Supersymmetry and trace formulas. Part I. Compact Lie groups

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ABSTRACT: In the context of supersymmetric quantum mechanics we formulate new supersymmetric localization principle, with application to trace formulas for a full thermal partition function. Unlike the standard localization principle, this new principle allows to compute the supertrace of non-supersymmetric observables, and is based on the existence of fermionic zero modes. We describe corresponding new invariant supersymmetric deformations of the path integral; they differ from the standard deformations arising from the circle action and require higher derivatives terms. Consequently, we prove that the path integral localizes to periodic orbits and not only on constant ones. We illustrate the principle by deriving bosonic trace formulas on compact Lie groups, including classical Jacobi inversion formula.

Contents

1   Introduction

Supersymmetry, a global symmetry between bosons and fermions, provides invaluable insights to the non-perturbative aspects of general strongly coupled quantum field theories, and is deeply related to various areas of mathematics. This was elucidated by E. Witten in papers [39] and [40], written from physics and mathematics perspective.

Specifically, the Hilbert space $\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^-$ of a supersymmetric quantum theory is graded by a fermion number and has an operator $(-1)^F$, which counts the number of fermions modulo two. The precise non-perturbative information about the ground states of a supersymmetric quantum Hamiltonian[1] $\hat{H}$ is given by the Witten index

$$I = \operatorname{Str} e^{-\beta \hat{H}} = \operatorname{Tr}(-1)^F e^{-\beta \hat{H}},$$

which mathematically is related to the index of the Dirac operator associated with the supersymmetry algebra.

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[1]We use notation $\hat{H}$ to distinguish from the classical Hamiltonian H.

Remarkably, this interpretation led to a beautiful derivation of the Atiyah-Singer formula for the index of a Dirac operator using supersymmetric quantum mechanics [2]. In this context, the index of a Dirac operator is the Witten index, represented as a path integral for a certain supersymmetric sigma model. Since the index does not depend on the parameter β (playing the role of inverse temperature), in the high-temperature limit $\beta \to 0$, the path integral localizes to the constant maps and becomes a finite-dimensional integral representing the $\hat{A}$-genus of a corresponding spin manifold. As it was eloquently explained by Atiyah [3], this approach admits a natural equivariant cohomological interpretation as an infinite-dimensional version of the Duistermaat-Heckman formula for the loop space of a spin manifold, where the fixed points of the circle action are constant loops. This is a paradigmatic example of localization of a path integral, where the original infinite dimensional path integral reduces to a finite dimensional integral around the fixed points.

Subsequently, the idea of localization was extended to quantum field theories and led to many discoveries. To name a few, earlier important milestones of localizations were on topological and cohomological quantum field theories: computation of Donaldson invariant from a twisted four-dimensional supersymmetric gauge theory [41], exact path integrals on 2d Yang-Mills theories [42] (see review [13]), and the computation of Seiberg-Witten prepotential [30] building upon [27, 29]. More recently, the localization has been successfully applied to non-topological supersymmetric field theories, starting from four dimensional $N = 2$ gauge theories [31], which then has been extended to three dimensions [24] and two dimensions [4, 5, 16], see [32] for more comprehensive reviews. These discoveries confirm the common belief that localization of a supersymmetric path integral is always a computation of the supertrace and not of the trace.

Indeed, in all these examples the localization has been applied to 'supersymmetric' observables, which are invariant under at least a single supercharge of the theory. However, it is clear that supersymmetric observables alone do not characterize the full physical theory. Therefore, a somewhat unorthodox question is the following: can one apply localization to the computation of a non-supersymmetric observable? In this paper, we focus on quantum mechanics and find that the answer is positive by extending the applicability of localization to a certain class of non-supersymmetric observables. Here we show that for some specific theories, it is possible to extend the localization principle to non-supersymmetric observables used to compute the partition function.

Namely, a fundamental object of a quantum theory is the partition function of a system,

$$Z(\beta) = \operatorname{Tr} e^{-\beta \hat{H}},$$

whose dependence on the inverse temperature β is highly non-trivial. In the spectral geometry one studies $Z(\beta)$ in case $\hat{H} = \frac{1}{2}\Delta$, the Laplace operator on a compact Riemannian manifold M, acting on the Hilbert space $L^2(M)$ of square-integrable functions with respect to the Riemannian volume form on M. By definition,

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta \lambda_n / 2},$$

where λ_n are the eigenvalues of Δ. In general, only asymptotics of $Z(\beta) = \operatorname{Tr} e^{-\frac{1}{2}\beta\Delta}$ and of the corresponding heat kernel are known as $\beta \to 0$, and no closed expression for $Z(\beta)$ exists in geometric terms.

In special cases when M is either a flat torus or a hyperbolic Riemann surface, the partition function $Z(\beta) = \operatorname{Tr} e^{-\frac{1}{2}\beta\Delta}$ can be computed exactly in terms of the underlying geometry of M. Thus in case $M = S^1$ classical Jacobi inversion formula for the theta-series [23] represents $Z(\beta)$ as a sum over closed geodesics of the flat metric on S^1. In case M is a hyperbolic compact Riemann surface, the corresponding result is the celebrated Selberg trace formula [37], which expresses $Z(\beta)$ as a sum over closed geodesics of the hyperbolic metric on M. These formulas can be thought of as a manifestation of the principle "Spectral trace = Matrix trace" in the abelian and non-abelian settings. An easier case is an explicit formula for the heat kernel of a Laplace operator for the bi-invariant metric on a compact semi-simple Lie group G, first obtained by L.D. Eskin [19], and later rediscovered and used by many authors [12, 17, 18, 20, 28, 33, 36].

One may ask when the matrix trace can be exactly computed by the path integral for a quantum particle on a Riemannian manifold M. This is obviously so in case of S^1, since the path integral is Gaussian and can be computed exactly. In case when $M = G$, a compact simple Lie group with a bi-invariant metric, M.S. Marinov and M.V. Terentyev [28] considered a semi-classical approximation of the path integral for a free quantum particle on G and obtained the Eskin formula for the heat kernel, generalizing the observations of the exactness of semi-classical approximation in $\operatorname{SU}(2)$ and $\operatorname{SU}(N)$ cases by L. Schulman and J.S. Dowker [18, 36]. It was indicated by R.F. Picken [33] that this remarkable property of purely bosonic path integral on G is related to the infinite-dimensional Duistermaat-Heckman formalism, outlined by Atiyah [3]. However, all these papers use an ad hoc addition of the so-called DeWitt term [15], a 'notorious' quantum correction to the Lagrangian for the path integral on curved spaces. Further, M. Gutzwiller [21] observed that when M is a hyperbolic Riemann surface, the critical points of the path integral in the semi-classical approximation are closed geodesics; the resulting formula, called 'Gutzwiller trace formula', is quite similar to the Selberg trace formula. However, Gutzwiller trace formula does not adequately reproduce the contribution of the identity element to the Selberg trace formula, and also uses an ad hoc addition of the DeWitt term.

Naturally, this calls for a question of whether there is a new localization principle that allows to compute pure bosonic partition function $Z(\beta)$ by localizing some supersymmetric path integral to the closed geodesics. At first glance, this sounds rather counter-intuitive since, unlike the Witten index I, $Z(\beta)$ non-trivially depends on β, so the standard localization principle does not apply. Nevertheless, here we introduce a new principle of supersymmetric localization, and use it to obtain trace formulas for S^1 and G by localizing on closed geodesics. Note that in this examples we have $I = 0$, and it is rather amusing that this vanishing of the Witten index provides a key for the answer!

Namely, consider the supersymmetric Lagrangian $\mathcal{L}$ with the action S, in which fermion degrees of freedom totally decouple, so the Hamiltonian H is purely bosonic. Then the fermion part (assuming that it is contains only first time derivatives) has zero modes and the Witten index I, the supersymmetric path integral with periodic boundary conditions, is

zero. Insertion of the fermion zero modes $\chi^1, \ldots, \chi^n$ into the path integral gives

$$I(\beta) = \mathrm{Tr} \ \hat{\chi}^1 \cdots \hat{\chi}^n (-1)^F e^{-\beta \hat{H}},$$

which is non-zero. If, in addition, the operator $\hat{\chi}^1 \cdots \hat{\chi}^n$ is proportional to $(-1)^F$, then the index with zero modes insertion $I(\beta)$ is equal to the partition function $Z(\beta)$! However, due to the presence of zero modes the functional $\chi^1 \cdots \chi^n e^{-S_E}$, where S_E is the Euclidean action, is no longer invariant under supersymmetry transformation, $\delta(\chi^1 \cdots \chi^n e^{-S_E}) \neq 0$. Thus the path integral of $\chi^1 \cdots \chi^n e^{-S_E + s \delta V}$, where V is standard supersymmetric deformation associated with the circle action, is no longer s-independent (otherwise it would localize on constant loops). However, in many cases there is another deformation V, containing higher derivatives of the fields, such that the path integral of $\chi^1 \cdots \chi^n e^{-S + sV}$ is s-independent and in the limit $s \to \infty$ localizes on the closed geodesics!

The present paper was greatly influenced by the work of J.-M. Bismut in [8–10], and it is illuminating to compare our approaches. Thus Bismut uses hypoelliptic deformation on the cotangent bundle of the manifold that smoothly interpolates between the Laplacian and the geodesic flow; as it is clearly explained in [8], such hypoelliptic deformation is a generalization of Witten's twist in Morse theory [40]. Henceforth such approach can be thought of as a generalization of the Hamiltonian approach to supersymmetry, and one needs to find its physics interpretation. On the other hand, our approach is purely Lagrangian and extends the equivariant cohomology used for supersymmetric path integrals.

The content of the paper is the following. In section 2.1 we recall standard basic facts on supersymmetric localization in the Hamiltonian (using equivariant cohomology) and in the Lagrangian (using path integral) approaches. In section 2.2 we explicitly formulate our new localization principle. Namely, we start with elementary finite-dimensional Lemma 1 and carefully list all necessary conditions. They are based on the zero Witten index and assume that the system has fermion zero modes satisfying conditions (i)–(ii). Conditions (A)–(B) represent requirements on the deformation V, necessary for the Proposition 1 on the independence of the path integral of the parameter s.

