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WHEELER–DEWITT UNIVERSE WAVE FUNCTION IN THE PRESENCE OF STIFF MATTER*

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We study the Wheeler–DeWitt (WDW) equation close to the Big Bang. We argue that an interaction-dominated fluid (speed of sound equal to the speed of light), if present, would dominate during such an early phase. Such a fluid with $p = \rho \propto 1/a^6$ generates a term in the potential of the wave function of the WDW equation proportional to $-1/a^2$. This very peculiar potential, which embodies a spontaneous breaking of dilatation invariance, has some very remarkable consequences for the wave function of the Universe: $\Psi(a)$ vanishes at the Big Bang: $\Psi(0) = 0$; the wave function $\Psi(a)$ is always real; a superselection rule assures that the system is confined to $a \geq 0$ without the need of imposing any additional artificial barrier for unphysical negative a. These results are valid for a continuous class of choices of the operator ordering of the WDW equation.

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1. Motivation

At the very beginning of the Universe evolution, just after the Big Bang, the energy density was extremely high. In a classical treatment, one has the so-called Big Bang singularity: the energy density diverges when the scale factor a defining the Friedmann–Robertson–Walker (FRW) metric vanishes. However, when assuming that a more general and quantum version of classical general relativity (GR) exists, quantum fluctuations can be as large as to deeply modify the early Universe evolution.

One of the first treatments of quantum gravity was put forward by Wheeler and DeWitt [1, 2]: it is a canonical approach, in which the Hamiltonian of general relativity is quantized, hence the wave function is a function of the (spacial) metric. A Schrödinger-like equation, called Wheeler–DeWitt (WDW) equation, emerges. Although we still do not know if this is the correct and/or the most efficient way to quantize gravity [3–5], it represents a useful approach to describe various problems in which both GR and QM merge. This is especially the case of quantum cosmology.

The WDW equation simplifies tremendously when a uniform and homogeneous FRW Universe is considered: the wave function is solely a function of the scale parameter a [hence, $\Psi = \Psi(a)$], see *e.g.* [6, 7]. However, it is not clear what fixes the boundary conditions (if any) associated with the WDW equation and a long debate has emerged in this context: while Hartle and Hawking find, within the so-called no-boundary proposal, a real wave function [8] (see also [9–11]) containing both ingoing and outgoing waves, Vilenkin [12] put forward a complex wave function corresponding solely to an outgoing wave. Usually, in such studies of the early time of the Universe, only the curvature and the constant cosmological terms are retained. For a recent description of the other possible components, such as matter and radiation, see Refs. [13, 14]. Indeed, the interest on the wave function of the Universe is very strong, as the recent lively and vibrating dispute on the effect of quantum gravitational perturbations in the early Universe shows [15–17].

Besides the problem of the explicit form of the wave function mentioned above, there are other issues connected to the WDW equation:

(i) What should be the wave function at the Big Bang, $\Psi(0)$? It is non-vanishing for both the Hartle–Hawing and Vilenkin solutions mentioned above.

(ii) How to implement the classical constraint $a \geq 0$ [18]? Usually, the transformation $a = \mathrm{e}^{\Omega}$ is performed [4], but this is merely a mathematical trick.

(iii) Should the wave function be real or complex?

(iv) Is there any influence of the so-called operator ordering problem?

In this work, we shall consider the effect of a stiff-matter interaction-dominated gas, for which the pressure equalizes the energy density, $p = \rho$

(the speed of sound is c, hence maximal). Namely, whatever degrees of freedom (d.o.f.) are present in the very early Universe, their strong interactions could generate such a gas. In the realm of quantum cosmology, a fluid with $p = \rho$ corresponds to a term of the type of $-1/a^2$ in the effective potential of the WDW equation. Such a potential, if present, necessarily dominates at small a (other possibilities are excluded since they would violate causality). This is indeed a *very peculiar quantum* potential that breaks all our naive expectations for a quantum system, see Ref. [19] and also Refs. [20, 21]. At first sight, it seems that no bound state should exist, since, if one exists, a continuum of bound states, one for each negative energy, would also exist. At a closer inspection, the system is much more interesting and its detailed treatment imposes to render the Hamiltonian self-adjoint [19, 22]: if the attraction is below a certain critical value, there is a single bound state, but above, there is an infinity of bound states (one of which with lower energy). In turn, this system provides a beautiful example of an anomaly: a characteristic length in the system emerges, which is, in a sense, similar to the development of the Yang–Mills energy scale in QCD.

