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QED in the exact renormalization group

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The functional flow equation and the quantum master equation are consistently solved in perturbation for chiral symmetric QED with and without four-fermi interactions. Due to the presence of a momentum cutoff, unconventional features related to gauge symmetry are observed even in our perturbative results. In the absence of the four-fermi couplings, a one-loop calculation gives us the standard results of anomalous dimensions and the beta function for the gauge coupling, and therefore the Ward identity, $Z_1 = Z_2$. This is a consequence of the regularization-scheme independence in the one-loop computation. We also find a photon mass term. When included, four-fermi couplings contribute to the beta function and the Ward identity is also modified, $Z_1 \neq Z_2$, due to a term proportional to the photon mass multiplied by the four-fermi couplings.

Subject Index B05, B31, B32, B39

1. Introduction

Recently, the exact renormalization group or functional renormalization group (ERG/FRG) approach to gauge theories has attracted a great deal of attention. The regularization scheme with a momentum cutoff Λ is not compatible with gauge invariance: the BRST transformation in its standard form is not a symmetry of the Wilsonian action. However, it has been shown that the BRST symmetry survives in a modified form [1–4]: the variation of a Wilsonian action S under appropriately modified BRST transformation defined at Λ is canceled by the Jacobian factor of the functional measure. This cancellation mechanism, the modified Ward–Takahashi (mWT) identity, is lifted to the quantum master equation (QME) [5–8] in the Batalin–Vilkovisky (BV) antifield formalism [9]. The QME and the flow equation are two basic equations to define a gauge theory in ERG/FRG. It has been a challenging problem to solve them consistently in appropriate truncation schemes.

In a previous work [10], the compatibility of two equations is discussed for Yang–Mills (YM) theory in a perturbative framework (see also Ref. [11]). The main results obtained there are as follows: firstly, two equations were combined to develop a BRST cohomology analysis [12–15] that uniquely determines the classical action of the first and second orders in gauge coupling; secondly, it was shown that the one-loop perturbative solution to the flow equation satisfies the QME or its Legendre transform, the modified Slavnov–Taylor (mST) identity [16,17], up to the third order in the coupling; and thirdly, the standard results are obtained for the beta function and anomalous dimensions as a consequence of the regularization-scheme-independent computation. This leads to the standard Slavnov–Taylor identities among the renormalization constants.

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In this paper, we consider the compatibility between the QME and the flow equation for a chiral invariant QED with four-fermi interactions [18]. This type of model has attracted interest in connection with the possible existence of a non-trivial UV-fixed point and associated chiral symmetry breaking [19–26]. In the light of the asymptotically safe scenario, there is new interest in finding a UV completion of QED [27] (see also Ref. [28]). In this paper we will not discuss such a non-perturbative structure. Instead, we take a perturbative approach in parallel to Ref. [10] and find how the higher-dimensional interactions affect the realization of BRST symmetry.

Here we mainly use the Legendre transform of the QME and flow equation to avoid redundancy arising from the one-particle reducible part of the Wilsonian action. The QME/mST is also best expressed in terms of Γ since its free part Γ_0 carries no regularization that simplifies the BRST cohomology analysis, although the Legendre transform of the measure contribution ΔS in the QME contains the inverse of the two-point function $\Gamma^{(2)}$, which is readily expandable perturbatively and does not cause any trouble.

We will show that, even in the presence of the four-fermi interactions, a perturbative solution to the flow equation satisfies the QME/mST up to the order of e^3 or $eG_{S,V}$ in a general covariant gauge.

After introducing the wave function renormalization factors via a canonical transformation of classical fields and their antifields, the beta function of the gauge coupling and anomalous dimensions are computed by using the flow equations.

Without four-fermi couplings, the standard perturbative results are obtained as in the case of YM theory and the Ward identity $Z_1 = Z_2$ holds. This is again a consequence of the regularization-scheme independence. The presence of the photon mass term proportional to $e^2\Lambda^2$ is observed. Once the four-fermi interactions are taken into account, the beta function acquires an extra term proportional to the photon mass term and we find $Z_1 \neq Z_2$ due to the mass term. Still, our perturbative solution satisfies the QME/mST by including the photon mass and the modified Ward identity among Z_1 and Z_2. We emphasize the fact that in ERG/FRG the BRST symmetry is realized in a modified form.

In the next section, we give a brief summary of the 1PI formulation in ERG/FRG. In Sect. 3, we show that a one-loop perturbative solution to the flow equation satisfies the QME/mST up to the third order in couplings. The beta function and anomalous dimensions are computed in Sect. 4. The summary and conclusions are given in Sect. 5.

2. Legendre transform of the QME and the flow equation

The Wilsonian action consists of free and interaction parts, $S = S_0 + S_I$. In the free action

$$S_0[\phi, \phi^*] = \frac{1}{2}\phi^A K^{-1} \Delta^{-1}_{AB} \phi^B + \phi^*_A K^{-1} R^A{}_B \phi^B, \tag{1}$$

the kinetic terms Δ^{-1}_{AB} are regularized by a UV cutoff function $K(p^2/\Lambda^2)$, satisfying the requirements that $K(0) = 1$ and $K(u) \to 0$ sufficiently fast as $u \to \infty$. Also included are free BRST transformations $R^A{}_B\phi^B$ for fields ϕ^A multiplied by their antifields ϕ^*_A and by an overall factor K^{-1}. By construction, the free BRST transformation satisfies the relation

$$\Delta^{-1}_{AC} R^C{}_B + \Delta^{-1}_{BC} R^C{}_A = 0. \tag{2}$$

$S_I[\phi, \phi^*]$ consists of interaction terms and some antifield-dependent terms with coupling constants. We use the condensed notation as in Refs. [4,10].

The regularized version of the antibracket and the measure operator can be defined as those in Ref. [10]:

$$(X, Y)_K = \frac{\partial^r X}{\partial \phi^A} K \frac{\partial^l Y}{\partial \phi^*_A} - \frac{\partial^r X}{\partial \phi^*_A} K \frac{\partial^l Y}{\partial \phi^A}\,, \tag{3}$$

and

$$\Delta_K X = (-)^{A+1} \frac{\partial^r}{\partial \phi^A} K \frac{\partial^r}{\partial \phi^*_A} X. \tag{4}$$

Here X and Y are arbitrary bosonic or fermionic functionals, and $(-)^A = (-)^{\epsilon_A}$ where ϵ_A is the Grassmann parity of ϕ^A. ϕ^*_A has the opposite Grassmann parity to ϕ^A. $\partial^{l(r)}$ denotes the derivative acting from the left (right). The BRST invariance of the Wilsonian action is expressed as the QME on the fields and their antifields

$$\Sigma = \frac{1}{2}(S, S)_K - \Delta_K S = 0. \tag{5}$$

The Wilsonian action S can be expressed as a tree-level expansion in terms of its 1PI part Γ_I [29–31]. The latter is related to $S_I[\phi, \phi^*]$ via a Legendre transformation

$$\Gamma_I[\Phi, \Phi^*] = S_I[\phi, \phi^*] - \frac{1}{2}(\phi - \Phi)^A\, \bar{\Delta}^{-1}_{AB}\, (\phi - \Phi)^B, \tag{6}$$

$$\frac{\partial^r}{\partial \phi^B} S_I[\phi, \phi^*] = (\phi - \Phi)^A \bar{\Delta}^{-1}_{AB} = \frac{\partial^r}{\partial \Phi^B} \Gamma_I[\Phi, \Phi^*], \tag{7}$$

where $\bar{\Delta}^{-1}_{AB}$ denote the inverse of the IR regulated propagators $\bar{\Delta}^{AB} = \bar{K}\Delta^{AB}$ with $\bar{K} = 1 - K$. For aesthetic reasons, we use the notation $\Phi^*_A = \phi^*_A$ for the 1PI effective action. By adding a free part, we introduce the 1PI effective action Γ as

$$\Gamma = \frac{1}{2}\Phi^A \Delta^{-1}_{AB} \Phi^B + \Phi^*_A R^A{}_B \Phi^B + \Gamma_I[\Phi, \Phi^*]. \tag{8}$$

We also define the *total* 1PI effective action with regularized kinetic terms as

$$\Gamma_{\rm tot} = \frac{1}{2}\Phi^A \bar{\Delta}^{-1}_{AB} \Phi^B + \Phi^*_A R^A{}_B \Phi^B + \Gamma_I[\Phi, \Phi^*]. \tag{9}$$

Note that the 1PI action Γ and $\Gamma_{\rm tot}$ differ only in the kinetic terms and the difference vanishes as the cutoff goes to zero.