In section 3 we use Proposition 1 to evaluate the partition function $Z(\beta)$ on the circle S^1 with the flat metric using supersymmetric localization, which yields classical Jacobi inversion formula for theta-series. In section 4 we derive the Eskin trace formula on the compact semi-simple Lie group G. Specifically, in section 4.1 we present necessary facts about supersymmetric particle on G, relegating standard details to appendices A, B and C. Finally, in section 4.2 we prove Theorem 1, which gives a supersymmetric derivation of the Eskin trace formula on G.

Acknowledgments

2 General remarks on new localization principle

2.1 Standard localization principle

We start by briefly recalling a finite-dimensional localization principle. Let M be compact orientable n-dimensional manifold with an action of the abelian group $U(1) = S^1$, and let v the vector field corresponding to this action. The corresponding equivariant differential is

$$D = d - \iota_v \tag{2.1}$$

where ι_v is the inner product operator with v. It satisfies

$$D^2 = -\mathscr{L}_v,$$

where $\mathscr{L}_v$ stands for the Lie derivative, and is a differential in the subcomplex $\Omega^\bullet_{S^1}(M)$ of the complex $\Omega^\bullet(M)$, consisting of S^1-invariant differential forms on M; $\alpha \in \Omega^\bullet_{S^1}(M)$ if $\mathscr{L}_v \alpha = 0$.

Let $\alpha \in \Omega^\bullet(M)$ be an equivariantly closed form, $D\alpha = 0$. The localization principle is the statement that for every $V \in \Omega^1_{S^1}(M)$ and $s \in \mathbb{R}$ we have

$$\int_M \alpha = \int_M \alpha \wedge e^{-sDV}. \tag{2.2}$$

By a suitable choice of V one can make $(DV)_0$, a component of DV in $\Omega^0(M)$, positive semi-definite, so in the limit $s \to \infty$ the integral $\int_M \alpha$ localizes onto the zero loci of $(DV)_0$. In case when M is even-dimensional and the circle action has only isolated fixed points, one gets Berline-Vergne localization formula [6].

The proof of (2.2) is very simple. We have

$$
\begin{aligned}
-\frac{d}{ds} \int_M \alpha \wedge e^{-sDV} &= \int_M \alpha \wedge DV \wedge e^{-sDV} \\
&= \int_M D(V \wedge \alpha \wedge e^{-sDV}) + \int_M V \wedge D(\alpha \wedge e^{-sDV}) \\
&= 0.
\end{aligned}
$$

Here the first integral in the second line is zero by the Stokes' theorem since the top component of an equivariantly exact form is exact, and the second integral is zero because of $D\alpha = 0$ and $D^2 V = 0$.

The infinite-dimensional case was eloquently explained by Atiyah [3], elaborating the observation by Witten. Namely, one replaces a finite-dimensional Riemannian manifold M by its loop space $LM = \text{Map}(S^1_\beta, M)$ where $S^1_\beta = \mathbb{R}/\beta\mathbb{Z}$ is a 'thermal' circle. The loop space LM is orientable when M is a spin manifold and has a natural circle action with the vector field v. The equivariant differential on LM has the same form (2.1), where now d stands for the de Rham differential on LM. The cotangent bundle T^*LM carries a natural S^1-invariant 1-form θ, dual to the vector field v with respect to the Riemannian metric on LM. The closed 2-form $\omega = d\theta$ plays the role of symplectic form on LM (which is degenerate along closed geodesics), and the circle action on LM is Hamiltonian: there is a function $H : LM \to \mathbb{R}$ such that

$$i_v \omega = -dH.$$

Since $-H + \omega = D\theta$, the differential form $\alpha = e^{-H+\omega} \in \Omega^\bullet(LM)$ is equivariantly closed, so analogously to (2.2) for any $V \in \Omega^1_{S^1}LM$ we have

$$\int_{LM} e^{-H+\omega} = \int_{LM} e^{-H+\omega-sDV}. \tag{2.3}$$

Here integral over LM stands for the integration of the top component of a differential form on the loop space. In particular, putting $V = \theta$, in the limit $s \to \infty$ we obtain the Atiyah-Singer formula for the index of the Dirac operator on M.

This approach can be also formulated using supersymmetric quantum mechanics, as was originally proposed by Witten. Namely, consider a classical supersymmetric mechanical system with the Lagrangian $\mathcal{L}$ and Hamiltonian H, having a single real supercharge Q satisfying $\{Q, Q\} = 2iH$, where $\{\,,\,\}$ is the graded Poisson bracket. After the quantization we get the simplest $N = 1/2$ supersymmetric quantum system with real supercharge $\hat{Q}$ satisfying $\hat{Q}^2 = \hat{H}$, where quantum Hamiltonian $\hat{H}$ acts in the Hilbert space[2] $\mathscr{H}$, naturally $\mathbb{Z}_2$-graded by a fermion number operator $(-1)^F$.

The Witten index is given by the path integral

$$I = \mathrm{Tr}(-1)^F e^{-\beta\hat{H}} = \int e^{-S_E[x,\psi]}\mathscr{D}x\mathscr{D}\psi, \tag{2.4}$$

where

$$S_E[x,\psi] = \int_0^\beta \mathcal{L}_E(x,\dot{x};\psi,\dot{\psi})d\tau$$

is the Euclidean action for the Lagrangian $\mathcal{L}$, and $\mathscr{D}x\mathscr{D}\psi$ is a suitable supersymmetric path integration measure for the bosonic and fermionic degrees of freedom. The integration in (2.4) goes over for the periodic boundary conditions and we have

$$\delta S_E = 0 \quad \text{and} \quad \delta(\mathscr{D}x\mathscr{D}\psi) = 0,$$

where δ is the Wick rotated (for Euclidean time) classical supersymmetry transformation generated by a supercharge Q — the analogue of the equivariant differential D in the loop space approach. Let $V[x,\psi]$ be an invariant deformation, a functional of classical fields satisfying

$$\delta^2 V = 0.$$

Formally repeating the proof of (2.2), we obtain that for all s

$$\int e^{-S_E}\mathscr{D}x\mathscr{D}\psi = \int e^{-S_E-s\delta V}\mathscr{D}x\mathscr{D}\psi. \tag{2.5}$$

In case $S_E = \delta V$ one can take $V = S_E$, so in the limit $s \to \infty$ the path integral localizes on the zero locus of S_E. The latter is nothing but the set of constant loops, arising from the standard kinetic term in the action. Finally, the relation $V = \delta\theta$ and the identification of δ with D establishes equivalence with the loop space approach.

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[2] Here it is assumed that we have even number of Majorana fermions.

2.2 New supersymmetric localization principle

We start by considering the following simple finite-dimensional example.

Let M be compact orientable n-dimensional manifold with circle action and let $N = M \times S^1$. The $n+1$-dimensional manifold N has an obvious circle action and let α be an equivariant closed form on N such that $i_u \alpha = 0$, where u is a vector filed on N which at a point $(p, \varphi) \in N$ is $d/d\varphi$, where φ mod 2π is the coordinate on S^1. In other words, α does not contain the 1-form $d\varphi$ as a factor and the top component of α has degree n. The standard localization principle applies to the integral $\int_N \alpha$, which is obviously 0, but does not apply to the integral $\int_N d\varphi \wedge \alpha$, since the form $d\varphi \wedge \alpha$ is not equivariantly closed. However, we have the following statement.

Lemma 1. *Let $V \in \Omega^1_{S^1}(N)$ be such that $i_u V = 0$ and $i_u DV = 0$. Then for all s*

$$\int_N d\varphi \wedge \alpha = \int_N d\varphi \wedge \alpha \wedge e^{-sDV}.$$

Proof. It follows the derivation of (2.2). Namely,

$$
\begin{aligned}
-\frac{d}{ds} \int_N d\varphi \wedge \alpha \wedge e^{-sDV} &= \int_N d\varphi \wedge \alpha \wedge DV \wedge e^{-sDV} \\
&= \int_N D(V \wedge d\varphi \wedge \alpha \wedge e^{-sDV}) + \int_N V \wedge D(d\varphi \wedge \alpha \wedge e^{-sDV}) \\
&= \int_N V \wedge D(d\varphi) \wedge \alpha \wedge e^{-sDV} - \int_N V \wedge d\varphi \wedge D(\alpha \wedge e^{-sDV}) \\
&= 0.
\end{aligned}
$$

As before, the first integral in the second line is zero by the Stokes' theorem, and the last integral in the third line is zero because $D\alpha = 0$ and $D^2 V = 0$. The new feature is first integral in the third line, which is zero because $D(d\varphi) = -1$ and conditions $i_u V = 0$ and $i_u DV = 0$ imply that the form $V \wedge \alpha \wedge e^{-sDV}$ does not contain $d\varphi$, so its top component is zero. $\qquad \square$

As in the previous section, in the infinite-dimensional case, we start with a classical supersymmetric mechanical system with the Lagrangian $\mathcal{L}$ and Hamiltonian H, having a single real supercharge Q satisfying $\{Q, Q\} = 2iH$, where $\{ \, , \, \}$ is the graded Poisson bracket. As a basic example, consider a supersymmetric particle on the n-dimensional Riemannian manifold M with the Lagrangian[3]

$$\mathcal{L} = \frac{1}{2}\langle \dot{x}, \dot{x} \rangle + \frac{i}{2}\langle \psi, \nabla_{\dot{x}} \psi \rangle. \tag{2.6}$$

Here $x(t)$ is a path in M expressed in terms of local coordinates $x = (x^1, \ldots, x^n)$, $\dot{x}$ is the velocity vector field along the path, $\langle \, , \, \rangle$ is the Riemannian inner product in $T_{x(t)}M$ and $\psi(t) = (\psi^1(t), \ldots, \psi^n(t)) \in \Pi T_{x(t)}M$ is a tangent vector field along the path with the reverse parity (i.e, with the anti-commuting values). Finally, ∇ in (2.6) is an arbitrary metric

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[3]Here we use the 'physical' time and imaginary i is needed to make quadratic expressions in ψ "real", that is, $\overline{i\psi^\mu \psi^\nu} = -i\psi^\nu \psi^\mu = i\psi^\mu \psi^\nu$.

connection in TM with totally anti-symmetric torsion. The latter condition guarantees the standard supersymmetry

$$\delta x^\mu = i\psi^\mu, \quad \delta\psi^\mu = -\dot{x}^\mu, \quad \mu = 1, \ldots, n, \tag{2.7}$$

with a real supercharge Q, obtained by the Noether theorem (see sections 3 and 4).

