectation values of the commutators, and in turn

$$\begin{aligned} \tilde{c}\left(\omega, \vec{k}\right)^* &= \tilde{\rho}_{A^{0a}P^a}\left(\omega, \vec{k}\right)^* = \tilde{\rho}_{A^{0a}P^a}\left(-\omega, -\vec{k}\right) = \tilde{c}\left(-\omega, -\vec{k}\right)\,, \\ \tilde{c}^P\left(\omega, \vec{k}\right)^* &= \tilde{\rho}_{P^aP^a}\left(\omega, \vec{k}\right)^* = -\tilde{\rho}_{P^aP^a}\left(-\omega, -\vec{k}\right) = -\tilde{c}^P\left(-\omega, -\vec{k}\right)\,, \end{aligned} \tag{A.18}$$

for the relevant spectral functions [see eq. (3.26)].

B Non-singlet axial Ward-Takahashi identities

In this appendix I review the derivation of the non-singlet axial Ward-Takahashi identities. In general, Ward-Takahashi identities are obtained observing that a change of integration variables in the path integral trivially leaves the result unchanged, and applying this observation to a change of variables through the infinitesimal form of some continuous symmetry but with x-dependent parameters.

In the case at hand, one upgrades the infinitesimal form of the non-singlet axial symmetry transformations, eq. (3.10), to the x-dependent transformations $\psi \to \psi + \delta_A\psi$ and $\bar{\psi} \to \bar{\psi} + \delta_A\bar{\psi}$, with

$$\delta_A\psi(x) = i\epsilon_a(x) t^a \gamma_5 \psi(x)\,, \qquad \delta_A\bar{\psi}(x) = i\epsilon_a(x) \bar{\psi}(x) t^a \gamma_5\,, \tag{B.1}$$

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with infinitesimal $\epsilon_a(x)$. Since the functional integration measure is invariant thanks to $\mathrm{tr}\, t^a = 0$, one obtains for any observable $\mathcal{O}$

$$\langle \delta_A \mathcal{O} \rangle_\beta = \langle \mathcal{O} \delta_A S_\mathrm{F} \rangle_\beta \,, \qquad \delta_A S_\mathrm{F} = i \int_\beta d^4x \, \epsilon_a(x) \left(-\partial_\mu A^a_\mu(x) + 2m P^a(x) \right) \,, \tag{B.2}$$

with A^a_μ the flavour non-singlet axial-vector currents, and P^a the flavour non-singlet pseudoscalar densities, defined in eq. (3.13). Since $\epsilon_a(x)$ is infinitesimal but otherwise arbitrary, one finds eq. (3.12),

$$\left\langle \left(-\partial_\mu A^a_\mu(x) + 2m P^a(x) \right) \mathcal{O} \right\rangle_\beta = \left\langle -i \frac{\delta_A \mathcal{O}}{\delta \epsilon_a(x)} \right\rangle_\beta \,. \tag{B.3}$$

In the case $\mathcal{O} = P^b(y)$, a straightforward calculation shows that

$$-i \frac{\delta_A P^b(y)}{\delta \epsilon_a(x)} = \delta^{(4)}(x-y) \left(\frac{1}{N_f} \delta^{ab} S(y) + d^{abc} S^c(y) \right) \,, \tag{B.4}$$

where the flavour singlet and flavour non-singlet scalar densities S and S^a read

$$S(x) \equiv \bar\psi(x) \psi(x) \,, \qquad S^a(x) \equiv \bar\psi(x) t^a \psi(x) \,, \tag{B.5}$$

with the totally symmetric symbol d^{abc} defined through

$$\left\{ t^a, t^b \right\} = \frac{1}{N_f} \delta^{ab} + d^{abc} t^c \,. \tag{B.6}$$

The four-dimensional Dirac delta in eq. (B.4) is periodic in time, see eq. (3.15). Invariance under vector flavour transformations implies that

$$\langle \bar\psi_{f_1}(0) \psi_{f_2}(0) \rangle_\beta \equiv \delta_{f_1 f_2} \Sigma \,, \tag{B.7}$$

with Σ the chiral condensate, and so

$$\langle S(0) \rangle_\beta = N_f \Sigma \,, \qquad \langle S^a(0) \rangle_\beta = 0 \,. \tag{B.8}$$

One then finds the Ward-Takahashi identity eq. (3.14),

$$-\partial_\mu \langle A^a_\mu(x) P^b(0) \rangle_\beta + 2m \langle P^a(x) P^b(0) \rangle_\beta = \delta^{(4)}(x) \delta^{ab} \Sigma \,. \tag{B.9}$$

Further exploiting vector flavour invariance, one has

$$\langle A^a_\mu(x) P^b(0) \rangle_\beta \equiv \delta^{ab} \mathcal{G}_{AP\,\mu}(x) \,, \qquad \langle P^a(x) P^b(0) \rangle_\beta \equiv \delta^{ab} \mathcal{G}_{PP}(x) \,, \tag{B.10}$$

and so eqs. (3.17) and (3.18) follow,

$$-\partial_\mu \mathcal{G}_{AP\,\mu}(x) + 2m \mathcal{G}_{PP}(x) = \delta^{(4)}(x) \Sigma \,, \tag{B.11}$$

$$i\omega_n \widetilde{\mathcal{G}}_{AP\,4}\left(\omega_n, \vec{k}\right) + i\vec{k} \cdot \vec{\widetilde{\mathcal{G}}}_{AP}\left(\omega_n, \vec{k}\right) + 2m \widetilde{\mathcal{G}}_{PP}\left(\omega_n, \vec{k}\right) = \Sigma \,. \tag{B.12}$$

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For future utility one defines also the flavour non-singlet vector currents, V^a_μ, as

$$V^a_\mu(x) \equiv \bar\psi(x)\gamma_\mu t^a \psi(x)\,. \tag{B.13}$$

Identities involving $\mathcal{O} = A^b_\nu(y), V^b_\nu(y)$, see eqs. (4.29) and (C.1), are obtained using

$$-i\frac{\delta_A A^b_\nu(y)}{\delta\epsilon_a(x)} = -\delta^{(4)}(x-y) i f^{abc} V^c_\nu(y)\,, \qquad -i\frac{\delta_A V^b_\nu(y)}{\delta\epsilon_a(x)} = -\delta^{(4)}(x-y) i f^{abc} A^c_\nu(y)\,. \tag{B.14}$$

In particular, eq. (4.29) follows since $\langle V^c_\nu\rangle_\beta = 0$ due to vector flavour symmetry (or to rotation and reflection symmetries).

A second set of Ward-Takahashi identities is obtained starting from the vector symmetry transformations eq. (3.9). Changing variables according to $\psi \to \psi + \delta_V\psi$ and $\bar\psi \to \bar\psi + \delta_V\bar\psi$, with

$$\delta_V\psi(x) = i\epsilon_a(x) t^a \psi(x)\,, \qquad \delta_V\bar\psi(x) = -i\epsilon_a(x)\bar\psi(x) t^a\,, \tag{B.15}$$

since the functional integration measure is invariant one obtains for any observable $\mathcal{O}$

$$\langle \delta_V \mathcal{O}\rangle_\beta = \langle \mathcal{O}\delta_V S_{\rm F}\rangle_\beta\,, \qquad \delta_V S_{\rm F} = i\int_\beta d^4x\, \epsilon_a(x)\left(-\partial_\mu V^a_\mu(x)\right)\,. \tag{B.16}$$

Since $\epsilon_a(x)$ is infinitesimal but otherwise arbitrary, one finds

$$\langle -\partial_\mu V^a_\mu(x)\mathcal{O}\rangle_\beta = \left\langle -i\frac{\delta_V\mathcal{O}}{\delta\epsilon_a(x)}\right\rangle_\beta\,. \tag{B.17}$$

C Renormalisation of the Ward-Takahashi identity

In this appendix I exploit the Ward-Takahashi identities eq. (B.3) [eq. (3.12)] to discuss the renormalisation of the relevant composite operators. The point of view is that explained in section 3.5: correlation functions are regularised by cutting off their Dirac mode decomposition at some UV cutoff Λ, which leads to violations of the Ward-Takahashi identities that nonetheless will vanish as $\Lambda \to \infty$. This is guaranteed by the existence of chiral-symmetry-respecting regularisations. One can then use the Ward-Takahashi identities to constrain and relate the various UV divergences. Since UV divergences are the same at zero and nonzero temperature (see, e.g., [126–128]), renormalisation of the zero-temperature theory is sufficient to make the finite-temperature theory finite as well. In what follows $T = 0$, i.e., $\beta = \infty$, unless specified otherwise. The line of reasoning is standard (see, e.g., refs. [105, 129]). It is assumed that the usual mass and coupling constant multiplicative renormalisations (additive mass renormalisation being forbidden by chiral symmetry) have already been carried out. In particular, the bare mass m_B and the renormalised mass m are related by $m_B = Z_m m$.

Multiplicative renormalisation. In the vector identities eq. (B.17), with $\mathcal{O}$ an arbitrary string of renormalised fundamental fields, one finds on the right-hand side only a sum of finite contact terms, and so the divergence $\partial_\mu V^a_{B\mu}$ of the bare vector currents is finite. Since $V^a_{B\mu}$ cannot mix with other operators of equal or lower dimension for symmetry reasons, the

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renormalised currents are simply $V^a_{B\mu} = Z_V V^a_\mu$, and finiteness of $\partial_\mu V^a_{B\mu}$ implies finiteness of Z_V, that can be set to $Z_V = 1$. In general, vector flavour invariance implies independence of the flavour index a, and SO(4) invariance implies independence of μ.

The bare non-singlet axial currents $A^a_{B\mu}$ and non-singlet pseudoscalar densities P^a_B, appearing in the axial Ward-Takahashi identities eq. (B.3), cannot mix with other operators, again due to symmetry reasons, so only multiplicative renormalisation may be required. Additive divergences, however, can still appear on the left-hand side of the identities in the form of contact terms at $x = 0$. These are discussed below in the case of interest. Let $A^a_{B\mu} = Z_A A^a_\mu$ and $P^a_B = Z_P P^a$ relate the bare and the renormalised axial-vector currents and pseudoscalar densities. Using eq. (B.3) with $\mathcal{O}$ a string of fundamental fields, finiteness of the contact terms on the right-hand side implies that $Z_A^{-1} Z_m Z_P$ is finite and can be set to 1. Alternatively, one could use eq. (B.9) at $x \neq 0$ to obtain the same result. Taking instead $\mathcal{O} = V^b_{B\nu}(y) A^c_{B\rho}(z)$, one finds the following identity for the renormalised fields,

$$\begin{aligned} &Z_A^2 \langle \Big(-\partial_\mu A^a_\mu(x) + 2mP^a(x) \Big) V^b_\nu(y) A^c_\rho(z) \rangle_\beta \\ &= -Z_A^2 \delta^{(4)}(x-y) i f^{abd} \langle A^d_\nu(y) A^c_\rho(z) \rangle_\beta - \delta^{(4)}(x-z) i f^{acd} \langle V^b_\nu(y) V^d_\rho(z) \rangle_\beta \,, \end{aligned} \tag{C.1}$$

valid for generic β and in particular for $\beta = \infty$. Taking $y \neq z$ and integrating in x over a domain containing z but not y, one finds a finite right-hand side, which implies that Z_A must be finite, and can be set to 1. One then concludes $Z_V = Z_A = Z_m Z_P = 1$.

