Published for SISSA by Springer

Received: January 5, 2024
Accepted: February 12, 2024
Published: March 7, 2024

On AdS/CFT duality in the twisted sector of string theory on $AdS_5 \times S^5/\mathbb{Z}_2$ orbifold background

Torben Skrzypek and Arkady A. Tseytlin[1]

Theoretical Physics Group, Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.

E-mail: t.skrzypek20@imperial.ac.uk, tseytlin@imperial.ac.uk

Abstract: We consider type IIB string theory on an $AdS_5 \times S^5/\mathbb{Z}_2$ orbifold background, which should be dual to 4d $\mathcal{N} = 2$ superconformal $\mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge theory with two bi-fundamental hypermultiplets. The correlator of two chiral BPS operators from the twisted sector of this quiver CFT exhibits non-trivial dependence on the 't Hooft coupling λ already in the planar limit. This dependence was recently determined using localisation and the expansion at large λ contains a subleading contribution proportional to $\zeta(3)\lambda^{-3/2}$. We address the question of how to reproduce this correction on the string theory side by starting with the $\zeta(3)\alpha'^3$ term in the type IIB string effective action. We find a regular solution of type IIB supergravity which represents a resolution of the $AdS_5 \times S^5/\mathbb{Z}_2$ orbifold and demonstrate that the relevant light twisted sector states may be identified as additional supergravity 2-form modes "wrapping" a finite 2-cycle in the resolution space. Reproducing the structure of the gauge theory result becomes more transparent in the large R-charge or BMN-like limit in which the resolved background takes a pp-wave form with the transverse space being a product of $\mathbb{R}^4$ and the Eguchi-Hanson space.

Keywords: AdS-CFT Correspondence, Spacetime Singularities, Extended Supersymmetry, Integrable Field Theories

ArXiv ePrint: 2312.13850

[1] Also at Lebedev Institute and ITMP, Moscow State University, Moscow, Russia.

Open Access, © The Authors.
Article funded by SCOAP[3].

https://doi.org/10.1007/JHEP03(2024)045

JHEP03(2024)045

Contents

1 Introduction

One of the simplest generalisations of the duality between $\mathcal{N} = 4$ SYM theory and type IIB superstring theory on $AdS_5 \times S^5$ background is based on taking its orbifold [1] (see also, e.g., [2–8]). In particular, in the case of a supersymmetric $\mathbb{Z}_2$-orbifold the duality is between the $\mathcal{N} = 2$ superconformal $\mathrm{SU}(N) \times \mathrm{SU}(N)$ quiver gauge theory (containing two vector multiplets and two bi-fundamental hypermultiplets) and string theory on $AdS_5 \times S^5/\mathbb{Z}_2$.

A way to check this duality is to compute "observables" on the gauge theory side for large N and any 't Hooft coupling λ, expand in large λ and then compare the result to the large-tension expansion of their counterparts on the string theory side. As the $\mathcal{N} = 2$ superconformal quiver arises as a $\mathbb{Z}_2$-orbifold of $\mathrm{SU}(2N)$ $\mathcal{N} = 4$ SYM [2], its "untwisted"

JHEP03(2024)045

sector observables, computed at the leading (planar) order in large N, are the same as in the SYM theory[1] with non-trivial corrections appearing at order $1/N^2$.

Recently, such leading $1/N^2$-corrections were studied for some of the simplest untwisted observables –the BPS circular Wilson loop and the free energy on the 4-sphere (see [9–11] and references therein). Localisation [12] allows one to compute their strong 't Hooft coupling expansion order by order in $1/N^2$. The comparison to string theory at order $1/N^2$ then requires knowledge of string loop corrections in $AdS_5 \times S^5/\mathbb{Z}_2$ which, unfortunately, is rather limited (see a discussion in [9, 11]).

At the same time, observables from the twisted sector may receive corrections already at the leading order in large N. These should be captured by the tree-level string theory and thus may be easier to analyse. In particular, the 2- and 3-point correlators of special twisted sector single-trace BPS operators with protected dimensions[2] may be computed at large N using localisation techniques (see, e.g., [11, 14–23]). The leading order large-λ terms in these correlators were matched [21–23] with the 6d low-energy effective action for the corresponding twisted sector string modes constructed in [5].

Our aim below is to attempt to extend this matching to the first sub-leading $\mathcal{O}\left(\lambda^{-3/2}\right)$-term in the 2-point twisted-state correlator [23]. We expect this term to be captured by the α'^3-correction to the string effective action.

1.1 Gauge theory results

If ϕ_0 and ϕ_1 denote the adjoint scalars in the two $\mathcal{N} = 2$ SU(N) vector multiplets, the simplest chiral BPS operators of dimension $\Delta = k$ belonging to untwisted and twisted sectors,respectively, are[3]

$$U_k(x) = \frac{1}{\sqrt{2k}} \left(\frac{2}{N}\right)^{\frac{k}{2}} \left(\operatorname{tr}\phi_0^k + \operatorname{tr}\phi_1^k\right), \qquad T_k(x) = \frac{1}{\sqrt{2k}} \left(\frac{2}{N}\right)^{\frac{k}{2}} \left(\operatorname{tr}\phi_0^k - \operatorname{tr}\phi_1^k\right). \tag{1.1}$$

As their conformal dimension is protected, the corresponding 2- and 3-point correlators are [23]

$$\langle \mathcal{O}_k(x_1)\bar{\mathcal{O}}_k(x_2)\rangle = \frac{\mathrm{G}_{\mathcal{O}_k}}{|x_1 - x_2|^{2k}}, \tag{1.2}$$

$$\langle \mathcal{O}_k(x_1)\mathcal{O}_l(x_2)\bar{\mathcal{O}}_{k+l}(x_3)\rangle = \frac{\mathrm{G}_{\mathcal{O}_k,\mathcal{O}_l,\bar{\mathcal{O}}_{k+l}}}{|x_1 - x_3|^{2k}|x_2 - x_3|^{2l}}, \tag{1.3}$$

where the operator $\mathcal{O}_k$ is either U_k or T_k in (1.1) and the constants G may depend on N and the 't Hooft coupling λ.

We shall focus on the leading order in large N. Then the correlators of untwisted BPS operators are the same as in $\mathcal{N} = 4$ SYM theory, i.e. they are protected by supersymmetry

[1] In general, to construct an orbifold one starts with a discrete subgroup Γ of the PSU$(2,2|4)$-symmetry. Γ-invariant states form the untwisted sector. In string theory, additional twisted sector states arise from strings that close only up to a Γ-transformation. These states correspond in the dual gauge theory to operators with an insertion of a Γ-symmetry operator.

[2] At the leading order in large N the spectrum of anomalous dimensions of non-BPS states may, in principle, be studied using integrability techniques (see [13] and references therein).

[3] We use a different normalisation than in [23].

JHEP03(2024)045

and are given by

$$\mathrm{G}_{U_k} = 1\,, \qquad\qquad \mathrm{G}_{U_k,U_l,\bar{U}_{k+l}} = \frac{1}{N}\sqrt{\frac{kl(k+l)}{2}}\,. \tag{1.4}$$

On the other hand, the correlators involving twisted sector operators are non-trivial functions of λ that can be found using localisation. Expanding in large λ one gets, in particular (see [17, 18, 20, 21] and [10, 11, 22, 23])[4]

$$\begin{aligned}\mathrm{G}_{T_k} = \frac{4\pi^2}{\lambda'}k(k-1)\left(\frac{\lambda'}{\lambda}\right)^k \Big[1+\tfrac{1}{2}(2k-1)(2k-2)(2k-3)\frac{\zeta(3)}{\lambda'^{\frac{3}{2}}} \\ -\tfrac{9}{16}(k-1)(2k-3)(2k-5)(4k^2-1)\frac{\zeta(5)}{\lambda'^{\frac{5}{2}}} \\ +\tfrac{1}{4}(k-1)(2k-1)(2k-3)(2k-5)(4k^2-20k-3)\frac{\zeta(3)^2}{\lambda'^3}+\mathcal{O}(\lambda'^{-7/2})\Big]\,,\end{aligned} \tag{1.5}$$

$$\mathrm{G}_{T_k,T_k,\bar{U}_{2k}} = \frac{k^{\frac{3}{2}}}{N}\cdot\frac{4\pi^2}{\lambda'}(k-1)^2\left(\frac{\lambda'}{\lambda}\right)^{k-\frac{1}{2}}\Big[1+\tfrac{1}{2}(2k-1)(2k-3)(2k-5)\frac{\zeta(3)}{\lambda'^{\frac{3}{2}}}+\mathcal{O}(\lambda'^{-5/2})\Big]\,, \tag{1.6}$$

$$\sqrt{\lambda'} \equiv \sqrt{\lambda}-4\log 2\,. \tag{1.7}$$

In general, these expressions depend on operator normalisations. Considering normalisation-independent ratios like $\frac{\mathrm{G}_{T_k,T_k,\bar{U}_{2k}}}{\sqrt{\mathrm{G}_{T_k}\mathrm{G}_{T_k}\mathrm{G}_{U_k}}}$, refs. [21, 22] successfully matched their leading large-λ behaviour to the predictions from the low-energy effective action [5] for the corresponding twisted sector string modes.

We shall attempt to understand the string origin of the subleading $\zeta(3)$-term in (1.5). Identifying $\frac{\sqrt{\lambda'}}{2\pi}$ with the effective string tension or $\frac{L^2}{2\pi\alpha'}$ (where L is the radius of both AdS_5 and S^5), the natural expectation is that this term should be reproduced by the first non-trivial α'^3-correction in the string effective action for the corresponding twisted sector modes in $AdS_5 \times S^5/\mathbb{Z}_2$, by analogy with the familiar $\alpha'^3\zeta(3)\mathcal{R}^4 + \ldots$ term for the standard massless string modes (cf. [24, 25]). This will require understanding how to construct a generalisation of the leading-order effective action for the twisted sector modes suggested in [5].

1.2 6d effective action for twisted sector modes

To recall, the $\Gamma = \mathbb{Z}_2$ orbifold on the string theory side acts on embedding coordinates (z_1, z_2, z_3) of $S^5 \subset \mathbb{C}^3$ as

$$\Gamma: \quad (z_1, z_2, z_3) \to (-z_1, -z_2, z_3)\,, \tag{1.8}$$

[4]Eqs. (1.5) and (1.6) are taken from [23]. The leading large-λ coefficients were found earlier in [20, 22]. The resummation in terms of λ' was introduced in [10]. Ref. [23] found the following relation between the 2- and 3-point coefficient functions:

$$\mathrm{G}_{T_k,T_k,\bar{U}_{2k}} = \frac{k}{\sqrt{2}}\left(\frac{N}{2}\right)^{k-1}(k+\lambda\partial_\lambda)\mathrm{G}_{T_k}\,.$$

JHEP03(2024)045

which breaks half of the maximal supersymmetry. The great circle of S^5 parametrised by $z_1 = z_2 = 0,\ z_3 = L\,e^{i\chi}$ is fixed under the action of Γ. The twisted sector strings, which close up to a Γ-transformation, extend around the orbifolded angles, so the lowest energy twisted states localise on the fixed circle, i.e. on the six-dimensional $AdS_5 \times S^1$ subspace of the 10d target space. They should then be described by an effective 6d action.

For the similar orbifold $\mathbb{R}^{1,5} \times \mathbb{C}^2/\Gamma$ of flat-space string theory the spectrum of twisted sector modes organises into tensor representations of 6d $\mathcal{N} = (2,0)$ supersymmetry [26] (see appendix A). The low-energy effective action for the light twisted modes could be reconstructed from correlators of the corresponding vertex operators (cf. [27]).

Ref. [26] provided an alternative interpretation of the twisted sector modes in terms of a resolution (or "blow-up") of the orbifold singularity. One may cut out a ball of size a around the singularity and glue in a smooth manifold, such that the total space $\mathcal{M}^4$ is asymptotically locally Euclidean with global $\mathbb{C}^2/\Gamma$ structure. For $\Gamma = \mathbb{Z}_2$ the smooth manifold is the Eguchi-Hanson space [28]. This resolution features three moduli and a non-trivial 2-cycle over which one may integrate the massless 2- and 4-form fields of type IIB supergravity generating extra light modes. In the limit $a \to 0$, the resolved space $\mathcal{M}^4$ approaches the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ with the moduli and extra modes (now localised at the singularity) to be taken into account. It turns out that this procedure reproduces the lightest states in the twisted sector spectrum as found directly from string theory. This suggests that one can access the light twisted sector modes using the 10d supergravity action expanded near the curved background representing a resolution of the orbifold.

In [5] it was suggested that this logic may apply also to the curved-space orbifold $S^5/\mathbb{Z}_2$. Close to the fixed circle, one may approximate $S^5/\mathbb{Z}_2$ by $\mathbb{C}^2/\mathbb{Z}_2 \times S^1$ and thus expect to get the same effective action for light twisted sector modes in terms of 6d tensor multiplets as in the flat-space case, corrected by contributions of the curvature and the F_5-flux of the $AdS_5 \times S^5/\mathbb{Z}_2$ background. One may then expand in Fourier modes on S^1, generating towers of fields in AdS_5 with masses labelled by the mode number ("KK level") k. These can then be put into correspondence with the dual BPS operators in the twisted sector of the gauge theory and turn out to have the required spectrum of conformal dimensions [5].

In particular, the twisted sector operator T_k in (1.1) is expected to be dual to a 5d mode representing a combination of B_2 and C_2 fields integrated over the 2-cycle the orbifold resolution. The relevant terms in the 10d type IIB supergravity action are (ignoring dependence on the dilaton and RR scalar)

$$S_{10} = -\frac{1}{2\kappa^2}\int d^{10}X\,\sqrt{-g}\left(\frac{1}{2\cdot 3!}H_3^2 + \frac{1}{2\cdot 3!}F_3^2 + \frac{1}{4\cdot 5!}\tilde{F}_5^2\right) - \frac{1}{4\kappa^2}\int B_2 \wedge F_3 \wedge F_5\,, \tag{1.9}$$

$$H_3 = dB_2\,, \qquad F_3 = dC_2\,, \qquad \tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3\,. \tag{1.10}$$

Let us set

$$B_2 = \beta(x,\chi)\,\Theta\,, \qquad C_2 = \gamma(x,\chi)\,\Theta\,, \tag{1.11}$$

where Θ is the anti-self-dual 2-form on the resolution of $\mathbb{C}^2/\mathbb{Z}_2$ [28] with a normalised integral over the resolution 2-cycle, χ is the fixed S^1 coordinate and x^i are AdS_5 coordinates. Using

JHEP03(2024)045

that $F_5 = 4\,(\mathrm{vol}_{AdS_5} + \mathrm{vol}_{S^5/\mathbb{Z}_2})$ we then arrive at the following effective 6d action for the fields β and γ [5, 7]

$$S_6 \sim \frac{1}{2} \int_{AdS_5\times S^1} \mathrm{d}^5x\, \mathrm{d}\chi\, \sqrt{-g_6}\, \Big[(\partial_i\beta)^2 + (\partial_\chi\beta)^2 + (\partial_i\gamma)^2 + (\partial_\chi\gamma)^2 - 8\,\beta\,\partial_\chi\gamma\Big]\,. \tag{1.12}$$

Expanding the fields β and γ in Fourier modes in χ (i.e. $\beta = \sum_k e^{ik\chi}\beta_k(x)$, $\gamma = \sum_k e^{ik\chi}\gamma_k(x)$) we get the following kinetic operator matrix for the $\beta_k(x)$ and $\gamma_k(x)$ fields on AdS_5

$$\begin{pmatrix} \nabla^2_{AdS_5} - k^2 & -4ik \\ 4ik & \nabla^2_{AdS_5} - k^2 \end{pmatrix}\,. \tag{1.13}$$

Diagonalising it gives the following masses

$$m^2_\pm = \Delta_\pm(\Delta_\pm - 4) = k(k \pm 4)\,. \tag{1.14}$$

The dual twisted-sector operators are then expected to be T_k in (1.1) with dimension $\Delta_- = k$ and the operator with dimension $\Delta_+ = k + 4$ represented by [5]

$$\mathcal{O}_F = \mathrm{tr}\left[\phi_0^k\,(F_0^2 + iF_0\tilde{F}_0)\right] - \mathrm{tr}\left[\phi_1^k\,(F_1^2 + iF_1\tilde{F}_1)\right]\,, \tag{1.15}$$

where F_0 and F_1 are the gauge fields from the two $\mathcal{N} = 2$ SU(N) gauge multiplets.