Remark 1. In case of even n, after the quantization we get $N = 1/2$ supersymmetric quantum particle on M with a single real supercharge $\hat{Q}$ and quantum Hamiltonian $\hat{H} = \frac{1}{2}\hat{Q}^2$ acting in the $\mathbb{Z}_2$-graded Hilbert space $\mathscr{H}$ with the fermion number operator

$$(-1)^F = 2^{n/2}\hat{\psi}^1 \cdots \hat{\psi}^n.$$

Here $\hat{\psi}^\mu$ are Majorana fermion operators satisfying canonical anti-commutation relations

$$[\hat{\psi}^\mu, \hat{\psi}^\nu] = \delta^{\mu\nu}\hat{I}, \tag{2.8}$$

where $[\,,\,]$ is a graded commutator and $\hat{I}$ is an identity operator acting on the Hilbert space. In the case of odd n it is well-known (see [14] for the recent discussion and the references therein) that Majorana fermions are anomalous in the sense that there is no Hilbert space interpretation.[4] To remedy this situation, we replace M by $M \times S^1$ and trivially add a single term $\psi^{n+1}\dot{\psi}^{n+1}$ to the Lagrangian, while keeping the kinetic term unchanged. Thus in this case we also have a graded Hilbert space $\mathscr{H}$ where $\hat{H}$ acts, and it what follows we will always assume such modification for odd n.

Like in the finite-dimensional example, our new localization principle applies to the systems with zero Witten index,

$$I = \text{Tr}(-1)^F e^{-\beta\hat{H}} = \int e^{-S_E[x,\psi]}\mathscr{D}x\mathscr{D}\psi = 0.$$

This situation is quite general: the index of the standard Dirac operator vanishes on arbitrary odd-dimensional spin manifolds, on compact Lie groups, etc. Specifically, we assume the following conditions.

(i) The vanishing of the index is solely due to the presence of certain fermionic zero modes $\chi_1, \ldots, \chi_n$ in the path integral.[5]

(ii) In the Hilbert space $\mathscr{H}$ the operators $\hat{\chi}_\mu$ satisfy (2.8) and

$$\hat{\chi}_1 \cdots \hat{\chi}_n = 2^{-n/2}(-1)^F,$$

where $n = \dim M$ if M is even-dimensional, and $n = \dim M + 1$ otherwise.

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[4]Nevertheless, such system with an odd number of Majorana fermions are still physically well-defined in terms of path integral formulation.

[5]In some cases, when one starts with a supersymmetric system with a vanishing I but without fermionic zero modes, the robustness of the index allows one to smoothly deform the system for which I vanishes exclusively due to the presence of fermionic zero modes.

It follows from (i)–(ii) that

$$2^{-n/2} \operatorname{Tr} e^{-\beta \hat{H}} = \operatorname{Str} \hat{\chi}_1 \cdots \hat{\chi}_n e^{-\beta \hat{H}} = \int \chi_1 \cdots \chi_n e^{-S_E[x,\psi]} \mathscr{D}x \mathscr{D}\psi, \qquad (2.9)$$

so the path integral with periodic boundary conditions for fermions, with insertion of the zero modes, computes the trace in the Hilbert space $\mathscr{H}$ and not the supertrace!

However, the standard supersymmetric localization discussed in section 2.1 no longer applies, because

$$\delta(\chi_1 \cdots \chi_n e^{-S_E}) \neq 0,$$

and the path integral in (2.9) non-trivially depends on β.

To formulate our new localization principle, we further assume that fermion zero modes satisfy the following conditions

$$\int \delta\chi_\mu d\chi_\mu = 0, \quad \mu = 1, \ldots, n, \qquad (2.10)$$

which mean that $\delta\chi_\mu$ does not contain fermion degree freedom χ_μ, which we will indicate as $\delta\chi_\mu \perp \chi_\mu$. The following definition is fundamental.

Definition. Invariant deformation for a supersymmetric system with fermion zero modes $\chi_1, \ldots, \chi_n$ satisfying (2.10) is a functional V with a Grassmannian odd parity, satisfying the following conditions.

(A) V is invariant,
$$\delta^2 V = 0.$$

(B) $V, \delta V \perp \chi_\mu$, i.e.
$$\int V d\chi_\mu = \int \delta V d\chi_\mu = 0, \quad \mu = 1, \ldots, n.$$

Note that condition (A) is standard, while condition (B), the absence of fermion zero modes in V and δV, is a completely new requirement. It is rather constraining and forces V to explicitly depend on the first time derivatives of fermion degrees of freedom. The analog of Lemma 1 is the following surprisingly simple statement.

Proposition 1. *Let S_E be the Euclidean action of the supersymmetric theory with fermion zero modes $\chi_1, \ldots, \chi_n$ satisfying (2.10). Then for all s we have*

$$\int \chi_1 \cdots \chi_n e^{-S_E} \mathscr{D}x \mathscr{D}\psi = \int \chi_1 \cdots \chi_n e^{-S_E - s\delta V} \mathscr{D}x \mathscr{D}\psi,$$

where V is a deformation satisfying conditions (A)–(B).

Proof. We have

$$-\frac{d}{ds} \int \chi_1 \cdots \chi_n e^{-S_E - s\delta V} \mathscr{D}x \mathscr{D}\psi = \int \chi_1 \cdots \chi_n \delta V e^{-S_E - s\delta V} \mathscr{D}x \mathscr{D}\psi$$

$$= (-1)^n \int \left[\delta(\chi_1 \cdots \chi_n V) - \delta(\chi_1 \cdots \chi_n) V \right] e^{-S_E - s\delta V} \mathscr{D}x \mathscr{D}\psi$$

$$= 0.$$

Here the integral of the first term in the second line is zero by Stokes' theorem, and the integral of the second term is zero since by conditions (2.10) and (A)–(B) it does not contain zero modes. □

As was remarked before, conditions (A)–(B) imply that localization V contains higher derivatives and when δV is positive semi-definite, the path integral localizes to the solutions of the second or higher order ordinary differential equations. In particular, such V provides a mechanism for localizing on the geodesic locus. We will consider this application of the new localization principle in sections 3 and 4.

Note that Proposition 1 is rather general and does not use condition (ii). Actually, one can relax the condition (ii) and still obtain the information about the partition function when the corresponding supersymmetric system has decoupled bosonic and fermionic degrees of freedom, so

$$\mathscr{H} = \mathscr{H}_B \otimes \mathscr{H}_F \quad \text{and} \quad \hat{H} = \hat{H}_B \otimes I_F + I_B \otimes \hat{H}_F,$$

where the fermion Hilbert space $\mathscr{H}_F$ is irreducible $2^{n/2}$-dimensional Clifford module (see Remark 1). In this case, the supertrace in (2.9) is proportional to bosonic trace,

$$\operatorname{Str} \hat{\chi}_1 \cdots \hat{\chi}_n e^{-\beta \hat{H}} = 2^{-n/2} \operatorname{Tr} e^{-\beta \hat{H}_B} \otimes e^{-\beta \hat{H}_F} = 2^{-n/2} \operatorname{Tr}_{\mathscr{H}_F} e^{-\beta \hat{H}_F} \cdot \operatorname{Tr}_{\mathscr{H}_B} e^{-\beta \hat{H}}.$$

In particular, if the supersymmetric system has free fermion part, then quantum Hamiltonian $\hat{H}$ acts as identity operator on the fermion Hilbert space $\mathscr{H}_F$ and

$$\operatorname{Str} \hat{\chi}_1 \cdots \hat{\chi}_n e^{-\beta \hat{H}} = \operatorname{Tr}_{\mathscr{H}_B} e^{-\beta \hat{H}}.$$

Using the representation (2.9) and Proposition 1, we obtain a pure bosonic trace formula by localizing the supersymmetric path integral (2.9) in the limit $s \to \infty$.

We will illustrate this general procedure in the next two sections by considering a simple case $M = S^1$ with the flat metric, and $M = G$, a compact connected semi-simple Lie group with the bi-invariant metric. In the first case the Lagrangian is quadratic so fermion degrees of freedom trivially decouple, while in the second case instead of the Levi-Civita connection we use invariant flat connection on G and corresponding Kostant cubic Dirac operator.

3 Localization proof of the trace formula on S^1

In this section, we consider the simplest application of the localization principle explained in section 2 by providing localization proof of the trace formula on S^1. This identity, the famous Jacobi inversion formula,[6] is related to the spectrum of the Laplace operator $\Delta = -\dfrac{d^2}{dx^2}$ on $\mathscr{H}_B = L^2(S^1)$, where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Namely, consider a free quantum particle of mass 1 on S^1 with the Hamiltonian $\hat{H} = \frac{1}{2}P^2 = \frac{1}{2}\Delta$, and let $Z(\beta) = \operatorname{Tr}_{\mathscr{H}_B} e^{-\beta \hat{H}}$ be the corresponding thermal partition function. We have

$$Z(\beta) = \sum_{n=-\infty}^{\infty} e^{-\beta n^2/2} = \sqrt{\frac{2\pi}{\beta}} \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2/\beta}, \quad \beta > 0. \tag{3.1}$$

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[6]It is a special case of general formula for Jacobi theta-function [23] and was already known to Poisson [34].

Remark 2. In the number theory, Jacobi inversion formula is usually written as

$$\theta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}}\,\theta(z), \quad \text{where} \quad \theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z} \quad \text{and} \quad \text{Im}\, z > 0$$

is the Jacobi theta series; the branch of the square root is defined by $\sqrt{1} = 1$. Of course, Jacobi inversion formula can be easily obtained from the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

where

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{2\pi i k x} dx.$$

Conversely, multiplying the Jacobi inversion formula for $z = it$ by a 'nice' function $g(t)$, integrating over t and denoting

$$f(x) = \int_0^{\infty} e^{-\pi t x^2} g(t) dt,$$

we get the Poisson summation formula.