Now we rewrite the QME in Eq. (5) in terms of the 1PI action. From Eqs. (6) and (7), we find

$$\frac{\partial^r S}{\partial \phi^A} K = \phi^B \Delta^{-1}_{BA} + \phi^*_B R^B{}_A + \frac{\partial^r S_I}{\partial \phi^A} K = \frac{\partial^r \Gamma}{\partial \Phi^A}, \quad \frac{\partial^l S_I}{\partial \phi^*_A} = \frac{\partial^l \Gamma_I}{\partial \Phi^*_A}. \tag{10}$$

Using Eqs. (2), (10), and (8), we find[1]

$$(S, S)_K = (\Gamma, \Gamma). \tag{11}$$

The antibracket on the r.h.s. is defined for arbitrary functionals of the classical fields Φ^A and their antifields Φ^*_A as

$$(Z, W) = \frac{\partial^r Z}{\partial \Phi^A} \frac{\partial^l W}{\partial \Phi^*_A} - \frac{\partial^r Z}{\partial \Phi^*_A} \frac{\partial^l W}{\partial \Phi^A}. \tag{12}$$

Note that the regulator function K is absent in the above expression.

[1] A more detailed derivation can be found in Ref. [10].

In rewriting the second term of the QME, $\Delta_K S$, we note that only the interaction action produces field-dependent contributions. Using Eq. (10), we obtain

$$\Delta_K S_I = \frac{\partial^r}{\partial\phi^A} K \frac{\partial^l S_I}{\partial\phi^*_A} = \frac{\partial^r}{\partial\Phi^B}\left(K\frac{\partial^l\Gamma_I}{\partial\Phi^*_A}\right)\frac{\partial^r\Phi^B}{\partial\phi^A} = \mathrm{Tr}\left(K\Gamma^{(2)}_{I*}\left[1+\bar{\Delta}\Gamma^{(2)}_I\right]^{-1}\right). \tag{13}$$

The last expression in Eq. (13) is reached by using the relation

$$\frac{\partial^r\Phi^A}{\partial\phi^B} = \left(\left[1+\bar{\Delta}\Gamma^{(2)}_I\right]^{-1}\right)^A_{\ B}, \tag{14}$$

which is derived from Eq. (7). Here we have used the notations

$$\left(\Gamma^{(2)}_I\right)_{AB} = \frac{\partial^l\partial^r}{\partial\Phi^A\partial\Phi^B}\Gamma_I, \tag{15}$$

as well as

$$\left(\Gamma^{(2)}_{I*}\right)^A_{\ B} = \frac{\partial^r\partial^l}{\partial\Phi^*_A\partial\Phi^B}\Gamma_I. \tag{16}$$

Finally, we find the modified Slavnov–Taylor (mST) identity as the Legendre transform of the QME:

$$\Sigma = \frac{1}{2}(\Gamma,\Gamma) - \mathrm{Tr}\left(K\Gamma^{(2)}_{I*}\left[1+\bar{\Delta}\Gamma^{(2)}_I\right]^{-1}\right) = 0. \tag{17}$$

It is also worth pointing out that the second functional derivative of Γ_{tot} appeared in the second term of Eq. (17) as

$$1+\bar{\Delta}\Gamma^{(2)}_I = \bar{\Delta}\Gamma^{(2)}_{\mathrm{tot}}. \tag{18}$$

In Eq. (17), it is interesting to find Γ in the first term and Γ_{tot} in the second term. Shortly we will find a similar trace structure in the flow equation written for the 1PI action.

The measure operator Δ similar to Eq. (4) defined in terms of Φ^A and Φ^*_A appears as the first-order part of Eq. (17):

$$\Delta\Gamma = \mathrm{Tr}\left(K\Gamma^{(2)}_{I*}\right). \tag{19}$$

Here an important remark is in order. The Legendre transformation (6) is not a canonical transformation from $\{\phi^A, \phi^*_A\}$ to $\{\Phi^A, \Phi^*_A\}$. Although the antibracket (12) in terms of $\{\Phi^A, \Phi^*_A\}$ is convenient to write the relation (11), one should not mix up two canonical structures in S-world and Γ-world.

Using the flow equation for S_I [32]

$$\dot{S}_I = \Lambda\partial_\Lambda S_I = -\frac{1}{2}\frac{\partial^r S_I}{\partial\phi^A}\dot{\bar{\Delta}}^{AB}\frac{\partial^l S_I}{\partial\phi^B} + \frac{1}{2}(-)^A\dot{\bar{\Delta}}^{AB}\frac{\partial^l\partial^r S_I}{\partial\phi^B\phi^A} \tag{20}$$

and the Legendre transformation (6), (7), and (14), we find that

$$\begin{aligned}\dot{\Gamma}_I &= \dot{S}_I + \frac{1}{2}(\phi-\Phi)^A\left(\bar{\Delta}^{-1}\dot{\bar{\Delta}}\bar{\Delta}^{-1}\right)(\phi-\Phi)^B \\ &= \dot{S}_I + \frac{1}{2}\frac{\partial^r S_I}{\partial\phi^A}\dot{\bar{\Delta}}^{AB}\frac{\partial^l S_I}{\partial\phi^B} = \frac{1}{2}(-)^A\dot{\bar{\Delta}}^{AB}\frac{\partial^l\partial^r S_I}{\partial\phi^B\phi^A}.\end{aligned} \tag{21}$$

Thus, we obtain the flow equation for Γ_I [29,33–35] as

$$\dot{\Gamma}_I = -\frac{1}{2}\mathrm{Str}\left(\dot{\bar{\Delta}}\bar{\Delta}^{-1}\left[1+\bar{\Delta}\Gamma^{(2)}_I\right]^{-1}\right). \tag{22}$$

The expression of the quantum master functional (QMF) Σ in Eq. (5) and the flow equation (20) are combined to give

$$\dot{\Sigma} = -\frac{1}{2}\frac{\partial^r S_I}{\partial\phi^A}\dot{\bar{\Delta}}^{AB}\frac{\partial^l\Sigma}{\partial\phi^B} + \frac{1}{2}(-)^A\dot{\bar{\Delta}}^{AB}\frac{\partial^l\partial^r\Sigma}{\partial\phi^B\phi^A}. \tag{23}$$

Table 1. The various properties of the (anti)fields, namely, Grassmann parity, ghost number, antighost/antifield number, pure ghost number = ghost number + antighost/antifield number, and mass dimension.

	ϵ	gh #	ag #	pure gh #	dimension
A_μ	0	0	0	0	1
C	1	1	0	1	1
$\Psi,\ \bar{\Psi}$	1	0	0	0	3/2
$\Psi^*,\ \bar{\Psi}^*$	0	−1	1	0	3/2
$\bar{C}$	1	−1	1	0	1
B	0	0	1	1	2
A^*_μ	1	−1	1	0	2
$\bar{C}^*$	0	0	0	0	2

That is, the QMF satisfies the linearized flow equation as a composite operator [36] (see also Ref. [4]). The QME is stable along the RG flow once it holds at some cutoff scale.

In the next section, we consider QED with chiral invariant four-fermi interactions and show that the QME/mST (17) and the flow equation (21) can be simultaneously solved in a perturbative expansion.