This identification relies heavily on supersymmetry and is supported by the successful matching of the leading order term in (1.5) demonstrated in [22]. To extend this matching to subleading order we need to put the above derivation of the effective action (1.12) on a firmer footing and then find α'^3-corrections to it using as an input the known structure of α'^3-terms in the type IIB string effective action.

1.3 Structure of the paper

We start in section 2 with presenting a regular solution of type IIB supergravity (depending on an extra parameter a) that is a resolution of the $S^5/\mathbb{Z}_2$ orbifold. It has $S^3 \times S^2$ topology and thus admits a non-trivial 2-cycle.

In section 3 we analyse solutions for the B_2 and C_2 fields in this background and identify the 10d analogues of the twisted sector states localised on the $AdS_5 \times S^1$ subspace, thus supporting the logic behind the derivation of the action (1.12) in [5].

In section 4 we generalise the discussion to the case when the starting point is not the supergravity action (1.9) but the type IIB string effective action including α'^3-corrections. We describe a strategy for reproducing the subleading $\lambda^{-3/2}$-term in the gauge theory result (1.5) for the two-point correlator.

In section 5 we focus specifically on reproducing the k^3 part of the $\lambda^{-3/2}$-term in (1.5) which dominates when the R-charge k is large. We note that the function (1.5) admits a regular BMN-like limit, i.e. for large k and λ with $\nu = \frac{k}{\sqrt{\lambda}}$ being fixed. This suggests to focus on the pp-wave limit of the resolved orbifold background which turns out to have the Eguchi-Hanson space as part of its "transverse" space. We suggest a candidate structure that should be part of the $\alpha'^3\zeta(3)(R^4 + \ldots)$ superinvariant and that may be responsible for reproducing the $\zeta(3)k^3\lambda^{-3/2}$-term.

JHEP03(2024)045

Some concluding remarks are made in section 6. In appendix A we review the spectrum of superstring theory on a flat-space orbifold. In appendix B we summarise some information about the structure of the leading α'^3-corrections to the tree-level type IIB supergravity action. Details of calculations in section 3 are presented in appendix C. Appendix D contains expressions for the Weyl tensor of the resolved orbifold background.

2 Resolution of the $S^5/\mathbb{Z}_2$ orbifold

In this section we find a particular resolution of the $AdS_5 \times S^5/\mathbb{Z}_2$ orbifold as a regular solution of type IIB supergravity depending on an extra "resolution parameter" a. In section 3 we expand near this background and identify the lightest modes corresponding to the twisted sector states. This suggests how to construct their low-energy effective action in the framework of type IIB supergravity.

To motivate the Ansatz for the resolved background we first review the resolution of the flat $\mathbb{C}^2/\mathbb{Z}_2$ orbifold represented by the Eguchi-Hanson (EH) space [28].

2.1 Eguchi-Hanson space as resolution of $\mathbb{C}^2/\mathbb{Z}_2$

The procedure of blowing up singularities usually involves glueing a projective $\mathbb{CP}^n$ space to the singularity and identifying appropriate subspaces. In the $\mathbb{C}^2/\mathbb{Z}_2$ case the orbifolding acts on the two complex coordinates as $(z_1, z_2) \to (-z_1, -z_2)$, resulting in a singularity at $(0, 0)$. Let us choose a parametrisation

$$z_1 = r \cos\tfrac{\theta}{2}\, e^{\frac{i}{2}(\psi+\phi)}, \qquad z_2 = r \sin\tfrac{\theta}{2}\, e^{\frac{i}{2}(\psi-\phi)}. \tag{2.1}$$

Then

$$\mathrm{d}s^2 = |dz_1|^2 + |dz_2|^2 = \mathrm{d}r^2 + r^2(\sigma_x^2 + \sigma_y^2 + \sigma_z^2), \tag{2.2}$$

where we introduced the SU(2) Cartan forms

$$\begin{aligned} &\sigma_x = \tfrac{1}{2}(\sin\psi\, \mathrm{d}\theta - \sin\theta\cos\psi\, \mathrm{d}\phi), \quad \sigma_y = \tfrac{1}{2}(-\cos\psi\, \mathrm{d}\theta - \sin\theta\sin\psi\, \mathrm{d}\phi), \quad \sigma_z = \tfrac{1}{2}(\mathrm{d}\psi + \cos\theta\, \mathrm{d}\phi), \\ &\mathrm{d}\sigma_x = 2\sigma_y\wedge\sigma_z, \qquad \mathrm{d}\sigma_y = 2\sigma_z\wedge\sigma_x, \qquad \mathrm{d}\sigma_z = 2\sigma_x\wedge\sigma_y. \end{aligned} \tag{2.3}$$

Here $\sigma_x^2 + \sigma_y^2 + \sigma_z^2$ represents the metric of S^3, parametrised as Hopf fibration over S^2 with $\mathrm{d}s^2_{S^2} = 4(\sigma_x^2 + \sigma_y^2) = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2$ and $\theta \in [0, \pi], \phi \in [0, 2\pi]$. For $\psi \in [0, 4\pi]$ it would cover the full S^3, but for $\psi \in [0, 2\pi]$ it only covers $S^3/\mathbb{Z}_2$, so that (2.2) represents the metric of $\mathbb{C}^2/\mathbb{Z}_2$ with a singularity at $r = 0$.

The resolution of the singularity is achieved by replacing (2.2) with the EH metric containing a function $V_0(r)$, which breaks the SO(4) symmetry to SO(3)

$$\mathrm{d}s^2_{\rm EH} = V_0(r)^{-1}\mathrm{d}r^2 + r^2\Big[\sigma_x^2 + \sigma_y^2 + V_0(r)\,\sigma_z^2\Big], \qquad V_0(r) = 1 - \frac{a^4}{r^4}. \tag{2.4}$$

This metric is Ricci-flat and its curvature form is self-dual.[5] Here we restrict r to the interval $r \in [a, \infty)$. For $r \to \infty$ we recover the $\mathbb{C}^2/\mathbb{Z}_2$ space asymptotically (i.e. EH is an ALE space).

[5]EH space is also a hyperkähler manifold, i.e. admits three complex structures that form an SU(2) triplet. In the 10d supergravity context this guarantees preservation of 16 supercharges and that a dimensional reduction on this space leads to 6d $\mathcal{N} = (2, 0)$ supergravity.

JHEP03(2024)045

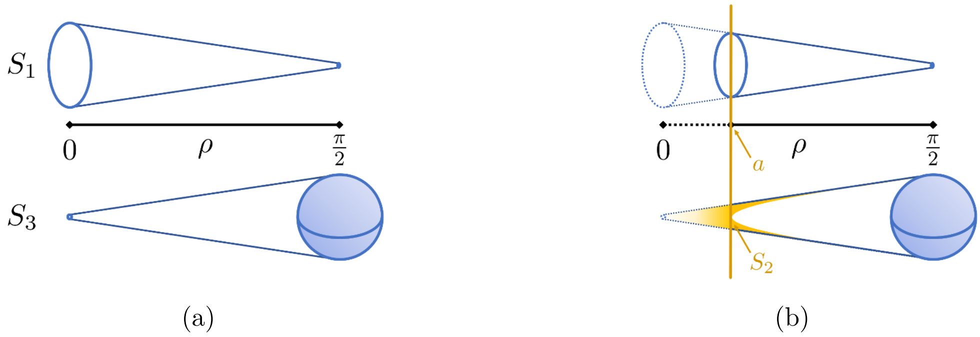

Figure 1. *(a)* Representation of S^5 as a fibration of $S^1 \times S^3$ over an interval. *(b)* Schematic representation of the resolution $\mathcal{M}^5$ (2.17) of the $S^5/\mathbb{Z}_2$ orbifold.

For $r \to a$ this space is regular as one can see by changing the coordinate $r \to u$ as

$$u^2 = r^2 V_0(r)\,. \tag{2.5}$$

Expanding (2.4) around the apparent singularity at $r = a$ or $u = 0$ yields

$$\mathrm{d}s^2_{\mathrm{EH}}\Big|_{u\to 0} \to \frac{1}{4}\Big[\mathrm{d}u^2 + a^2\,\mathrm{d}s^2_{S^2} + u^2(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2\Big]\,. \tag{2.6}$$

For fixed θ and ϕ and with $\psi \in [0, 2\pi]$ the point at $u = 0$ is just a coordinate singularity: the local geometry near $u = 0$ is that of an $\mathbb{R}^2$-bundle over S^2. The orbifold singularity of $\mathbb{C}^2/\mathbb{Z}_2$ is recovered in the limit $a \to 0$.

2.2 $S^5/\mathbb{Z}_2$ orbifold and its resolution $\mathcal{M}^5$

Let us represent the unit-radius S^5 metric as that of $S^1 \times S^3$ fibered over an interval $\rho \in [0, \frac{\pi}{2}]$

$$\mathrm{d}s^2_{S^5} = \mathrm{d}\rho^2 + \cos^2\rho\,\mathrm{d}s^2_{S^1} + \sin^2\rho\,\mathrm{d}s^2_{S^3} = \mathrm{d}\rho^2 + \cos^2\rho\,\mathrm{d}\chi^2 + \sin^2\rho\Big(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\Big)\,. \tag{2.7}$$

Here we parametrised S^1 by $\chi \in [0, 2\pi]$ and S^3 as in (2.2) with $\psi \in [0, 4\pi]$. A schematic picture of this parametrisation is given in figure 1a. At $\rho = 0$ (the "north pole") S^3 shrinks to a point and we recover a local $\mathbb{R}^4 \times S^1$ geometry. At the "south pole" $\rho = \frac{\pi}{2}$ where S^1 shrinks to a point the local geometry is $\mathbb{R}^2 \times S^3$.

To get the metric of the $S^5/\mathbb{Z}_2$ orbifold we restrict ψ to $[0, 2\pi]$. This space is then singular at $\rho = 0$ with S^1_χ at the north pole being the fixed circle. Thus, $S^5/\mathbb{Z}_2$ looks like $S^1 \times S^3/\mathbb{Z}_2$ fibered over the ρ-interval.

Motivated by the comparison of the S^3 parts of (2.2) and (2.7), and by the Eguchi-Hanson resolution (2.4) of the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold let us consider the following Ansatz for a resolution $\mathcal{M}^5$ of $S^5/\mathbb{Z}_2$

$$ds^2_{\mathcal{M}^5} = V(\rho)^{-1}\mathrm{d}\rho^2 + \cos^2\rho\,\mathrm{d}\tilde{\chi}^2 + \sin^2\rho\left[\sigma_x^2 + \sigma_y^2 + V(\rho)\,\tilde{\sigma}_z^2\right]. \tag{2.8}$$

Here we "deformed" (2.7) by introducing a function $V(\rho)$ and defined

$$\tilde{\chi} = p\,\chi\,, \qquad\qquad \tilde{\sigma}_z = q\,\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\,, \tag{2.9}$$

JHEP03(2024)045

where p and q are two constants which are to be fixed momentarily. To provide a resolution of the orbifold singularity, the function $V(\rho)$ should behave as the Eguchi-Hanson one (2.4) near $\rho \sim 0$, i.e. $V(\rho) \sim 1 - (\frac{a}{\rho})^4$, and it should approach a constant for $\rho \sim \frac{\pi}{2}$. This is accomplished by the following choice

$$V(\rho) = 1 - \left(\frac{\sin a}{\sin \rho}\right)^4, \qquad a \in \left(0, \frac{\pi}{2}\right). \tag{2.10}$$

Remarkably, (2.8) with (2.10) satisfies the same 5d Einstein equation as the undeformed metric (2.7):

$$\mathcal{R}_{ab} = 4 g_{ab}, \qquad \mathcal{R} = 20. \tag{2.11}$$

Let us note that although we can use this metric (2.8) as a base for a Ricci-flat 6d cone, there is no associated Kähler form, so this resolution is not a Sasaki-Einstein space and breaks supersymmetry completely.

Let us demonstrate that $\mathcal{M}^5$ is indeed non-singular despite an apparent singularity at $\rho = a$. Applying the following coordinate transformation from ρ to u (cf. (2.5))

$$u^2 = \sin^2 \rho \, V(\rho), \tag{2.12}$$

implies that for $u \to 0$ (or $\rho \to a$) the metric becomes

$$\mathrm{d}s^2_{\mathcal{M}^5}\Big|_{u\to 0} \to \frac{1}{4}\left[\frac{1}{\cos^2 a}\,\mathrm{d}u^2 + 4\cos^2 a\,\mathrm{d}\tilde{\chi}^2 + \sin^2 a\,\mathrm{d}s^2_{S^2} + u^2(q\,\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2\right]. \tag{2.13}$$

For this to be smooth at $u = 0$ we have to choose

$$q = (\cos a)^{-1}. \tag{2.14}$$

In this case the original orbifold singularity at $\rho = 0$ is removed like in the above EH example.

To study the vicinity of the "south pole" at $\rho = \frac{\pi}{2}$ let us introduce the coordinate $v = \frac{\pi}{2} - \rho$ so that the metric becomes

$$ds^2_{\mathcal{M}^5}\Big|_{\rho\to\frac{\pi}{2}} \to \frac{1}{1-\sin^4 a}dv^2 + v^2 p^2\,\mathrm{d}\chi^2 + \frac{1}{4}\left[\mathrm{d}s^2_{S^2} + (1-\sin^4 a)\left(\frac{1}{\cos a}\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\right)^2\right]. \tag{2.15}$$

For this to be smooth at $v = 0$ we need to choose

$$p = (1 - \sin^4 a)^{-1/2}. \tag{2.16}$$

Note that both q (2.14) and p become 1 at $a = 0$.