It is rather remarkable that one can get Jacobi inversion formula (3.1) as an elementary application of the new supersymmetric localization principle, formulated in section 2.2. Namely, consider the simplest supersymmetric Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + i\psi\dot{\psi}) \tag{3.2}$$

which has a standard $N = 1/2$ supersymmetry (2.7). We have

$$\delta\mathcal{L} = \frac{1}{2}(2\delta\dot{x}\,\dot{x} + i\delta\psi\dot{\psi} - i\psi\delta\dot{\psi}) = \frac{i}{2}\frac{d}{dt}(\dot{x}\psi),$$

so by the Noether theorem,

$$iQ = \delta x \frac{\partial\mathcal{L}}{\partial\dot{x}} + \delta\psi\frac{\partial\mathcal{L}}{\partial\dot{\psi}} - \frac{i}{2}(\dot{x}\psi) = i\dot{x}\psi,$$

where $\dfrac{\partial}{\partial\dot{\psi}}$ stands for the left partial derivative in Grassmann variable. Using the graded Poisson bracket

$$\{\psi, \psi\} = i$$

for Majorana fermion, we get the classical Hamiltonian

$$H = -\frac{i}{2}\{Q, Q\} = \frac{1}{2}p^2, \tag{3.3}$$

where $p = \dot{x}$ by the Legendre transform.

The corresponding Witten index I is given by (2.4), where the Euclidean action is

$$S_E = \int_0^{\beta} \frac{1}{2}(\dot{x}^2 + \psi\dot{\psi})d\tau,$$

where now dot stands for the derivative to the Euclidean time τ; due to the presence of the zero mode

$$\chi = \frac{1}{\beta} \int_0^\beta \psi \, d\tau$$

we have $I = 0$. Since after the quantization we have $\hat{\chi}^2 = \frac{1}{2}I$, conditions (i)–(ii) of our localization principle are satisfied (here we are using Remark 1). Moreover, $\hat{H} = \frac{1}{2}\hat{Q}^2 = \frac{1}{2}\Delta$, so the fermion degrees of freedom totally decouple and we have

$$\text{Str}\,\hat{\chi} e^{-\beta \hat{H}} = \text{Tr}_{\mathscr{H}_B}\, e^{-\beta \hat{H}} = Z(\beta). \tag{3.4}$$

On the path integral side, we have

$$I(\chi, \beta) = \int_{\Pi T L S^1} \chi e^{-S_E} \mathscr{D}x \mathscr{D}\psi. \tag{3.5}$$

Here $e^{ix(\tau)} \in LS^1 = \text{Map}(S_\beta^1, S^1)$ is the loop group of the circle S^1, where $S_\beta^1 = \mathbb{R}/\beta\mathbb{Z}$ the 'thermal' circle, $\psi(\tau) \in \Pi L\mathbb{R}$ are β-periodic functions with anticommuting values, and integration goes over $\Pi T L S^1 = LS^1 \times \Pi L\mathbb{R}$, the tangent bundle of LS^1 with the reverse parity of the fibers, and $\mathscr{D}x \mathscr{D}\psi$ denotes the path integral 'measure' in $\Pi T L S^1$, defined as follows. Lett $u_0(\tau) = \frac{1}{\sqrt{\beta}}$, $u_n(\tau) = \sqrt{\frac{2}{\beta}} \sin \omega_n \tau$ for $n > 0$ and $u_n(\tau) = \sqrt{\frac{2}{\beta}} \cos \omega_n \tau$ for $n < 0$, where $\omega_n = \frac{2\pi n}{\beta}$, be the orthonormal eigenfunctions of the operator $-\frac{d^2}{d\tau^2}$ in the real Hilbert space $L^2(S_\beta^1; \mathbb{R})$. Consider the eigenfunction expansions

$$x(\tau) = \sum_{n=-\infty}^{\infty} c_n u_n(\tau) \quad \text{and} \quad \psi(\tau) = \sum_{n=-\infty}^{\infty} \psi_n u_n(\tau),$$

where[7] $\bar{c}_n = c_n$, $\bar{\psi}_n = \psi_n$, and put

$$\mathscr{D}x\mathscr{D}\psi = dc_0 \prod_{n=1}^{\infty} dc_n dc_{-n} \, d\psi_0 \prod_{n=1}^{\infty} d\psi_{-n} d\psi_n. \tag{3.6}$$

As the result, bosonic and fermionic Gaussian path integrals with respect to $\mathscr{D}x\mathscr{D}\psi$ are expressed in terms of the regularized determinants and Pfaffians of the corresponding differential operators in the real Hilbert space $L^2(S_\beta^1; \mathbb{R})$. In particular, since the eigenvalues of $-\frac{d^2}{d\tau^2}$ are ω_n^2, we have by the standard zeta function regularization

$$\det(-\partial_\tau^2) = \prod_{n=1}^{\infty} \omega_n^4 = \beta^2, \tag{3.7}$$

which coincides with $\det(-\partial_\tau^2)$ in the complex Hilbert space $L^2(S_\beta^1)$. For the skew-symmetric operator ∂_τ in $L^2(S_\beta^1; \mathbb{R})$ we have

$$\partial_\tau \begin{pmatrix} u_n \\ u_{-n} \end{pmatrix} = \begin{pmatrix} 0 & \omega_n \\ -\omega_n & 0 \end{pmatrix} \begin{pmatrix} u_n \\ u_{-n} \end{pmatrix}, \quad n > 0,$$

---

[7]This reflects the fact that $x(\tau)$ are real-valued and $\psi(\tau)$ is Majorana fermion.

so

$$\mathrm{Pf}(\partial_\tau^{2l+1}) = \prod_{n=1}^\infty \omega_n^{2l+1} = \beta^{l+\frac{1}{2}}. \tag{3.8}$$

Remark 3. The bosonic and fermionic measures $\mathscr{D}x$ and $\mathscr{D}\psi$ have remarkable scaling property: they are invariant under the change of variables $c_0 \mapsto ac_0$ and $c_n \mapsto ac_n$ for any $a > 0$, and $\psi_0 \mapsto b\psi_0$ and $\psi_n \mapsto b\psi_n$ for any $b > 0$. Indeed, the Jacobian for the bosonic change of variables is

$$a \prod_{n=1}^\infty a^2 = a\, e^{2\zeta(0)\log a} = 1,$$

since for the Riemann zeta-function $\zeta(0) = -1/2$, and similarly for the fermionic variables. We choose $a = b = 1/\sqrt{\beta}$, so that ac_0 varies from 0 to $2\pi = \mathrm{Vol}(S^1)$, and $\chi = a\psi_0$ satisfies condition (ii) in section 2.2.

It follows from the discussion in section 2.2. that the simplest invariant deformation of $I(\chi, \beta)$ is given by the following higher-derivative functional[8]

$$V = -\frac{1}{2} \int_0^\beta \dot{\psi}\ddot{x}\, d\tau, \tag{3.9}$$

so

$$\delta V = \frac{1}{2} \int_0^\beta (\ddot{x}^2 + \dot{\psi}\ddot{\psi})d\tau, \tag{3.10}$$

where we are using Euclidean version of formulas (2.7),

$$\delta x = \psi \quad \text{and} \quad \delta\psi = -\dot{x}. \tag{3.11}$$

Indeed, it is elementary to verify that $\delta\chi \perp \chi$ and $V, \delta V \perp \chi$. Thus

$$I(\chi, \beta) = \int_{\Pi T L S^1} \chi e^{-S_E - s\delta V} \mathscr{D}x \mathscr{D}\psi \tag{3.12}$$

and according to (3.10), in the limit $s \to \infty$ the path integral localizes on closed geodesics satisfying $\ddot{x} = 0$, i.e., $x(\tau) = c + \omega_n \tau$.

Let ΩS^1 be the space of based loops,

$$\mathscr{D}'x\mathscr{D}'\psi = \prod_{n=1}^\infty dc_n dc_{-n} \prod_{n=1}^\infty d\psi_{-n} d\psi_n$$

be the corresponding path integral measure on $\Pi T \Omega S^1$, and $x_{cl}(\tau) = \omega_n \tau$ be isolated geodesics in ΩS^1. Using functional analog of the elementary formula

$$\delta(f(x)) = \sum_{f(x_l)=0} \frac{\delta(x - x_l)}{|f'(x_l)|},$$

---

[8]It is noteworthy that such higher-derivative term resembles the one in the linearized Schwarzian action as in [38].

translation invariance of the action, $\text{Vol}(S^1) = 2\pi$, and formulas (3.7)–(3.8), we easily compute

$$
\begin{aligned}
I(\chi, \beta) &= 2\pi \lim_{s \to \infty} \int_{\Pi T \Omega S^1} e^{-S_E - s\delta V} \mathscr{D}'x \mathscr{D}'\psi \\
&= 2\pi \cdot (2\pi)^{\zeta(0)} \int_{\Pi T \Omega S^1} e^{-S_E[x,\psi]} \delta(\ddot{x}) \delta(\psi) \text{Pf}(-\partial_\tau^3) \mathscr{D}'x \mathscr{D}'\psi \\
&= 2\pi \cdot (2\pi)^{\zeta(0)} \int_{\Omega S^1} e^{-S_E[x,0]} \sum_{x_{cl}} \frac{\delta(x - x_{cl})}{|\det(\partial_\tau^2)|} \text{Pf}(-\partial_\tau^3) \mathscr{D}'x \\
&= 2\pi \cdot (2\pi)^{\zeta(0)} \sum_{x_{cl}} e^{-\frac{1}{2} \int_0^\beta \dot{x}_{cl}^2 d\tau} \frac{\text{Pf}(-\partial_\tau^3)}{\det(-\partial_\tau^2)} \\
&= \sqrt{\frac{2\pi}{\beta}} \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2/\beta}.
\end{aligned}
$$

Remark 4. One can think of $2\zeta(0) = 2\sum_{n=1}^{\infty} 1 = -1$ as the regularized dimension of ΩS^1, so the prefactor $(2\pi)^{\zeta(0)}$ is analogous to the prefactor $(2\pi)^{\dim M/2}$ in the finite-dimensional Gaussian integration.