3. 1PI effective action in QED and the QME/mST

3.1 *The classical effective action*

We consider 1PI effective action for QED with a massless Dirac fermion. The free part of the covariantly gauge-fixed action contains kinetic terms for the photon A_μ, the Dirac field Ψ, $\bar{\Psi}$, and the FP ghost fields C and $\bar{C}$: the auxiliary field B and the gauge parameter ξ are introduced accordingly.[2] In addition, here we also include antifields A^*_μ and $\bar{C}^*$ as sources for the free BRST transformations of A_μ and the antighost $\bar{C}$:

$$\Gamma_0 = \int_x \left[\frac{1}{2}\left\{(\partial_\mu A_\nu)^2 - (\partial\cdot A)^2\right\} + \bar{\Psi} i\partial\!\!\!/ \Psi + \left(A^*_\mu - i\partial_\mu \bar{C}\right)\partial_\mu C + \frac{1}{2}\xi B^2 + \left(\bar{C}^* - i\partial\cdot A\right)B\right]. \tag{24}$$

Starting from Γ_0 in Eq. (24), we construct a 1PI effective action that satisfies the classical master equation

$$(\Gamma_{\rm cl},\ \Gamma_{\rm cl}) = 0 \tag{25}$$

up to $\mathcal{O}(e^2)$, $\Gamma_{\rm cl} = \Gamma_0 + \Gamma_1 + \Gamma_2$. The lower index is for the order of the gauge coupling. The quantum part Γ_q will be discussed later. In order to solve Eq. (25) we utilize the BRST cohomology argument [12–15] that was applied earlier to Yang–Mills theory in ERG [10].

From $(\Gamma_1, \Gamma_0) = 0$, we will uniquely determine Γ_1, up to some normalization factors to be discussed later. We decompose Γ_1 into parts of definite antighost numbers. Table 1 lists various gradings of (anti)fields. Looking for local field combinations with the highest antighost number, mass dimension four, and of vanishing fermion and ghost numbers, we find that the highest antighost number is one and $\Gamma^1_1 = \int_x [c_1 \Psi^* \Psi C + c_2 \bar{\Psi}\bar{\Psi}^* C]$ with coefficients c_1 and c_2 to be determined shortly. The superscript of Γ^1_1 indicates the antighost number. The only candidate for Γ^0_1 is the minimal gauge interaction term with the coupling e, $\Gamma^0_1 = -e\int_x \bar{\Psi} A\!\!\!/ \Psi$. Now the requirement $(\Gamma^0_1 + \Gamma^1_1, \Gamma_0) = 0$ fixes the coefficients c_1 and c_2 in Γ^1_1 as $c_1 = -c_2 = -ie$. In this

[2] We take the gauge-fixed basis for antifields [10].

manner, we find

$$\Gamma_1 = \int_x \Big[-e\bar{\Psi}\not{A}\Psi - ie\Psi^*\Psi C + ie\bar{\Psi}\bar{\Psi}^* C \Big]. \tag{26}$$

All contained in $\Gamma_0 + \Gamma_1$ are marginal terms and Λ independent.

We also include chiral invariant four-fermi interactions as irrelevant terms:

$$\Gamma_{2,\mathrm{cl}} = \int_x \Bigg[\frac{G_S}{2\Lambda^2} \Big\{ \left(\bar{\Psi}\Psi\right)\left(\bar{\Psi}\Psi\right) - \left(\bar{\Psi}\gamma_5\Psi\right)\left(\bar{\Psi}\gamma_5\Psi\right) \Big\} + \frac{G_V}{2\Lambda^2} \Big\{ \left(\bar{\Psi}\gamma_\mu\Psi\right)\left(\bar{\Psi}\gamma_\mu\Psi\right) + \left(\bar{\Psi}\gamma_5\gamma_\mu\Psi\right)\left(\bar{\Psi}\gamma_5\gamma_\mu\Psi\right) \Big\} \Bigg]. \tag{27}$$

It is easy to confirm that $\Gamma_{\mathrm{cl}} = \Gamma_0 + \Gamma_1 + \Gamma_2$ satisfies the classical master equation (25). In our perturbative expansion, we regard $G_S,\ G_V$ at the order of e^2.

The one-loop correction to the 1PI effective action is given as the closed-form solution to Eq. (22):

$$\Gamma_{\mathrm{q}} = \frac{1}{2}\mathrm{Str}\log\left(\bar{\Delta}^{-1} + \Gamma^{(2)}_{I,\mathrm{cl}}\right), \tag{28}$$

where $\Gamma^{(2)}_{I,\mathrm{cl}}$ is the classical part of Eq. (15), the second functional derivative of $\Gamma_1 + \Gamma_2$. $\bar{\Delta}$ in Eq. (28) are the IR-regularized propagators,

$$\bar{\Delta}_{\mu\nu} = (P^T_{\mu\nu} + \xi P^L_{\mu\nu})\bar{\boldsymbol{\Delta}},\ \bar{\Delta}_{\alpha\hat{\beta}} = (i\not{\partial})_{\alpha\hat{\beta}}\bar{\boldsymbol{\Delta}}, \tag{29}$$

for the gauge and Dirac fields respectively. Here, $\bar{\boldsymbol{\Delta}} = (1-K)/(-\partial^2)$; $P^T_{\mu\nu}$ and $P^L_{\mu\nu}$ are the transverse and longitudinal projection operators. The lowest-order quantum correction is simply

$$\Gamma_{1,\mathrm{q}} = \frac{1}{2}\mathrm{Str}\big(\bar{\Delta}\Gamma^{(2)}_1\big). \tag{30}$$

The r.h.s. of Eq. (30) is evaluated with the first two vertices in Eq. (A1) of Appendix A and turns out to be zero. The perturbative expansion of Eq. (28) starts from the $\mathcal{O}(e^2)$ term.

We expand the QMF according to the order of couplings as

$$\Sigma = \Sigma_0 + \Sigma_1 + \Sigma_2 + \Sigma_3 + \cdots, \tag{31}$$

and we find $\Sigma_0 = (\Gamma_0, \Gamma_0)/2 - \Delta\Gamma_0 = 0$ and $\Sigma_1 = (\Gamma_1, \Gamma_0) - \Delta\Gamma_1 = 0$ with Δ defined in Eq. (19).

In the following two subsections, we evaluate $\Sigma_{2,3}$, higher-order terms in Eq. (31), after obtaining quantum corrections, $\Gamma_{2,\mathrm{q}}$ and $\Gamma_{3,\mathrm{q}}$.

3.2 *Second order in gauge coupling*

Expanding Eq. (28) to the orders of e^2 and $G_{S,V}$, we obtain a quantum part of the action

$$\Gamma_{2,\mathrm{q}} = \frac{1}{2}\mathrm{Str}\Big(\bar{\Delta}\Gamma^{(2)}_{2,\mathrm{cl}}\Big) + \mathrm{Str}\Big(-\frac{1}{4}\bar{\Delta}\Gamma^{(2)}_1\bar{\Delta}\Gamma^{(2)}_1\Big), \tag{32}$$

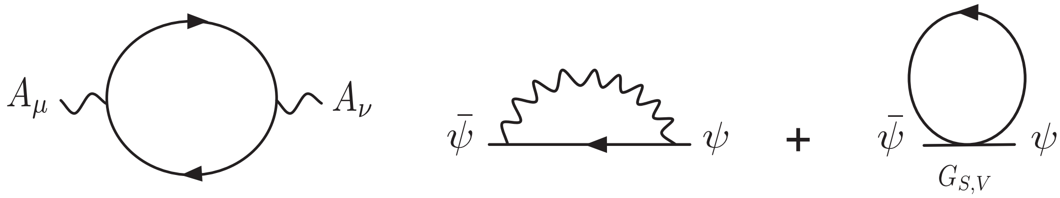
which has gauge and fermion fields contributions. We write them separately as

$$\Gamma^{AA}_{2,\mathrm{q}} = \frac{1}{2}\bar{\Delta}_{\alpha\hat{\alpha}}\tau^{(-\not{A})}_{\hat{\alpha}\beta}\bar{\Delta}_{\beta\hat{\beta}}\tau^{(-\not{A})}_{\hat{\beta}\alpha} = \frac{1}{2}e^2\Big[(i\not{\partial})\bar{\boldsymbol{\Delta}}\not{A}(i\not{\partial})\bar{\boldsymbol{\Delta}}\not{A}\Big] \tag{33}$$

and

$$\Gamma^{\bar{\Psi}\Psi}_{2,\mathrm{q}} = -\bar{\Delta}_{\alpha\hat{\beta}}\tau^{(\bar{\Psi}\Psi)}_{\hat{\beta}\alpha} + \bar{\Delta}_{\alpha\hat{\alpha}}\tau^{(-\gamma\Psi)}_{\hat{\alpha}\mu}\bar{\Delta}_{\mu\nu}\tau^{(-\bar{\Psi}\gamma)}_{\nu\alpha} = -e^2\Big[\bar{\Delta}_{\mu\nu}\bar{\Psi}\gamma_\nu(i\not{\partial})\bar{\boldsymbol{\Delta}}\gamma_\mu\Psi\Big]. \tag{34}$$

Here the quantities τ are the vertices obtained from $\Gamma^{(2)}$ and are listed in Appendix A. Figure 1 shows graphical representations of $\Gamma^{AA}_{2,\mathrm{q}}$ and $\Gamma^{\bar{\Psi}\Psi}_{2,\mathrm{q}}$ in Eqs. (33) and (34). Fields in τ are attached to interaction vertices without external lines.