To summarise, using (2.8), (2.10), (2.14), (2.16) we thus find a smooth resolution $\mathcal{M}^5$ of $S^5/\mathbb{Z}_2$ which is an Einstein space with the metric

$$\mathrm{d}s^2_{\mathcal{M}^5} = V(\rho)^{-1}\,\mathrm{d}\rho^2 + \frac{\cos^2\rho}{1-\sin^4 a}\,\mathrm{d}\chi^2 + \frac{1}{4}\sin^2\rho\left[\mathrm{d}s^2_{S^2} + V(\rho)\left(\frac{1}{\cos a}\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\right)^2\right], \tag{2.17}$$

$$\mathrm{d}s^2_{S^2} = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2, \quad \rho \in [a, \frac{\pi}{2}], \quad \chi \in [0, 2\pi], \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi], \quad \psi \in [0, 2\pi].$$

JHEP03(2024)045

As an illustration of regularity of (2.17) let us note that the square of its curvature is given by

$$\mathcal{R}_{abcd}\mathcal{R}^{abcd} = 40 + 24\frac{\sin^8 a}{\sin^8\rho}\left(3 + 16\frac{\cos^2\rho}{\sin^4\rho}\right), \tag{2.18}$$

which is finite for $\rho \in (a, \frac{\pi}{2}]$. If we consider the limit $\rho \to a$ and then $a \to 0$ we recover the orbifold singularity as $\mathcal{R}_{abcd}\mathcal{R}^{abcd} \to 384\, a^{-4}$.

The volume form and the volume of $\mathcal{M}^5$ are given by

$$\begin{aligned} \mathrm{vol}_{\mathcal{M}^5} &= \frac{\sqrt{1+\sin^2 a}}{8\,(1-\sin^4 a)} \sin^3\rho\cos\rho\sin\theta\, \mathrm{d}\rho\wedge\mathrm{d}\chi\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\mathrm{d}\psi\,, \\ \mathrm{V}_{\mathcal{M}^5} &= \int_{\mathcal{M}^5}\mathrm{vol}_{\mathcal{M}^5} = \frac{\pi^3}{2}\sqrt{1+\sin^2 a}\,. \end{aligned} \tag{2.19}$$

In the limit $a \to 0$ we recover the volume of the orbifold $\mathrm{V}_{S^5/\mathbb{Z}_2} = \frac{1}{2}\mathrm{V}_{S^5} = \frac{\pi^3}{2}$.

Let us comment also on the limit $a \to \frac{\pi}{2}$. In terms of the coordinate $v = \frac{\pi}{2} - \rho \in [0, b]$ in (2.15) where $b = \frac{\pi}{2} - a$ we can write (2.17) as

$$\mathrm{d}s^2_{\mathcal{M}^5} = V(v)^{-1}\mathrm{d}v^2 + \frac{\sin^2 v}{1-\cos^4 b}\,\mathrm{d}\chi^2 + \frac{1}{4}\cos^2 v\left[\mathrm{d}s^2_{S^2} + 4V(v)\tilde{\sigma}_z^2\right], \quad V(v) = 1 - \left(\frac{\cos b}{\cos v}\right)^4. \tag{2.20}$$

Setting $v = b\cos\eta$ and then taking the limit $b \to 0$ we get

$$\mathrm{d}s^2_{\mathcal{M}^5}\Big|_{a=\frac{\pi}{2}} = \frac{1}{4}\Big[2\mathrm{d}\eta^2 + 2\cos^2\eta\,\mathrm{d}\chi^2 + \mathrm{d}s^2_{S^2} + 2\sin^2\eta\,\mathrm{d}\psi^2\Big], \tag{2.21}$$

which is the metric of $S^3 \times S^2$. Thus for a changing from 0 to $\frac{\pi}{2}$ the metric (2.17) of $\mathcal{M}^5$ interpolates between $S^5/\mathbb{Z}_2$ and $S^3 \times S^2$ and is in general non-singular with $S^3 \times S^2$ topology (cf. figure 1b).

In view of (2.11), we conclude that the resolved space $AdS_5 \times \mathcal{M}^5$ gives a regular solution of type IIB supergravity if we supplement it by a direct generalisation of the standard RR 5-form

$$F_5 = 4\,(\mathrm{vol}_{AdS_5} + \mathrm{vol}_{\mathcal{M}^5})\,. \tag{2.22}$$

3 Twisted sector modes from 2-form fields in $AdS_5 \times \mathcal{M}^5$

As discussed in the introduction, the twisted sector operator T_k in (1.1) should be dual to a scalar mode in $AdS_5 \times S^5/\mathbb{Z}_2$ localised on $AdS_5 \times S^1$ and having k units of momentum along the S^1. Motivated by the idea of identifying twisted sector fields using a blow-up procedure of orbifold singularities [26], we expect this mode to originate from a combination of fluctuations of B_2 and C_2 fields in (1.9) that are "wrapping" the blow-up 2-cycle in the resolved space $AdS_5 \times \mathcal{M}^5$ (cf. (1.11)).

To justify this picture, we first study the type IIB supergravity equations for fluctuations of B_2 and C_2 fields in the $AdS_5 \times \mathcal{M}^5$ background. We identify modes which appear due to the resolution and correspond to the lightest twisted sector states in the orbifold limit $a \to 0$. We then discuss the construction of their 6d effective action, which should reproduce the action (1.12) of [5].

JHEP03(2024)045

3.1 Solution of supergravity equations for 2-form fields

The type IIB supergravity equations for the B_2 and C_2 fields following from (1.9) may be written in equivalent real and complex forms as

$$\mathrm{d} * F_3 - F_5 \wedge H_3 = 0\,, \qquad \mathrm{d} * H_3 + F_5 \wedge F_3 = 0\,, \tag{3.1}$$

$$\mathrm{d} * G_3 + iF_5 \wedge G_3 = 0\,, \qquad G_3 \equiv F_3 + iH_3 = \mathrm{d}A_2\,, \qquad A_2 = C_2 + iB_2\,. \tag{3.2}$$

We study these equations in the background of the $AdS_5 \times \mathcal{M}^5$ metric (2.17) (of radius $L = 1$) supported by the 5-form flux given in (2.22).

Assuming 5+5 separation of coordinates, let us choose the following Ansatz for the potential A_2 in (3.2)[6]

$$A_2(x, y) = \varphi(x)\,\Omega(y)\,, \tag{3.3}$$

where $\Omega(y)$ is a complex 2-form on $\mathcal{M}^5$ and $\varphi(x)$ is a real scalar function solving a free massive scalar equation in AdS_5

$$\nabla^2_{AdS_5}\varphi = m^2\varphi\,. \tag{3.4}$$

Then (3.1) is solved if $\Omega(y)$ on $\mathcal{M}^5$ satisfies[7]

$$\mathrm{d} \star \Omega = 0\,, \qquad \mathrm{d} \star \mathrm{d}\Omega - 4i\mathrm{d}\Omega + m^2 \star \Omega = 0\,. \tag{3.5}$$

These equations are satisfied if we express Ω in terms of a closed 3-form ω as

$$\Omega = \star\omega\,, \qquad \mathrm{d}\omega = 0\,, \qquad \Big[(\mathrm{d}\star)^2 - 4i(\mathrm{d}\star) + m^2\Big]\omega = 0\,. \tag{3.6}$$

Equivalently, the closed 3-form ω on $\mathcal{M}^5$ should satisfy

$$\mathrm{d} \star \omega = -iM\omega\,, \qquad M(M+4) = m^2\,. \tag{3.7}$$

The complex conjugate field $\bar{A}_2 = \varphi(x)\bar{\Omega}(y)$ should solve the complex conjugate equations, so that $\bar{\Omega}(y)$ is expressed in terms of $\bar{\omega}$ as (we assume that m is real)

$$\mathrm{d} \star \bar{\omega} = -iM\bar{\omega}\,, \qquad M(M-4) = m^2\,. \tag{3.8}$$

The metric (2.17) of $\mathcal{M}^5$ has an isometry along the S^1 parametrised by χ so we may expand in Fourier modes

$$\omega(y) = \sum_{k=-\infty}^{\infty} e^{ik\chi}\,\omega_k(\rho, \theta, \phi, \psi)\,, \tag{3.9}$$

and, for the time being, focus on one particular mode ω_k with positive k.[8]

---

[6] We use coordinates x^i with indices $(i, j, \ldots)$ for AdS_5 and y^a with indices $(a, b, \ldots)$ for $\mathcal{M}^5$. In general, we shall use capital latin indices for coordinates of a generic 10d spacetime. Small latin indices from the beginning of the alphabet $(a, b, c, \ldots)$ label coordinates of internal 5-space and latin indices from the middle of the alphabet $(i, j, k, \ldots)$ label coordinates of non-compact 5-space or AdS_5.

[7] Here and below $\star$ stands for the Hodge-dual form in $\mathcal{M}^5$.

[8] Note that ω and $\bar{\omega}$ solve the same differential equation (3.7), so we may consider only ω_k with $k > 0$ and relegate $k < 0$ to modes of $\bar{\omega}$.

JHEP03(2024)045

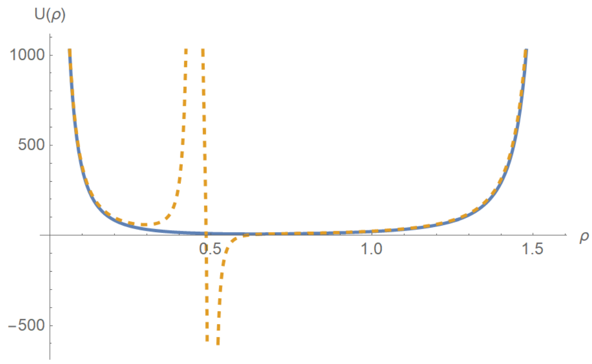

Figure 2. Potential $\mathrm{U}(\rho)$ for $k = -M = 3$ and (i) $a = 0$ (solid curve) and (ii) $a = \frac{1}{2}$ (dotted curve).

As we are interested in the analogues of the twisted sector modes, which are to be localised near the north pole $\rho = 0$ of the orbifold we may restrict our attention to the lowest harmonics on the deformed S^3 part of (2.17). Explicitly, we may assume no dependence on ψ and a spherical symmetry in (θ, ϕ). A general Ansatz for such ω then takes the form[9]

$$\omega_k = f_1(\rho)\, \mathrm{d}\rho \wedge \mathrm{d}\tilde{\chi} \wedge \tilde{\sigma}_z + f_2(\rho)\, \mathrm{d}\rho \wedge \sigma_x \wedge \sigma_y + f_3(\rho)\, \mathrm{d}\tilde{\chi} \wedge \sigma_x \wedge \sigma_y \,, \tag{3.10}$$

where $\mathrm{d}\tilde{\chi}$ and $\tilde{\sigma}_z$ were defined in (2.9), (2.14), (2.16). Then the equation (3.7) relates f_1 and f_2 to f_3 with the latter being subject to a 2nd order ordinary differential equation (see appendix C). This equation can be put into the Schrödinger-type form

$$\tilde{f}_3''(\rho) - \mathrm{U}(\rho)\ \tilde{f}_3(\rho) = 0\,, \tag{3.11}$$

where $\tilde{f}_3$ is related to f_3 via rescaling by a function of ρ (see (C.10)). The potential $\mathrm{U}(\rho)$ is depicted in figure 2 for some special values of the parameters.

At $a = 0$ the potential has the form of a well between two second-order poles and the equation (3.11) can be solved explicitly, generating a discrete spectrum of solutions. In the case of

$$M = -k\,, \tag{3.12}$$

we find for the 2-form Ω in (3.6), (3.9), (3.10)

$$\Omega = e^{ik\chi}\,\Omega_k\,, \quad \Omega_k = \frac{\cos^k\rho}{\sin^2\rho}\left[(2+k\sin^2\rho)\,\sigma_x\wedge\sigma_y + ik\sin^2\rho\,\mathrm{d}\chi\wedge\sigma_z - \frac{2\cos^2\rho+k\sin^2\rho}{\cos\rho\sin\rho}\,\mathrm{d}\rho\wedge\sigma_z\right]. \tag{3.13}$$

This solution is not normalisable as it diverges near $\rho \to 0$. It does not have an analogue in the KK spectrum [29] of the usual S^5 compactification, which starts with the first normalisable solution at $M = k + 2$ (see appendix C).

For $a > 0$ the potential $\mathrm{U}(\rho)$ develops a pole at $\rho = a$ (cf. figure 2):

$$\mathrm{U}(a+\epsilon) = -\frac{1}{4\epsilon^2} - \frac{\kappa\mu}{\epsilon} + \frac{\mu^2}{4} + \mathcal{O}(\epsilon)\,, \qquad \kappa \sim \frac{1}{2\sqrt{5}} + \mathcal{O}\big(a^2\big)\,, \qquad \mu \sim \frac{16\sqrt{5}}{k^2a^3} + \mathcal{O}\big(a^{-1}\big)\,, \tag{3.14}$$

[9]We do not include $f_0(\rho)\,\sigma_x \wedge \sigma_y \wedge \tilde{\sigma}_z$ as this form is not closed.

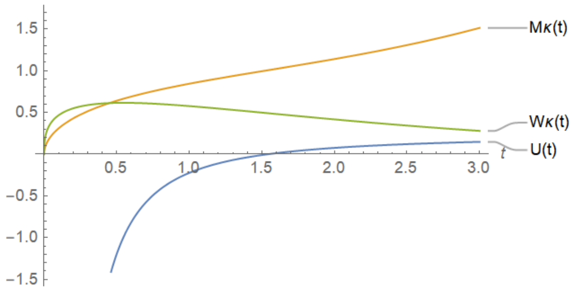

Figure 3. Whittaker equation potential and solutions for small a.

where we gave the small-a expansions of the coefficient functions $\kappa(a,k,M)$ and $\mu(a,k,M)$ entering the expression for $\mathrm{U}(\rho)$. Near $\rho = a$ the equation (3.11) can be transformed into the standard Whittaker equation form by a rescaling $\epsilon = \rho - a \to \mu^{-1}t$ (here the derivatives are w.r.t. t)

$$\tilde{f}_3''(t) + \left(-\frac{1}{4} + \frac{\kappa}{t} + \frac{1}{4t^2}\right)\tilde{f}_3(t) = 0\,. \tag{3.15}$$

This equation is solved by the Whittaker functions $\mathrm{M}_{\kappa,0}(t)$ and $\mathrm{W}_{\kappa,0}(t)$, which may be expressed in terms of confluent hypergeometric functions. We plot these functions and the potential $\mathrm{U}(t)$ of (3.15) in figure 3 . Both functions vanish asymptotically at $\rho \to a$, i.e. the resolution provides a regularisation of the $\rho = 0$ singularity present in the orbifold limit.

In general, as the potential $\mathrm{U}(\rho)$ in (3.11) is smooth, there are two solutions of (3.11). Close to $\rho = a$ they look like the Whittaker functions, while away from this point they look like the solutions found in the $a = 0$ case (cf. (3.13)). The full solution may be constructed numerically.

This implies that a full solution generalizing the one in (3.13) found for $a = 0$ (which was divergent at $\rho = 0$ and thus potentially discarded as non-normalisable) is regular and normalisable for $a \neq 0$. In the limit $a \to 0$ such solutions appear to be localised near $\rho = 0$. This suggests that taking the limit $a \to 0$ we need to keep these modes in the spectrum, and they should represent the light twisted sector states.