Contact terms. Renormalisation of the Ward-Takahashi identity eq. (B.9) requires also the subtraction of divergent contact terms at $x = 0$. In general (at least in perturbation theory), UV divergences must be polynomial in m as a consequence of locality. Dimensional analysis and the symmetries of the theory then constrain the divergent contact terms $\delta^{ab}\mathrm{CT}_{AP}$ and $\delta^{ab}\mathrm{CT}_{PP}$, appearing respectively in $\langle A^a_{B\mu}(x) P^b_B(0) \rangle$ and $\langle P^a_B(x) P^b_B(0) \rangle$, to be of the following form,

$$\begin{aligned} \mathrm{CT}_{AP}(x) &= \partial_\mu \delta^{(4)}(x) m K_{AP} \,, \\ \mathrm{CT}_{PP}(x) &= \Big(\delta^{(4)}(x) \Big(\Lambda^2 K^{(1)}_{PP} + m^2 K^{(2)}_{PP} \Big) + \Box \delta^{(4)}(x) K^{(3)}_{PP} \Big) \,, \end{aligned} \tag{C.2}$$

with K_{AP}, $K^{(1,2,3)}_{PP}$ dimensionless quantities, depending logarithmically on Λ. Since the pseudoscalar-pseudoscalar correlator appears multiplied by m, all these contact terms drop from eq. (B.9) in the chiral limit. The spacetime and flavour structure are determined by SO(4) and reflection invariance, and by the unbroken vector flavour symmetry, respectively. The dependence on m is dictated by the fact that a "$\mathcal{R}_5$-parity" transformation [108],

$$\psi \to \gamma_5 \psi \,, \qquad \bar{\psi} \to -\bar{\psi} \gamma_5 \,, \tag{C.3}$$

which is an element of the non-anomalous $\mathrm{SU}(N_f)_V \times \mathrm{SU}(N_f)_A$ symmetry group in the chiral limit, is equivalent to changing the sign of the fermion mass. This requires the expectation value of operators that are even (respectively odd) under $\mathcal{R}_5$-parity to be even (respectively odd) under $m \to -m$. Since the axial currents are odd while the pseudoscalar densities are even, the mass dependence in eq. (C.2) follows.

The contact term on the right-hand side of eq. (B.9) is proportional to the chiral condensate, i.e., the expectation value of the scalar density. This operator can mix with

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the identity operator, and so requires both additive and multiplicative renormalisation, $S_B = Z_S S + Z_1 \mathbf{1}$. The divergent part of the mixing coefficient, Z_1, is determined by the same type of argument used above to be of the form

$$Z_1 = N_f \left(m\Lambda^2 K_S^{(1)} + m^3 K_S^{(2)} \right) , \tag{C.4}$$

with dimensionless coefficients $K_S^{(1,2)}$ (again depending logarithmically on Λ), since the scalar density is odd under $\mathcal{R}_5$-parity. Also these terms drop from eq. (B.9) in the chiral limit, while matching the two sides at finite m one finds the relations $K_{AP} = 2K_{PP}^{(3)}$, $2K_{PP}^{(1)} = K_S^{(1)}$, and $2K_{PP}^{(2)} = K_S^{(2)}$.

Renormalised correlation functions. Defining now the fully renormalised correlation functions and chiral condensate via

$$\begin{aligned} Z_A Z_P \mathcal{G}_{AP\,\mu}(x)\delta^{ab} &= Z_A Z_P \langle A_\mu^a(x) P^b(0)\rangle_\beta = \langle A_{B\mu}^a(x) P_B^b(0)\rangle_\beta - \delta^{ab}\mathrm{CT}_{AP}(x) , \\ Z_P^2 \mathcal{G}_{PP}(x)\delta^{ab} &= Z_P^2 \langle P^a(x) P^b(0)\rangle_\beta = \langle P_B^a(x) P_B^b(0)\rangle_\beta - \delta^{ab}\mathrm{CT}_{PP}(x) , \\ Z_S \Sigma &= \frac{1}{N_f} Z_S \langle S(0)\rangle_\beta = \frac{1}{N_f}(\langle S_B(0)\rangle_\beta - Z_1) , \end{aligned} \tag{C.5}$$

one can use again the Ward-Takahashi identity to fix the multiplicative renormalisation constant Z_S. After renormalisation and integration over spacetime, one finds from eq. (B.9)

$$Z_m Z_P^2\, 2m \int_\beta d^4x\, \langle P^a(x) P^b(0)\rangle_\beta = \delta^{ab} Z_S \Sigma , \tag{C.6}$$

since for finite m the integral of the divergence of the axial currents gives zero contribution. One then concludes $Z_S = Z_P = Z_m^{-1}$.

It should be noted that subleading terms in the regularised bare chiral condensate $\frac{1}{N_f}\langle S_B(0)\rangle_\beta$, that vanish as $\Lambda \to \infty$, could conspire with the UV divergences in Z_S to give a finite contribution, generating further finite but more singular contact terms on the right-hand side of eq. (B.9). However, the only other possible term allowed by locality, SO(4) invariance, behaviour under $\mathcal{R}_5$-parity and dimensional analysis is $m\Box\delta^{(4)}(x)$, which vanishes in the chiral limit and is therefore irrelevant for the purposes of this paper.[32]

D Euclidean Goldstone theorem in coordinate space

In this appendix I give another proof of Goldstone's theorem at finite temperature, based on the Ward-Takahashi identity eq. (3.14) [eq. (B.9)], obtained by working in coordinate space. To this end, one defines the integrated correlation functions

$$\begin{aligned} \mathrm{Q}(t) &\equiv \lim_{V\to\infty} \int_V d^3x\, \mathcal{G}_{AP\,4}(t,\vec{x}) , \\ \mathrm{B}(t) &\equiv \lim_{V\to\infty} \int_V d^3x\, \vec{\nabla}\cdot\vec{\mathcal{G}}_{AP}(t,\vec{x}) = \lim_{V\to\infty} \int_{\partial V} d^2\vec{\Sigma}\cdot\vec{\mathcal{G}}_{AP}(t,\vec{x}) , \\ \mathrm{P}(t) &\equiv \lim_{V\to\infty} \int_V d^3x\, 2m\mathcal{G}_{PP}(t,\vec{x}) = \lim_{V\to\infty} \int_V d^3x\, \mathcal{R}(t,\vec{x}) . \end{aligned} \tag{D.1}$$

[32] At finite m, in momentum space this term becomes simply $m(\omega^2 + \vec{k}^{\,2})$ and so vanishes in the limit of zero frequency and zero spatial momentum.

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An infrared cutoff is imposed on the spatial integral in the form of a finite volume V with boundary ∂V (with outward-oriented infinitesimal surface element $d^2\vec{\Sigma}$), which is removed only at the end of the calculation. In particular, in the chiral limit this is done *after* the limit $m \to 0$ has already been taken. Notice that this is only a cutoff on the integral and not on the full theory, which is defined in infinite volume. Integrating eq. (B.9) over space one finds

$$-\partial_t \mathrm{Q}(t) - \mathrm{B}(t) + \mathrm{P}(t) = \delta_P(t)\Sigma\,, \tag{D.2}$$

where the periodic Dirac delta is defined in eq. (3.15).

Continuity properties of the integrated correlators. To proceed further one needs to discuss first the continuity properties at $t = 0$ of the integrated correlators, eq. (D.1). Here the symmetry properties of the Euclidean correlators under time reflection, eqs. (3.21) and (3.22), are used. For $\vec{x} \neq \vec{0}$, $\mathcal{G}_{AP\,\mu}(z, \vec{x})$ is analytic for complex $z = t - i\tau$ also for $t = 0$ if $|\tau| < |\vec{x}|$, so in particular $\mathcal{G}_{AP\,\mu}(t, \vec{x})$ is continuous at $t = 0$. For $\mathcal{G}_{AP\,4}$ at $t = 0$ one finds

$$\mathcal{G}_{AP\,4}(0, \vec{x}) = -\mathcal{G}_{AP\,4}(\beta, \vec{x})\,. \tag{D.3}$$

Combining eq. (D.3) with periodicity one concludes $\mathcal{G}_{AP\,4}(0, \vec{x}) = 0$ for $\vec{x} \neq \vec{0}$, while for $\vec{x} = \vec{0}$ this does not follow since continuity at $t = 0$ is not guaranteed. In general then $\mathrm{Q}(t)$ satisfies $\mathrm{Q}(\beta - t) = -\mathrm{Q}(t)$ but need not be continuous at $t = 0$. On the other hand, $\mathcal{G}_{AP\,\mu}$ is continuous (in fact, analytic) at $t = \frac{\beta}{2}$ for any $\vec{x}$, and so

$$\mathcal{G}_{AP\,4}\left(\tfrac{\beta}{2}, \vec{x}\right) = -\mathcal{G}_{AP\,4}\left(\tfrac{\beta}{2}, \vec{x}\right) = 0\,, \qquad \mathrm{Q}\left(\tfrac{\beta}{2}\right) = -\mathrm{Q}\left(\tfrac{\beta}{2}\right) = 0\,. \tag{D.4}$$

For the spatial components $\vec{\mathcal{G}}_{AP}$, eq. (3.21) and periodicity imply $\vec{\mathcal{G}}_{AP}(t, \vec{x}) = \vec{\mathcal{G}}_{AP}(\beta - t, \vec{x}) = \vec{\mathcal{G}}_{AP}(-t, \vec{x})$ for all $\vec{x}$, which already follows from continuity for $\vec{x} \neq 0$. If the limits for $t \to 0^{\pm}$ exist also for $\vec{x} = \vec{0}$, then $\vec{\mathcal{G}}_{AP}(t, \vec{0})$ must be continuous at $t = 0$.

More interestingly, and independently of time-reflection symmetry, if $\vec{\mathcal{G}}_{AP}$ vanishes sufficiently fast at spatial infinity then $\mathrm{B}(t)$ is continuous at $t = 0$, since the point $\vec{x} = \vec{0}$ is not involved in the integral. Moreover, as a consequence of the regularity condition eq. (4.7) discussed in section 4.2, one finds that $\mathrm{B}(t)$ is constant in (Euclidean) time. In fact, $\mathrm{B}(t)$ is obtained by summing $\vec{k} \cdot \vec{\tilde{\mathcal{G}}}_{AP}$ over Matsubara frequencies and taking the limit $\vec{k} \to 0$ [see eq. (2.7)],

$$\mathrm{B}\,(t) = \lim_{\vec{k}\to 0} \frac{1}{\beta} \sum_n e^{-i\omega_n t} \left(-i\vec{k}\right) \cdot \vec{\tilde{\mathcal{G}}}_{AP}\left(\omega_n, \vec{k}\right) = -\frac{i}{\beta} \lim_{\vec{k}\to 0} \vec{k} \cdot \vec{\tilde{\mathcal{G}}}_{AP}\left(0, \vec{k}\right)\,, \tag{D.5}$$

which is time-independent. Notice that for $\omega_0 = 0$ there is no requirement from eq. (4.7), so that $\mathrm{B}(t) = \mathrm{B}$ need not be zero in general.

Finally, eq. (3.22) implies that $\mathcal{G}_{PP}(t, \vec{x})$ is continuous at $t = 0$ for $\vec{x} \neq \vec{0}$, and also for $\vec{x} = 0$ if the limits $t \to 0^{\pm}$ exist, like for $\vec{\mathcal{G}}_{AP}$. The quantity $\mathrm{P}(t)$ may not be continuous at $t = 0$ if $\mathcal{G}_{PP}$ is divergent at $x = 0$, but for our purposes it suffices to assume that at $t = 0$ it develops at most an integrable singularity in t.