4 Localization proof of the Eskin trace formula on G

4.1 Supersymmetric particle on G

Here we describe classical and quantum aspects of the supersymmetric particle on a compact semi-simple Lie group G. We summarized some basic facts on G in the appendix A.2. As in section 2.2, let $x^\mu(t)$ be local coordinates in the neighborhood of $g(t)$ for a path $g(t)$ in G, and let $\psi(t) \in \Pi T_{g(t)} G$ be a vector field along the path with the reverse parity. We consider the following Lagrangian $\mathcal{L}$ of a free supersymmetric particle on G endowed with the connection with totally anti-symmetric parallelizing torsion,

$$
\mathcal{L} = \frac{1}{2}(\dot{x}, \dot{x}) + \frac{i}{2}(\psi, \nabla_{\dot{x}}^- \psi), \tag{4.1}
$$

where ∇^- is a flat connection on G,[9] defined by the condition that the elements of $\mathfrak{g}$ are parallel vector fields with the parallel transport given by the left translations $(L_g)_*$. In terms of the local coordinates the Lagrangian takes the form

$$
\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu} (\psi^\mu \dot{\psi}^\nu + \Gamma^\nu_{\lambda\rho} \dot{x}^\lambda \psi^\mu \psi^\rho), \tag{4.2}
$$

where Christoffel symbols are given by $\Gamma^\nu_{\rho\lambda}$ are

$$
\Gamma^\nu_{\lambda\mu} = \theta_a^\nu \partial_\lambda \theta_\mu^a \tag{4.3}
$$

(see appendix A.2 for notations). The supersymmetry transformation[10] is given by standard formulas (2.7),

$$
\delta x^\mu = i\psi^\mu, \quad \delta \psi^\mu = -\dot{x}^\mu, \quad \mu = 1, \dots, n. \tag{4.4}
$$

---

[9]Equivalently, we could start with the choice of the right-invariant flat connection ∇^+.

[10]Here we use the Lorentzian time and imaginary i is needed to make quadratic expressions in ψ "real", that is, $\overline{i\psi^\mu \psi^\nu} = -i\psi^\nu \psi^\mu = i\psi^\mu \psi^\nu$.

In terms of the current $J = g^{-1}\dot{g} = J^a T_a \in \mathfrak{g}$ and $\psi = \psi^a T_a \in \Pi\mathfrak{g}$, where $\psi^a = \theta^a_\mu \psi^\mu$, the supersymmetry transformation takes the following form

$$\delta g = ig\psi, \tag{4.5}$$

$$\delta\psi = -J - i\psi\psi, \tag{4.6}$$

$$\delta J = i(\partial_t + \text{ad}_J)\psi, \tag{4.7}$$

where the formula for δJ is a direct consequence of the first two. In the formula (4.5) it is understood that $g\psi = (L_g)_*\psi$, and in (4.6)

$$\psi\psi = \frac{1}{2}\psi^a\psi^b(T_a T_b - T_b T_a) = \frac{1}{2}[\psi, \psi].$$

Indeed, formula (4.5) immediately follows from the first formula in (4.4), while for (4.6) we have the following simple computation, using (A.3):

$$
\begin{aligned}
\delta\psi^a &= -\theta^a_\mu \dot{x}^\mu + i\partial_\nu \theta^a_\mu \psi^\nu \psi^\mu = -J^a + \frac{i}{2}(\partial_\nu \theta^a_\mu - \partial_\mu \theta^a_\nu)\psi^\nu \psi^\mu \\
&= -J^a - \frac{i}{2} f^a_{bc} \theta^b_\nu \theta^c_\mu \psi^\nu \psi^\mu = -J^a - \frac{i}{2} f^a_{bc} \psi^b \psi^c \\
&= -J^a - i(\psi\psi)^a.
\end{aligned}
$$

In the Hamiltonian formalism, Legendre transform associated with the Lagrangian (4.1) formally gives[11]

$$\tilde{p}_\mu = \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = g_{\mu\nu}\dot{x}^\nu + \frac{i}{2}g_{\rho\nu}\Gamma^\nu_{\lambda\mu}\psi^\rho\psi^\lambda \quad \text{and} \quad \tilde{\pi}_\mu = \frac{\partial\mathcal{L}}{\partial\dot{\psi}^\mu} = -\frac{i}{2}g_{\mu\nu}\psi^\nu \tag{4.8}$$

as canonically conjugated variables to x^μ and ψ^μ. However, we have constraints

$$\tilde{\Phi}_a = \tilde{\pi}_\mu + \frac{i}{2}g_{\mu\nu}\psi^\nu = 0,$$

and one needs to use the Dirac formalism of Poisson brackets with constraints (see, e.g., [11, 35]). Due to the simple nature of the flat connection ∇^-, it is convenient to use variables ψ^a instead of ψ^μ. Namely, using (4.3), we have

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}g_{ab}\psi^a\dot{\psi}^b, \tag{4.9}$$

so defining

$$p_\mu = \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = g_{\mu\nu}\dot{x}^\nu \quad \text{and} \quad \pi_a = \frac{\partial\mathcal{L}}{\partial\dot{\psi}^a} = -\frac{i}{2}g_{ab}\psi^b$$

we see that the fermion degrees of freedom decouple. The corresponding constraints are

$$\Phi_a = \pi_a + \frac{i}{2}g_{ab}\psi^b = 0,$$

---

[11]Here $\dfrac{\partial}{\partial\dot{\psi}^\mu}$ stands for the left partial derivative in Grassmann variables.

and the matrix

$$C_{ab} = \{\Phi_a, \Phi_b\} = i g_{ab}$$

does not depends on the coordinates x^μ. Thus in the Dirac formalism the canonical coordinates of the reduced phase space are p_μ, x^μ and ψ^a with the following non-vanishing Poisson brackets

$$\{p_\mu, x^\nu\} = \delta^\nu_\mu \quad \text{and} \quad \{\psi^a, \psi^b\} = i g^{ab}. \tag{4.10}$$

For the corresponding Hamiltonian, we have from (4.9)

$$H = \dot{x}^\mu p_\mu + \dot{\psi}^a \pi_a - \mathcal{L} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu, \tag{4.11}$$

and as in section C we obtain

$$H = \frac{1}{2} g^{ab} l_a l_b. \tag{4.12}$$

To establish the classical supersymmetry algebra, it is convenient to write

$$\mathcal{L} = \frac{1}{2}(J, J) + \frac{i}{2}(\psi, \dot{\psi}), \tag{4.13}$$

where $(u, v) = g_{ab} u^a v^b$ and to represent ad-invariance of the Killing form as

$$([x, y], z) = (x, [y, z]), \tag{4.14}$$

where $x, y, z \in \mathfrak{g}$ can be even or odd elements. Also, when x, y are odd elements of $\mathfrak{g}$, we have

$$[x, y] = [y, x] \quad \text{and} \quad [x, [x, x]] = 0,$$

so

$$([x, x], [x, x]) = (x, [x, [x, x]]) = 0. \tag{4.15}$$

Using equations (4.6)–(4.7) and (4.14), we readily compute

$$\begin{aligned}
\delta\mathcal{L} &= \frac{1}{2}(\delta J, J) + \frac{1}{2}(J, \delta J) + \frac{i}{2}(\delta\psi, \dot{\psi}) - \frac{i}{2}(\psi, \delta\dot{\psi}) \\
&= \frac{1}{2}\frac{d}{dt}\left(i(\psi, J) - \frac{1}{6}(\psi, [\psi, \psi]) \right),
\end{aligned}$$

so by the Noether theorem, the supercharge Q is given by

$$\begin{aligned}
iQ &= \delta x^\mu \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} + \delta\psi^a \frac{\partial\mathcal{L}}{\partial\dot{\psi}^a} - \frac{1}{2}\left(i(\psi, J) - \frac{1}{6}(\psi, [\psi, \psi]) \right) \\
&= i(\psi, J) - \frac{1}{6}(\psi, [\psi, \psi]).
\end{aligned}$$

Using the Legendre transform, we get the following formula for the supercharge in the phase space

$$Q = (\psi, L) + \frac{i}{6}(\psi, [\psi, \psi]), \tag{4.16}$$

where, according to (C.4),

$$L = l^a T_a, \quad l^a = g^{ab} l_b.$$

It follows from (C.6) that

$$\{Q, r_a\} = 0.$$

The following result manifests classical supersymmetry.

Lemma 2. *The following classical supersymmetry algebra relation holds*

$$\{Q, Q\} = 2iH. \tag{4.17}$$

Proof. Indeed, we have, using (4.10), (4.12) and (4.15),

$$
\begin{aligned}
\{Q, Q\} &= \{\psi^a l_a, \psi^b l_b\} + \frac{i}{6} \left(l_a \{\psi^a, (\psi, [\psi, \psi])\} + l_a \{(\psi, [\psi, \psi]), \psi^a\} \right) \\
&\quad - \frac{1}{36} \{(\psi, [\psi, \psi]), (\psi, [\psi, \psi])\} \\
&= i g^{ab} l_a l_b + ([\psi, \psi], L) - ([\psi, \psi], L) - \frac{1}{4}([\psi, \psi], [\psi, \psi]) \\
&= 2iH. \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \square
\end{aligned}
$$

Since bosonic and fermionic degrees of freedom in the Lagrangian (4.9) are totally decoupled, quantization of the classical system is straightforward.

First, the Hilbert space is

$$\mathscr{H} = L^2(G) \otimes \mathscr{H}_F, \tag{4.18}$$

where, according to the Remark 1, $\mathscr{H}_F$ is irreducible Clifford algebra module of dimension $2^{n/2}$.