3.2 Effective action for twisted sector modes

The localised modes of 2-form fields discussed above propagate on the $AdS_5 \times S^1$ subspace at the north pole of $AdS_5 \times \mathcal{M}^5$. Expanding in modes of $\mathcal{M}^5$ in general yields a KK tower of massive fields in AdS_5 (cf. (3.4)). We focus on the twisted sector solutions corresponding to (3.13) for which

$$M = \pm k\,, \qquad m^2 = k^2 \pm 4k\,, \tag{3.16}$$

where k is the S^1 mode number. As the "transverse" part $\Omega(y)$ of the field $A_2 = C_2 + iB_2$ in (3.3) depends on k we label the corresponding AdS_5 part as $\varphi_k(x)$.

We may then reconstruct the corresponding 5d effective action for $\varphi_k(x)$ by starting with the 10d supergravity action in (1.9). Written in terms of G_3 defined in (3.2) the

JHEP03(2024)045

relevant part reads

$$S_{10} = -\frac{1}{4\kappa^2}\int \left[\bar{G}_3 \wedge *G_3 + i\bar{A}_2 \wedge G_3 \wedge F_5\right]. \tag{3.17}$$

Inserting here the Ansatz (3.3) with Ω given by (3.6), (3.7) and integrating over $\mathcal{M}^5$ we get the effective action for φ_k propagating in AdS_5

$$S_5 \sim \tfrac{1}{2}\mathcal{V}\int \mathrm{d}^5x\sqrt{-g_{AdS_5}}\;\varphi_k\,(\nabla^2_{AdS_5} - m^2)\,\varphi_k\,, \qquad m^2 = k(k-4)\,, \tag{3.18}$$

$$\mathcal{V} \equiv \int_{\mathcal{M}^5} \bar{\omega}_k \wedge \star\omega_k\,. \tag{3.19}$$

For every real φ_k there is also mode φ_{-k} arising from $\bar{\omega}$ (3.8) with $m^2 = k(k+4)$. For the Ansatz (3.10) the prefactor (3.19) becomes

$$\mathcal{V} = \tfrac{1}{8}\int \sin\theta\,\mathrm{d}\chi\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\psi \int_a^{\frac{\pi}{2}} \mathrm{d}\rho\,\frac{\sin^2\rho|f_1(\rho)|^2 + \cos^2\rho\,V(\rho)|f_2(\rho)|^2 + |f_3(\rho)|^2}{\cos\rho\sin\rho}\,. \tag{3.20}$$

Inserting the solution (3.13) corresponding to the twisted sector mode of interest we find for $a \to 0$

$$\mathcal{V} \sim \frac{4\pi^3}{a^4} + \mathcal{O}\!\left(a^{-2}\right). \tag{3.21}$$

This suggests that we should rescale φ_k by a^2, getting $\mathcal{V} \to a^4\mathcal{V}$ and thus a finite action in the $a \to 0$ limit.[10]

Alternatively, instead of starting with the 5+5 Ansatz (3.3) we may follow the idea of [5] and attempt to construct an effective 6d action for twisted modes localised on $AdS_5 \times S^1$. Locally, near $\rho = 0$, we may approximate $S^5/\mathbb{Z}_2$ as $\mathbb{C}^2/\mathbb{Z}_2 \times S^1$, so that the S^1 dependence factorises. One may try to justify this approach by using the solution on the resolved manifold $\mathcal{M}^5$ found above. The solution for the 2-form Ω in (3.3) given by (3.13) is regular everywhere apart from $\rho = 0$, where

$$\Omega \overset{\rho\to 0}{\sim} \frac{2e^{ik\chi}}{\sin^2\rho}\left(\sigma_x \wedge \sigma_y - \frac{\mathrm{d}(\sin\rho)}{\sin\rho} \wedge \sigma_z\right) \to e^{ik\chi}\,\Theta(\rho,\theta,\phi,\psi)\,, \qquad \Theta \equiv \mathrm{d}\left(\frac{1}{\sin^2\rho}\sigma_z\right). \tag{3.22}$$

In the vicinity of $\rho = 0$ we thus find that Ω is an exact 3-form up to an $e^{ik\chi}$ phase factor. This suggests that for $a \to 0$ (when the relevant modes localise close to $\rho = 0$) we may effectively decouple the S^1_χ from the rest of $\mathcal{M}^5$. Then starting with the factorised form of the field A_2 in (3.3) and summing over positive and negative k-modes we find that the part of A_2 that is divergent for $\rho \to 0$ factorises as

$$A_2 = \sum_{k=-\infty}^{\infty} \varphi_k(x)e^{ik\chi}\Omega_k(\rho,\theta,\phi,\psi) \overset{\rho\to 0}{\sim} \hat{\varphi}(x,\chi)\,\Theta(\rho,\theta,\phi,\psi)\,, \tag{3.23}$$

$$\hat{\varphi}(x,\chi) = \sum_{k=-\infty}^{\infty} e^{ik\chi}\,\varphi_k(x)\,. \tag{3.24}$$

[10] With this a^4 factor added, the integrand of the ρ-integral in the region where $\rho \to a \to 0$ takes the form of $\delta(\rho - a)$ restricting the integration to a small $S^3/\mathbb{Z}_2$-shell around $\rho = 0$. This may be interpreted as the orbifold singularity being cut off at scale a.

JHEP03(2024)045

Assuming that this divergent part which is regularised by the blow-up procedure dominates over contributions coming from the regular part of Ω_k in (3.13) we may thus approximate our 2-form fields as in (1.11) by

$$B_2 = \beta(x,\chi)\,\Theta\,, \qquad C_2 = \gamma(x,\chi)\,\Theta\,, \qquad \hat{\varphi}(x,\chi) = \beta(x,\chi) + i\gamma(x,\chi)\,, \tag{3.25}$$

where the real 2-form Θ is independent of the S^1 coordinate χ. The resulting action for the 6d fields β and γ following from (1.9) or (3.17) has the following structure

$$S_6 \sim \tfrac{1}{2}\int_{AdS_5\times S^1} \mathrm{d}^5x\,\mathrm{d}\chi\,\sqrt{-g_6}\left[c_1(\partial_i\beta)^2 + c_2(\partial_\chi\beta)^2 + c_1(\partial_i\gamma)^2 + c_2(\partial_\chi\gamma)^2 - 8c_3\,\beta\,\partial_\chi\,\gamma\right], \tag{3.26}$$

$$c_1 = \pi\int \mathrm{d}^4y\sqrt{g}g^{ac}g^{bd}\Theta_{ab}\Theta_{cd}\,, \quad c_2 = \pi\int \mathrm{d}^4y\sqrt{g}g^{ac}g^{bd}\Theta_{ab}\Theta_{cd}g^{\chi\chi}\,, \quad c_3 = -2\pi\int \Theta\wedge\Theta\,, \tag{3.27}$$

where the integrals go over the local factor-space $\mathcal{M}^5/S^1$.

In the case of the flat-space orbifold we may choose an anti-self-dual Θ and $g^{\chi\chi} = 1$ so that $c_1 = c_2 = c_3$. In the $S^5/\mathbb{Z}_2$ orbifold case using the approximation (3.25) and the definition of Θ in (3.22) as well as (2.8),[11] we find that integrands of c_1, c_2 and c_3 in (3.27) diverge as ρ^{-5} for $\rho \to a \to 0$ implying that

$$c_1 \sim c_2 \sim c_3 \sim \frac{4\pi^3}{a^4} + \mathcal{O}\Big(a^{-3}\Big)\,. \tag{3.28}$$

Notice that this divergence is of the same order as in (3.21) and should be treated similarly. As a result, we get essentially the same expression as in (1.12) for the leading contribution to the 6d effective action

$$S_6 \sim \tfrac{1}{2}\mathcal{V}\int_{AdS_5\times S^1} \mathrm{d}^5x\,\mathrm{d}\chi\,\sqrt{-g_6}\left[(\partial_i\beta)^2 + (\partial_\chi\beta)^2 + (\partial_i\gamma)^2 + (\partial_\chi\gamma)^2 - 8\,\beta\,\partial_\chi\gamma\right]. \tag{3.29}$$

Expanding β and γ in Fourier modes along χ and diagonalising the action (cf. (1.13)) results in the same spectrum and 5d action as in (3.16), (3.18).

4 Matching to gauge theory: leading and subleading corrections

Let us now address the question of reproducing the gauge theory prediction for the 2-point correlator of twisted sector modes in (1.5). We shall first comment on the matching at the leading order in strong coupling [22] based on the low-energy action (1.12) and then discuss how to modify this action to include leading α'^3-corrections.

[11] Explicitly, one has

$$\frac{1}{2}g^{ac}g^{bd}\Theta_{ab}\Theta_{cd} = \frac{8}{\sin^8\rho}, \quad g^{\chi\chi} = \frac{1}{\cos^2\rho}\,, \quad \sqrt{g} = \frac{1}{8}\sin^3\rho\cos\rho\sin\theta\,.$$

JHEP03(2024)045

4.1 Normalisation factors

Let us start with an action for a massive scalar in AdS_5

$$S[\varphi] = \frac{1}{2}\mathcal{V}_\varphi \int d^5x \sqrt{-g_{AdS_5}}\, \varphi\, (\nabla^2_{AdS_5} - m^2)\varphi\,, \qquad \Delta(\Delta-4) = m^2\,, \tag{4.1}$$

where for generality we introduced a normalisation factor $\mathcal{V}_\varphi$. Fixing the Dirichlet boundary condition $\varphi|_{\partial AdS} = \varphi_0$ one gets for the value of the action [30, 31]

$$S[\varphi_0] = \frac{1}{2}\mathcal{V}_\varphi\, \mathrm{n}_\Delta \int_{\partial AdS_5} d^4x d^4x' \, \frac{\varphi_0(x)\varphi_0(x')}{|x-x'|^{2\Delta}}\,, \qquad \mathrm{n}_\Delta = \frac{2}{\pi^2}\frac{\Delta-2}{\Delta}\frac{\Gamma(\Delta+1)}{\Gamma(\Delta-2)}\,. \tag{4.2}$$

We assume that the generating functional for the correlators of the corresponding dual operator $\mathcal{O}$ contains the source term $C_\mathcal{O}\int_{\partial AdS_5}\varphi_0\mathcal{O}$. The resulting prefactor in the 2-point function of $\mathcal{O}$ is given by (cf. (1.2))

$$\mathrm{G}_\mathcal{O} = \mathrm{n}_\Delta\, \mathcal{V}_\varphi\, C^2_\mathcal{O}\,. \tag{4.3}$$

In general, the value of $\mathrm{G}_\mathcal{O}$ is ambiguous depending on normalisations of the field φ and the dual operator $\mathcal{O}$.

In ref. [20] the dimensionless action normalisation constants $\mathcal{V}_\varphi$ in the 5d actions corresponding to the untwisted and twisted scalar fields dual to the operators U_k and T_k in (1.1) were given as[12]

$$\mathcal{V}_{\varphi_{U_k}} = \frac{1}{2\kappa^2}\cdot L^3 \cdot \frac{\pi^3}{2}L^5 \cdot s_k = \frac{N^2}{2^{k-4}\pi^2}\frac{k(k-1)}{(k+1)^2}\,, \tag{4.4}$$

$$\mathcal{V}_{\varphi_{T_k}} = \frac{1}{2\kappa^2}(\pi\alpha')^2\cdot L^3\cdot 2\pi L = \frac{N^2}{\lambda\,\pi^2}\,. \tag{4.5}$$

In (4.4) the factor $\frac{\pi^3}{2}L^5$ is the volume of $S^5/\mathbb{Z}_2$ and s_k comes from the KK-mode overlap integral on the compact space (cf. [32]). With the choice of normalisation (1.1) of U_k (corresponding to a particular $C_{\varphi_{U_k}}$ in (4.3)) we have $\mathrm{G}_{U_k} = 1$ in (1.4).

The presence of the factor $(\pi\alpha')^2$ in the twisted field normalisation (4.5) appears to be an ad hoc choice required to explain the presence of the non-trivial λ^{-1} prefactor in G_{T_k} in (1.5).[13] It may be attributed to the "stringy" nature of the twisted sector modes described by a 6d action (1.12) that should have an overall normalisation fixed directly from the string theory computation involving twisted-state vertex operators normalised in a particular way. It should be related to the localisation of the twisted-sector modes to the fixed 6d subspace with the factor α'^2 effectively replacing the "transverse" 4-volume factor L^4 in the untwisted case (4.4).

In our present approach where the starting point is the 10d supergravity action expanded near the resolved orbifold background $AdS_5\times\mathcal{M}^5$ the role of this extra scale factor is effectively played by the resolution parameter a. Indeed, as we discussed above, compactifying from

[12] Here L is the scale of AdS_5 and $\frac{1}{2\kappa^2} = \frac{1}{(2\pi)^7\alpha'^4 g_s^2} = \frac{4(2N)^2}{(2\pi)^5 L^8}$, $\quad s_k = 2^{6-k}\frac{k(k-1)}{(k+1)^2}$, where $2N$ is the rank of the gauge group before orbifolding.

[13] In [20], the factor $(2\pi\alpha')^2$ was introduces as a rescaling to make the boundary value of the field φ_0 dimensionless (with extra 4 to cancel 1/4 factor in the 6d action there).

JHEP03(2024)045

10d to 5d we get the twisted-mode action (3.18) with a^{-4} scaling (3.21) representing the delta-function in 4 transverse directions (see also footnote 10) that can be eliminated by a rescaling $\varphi \to a^2\varphi$. Interpreting a as an effective counterpart of the string scale $\sqrt{\alpha'}$ in a direct string theory computation, this produces an extra $(\frac{\alpha'}{L^2})^2$ factor in the twisted case (4.5) relative to the untwisted one (4.4).

Going beyond the leading order in large λ the form of the localisation result for G_{T_k} in (1.5) implies that one is to replace $\sqrt{\lambda}$ by $\sqrt{\lambda'}$ according to (1.7) [10, 11]. This redefinition may be interpreted on the string side as being related to a renormalisation of the effective string tension or of the AdS_5 radius which should be due to the fact that orbifolding breaks half of the maximal supersymmetry of $AdS_5 \times S^5$, i.e.

$$\sqrt{\lambda} = \frac{L^2}{\alpha'} \;\; \to \sqrt{\lambda'} = \frac{L^2}{\alpha'} - 4\log 2\,. \tag{4.6}$$

The extra factor $\left(\frac{\lambda'}{\lambda}\right)^k$ in (1.5) may be absorbed into the normalisation of the twisted sector operators. Up to the overall factor the subleading corrections in the strong-coupling expansion (1.5) of G_{T_k} have the same pattern as $c_1\alpha'^3 + c_2\alpha'^5 + \ldots$ corrections in type IIB string theory. The first subleading term is

$$\mathrm{G}_{T_k} \sim 1 + \tfrac{1}{2}(2k-1)(2k-2)(2k-3)\frac{\zeta(3)}{\lambda'^{\frac{3}{2}}} + \mathcal{O}(\lambda'^{-5/2})\,. \tag{4.7}$$

Let us now discuss a possible tree-level string theory origin of this large-N strong coupling correction.