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Chiral limit. The properties discussed above are expected to hold also in the chiral limit. In particular, as in section 4.2, it is assumed that the regularity condition holds as $V \to \infty$ in the spatial integral (i.e., $\vec{k} \to 0$ in momentum space) also after the chiral limit has been taken. Taking now the chiral limit in the integrated Ward-Takahashi identity, eq. (D.2), one finds

$$-\partial_t \mathrm{Q}_*(t) - \mathrm{B}_* + \mathrm{P}_*(t) = \delta_P(t)\Sigma_* \,, \tag{D.6}$$

where I used eq. (D.5), and all quantities are computed in the chiral limit (taken before $V \to \infty$ in the integral), as denoted by the subscript $*$. Further integrating over time between $\frac{\beta}{2}$ and $t \in (0, \beta)$, and using $\mathrm{Q}(\frac{\beta}{2}) = 0$ following from antisymmetry [see eq. (D.4)], one finds

$$\mathrm{Q}_*(t) = -\mathrm{B}_* \cdot \left(t - \frac{\beta}{2}\right) + \int_{\frac{\beta}{2}}^{t} dt' \, \mathrm{P}_*(t') \,, \qquad t \in (0, \beta) \,. \tag{D.7}$$

This is then repeated periodically as $\mathrm{Q}_*(t + n\beta) = \mathrm{Q}_*(t)$, $\forall n \in \mathbb{Z}$.

The value of B_* is determined by the contact term at $t = 0$ in eq. (D.6). Integrating this equation in an infinitesimal neighbourhood $[-\epsilon, \epsilon]$ of $t = 0$, using the property $\mathrm{Q}_*(-t) = \mathrm{Q}_*(\beta - t) = -\mathrm{Q}_*(t)$ which follows from periodicity and antisymmetry under temporal reflection, and using integrability of P_* at $t = 0$, one finds

$$\lim_{\epsilon \to 0} -[\mathrm{Q}_*(\epsilon) - \mathrm{Q}_*(-\epsilon)] = \lim_{\epsilon \to 0} -2\mathrm{Q}_*(\epsilon) = \Sigma_* \,. \tag{D.8}$$

One the other hand, using the solution eq. (D.7) and the property $\mathrm{P}_*(t) = \mathrm{P}_*(\beta - t)$ one obtains

$$-\Sigma_* = \lim_{\epsilon \to 0} 2\mathrm{Q}_*(\epsilon) = \mathrm{B}_*\beta + 2\int_{\frac{\beta}{2}}^{0} dt' \, \mathrm{P}_*(t') = \mathrm{B}_*\beta - 2\int_{0}^{\beta} dt' \, \mathrm{P}_*(t') \equiv \mathrm{B}_*\beta - \Xi_* \,. \tag{D.9}$$

Plugging this into eq. (D.7) one finally obtains

$$\mathrm{Q}_*(t) = -(\Sigma_* - \Xi_*)\left(\frac{t}{\beta} - \frac{1}{2}\right) + \int_{\frac{\beta}{2}}^{t} dt' \, \mathrm{P}_*(t') \,, \qquad t \in (0, \beta) \,. \tag{D.10}$$

In the standard case $\mathrm{P}_* = 0$, the resulting function manifestly satisfies the analyticity, continuity and bounded-growth hypotheses of the reconstruction theorem of real-time Green functions discussed in ref. [69]. This will hold also if $\mathrm{P}_* \neq 0$ under reasonable analyticity and bounded-growth assumptions for this quantity.

Analytic continuation. The analytic continuation required to obtain the spectral function is most clearly done exploiting again the antisymmetry of Q, extended by analytic continuation. One finds

$$\begin{aligned} \mathrm{Q}_*(\epsilon + it) - \mathrm{Q}_*(-\epsilon + it) &= \mathrm{Q}_*(\epsilon + it) + \mathrm{Q}_*(\epsilon - it) \\ &= -(\Sigma_* - \Xi_*) + \int_{\frac{\beta}{2}}^{\epsilon+it} dt' \, \mathrm{P}_*(t') + \int_{\frac{\beta}{2}}^{\epsilon-it} dt' \, \mathrm{P}_*(t') \,, \end{aligned} \tag{D.11}$$

where the complex paths appearing in the last two terms are chosen to run along the real t' axis from $\frac{\beta}{2}$ to ϵ, and then on the axis $z = \epsilon + it'$ in the positive or negative t' direction. If

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$\mathrm{P}_*(z)$ is free of singularities for $\mathrm{Re}\, z \in (0,\beta)$, then the analytic continuation is unambiguous in this strip. Exploiting the symmetry properties of P_*, one finds for the last two terms in eq. (D.11)

$$\begin{aligned}\int_{\frac{\beta}{2}}^{\epsilon+it} dt'\, \mathrm{P}_*(t') + \int_{\frac{\beta}{2}}^{\epsilon-it} dt'\, \mathrm{P}_*(t') &= -\Xi_* + i\int_0^t dt'\ \left[\mathrm{P}_*(\epsilon+it') - \mathrm{P}_*(\epsilon-it')\right] \\ &= -\Xi_* + i\int_0^t dt'\ \left[\mathrm{P}_*(\epsilon+it') - \mathrm{P}_*(-\epsilon+it')\right] \\ &= -\Xi_* + i\int_0^t dt'\, \Lambda_*(t')\,,\end{aligned} \tag{D.12}$$

where [recall eq. (3.25)]

$$\Lambda_*(t) \equiv \mathrm{P}_*(\epsilon+it') - \mathrm{P}_*(-\epsilon+it') = \lim_{V\to\infty}\int_V d^3x\, \lim_{m\to 0} 2m \langle\!\langle \left[\hat{P}^a(t,\vec{x}), \hat{P}^a(0)\right]\rangle\!\rangle_\beta\,. \tag{D.13}$$

One then obtains [recall again eq. (3.25)]

$$\begin{aligned}\mathrm{Q}_*(\epsilon+it) - \mathrm{Q}_*(-\epsilon+it) &= \lim_{V\to\infty}\int_V d^3x\, \lim_{m\to 0} \langle\!\langle \left[\hat{A}^{0a}(t',\vec{x}), \hat{P}^a(0)\right]\rangle\!\rangle_\beta \\ &= -\Sigma_* + i\int_0^t dt'\, \Lambda_*(t')\,.\end{aligned} \tag{D.14}$$

In the standard case $\mathrm{P}_* = 0$ (i.e., $\mathcal{R}_* = 0$, or no $1/m$ divergence in the pseudoscalar-pseudoscalar correlator), it is easy to see that the axial-vector-pseudoscalar spectral function contains a Dirac-delta term proportional to Σ_*:

$$\lim_{\vec{k}\to 0} \tilde{c}_*\left(\omega,\vec{k}\right) = \lim_{V\to\infty}\int dt \int_V d^3x\, e^{i\omega t}\, \lim_{m\to 0} \langle\!\langle \left[\hat{A}^{0a}(t,\vec{x}), \hat{P}^a(0)\right]\rangle\!\rangle_\beta = -2\pi\Sigma_*\delta(\omega)\,. \tag{D.15}$$

In the case $\mathrm{P}_* \neq 0$, one still generally finds a Dirac-delta contribution to the spectral function, whose coefficient depends on the large-time behaviour of the integral in eq. (D.14). This can be related back to the Euclidean correlator by means of a path deformation argument analogous to the one used in ref. [68] and above in appendix A. Consider

$$I = \int_{\mathcal{C}_>} dz\, \mathrm{P}_*(iz)\,, \tag{D.16}$$

where the path $\mathcal{C}_>$ is shown in figure 1. Shrinking the path towards the imaginary axis one finds

$$I = i\int_0^\infty dt\ \left[\mathrm{P}_*(\epsilon+it) - \mathrm{P}_*(-\epsilon+it)\right] = i\int_0^\infty dt\, \Lambda_*(t)\,. \tag{D.17}$$

On the other hand, periodicity implies that the integrals on the parts of the path reaching to infinity cancel each other out, and so

$$I = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt\, \mathrm{P}_*(t) = \int_0^\beta dt\, \mathrm{P}_*(t) = \Xi_*\,, \tag{D.18}$$

assuming that $\mathrm{P}_*(z)$ vanishes for $\mathrm{Im}\, z \to \infty$, so that the part of $\mathcal{C}_>$ at large $\mathrm{Im}\, z$ does not contribute. Then

$$\lim_{t\to+\infty} \mathrm{Q}_*(\epsilon+it) - \mathrm{Q}_*(-\epsilon+it) = -(\Sigma_* - \Xi_*)\,, \tag{D.19}$$

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and by antisymmetry

$$\lim_{t\to-\infty} \mathrm{Q}_*(\epsilon+it) - \mathrm{Q}_*(-\epsilon+it) = -\lim_{t\to+\infty} \mathrm{Q}_*(-\epsilon+it) - \mathrm{Q}_*(\epsilon+it) = -(\Sigma_* - \Xi_*)\,. \tag{D.20}$$

Isolating the contribution of the constant large-$|t|$ behaviour one concludes that

$$\begin{aligned}\lim_{\vec{k}\to0} \tilde{c}_*\left(\omega,\vec{k}\right) &= \lim_{V\to\infty}\int dt\int_V d^3x\, e^{i\omega t}\lim_{m\to0}\langle\!\langle\left[\hat{A}^{0a}(t,\vec{x}),\hat{P}^a(0)\right]\rangle\!\rangle_\beta \\ &= -2\pi(\Sigma_*-\Xi_*)\delta(\omega) + \text{less singular}\,.\end{aligned} \tag{D.21}$$

Since $\Xi_* = \mathrm{R}_*(0)$ in the notation of section 4.3 [see eqs. (4.8) and (4.15)], the result eq. (4.26) is reproduced.

If $\mathrm{P}_*(z)$ has a finite nonzero limit as $\mathrm{Im}\, z \to \infty$, then under suitable boundedness conditions in the strips $\mathrm{Re}\, z \in (0, \frac{\beta}{2}]$ and $\mathrm{Re}\, z \in [-\frac{\beta}{2}, 0)$, the limits $\lim_{\tau\to\infty} \mathrm{P}_*(\pm t + i\tau)$, $t \in (0, \frac{\beta}{2}]$, are independent of t due to the Phragmén-Lindelöf theorem (see, e.g., ref. [130]), and they also do not depend on the sign due to periodicity fixing $\lim_{\tau\to\infty} \mathrm{P}_*(-\frac{\beta}{2} + i\tau) = \lim_{\tau\to\infty} \mathrm{P}_*(\frac{\beta}{2} + i\tau)$. One then finds an extra contribution to I, corresponding to the effect of a transport peak in the spectral density [see eq. (A.6) and following discussion, and footnote 12],

$$\begin{aligned}I - \Xi_* &= -\beta \mathrm{P}_*\left(\tfrac{\beta}{2} + i\infty\right) = -\beta\int d^3x \lim_{m\to0} 2m\mathcal{G}_{PP}\left(\tfrac{\beta}{2}+i\infty\right) \\ &= -\lim_{\vec{k}\to0}\lim_{m\to0} 2mA_{PP}\left(\vec{k}\right) = -A_*\,,\end{aligned} \tag{D.22}$$

and so $\lim_{\vec{k}\to0} \tilde{c}_*\left(\omega,\vec{k}\right)|_{\text{singular}} = -2\pi\left(\Sigma_* - \Xi_* + A_*\right)\delta\left(\omega\right)$ (see footnote 12).