Quite remarkably, the unexpected features of the potential $-1/a^2$ in the WDW equation may help to relieve in an elegant way the problems of the WDW approach listed above:

(i) The wave function vanishes at the Big Bang: $\Psi(a = 0) = 0$. This condition reminds the old idea of DeWitt according to which a vanishing wave function could represent a solution of the problem of the singularity [1, 4]. Indeed, it has been also discussed as an example of ‘Planck potential’ in Ref. [23], where the WKB approximation is adopted to show that not only the wave function vanishes at the Big Bang but that it leads to an intrinsic symmetric initial condition which removes the dependence of the wave function on all the matter degrees of freedom at the singularity.

(ii) It generates a superselection rule according to which only positive (or negative) values of a are allowed. Hence, once $a > 0$ is chosen, the wave function is automatically nonzero only on the r.h.s. and there is no need of any further artificial restriction.

(iii) The wave function is *real* in agreement with the result of Hartle and Hawking [8].

(iv) An important technical aspect concerns the choice of the operator ordering. The results are qualitatively independent on a large and continuous class of choices (but not on all of them), see the detailed discussion later on.

In previous works, a component of the form of $p = \rho$ has been taken into account for classical cosmologies (*e.g.* Refs. [24, 25]). Moreover, an equivalent term in the classical FRW equation appears also in the presence of the so-called kination domain [26–28], in which a massless (and homogeneous) scalar field ϕ is considered. The investigation of the quantum version of a massless and homogeneous scalar field was the topic of various publications, *e.g.* Refs. [29–33]. The corresponding wave function $\Psi(a, \phi)$ is subject to a potential of the 'stiff-matter' type $1/a^2$. Interestingly, in those works, the possibility to avoid the initial singularity is discussed in great detail: as a consequence, in many of the discussed solutions dealing with stiff matter, the wave function Ψ vanishes for $a \to 0$, in agreement with our results. Indeed, also the problem of 'self-adjointness' mentioned above has been discussed [30, 31], but a different prescription than the one discussed in Ref. [19] is employed. In our work, we use the results of Ref. [19], which lead to different solutions of $\Psi(a)$ for small a.

Another interesting theoretical aspect concerns the quantum field theory which generates stiff matter: in Ref. [34] (see also references therein and Ref. [35]), it is shown that the case of $p = \rho$ is obtained in the presence of a quantum field theory in which the interaction dominates over the kinetic terms. This is quite different from the case of a massless scalar field $\phi(t, x)$ (in which the space-dependence is retained), which in flat space should generate a gas of the type of $p = \rho/3$ when quantum fluctuations of the fields are included. Yet, the description of a strongly interacting quantum field theory in early quantum cosmology is at present not possible thus, one needs to either start with $p = \rho$ (as we do in our work) or consider an homogenous scalar field, as in Refs. [29–33, 36]. Another possibility is suggested within the context of loop quantum cosmology: in Ref. [37], it has been shown that non-perturbative quantum geometric effects manifest themselves in the form of a definite negative ρ^2 modification of the Friedmann equation. In this case, a 'dark plus visible' matter component (with a dust-like equation of state) would again produce a potential $\propto 1/a^2$ in the WDW equation. It would be interesting to investigate whether this correction could also be positive definite such as to lead to an attractive potential in the WDW equation as the one we are going to discuss here.