(a) Graphical representation of (33)

(b) Graphical representation of (34)

Fig. 1. Contributions to $\Gamma_{2,\mathrm{q}}$. τ consist of vertices and fields with no external lines. Internal lines are IR-regularized propagators $\bar\Delta$.

Note that the four-fermi interactions give vanishing contributions in Eq. (34): in momentum space, it becomes

$$\bar\Delta_{\alpha\hat\beta}\tau^{(\bar\Psi\Psi)}_{\hat\beta\alpha} \to \frac{2}{\Lambda^2}(G_S - 4G_V)\int_{p,q}\bar\Psi(-p)\gamma_\mu\Psi(p)\frac{1-K(q)}{q^2}q_\mu = 0. \tag{35}$$

The integral over q vanishes due to the Lorentz covariance.[3]

Having constructed $\Gamma_{2,q}$, we may calculate the QMF at $\mathcal{O}(e^2)$,

$$\Sigma_2 = (\Gamma_0, \Gamma_{2,\mathrm{q}}) + \left[K\Gamma^{(2)}_{1*}\bar\Delta\Gamma^{(2)}_1\right], \tag{36}$$

where $\Gamma^{(2)}_{1*}$ is the $\mathcal{O}(e)$ part of $\Gamma^{(2)}_{I*}$. It turns out that both terms of Σ_2 are proportional to $A_\mu C$, the gauge field multiplied by the ghost. The second term in Eq. (36) becomes

$$\begin{aligned}\Sigma_2|_K &= \left[K\Gamma^{(2)}_{1*}\bar\Delta\Gamma^{(2)}_1\right] = \left[K\tau^C_{*\alpha\beta}\bar\Delta_{\beta\hat\beta}\tau^{(-\not A)}_{\hat\beta\alpha} + K\tau^C_{*\hat\alpha\hat\beta}\bar\Delta^t_{\hat\beta\alpha}\tau^{(\not A^t)}_{\alpha\hat\alpha}\right]\\ &= e^2\left[KC\not\partial\bar{\mathbf{\Delta}}\not A - \not A\not\partial\bar{\mathbf{\Delta}}CK\right] = 8e^2\left[KC\partial_\mu\bar{\mathbf{\Delta}}A_\mu\right].\end{aligned} \tag{37}$$

On the other hand, as shown in Appendix B, the first term of Eq. (36) becomes

$$\begin{aligned}\Sigma_2|_{(\Gamma_0,\Gamma_{2,\mathrm{q}})} &= -\frac{e^2}{2}\mathrm{tr}\big(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\big)\left[\partial_\mu\bar{\mathbf{\Delta}}A_\nu\partial_\rho\bar{\mathbf{\Delta}}\partial_\sigma C + \partial_\mu\bar{\mathbf{\Delta}}\partial_\nu C\partial_\rho\bar{\mathbf{\Delta}}A_\sigma\right]\\ &= 8e^2\left[(1-K)C\partial_\nu\bar{\mathbf{\Delta}}A_\nu\right].\end{aligned} \tag{38}$$

Therefore,

$$\begin{aligned}\Sigma_2 &= \Sigma_2|_{(\Gamma_0,\Gamma_2)} + \Sigma_2|_K = 8e^2\left[C\partial_\mu\bar{\mathbf{\Delta}}A_\mu\right]\\ &= -8ie^2\int_{p,q}C(p)\frac{[1-K(q)]}{q^2}q_\mu A_\mu(-p) = 0.\end{aligned} \tag{39}$$

We have shown that the QME and the flow equation are consistently solved at $\mathcal{O}(e^2)$ and $\mathcal{O}(G_{S,V})$.

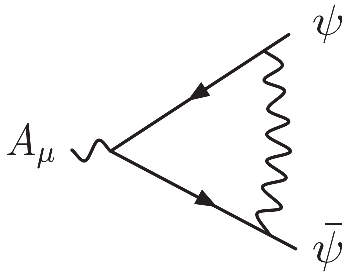
$\Gamma^{AA}_{2,\mathrm{q}}$ given in Eq. (33) may be written as

$$\Gamma^{AA}_{2,\mathrm{q}} = \frac{1}{2}e^2\int_p A_\mu(-p)\left[P^T\mathcal{T}(p) + P^L\mathcal{L}(p)\right]A_\nu(p). \tag{40}$$

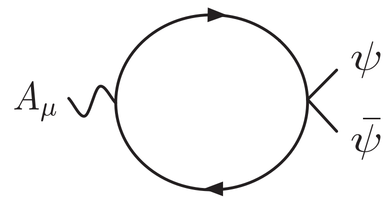
Its longitudinal part $\mathcal{L}$

$$\mathcal{L}(p) = -8\int_q K(p+q)[1-K(q)]\frac{(p\cdot q)}{p^2q^2} \tag{41}$$

[3]Strictly speaking, this happens due to a cancellation of divergent contributions, because $\int_q 1/q^2$ is UV divergent. We may regularize the q^2 integral $\int_0^\infty dq^2$ as $\int_0^{\Lambda_0^2} dq^2$ with a UV cutoff Λ_0 or we may instead use dimensional regularization.



(a) Graphical representation of (44)



(b) Graphical representation of (45)

Fig. 2. Contributions to $\Gamma_{3,\mathrm{q}}$. τ consist of vertices and fields with no external lines. Internal lines are IR-regularized propagators $\bar{\Delta}$.

is necessary to satisfy the QME at $\mathcal{O}(e^2)$, $\Sigma_2 = 0$. Once we set the IR cutoff $K = 0$ by sending $\Lambda \to 0$, we recover the standard WT relation $\mathcal{L}(p) = 0$.

3.3 *Third order in gauge coupling*

Expanding Eq. (28) to $\mathcal{O}(e^3)$ and $\mathcal{O}(eG_{S,V})$, we obtain

$$\Gamma_{3,\mathrm{q}} = \frac{1}{6}\mathrm{Str}\Big(\bar{\Delta}\Gamma_1^{(2)}\bar{\Delta}\Gamma_1^{(2)}\bar{\Delta}\Gamma_1^{(2)}\Big) - \frac{1}{4}\mathrm{Str}\Big(\bar{\Delta}\Gamma_1^{(2)}\bar{\Delta}\Gamma_{2,\mathrm{cl}}^{(2)}\Big). \tag{42}$$