E Exponentially localised modes

In this appendix I repeat the large-volume estimate of the contributions of localised modes to $C^{\Gamma}_{s\,V,\Lambda}$ done in section 5.2, using the more accurate characterisation of localised modes as exponentially decreasing in space, rather than strictly confined in a finite region. This improves over the x-independent estimate eq. (5.14), showing that an exponential decay at large separation is to be expected.

Assume that localised modes are bounded by an envelope that exponentially decays in space starting from some localisation centre $\vec{x}_{0n}$,

$$\|\psi_n(x)\|^2 \leq \frac{K_n}{\ell_n^3} e^{-\frac{|\vec{x}-\vec{x}_{0n}|}{\ell_n}}\,, \tag{E.1}$$

with ℓ_n the localisation length of the mode, and K_n a positive constant. Using the bound [see the second line of eq. (5.10)]

$$\begin{aligned}\left|\langle\mathcal{O}^{\Gamma}_{nn}(x)\mathcal{O}^{\Gamma}_{nn}(0)\rangle_\beta\right| &\leq \langle\left|\mathcal{O}^{\Gamma}_{nn}(x)\right|\left|\mathcal{O}^{\Gamma}_{nn}(0)\right|\rangle_\beta \leq \langle\|\psi_n(x)\|^2\|\psi_n(0)\|^2\rangle_\beta \\ &= \frac{1}{\beta V}\int_\beta d^4y\,\langle\|\psi_n(x+y)\|^2\|\psi_n(y)\|^2\rangle_\beta\,,\end{aligned} \tag{E.2}$$

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where in the last passage I used translation invariance, together with eq. (E.1), one finds

$$\left|\langle \mathcal{O}^{\Gamma}_{nn}(x)\mathcal{O}^{\Gamma}_{nn}(0)\rangle_\beta\right| \leq \frac{1}{\beta V}\int_\beta d^4y \left\langle \frac{K_n^2}{\ell_n^6} e^{-\frac{|\vec{x}+\vec{y}-\vec{x}_{0n}|+|\vec{y}-\vec{x}_{0n}|}{\ell_n}}\right\rangle_\beta = \frac{1}{V}\langle K_n^2 R(\vec{x},\ell_n)\rangle_\beta\,, \tag{E.3}$$

where

$$R(\vec{x},\ell) \equiv \frac{1}{\ell^6}\int d^3y\, e^{-\frac{|\vec{x}+\vec{y}|+|\vec{y}|}{\ell}}\,. \tag{E.4}$$

An explicit calculation shows that

$$R(\vec{x},\ell) = \frac{\pi}{\ell^3} e^{-\frac{|\vec{x}|}{\ell}}\left(1+\frac{|\vec{x}|}{\ell}+\frac{1}{3}\frac{|\vec{x}|^2}{\ell^2}\right). \tag{E.5}$$

One then qualitatively expects an exponential suppression of the spectral correlators at large distance.

Under the further assumption that the support of the local probability distribution of the localisation length at a given point in the spectrum is bounded from above by some $\xi(\lambda)$, and if K_n is also locally bounded by some $K(\lambda)$, then since

$$\frac{\partial}{\partial\ell}\frac{1}{\ell^3}e^{-\frac{|\vec{x}|}{\ell}}\left(1+\frac{|\vec{x}|}{\ell}+\frac{1}{3}\frac{|\vec{x}|^2}{\ell^2}\right) = \frac{1}{3\ell^4}e^{-\frac{|\vec{x}|}{\ell}}\left(\frac{|\vec{x}|^3}{\ell^3}-2\frac{|\vec{x}|^2}{\ell^2}-9\frac{|\vec{x}|}{\ell}-9\right), \tag{E.6}$$

which is positive for $|\vec{x}| \geq c_0\ell$, $c_0 \simeq 4.466$, one obtains the exponential bound

$$\left|\left\langle \sum_n \delta(\lambda-\lambda_n)\mathcal{O}^{\Gamma}_{nn}(x)\mathcal{O}^{\Gamma}_{nn}(0)\right\rangle_\beta\right| \leq \rho_{B,V}(\lambda)\frac{\pi K(\lambda)^2}{\xi(\lambda)^3}e^{-\frac{|\vec{x}|}{\xi(\lambda)}}\left(1+\frac{|\vec{x}|}{\xi(\lambda)}+\frac{1}{3}\frac{|\vec{x}|^2}{\xi(\lambda)^2}\right), \tag{E.7}$$

valid at large $|\vec{x}| \geq c_0\xi(\lambda)$.

F Pseudoscalar-pseudoscalar correlator in the chiral limit

In this appendix I discuss the evaluation of the various contributions to the pseudoscalar-pseudoscalar correlator in the chiral limit. For the contributions involving large eigenvalues, I show that independently of the subtraction procedure employed to deal with additive divergences, the remaining finite contributions as $\Lambda \to \infty$ stay finite also as $m \to 0$. I then identify what contributions from the small eigenvalues can lead to a finite $\mathcal{R}_*$ in the chiral limit.

It is assumed that finite limits exist as the argument λ of the spectral correlators $C^{\Gamma}_{\rm loc}$, and either or both of the arguments λ, λ' of the spectral correlators $\bar{C}^{\Gamma}$, tend to zero. It is also assumed that such limits are approached at least as fast as some power law. Finally, it is assumed that the limit $m \to 0$ of the spectral correlators is well defined.

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Bounds on spectral integrals. The following inequalities are used to derive bounds on the asymptotic m-dependence of the relevant spectral integrals as $m \to 0$,

$$\left|\int_0^\mu d\lambda\, \frac{f(\lambda)}{\lambda^2+m^2}\right| \le \left(\frac{\pi}{2m}\right)^{1-\frac{1}{2^N}} \left(\int_0^\mu d\lambda\, \frac{|f(\lambda)|^{2^N}}{\lambda^2+m^2}\right)^{\frac{1}{2^N}}, \tag{F.1}$$

$$\left|\int_0^\mu d\lambda\, \frac{f(\lambda)}{(\lambda^2+m^2)^2}\right| \le \left(\frac{1}{2\mu m^2}+\frac{\pi}{4m^3}\right)^{1-\frac{1}{2^N}} \left(\int_0^\mu d\lambda\, \frac{|f(\lambda)|^{2^N}}{(\lambda^2+m^2)^2}\right)^{\frac{1}{2^N}}, \tag{F.2}$$

as well as

$$\begin{aligned} &\left|\int_0^\mu d\lambda \int_0^\mu d\lambda'\, \frac{F(\lambda,\lambda')}{(\lambda^2+m^2)(\lambda'^2+m^2)}\right| \\ &\le \left(\frac{\pi}{2m}\right)^{2\left(1-\frac{1}{2^N}\right)} \left(\int_0^\mu d\lambda \int_0^\mu d\lambda'\, \frac{|F(\lambda,\lambda')|^{2^N}}{(\lambda^2+m^2)(\lambda'^2+m^2)}\right)^{\frac{1}{2^N}}, \end{aligned} \tag{F.3}$$

where N is an arbitrary non-negative integer number. These inequalities follow from repeated application of the Cauchy-Schwarz inequality, and from the following elementary results,

$$\begin{aligned} \int_0^\mu d\lambda\, \frac{1}{\lambda^2+m^2} &= \frac{1}{m}\arctan\frac{\mu}{m} \le \frac{\pi}{2m}, \\ \int_0^\mu d\lambda\, \frac{1}{(\lambda^2+m^2)^2} &= \frac{1}{2m^2}\left(\frac{\mu}{\mu^2+m^2}+\frac{1}{m}\arctan\frac{\mu}{m}\right) \le \frac{1}{2m^3}\left(\frac{\pi}{2}+\frac{m}{\mu}\right). \end{aligned} \tag{F.4}$$

As an example, eq. (F.1) is obtained by noticing that the case $N = 0$ is obvious, and using the Cauchy-Schwarz inequality to show that

$$\begin{aligned} \left(\int_0^\mu d\lambda\, \frac{|f(\lambda)|^{2^N}}{\lambda^2+m^2}\right)^{\frac{1}{2^N}} &\le \left[\left(\int_0^\mu d\lambda\, \frac{|f(\lambda)|^{2^{N+1}}}{\lambda^2+m^2}\right)^{\frac{1}{2}} \left(\int_0^\mu d\lambda\, \frac{1}{\lambda^2+m^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2^N}} \\ &\le \left(\frac{\pi}{2m}\right)^{\frac{1}{2^{N+1}}} \left(\int_0^\mu d\lambda\, \frac{|f(\lambda)|^{2^{N+1}}}{\lambda^2+m^2}\right)^{\frac{1}{2^{N+1}}}, \end{aligned} \tag{F.5}$$

from which the result follows since $1 - 2^{-N} + 2^{-(N+1)} = 1 - 2^{-(N+1)}$.

Let now $f(\lambda)$ vanish at least as fast as λ^γ for $\lambda \to 0$, i.e., $\lim_{\lambda\to 0} \lambda^{-\gamma} f(\lambda) < \infty$, with $\gamma > 0$. Using eq. (F.1), one finds

$$\begin{aligned} \left|\int_0^\mu d\lambda\, \frac{m f(\lambda)}{\lambda^2+m^2}\right| &\le m\left(\frac{\pi}{2m}\right)^{1-\frac{1}{2^N}} \left(\int_0^\mu d\lambda\, \frac{|f(\lambda)|^{2^N}}{\lambda^2+m^2}\right)^{\frac{1}{2^N}} \\ &\le m^{\frac{1}{2^N}} \left(\frac{\pi}{2}\right)^{1-\frac{1}{2^N}} \left(\int_0^\mu d\lambda\, \frac{|f(\lambda)|^{2^N}}{\lambda^2}\right)^{\frac{1}{2^N}} = o(1), \end{aligned} \tag{F.6}$$

if N is chosen such that $2^N\gamma > 1$, so that the last integral is convergent. Similarly, using eq. (F.2), one finds

$$\begin{aligned} \left|\int_0^\mu d\lambda\, \frac{m^3 f(\lambda)}{(\lambda^2+m^2)^2}\right| &\le m^3\left(\frac{1}{2\mu m^2}+\frac{\pi}{4m^3}\right)^{1-\frac{1}{2^N}} \left(\int_0^\mu d\lambda\, \frac{|f(\lambda)|^{2^N}}{(\lambda^2+m^2)^2}\right)^{\frac{1}{2^N}} \\ &\le m^{\frac{3}{2^N}} \left(\frac{m}{2\mu}+\frac{\pi}{4}\right)^{1-\frac{1}{2^N}} \left(\int_0^\mu d\lambda\, \frac{|f(\lambda)|^{2^N}}{\lambda^4}\right)^{\frac{1}{2^N}} = o(1), \end{aligned} \tag{F.7}$$

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provided $2^N\gamma > 3$. Finally, for $F(\lambda,\lambda')$ vanishing at least as fast as λ^γ (respectively $\lambda^{\gamma'}$) for $\lambda \to 0$ (respectively $\lambda' \to 0$), using eq. (F.3) one finds

$$\begin{aligned}\Big|\int_0^\mu d\lambda \int_0^\mu d\lambda' \frac{m^2 F(\lambda,\lambda')}{(\lambda^2+m^2)(\lambda'^2+m^2)}\Big| \\ \leq m^{\frac{1}{2^{N-1}}}\left(\frac{\pi}{2}\right)^{2\left(1-\frac{1}{2^N}\right)}\left(\int_0^\mu d\lambda\int_0^\mu d\lambda' \frac{|F(\lambda,\lambda')|^{2^N}}{(\lambda^2+m^2)(\lambda'^2+m^2)}\right)^{\frac{1}{2^N}} \\ \leq m^{\frac{1}{2^{N-1}}}\left(\frac{\pi}{2}\right)^{2\left(1-\frac{1}{2^N}\right)}\left(\int_0^\mu d\lambda\int_0^\mu d\lambda' \frac{|F(\lambda,\lambda')|^{2^N}}{\lambda^2\lambda'^2}\right)^{\frac{1}{2^N}} = o(1)\,,\end{aligned} \tag{F.8}$$

provided $2^N \min(\gamma,\gamma') > 1$.