2. WDW equation in cosmology: brief review and consequences of stiff matter

First, we recall how the WDW equation emerges in cosmology. First, we consider the scale factor $a \equiv a(t)$ as a field with dimension length subject to the classical Lagrangian

$$L_{\rm FRW} = -Ca^3\left[\left(\frac{\dot a}{a}\right)^2 - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} + \frac{8\pi G}{3c^2}\rho\right] \quad \text{with} \quad C = \frac{3\pi c^2}{4G}\,, \tag{1}$$

where k and Λ parametrize the curvature and the cosmological constant contributions to the Universe's evolution.

The energy density ρ describes the contribution of matter and energy. Here, we shall consider that each component fulfills the EoS $p = w\rho$, which has a constant speed of sound $v_{\rm sound} = c\sqrt{{\rm d}p/{\rm d}\rho} = c\sqrt{w} \leq c$. The adiabatic expansion ${\rm d}E + p\,{\rm d}V = 0$ translates into ${\rm d}(\rho a^3) + p\,{\rm d}(a^3) = 0$, then

$$a\frac{{\rm d}\rho}{{\rm d}a} = -3(\rho + p) = -3(1+w)\rho \implies \rho(a) = \frac{A_w}{a^{3(w+1)}}\,. \tag{2}$$

As renowned [38], for $w = 0$, a Universe dominated by dust is obtained ($\rho_{\rm dust} \propto a^{-3}$, dark plus visible matter, about 30% of contribution to the present state of the expansion, the rest being the present cosmic inflation). A radiation-dominated Universe is found by setting $w = 1/3$ ($\rho_{\rm radiation} \propto a^{-4}$); this was relevant in the radiation-dominated era of the Universe. Of course, the use of a constant w is an approximation, since a relativistic plasma with $w \simeq 1/3$ turns into a non-relativistic gas $w \simeq 0$ when the Universe cools down. Moreover, at a given time, different disjunct components of the fluid can follow their own EoS, leading to

$$\rho = \rho_{\rm dust} + \rho_{\rm radiation} + \ldots \tag{3}$$

Here, we argue that at the very beginning of the Universe, an interaction-dominated gas whose EoS is given by $w = 1$ could have been present (whatever d.o.f. were relevant, see *e.g.* Ref. [39] and references therein). For this fluid,

$$p = \rho \to v_{\rm sound} = 1 \quad \text{and} \quad \rho_{\text{int-dom}} = \frac{A_{\text{int-dom}}}{a^6}\,. \tag{4}$$

Clearly, this component can be relevant only at a very stage of the expansion, since

(i) it decreases very fast for increasing a and

(ii) the strong interaction generating it weakens down and transforms this fluid into a more conventional component [hence, w decreases from 1 to 1/3 (or even smaller)].

The first Friedmann equation is obtained by imposing that the Hamiltonian vanishes

$$H_{\rm FRW} = p\dot a - L_{\rm FRW} = 0 \quad \text{with} \quad p = \frac{\partial L_{\rm FRW}}{\partial \dot a} - 2Ca\dot a\,. \tag{5}$$

This constraint follows from the invariance under coordinate transformations of GR. In terms of a and $\dot{a}$, Eq. (5) gives the first Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{\Lambda c^2}{3} - \frac{8\pi G}{3c^2}\rho = 0 \,. \tag{6}$$

(The second Friedmann equation is obtained by studying the equation of motion of L_{FRW} together with the continuity equation, see details in [6, 13, 14].) As a function of p and a, the Hamiltonian reads

$$H_{\mathrm{FRW}} = \frac{-1}{4C}\frac{p^2}{a} + C\left(-kc^2 a + \frac{\Lambda c^2}{3}a^3 + \frac{8\pi G}{3c^2}\rho a^3\right) . \tag{7}$$

When promoting H_{FRW} as an operator via $a \to a$ and $p \to -\mathrm{i}\hbar\partial_a$ and by choosing the ordering $\frac{p^2}{a} = \frac{1}{a}p^2$ (the following results do not depend on this choice as we will explain later), one obtains the stationary Schrödinger equation with zero energy

$$\left[-\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}a^2} + V_{\mathrm{eff}}(a)\right]\Psi(a) = 0$$
$$\text{with} \quad V_{\mathrm{eff}}(a) = 4C^2\left(kc^2a^2 - \frac{\Lambda c^2}{3}a^4 - \frac{8\pi G}{3c^2}\rho a^4\right) . \tag{8}$$

This is the famous WDW equation. It is a timeless equation: a discussion about the emergence of time can be found in the literature [3, 18, 40, 41].