Second, classical variables l_a and ψ^a are replaced by quantum operators $\hat{l}^a$ and $\hat{\psi}^a$ acting on $\mathscr{H}$ and satisfying the following graded commutation relations

$$[\hat{\psi}^a, \hat{\psi}^b] = g^{ab} \quad \text{and} \quad [\hat{l}_a, \hat{l}_a] = -i f^c_{ab} \hat{l}_c \tag{4.19}$$

— quantization of the graded Poisson brackets in appendices B and C. In particular, $\hat{l}_a = -i e_a$, where e_a are left-invariant vector fields on G, realized as the first order differential operators on G. Correspondingly,

$$\Delta = g^{ab} \hat{l}_a \hat{l}_b = -g^{ab} e_a e_b \tag{4.20}$$

is the Laplace operator on G, the quadratic Casimir operator C_2 on the universal enveloping algebra $U\mathfrak{g}$, realized as algebra of differential operators on $L^2(G)$.

The quantum supercharge $\hat{Q}$ is defined unambiguously (i.e., there is no problem of ordering the quantum operators) by the following formula

$$\hat{Q} = \hat{\psi}^a \hat{l}_a + \frac{i}{6} f_{abc} \hat{\psi}^a \hat{\psi}^b \hat{\psi}^c, \quad f_{abc} = g_{ae} f^e_{bc}. \tag{4.21}$$

Since $\hat{l}_a$ and $\hat{r}_a = -i f_a$ commute, where f_a are right-invariant vector fields on G, realized as the first order differential operators on G, we have

$$[\hat{Q}, \hat{r}_a] = 0.$$

The following result is fundamental.

Proposition 2. *Defined by the supersymmetry algebra the quantum Hamiltonian $\hat{H} = \hat{Q}^2$ is given by*

$$\hat{H} = \frac{1}{2} g^{ab} \hat{l}_a \hat{l}_b + \frac{1}{48} f_{abc} f^{abc} \hat{I} = \frac{1}{2} \Delta + \frac{R}{12} \hat{I}, \qquad (4.22)$$

where Δ is the Laplace operator on $L^2(G)$ and

$$R = \frac{1}{4} f_{abc} f^{abc} = \frac{n}{4}$$

is the scalar curvature of the Cartan-Killing metric on G.

Proof. From the commutation relations in (4.19) we readily obtain

$$[\hat{\psi}^a \hat{l}_a, \hat{\psi}^b \hat{l}_b] = g^{ab} \hat{l}_a \hat{l}_b - i f^c_{ab} \hat{l}_c \hat{\psi}^a \hat{\psi}^b,$$

$$[\hat{\psi}^a \hat{l}_a, f_{bcd} \hat{\psi}^b \hat{\psi}^c \hat{\psi}^d] = 3 f^a_{bc} \hat{l}_a \hat{\psi}^b \hat{\psi}^c,$$

$$[f_{abc} \hat{\psi}^a \hat{\psi}^b \hat{\psi}^c, f_{def} \hat{\psi}^d \hat{\psi}^e \hat{\psi}^f] = -\frac{3}{2} f_{abc} f^{abc}.$$

Using these relations, it is straightforward to verify that

$$\hat{Q}^2 = \frac{1}{2} [\hat{Q}, \hat{Q}] = \frac{1}{2} [\hat{\psi}^a \hat{l}_a + \frac{i}{6} f_{abc} \hat{\psi}^a \hat{\psi}^b \hat{\psi}^c, \hat{\psi}^d \hat{l}_d + \frac{i}{6} f_{def} \hat{\psi}^d \hat{\psi}^e \hat{\psi}^f]$$

$$= \frac{1}{2} g^{ab} \hat{l}_a \hat{l}_b + \frac{1}{48} f_{abc} f^{abc} \hat{I}. \qquad \square$$

Remark 5. It should be noted that in representation theory the quantum supercharge $\hat{Q}$ is just the Kostant's cubic Dirac operator on G, introduced in [26] (see also [1]). According to [7], it is just a G-invariant Dirac operator, associated with the invariant flat connection ∇^-. In the physics literature, such operator was first obtained by H.W. Braden [11] for canonical quantization of the $N = 1/2$ supersymmetric sigma model. Our elementary derivation of the supersymmetry algebra (cf. with [9, 11]) is possible due to commutation relations (4.19).

Remark 6. It is quite remarkable that proportional to the scalar curvature innocuously looking constant term in $\hat{H}$ is precisely the well-known 'notorious' DeWitt term for the bosonic quantum particle on curved manifold [15]. It is proportional to $\hbar^2$ quantum correction to the action in the path integral for the propagator of a pure bosonic quantum particle moving on a curved manifold, and is obtained by a tedious analysis of the discretized path integral. It is really striking that such term naturally appears through the quantum supersymmetry algebra! This clearly supports the idea of using supersymmetry to derive pure bosonic trace formulas, and will be a crucial ingredient in the derivation of the trace formula on G.

4.2 Eskin's trace formula on G

The supersymmetric particle on G with the connection ∇^- considered satisfies all conditions of our new localization principle formulated in section 2.2. Namely, it has n fermion zero modes

$$\chi^a = \frac{1}{\beta} \int_0^\beta \psi^a \, d\tau$$

such that

$$\hat{\chi}^1 \cdots \hat{\chi}^n = 2^{-n/2} (-1)^F,$$

so according to (4.22), bosonic and fermionic degrees of freedom are totally decoupled and

$$\text{Str}\left(\hat{\chi}^1 \dots \hat{\chi}^n e^{-\beta \hat{H}}\right) = e^{-\frac{1}{12}\beta R} \,\text{Tr}\, e^{-\frac{1}{2}\beta \Delta}.$$

Since Δ commutes with left and right translations, one can also express the heat kernel on G as a supertrace. Namely, we recall that the heat kernel is a fundamental solution $K_\tau(g_1, g_2)$ of the heat equation on G

$$\frac{\partial K}{\partial \tau} = -\frac{1}{2}\Delta K$$

with respect to g_1, satisfying

$$\lim_{\tau \to 0} K_\tau(g_1, g_2) = \delta_G(g_1 g_2^{-1}),$$

where δ_G is the Dirac delta-function on G with respect to the Cartan-Killing volume form. Fix a Cartan subgroup T in G and corresponding Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{g}$, $\dim \mathfrak{t} = r$, the rank of $\mathfrak{g}$. It follows from the bi-invariance of the heat kernel that it only depends on $g_1 g_2^{-1} \in T$, and we will denote it by $k_t(e^h)$, where $h \in \mathfrak{t}$ and $e^h \in T$. Using Dirac notation,

$$k_\tau(e^h) = K_\tau(e^h, 1) = \langle e^h | e^{-\frac{1}{2}\tau \Delta} | 1 \rangle,$$

where 1 is the identity element in G.

Correspondingly, $\text{Tr}\, e^{-\frac{1}{2}\beta \Delta} = V_G k_\beta(1)$, where V_G is the volume of G, and more generally

$$\text{Tr}\, e^{-\frac{1}{2}\beta \Delta + i(h,\hat{r})} = V_G k_\beta(e^h),$$

where $\hat{r} = \hat{r}^a T_a$. The extra term $i(h, \hat{r})$ in the exponent can be thought as a (imaginary) 'chemical potential' added to the Hamiltonian $\hat{H}$. Since the operators $\hat{r}^a$ commute with $\hat{Q}$, we have

$$\text{Str}\, \hat{\chi}^1 \dots \hat{\chi}^n e^{-\beta \hat{H} + i(h,\hat{r})} = V_G e^{-\frac{1}{12}\beta R} k_\beta(e^h). \tag{4.23}$$

Here the supertrace is given by the following path integral

$$\int_{\Pi T L G} \chi^1 \dots \chi^n e^{-S_E^h} \,\mathscr{D}g \mathscr{D}\psi \tag{4.24}$$

with the Euclidean action

$$S_E^h = \frac{1}{2}\int_0^\beta \left((J, J) + (\psi, \dot{\psi})\right) d\tau + \frac{1}{\beta}\int_0^\beta (\text{Ad}_{g^{-1}}h, J) d\tau + \frac{1}{2\beta}(h, h) \tag{4.25}$$

and G-invariant 'measure' $\mathscr{D}g$. Because of the additional term $(\text{Ad}_{g^{-1}}h, J)$ in the action, it is easy to verify that S_E^h is now invariant under the modified supersymmetry transformation δ_h, which in the Euclidean time for fixed $h \in \mathfrak{t}$ has the form

$$\delta_h g = g\psi,$$
$$\delta_h \psi = -J^h - \psi\psi,$$
$$\delta_h J^h = (\partial_\tau + \text{ad}_{J^h})\psi,$$

where $J^h = J + \frac{1}{\beta}\text{Ad}_{g^{-1}}h$.

The spectral representation of $k_\beta(e^h)$ is well-known since $\Delta = C_2$, the quadratic Casimir operator. Namely, we have

$$k_\beta(e^h) = \frac{1}{V_G} \sum_{\lambda \in \text{Irrep}\, G} d_\lambda\, \chi_\lambda(h) e^{-\frac{1}{2}\beta C_2(\lambda)}, \tag{4.26}$$

where χ_λ is the character for an irreducible representation λ of G, d_λ is its dimension, and $C_2(\lambda)$ is the eigenvalue of C_2. We have (see appendix A.1 for notation)

$$C_2(\lambda) = \langle \lambda + \rho, \lambda + \rho \rangle - \langle \rho, \rho \rangle, \quad \text{where} \quad \rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha \in P_+,$$

where R_+ is the set of positive roots and P_+ is the set of dominant weights. Using Weyl dimension formula

$$d_\lambda = \prod_{\alpha \in R_+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle},$$

one can rewrite (4.26) as

$$k_\beta(e^h) = \frac{1}{V_G} e^{\frac{1}{2}\beta\langle\rho,\rho\rangle} \sum_{\lambda \in P_+} \chi_\lambda(h) \prod_{\alpha \in R_+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} e^{-\frac{1}{2}\beta(\lambda+\rho,\lambda+\rho)}. \tag{4.27}$$

The Freudenthal-de Vries's 'strange formula'

$$\langle \rho, \rho \rangle = \frac{n}{24},$$

ensures that the prefactor in (4.27) exactly cancels the factor $e^{-\frac{1}{2}\beta R} = e^{-\frac{1}{48}n\beta}$ in (4.23)!