Large-λ contributions. Additive divergences originating from the large-eigenvalue region appear in $\mathcal{G}^{(12)}_{PPB}$, $\mathcal{G}^{(21)}_{PPB}$, and $\mathcal{G}^{(22)}_{PPB}$. These are removed by subtracting the leading contributions from the factors $(\lambda^2+m^2)^{-1}$ and $(\lambda'^2+m^2)^{-1}$ appearing in eq. (5.26) when the integration range is $I_2 = [\mu, \Lambda/Z_m]$. One has

$$\frac{1}{\lambda^2+m^2} = \left(\sum_{n=0}^{N-1}(-1)^n\frac{m^{2n}}{\lambda^{2(n+1)}}\right) + \frac{(-m^2)^N}{\lambda^{2N}(\lambda^2+m^2)}\,, \tag{F.9}$$

with the case of no subtraction corresponding to setting $N = 0$. This shows that terms more suppressed in λ^2 contain higher powers of m^2, and that the remainder is suppressed by powers of m^2 as well. For sufficiently large N the integral containing the remainder is convergent. Independently of the specific procedure employed to subtract the divergent part, the remaining finite contributions coming from the N leading terms in the expansion eq. (F.9) are then suppressed by powers of m, and can produce at most a constant term in the chiral limit. One similarly shows that the same holds true for the contribution of the remainders. It is assumed that the residual m dependence of the remaining integrals through the spectral correlators is regular, with finite limits as $m \to 0$.[33]

To show this in detail, for $\mathcal{G}^{(12)}_{PPB} = \mathcal{G}^{(21)}_{PPB}$ one writes

$$\begin{aligned}-Z_m^2\mathcal{G}^{(12)}_{PPB} = \sum_{n=0}^{N-1}(-1)^n m^{2n}\int_0^\mu d\lambda\int_\mu^{\frac{\Lambda}{Z_m}} d\lambda' \frac{m^2\bar{C}^+(\lambda,\lambda';x;m)+\lambda\lambda'\bar{C}^-(\lambda,\lambda';x;m)}{(\lambda^2+m^2)\lambda'^{2(n+1)}} \\ + m^{2N}\int_0^\mu d\lambda\int_\mu^\infty d\lambda' \frac{m^2\bar{C}^+(\lambda,\lambda';x;m)+\lambda\lambda'\bar{C}^-(\lambda,\lambda';x;m)}{(\lambda^2+m^2)(\lambda'^2+m^2)\lambda'^{2N}}\,,\end{aligned} \tag{F.10}$$

where I set $\bar{C}^\pm \equiv \bar{C}^{\mathbf{1}} \pm \bar{C}^{\gamma_5}$, and terms that vanish as $\Lambda \to \infty$ have been omitted. The integral over λ' in the first N terms is divergent as $\Lambda \to \infty$, but it does not introduce any further mass dependence besides that of the spectral correlators, which is assumed to be sufficiently regular. The further mass dependence introduced by the integral over λ is

[33] More precisely, the requirement is that the coefficients of the expansion of $Z_m^2\mathcal{G}^{(ij)}_{PPB}$ in powers of Λ and of nested logarithms of Λ/μ (including negative powers) remain finite as $m \to 0$.

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harmless, since

$$\begin{aligned}&\left|\int_0^\mu d\lambda \int_\mu^{\frac{\Lambda}{Z_m}} d\lambda' \, \frac{m^2\bar{C}^+(\lambda,\lambda';x;m)+\lambda\lambda'\bar{C}^-(\lambda,\lambda';x;m)}{(\lambda^2+m^2)\lambda'^{2(n+1)}}\right| \\ &\leq \int_0^\mu d\lambda \int_\mu^{\frac{\Lambda}{Z_m}} d\lambda' \, \frac{|\bar{C}^+(\lambda,\lambda';x;m)|}{\lambda'^{2(n+1)}} + \int_0^\mu d\lambda \int_\mu^{\frac{\Lambda}{Z_m}} d\lambda' \, \frac{|\bar{C}^-(\lambda,\lambda';x;m)|}{\lambda\lambda'^{2n-1}} \,,\end{aligned} \tag{F.11}$$

where the integral over λ in the second term on the second line is convergent, since one has $\bar{C}^-(0,\lambda';x;m)=0$ [see eq. (5.8)], and since it is assumed that the limit $\lambda\to 0$ is reached at least as fast as some power law. These contributions have then a regular, at most $O(1)$ chiral limit.[34] This must hold separately for the divergent and finite parts, whose contributions to $Z_m^2\mathcal{G}^{(12)}_{PPB}$ are then constant or suppressed by powers of m in the chiral limit. The exact same argument with $n\to N$ works for the remainder term, since one can bound $(\lambda'^2+m^2)^{-1}\leq\lambda'^{-2}$.

For $\mathcal{G}^{(22)}_{PPB}$, one splits $\mathcal{G}^{(22)}_{PPB}=\mathcal{G}^{(22),1}_{PPB}+\mathcal{G}^{(22),2}_{PPB}$, with

$$\begin{aligned}-Z_m^2\mathcal{G}^{(22),1}_{PPB} &= \sum_{n=0}^{N-1}(-1)^n m^{2n}\int_\mu^{\frac{\Lambda}{Z_m}} d\lambda \left(\frac{C^{\mathbf{1}}_{\mathrm{loc}}(\lambda;x;m)}{\lambda^{2(n+1)}}+\frac{(m^2-\lambda^2)C^{\gamma_5}_{\mathrm{loc}}(\lambda;x;m)}{\lambda^{2(n+1)}(\lambda^2+m^2)}\right) \\ &\quad + m^{2N}\int_\mu^\infty d\lambda \left(\frac{C^{\mathbf{1}}_{\mathrm{loc}}(\lambda;x;m)}{\lambda^{2N}(\lambda^2+m^2)}+\frac{(m^2-\lambda^2)C^{\gamma_5}_{\mathrm{loc}}(\lambda;x;m)}{\lambda^{2N}(\lambda^2+m^2)^2}\right),\end{aligned} \tag{F.12}$$

where I expanded only one power of $(\lambda^2+m^2)^{-1}$ in the second term under the integrals, and

$$\begin{aligned}-Z_m^2\mathcal{G}^{(22),2}_{PPB} &= \sum_{n=0}^{N-1}\sum_{n'=0}^{N-1}(-1)^{n+n'}m^{2(n+n')}\int_\mu^{\frac{\Lambda}{Z_m}} d\lambda\int_\mu^{\frac{\Lambda}{Z_m}} d\lambda' \, \frac{W(\lambda,\lambda';x;m)}{\lambda^{2(n+1)}\lambda'^{2(n'+1)}} \\ &\quad + \sum_{n=0}^{N-1}(-1)^n m^{2n}\int_\mu^{\frac{\Lambda}{Z_m}} d\lambda\int_\mu^{\frac{\Lambda}{Z_m}} d\lambda' \, \frac{W(\lambda,\lambda';x;m)+W(\lambda',\lambda;x;m)}{\lambda^{2(n+1)}(\lambda'^2+m^2)\lambda'^{2N}} \\ &\quad + m^{4N}\int_\mu^\infty d\lambda\int_\mu^\infty d\lambda' \, \frac{W(\lambda,\lambda';x;m)}{(\lambda^2+m^2)(\lambda'^2+m^2)\lambda^{2N}\lambda'^{2N}} \,,\end{aligned} \tag{F.13}$$

where I set $W=m^2\bar{C}^+ + \lambda\lambda'\bar{C}^-$ for brevity and exploited the symmetry under $\lambda\leftrightarrow\lambda'$ of the spectral correlators. Terms that vanish as $\Lambda\to\infty$ are again ignored. In principle different choices of N may be needed for the two contributions from localised modes; the generalisation is straightforward. Clearly, all integrals are convergent (at finite Λ) if one lets $m\to 0$ in the denominators, and so $O(1)$ in the chiral limit. This holds for divergent and finite parts separately, whose contributions to $Z_m^2\mathcal{G}^{(22)}_{PPB}$ are then constant or suppressed by powers of m in the chiral limit. One then concludes that, after subtraction of the divergent parts and multiplicative renormalisation, the contributions of $\mathcal{G}^{(12)}_{PPB}$, $\mathcal{G}^{(21)}_{PPB}$, and $\mathcal{G}^{(22)}_{PPB}$ to the renormalised pseudoscalar-pseudoscalar correlator tend to a constant in the chiral limit [see eq. (5.28)].

Notice that for finite m, the finite parts in the subtraction terms used for $\mathcal{G}_{PP}$ should be matched with those employed in the renormalisation of the chiral condensate and of

[34] One can actually show that the contribution from $\bar{C}^+$ is $O(m)$.

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$\mathcal{G}_{AP\,\mu}$, in order to ensure that the Ward-Takahashi identity eq. (B.9) [eq. (3.14)] holds after renormalisation. This also determines how far in the expansion eq. (F.9) one has to go. In any case, these finite terms remain finite also as $m \to 0$ and so are irrelevant for the chiral limit of $\mathcal{R} = 2m\mathcal{G}_{PP}$.

Small-λ contributions. I now discuss the contributions of small eigenvalues ($|\lambda| \leq \mu$) to $\mathcal{R}_*$ under the assumptions of section 5.5, see eq. (5.31). In this case one finds three types of contributions. The first contribution, eq. (5.32), comes from localised near-zero modes. One sets

$$\begin{aligned} C^{\Gamma}_{\text{loc}}(\lambda; x; m) &= C^{\Gamma}_{\text{loc}}(0; x; m) + \left(C^{\Gamma}_{\text{loc}}(\lambda; x; m) - C^{\Gamma}_{\text{loc}}(0; x; m)\right) \\ &= C^{\mathbf{1}}_{\text{loc}}(0; x; m) + f^{\Gamma}(\lambda; x; m)\,, \end{aligned} \tag{F.14}$$

and assuming that f^{Γ} vanishes at least as fast as some power law as $\lambda \to 0$, one finds

$$\begin{aligned} \int_0^{\lambda_c} d\lambda\, \frac{C^{\mathbf{1}}_{\text{loc}}(\lambda; x; m)}{\lambda^2 + m^2} &= C^{\mathbf{1}}_{\text{loc}}(0; x; m) \int_0^{\lambda_c} d\lambda\, \frac{1}{\lambda^2 + m^2} + \frac{1}{m} \int_0^{\lambda_c} d\lambda\, \frac{m f^{\mathbf{1}}(\lambda; x; m)}{\lambda^2 + m^2} \\ &= C^{\mathbf{1}}_{\text{loc}}(0; x; m) \frac{1}{m} \arctan \frac{\lambda_c}{m} + o(1/m)\,, \end{aligned} \tag{F.15}$$

having used eq. (F.6), and

$$\begin{aligned} \int_0^{\lambda_c} d\lambda\, \frac{(m^2 - \lambda^2) C^{\gamma_5}_{\text{loc}}(\lambda; x; m)}{(\lambda^2 + m^2)^2} &= C^{\gamma_5}_{\text{loc}}(0; x; m) \int_0^{\lambda_c} d\lambda\, \frac{m^2 - \lambda^2}{(\lambda^2 + m^2)^2} \\ &\quad - \frac{1}{m} \int_0^{\lambda_c} d\lambda\, \frac{m f^{\gamma_5}(\lambda; x; m)}{\lambda^2 + m^2} + \frac{2}{m} \int_0^{\lambda_c} d\lambda\, \frac{m^3 f^{\gamma_5}(\lambda; x; m)}{(\lambda^2 + m^2)^2} \\ &= C^{\gamma_5}_{\text{loc}}(0; x; m) \frac{1}{m} \frac{\frac{\lambda_c}{m}}{1 + \frac{\lambda_c^2}{m^2}} + o(1/m)\,, \end{aligned} \tag{F.16}$$

having used eq. (F.7). The $o(1/m)$ estimate for the behaviour of the omitted terms is correct even if λ_c is not finite but vanishes in the chiral limit. In fact, since the omitted terms are already $o(1/m)$ if λ_c is treated as an independent variable, then taking into account its dependence on m can only make them less divergent, as they involve integrals that vanish as $\lambda_c \to 0$. This means that they cannot produce a divergence as strong as or stronger than $1/m$ in any case, and a contribution to $\mathcal{R}_*$ can only come from the explicitly computed terms.