Here, we are interested in the very early time evolution, therefore, we consider $\rho = \rho_{\text{int-dom}} = A_{\text{int-dom}}/a^6$ (we neglect dust and radiation as well as other contributions, which become important at later stages of the evolution). Thus, our final form for the effective potential reads

$$V_{\mathrm{eff}}(a) = 4C^2\left(kc^2a^2 - \frac{\Lambda c^2}{3}a^4\right) - \frac{\alpha\hbar^2}{a^2} \,. \tag{9}$$

The first term in the parenthesis is the one usually studied for the early quantum cosmology [8, 9, 12, 42], and the second piece represents the additional part being the main subject of the present work. It is parametrized by the dimensionless coupling α

$$\alpha = 4C^2\frac{8\pi G}{3c^2\hbar^2}A_{\text{int-dom}} = 6\pi^3\frac{c^2}{G\hbar^2}A_{\text{int-dom}} \,. \tag{10}$$

Thus, for a very small, the term $-\alpha\hbar^2/a^2$ dominates, leading to the WDW equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}a^2} + \frac{\alpha}{a^2}\right]\Psi(a) = 0 \qquad \text{for } a \text{ very small} \,. \tag{11}$$

The potential $-1/a^2$ has very remarkable properties that have been studied in detail in Refs. [19–21]. Since α in Eq. (10) is dimensionless, there is no typical energy scale in the problem: writing the eigenvalue equation for this potential, one can infer that — if it admits a bound state — there are infinite bound states, or in other terms, there is no ground state. Actually, this simple argument fails when imposing $H_{\rm FRW}$ to be self-adjoint, as we will see in the following. Independently from the self-adjointness of $H_{\rm FRW}$, Eq. (11) admits two independent solutions which in the limit of $a \to 0$ scale as $\sqrt{a}\, \mathrm{e}^{\pm \mathrm{i} g \log a}$, where $g = \sqrt{\alpha - 1/4}$. Since both of them vanish in the limit of $a \to 0$ also $\Psi(a = 0) = 0$.

This is the first important and general result of our study (which is solely a consequence of Eq. (11) and is independent of the use of self-adjointness requirement introduced below): when considering an interaction-dominated fluid that could have appeared just after the Big Bang, the wave function of the Universe fulfills the requirement postulated long ago by DeWitt to solve the problem of the Big Bang singularity.

A second property of the $1/a^2$ attractive potential concerns a superselection rule imposed on the allowed range of the variable a. As shown in [20], the quantum system is confined to $a > 0$ (or $a < 0$) [20]; in other words, there is no linear superposition of wave functions which live at $a > 0$ with the ones at $a < 0$. Hence, there is no problem with negative values of a. Next, we turn to explicit solutions in order to show that the wave function is real.

When interpreting $H_{\rm FRW}$ as an operator associated to a physical observable (*i.e.* the Hamiltonian), one has to verify that the operator is a self-adjoint operator *i.e.* it is symmetric and the domain of it coincides with the domain of its adjoint. In general, self-adjointness is a standard requirement in Quantum Mechanics [43], that we also impose in the framework of the WDW equation. It is not yet clear if this is a criterion that should be imposed to the WDW equation as well, but due to its importance in QM, it is interesting to study the implications of self-adjointness also in a quantum cosmological system. Yet, it should be mentioned that some very interesting ideas have been presented on the possibility to represent physical observables also with non-symmetric (non-Hermitian) operators, see, for instance, the seminal paper on the spectrum of non-Hermitian Hamiltonians [44].