This gives quantum corrections to the $\bar{\Psi}\not{A}\Psi$ vertex. Two terms in Eq. (42) are proportional to e^3 and $eG_{S,V}$ respectively:

$$\Gamma_{3,\mathrm{q}} = \Gamma_{3,\mathrm{q}}|_{e^3} + \Gamma_{3,\mathrm{q}}|_{eG}, \tag{43}$$

where

$$\begin{aligned}\Gamma_{3,\mathrm{q}}|_{e^3} &= \frac{1}{3}\Big(-\bar{\Delta}_{\alpha\hat{\alpha}}\tau_{\hat{\alpha}\mu}^{(-\gamma\Psi)}\bar{\Delta}_{\mu\nu}\tau_{\nu\beta}^{(-\bar{\Psi}\gamma)}\bar{\Delta}_{\beta\hat{\beta}}\tau_{\hat{\beta}\alpha}^{(-\not{A})} \\ &\quad - \bar{\Delta}_{\alpha\hat{\alpha}}\tau_{\hat{\alpha}\beta}^{(-\not{A})}\bar{\Delta}_{\beta\hat{\beta}}\tau_{\hat{\beta}\mu}^{(-\gamma\Psi)}\bar{\Delta}_{\mu\nu}\tau_{\nu\alpha}^{(-\bar{\Psi}\gamma)} + \bar{\Delta}_{\mu\nu}\tau_{\nu\alpha}^{(-\bar{\Psi}\gamma)}\bar{\Delta}_{\alpha\hat{\alpha}}\tau_{\hat{\alpha}\beta}^{(-\not{A})}\bar{\Delta}_{\beta\hat{\beta}}\tau_{\hat{\beta}\mu}^{(-\gamma\Psi)}\Big) \\ &= -e^3\Big[\bar{\Delta}_{\mu\nu}\bar{\Psi}\gamma_\nu(i\not{\partial})\bar{\boldsymbol{\Delta}}\not{A}(i\not{\partial})\bar{\boldsymbol{\Delta}}\gamma_\mu\Psi\Big],\end{aligned} \tag{44}$$

$$\begin{aligned}\Gamma_{3,\mathrm{q}}|_{eG} &= \bar{\Delta}_{\alpha\hat{\alpha}}\tau_{\hat{\alpha}\beta}^{(-\not{A})}\bar{\Delta}_{\beta\hat{\beta}}\tau_{\hat{\beta}\alpha}^{\bar{\Psi}\Psi} \\ &= 2e\frac{G_S}{\Lambda^2}\Big[\bar{\Psi}(i\not{\partial})\bar{\boldsymbol{\Delta}}\not{A}(i\not{\partial})\bar{\boldsymbol{\Delta}}\Psi\Big] + 2e\frac{G_V}{\Lambda^2}\Big[\bar{\Psi}\gamma_\mu(i\not{\partial})\bar{\boldsymbol{\Delta}}\not{A}(i\not{\partial})\bar{\boldsymbol{\Delta}}\gamma_\mu\Psi\Big], \\ &\quad - 4e\frac{G_V}{\Lambda^2}\Big[i\partial_\rho\bar{\boldsymbol{\Delta}}A_\rho i\partial_\mu\bar{\boldsymbol{\Delta}} - i\partial_\rho\bar{\boldsymbol{\Delta}}A_\mu i\partial_\rho\bar{\boldsymbol{\Delta}} + i\partial_\mu\bar{\boldsymbol{\Delta}}A_\rho i\partial_\rho\bar{\boldsymbol{\Delta}}\Big]\Big(\bar{\Psi}\gamma_\mu\Psi\Big).\end{aligned} \tag{45}$$

Figure 2 shows graphical representations of $\Gamma_{3,\mathrm{q}}|_{e^3}$ and $\Gamma_{3,\mathrm{q}}|_{eG}$ in Eqs. (44) and (45). As stated earlier, fields in τ are attached to interaction vertices without external lines.

Now we may calculate the QME at $\mathcal{O}(e^3)$ and $\mathcal{O}(eG_{S,V})$ as

$$\Sigma_3 = (\Gamma_{3,\mathrm{q}}, \Gamma_0) + (\Gamma_1, \Gamma_{2,\mathrm{q}}) + \Big[K\Gamma_{1*}^{(2)}\bar{\Delta}\Gamma_{2,\mathrm{cl}}^{(2)}\Big] - \Big[K\Gamma_{1*}^{(2)}\bar{\Delta}\Gamma_1^{(2)}\bar{\Delta}\Gamma_1^{(2)}\Big]. \tag{46}$$

All the terms in Σ_3 are proportional to $\bar{\Psi}\Psi C$ with coefficients of $\mathcal{O}(e^3)$ or $\mathcal{O}(eG)$: $\Sigma_3 = \Sigma_{3,e^3} + \Sigma_{3,eG}$. There are three $\mathcal{O}(e^3)$ terms,

$$\Sigma_{3,e^3} = \Sigma_{3,e^3}|_K + \Sigma_{3,e^3}|_{(\Gamma_0,\Gamma_{3,\mathrm{q}})} + \Sigma_{3,e^3}|_{(\Gamma_1,\Gamma_{2,\mathrm{q}})}, \tag{47}$$

where

$$
\begin{aligned}
\Sigma_{3,e^3}|_K &= -\Big[K\Gamma^{(2)}_{1*}\bar{\Delta}\Gamma^{(2)}_1\bar{\Delta}\Gamma^{(2)}_1\Big] \\
&= -\Big[K\tau^C_{*\alpha\beta}\bar{\Delta}_{\beta\hat{\alpha}}\tau^{(-\gamma\Psi)}_{\hat{\alpha}\mu}\bar{\Delta}_{\mu\nu}\tau^{(-\bar{\Psi}\gamma)}_{\nu\alpha} + K\tau^C_{*\hat{\alpha}\hat{\beta}}\bar{\Delta}_{\hat{\beta}\alpha}\tau^{(\bar{\Psi}\gamma)}_{\alpha\mu}\bar{\Delta}_{\mu\nu}\tau^{(\gamma\Psi)}_{\nu\hat{\alpha}}\Big] \\
&= e^3\Big[\bar{\Psi}\gamma_\nu KC\partial\!\!\!/\bar{\mathbf{\Delta}}\gamma_\mu\Psi\bar{\Delta}_{\mu\nu}\Big] - e^3\Big[\bar{\Psi}\gamma_\mu\partial\!\!\!/\bar{\mathbf{\Delta}}CK\gamma_\nu\Psi\bar{\Delta}_{\mu\nu}\Big],
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
\Sigma_{3,e^3}|_{(\Gamma_1,\Gamma_{2,q})} &= -\frac{\partial\Gamma_1}{\partial\Psi^*}\frac{\partial^l\Gamma_{2,q}}{\partial\Psi} - \frac{\partial\Gamma_1}{\partial\bar{\Psi}^*}\frac{\partial^l\Gamma_{2,q}}{\partial\bar{\Psi}} \\
&= e^3\Big[\bar{\Psi}\gamma_\nu\partial\!\!\!/\bar{\mathbf{\Delta}}C\gamma_\mu\Psi\bar{\Delta}_{\mu\nu}\Big] - e^3\Big[\bar{\Psi}\gamma_\nu C\partial\!\!\!/\bar{\mathbf{\Delta}}\gamma_\mu\Psi\bar{\Delta}_{\mu\nu}\Big],
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
\Sigma_{3,e^3}|_{(\Gamma_0,\Gamma_{3,q})} &= -e^3\Big[\bar{\Psi}\gamma_\nu\partial\!\!\!/\bar{\mathbf{\Delta}}\partial\!\!\!/C\partial\!\!\!/\bar{\mathbf{\Delta}}\gamma_\mu\Psi\bar{\Delta}_{\mu\nu}\Big] \\
&= e^3\Big[\bar{\Psi}\gamma_\nu(1-K)C\partial\!\!\!/\bar{\mathbf{\Delta}}\gamma_\mu\Psi\bar{\Delta}_{\mu\nu}\Big] - e^3\Big[\bar{\Psi}\gamma_\nu\partial\!\!\!/\bar{\mathbf{\Delta}}C(1-K)\gamma_\mu\Psi\bar{\Delta}_{\mu\nu}\Big].
\end{aligned} \tag{50}
$$

The above results lead to

$$
\Sigma_{3,e^3} = 0. \tag{51}
$$

As for the $\mathcal{O}(eG)$ terms, we find two contributions:

$$
\Sigma_{3,eG} = \Sigma_{3,eG}|_K + \Sigma_{3,eG}|_{(\Gamma_0,\Gamma_{3,q})}. \tag{52}
$$