On the other hand, if $\lambda_c \to 0$ as $m \to 0$, it is not guaranteed that these terms are actually the leading terms for the pseudoscalar-pseudoscalar correlator. In general, for $\lambda \in [0, \lambda_c]$ one can bound $|f^{\Gamma}| \leq a_{\Gamma} \lambda^{\tau_{\Gamma}}$, for some a_{Γ} independent of m for sufficiently small m and some $\tau_{\Gamma} > 0$. One has

$$\begin{aligned} \left| \int_0^{\lambda_c} d\lambda\, \frac{m f^{\mathbf{1}}(\lambda; x; m)}{\lambda^2 + m^2} \right| &\leq \frac{a_{\mathbf{1}}}{\tau_{\mathbf{1}} + 1} \lambda_c^{\tau_{\mathbf{1}}} \frac{\lambda_c}{m}\,, \\ \left| \int_0^{\lambda_c} d\lambda\, \frac{(m^2 - \lambda^2) m f^{\gamma_5}(\lambda; x; m)}{(\lambda^2 + m^2)^2} \right| &\leq \frac{a_{\gamma_5}}{\tau_{\gamma_5} + 1} \lambda_c^{\tau_{\gamma_5}} \frac{\lambda_c}{m} \left(1 + \frac{\lambda_c^2}{m^2}\right)\,, \end{aligned} \tag{F.17}$$

and so if λ_c vanishes and λ_c/m vanishes or remains constant, one has that the omitted terms are actually $o(\lambda_c/m^2)$, and so negligible compared to the explicitly computed ones.

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If instead λ_c/m diverges in the chiral limit, the first term in eq. (F.15) is precisely $O(1/m)$, and so the omitted terms are surely subleading. The first term in eq. (F.16), instead, diverges only like $1/\lambda_c$ in this case, and so more slowly than $1/m$, thus not contributing to $\mathcal{R}_*$. While an explicit estimate shows that the omitted terms may actually be leading in eq. (F.16), they are nonetheless inconsequential for $\mathcal{R}_*$, and for the qualitative fact that the pseudoscalar-pseudoscalar correlator diverges as $m \to 0$, as this follows already from the explicitly computed term.

The second contribution, eq. (5.32), is from localised modes separated from the origin of the spectrum. One easily finds

$$\left| \int_{\lambda_c'}^{\mu} d\lambda \left(\frac{C_{\text{loc}}^{\mathbf{1}}(\lambda;x;m)}{\lambda^2+m^2} + \frac{(m^2-\lambda^2)C_{\text{loc}}^{\gamma_5}(\lambda;x;m)}{(\lambda^2+m^2)^2} \right) \right| \leq 2 \int_{\lambda_c'}^{\mu} d\lambda \, \frac{C_{\text{loc}}^{\mathbf{1}}(\lambda;x;m)}{\lambda^2} \,, \tag{F.18}$$

which is at most $O(1)$ under the assumption that λ_c' remains separated from the origin in the chiral limit. This follows from $|C_{\text{loc}}^{\gamma_5}| \leq C_{\text{loc}}^{\mathbf{1}}$, which is a consequence of the bound eq. (5.9). Therefore, these modes cannot contribute to $\mathcal{R}_*$.

Finally, the third contribution, eq. (5.32), comes from both localised and delocalised modes. One sets

$$\begin{aligned}
\bar{C}^{\Gamma}(\lambda,\lambda';x;m) &= c(x;m) + f(\lambda;x;m) + f(\lambda';x;m) + F^{\Gamma}(\lambda,\lambda';x;m) \,, \\
c(x;m) &= \bar{C}^{\Gamma}(0,0;x;m) \,, \\
f(\lambda;x;m) &= \bar{C}^{\Gamma}(\lambda,0;x;m) - \bar{C}^{\Gamma}(0,0;x;m) = \bar{C}^{\Gamma}(0,\lambda;x;m) - \bar{C}^{\Gamma}(0,0;x;m) \,, \\
F^{\Gamma}(\lambda,\lambda';x;m) &= \bar{C}^{\Gamma}(\lambda,\lambda';x;m) - f(\lambda) - f(\lambda') \,,
\end{aligned} \tag{F.19}$$

where c and f do not depend on the choice of Γ due to eq. (5.8), and again it is assumed that the limits of vanishing λ or λ' are approached at least as fast as some power law. One then finds

$$\int_0^{\mu} d\lambda \int_0^{\mu} d\lambda' \, \frac{(m^2+\lambda\lambda')\bar{C}^{\mathbf{1}}(\lambda,\lambda';x;m) + (m^2-\lambda\lambda')\bar{C}^{\gamma_5}(\lambda,\lambda';x;m)}{(\lambda^2+m^2)(\lambda'^2+m^2)} = I_0 + I_1 + I_+ + I_- \,, \tag{F.20}$$

where

$$I_0 = 2\, c(x;m) \left(\int_0^{\mu} d\lambda \, \frac{m}{\lambda^2+m^2} \right)^2 = 2\, \bar{C}^{\mathbf{1}}(0,0;x;m) \left(\arctan \frac{\mu}{m} \right)^2 \,, \tag{F.21}$$

is obtained explicitly and is $O(1)$,

$$I_1 = 4 \int_0^{\mu} d\lambda \, \frac{m}{\lambda^2+m^2} \int_0^{\mu} d\lambda' \, \frac{m f(\lambda';x;m)}{\lambda'^2+m^2} = 4 \arctan \frac{\mu}{m} \cdot o(1) = o(1) \,, \tag{F.22}$$

having used eq. (F.6)

$$I_+ = \int_0^{\mu} d\lambda \int_0^{\mu} d\lambda' \, \frac{m^2 \left(F^{\mathbf{1}}(\lambda,\lambda';x;m) + F^{\gamma_5}(\lambda,\lambda';x;m) \right)}{(\lambda^2+m^2)(\lambda'^2+m^2)} = o(1) \,, \tag{F.23}$$

having used eq. (F.8), and

$$I_- = \int_0^{\mu} d\lambda \int_0^{\mu} d\lambda' \, \frac{\lambda\lambda' \left(F^{\mathbf{1}}(\lambda,\lambda';x;m) - F^{\gamma_5}(\lambda,\lambda';x;m) \right)}{(\lambda^2+m^2)(\lambda'^2+m^2)} = O(1) \,, \tag{F.24}$$

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since

$$F^{\mathbf{1}}(\lambda,\lambda';x;m) - F^{\gamma_5}(\lambda,\lambda';x;m) = \bar{C}^{\mathbf{1}}(\lambda,\lambda';x;m) - \bar{C}^{\gamma_5}(\lambda,\lambda';x;m)\,, \tag{F.25}$$

and so the integral remains convergent if one sets $m = 0$ in the denominator. Also in this case there is no contribution to $\mathcal{R}_*$.

G Renormalisation of the spectral correlators

In this appendix I show that the renormalised spectral correlators [see eqs. (5.5) and (5.27)],

$$C^{\Gamma}(\lambda,\lambda';x;m) \equiv \lim_{\Lambda\to\infty}\lim_{V\to\infty} Z_m^2 C^{\Gamma}_{\Lambda,V}(Z_m\lambda, Z_m\lambda'; x; Z_m m)\,, \qquad \Gamma = \mathbf{1},\gamma_5\,, \tag{G.1}$$

are finite functions of λ, λ', and of the renormalised mass m (and of x). The proof follows closely the strategy of refs. [114, 115], and relies on the renormalisation properties of the so-called density chain correlation functions. As mentioned in appendix C, renormalisation properties at finite temperature are identical to those of the zero-temperature theory.

Density chain correlation functions. In a partially quenched gauge theory with $2\bar{N}_f$ extra pairs of "valence" fermion fields $\psi_i, \bar{\psi}_i$, $i = 1,\ldots,2\bar{N}_f$, and corresponding $2\bar{N}_f$ pseudofermion fields exactly cancelling out their contribution to the fermionic determinant, all with the same mass as the original N_f fermions, one defines the (bare) density operators

$$X^{\Gamma}_{ij\,B} \equiv \bar{\psi}_i(x)\Gamma\psi_j(x)\,, \qquad \Gamma = \mathbf{1},\gamma_5\,, \quad i,j = 1,\ldots,2\bar{N}_f\,. \tag{G.2}$$

These composite fields renormalise in the standard way, i.e., for $i \neq j$,

$$X^{\Gamma}_{ij\,B}(x) = Z_\Gamma X_{ij}(x)\,, \tag{G.3}$$

with flavour-independent renormalisation constants Z_Γ that can be taken equal to those obtained in the $\bar{N}_f = 0$ case [131], i.e., $Z_{\mathbf{1}} = Z_S$ and $Z_{\gamma_5} = Z_P$. In a regularisation that preserves (some form of) chiral symmetry, one further has $Z_S = Z_P = Z_m^{-1}$, so $Z_\Gamma = Z_m^{-1}$. As shown in refs. [114, 115, 132, 133], the density-chain correlation functions,

$$\mathscr{X}^{\Gamma_1\ldots\Gamma_n}(x_1,\ldots,x_{n-1}) \equiv \langle X^{\Gamma_1}_{n1\,B}(x_1) X^{\Gamma_2}_{12\,B}(x_2)\ldots X^{\Gamma_{n-1}}_{n-2\,n-1\,B}(x_{n-1}) X^{\Gamma_n}_{n-1\,n\,B}(0)\rangle_\beta\,, \tag{G.4}$$

renormalise multiplicatively, i.e.,

$$Z^{-1}_{\Gamma_1}\ldots Z^{-1}_{\Gamma_n}\mathscr{X}^{\Gamma_1\ldots\Gamma_n}(x_1,\ldots,x_{n-1}) = (Z_m)^n\,\mathscr{X}^{\Gamma_1\ldots\Gamma_n}(x_1,\ldots,x_{n-1}) \tag{G.5}$$

is a renormalised quantity after the usual mass and coupling renormalisation. Moreover, their short-distance singularities when the x_i get close to each other or to 0 are integrable if $n \geq 5$.