As discussed in [19], the property of self-adjointness for the $-1/a^2$ potential is obtained by imposing a specific boundary condition on the wave function (see Eq. (73) and (81) of Ref. [19], see also Ref. [22]). For $g = \sqrt{\alpha - 1/4} \neq 0$,

$$\sqrt{a}\left[\mathrm{e}^{2\mathrm{i}g\log\frac{a}{a_0}}-1\right]\frac{\mathrm{d}\Psi^*(a)}{\mathrm{d}a}-\frac{1}{\sqrt{a}}\left[\left(\frac{1}{2}+\mathrm{i}g\right)\mathrm{e}^{2\mathrm{i}g\log\frac{a}{a_0}}-\left(\frac{1}{2}-\mathrm{i}g\right)\right]\Psi^*(a)\to 0\,, \tag{12}$$

while for $g=\sqrt{\alpha-1/4}=0$,

$$\sqrt{a}\log\frac{a}{a_0}\frac{\mathrm{d}\Psi(a)}{\mathrm{d}a}-\frac{1}{\sqrt{a}}\left[1+\frac{1}{2}\log\frac{a}{a_0}\right]\Psi(a)\to 0\,. \tag{13}$$

The general solution of Eq. (11) is a linear combination of two independent functions with, in general, complex coefficient. However, when imposing Eqs. (12) and (13) for the cases of $g\neq 0$ and $g=0$, respectively, one finds that only real wave functions are allowed. Let us discuss the case of $g=0$: the solution is $\Psi(a)=\sqrt{a}(c_1+c_2\log(a/a_0))$, where c_1, c_2 are complex numbers. Equation (13) leads to the conditions $c_1=0$, and c_2 can be chosen as a real number, thus the wave function is real. Similarly, one can compute the solution in the general case of $g\neq 0$: $\Psi(a)=\sqrt{a}a^{-\mathrm{i}g}(c_1+c_2a^{2\mathrm{i}g})$ and impose the limit of Eq. (12). Notice that both independent solutions vanish in the limit of $a\to 0$.

After a straightforward calculation, the solutions for the wave function very close to the Big Bang can be recast in the following compact form:

$$\Psi(a)=\begin{cases} N\sqrt{a}\sin\left[g\log(a/a_0)\right] & \text{for } g>0 \ (\textit{i.e.},\ \alpha>1/4)\\ N\sqrt{a}\log(a/a_0) & \text{for } g=0 \ (\textit{i.e.},\ \alpha=1/4)\\ N\sqrt{a}\left[(a/a_0)^{-\tilde{g}}-(a/a_0)^{\tilde{g}}\right] & \\ \text{for } g=\mathrm{i}\tilde{g} \text{ with } 0<\tilde{g}<1/2\ (\textit{i.e.},\ 0<\alpha<1/4) & \end{cases}. \tag{14}$$

These solutions summarize the results of this paper. As anticipated previously, $\Psi(a=0)=0$: the wave function vanishes at the Big Bang, thus offering a possible solution of the singularity problem (this is why α cannot be negative, otherwise $\Psi(a)$, although formally still given by the last line of Eq. (14), would be divergent for $a\to 0$).

Some comments on these results are in order. It is important to stress that, in presence of the $-1/a^2$ potential, there is a superselection rule that separates the quantum states at $a>0$ from the states at $a<0$ (see Ref. [30]). In this respect, there is no need to send $V(a)$ to infinity for $a<0$ in order to "exclude" this part of the space. Moreover, the fact that the wave function vanishes at $a=0$ is *not* an arbitrary boundary condition, but originates from the presence of a regular singular point for the differential equation (the stationary Schrödinger equation with zero eigenvalue) at $a=0$. Indeed, there are two independent solutions, but both of them vanish in the limit of

$a \to 0$ as one can easily verify. The requirement of self-adjointness only fixes a specific linear combination of these two independent solutions (Eq. (15)) which, in turn, allows to choose a wave function which is real.