We may calculate them as

$$
\begin{aligned}
\Sigma_{3,eG}|_K &= \Big[K\Gamma^{(2)}_{1*}\bar{\Delta}\Gamma^{(2)}_{2,\mathrm{cl}}\Big] = \Big[K\tau^C_{*\alpha\beta}\bar{\Delta}_{\beta\hat{\alpha}}\tau^{(\bar{\Psi}\Psi)}_{\hat{\alpha}\alpha} + K\tau^C_{*\hat{\alpha}\hat{\beta}}\bar{\Delta}^T_{\hat{\beta}\alpha}\left(\tau^{(\bar{\Psi}\Psi)}\right)^T_{\alpha\hat{\alpha}}\Big] \\
&= \frac{2e(G_S-2G_V)}{\Lambda^2}\Big[\bar{\Psi}\Big(KC\partial\!\!\!/\bar{\mathbf{\Delta}} - \partial\!\!\!/\bar{\mathbf{\Delta}}CK\Big)\Psi\Big] \\
&\quad - \frac{4eG_V}{\Lambda^2}\Big[\Big(KC\partial_\mu\bar{\mathbf{\Delta}} - \partial_\mu\bar{\mathbf{\Delta}}CK\Big)(\bar{\Psi}\gamma_\mu\Psi)\Big],
\end{aligned} \tag{53}
$$

and

$$
\begin{aligned}
\Sigma_{3,eG}|_{(\Gamma_0,\Gamma_{3,q})} &= \frac{2e(G_S-2G_V)}{\Lambda^2}\Big[\bar{\Psi}\Big((1-K)C\partial\!\!\!/\bar{\mathbf{\Delta}} - \partial\!\!\!/\bar{\mathbf{\Delta}}C(1-K)\Big)\Psi\Big] \\
&\quad - \frac{4eG_V}{\Lambda^2}\Big[\Big((1-K)C\partial_\mu\bar{\mathbf{\Delta}} - \partial_\mu\bar{\mathbf{\Delta}}C(1-K)\Big)(\bar{\Psi}\gamma_\mu\Psi)\Big].
\end{aligned} \tag{54}
$$

Equations (53) and (54) sum up to give a vanishing result:

$$
\Sigma_{3,eG} = \frac{2e(G_S-2G_V)}{\Lambda^2}\Big[\bar{\Psi}\big(C\partial\!\!\!/\bar{\mathbf{\Delta}} - \partial\!\!\!/\bar{\mathbf{\Delta}}C\big)\Psi\Big] - \frac{4eG_V}{\Lambda^2}\Big[\big(C\partial_\mu\bar{\mathbf{\Delta}} - \partial_\mu\bar{\mathbf{\Delta}}C\big)(\bar{\Psi}\gamma_\mu\Psi)\Big] = 0. \tag{55}
$$

From Eqs. (51) and (55), we finally obtain the result

$$
\Sigma_3 = 0. \tag{56}
$$

We have confirmed that the QME/mST can be solved consistently with the flow equation up to the orders of e^3 and $eG_{S,\,V}$. We have seen that the four-fermi interactions generate quantum corrections to the $\bar{\Psi}A\!\!\!/\Psi$ vertex function and, in Σ_3, their free BRST transformation and the measure factor cancel each other.

4. Wave function renormalization constants and β functions

In order to take account of Λ evolution, we introduce renormalization constants for fields and couplings. The corresponding Z factors are defined as

$$\begin{aligned} &A_\mu \to Z_3^{1/2} A_\mu, \qquad \Psi \to Z_2^{1/2}\Psi, \qquad \bar\Psi \to Z_2^{1/2}\bar\Psi, \qquad C \to Z_3^{1/2} C, \\ &\bar C \to Z_3^{-1/2}\bar C, \qquad B \to Z_3^{-1/2} B, \qquad A^*_\mu \to Z_3^{-1/2} A^*_\mu, \\ &\Psi^* \to Z_2^{-1/2}\Psi^*, \qquad \bar\Psi^* \to Z_2^{-1/2}\Psi^*, \qquad \bar C^* \to Z_3^{1/2}\bar C^*. \end{aligned} \tag{57}$$

For the gauge coupling, four-fermi couplings, and gauge parameter, we set $e \to e_\Lambda = Z_e e$, $G_{S(V)} \to G_{S(V),\,\Lambda} = Z_{S(V)} G_{S(V)}$, and $\xi \to \xi_\Lambda = Z_3 \xi$. As for the wave function rescalings in Eq. (57), we require that they are due to canonical transformations so that the fields and antifields are rescaled in opposite directions [10]. Two comments are in order: no quantum corrections are present in the ghost kinetic term $-i\partial_\mu \bar C \partial_\mu C$ in QED; and the scaling of the B field cancels the scaling of the longitudinal mode of the photon.

The 1PI effective action is expressed as

$$\begin{aligned} \Gamma_0 &= \int_x \left[\frac{Z_3}{2} A_\mu\big(-\partial^2 P^T_{\mu\nu}\big) A_\nu + Z_2 \bar\Psi i \partial\!\!\!/ \Psi + \big(A^*_\mu - i\partial_\mu \bar C\big)\partial_\mu C + \frac{\xi}{2} B^2 + \big(\bar C^* + i\partial\cdot A\big) B \right], \\ \Gamma_1 &= \int_x \left[-e(Z_e Z_3^{1/2} Z_2) \bar\Psi A\!\!\!/ \Psi - ie(Z_e Z_3^{1/2})\Psi^*\Psi C + ie(Z_e Z_3^{1/2})\bar\Psi\bar\Psi^* C \right], \\ \Gamma_2 &= \int_x \Bigg[(Z_S Z_2^2)\frac{G_S}{2\Lambda^2}\Big\{ \big(\bar\Psi\Psi\big)\big(\bar\Psi\Psi\big) - \big(\bar\Psi\gamma_5\Psi\big)\big(\bar\Psi\gamma_5\Psi\big) \Big\} \\ &\quad + (Z_V Z_2^2)\frac{G_V}{2\Lambda^2}\Big\{ \big(\bar\Psi\gamma_\mu\Psi\big)\big(\bar\Psi\gamma_\mu\Psi\big) + \big(\bar\Psi\gamma_5\gamma_\mu\Psi\big)\big(\bar\Psi\gamma_5\gamma_\mu\Psi\big) \Big\} \Bigg]. \end{aligned} \tag{58}$$

At the one-loop level anomalous dimensions of the photon and Dirac fields are expressed as

$$Z_{2,3} = 1 - \eta_{\Psi,A} \log(\Lambda/\mu). \tag{59}$$

For the gauge coupling, its beta function is expressed as $\beta_e = \eta_e e$, where

$$Z_e = 1 + \eta_e \log(\Lambda/\mu). \tag{60}$$

We first compute $\eta_{A,\Psi,e}$ in the absence of the four-fermi interactions, $G_S = G_V = 0$. For the photon two-point functions, it follows from Eq. (33) in momentum space:

$$\begin{aligned} \dot\Gamma^{AA} &= e_\Lambda^2 \int_{p,q} \mathrm{Tr}\left\{ \frac{\dot{\bar K}(q)\bar K(p+q)}{q^2(p+q)^2} \big[q\!\!\!/ A\!\!\!/(-p)(p\!\!\!/ + q\!\!\!/)A\!\!\!/(p) \big] \right\} \\ &= \frac{e_\Lambda^2}{2}\int_p A_\mu(-p)\dot{\mathcal{A}}_{\mu\nu}(p) A_\nu(p). \end{aligned} \tag{61}$$

Expanding $\dot{\mathcal{A}}_{\mu\nu}(p)$ in the external momentum up to $\mathcal{O}(p^2)$, $\dot{\mathcal{A}}_{\mu\nu}(p) = 2M_A^2\delta_{\mu\nu} - \eta_A(p^2\delta_{\mu\nu} - p_\mu p_\nu) + \cdots$, we find the photon mass term

$$M_A^2 = \frac{e_\Lambda^2}{4\pi^2}\Lambda^2 \int_0^\infty du\, u\bar K'(u)\bar K(u), \tag{62}$$

and the anomalous dimension

$$\eta_A = -\frac{e_\Lambda^2}{6\pi^2}\int_0^\infty du \Big\{ \big[(u\bar K(u)')^2\big]' - (\bar K^2(u))' \Big\} = \frac{e_\Lambda^2}{6\pi^2}\,. \tag{63}$$