4.2 α'^3-corrections

Our strategy is to find a higher-derivative correction to the 6d action (3.29) quadratic in the twisted sector modes by starting with the tree level type IIB string effective action including $\alpha'^3\zeta(3)\mathcal{R}^4 + \ldots$ terms and repeating the procedure that led from the 10d supergravity action to the action (3.29).

In section 3 we have shown that the fields B_2 and C_2 develop additional normalisable modes on the resolved orbifold background $\mathcal{M}^5$. Sending the resolution parameter a to zero, we observed that these modes localise close to the emerging orbifold singularity and can be described by the effective 6d action (3.29). We now expand the relevant α'^3-terms which are quadratic in $A_2 = B_2 + iC_2$ near the deformed orbifold background $AdS_5 \times \mathcal{M}^5$ and use that the relevant modes localise on $AdS_5 \times S^1$ to integrate over the internal 4-space. This should result in α'^3-corrections to the 6d action (3.29) responsible for the subleading $\zeta(3)$-term in (4.7).

To recall, the tree-level type IIB string low-energy effective action has the following schematic form (see, e.g., [33–35])

$$S_{\text{eff}} = S_{10} + \alpha'^3\zeta(3)\int \mathrm{d}^{10}x\,\sqrt{-g}\Big[\mathcal{R}^4 + \mathcal{L}_8(\mathcal{R}, F_5, G_3)\Big] + \mathcal{O}\Big(\alpha'^5\Big)\,. \tag{4.8}$$

Here S_{10} is the type IIB supergravity action, $\mathcal{R}^4$ indicates the curvature-dependent invariant and $\mathcal{L}_8$ depends on RR fields (we ignore numerical constants and dependence on dilaton, RR scalar and fermions). Expanded near flat space, $\mathcal{L}_8$ involves at least four fields and eight

JHEP03(2024)045

derivatives as the 2-point and 3-point string amplitudes for the massless modes do not receive α'-corrections. The explicit form of $\mathcal{L}_8$ should be fixed by supersymmetry but is presently not known (cf. appendix B) so our discussion below is partly qualitative.

Assuming a scheme (field redefinition) choice in which the curvature dependence of the string effective action is expressed in terms of the 10d Weyl tensor $\mathcal{C}$ (i.e. replacing $\mathcal{R}^4$ in (4.8) by $\mathcal{C}^4$, etc.) one concludes that the 2-point and 3-point functions of the massless string modes do also not receive corrections near the conformally flat $AdS_5 \times S^5$ background (cf. [24, 25]). The same then applies to the untwisted sector BPS modes in the $AdS_5 \times S^5/\mathbb{Z}_2$ orbifold case (the Weyl tensor here is again zero away from the orbifold singularity).

To find the correction to the action (3.29) of the twisted sector modes we need to consider (4.8) expanded near the resolved $AdS_5 \times \mathcal{M}^5$ background and determine terms quadratic in the A_2-field that survive in the $a \to 0$ limit. For $AdS_5 \times \mathcal{M}^5$ the Ricci tensor and F_5 have the same structure as for $AdS_5 \times S^5$ (see (2.11), (2.22))

$$\mathcal{R}_{ij} = -\frac{4}{L^2} g_{ij}\,, \qquad \mathcal{R}_{ab} = \frac{4}{L^2} g_{ab}\,, \qquad F_5 = \frac{4}{L}\big(\,\mathrm{vol}_{AdS_5} + \mathrm{vol}_{\mathcal{M}^5}\,\big)\,, \tag{4.9}$$

but the Weyl tensor of $\mathcal{M}^5$ is no longer zero (cf. (2.18), (D.7))

$$\mathcal{C}_{abcd}(\mathcal{M}^5) \sim \frac{\sin^4 a}{\sin^6 \rho}\,. \tag{4.10}$$

The leading correction to the term quadratic in $G_3 = dA_2$ should then come from the structures in $\mathcal{L}_8$ in (4.8) that are at least linear in the Weyl tensor $\mathcal{C}(\mathcal{M}^5)$, i.e.

$$\mathcal{L}_8 \sim \mathcal{C}\, F_5\, \bar{G}_3 \nabla^3 G_3 + \dots\,. \tag{4.11}$$

Here we indicated only the term with highest possible (third) power of covariant derivatives as one can see on dimensional grounds. This term may be related to the k^3-term in (4.7). Its detailed index structure is discussed below.

In general, assuming that the relevant modes of B_2 and C_2 "localise" to 6d space as in (3.25) we are led to the following correction to the 6d action (3.26)

$$\Delta S_6 \sim \alpha'^3 \zeta(3)\, \mathcal{V} \int_{AdS_5 \times S^1} \mathrm{d}^5x\, \mathrm{d}\chi\, \sqrt{-g_6}\, \Big(\beta\, \mathcal{K}_1\, \beta + \gamma\, \mathcal{K}_2\, \gamma + \beta\, \mathcal{K}_3\, \gamma\Big)\,, \tag{4.12}$$

$$\mathcal{K}_r = \sum_{n=0}^{2} \sum_{l=0}^{5-2n} \mathrm{k}_{nl,r}\, (\nabla^2_{AdS_5})^n\, (\partial_\chi)^l\,. \tag{4.13}$$

The masses of the β and γ fields are expected to be protected by supersymmetry so $\mathcal{K}_1$ and $\mathcal{K}_2$ should depend only on the 6d covariant combination $\nabla^2_{AdS_5} + \partial^2_\chi$. The terms with second power of $\nabla^2_{AdS_5}$ in (4.13) can then be eliminated using field redefinitions (cf. (3.26)) so we may ignore them. Furthermore, the mixing of β and γ in (4.12) may only affect normalisations at subleading order in α'. We may then assume that the only relevant effect of adding the correction ΔS_6 to the leading-order action (3.29) is a possible change of the overall normalisation due to an extra operator $P(\partial_\chi) = p_3 \partial_\chi^3 + p_2 \partial_\chi^2 + p_1 \partial_\chi + p_0$ factor in $\mathcal{K}_r$ operators, i.e.

$$\mathcal{K}_1 = \mathcal{K}_2 = P(\partial_\chi)\Big(\nabla^2_{AdS_5} + \partial^2_\chi\Big), \qquad \mathcal{O}_{\beta\gamma} = 8P(\partial_\chi)\, \partial_\chi\,. \tag{4.14}$$

JHEP03(2024)045

After expanding in S^1_χ modes we have $\partial_\chi \to ik$ so to reproduce the correction in (4.7) we need

$$P(\partial_\chi) \sim \tfrac{1}{2}(2i\partial_\chi + 1)(2i\partial_\chi + 2)(2i\partial_\chi + 3)\,. \tag{4.15}$$

This peculiar structure should be dictated by supersymmetry.

5 Matching the k^3-term

Let us now try to substantiate the above procedure and fix the required structure of the correction in (4.11) by focussing on the leading ∂^3_χ-term in (4.15) that should reproduce the k^3-term in (4.7).

This term is dominant in the large R-charge limit $k \to \infty$ which, combined with the large-λ expansion in (1.5), should be analogous to the familiar BMN limit [36]. Indeed, the strong coupling expansion of the gauge theory expression in (1.5) expanded also at large k admits a regular limit

$$\mathrm{G}_{T_k}\Big|_{\lambda,\,k\to\infty} = (2\pi\nu)^2\, e^{-8\nu\log 2}\Big[1 + 4\zeta(3)\,\nu^3 - 9\zeta(5)\,\nu^5 + 8\zeta(3)^2\,\nu^6 + \dots\Big]\,, \qquad \nu \equiv \frac{k}{\sqrt{\lambda}}\,. \tag{5.1}$$

This gives a hint that one may be able to reproduce (5.1) on the string theory side by starting with a pp-wave limit of the orbifold background.[14] The parameter $\nu = \frac{k}{\sqrt{\lambda}}$ is the analogue of the semiclassical BMN-momentum along S^1_χ which is fixed in the large-k, large-λ limit.

Below we suggest a strategy to reproduce the ν^3-term in (5.1) by starting with the pp-wave limit of the resolved orbifold background $AdS_5 \times \mathcal{M}^5$. This provides a substantial simplification allowing one to see more explicitly that α'^3-corrections as in (4.11) indeed lead to the $k^3\zeta(3)$-term in (1.5), (4.7) or the $\nu^3\zeta(3)$-term in (5.1).

At the end of this section we shall return to the case of the original $AdS_5 \times \mathcal{M}^5$ and identify a particular structure in (4.11) that may correspond to the k^3-term without first taking the pp-wave limit.

5.1 Large-k limit: pp-wave analogue of the resolved orbifold

In the familiar $AdS_5 \times S^5$ case the Penrose limit [37] corresponds to focussing on states with a large momentum k along $S^1 \subset S^5$, i.e. expanding near a light-like geodesic along the time direction of AdS_5 and an isometry circle of the S^5. This is equivalent to a scaling limit

$$\frac{L}{\sqrt{\alpha'}} \to \infty\,, \quad k \to \infty\,, \qquad \nu = \tfrac{k}{\sqrt{\lambda}} = \tfrac{\alpha'}{L^2}k = \text{fixed}\,. \tag{5.2}$$

In the present orbifold case we may consider a similar limit of $AdS_5 \times \mathcal{M}^5$ with S^1 being the fixed χ-circle. Starting with

$$\mathrm{d}s^2_{10} = L^2\Big(\mathrm{d}s^2_{AdS_5} + \mathrm{d}s^2_{\mathcal{M}^5}\Big), \qquad \mathrm{d}s^2_{AdS_5} = -\mathrm{d}t^2\cosh^2 r + \mathrm{d}r^2 + \sinh^2 r\,\mathrm{d}s^2_{S^3}\,, \tag{5.3}$$

[14] While this may work for the 2-point correlator, in general, the pp-wave limit may not be enough for reproducing 3-point functions. Still, it is interesting to note that according to (1.4) and (1.6) in the large-k limit, we get $\mathrm{G}_{U_k,U_k,\bar U_{2k}} \to \frac{k^{3/2}}{N}$, $\mathrm{G}_{T_k,T_k,\bar U_{2k}} \to \frac{k^{3/2}}{N}\,(2\pi\nu)^2\,e^{-8\nu\log 2}\big[1 + 4\zeta(3)\nu^3 + \mathcal{O}(\nu^5)\big]$. Thus the ratio $\mathrm{G}_{T_k,T_k,\bar U_{2k}}/\mathrm{G}_{U_k,U_k,\bar U_{2k}}$ has a well-defined limit depending only on ν.

JHEP03(2024)045

where $\mathrm{d}s^2_{\mathcal{M}^5}$ is given by (2.17), we perform the rescaling

$$x^- \to \frac{1}{L^2}x^- , \qquad r \to \frac{1}{L}r , \qquad \rho \to \frac{1}{L}\rho , \qquad a \to \frac{1}{L}a , \qquad x^{\pm} \equiv \frac{1}{2}(t \pm \chi) . \tag{5.4}$$

At leading order in large L this results in the following pp-wave metric

$$\mathrm{d}s^2_{10} = -4\mathrm{d}x^+\mathrm{d}x^- - (x^2+\rho^2)(\mathrm{d}x^+)^2 + \mathrm{d}x^i\mathrm{d}x^i + V_0(\rho)^{-1}\mathrm{d}\rho^2 + \rho^2\left[\sigma_x^2+\sigma_y^2+V_0(\rho)\,\sigma_z^2\right], \tag{5.5}$$

$$V_0(\rho) = 1 - \tfrac{a^4}{\rho^4} , \tag{5.6}$$

where x^i $(i = 1, 2, 3, 4)$ are originating from the AdS_5 coordinates. This background is a pp-wave with the transverse space being the product of $\mathbb{R}^4$ and the EH space with metric (2.4). The 5-form (2.22) becomes

$$F_5 = 4\,\mathrm{d}x^+ \wedge (\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 - \rho^3\mathrm{d}\rho \wedge \sigma_x \wedge \sigma_y \wedge \sigma_z) . \tag{5.7}$$

One can check directly that this background solves the 10d supergravity equations and preserves half of the maximal supersymmetry.

Alternatively, we may arrive at this background by first taking the Penrose limit of the orbifold $AdS_5 \times S^5/\mathbb{Z}_2$, resulting in an orbifolded pp-wave background [38–42] with the transverse space being $\mathbb{R}^4 \times \mathbb{C}^2/\mathbb{Z}_2$. To resolve the orbifold singularity [41] we may replace $\mathbb{C}^2/\mathbb{Z}_2$ by its EH resolution (2.4). One is then to check that the resulting pp-wave metric satisfies the supergravity equations. As the EH metric is Ricci flat, this requires that the coefficient $H(x, \rho)$ of the $(\mathrm{d}x^+)^2$-term in the pp-wave metric should satisfy the Laplace equation $\nabla^2 H = 0$ on the transverse space $\mathbb{R}^4 \times \mathrm{EH}_4$. Choosing $H = x^2 + h(\rho)$ one finds that h should solve the Poisson equation on the EH space

$$\nabla^2_{\mathrm{EH}} h(\rho) = 8 . \tag{5.8}$$

It has $h(\rho) = \rho^2$ as its simplest solution which reproduces (5.5).[15]

Next, we also need to find a similar limit in the solution (3.3), (3.13) for the twisted sector mode. The $e^{ik\chi}$ mode in (3.13) should correspond to a particle moving fast along the S^1. We get

$$\Omega \to L^2\, e^{-2i\bar{\nu}x^-}\, \frac{e^{-\frac{\bar{\nu}}{2}\rho^2}}{\rho^2}\left[(2+\bar{\nu}\rho^2)\sigma_x\wedge\sigma_y + i\bar{\nu}\rho^2\mathrm{d}x^+\wedge\sigma_z - \frac{2+\bar{\nu}\rho^2}{\rho}\mathrm{d}\rho\wedge\sigma_z\right], \qquad \bar{\nu} \equiv \frac{k}{L^2} = \frac{\nu}{\alpha'} . \tag{5.9}$$

Indeed, starting with the analogue of the supergravity equations (3.1), (3.2) in the pp-wave background (5.5), (5.7) we get for the pp-wave analogue of the solution (3.3), (3.13)

$$A_2 = \varphi(x^+, x^i)\,\Omega , \qquad (2i\bar{\nu}\partial_+ + \partial_i\partial^i - \bar{\nu}^2x^2 - 4\bar{\nu})\varphi = 0 . \tag{5.10}$$

The x^i dependence is found as in the harmonic oscillator problem. For the ground-state solution we get the dispersion relation corresponding to a particle with $m^2 = \bar{\nu}^2 - 4\bar{\nu}$.

Like the original solution (3.13), the pp-wave solution (5.9) diverges for $\rho \to 0$ with the leading term being (cf. (3.22))

$$\Omega \overset{\rho\to 0}{\sim} L^2 e^{-2i\bar{\nu}x^-}\Theta_{\mathrm{EH}} , \qquad \Theta_{\mathrm{EH}} = \mathrm{d}(\rho^{-2}\,\sigma_z) , \tag{5.11}$$

[15] Note that ref. [41] discussed a different solution for $h(\rho)$.