It is interesting to notice that a length scale a_0 appears in the formulae for the wave function that was not at all present in the Hamiltonian. Indeed, while the Hamiltonian features scale invariance, the procedure of imposing the self-adjointness, similarly to the case of the regularization and renormalization procedure in quantum field theory, leads to the appearance of a new length (energy) scale. Therefore, this is an example of 'anomalous symmetry breaking' in the context of non-relativistic quantum mechanics [19]. However, the value of the constant a_0 cannot be determined similarly to the case of Λ_{QCD} in quantum chromodynamics. At first, it seems natural to set $a_0 \simeq \sqrt{\frac{\hbar G}{c^3}} = l_{\mathrm{P}} \simeq 10^{-33}$ cm, but there is actually no compelling reason for that. The eventual role of this new fundamental length a_0 and the fact $\Psi(a = a_0) = 0$ should be investigated in the future. The constant N is a normalization constant which can be always taken as real, therefore, $\Psi(a)$ is real. There is no problem for $a \to 0$, and also no problem for $a < 0$ (it never goes to $a < 0$ without imposing any additional requirement).

Finally, it is important to discuss how does the ordering of the operators affect our results. For a large class of choices, other prescriptions induce a shift of the critical value of α, but there is no qualitative change of our discussion. For instance, for $\frac{p^2}{a} \to \hat{p}\frac{1}{a}\hat{p}$, one still finds that $\Psi(a \to 0) = 0$ for each $\alpha > 0$. Upon defining $\Psi^{\mathrm{new}}(a) = \Psi(a)/\sqrt{a}$, Eq. (11) is re-obtained for a shifted α

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}a^2} + \frac{\alpha^{\mathrm{new}}}{a^2}\right]\Psi^{\mathrm{new}}(a) = 0 \tag{15}$$

with $\alpha^{\mathrm{new}} = \alpha - \frac{3}{4}$. Thus, being $\Psi^{\mathrm{new}}(a)$ real, $\Psi(a)$ is also such. For the critical value $\alpha^{\mathrm{new}} = 1/4$ (hence, $\alpha = 1$), $\Psi(a) = Na\log(a/a_0)$. (For $\alpha < 3/4$, $\alpha^{\mathrm{new}} < 0$, $\Psi^{\mathrm{new}}(a)$ is still given as the last line of Eq. (14), but for $\tilde{g} > 1/2$, hence $\Psi^{\mathrm{new}}(a \to 0)$ diverges. There is, however, no problem since this divergence is compensated by $\sqrt{a}$ as long as $\alpha > 0$. The wave function $\Psi(a)$ always vanishes at the Big Bang for positive α.)

Next, let us consider a general two-parameter expression for the operator ordering [45]

$$\frac{p^2}{a} \to \frac{1}{a^i}\,\hat{p}\,\frac{1}{a^j}\,\hat{p}\,\frac{1}{a^{1-i-j}}\,. \tag{16}$$

Our case corresponds to $i = 1$, $j = 0$ (and it is also a typical choice in various works, see *e.g.* Ref. [6]), DeWitt's choice to $i = 1/4$, $j = 1/2$ [2], Vilenkin's choice to $i = 2$, $j = -1$; finally, the parametrization $j = 1 - i$ was studied in Ref. [46]. For the general case of Eq. (16), the WDW Eq. (11) takes the form of

$$\left[\frac{1}{a^i}\frac{\mathrm{d}}{\mathrm{d}a}\frac{1}{a^j}\frac{\mathrm{d}}{\mathrm{d}a}\frac{1}{a^{1-i-j}}+\frac{\alpha}{a^3}\right]\Psi(a)=0 \qquad \text{for } a \text{ very small}\,. \tag{17}$$