Spectral correlators from density chain correlation functions. The relevant density-chain correlation functions for the problem at hand are the bare quantities

$$\mathcal{M}^{\Gamma}_{\ell_1\ell_2\,B}(x;m_B) \equiv \int d^4x_1\ldots\int d^4x_{2(\ell_1+\ell_2)+1}\,\delta^{(4)}(x - x_{2\ell_2+1}) \times \mathscr{X}^{\overbrace{\gamma_5\ldots\gamma_5}^{2\ell_2}\Gamma\overbrace{\gamma_5\ldots\gamma_5}^{2\ell_1}\Gamma}(x_1,\ldots,x_{2(\ell_1+\ell_2)+1})\,, \tag{G.6}$$

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defined in a finite volume and in the UV-regularised theory, and their renormalised counterparts $\mathcal{M}^{\Gamma}_{\ell_1\ell_2}$,

$$\mathcal{M}^{\Gamma}_{\ell_1\ell_2}(x;m) = \lim_{\Lambda\to\infty}\lim_{V\to\infty} Z_m^{2(\ell_1+\ell_2+1)}\mathcal{M}^{\Gamma}_{\ell_1\ell_2\,B}(x;Z_m m)\,. \tag{G.7}$$

In a finite volume the Dirac spectrum is discrete, and one can use the decomposition of the quark propagator in Dirac eigenmodes to get with a straightforward calculation[35]

$$\mathcal{M}^{\Gamma}_{\ell_1\ell_2\,B}(x;m_B) = -\int_{-\Lambda}^{\Lambda} d\lambda \int_{-\Lambda}^{\Lambda} d\lambda' \frac{(m_B^2-\lambda\lambda')C^{\Gamma}_{V,\Lambda}(\lambda,\lambda';x;m_B)}{(\lambda^2+m_B^2)^{1+\ell_1}(\lambda'^2+m_B^2)^{1+\ell_2}}\,. \tag{G.8}$$

Here $C^{\Gamma}_{V,\Lambda}$ is computed in the partially quenched theory with $\bar{N}_f = \ell_1+\ell_2+1$, but since it is obtained by averaging the Dirac spectrum over gauge configurations only, and since the weight of a configuration is independent of $\bar{N}_f$ due to partial quenching, one obtains the same result as in the standard ($\bar{N}_f = 0$) theory. One can then vary the number of extra valence quarks as demanded by the left-hand side of eq. (G.8) in order to freely vary $\ell_{1,2}$, without changing $C^{\Gamma}_{V,\Lambda}$ appearing on the right-hand side.

Next, one defines the resolvents,

$$R^{\Gamma}_{r\,B}(z,z';x;m_B) \equiv \int d\lambda \int d\lambda' \frac{(m_B^2-\lambda\lambda')C^{\Gamma}_{V,\Lambda}(\lambda,\lambda';x;m_B)}{(\lambda^2+m_B^2)^r(\lambda'^2+m_B^2)^r}\frac{1}{\lambda^2+m_B^2-z}\frac{1}{\lambda'^2+m_B^2-z'}\,. \tag{G.9}$$

These functions are analytic in z and z' in the entire complex plane, except for cuts at $|\mathrm{Re}\,z|, |\mathrm{Re}\,z'| \geq m_B$. The integer r may have to be set to a nonzero value to guarantee convergence at large λ, λ', and to avoid non-integrable short distance singularities. It is simple to show that $C^{\Gamma}_{V,\Lambda}(\lambda,\lambda';x;m_B)$ can be recovered from the following discontinuity of the resolvents,

$$\mathscr{D}^{\Gamma}_{r\,B}(\lambda,\lambda';x;m_B) \equiv \lim_{\epsilon\to 0} \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' R^{\Gamma}_{r\,B}\left(m_B^2+\lambda^2+i\sigma\epsilon, m_B^2+\lambda'^2+i\sigma'\epsilon; x; m_B\right)\,. \tag{G.10}$$

One has explicitly for $\lambda\lambda' \geq 0$

$$\begin{aligned} C^{\Gamma}_{V,\Lambda}(\lambda,\lambda';x;m_B) = {} & F_r^-(\lambda,\lambda';x;m_B)\mathscr{D}^{\Gamma}(\lambda,\lambda';x;m_B) \\ & + F_r^+(\lambda,\lambda';x;m_B)\mathscr{D}^{\gamma_5\Gamma}(\lambda,\lambda';x;m_B)\,, \end{aligned} \tag{G.11}$$

with

$$F_r^{\pm}(\lambda,\lambda';x;m_B) = -\frac{1}{8\pi^2 m_B^2}\left(\lambda^2+m_B^2\right)^r\left(\lambda'^2+m_B^2\right)^r\left(\lambda\lambda'\pm m_B^2\right)\,. \tag{G.12}$$

The case $\lambda\lambda' < 0$ is obtained using the symmetry property $C^{\Gamma}_{V,\Lambda}(-\lambda,\lambda') = C^{\gamma_5\Gamma}_{V,\Lambda}(\lambda,\lambda')$, eq. (5.8). Expanding the resolvent in powers of z, z' one finds

$$R^{\Gamma}_{r\,B}(z,z';x;m_B) = \sum_{\ell,\ell'=0}^{\infty} z^{\ell}z'^{\ell'}\mathcal{M}^{\Gamma}_{r+\ell\;r+\ell'\,B}(x;m_B)\,, \tag{G.13}$$

[35] The exchange of the order of the various integrals has been justified in section 5.6.

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and using eq. (G.7) one finds that the quantity

$$\begin{aligned} R_r^\Gamma\left(\zeta,\zeta';x;m\right) &\equiv \lim_{\Lambda\to\infty}\lim_{V\to\infty} Z_m^{2(2r+1)} R_{r\,B}^\Gamma\left(Z_m^2\zeta, Z_m^2\zeta';x;Z_m m\right) \\ &= \sum_{\ell,\ell'=0} \zeta^\ell \zeta'^{\,\ell'} \mathcal{M}^\Gamma_{r+\ell\; r+\ell'}(x;m) \end{aligned} \tag{G.14}$$

is a finite function of ζ, ζ', and m. Since the number of density operators $2(2r+\ell+\ell'+1) \geq 4r+1$, $r=1$ suffices to ensure the absence of non-integrable short-distance singularities. Equation (G.1) then follows from eqs. (G.10)–(G.12) and (G.14).

Renormalisation of the mobility edge. In the mode-sum representation of C^Γ, it is convenient to separate the contribution from $\lambda_n = \pm\lambda_{n'}$ from the rest and write[36] [see eqs. (5.6) and (5.7)]

$$\begin{aligned} C^\Gamma_{V,\Lambda}(\lambda,\lambda';x;m_B) &= \delta(\lambda-\lambda')C^\Gamma_{s\,V,\Lambda}(\lambda;x;m_B) + \delta(\lambda+\lambda')C^{\gamma_5\Gamma}_{s\,V,\Lambda}(\lambda;x;m_B) \\ &\quad + \bar{C}^\Gamma_{V,\Lambda}(\lambda,\lambda';x;m_B)\,, \end{aligned} \tag{G.15}$$

with

$$\begin{aligned} C^\Gamma_{s\,V,\Lambda}(\lambda;x;m_B) &\equiv \left\langle \sum_n \delta(\lambda-\lambda_n)\mathcal{O}^\Gamma_{nn}(x)\mathcal{O}^\Gamma_{nn}(0)\right\rangle_\beta, \\ \bar{C}^\Gamma_{V,\Lambda}(\lambda,\lambda';x;m_B) &\equiv \left\langle \sum_{\substack{n,n'\\ n\neq\pm n'}} \delta(\lambda-\lambda_n)\delta(\lambda'-\lambda_{n'})\mathcal{O}^\Gamma_{n'n}(x)\mathcal{O}^\Gamma_{nn'}(0)\right\rangle_\beta, \end{aligned} \tag{G.16}$$

with $\bar{C}^\Gamma_{V,\Lambda}$ regular as $\lambda\to\pm\lambda'$. Using eq. (G.1) one finds

$$\begin{aligned} C^\Gamma(\lambda,\lambda';x;m) &= \lim_{\Lambda\to\infty}\lim_{V\to\infty}\Big\{\delta(\lambda-\lambda')Z_m C^\Gamma_{s\,V,\Lambda}(Z_m\lambda;x;Z_m m) \\ &\qquad + \delta(\lambda+\lambda')Z_m C^{\gamma_5\Gamma}_{s\,V,\Lambda}(Z_m\lambda;x;Z_m m) \\ &\qquad + Z_m^2\bar{C}^\Gamma_{V,\Lambda}(Z_m\lambda,Z_m\lambda';x;Z_m m)\Big\} \\ &\equiv \delta(\lambda-\lambda')C^\Gamma_{\rm loc}(\lambda;x;m) + \delta(\lambda+\lambda')C^{\gamma_5\Gamma}_{\rm loc}(\lambda;x;m) + \bar{C}^\Gamma(\lambda,\lambda';x;m)\,, \end{aligned} \tag{G.17}$$

with each term on the right-hand side separately finite due their different degree of singularity for $\lambda\to\pm\lambda'$.

In eq. (G.17) the results of section 5.2 have been used, that show that $C^\Gamma_{\rm loc}$ has support only in spectral regions where modes are localised. These are separated by mobility edges $\lambda^{(i)}_{c\,B}$ $(i=1,\ldots)$ from regions where modes are delocalised. The renormalisation properties of $C^\Gamma_{\rm loc}$ then imply that $\lambda^{(i)}_c \equiv Z_m^{-1}\lambda^{(i)}_{c\,B}$ are finite, renormalised quantities. Formally, the unrenormalised spectral correlator in infinite volume, $C^\Gamma_{{\rm loc}\,\Lambda}(\lambda;x;m_B)$, reads

$$C^\Gamma_{{\rm loc}\,\Lambda}(\lambda;x;m_B) = \sum_i \chi_{I^{(i)}_B}(\lambda) f^\Gamma_{(i)\,\Lambda}(\lambda;x;m_B)\,, \tag{G.18}$$

[36] Accidental degeneracies of eigenvalues can be ignored. See footnotes 15 and 20.

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with $\chi_{I_B^{(i)}}$ the characteristic functions of the disjoint spectral regions $I_B^{(i)} = [\lambda_{c\,B}^{(2i+1)}, \lambda_{c\,B}^{(2i+2)}]$ $(i = 0, \ldots)$ where modes are localised, delimited by the lower and upper mobility edges $\lambda_{c\,B}^{(2i+1)}$ and $\lambda_{c\,B}^{(2i+2)}$. After renormalisation

$$\begin{aligned} C^{\Gamma}_{\mathrm{loc}}(\lambda; x; m) &= \lim_{\Lambda\to\infty} \sum_i \chi_{I_B^{(i)}}(Z_m\lambda) Z_m f^{\Gamma}_{(i)\,\Lambda}(Z_m\lambda; x; Z_m m) \\ &= \lim_{\Lambda\to\infty} \sum_i \chi_{Z_m^{-1} I_B^{(i)}}(\lambda) Z_m f^{\Gamma}_{(i)\,\Lambda}(Z_m\lambda; x; Z_m m)\,, \end{aligned} \tag{G.19}$$

and finiteness of the left-hand side requires that the quantities

$$f^{\Gamma}_{(i)}(\lambda; x; m) = \lim_{\Lambda\to\infty} Z_m f^{\Gamma}_{(i)\,\Lambda}(Z_m\lambda; x; Z_m m) \tag{G.20}$$

are finite for each i, since they have disjoint support. More importantly, it also implies that the renormalised spectral regions

$$I^{(i)} \equiv \left[\lambda_c^{(2i+1)}, \lambda_c^{(2i+2)}\right] = \lim_{\Lambda\to\infty} Z_m^{-1} I_B^{(i)} = \lim_{\Lambda\to\infty} \left[Z_m^{-1}\lambda_{c\,B}^{(2i+1)}, Z_m^{-1}\lambda_{c\,B}^{(2i+2)}\right] \tag{G.21}$$

are delimited by finite renormalised mobility edges $\lambda_c^{(i)} \equiv \lim_{\Lambda\to\infty} Z_m^{-1}\lambda_{c\,B}^{(i)}$. In other words, the mobility edges renormalise like the fermion mass, so that $\lambda_{c\,B}^{(i)}/m_B$ are renormalisation-group-invariant quantities, up to terms that vanish as $\Lambda \to \infty$.