JHEP03(2024)045

where $\Theta_{\rm EH}$ is the anti-self-dual exact 2-form on the EH space with non-zero integral over the resolution cycle [28]. Rescaling A_2 by a^2 we thus get

$$A_2 = L^2 a^2\, \hat{\varphi}(x^-, x^+, x^i)\, \Theta_{\rm EH}\,, \qquad \hat{\varphi}(x^-, x^+, x^i) = e^{-2i\bar{\nu}x^-}\, \varphi(x^+, x^i)\,, \tag{5.12}$$

which satisfies the equations of motion up to terms of order ρ^2

$$\nabla^2 A_{AB} + iF_{AB}{}^{CDE}\nabla_C A_{DE} = (\nabla^2 - 2i\partial_-)A_{AB} = \frac{\rho^2}{4}\partial_-^2 A_{AB}\,. \tag{5.13}$$

This is not an issue for the interpretation of A_2 as the origin of the twisted modes as according to the discussion in section 3.2 these modes localise near $\rho = 0$. However, we will see below that the ρ^2-terms are still important to render the α'^3-correction finite.

Let us return to the analysis of α'^3-corrections (4.11) in section 4.2 now using the pp-wave background. The Weyl tensor corresponding to the pp-wave metric (5.5) (see appendix D) splits into two parts

$$\mathcal{C} = \mathcal{C}_{\rm EH} + \mathcal{C}_{\rm mix}\,, \qquad \mathcal{C}_{\rm EH} \sim \mathcal{O}\left(\frac{a^4}{\rho^6}\right) \qquad \mathcal{C}_{\rm mix} \sim \mathcal{O}\left(\frac{a^4}{\rho^4}\right)\,, \tag{5.14}$$

where $\mathcal{C}_{\rm EH} = \mathcal{C}_{bcde}$ is the Weyl tensor of the EH space and $\mathcal{C}_{\rm mix}$ has non-zero components of the form

$$(\mathcal{C}_{\rm mix})_{+b+b} = \pm\frac{a^4}{\rho^4}\,, \tag{5.15}$$

where $b = 1, \dots, 4$ is a Vierbein index of the EH space. The invariant in (4.11) involves one power of the Weyl tensor and two powers of the $G_3 = 2A_2$ field, which according to (5.12) is proportional to $\Theta_{\rm EH}$. One can check that contractions of the form $\mathcal{C}_{\rm mix}\Theta_{\rm EH}\Theta_{\rm EH}$ vanish. We thus need to consider only contractions with $\mathcal{C}_{\rm EH} = (\mathcal{C}_{bcde})$ with the relevant ones being

$$\mathcal{C}_{bcde}\,\Theta_{\rm EH}^{de} = 16\frac{a^4}{\rho^6}\Theta_{{\rm EH},bc}\,, \qquad \mathcal{C}_{bdce}\,\Theta_{\rm EH}^{de} = 8\frac{a^4}{\rho^6}\Theta_{{\rm EH},bc}\,. \tag{5.16}$$

The EH Weyl tensor diverges as $\mathcal{C}_{\rm EH} \sim \mathcal{O}(a^{-2})$ in the orbifold limit $\rho \to a \to 0$. Therefore, an insertion of $\mathcal{C}_{\rm EH}$ should be accompanied by an additional factor proportional to a^2 or ρ^2 to get a finite correction. According to (5.13) such a factor may come from a $(\nabla^2 - 2i\partial_-)A_2$-term.

We can now specify more explicitly the conjectured structure of the α'^3-invariant in (4.11) required to reproduce the leading k^3-term in (4.7). Staring with the term $F^{ABCDE}\mathcal{C}_{DE}{}^{FG}\nabla_A\bar{G}_{BC}{}^H\Box G_{FGH}$ we need to add to it other terms with a smaller number of covariant derivatives in order to get the combination $\nabla^2 - 2i\partial_-$ and to reproduce the right spectrum. This is achieved by starting with

$$\begin{aligned}\mathcal{L}_8 \sim F^{ABCDE}\mathcal{C}_{DE}{}^{FG}\Big(&i\nabla_A\bar{G}_{BC}{}^H\nabla^2 G_{FGH} + \tfrac{1}{2}F_{FG}{}^{HIJ}\nabla_A\bar{G}_{BC}{}^K\nabla_H G_{IJK}\\ &+ F_{FG}{}^{HIJ}\bar{G}_{ABC}\nabla^2 G_{HIJ} - \tfrac{i}{2}F_{FG}{}^{HIJ}F_{IJ}{}^{KMN}\bar{G}_{ABC}\nabla_H G_{KMN} + \text{c.c.}\Big)\,.\end{aligned} \tag{5.17}$$

Using the equations of motion and integrating over the EH space we then find the following α'^3-correction to the 6d effective action (cf. (4.12))[16]

$$\Delta S_6 \sim \alpha'^3\zeta(3)\,\mathcal{V}\int_{AdS_5\times S^1} \mathrm{d}^5x\,\mathrm{d}\chi\,\sqrt{-g_6}\,\Big(i\nabla^A\bar{\hat{\varphi}}\nabla_-^3\nabla_A\hat{\varphi} + 4\bar{\hat{\varphi}}\nabla_-^4\hat{\varphi} + \text{c.c.}\Big)\,. \tag{5.18}$$

[16]Note that any contraction of F_5 with $\Theta_{\rm EH}$, $\mathcal{C}$ and another F_5 can only involve indices from the EH space, leaving at least the $+$ index of F_5 uncontracted. This then requires a contraction with ∇_-.

JHEP03(2024)045

This correction has the same structure as the leading-order action in (1.12) (with $\hat{\varphi} = \beta + i\gamma$ as in (3.25)) but with an extra insertion of the operator ∇_-^3. It thus corresponds to the ∂_χ^3-term in (4.15). Acting on $\hat{\varphi}$ in (5.12) it produces the expected ν^3-term in the 2-point function (4.7).

Let us note that the suggested structure (5.17) of the required α'^3-term is not unique. For example we could use other contractions of internal indices, like the second option in (5.16), or a different ordering of the F_5 and $\mathcal{C}$ factors. The structure of $\mathcal{L}_8$ in (4.8) dictated by supersymmetry should be a combination of all such terms that leads to (5.18) with the right overall coefficient to match the one in (4.7).

5.2 Back to $AdS_5 \times \mathcal{M}^5$

Let us now see how to reproduce the k^3-term in (4.7) without first taking the pp-wave limit, i.e. by starting with the string effective action expanded near $AdS_5 \times \mathcal{M}^5$

The Weyl tensor $AdS_5 \times \mathcal{M}^5$ written in Vielbein components (see appendix D) can be split, like in (5.14), into two parts

$$\mathcal{C} = \mathcal{C}_{\text{int}} + \mathcal{C}_{\text{mix}}\,, \qquad \mathcal{C}_{\text{int}} \sim \mathcal{O}\left(\frac{\sin^4 a}{\sin^6 \rho}\right), \qquad \mathcal{C}_{\text{mix}} \sim \mathcal{O}\left(\frac{\sin^4 a}{\sin^4 \rho}\right), \tag{5.19}$$

where $\mathcal{C}_{\text{int}}$ resembles the EH Weyl tensor $\mathcal{C}_{\text{EH}}$ for small ρ

$$\mathcal{C}_{\text{int}} \overset{\rho\to 0}{\sim} \mathcal{C}_{\text{EH}} + \mathcal{O}\left(\frac{a^4}{\rho^4}\right), \tag{5.20}$$

and $\mathcal{C}_{\text{mix}}$ only has entries of the form

$$(\mathcal{C}_{\text{mix}})_{\chi b \chi b} = \pm \frac{\sin^4 a}{\sin^4 \rho}\,. \tag{5.21}$$

Expanding the approximate solution for $\Omega = e^{ik\chi}\Theta$ in (3.22) for small ρ yields Θ_{EH} as the leading $\mathcal{O}(\rho^{-2})$-term in Θ

$$\Omega = e^{ik\chi}\Theta \overset{\rho\to 0}{\sim} e^{ik\chi}\,\Theta_{\text{EH}} + \mathcal{O}\left(\rho^0\right). \tag{5.22}$$

We check that like in the pp-wave limit in (5.13), the field $A_2 = \hat{\varphi}(x,\chi)\Theta$ in (3.23) satisfies the equations of motion (3.2) up to $\mathcal{O}(\rho^2)$-terms

$$\nabla^2 A_{AB} + iF_{AB}{}^{CDE}\nabla_C A_{DE} = -k^2\rho^2 A_{AB} + \mathcal{O}\left(\rho^4 A\right) = \rho^2\partial_\chi^2 A_{AB} + \mathcal{O}\left(\rho^4 A\right). \tag{5.23}$$

Given that only the leading term in $\rho \to 0$ should contribute to the relevant part of the action, we may use the same α'^3-combination as in (5.17), now starting with the $AdS_5 \times \mathcal{M}^5$ background. It evaluates to the same 6d correction (5.18) reproducing again the k^3-term in (4.7).

In addition, we expect also other corrections that should correspond to $\zeta(3)$-terms with lower powers of k in (4.7). Various cancellations that prevent divergence of the α'^3-corrections in the pp-wave limit should still occur here, but only up to subleading finite terms.[17] To

[17] Consider, for example, the $\mathcal{C}_{\text{int}}^2$-term which diverges as $\frac{a^8}{\rho^{12}}$. We expect this divergence to cancel algebraically up to order $\frac{a^8}{\rho^{10}}$, where finite contributions can arise when multiplied with the operator in (5.23). These finite contributions did not arise in the large-k limit, so they should be due to the ρ^2-corrections to the $AdS_5 \times \mathcal{M}^5$ quantities.

illustrate the cancellation structure we expect, let us consider the contraction

$$\sqrt{-g}\,\mathcal{C}^{ABCD}\mathcal{C}_{CD}{}^{EF}(\bar{\Omega}_{AB}\Omega_{EF}-8\bar{\Omega}_{AE}\Omega_{BF})\,. \tag{5.24}$$

As this term involves two Weyl tensors $\mathcal{C}$ and two 2-forms Ω, we would expect a small-ρ divergence of order $\frac{a^{12}}{\rho^{17}}$. In fact, this contraction vanishes on the pp-wave background of section 5.1 and behaves as $\frac{a^{12}}{\rho^{13}}$ in the $AdS_5\times\mathcal{M}^5$ case. Supplementing (5.24) with other terms involving factors of derivatives and F_5 we may be able to build an invariant like (5.17) that reproduces the k^2-term in (4.7) but vanishes when evaluated on the pp-wave background.

At this point, the appearance of such cancellations in an eventual complete description of the α'^3-terms is speculative. However, let us mention that precisely this cancellation pattern is observed in the explicitly known $\alpha'^3\zeta(3)\mathcal{R}^4$ term in (4.8): despite involving four powers of the Weyl tensor it vanishes on the pp-wave background and is finite in the $\rho\to a\to 0$ limit on $AdS_5\times\mathcal{M}^5$.

6 Concluding remarks

In this paper we suggested a strategy of matching the large-coupling expansion of twisted sector correlators in planar 4d $\mathcal{N}=2$ superconformal SU(N) $\times$ SU(N) quiver gauge theory to α'-corrections in the dual orbifold string theory. Specifically, we considered the 2-point function of the twisted sector operators T_k in (1.1).

The corresponding twisted sector string modes localise on the fixed $AdS_5\times S^1$ subspace. To access these localised modes, we proposed an explicit resolution of the $S^5/\mathbb{Z}_2$ orbifold singularity (2.17), represented by a (non-supersymmetric) solution of 10d type IIB supergravity. The resolved space has a non-trivial 2-cycle on which the 2-form supergravity fields can "wrap".

These extra modes should effectively represent the lightest twisted sector string modes that should be present in the first-principles string theory approach (which is possible in flat space but is not directly available in the $AdS_5\times S^5/\mathbb{Z}_2$ case). We derived an effective 6d action for these 2-form modes following from the supergravity action expanded near the resolved background. An analogous treatment for other light modes in the twisted sector should be possible, too (cf. [5]).

We then suggested how the inclusion of the α'^3-corrections to the type IIB effective action may allow one to match the subleading term in the localisation result (1.5). Our discussion remained at a qualitative level due to lack of knowledge about the full expression for the supersymmetric completion of the α'^3R^4 invariant. We pointed out the important simplification that happens in the large R-charge limit $k\to\infty$ when the resolved orbifold background simplifies to a pp-wave one.

It would be interesting to extend our approach to the 3-point function (1.6) and understand the string theory origin of the relation between the 2-point and 3-point coefficients mentioned in footnote 4. Another possible extension is to the case of the $AdS_5\times S^5/\mathbb{Z}_\text{L}$ orbifold dual to the L-node quiver in which case the generalisation of the expansions in (1.5), (1.6) where recently found (see [43] and refs. there). Ref. [43] observed interesting simplifications occurring in the large-L limit correlated with taking k or λ large; this may be suggesting the existence of a similar well-defined limit on the string theory side.

JHEP03(2024)045

Acknowledgments

We would like to thank Matteo Beccaria, Nikolay Bobev, Gregory Korchemsky, Elli Pomoni and Bogdan Stefanski for helpful and encouraging discussions. TS acknowledges funding by the President's PhD Scholarship of Imperial College London. AAT is supported in part by the STFC Consolidated Grants ST/T000791/1 and ST/X000575/1.

A String spectrum for a flat-space orbifold

Here we review some facts about the superstring spectrum on a particular $R^{1,6} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold (see [26, 44, 45]). Starting with $\mathbb{R}^{1,9}$ we identify coordinates as $(x_5, x_6, x_7, x_8) \sim (-x_5, -x_6, -x_7, -x_8)$.[18] We shall use the Green-Schwarz formulation in light-cone gauge with the action $(i = 1, 2 \dots, 8)$

$$\mathcal{S} = \frac{1}{\pi\alpha'} \int \mathrm{d}^2\sigma \left(\partial_+ x^i \partial_- x^i + i S_R^\alpha \partial_+ S_R^\alpha + i S_L^\alpha \partial_- S_L^\alpha\right). \tag{A.1}$$

We are to keep only states that are invariant with respect to the $\mathbb{Z}_2$-symmetry. In particular, states localised at the singularity (i.e. with $x_0^a = p_0^a = 0$, where $a \in \{5, 6, 7, 8\}$) can only have even numbers of $\mathbb{Z}_2$-odd excitations, as, for example, the bosonic modes α_{-n}^a and $\tilde{\alpha}_{-n}^a$. To analyse the fermionic modes, we need to decompose the two SO(8)-spinors S_L and S_R, which in type IIB theory have equal chirality, according to the splitting rules

$$\begin{aligned}
&\mathrm{SO}(8) \to \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2), \\
&\mathbf{8_v} \to (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}), \quad \mathbf{8_s} \to (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2}), \quad \mathbf{8_c} \to (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) \oplus (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}).
\end{aligned} \tag{A.2}$$

The orbifold action in this decomposition is represented by

$$\Gamma = \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.3}$$

We interpret the first two quantum numbers as determining the representation in the 4d-space spanned by x^i with $i \in 1, 2, 3, 4$. The remaining untwisted SU(2) will become an $\mathcal{R}$-symmetry in an eventual compactification.