Through a redefinition

$$\Psi^{\mathrm{new}}(a)=a^{-\delta}\Psi(a)\,, \tag{18}$$

Eq. (15) is obtained for δ and α^{new} given by

$$\alpha^{\mathrm{new}}=\alpha-\frac{j}{4}(2+j)\,, \qquad \delta=\frac{2-2i-j}{2}\,. \tag{19}$$

After a straightforward algebraic evaluation, one can show that, under the conditions

$$i+j\leq 1\,, \qquad i\leq 2\,, \tag{20}$$

the wave function is such that

$$\Psi(a\to 0)=0 \quad \text{for any} \quad \alpha>0\,. \tag{21}$$

In this case, all previous conclusions are still valid. (For instance, for the critical value $\alpha^{\mathrm{new}}=1/4$, $\Psi(a)=a^{\frac{3-2i-j}{2}}\log(a/a_0)$, where $\frac{3-2i-j}{2}$ is always positive in the chosen ranges for i and j. Again, $\Psi^{\mathrm{new}}(a)$ is real, then also $\Psi(a)$ is real.)

In other words, there is a continuous class of (qualitatively) equivalent choices of the operator ordering, but there is still a limitation on it. Admittedly, at the present stage, it is still not known which is the correct ordering of Eq. (16). Indeed, an intuitively appealing requirement is to impose that the factor $1/a$ is split into three different parts, each with a positive power

$$\frac{1}{a}=\frac{1}{a^i}\frac{1}{a^j}\frac{1}{a^{k=1-i-j}}\,, \tag{22}$$

with

$$0\leq i\leq 1\,, \qquad 0\leq j\leq 1\,, \qquad 0\leq 1-i-j\leq 1\,. \tag{23}$$

These ranges are a subset of those of Eq. (20), thus also for them the wave function vanishes at the Big Bang. The simplest choices are $i=1$, $j=0$, or $i=1$, $j=0$ or $i=j=0$, since they amount to put the factor $1/a$ either on the left, in the middle, or on the right of $\hat{p}^2=\hat{p}\hat{p}$ without splitting it into pieces (see Eq. (16)). These possibilities represent the vertices of a triangular area given by Eq. (23).

Let us discuss now one example in which conditions (21) are not fulfilled: by taking

$$2i + j = 3\,, \tag{24}$$

it is possible to transform Eq. (17) into an equation of the type of

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k\right]\Psi(x) = 0\,, \quad \text{where} \quad x = \log a \quad \text{and} \quad k = \alpha - (2-i)^2\,. \tag{25}$$

For this equation, the wave function does not vanish at the Big Bang. This is only one of the possible choices which, however, will not be considered in this work. On the other hand, when studying the limit of large a, one can use the WKB approximation and the ordering problem is relieved.

Notice also that the sign of k is not fixed: $\Psi(x \to -\infty)$ is divergent for $k < 0$, and is finite (but non-necessarily zero) for $k \geq 0$.

In conclusion, this analysis shows that there are areas on the parameter plane spanned by the variables i and j for which different results of the wave function at the Big Bang are obtained. In this respect, this discussion offers also a different view to look at the problem of the operator ordering in a specific case; definitely, future studies on this issue are needed. Here, we have found that a positive contribution of stiff matter ($\alpha > 0$) together with the rather intuitive requirement of Eq. (23) (or, eventually with the less restrictive but less intuitive constraint of Eq. (20)) assure that the wave function vanishes at the Big Bang. Moreover, the condition of self-adjointness imposes that the wave function is real and a natural superselection to $a \geq 0$ applies.