H Bound for localised modes

In this appendix I argue that the $1/m$-divergent part of the pseudoscalar-pseudoscalar correlator, that originates from localised near-zero modes and leads to a finite P_* in chiral limit, is expected to have fast decay properties at large spatial distance, that allow to interchange spacetime integration and chiral limit. Since

$$\left| \int_0^{\lambda_c} d\lambda \left(\frac{(m^2-\lambda^2) C^{\gamma_5}_{\mathrm{loc}}(\lambda; x; m)}{(\lambda^2+m^2)^2} \right) \right| \leq \int_0^{\lambda_c} d\lambda \, \frac{C^{\mathbf{1}}_{\mathrm{loc}}(\lambda; x; m)}{\lambda^2+m^2}\,, \tag{H.1}$$

it suffices to check the contribution of the spectral correlator $C^{\mathbf{1}}_{\mathrm{loc}}$. It is reasonable to assume that it decays exponentially with the spatial distance, up to power-law corrections that do not affect the following argument (see appendix E). Since the temporal direction is compact, one could maximise the right-hand side of eq. (H.1) over time. However, since I am only interested in justifying the exchange of chiral limit and spacetime integration, and integration over time causes no problem there, it suffices and is practically more convenient to integrate over time. One then considers

$$\mathrm{C}_{\mathrm{loc}}(\vec{x}; m) \equiv \int_0^{\lambda_c} d\lambda \, \frac{2m}{\lambda^2+m^2} \, \mathrm{L}_{\mathrm{loc}}(\lambda; \vec{x}; m)\,, \qquad \mathrm{L}_{\mathrm{loc}}(\lambda; \vec{x}; m) \equiv \int_0^{\beta} dt \, C^{\mathbf{1}}_{\mathrm{loc}}(\lambda; x; m)\,, \tag{H.2}$$

from which $\mathrm{R}_*(0)$ is obtained as follows,

$$\mathrm{R}_*(0) = \int d^3x \lim_{m\to 0} \mathrm{C}_{\mathrm{loc}}(\vec{x}; m)\,. \tag{H.3}$$

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I now assume that $\mathrm{L}_{\mathrm{loc}}$ takes the following form,

$$\mathrm{L}_{\mathrm{loc}}(\lambda;\vec{x};m) = A(\lambda)e^{-\frac{|\vec{x}|}{\xi(\lambda)}}\,, \tag{H.4}$$

where $\xi(\lambda)$ is a λ-dependent (and possibly m-dependent) correlation length. As discussed in section 5.6, interchanging spacetime integration and the various limits is expected to be justified at finite fermion mass, and in this case using eq. (5.44) one finds that

$$\rho(\lambda,m) = \int_\beta d^4x\, C^{\mathbf{1}}_{\mathrm{loc}}(\lambda;x;m) = \int d^3x\, \mathrm{L}_{\mathrm{loc}}(\lambda;\vec{x};m) = A(\lambda)8\pi\xi(\lambda)^3\,, \tag{H.5}$$

and so

$$\mathrm{C}_{\mathrm{loc}}(\vec{x};m) = \int_0^{\lambda_c} d\lambda\,\rho(\lambda,m)\frac{2m}{\lambda^2+m^2}\,\frac{1}{8\pi\xi(\lambda)^3}e^{-\frac{|\vec{x}|}{\xi(\lambda)}}\,. \tag{H.6}$$

Equation (H.6) is expected to provide at least a qualitative understanding of how the localisation properties of the modes are transferred to the correlator.

Further assumptions are needed to proceed. I consider first the case of a maximal correlation length. Since

$$\frac{\partial}{\partial\xi}\frac{1}{\xi^3}e^{-\frac{|\vec{x}|}{\xi}} = \frac{1}{\xi^4}e^{-\frac{|\vec{x}|}{\xi}}\left(\frac{|\vec{x}|}{\xi}-3\right)\,, \tag{H.7}$$

if $\xi(\lambda)\leq\xi_{\max}$ for $\lambda\in[0,\lambda_c]$ (at least for sufficiently small m), then for $|\vec{x}|\geq 3\xi_{\max}$ one has

$$\mathrm{C}_{\mathrm{loc}}(\vec{x};m) \leq \frac{1}{8\pi\xi_{\max}^3}e^{-\frac{|\vec{x}|}{\xi_{\max}}}\int_0^{\lambda_c} d\lambda\,\rho(\lambda,m)\frac{2m}{\lambda^2+m^2} \underset{m\to 0}{\to} \frac{\rho(0;0)}{4\pi\xi_{\max}^3}e^{-\frac{|\vec{x}|}{\xi_{\max}}}\arctan\tfrac{\lambda_c}{m}\,. \tag{H.8}$$

In this case, the fast decay of eq. (H.8) allows one to use the dominated convergence theorem to justify interchanging spacetime integration and chiral limit.

On the other hand, since the localisation length diverges at the mobility edge [38–43], one could find that also $\xi(\lambda)$ diverges. As an alternative possibility, I now assume that

$$\xi(\lambda) = \xi_0\left|1-\tfrac{\lambda}{\lambda_c}\right|^{-\nu}\,, \tag{H.9}$$

with ν a suitable exponent, which could be, e.g., the localisation length critical exponent appropriate for the symmetry class of the Dirac operator for the given gauge group [134]. I also assume that ξ_0 remains finite in the chiral limit: in this way, even if $\lambda_c\to 0$, it still makes sense to speak of localised modes at the origin. Notice that while the localisation length of the localised modes ψ_n diverges at λ_c, the correlation length of $\langle\|\psi_n(x)\|^2\|\psi_n(0)\|^2\rangle_\beta$ could remain finite, due to other long-distance effects related to the averaging over gauge fields; and if it diverges, an algebraic decay is expected at criticality [41]. Using eq. (H.9) should then lead at least to an upper bound on the large-distance behaviour of $\mathrm{C}_{\mathrm{loc}}$.[37]

[37] It should also be noted that the localisation length, characterising the exponential fall-off of localised modes and diverging at the Anderson transition, is in general different from the typical size of a localised mode (see, e.g., ref. [40]), and so even assuming that eq. (H.9) holds, it is not clear what exponent one should use. However, this does not affect the qualitative features of the following argument.

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Working under the assumption eq. (H.9), if λ_c remains finite or vanishes more slowly than m, so that $\lambda_c/m \to \infty$ as $m \to 0$, one can change variables to $\lambda = mz$ and write

$$\mathrm{C}_{\mathrm{loc}}(\vec{x};m) = \frac{1}{4\pi\xi_0^3}\int_0^{\frac{\lambda_c}{m}} dz\,\rho(mz,m)\frac{\left|1-\frac{m}{\lambda_c}z\right|^\nu}{z^2+1}\,e^{-\frac{|\vec{x}|}{\xi_0}\left|1-\frac{m}{\lambda_c}z\right|^\nu} \underset{m\to 0}{\to} \frac{\rho(0,0)}{8\xi_0^3}e^{-\frac{|\vec{x}|}{\xi_0}}\,, \tag{H.10}$$

and so $\mathrm{C}_{\mathrm{loc}}$ can again be bounded exponentially in $|\vec{x}|$, uniformly in m for sufficiently small m, and dominated convergence allows one to exchange chiral limit and spacetime integration. On the other hand, setting $\chi = (|\vec{x}|/\xi_0)^{1/\nu}$, one can generally change variables to $\lambda = \lambda_c(1-z/\chi)$ to write

$$\begin{aligned}\mathrm{C}_{\mathrm{loc}}(\vec{x};m) &= \frac{\lambda_c}{m}\frac{1}{4\pi\xi_0^3}\left(\frac{\xi_0}{|\vec{x}|}\right)^{3+\frac{1}{\nu}}\int_0^\chi dz\,\frac{\rho\left(\lambda_c\left(1-\frac{z}{\chi}\right),m\right)}{1+\frac{\lambda_c^2}{m^2}\left(1-\frac{z}{\chi}\right)^2}z^{3\nu}e^{-z^\nu} \\ &\leq \frac{\lambda_c}{m}\frac{\rho_{\max}}{4\pi|\vec{x}|^3}\left(\frac{\xi_0}{|\vec{x}|}\right)^{\frac{1}{\nu}}\int_0^\chi dz\,z^{3\nu}e^{-z^\nu} \leq \frac{\lambda_c}{m}\frac{\rho_{\max}}{4\pi|\vec{x}|^3}\left(\frac{\xi_0}{|\vec{x}|}\right)^{\frac{1}{\nu}}\frac{1}{\nu}\Gamma\left(3+\frac{1}{\nu}\right)\,,\end{aligned} \tag{H.11}$$

where $\rho_{\max}$ is a bound on the mode density in the spectral region of localised modes for small mass. The bound is integrable (at large distances) in three spatial dimensions, and so, if λ_c/m does not diverge, it can be used to invoke dominated convergence.

While only qualitative, the estimates above are quite robust under refinements. For example, adding power corrections $(|\vec{x}|/\xi)^k$ to the exponential behaviour of the spectral correlator only modifies the large-$|\vec{x}|$ behaviour of $\mathrm{C}_{\mathrm{loc}}$ by similar power corrections $(|\vec{x}|/\xi_0)^k$ in eq. (H.8) (at sufficiently large distance) and in eq. (H.10). In eq. (H.11), instead, it only modifies the integrand in the last passage, and so the numerical prefactor, but not the large-$|\vec{x}|$ behaviour of $\mathrm{C}_{\mathrm{loc}}$. The exchange of chiral limit and spacetime integration seems then justified in the calculation of the anomalous remnant $\mathrm{R}_*(0)$ originating from localised near-zero modes.

It is worth commenting on how eqs. (H.10) and (H.11) compare with the scenarios discussed in section 5.5. Equation (H.11) leads one to expect that if λ_c/m vanishes in the chiral limit, so does $\mathrm{C}_{\mathrm{loc}}$. One would then expect no anomalous remnant, in agreement with what was stated above in section 5.5 for scenario (i). Moreover, eq. (H.11) shows that if λ_c/m vanishes faster than m, then one expects the pseudoscalar correlator to be regular in the chiral limit, as in scenario (i-a), while if it vanishes more slowly than m one expects a divergent correlator, as in scenario (i-b). If λ_c/m remains constant in the chiral limit, corresponding to scenario (ii), then eq. (H.11) still allows to bound $\mathrm{C}_{\mathrm{loc}}$ with an algebraic but integrable decay $|\vec{x}|^{-3-\frac{1}{\nu}}$ instead of an exponential one in the chiral limit. This may actually be the qualitative behaviour of the correlator in this case, and so a distinguishing feature for scenario (ii) as compared to the standard scenario, if the correlation length diverges at the mobility edge. This is clearly speculative at this stage, and a better understanding of the spectral correlators is required to make a definite statement. Finally, scenario (iii) is covered by eq. (H.10), showing that an exponential bound should be expected.

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