In momentum space the $\Lambda\partial_\Lambda = \partial_t$ derivative of the fermion two-point function (34) takes the form

$$\dot{\Gamma}^{\bar{\Psi}\Psi} = -e_\Lambda^2 \int_p \bar{\Psi}(-p)\dot{\Gamma}^{\bar{\Psi}\Psi}(p)\Psi(p), \tag{64}$$

where

$$\dot{\Gamma}^{\bar{\Psi}\Psi}(p) = \int_q \gamma_\nu(\not{p}+\not{q})\gamma_\mu \frac{\dot{\bar{K}}(q)\bar{K}(p+q)+\bar{K}(q)\dot{\bar{K}}(p+q)}{q^2(p+q)^2}\Big(\delta_{\mu\nu}+(\xi_\Lambda-1)\frac{q_\mu q_\nu}{q^2}\Big). \tag{65}$$

It gives

$$\frac{\not{p}}{p^2}\dot{\Gamma}^{\bar{\Psi}\Psi}(p)|_{p^2=0} = \frac{\xi_\Lambda}{8\pi^2}\int_0^\infty du[\bar{K}^2(u)]' = \frac{\xi_\Lambda}{8\pi^2}. \tag{66}$$

Therefore, the anomalous dimension for the Dirac fields is

$$\eta_\Psi = \frac{e_\Lambda^2\xi_\Lambda}{8\pi^2}. \tag{67}$$

For the gauge interaction vertex, we have

$$\dot{\Gamma}^{\bar{\Psi}A\Psi} = -e_\Lambda^3 \int_{p,q,r} \dot{\Gamma}_\rho^{\bar{\Psi}A\Psi}(p,q,r)\bar{\Psi}(p)A_\rho(q)\Psi(r)\delta(p+q+r), \tag{68}$$

where

$$\begin{aligned}\dot{\Gamma}_\rho^{\bar{\Psi}A\Psi}(0,0,0) &= \frac{\partial}{\partial t}\int_q \frac{\bar{K}^3(q)}{q^6}\gamma_\nu\not{q}\gamma_\rho\not{q}\gamma_\mu\Big(\delta_{\mu\nu}+(\xi_\Lambda-1)\frac{q_\mu q_\nu}{q^2}\Big)\\ &= -\frac{2\xi_\Lambda\gamma_\rho}{(4\pi)^2}\int_0^\infty du\,(\bar{K}^3(u))' = -\frac{\xi_\Lambda\gamma_\rho}{8\pi^2}.\end{aligned} \tag{69}$$

Therefore,

$$\dot{\Gamma}^{\bar{\Psi}A\Psi} = \frac{e_\Lambda^3\xi}{8\pi^2}\bar{\Psi}\not{A}\Psi. \tag{70}$$

The flow equation for the gauge interaction vertex takes the form

$$e_\Lambda\left(\frac{1}{2}\eta_A+\eta_\Psi-\eta_e\right) = \frac{e_\Lambda^3\xi_\Lambda}{8\pi^2}, \tag{71}$$

which leads to

$$\eta_e = \frac{1}{2}\eta_A = \frac{e_\Lambda^2}{12\pi^2}. \tag{72}$$

This is equivalent to the well-known Ward identity,

$$Z_1 = Z_3^{1/2}Z_eZ_2 = Z_2, \tag{73}$$

in our one-loop computation.

The beta function for the gauge coupling is given by

$$\dot{e}_\Lambda = \beta_e = \eta_e e_\Lambda = \frac{e_\Lambda^3}{12\pi^2}. \tag{74}$$

We stress that the anomalous dimensions $\eta_{A,\Psi}$ and the beta function β_e are universal, being independent of the choice of cutoff function K. They are the same as those obtained by using gauge invariant regularization as the dimensional regularization in the standard perturbation theory.

We now include the four-fermi interactions, G_S and G_V. The presence of these couplings leaves the anomalous dimensions $\eta_{A,\Psi}$ unchanged, while there arises an additional contribution in the r.h.s. of the flow equation (71). Instead of Eq. (72), we have

$$\eta_e = \frac{1}{2}\eta_A - \frac{1}{4\pi^2}(G_{S,\Lambda}-4G_{V,\Lambda})\int_0^\infty du\,u\bar{K}(u)\bar{K}(u)'. \tag{75}$$

The coefficient of $G_{S,\Lambda} - 4G_{V,\Lambda}$ depends on the choice of cutoff function, but is related to the photon mass term. Therefore, we obtain

$$\eta_e = \frac{1}{2}\eta_A - \bar{M}_A^2(G_{S,\Lambda} - 4G_{V,\Lambda}), \tag{76}$$

where $\bar{M}_A^2 = M_A^2/(e_\Lambda^2\Lambda^2)$.

Contributions of four-fermi couplings to the beta function were observed earlier in a non-perturbative study with a truncated 1PI action by using the modified WT identity [17]. Though Eq. (76) is a perturbative result, the gauge symmetry is kept intact via the QME.

5. Summary and discussion

In the ERG with a momentum cutoff, the flow equation generates gauge non-invariant quantum corrections such as a photon mass term. BRST transformation of these symmetry breaking corrections are systematically canceled, if the QME is fulfilled. Using the 1PI formulation, we have shown that the perturbative solutions to the flow equation also solve the QME/mST for QED with four-fermi interactions in a general covariant gauge.

As for Λ evolution, we obtain the standard anomalous dimensions and the standard beta function of the gauge coupling together with the $Z_1 = Z_2$ relation when removing four-fermi interactions. This reflects the regularization-scheme independence in the one-loop computation for these objects. When included, the four-fermi terms yield a new contribution to the beta function. Its coefficient depends on the choice of cutoff function and is expressed in terms of the photon mass term. Even for $Z_1 \neq Z_2$, BRST symmetry is unbroken because the QME/mST remains intact for the rescaled 1PI action (58). This is a consequence of the invariance of the QME/mST under the canonical transformation used in introducing wave function renormalization factors.

The use of such a canonical transformation will induce an undesirable Λ evolution in the $eZ_eZ_3^{1/2}\Psi^*\Psi C$ and $eZ_eZ_3^{1/2}\bar{\Psi}^*\bar{\Psi}C$ vertices for $Z_1 \neq Z_2$, i.e., $Z_eZ_3^{1/2} \neq 1$. Hence, RG flows should be computed suppressing the antifields at the final stage. Then, Eq. (23) is satisfied, and the RG flows stay in a BRST invariant submanifold of the theory space.

Let us consider the cutoff removing limit $K \to 0$ ($\Lambda \to 0$). In the one-loop formula (28), the IR-regulated propagators $\bar{\Delta}^{AB}$ are replaced with unregularized ones Δ^{AB}. The quantum actions $\Gamma_{2,q}$ and $\Gamma_{3,q}$ whose UV divergences are removed using dimensional regularization satisfy the Zinn–Justin equations: $(\Gamma_0, \Gamma_{2,q}) = 0$, $(\Gamma_0, \Gamma_{3,q}) + (\Gamma_1, \Gamma_{2,q}) = 0$. The first equation leads to vanishing quantum corrections in the longitudinal part of the photon two-point functions, $\mathcal{L} = 0$. The second equation gives the standard relation between the mass operator of the fermion two-point functions and the gauge interaction vertices. Since the four-fermi interactions yield no contribution to β_e, as seen from Eq. (76) with $M_A = 0$, they do not affect the WT relation $Z_1 = Z_2$. Therefore, we observe that the classical BRST symmetry persists at the quantum level, irrespective of the presence of higher-dimensional operators such as the four-fermi interactions. These results should be compared with those for $\Lambda \neq 0$.

Our perturbative results imply that a non-perturbative study of the chiral invariant QED will certainly observe a similar modification of the Ward identity if we include the four fermi interactions.