3. A numerical example

For larger values of a, the terms proportional to Λ and k become important. It is then instructive to study a numerical case which is reminiscent of the potentials studies in Refs. [8, 9, 12, 42], with the inclusion of the additional short-range $-1/a^2$ potential. To this end, we start by rewriting the WDW equation in natural units and in terms of the dimensionless $a' = a/\sqrt{G}$

$$-\frac{\mathrm{d}^2\Psi}{\mathrm{d}a'^2} + \tilde{V}_{\mathrm{eff}}(a')\Psi = 0\,, \quad \text{where} \quad \tilde{V}_{\mathrm{eff}}(a') = \tilde{k}a'^2 - \lambda a'^4 - \frac{\alpha}{a'^2}\,, \tag{26}$$

with dimensionless constants $\tilde{k} = \frac{9\pi^2 k}{4}$, $\lambda = \frac{3\pi^2}{4}\Lambda G$, and α already introduced in Eq. (10). In general, we could not find analytic solutions of this equation and we did not derive the conditions of self-adjointness of this new operator. However, the behaviour of the wave function close to $a' = 0$ should be anyway dominated by the $1/a'^2$ term and thus, we will again have solutions which vanish in $a' = 0$ and are real, see Fig. 1 for an illustrative example.

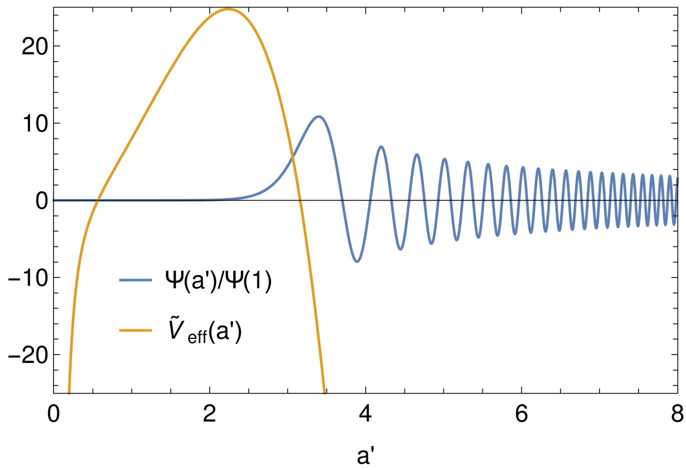

Fig. 1. Example of one specific choice of the parameters of the effective potential $\tilde{V}_{\rm eff}$ ($\tilde{k} = 10$, $\lambda = 1$, $\alpha = 1$) and the corresponding wave-function solution of the WDW equation (normalized to the value at $a' = 1$).

It is easy to prove that the wave function $\in L^2(0, +\infty)$: indeed in the limit of $a' \to \infty$, Eq. (26) admits a solution of the type of $\sqrt{a'}\,\mathrm{BesselJ}\,(\pm 1/6, \sqrt{\lambda}a'^3/3)$ which scales as $1/a'$ at infinity. We thus find a solution which interpolates between the analytical one in Eq. (14) and the one of Hartle and Hawking (real and normalizable).

Last, while in the general case no analytic solutions could be found, one can show that for $\tilde{k} = 0$ and $\alpha = 1/4$, only real solutions emerge. In fact, the general solution is a combination of $\sqrt{a'}\,\mathrm{BesselJ}\,(0, \sqrt{\lambda}a'^3/3)$ and $\sqrt{a'}\,\mathrm{BesselY}\,(0, \sqrt{\lambda}a'^3/3)$, but for small a', the former reduces to $\sqrt{a'}$, which should be rejected due to the self-adjointness requirement for small a', and the latter reduces to $\sqrt{a'}\log(a')$, which is then the only physically acceptable solution and can be taken as real.

4. Conclusions

In this work, we have studied the effect of a stiff-matter component in the very early phase of the Universe. The corresponding potential in the WDW equation is proportional to $-1/a^2$. This very interesting and atypical potential has some remarkable features for the WDW equation: the wave function vanishes at the origin, it is defined only for positive a, and it is real. Moreover, our qualitative results do not depend on a large class of choices of the operator ordering of the WDW equation. In the future, more detailed and more realistic numerical studies which take into account such an interaction-dominated gas as well as additional terms are needed. The investigation of possible phenomenological implication of an initial stiff matter on the early inflation and on present cosmological observables represents a promising outlook of the present work.

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