It is remarkable, that there is another representation for $k_\beta(e^h)$ as a sum over so-called characteristic lattice, obtained by L.D. Eskin [19]. Namely, suppose (which is the main assumption) that Cartan element h is regular, which means that $\langle h, \alpha \rangle \notin 2\pi i\mathbb{Z}$ for any root $\alpha \in R$. Then one has the following representation

$$k_\beta(e^h) = \frac{e^{\frac{1}{2}\beta\langle\rho,\rho\rangle}}{(2\pi\beta)^{n/2}} \sum_{\gamma \in \Gamma} \prod_{\alpha \in R_+} \frac{\frac{1}{2}\langle \alpha, h+\gamma \rangle}{\sinh\frac{1}{2}\langle \alpha, h+\gamma \rangle} e^{-\frac{1}{2\beta}(h+\gamma,h+\gamma)}, \tag{4.28}$$

where $\Gamma = \{\gamma \in \mathfrak{t} : e^\gamma = 1\}$ is the characteristic lattice,[12] which is related to the maximal torus of G by $T = \mathfrak{t}/\Gamma$. Comparing formulas (4.27) and (4.28) yields Eskin trace formula.[13]

Up to a DeWitt term $e^{\frac{1}{2}\beta\langle\rho,\rho\rangle}$, the right hand side in (4.28) exhibits a nature of the exact semi-classical approximation with geodesics $\exp((h+\gamma)\tau/\beta)$, $\gamma \in \Gamma$, connecting 1 and e^h. The regularity condition ensures that each denominator in (4.28) is non zero, since for $\gamma \in \Gamma$

$$\prod_{\alpha \in R_+} \sinh\tfrac{1}{2}\langle \alpha, h+\gamma \rangle = \pm \prod_{\alpha \in R_+} \sinh\tfrac{1}{2}\langle \alpha, h \rangle. \tag{4.29}$$

Here we show that the Eskin trace formula can be obtained very naturally as a result of supersymmetric localization of the path integral (4.24). Namely, we have the following result.

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[12]In case G is simple and simply connected, $\Gamma = 2\pi i Q^\vee$ where $Q^\vee$ is a coroot lattice.

[13]Note that Eskin was using the convention that $R \subset \mathfrak{t}$, which explains the difference between hyperbolic sine functions in (4.28) and trigonometric sine functions in [19].

Theorem 1. *We have the following exact localization formula*

$$\int\limits_{\Pi TLG} \chi^1 \dots \chi^n e^{-S_E^h} \mathscr{D}g\mathscr{D}\psi = \frac{V_G}{(2\pi\beta)^{n/2}} \sum_{\gamma \in \Gamma} \prod_{\alpha \in R_+} \frac{\frac{1}{2}\langle \alpha, h+\gamma \rangle}{\sinh \frac{1}{2}\langle \alpha, h+\gamma \rangle} e^{-\frac{1}{2\beta}(h+\gamma, h+\gamma)}.$$

Proof. Put

$$V = -\frac{1}{2} \int_0^\beta (J^h, \dot{\psi}) \, d\tau, \tag{4.30}$$

so

$$\delta_h V = \frac{1}{2} \int_0^\beta \left((J^h, J^h) + (\dot{\psi}, (\partial_\tau + \mathrm{ad}_{J^h})\dot{\psi}) \right) \, d\tau. \tag{4.31}$$

It is straightforward to check that $\delta_h^2 V = 0$ and $V, \delta_h V \perp \chi^a$, $a = 1, \dots, n$. Hence V is an invariant deformation to the path integral (4.24), and according to Proposition 1,

$$\int\limits_{\Pi TLG} \chi^1 \dots \chi^n e^{-S_E^h} \mathscr{D}g\mathscr{D}\psi = \int\limits_{\Pi TLG} \chi^1 \dots \chi^n e^{-S_E^h - s\delta_h V} \mathscr{D}g\mathscr{D}\psi. \tag{4.32}$$

Moreover, purely bosonic part of $\delta_h V$ is positive semi-definite with zeros at $\dot{J}^h = 0$, so in the limit $s \to \infty$, the path integral (4.32) localizes on the locus $\dot{J}^h = 0$. To facilitate the computation, we use the left G-invariance and integrate over fermionic zero modes to obtain

$$\int\limits_{\Pi TLG} \chi^1 \dots \chi^n e^{-S_E^h - s\delta_h V} \mathscr{D}g\mathscr{D}\psi = V_G \int\limits_{\Pi T\Omega G} e^{-S_E^h - s\delta_h V} \mathscr{D}'g\mathscr{D}'\psi, \tag{4.33}$$

where ΩG is the space of based loops on G, $g(0) = g(\beta) = 1$. Since $h \in \mathfrak{t}$ is regular, solutions of the equation $\dot{J}^h = 0$ in ΩG are isolated geodesics parameterized by the lattice Γ,

$$g_\gamma(\tau) = e^{\frac{1}{\beta}\tau\gamma} \quad \text{and} \quad J_\gamma^h = \frac{h+\gamma}{\beta}, \quad \gamma \in \Gamma. \tag{4.34}$$

Indeed, we have

$$J^h = g^{-1}\left(\dot{g} + \frac{1}{\beta}hg\right) = \tilde{g}^{-1}\dot{\tilde{g}},$$

where $\tilde{g}(\tau) = e^{h\tau/\beta}g(\tau)$. Hence equation $\dot{J}^h = 0$ implies $J^h(\tau) = c/\beta$ with some $c \in \mathfrak{g}$, so $\tilde{g}(\tau) = \tilde{g}(0)e^{c\tau/\beta}$ and $g(\tau) = e^{-h\tau/\beta}g(0)e^{c\tau/\beta}$. The condition $g \in \Omega G$ gives $e^h = e^c$, and $h \in \mathfrak{t}$ being regular implies that $c \in \mathfrak{t}$ (see Lemma 3 in appendix A.1). Thus $e^{c-h} = 1$ and $c - h = g\tilde{\gamma}g^{-1}$ for some $\tilde{\gamma} \in \Gamma$ and $g \in G$, and since $c - h \in \mathfrak{t}$, we have $g \in W$, the Weyl group of G, and $\gamma = g\tilde{\gamma}g^{-1} \in \Gamma$.

Similarly, since h is nonsingular, it is easily to see that solution of the equation

$$\partial_\tau(\partial_\tau + \mathrm{ad}_{J^h})\dot{\psi} = 0$$

in $\Pi\Omega\mathfrak{g}$ gives $\psi_{cl} = 0$.

Now as in section 3, using (4.31) we readily obtain

$$\lim_{s\to\infty}\int\limits_{\Pi T\Omega G} e^{-S_E^h-s\delta_h V}\mathscr{D}'g\mathscr{D}'\psi$$

$$=\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\Pi T\Omega G} e^{-S_E^h[g,\psi]}\delta(\dot{J}^h)\delta(\psi)\mathrm{Pf}(-\partial_\tau^3-\mathrm{ad}_{J^h}\partial_\tau^2)\mathscr{D}'g\mathscr{D}'\psi$$

$$=\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\Omega G}\sum_{\gamma\in\Gamma} e^{-S_E^h[g,0]}\frac{\delta(gg_\gamma^{-1})}{|\det(\mathrm{d}\dot{J}^h)|}\mathrm{Pf}(-\partial_\tau^3-\mathrm{ad}_{J^h}\partial_\tau^2)\mathscr{D}'g$$

$$=\frac{1}{(2\pi)^{\frac{n}{2}}}\sum_{\gamma\in\Gamma}\frac{\mathrm{Pf}(-\partial_\tau^3-\mathrm{ad}_{(h+\gamma)/\beta}\partial_\tau^2)}{|\det(\mathrm{d}\dot{J}^h)|}e^{-\frac{(h+\gamma,h+\gamma)}{2\beta}},\qquad(4.35)$$

where the linear operator $\mathrm{d}\dot{J}^h:\Omega\mathfrak{g}\to\Omega\mathfrak{g}$ is the differential of the functional $\dot{J}^h(\tau)$ on ΩG at the critical point J_γ^h.

By definition of J^h we have

$$\mathrm{d}J^h(\tau)=\partial_\tau+\mathrm{ad}_{J^h(\tau)},\qquad(4.36)$$

so

$$\mathrm{d}\dot{J}^h(\tau)=\partial_\tau^2+\partial_\tau\mathrm{ad}_{J^h(\tau)}.\qquad(4.37)$$

Indeed, put $\delta g=gX$, $X\in\mathfrak{g}$, so

$$\delta\dot{g}=\dot{g}X+g\dot{X}$$

and for $\delta J^h=\mathrm{d}\dot{J}^h(\tau)(X)$ we have

$$\delta J^h=-g^{-1}\delta gg^{-1}\dot{g}+g^{-1}\delta\dot{g}+\frac{1}{\beta}(g^{-1}h\delta g-g^{-1}\delta gg^{-1}hg)=\dot{X}+[J^h,X],$$

which proves (4.36). Thus for each critical point $J^h(\tau)=(h+\gamma)/\beta$ we obtain

$$\mathrm{d}\dot{J}^h(\tau)=\partial_\tau^2+\mathrm{ad}_{(h+\gamma)/\beta}\partial_\tau.\qquad(4.38)$$

To compute the functional determinants in (4.35), we use the Cartan-Weyl basis (A.1) for $\mathfrak{g}$ (cf. [33]) and as in section 3 consider the expansions

$$X(\tau)=\sum_{n\neq0}\left(\sum_{j=1}^r c_{j,n}iH_j+\sum_{\alpha\in R}c_{\alpha,n}^{(1)}X_\alpha+\sum_{\alpha\in R}c_{\alpha,n}^{(2)}Y_\alpha\right)u_n(\tau)\in L\mathfrak{g}/\mathfrak{g}\simeq\Omega\mathfrak{g},\qquad(4.39)$$

$$\psi(\tau)=\sum_{n\neq0}\left(\sum_{j=1}^r \psi_{j,n}iH_j+\sum_{\alpha\in R}\psi_{\alpha,n}^{(1)}X_\alpha+\sum_{\alpha\in R}\psi_{\alpha,n}^{(2)}Y_\alpha\right)u_n(\tau)\in\Pi\Omega\mathfrak{g},\qquad(4.40)$$

where we put $X_\alpha=E_\alpha-E_{-\alpha}$ and $Y_\alpha=i(E_\alpha+E_{-\alpha})$. It follows from the commutation relations

$$[H,X_\alpha]=-i\langle H,\alpha\rangle Y_\alpha\quad\text{and}\quad[H,Y_\alpha]=i\langle H,\alpha\rangle X_\alpha,\quad\text{where}\quad H\in i\mathfrak{t},$$

(see appendix A.1) that $iH_ju_n(\tau)$ and $X_\alpha u_n(\tau)+Y_\alpha u_{-n}(\tau)$ are the eigenfunctions of the operator $-\partial_\tau^2-\mathrm{ad}_{(h+\gamma)/\beta}\partial_\tau$ with the eigenvalues ω_n^2 and $\omega_n^2+i\dfrac{\langle h+\gamma,\alpha\rangle}{\beta}\omega_n$.