By adding four fermi couplings, we have found an extra contribution to the gauge-coupling beta function and the textbook Ward identity is modified even in our perturbative calculation.

This is a price to pay for using a momentum cutoff as regularization. Yet the presence of the gauge symmetry is guaranteed by the QME.

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Appendix A

To calculate the r.h.s. of Eqs. (17) and (22), we need to find field-dependent parts of $\Gamma_I^{(2)}$ and $\Gamma_{I*}^{(2)}$, the vertices denoted as τ.

From Γ_1 we find the following vertices:

$$\begin{aligned}
\tau_{\hat{\alpha}\beta}^{(-\not{A})}(x,y) &= \frac{\partial^l \partial^r \Gamma_1}{\partial \bar{\Psi}_{\hat{\alpha}}(x) \partial \Psi_\beta(y)} = -e(\not{A})_{\hat{\alpha}\beta}(x)\delta(x-y), \\
\tau_{\alpha\hat{\beta}}^{(\not{A}^T)}(x,y) &= \frac{\partial^l \partial^r \Gamma_1}{\partial \Psi_\alpha(x) \partial \bar{\Psi}_{\hat{\beta}}(y)} = +e(\not{A}^T)_{\alpha\hat{\beta}}(x)\delta(x-y), \\
\tau_{\hat{\alpha}\mu}^{(-\gamma\Psi)}(x,y) &= \frac{\partial^l \partial^r \Gamma_1}{\partial \bar{\Psi}_{\hat{\alpha}}(x) \partial A_\mu(y)} = -e(\gamma_\mu \Psi)_{\hat{\alpha}}(x)\delta(x-y) = -\tau_{\mu\hat{\alpha}}^{(\gamma\Psi)}(x,y), \\
\tau_{\mu\beta}^{(-\bar{\Psi}\gamma)}(x,y) &= \frac{\partial^l \partial^r \Gamma_1}{\partial A_\mu(x) \partial \Psi_\beta(y)} = -e(\bar{\Psi}\gamma_\mu)_\beta(x)\delta(x-y) = -\tau_{\beta\mu}^{(\bar{\Psi}\gamma)}(x,y).
\end{aligned} \tag{A1}$$

Here the superscripts of τ indicate structures of vertices. Similarly, from $\Gamma_{2,\mathrm{cl}}^{(2)}$, we have

$$\begin{aligned}
\tau_{\hat{\alpha}\beta}^{(\bar{\Psi}\Psi)}(x,y) &= \left[\frac{\partial^l \partial^r \Gamma_{2,\mathrm{cl}}[\Phi]}{\partial \bar{\Psi}_{\hat{\alpha}}(x) \partial \Psi_\beta(y)}\right]_{A=0} \\
&= G_S \delta(x-y) \Big\{ \Big[\delta_{\hat{\alpha}\beta}\big(\bar{\Psi}(x)\Psi(x)\big) - (\gamma_5)_{\hat{\alpha}\beta}\big(\bar{\Psi}(x)\gamma_5\Psi(x)\big) \Big] \\
&\quad - \Big[\bar{\Psi}_\beta(x)\Psi_{\hat{\alpha}}(x) - (\bar{\Psi}\gamma_5)_\beta(x)(\gamma_5\Psi)_{\hat{\alpha}}(x) \Big] \Big\} \\
&\quad + G_V \delta(x-y) \Big\{ \Big[(\gamma_\mu)_{\hat{\alpha}\beta}\big(\bar{\Psi}(x)\gamma_\mu\Psi(x)\big) + (\gamma_5\gamma_\mu)_{\hat{\alpha}\beta}\big(\bar{\Psi}(x)\gamma_5\gamma_\mu\Psi(x)\big) \Big] \\
&\quad - \Big[\big(\bar{\Psi}(x)\gamma_\mu\big)_\beta\big(\gamma_\mu\Psi(x)\big)_{\hat{\alpha}} + \big(\bar{\Psi}(x)\gamma_5\gamma_\mu\big)_\beta\big(\gamma_5\gamma_\mu\Psi(x)\big)_{\hat{\alpha}} \Big] \Big\} \\
&= -\big(\tau^{(\bar{\Psi}\Psi)}\big)^T_{\beta\hat{\alpha}}(y,x).
\end{aligned} \tag{A2}$$

As for $\Gamma_{I*}^{(2)}$, we need vertices out of Γ_1:

$$\begin{aligned}
\tau_{*\alpha\beta}^{C} &= \frac{\partial}{\partial \Psi_\alpha^*(x)} \frac{\partial^r}{\partial \Psi_\beta(y)} \Gamma_1 = +ie\delta_{\alpha\beta}C(x)\delta(x-y), \\
\tau_{*\hat{\alpha}\hat{\beta}}^{C} &= \frac{\partial}{\partial \bar{\Psi}_{\hat{\alpha}}^*(x)} \frac{\partial^r}{\partial \bar{\Psi}_{\hat{\beta}}(y)} \Gamma_1 = -ie\delta_{\hat{\alpha}\hat{\beta}}C(x)\delta(x-y).
\end{aligned} \tag{A3}$$

Appendix B

The following example shows our notation for computing the QMF:

$$\begin{aligned}
\mathrm{Tr}\Big[KC\partial\!\!\!/\bar{\mathbf{\Delta}}A\!\!\!/\Big] &= \int_{x,y} K(x-y)C(y)\mathrm{tr}\Big[\partial\!\!\!/_y\bar{\mathbf{\Delta}}(y-x)A\!\!\!/(x)\Big] \\
&= \int_{x,y} \mathrm{tr}\Big[A\!\!\!/^T(x)\partial\!\!\!/_x^T\bar{\mathbf{\Delta}}(y-x)\Big]C(y)K(x-y) \\
&= -\int_{x,y} \mathrm{tr}\Big[A\!\!\!/(x)\partial\!\!\!/_x\bar{\mathbf{\Delta}}(x-y)\Big]C(y)K(y-x) \\
&= -\Big[A\!\!\!/\partial\!\!\!/\bar{\mathbf{\Delta}}CK\Big],
\end{aligned} \tag{B1}$$

where the trace is taken for γ matrices. We have also used the charge conjugation relation $C\gamma_\mu^T C^{-1} = -\gamma_\mu$, and symmetry properties $\bar{\mathbf{\Delta}}(x-y) = \bar{\mathbf{\Delta}}(y-x)$ and $K(x-y) = K(y-x)$.

In this notation, we obtain Eq. (38) by making integration by parts as

$$\begin{aligned}
\Sigma_2|_{(\Gamma_0,\Gamma_{2,q})} &= -\frac{e^2}{2}\Big[\partial\!\!\!/\bar{\mathbf{\Delta}}A\!\!\!/\partial\!\!\!/\bar{\mathbf{\Delta}}\partial\!\!\!/C + \partial\!\!\!/\bar{\mathbf{\Delta}}\partial\!\!\!/C\partial\!\!\!/\bar{\mathbf{\Delta}}A\!\!\!/\Big] = -e^2\Big[\partial\!\!\!/\bar{\mathbf{\Delta}}\partial\!\!\!/C\partial\!\!\!/\bar{\mathbf{\Delta}}A\!\!\!/\Big] \\
&= e^2\int_{x,y}\mathrm{tr}\Big[\partial\!\!\!/_x\partial\!\!\!/_y\bar{\mathbf{\Delta}}(x-y)C(y)\partial\!\!\!/_y\bar{\mathbf{\Delta}}(y-x)A\!\!\!/(x) + \partial\!\!\!/_x\bar{\mathbf{\Delta}}(x-y)C(y)\partial\!\!\!/_y^2\bar{\mathbf{\Delta}}(y-x)A\!\!\!/(x)\Big] \\
&= e^2\Big[(1-K)C\partial\!\!\!/\bar{\mathbf{\Delta}}A\!\!\!/ - \partial\!\!\!/\bar{\mathbf{\Delta}}C(1-K)A\!\!\!/\Big] = 2e^2\Big[(1-K)C\partial\!\!\!/\bar{\mathbf{\Delta}}A\!\!\!/\Big].
\end{aligned} \tag{B2}$$

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