The localised states belong to two distinct sectors, the untwisted one, which is just the $\mathbb{Z}_2$-invariant part of the usual type IIB spectrum, and the twisted one which closes up to a Γ-transformation. Let us analyse them in turn.

Untwisted sector. We can follow the usual IIB construction and use (A.2) to split

$$(\mathbf{8_c} \oplus \mathbf{8_v}) \to \big((\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})\big)_1 \oplus \big((\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) \oplus (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})\big)_{-1}, \tag{A.4}$$

where the subscript denotes the eigenvalue under Γ. When we combine left- and right-movers, we only keep Γ-invariant states, so we can ignore cross-terms and drop the **3**-representations of the last SU(2). This results in the spectrum

$$\begin{aligned}
I: &\quad (\mathbf{3},\mathbf{3};\mathbf{1}) \oplus (\mathbf{2},\mathbf{3};\mathbf{2}) \oplus (\mathbf{2},\mathbf{3};\mathbf{2}) \oplus (\mathbf{1},\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3};\mathbf{1}) \oplus (\mathbf{1},\mathbf{3};\mathbf{3}), \\
II: &\quad (\mathbf{3},\mathbf{1};\mathbf{1}) \oplus (\mathbf{2},\mathbf{1};\mathbf{2}) \oplus (\mathbf{2},\mathbf{1};\mathbf{2}) \oplus (\mathbf{1},\mathbf{1};\mathbf{1}) \oplus (\mathbf{1},\mathbf{1};\mathbf{1}) \oplus (\mathbf{1},\mathbf{1};\mathbf{3}), \\
III: &\quad (\mathbf{3},\mathbf{1};\mathbf{1}) \oplus (\mathbf{2},\mathbf{1};\mathbf{2}) \oplus (\mathbf{2},\mathbf{1};\mathbf{2}) \oplus (\mathbf{1},\mathbf{1};\mathbf{1}) \oplus (\mathbf{1},\mathbf{1};\mathbf{1}) \oplus (\mathbf{1},\mathbf{1};\mathbf{3}).
\end{aligned} \tag{A.5}$$

[18] We can also describe this as simultaneously rotating two $\mathbb{C}$-planes by an angle π.

JHEP03(2024)045

Here, we dropped the last SU(2) factor (all fields are in the representation **1**) and separated the $\mathcal{R}$-symmetry. The three sets form representations of 6d $\mathcal{N} = (2,0)$ supergravity, namely the gravity multiplet I and two tensor multiplets II and III. Alternatively, we could have arrived at these multiplets by expanding the known type IIB spectrum in SU(2) representations and projecting upon the **1** of the fourth SU(2) factor.

Twisted sector. In the twisted sector, half of the bosonic and fermionic oscillators have to be glued with antisymmetric boundary conditions. In general, such twisted boundary conditions can affect the normal ordering constant, but here it receives equal contributions from fermions and bosons and therefore vanishes. However, only the fermions with symmetric boundary conditions have zero modes, so at the massless level, the usual ground-ground-statestate degeneracy is generated only by one spinor $\hat{S}_0^\alpha$ in the representation $(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})$. Starting with a Γ-charged state $|a\rangle \in (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})$, we can act with $\hat{S}_0^\alpha$ and create

$$|\dot{\beta}\rangle = \Gamma^a_{\dot{\beta}\alpha}\hat{S}_0^\alpha\,|a\rangle \in (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})\,. \tag{A.6}$$

Another action with $\hat{S}_0^\alpha$ leads back to $(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})$. Therefore the massless states generated in both left- and right-moving sector furnish the reduced representation

$$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) \oplus (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})\,. \tag{A.7}$$

Combining the left- and right-moving states we get

$$T: \quad (\mathbf{3},\mathbf{1};\mathbf{1}) \oplus (\mathbf{2},\mathbf{1};\mathbf{2}) \oplus (\mathbf{2},\mathbf{1};\mathbf{2}) \oplus (\mathbf{1},\mathbf{1};\mathbf{1}) \oplus (\mathbf{1},\mathbf{1};\mathbf{1}) \oplus (\mathbf{1},\mathbf{1};\mathbf{3})\,, \tag{A.8}$$

which is another tensor multiplet of 6d $\mathcal{N} = (2,0)$ supergravity.

B Comments on α'^3-terms in the type IIB string effective action

The familiar R^4-term in the tree level type II superstring effective Lagrangian may be written as[19]

$$\mathcal{L} = \frac{\alpha'^3}{3\cdot 2^{11}}\,\zeta(3)\,\left(t_8 t_8 - \frac{1}{4}\epsilon_8\epsilon_8\right)\mathcal{R}^4\,. \tag{B.1}$$

Here the t_8t_8-term is fixed from 4-graviton amplitude [34] while the presence of the ϵ-term may be deduced from sigma model considerations [46, 47] or by computing a 5-point amplitude (see a review in [48]). Other α'^3-terms should be related to (B.1) by supersymmetry. We are interested, in particular, in the terms quadratic in $G_3 = H_3 + iF_3$ but they appear to be not known completely.

Ref. [49] discussed some subset of terms with $\mathcal{R}^4 \to \mathcal{R}^4 + 6\mathcal{R}^2|\nabla G|^2 + \ldots$ in (B.1) and extra $|\nabla G|^4$ that have different index structure. Another approach followed in [50] was to use contraction tensors t_m which are expected to arise from 10d superspace integrals. In

[19] We shall ignore the dependence on the dilaton and set $\epsilon_8\epsilon_8 = -\frac{1}{2}\epsilon_{10}\epsilon_{10}$.

JHEP03(2024)045

particular, they considered the following subset of terms (c_n and c_5 are numerical constants)

$$\mathcal{L} \sim \sum_{n=0}^{4} c_n \, t_{24} \, G_3^{2n} \bar{\mathcal{R}}_{(6)}^{4-n} \,, \tag{B.2}$$

$$\bar{\mathcal{R}}_{ABCDEK} = \tfrac{1}{8} g_{CK} \mathcal{C}_{ABDE} + \tfrac{i}{48} \nabla_A F_{BCDEK} \tag{B.3}$$

$$+ \tfrac{1}{768} \Big(F_{ABCLM} F_{DEK}{}^{LM} - 3 F_{ABKLM} F_{CDE}{}^{LM} \Big) + c_5 G_{ABC} \bar{G}_{DEK} \,, \tag{B.4}$$

where $\mathcal{C}_{ABCD}$ is the Weyl tensor and F_5 is the RR 5-form. We refer to [50] for details.

C Solution of the 2-form equations of motion

In this appendix we solve the equation (3.7) by imposing the spherically symmetric Ansatz (3.10). At finite value of the resolution parameter a it is useful to define $\tilde{k} = p^{-1} k$ such that $k\chi = \tilde{k}\tilde{\chi}$ (cf. (2.9), (2.16)). In the orbifold limit $a \to 0$ the distinction between $\tilde{k}$ and k disappears.

Demanding closure of the 3-form ω results in the equation

$$2 f_1(\rho) - i\tilde{k} f_2(\rho) + f_3'(\rho) = 0 \,. \tag{C.1}$$

Eq. (3.7) takes the form of 3 coupled ordinary differential equations

$$-iM f_1(\rho) = \partial_\rho \big[\cot\rho \, V(\rho) \, f_2(\rho) \big] + i\tilde{k} \frac{1}{\cos\rho \sin\rho} f_3(\rho) \,, \tag{C.2}$$

$$-iM f_2(\rho) = \partial_\rho \big[\tan\rho \, f_1(\rho) \big] + \frac{2}{\cos\rho \sin\rho} f_3(\rho) \,, \tag{C.3}$$

$$-iM f_3(\rho) = i\tilde{k} \tan\rho \, f_1(\rho) - 2 \cot\rho \, V(\rho) \, f_2(\rho) \,, \tag{C.4}$$

which are consistent with (C.1). We can solve algebraically for f_1 and f_2

$$f_1(\rho) = -\frac{\tilde{k} M \tan\rho \, f_3(\rho) + 2V(\rho) f_3'(\rho)}{W(\rho)} \,, \qquad f_2(\rho) = i \frac{2M \tan\rho \, f_3(\rho) - \tilde{k} \tan^2\rho \, f_3'(\rho)}{W(\rho)} \,, \tag{C.5}$$

$$W(\rho) = \tilde{k}^2 \tan^2\rho + 4V(\rho) \,. \tag{C.6}$$

We then get a second order linear ODE for f_3 of the form

$$f_3''(\rho) + P(\rho) f_3'(\rho) + Q(\rho) f_3(\rho) = 0 \,, \tag{C.7}$$

$$P(\rho) = \frac{1}{\cos\rho \sin\rho} - \frac{W'(\rho)}{W(\rho)} + \frac{V'(\rho)}{V(\rho)} \,, \tag{C.8}$$

$$Q(\rho) = \frac{1}{V(\rho)} \left[-\frac{W(\rho)}{\sin^2\rho} + M^2 - \tilde{k} M \left(\frac{\tan\rho}{2} \frac{W'(\rho)}{W(\rho)} - \frac{1}{\cos^2\rho} \right) \right] . \tag{C.9}$$

Introducing $\tilde{f}_3(\rho)$ defined as

$$f_3(\rho) = \exp\left[-\frac{1}{2} \int P(\rho) \right] \tilde{f}_3(\rho) = \sqrt{\cot\rho \frac{W(\rho)}{V(\rho)}} \; \tilde{f}_3(\rho) \,, \tag{C.10}$$

JHEP03(2024)045

we get for it the Schrödinger type equation (3.11), with the potential

$$\mathrm{U}(\rho) = \frac{1}{4}P^2(\rho) + \frac{1}{2}P'(\rho) - Q(\rho)\,. \tag{C.11}$$

$\mathrm{U}(\rho)$ has the form of a smooth well between two poles at $\rho = 0$ and $\rho = \frac{\pi}{2}$ $(\epsilon \to 0)$

$$\mathrm{U}(0+\epsilon) \sim \frac{15}{4\epsilon^2} + \mathcal{O}(1)\,, \qquad \mathrm{U}\left(\frac{\pi}{2} - \epsilon\right) \sim \frac{k^2 - \frac{1}{4}}{\epsilon^2} + \mathcal{O}(1)\,. \tag{C.12}$$

For $a > 0$ we also find a pole at $\rho = a$ as is evident from (3.14) and is illustrated in 2.

Imposing appropriate boundary conditions restricts the value of the parameter M in (C.9) to a discrete set for every given $\tilde{k}$.

For $a = 0$ the spectrum of normalisable modes corresponds to a subsector of the usual Kaluza-Klein spectrum on the (orbifolded) sphere [29]. It is instructive to study this case in more detail.

Solutions for $\boldsymbol{a = 0}$. At $a = 0$ the coefficient functions P and Q take the form

$$P_0(\rho) = \frac{1}{\cos\rho\sin\rho}\left(1 - 2k^2\frac{\tan^2\rho}{W_0(\rho)}\right), \qquad Q_0(\rho) = -\frac{W_0(\rho)}{\sin^2\rho} + M^2 + \frac{4kM}{W_0(\rho)\cos^2\rho}\,, \tag{C.13}$$

where $W_0(\rho) = k^2\tan^2\rho + 4$ is the value of $W(\rho)$ in (C.6) at $a = 0$. We can put (C.7) in the Sturm-Liouville form

$$\partial_\rho\left[\frac{\tan\rho}{W_0(\rho)}\partial_\rho f_3(\rho)\right] + \frac{\tan\rho}{W_0(\rho)}Q_0(\rho)f_3(\rho) = 0\,. \tag{C.14}$$

The corresponding norm of f_3 is then

$$\|f_3\|^2 = \int_0^{\frac{\pi}{2}} \mathrm{d}\rho\,\frac{\tan\rho}{W_0(\rho)}f_3(\rho)\bar{f}_3(\rho)\,. \tag{C.15}$$

Normalisability is guaranteed if f_3 vanishes at both singularities. Close to the singularities, (C.7) takes the form of the Bessel equation, and we can extract the leading asymptotics of the solution as

$$f_3(\rho) \overset{\rho\to 0}{\sim} c\sin^2\rho + \tilde{c}\sin^{-2}\rho\,, \qquad f_3(\rho) \overset{\rho\to\frac{\pi}{2}}{\sim} d\cos^k\rho + \tilde{d}\cos^{-k}\rho\,. \tag{C.16}$$

Then normalisability restricts $\tilde{c} = \tilde{d} = 0$. Introducing $h(\rho)$ defined by

$$f_3(\rho) = \sin^2\rho\cos^k\rho\, h(\rho)\,, \tag{C.17}$$

we get for it the following equation

$$\partial_\rho\Big[\mu(\rho)\partial_\rho h(\rho)\Big] + \mu(\rho)(M-k-2)\left[(M+k+2) + \frac{4k}{W_0\cos^2\rho}\right]h(\rho) = 0\,, \tag{C.18}$$

where

$$\mu(\rho) = \frac{\cos^{2k-1}\rho\sin^5\rho}{W_0(\rho)}\,, \tag{C.19}$$

JHEP03(2024)045

is again the relevant measure that defines the Sturm-Liouville norm. We shall assume Neumann boundary conditions.

The potential in (C.18) vanishes for $M = k + 2$, in which case this equation is solved by constant $h(\rho)$. This corresponds to the effective 5d masses

$$m^2 = (k+2)(k+6)\,, \qquad \bar{m}^2 = (k+2)(k-2)\,, \tag{C.20}$$

which match the spectrum found in [29]. We also find normalisable modes at every successive value $M = k + 2(n + 1)$, where $n \in \mathbb{N}_0$ denotes the number of zeros of the function $h(\rho)$. This is the KK tower of 2-form excitations obeying our symmetry requirements. If we try to extend this tower to $n < 0$ we instead find that the boundary asymptotics change to $f_3(\rho) \sim \sin^{-2}\rho$. We also find a tower of non-normalisable solutions at $M = k - 2n$, $n \in \mathbb{N}_0$.

Twisted sector solutions. Turning to the discussion of solutions corresponding to the twisted sector, let us now pretend that the boundary conditions at $\rho = 0$ can be violated and $f_3 \sim \sin^{-2}\rho$ is still a valid solution. We can then again extract a factor from f_3 by setting

$$f_3(\rho) = \frac{\cos^k \rho}{\sin^2 \rho}\,\tilde{h}(\rho)\,, \tag{C.21}$$

which results in the following equation for $\tilde{h}$

$$\partial_\rho\Big(\tilde{\mu}(\rho)\partial_\rho\tilde{h}(\rho)\Big) + \tilde{\mu}(\rho)(M+k-2)\left((M-k+2) + \frac{4k}{W_0\cos^2\rho}\right)\tilde{h}(\rho) = 0\,, \tag{C.22}$$

$$\tilde{\mu}(\rho) = \frac{\cos^{2k-1}\rho\,\sin^{-3}\rho}{W_0(\rho)}\,, \tag{C.23}$$

which is again solved with Neumann boundary conditions.

To make contact with the relevant solution in (3.22), we need to require that the component

$$\Omega_{\chi z} = V(\rho)\,\cot\rho\, f_2(\rho) \overset{a=0}{=} \cot\rho\, f_2(\rho) \tag{C.24}$$

is finite at $\rho \to 0$. This singles out the non-normalisable solution at $M = -k$ given by[20]

$$f_3 = \frac{\cos^k\rho}{\sin^2\rho}\,(2\cos^2\rho + k\sin^2\rho)\,, \tag{C.25}$$

which reproduces the spectrum of twisted states dual to the T_k operators in (1.1). The other two functions in the full Ansatz (3.10) are given by

$$f_1(\rho) = \frac{\cos^{k+1}\rho}{\sin^3\rho}(2 + k\sin^2\rho)\,, \qquad f_2(\rho) = ik\cos^{k-1}\rho\sin\rho\,. \tag{C.26}$$

Then

$$\Omega = e^{ik\chi}\left(\tan\rho\, f_1(\rho)\,\sigma_x\wedge\sigma_y + \cot\rho\, f_2(\rho)\,\mathrm{d}\chi\wedge\sigma_z - \frac{f_3(\rho)}{\cos\rho\sin\rho}\,\mathrm{d}\rho\wedge\sigma_z\right)\,, \tag{C.27}$$

takes the form given in (3.13).

[20] From the previous perspective, this is a diagonal cross-section through the tower $M = k - 2n$ at $n = k$.

JHEP03(2024)045

D Weyl tensors in Vielbein basis

pp-wave background. We can write the metric (5.5) in a partial Vielbein basis

$$e^i = \mathrm{d}y^i\,, \qquad e^5 = \frac{1}{\sqrt{V_0(\rho)}}\mathrm{d}\rho\,, \qquad e^6 = \rho\,\sigma_x\,, \qquad e^7 = \rho\,\sigma_y\,, \qquad e^8 = \rho\sqrt{V_0(\rho)}\,\sigma_z\,, \tag{D.1}$$

$$\mathrm{d}s^2_{10} = -4\mathrm{d}x^+\mathrm{d}x^- - \left(y^2+\rho^2\right)(\mathrm{d}x^+)^2 + \sum_{A=1}^{8}\mathrm{d}e^A\mathrm{d}e^A\,. \tag{D.2}$$

The corresponding components of F_5 in (5.7) are then $F_{+1234} = F_{+5678} = 4$. In this basis, the non-vanishing components of the Weyl tensor $\mathcal{C}$ are

$$\mathcal{C}_{5656} = \mathcal{C}_{5757} = \mathcal{C}_{6868} = \mathcal{C}_{7878} = -\mathcal{C}_{5678} = \mathcal{C}_{5768} = -2\frac{a^4}{\rho^6}\,, \quad \mathcal{C}_{5858} = \mathcal{C}_{6767} = -\mathcal{C}_{5867} = 4\frac{a^4}{\rho^6}\,, \tag{D.3}$$

$$\mathcal{C}_{+5+5} = -\mathcal{C}_{+6+6} = -\mathcal{C}_{+7+7} = \mathcal{C}_{+8+8} = \frac{a^4}{\rho^4}\,, \tag{D.4}$$

and other components related by the usual symmetries $\mathcal{C}_{ABCD} = -\mathcal{C}_{BACD} = -\mathcal{C}_{ABDC} = \mathcal{C}_{CDAB}$. Eqs. (D.3) represent the Weyl tensor $\mathcal{C}_{\mathrm{EH}}$ of Eguchi-Hanson space and (D.4) are the mixed components $\mathcal{C}_{\mathrm{mix}}$. This Weyl tensor vanishes in the orbifold limit $a \to 0$.

Resolved background $\mathcal{M}^5$. Similarly, we may introduce the following Vielbein-basis for the metric (2.17) (see (2.3) and (2.9))

$$\begin{aligned} e^\chi &= \frac{\cos\rho}{\sqrt{1-\sin^4 a}}\mathrm{d}\chi\,, \qquad e^5 = \frac{1}{\sqrt{V(\rho)}}\mathrm{d}\rho\,, \\ e^6 &= \sin\rho\,\sigma_x\,, \qquad e^7 = \sin\rho\,\sigma_y\,, \qquad e^8 = \sin\rho\sqrt{V(\rho)}\,\tilde{\sigma}_z\,, \end{aligned} \tag{D.5}$$

$$\mathrm{d}s^2_{10} = \mathrm{d}s^2_{AdS_5} + e^\chi e^\chi + \sum_{a=5}^{8} e^a e^a\,. \tag{D.6}$$

The 5-form components on $\mathcal{M}^5$ are then $F_{\chi 5678} = 4$.

We find that the corresponding Weyl tensor expanded in small ρ is given by $\mathcal{C}_{\mathrm{EH}}$ (D.3) of the Eguchi-Hanson space plus higher order corrections $\mathcal{O}(\rho^{-4})$. The remaining mixed components representing $\mathcal{C}_{\mathrm{mix}}$ are

$$\mathcal{C}_{\chi 5\chi 5} = -\mathcal{C}_{\chi 6\chi 6} = -\mathcal{C}_{\chi 7\chi 7} = \mathcal{C}_{\chi 8\chi 8} = \frac{\sin^4 a}{\sin^4\rho}\,. \tag{D.7}$$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

[1] S. Kachru and E. Silverstein, *4-D conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [hep-th/9802183] [INSPIRE].

[2] A.E. Lawrence, N. Nekrasov and C. Vafa, *On conformal field theories in four-dimensions*, *Nucl. Phys. B* **533** (1998) 199 [hep-th/9803015] [INSPIRE].

JHEP03(2024)045

[3] M. Bershadsky, Z. Kakushadze and C. Vafa, *String expansion as large N expansion of gauge theories*, *Nucl. Phys. B* **523** (1998) 59 [hep-th/9803076] [INSPIRE].

[4] M. Bershadsky and A. Johansen, *Large N limit of orbifold field theories*, *Nucl. Phys. B* **536** (1998) 141 [hep-th/9803249] [INSPIRE].

[5] S. Gukov, *Comments on N = 2 AdS orbifolds*, *Phys. Lett. B* **439** (1998) 23 [hep-th/9806180] [INSPIRE].

[6] I.R. Klebanov and E. Witten, *Superconformal field theory on three-branes at a Calabi-Yau singularity*, *Nucl. Phys. B* **536** (1998) 199 [hep-th/9807080] [INSPIRE].

[7] I.R. Klebanov and N.A. Nekrasov, *Gravity duals of fractional branes and logarithmic RG flow*, *Nucl. Phys. B* **574** (2000) 263 [hep-th/9911096] [INSPIRE].

[8] A. Gadde, E. Pomoni and L. Rastelli, *The Veneziano Limit of N = 2 Superconformal QCD: Towards the String Dual of N = 2 SU(N(c)) SYM with N(f) = 2 N(c)*, arXiv:0912.4918 [INSPIRE].

[9] M. Beccaria and A.A. Tseytlin, *$1/N$ expansion of circular Wilson loop in $\mathcal{N} = 2$ superconformal* $\mathrm{SU}(N) \times \mathrm{SU}(N)$ *quiver*, *JHEP* **04** (2021) 265 [*Erratum ibid.* **01** (2022) 115] [arXiv:2102.07696] [INSPIRE].

[10] M. Beccaria, G.P. Korchemsky and A.A. Tseytlin, *Strong coupling expansion in $\mathcal{N} = 2$ superconformal theories and the Bessel kernel*, *JHEP* **09** (2022) 226 [arXiv:2207.11475] [INSPIRE].

[11] M. Beccaria, G.P. Korchemsky and A.A. Tseytlin, *Non-planar corrections in orbifold/orientifold $\mathcal{N} = 2$ superconformal theories from localization*, *JHEP* **05** (2023) 165 [arXiv:2303.16305] [INSPIRE].

[12] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[13] T. Skrzypek, *Integrability treatment of AdS/CFT orbifolds*, *J. Phys. A* **56** (2023) 345401 [arXiv:2211.03806] [INSPIRE].

[14] A. Pini, D. Rodriguez-Gomez and J.G. Russo, *Large N correlation functions $\mathcal{N} = 2$ superconformal quivers*, *JHEP* **08** (2017) 066 [arXiv:1701.02315] [INSPIRE].

[15] M. Billò et al., *Two-point correlators in N = 2 gauge theories*, *Nucl. Phys. B* **926** (2018) 427 [arXiv:1705.02909] [INSPIRE].

[16] M. Beccaria et al., *$\mathcal{N} = 2$ Conformal SYM theories at large $\mathcal{N}$*, *JHEP* **09** (2020) 116 [arXiv:2007.02840] [INSPIRE].

[17] F. Galvagno and M. Preti, *Chiral correlators in $\mathcal{N} = 2$ superconformal quivers*, *JHEP* **05** (2021) 201 [arXiv:2012.15792] [INSPIRE].

[18] M. Beccaria et al., *Exact results in a $\mathcal{N} = 2$ superconformal gauge theory at strong coupling*, *JHEP* **07** (2021) 185 [arXiv:2105.15113] [INSPIRE].

[19] B. Fiol and A.R. Fukelman, *The planar limit of $\mathcal{N} = 2$ chiral correlators*, *JHEP* **08** (2021) 032 [arXiv:2106.04553] [INSPIRE].

[20] M. Billò et al., *Strong-coupling results for $\mathcal{N} = 2$ superconformal quivers and holography*, *JHEP* **10** (2021) 161 [arXiv:2109.00559] [INSPIRE].

[21] M. Billò et al., *Structure Constants in N = 2 Superconformal Quiver Theories at Strong Coupling and Holography*, *Phys. Rev. Lett.* **129** (2022) 031602 [arXiv:2206.13582] [INSPIRE].

JHEP03(2024)045

[22] M. Billò et al., *Localization vs holography in 4d $\mathcal{N}=2$ quiver theories*, JHEP **10** (2022) 020 [arXiv:2207.08846] [INSPIRE].

[23] M. Billò et al., *Strong coupling expansions in $\mathcal{N}=2$ quiver gauge theories*, JHEP **01** (2023) 119 [arXiv:2211.11795] [INSPIRE].

[24] T. Banks and M.B. Green, *Nonperturbative effects in $AdS_5 \times S^5$ string theory and $d=4$ SUSY Yang-Mills*, JHEP **05** (1998) 002 [hep-th/9804170] [INSPIRE].

[25] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, *Coupling constant dependence in the thermodynamics of $N=4$ supersymmetric Yang-Mills theory*, Nucl. Phys. B **534** (1998) 202 [hep-th/9805156] [INSPIRE].

[26] M.R. Douglas and G.W. Moore, *D-branes, quivers, and ALE instantons*, hep-th/9603167 [INSPIRE].

[27] D. Berenstein and R.G. Leigh, *Discrete torsion, AdS / CFT and duality*, JHEP **01** (2000) 038 [hep-th/0001055] [INSPIRE].

[28] T. Eguchi and A.J. Hanson, *Selfdual Solutions to Euclidean Gravity*, Annals Phys. **120** (1979) 82 [INSPIRE].

[29] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, *The Mass Spectrum of Chiral $N=2$ $D=10$ Supergravity on S^5*, Phys. Rev. D **32** (1985) 389 [INSPIRE].

[30] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253 [hep-th/9802150] [INSPIRE].

[31] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT_d / AdS_{d+1} correspondence*, Nucl. Phys. B **546** (1999) 96 [hep-th/9804058] [INSPIRE].

[32] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three point functions of chiral operators in $D=4$, $N=4$ SYM at large N*, Adv. Theor. Math. Phys. **2** (1998) 697 [hep-th/9806074] [INSPIRE].

[33] M.B. Green, J.H. Schwarz and L. Brink, *$N=4$ Yang-Mills and $N=8$ Supergravity as Limits of String Theories*, Nucl. Phys. B **198** (1982) 474 [INSPIRE].

[34] D.J. Gross and E. Witten, *Superstring Modifications of Einstein's Equations*, Nucl. Phys. B **277** (1986) 1 [INSPIRE].

[35] N. Sakai and Y. Tanii, *One Loop Amplitudes and Effective Action in Superstring Theories*, Nucl. Phys. B **287** (1987) 457 [INSPIRE].

[36] D.E. Berenstein, J.M. Maldacena and H.S. Nastase, *Strings in flat space and pp waves from $N=4$ superYang-Mills*, JHEP **04** (2002) 013 [hep-th/0202021] [INSPIRE].

[37] M. Blau, J.M. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, JHEP **01** (2002) 047 [hep-th/0110242] [INSPIRE].

[38] N. Itzhaki, I.R. Klebanov and S. Mukhi, *PP wave limit and enhanced supersymmetry in gauge theories*, JHEP **03** (2002) 048 [hep-th/0202153] [INSPIRE].

[39] M. Alishahiha and M.M. Sheikh-Jabbari, *The pp wave limits of orbifolded $AdS_5 \times S^5$*, Phys. Lett. B **535** (2002) 328 [hep-th/0203018] [INSPIRE].

[40] N. Kim, A. Pankiewicz, S.-J. Rey and S. Theisen, *Superstring on PP wave orbifold from large N quiver gauge theory*, Eur. Phys. J. C **25** (2002) 327 [hep-th/0203080] [INSPIRE].

[41] E. Floratos and A. Kehagias, *Penrose limits of orbifolds and orientifolds*, *JHEP* **07** (2002) 031 [hep-th/0203134] [INSPIRE].

[42] E.M. Sahraoui and E.H. Saidi, *Metrics building of pp wave orbifold geometries*, *Phys. Lett. B* **558** (2003) 221 [hep-th/0210168] [INSPIRE].

[43] M. Beccaria and G.P. Korchemsky, *Four-dimensional $\mathcal{N} = 2$ superconformal long circular quivers*, arXiv:2312.03836 [INSPIRE].

[44] L.J. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on Orbifolds*, *Nucl. Phys. B* **261** (1985) 678 [INSPIRE].

[45] K. Becker, M. Becker and J.H. Schwarz, *String theory and M-theory: A modern introduction*, Cambridge University Press (2006) [DOI:10.1017/CBO9780511816086] [INSPIRE].

[46] M.T. Grisaru and D. Zanon, *σ Model Superstring Corrections to the Einstein-hilbert Action*, *Phys. Lett. B* **177** (1986) 347 [INSPIRE].

[47] M.D. Freeman, C.N. Pope, M.F. Sohnius and K.S. Stelle, *Higher Order σ Model Counterterms and the Effective Action for Superstrings*, *Phys. Lett. B* **178** (1986) 199 [INSPIRE].

[48] J.T. Liu and R. Minasian, *Higher-derivative couplings in string theory: dualities and the B-field*, *Nucl. Phys. B* **874** (2013) 413 [arXiv:1304.3137] [INSPIRE].

[49] J.T. Liu and R. Minasian, *Higher-derivative couplings in string theory: five-point contact terms*, *Nucl. Phys. B* **967** (2021) 115386 [arXiv:1912.10974] [INSPIRE].

[50] J.T. Liu, R. Minasian, R. Savelli and A. Schachner, *Type IIB at eight derivatives: insights from Superstrings, Superfields and Superparticles*, *JHEP* **08** (2022) 267 [arXiv:2205.11530] [INSPIRE].

JHEP03(2024)045