Similarly, in the two-dimensional subspaces spanned by $iH_j u_n(\tau), iH_j u_{-n}(\tau)$ and by $X_\alpha u_n(\tau) + Y_\alpha u_{-n}(\tau), X_\alpha u_{-n}(\tau) - Y_\alpha u_n(\tau)$, where $n > 0$, the skew-symmetric operator $-\partial_\tau^3 - \mathrm{ad}_{(h+\gamma)/\beta} \partial_\tau^2$ acts by the following 2×2 matrices

$$\begin{pmatrix} 0 & \omega_n^3 \\ -\omega_n^3 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \omega_n^3 + i\omega_n^2 \dfrac{\langle h + \gamma, \alpha \rangle}{\beta} \\ -\omega_n^3 - i\omega_n^2 \dfrac{\langle h + \gamma, \alpha \rangle}{\beta} & 0 \end{pmatrix}.$$

Finally, for the ratio of the functional determinants in (4.35) we have, using the standard zeta function regularization,

$$\begin{aligned}
&\frac{\mathrm{Pf}(-\partial_\tau^3 - \mathrm{ad}_{(h+\gamma)/\beta} \partial_\tau^2)}{\det(-\partial_\tau^2 - \mathrm{ad}_{(h+\gamma)/\beta} \partial_\tau)} \\
&= \left(\prod_{n=1}^\infty \omega_n^3 \right)^r \prod_{\alpha \in R} \prod_{n=1}^\infty \left(\omega_n^3 + i \frac{\langle h + \gamma, \alpha \rangle}{\beta} \omega_n^2 \right) \bigg/ \left(\prod_{n=1}^\infty \omega_n^4 \right)^r \prod_{\alpha \in R} \prod_{n=1}^\infty \left(\omega_n^2 + i \frac{\langle h + \gamma, \alpha \rangle}{\beta} \omega_n \right)^2 \\
&= \left(\prod_{n=1}^\infty \omega_n \right)^{-r} \prod_{\alpha \in R} \prod_{n=1}^\infty \left(\omega_n + i \frac{\langle h + \gamma, \alpha \rangle}{\beta} \right)^{-1} \\
&= \beta^{-\frac{n}{2}} \prod_{\alpha \in R_+} \frac{\frac{1}{2} \langle \alpha, h + \gamma \rangle}{\sinh \frac{1}{2} \langle \alpha, h + \gamma \rangle}. \qquad \qquad \Box
\end{aligned}$$

Remark 7. Using (4.23), (4.24) and (4.27), the statement of the theorem can be rewritten as the equality "Spectral Trace = Matrix Trace" for the operator $e^{-\frac{1}{2}\beta\Delta}$,

$$\begin{aligned}
&e^{\frac{1}{2}\beta\langle \rho, \rho \rangle} \sum_{\lambda \in P_+} \chi_\lambda(h) \prod_{\alpha \in R_+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} e^{-\frac{1}{2}\beta\langle \lambda + \rho, \lambda + \rho \rangle} \\
&= V_G \frac{e^{\frac{1}{2}\beta\langle \rho, \rho \rangle}}{(2\pi\beta)^{n/2}} \sum_{\gamma \in \Gamma} \prod_{\alpha \in R_+} \frac{\frac{1}{2} \langle \alpha, h + \gamma \rangle}{\sinh \frac{1}{2} \langle \alpha, h + \gamma \rangle} e^{-\frac{1}{2\beta}(h+\gamma, h+\gamma)},
\end{aligned}$$

which generalizes Jacobi inversion formula. After trivial rewriting, this formula is precisely the Eskin trace formula! It should also be noted that when G is simply connected the sign in formula (4.29) is plus for all $\gamma \in \Gamma$, and Eskin formula is proved in Theorem 4.3 in [9]. The latter is obtained as a special case of Theorem 4.3.4 in [20], when one of Cartan elements goes to 0.

5 Singular trace formula on G

Since the spectral side in Eskin formula is a smooth function of $h \in \mathfrak{t}$, it is natural to relax the regularity condition on h. In particular, for $h = 0$ the spectral side is just $\mathrm{Tr}\, e^{-\frac{1}{2}\beta\Delta}$. However, for singular h the sum over Γ has vanishing denominators, and it was observed in [19] that one needs to sum over the orbits of the Weyl group of G, and only then take the limit $h \to 0$. Physically, such singular behavior indicates that geodesics are no-longer isolated, so the localization formalism should be modified. In this section, we consider the most singular case $h = 0$; generalization to other singular cases is straightforward.

Let's start from the localization formula established in the previous section in the case of $h = 0$.

$$\frac{1}{V_G} e^{-\frac{1}{12}\beta R} \operatorname{Tr} e^{-\frac{1}{2}\beta\Delta} = \int_{\Pi T\Omega G} e^{-S_E - s\delta V} \mathscr{D}g\mathscr{D}\psi \equiv I(s) \tag{5.1}$$

We write the path integral as

$$I(s) = \int_{\Pi T\Omega G} F(J,\psi) \exp\left\{ -\frac{s}{2}\int_0^\beta \left((\dot{J},\dot{J}) + (\dot{\psi},(\partial_\tau + \operatorname{ad}_J)\dot{\psi}) \right) d\tau \right\} \mathscr{D}g\mathscr{D}\psi$$

where

$$F(J,\psi) = \exp\left\{ -\frac{1}{2}\int_0^\beta \left((J,J) + (\psi,\dot{\psi}) \right) d\tau \right\},$$

We first analyze the critical points under the localization limit $s \to \infty$.

5.1 Critical points

We have $\dot{J} = 0$, where $J = g^{-1}\dot{g}$, so equation $J = \text{const}$ and $g(0) = g(\beta) = 1$ give $g(\tau) = e^{u\tau}$, where $e^{u\beta} = 1$. Therefore $u = \frac{1}{\beta}v\gamma v^{-1}$, where $\gamma \in \Gamma$, the characteristic lattice, and $v \in G/C_\gamma$, where

$$C_\gamma = \{ g \in G : g\gamma g^{-1} = \gamma \}$$

is the centralizer of γ in G. Thus $u \in \mathcal{O}_\gamma$, the orbit of γ/β in $\mathfrak{g}$ under the adjoint action of G. Clearly, $\mathcal{O}_\gamma = \mathcal{O}_{\gamma'}$ if $\gamma' = w\gamma w^{-1}$, where $w \in W$, the Weyl group of G. Thus connected components for bosonic critical points are the sets $\mathcal{M}_\gamma$ in ΩG,

$$g_{cl}(\tau) = v e^{\frac{\tau}{\beta}\gamma} v^{-1}, \quad J_{cl} = \frac{1}{\beta}v\gamma v^{-1}, \tag{5.2}$$

parameterized by $v \in G/C_\gamma$, where $\gamma \in \Gamma/W$. In other words, $J_{cl} \in \mathcal{O}_\gamma$.

As in section 4.2, for $g \in \Omega G$ we put $X = g^{-1}\delta g$ and rewrite the expansion (4.39) in the form[14]

$$X(\tau) = \sum_{n,j} z_{j,n} H_j e^{i\omega_n\tau} + \sum_{n,\alpha\in R} z_{\alpha,n} E_\alpha e^{i\omega_n\tau} \in \Omega\mathfrak{g}, \tag{5.3}$$

where $\overline{z}_{j,n} = -z_{j,-n}$, $\overline{z}_{\alpha,n} = -z_{-\alpha,-n}$ and $H_1,\ldots H_r$, E_α, $\alpha \in R$, is the orthonormal Cartan-Weyl basis (see appendix A.1). Condition that X is a tangent vector to the image of ΩG under the map $g \to J = g^{-1}\dot{g}$ is $X(0) = 0$, so the path integral measure $\mathscr{D}'g$ becomes proportional to

$$\prod_j \delta\left(\sum_{n\in\mathbb{Z}} z_{j,n}\right) \prod_{\alpha\in R} \delta\left(\sum_{n\in\mathbb{Z}} z_{\alpha,n}\right)$$

$$\times \prod_j dz_{j,0} \prod_{n=1}^\infty dz_{j,n} dz_{j,-n} \prod_{\alpha\in R} dz_{\alpha,0} \prod_{n=1}^\infty dz_{\alpha,n} dz_{\alpha,-n}.$$

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[14]In this section, we are not going to use the isomorphism $\Omega\mathfrak{g} \simeq L\mathfrak{g}/\mathfrak{g}$.

For notational convenience, we put $\gamma = 2\pi i \nu$.[15] Writing $v = e^{\sum u^\alpha E_\alpha}$ with $\overline{u}^\alpha = -u^{-\alpha}$, where summation goes over α such that $\langle \alpha, \nu \rangle \neq 0$ and using

$$[\gamma, E_\alpha] = 2\pi i \langle \alpha, \nu \rangle E_\alpha, \tag{5.4}$$

we have for the corresponding tangent vector along $\mathcal{M}_\gamma$

$$\begin{aligned} g_\gamma^{-1}\delta g_\gamma &= u^\alpha \left(e^{-\frac{\tau}{\beta}\gamma} E_\alpha e^{\frac{\tau}{\beta}\gamma} - E_\alpha \right) \\ &= u^\alpha (e^{-i\omega\langle \alpha, \nu \rangle \tau} - 1)E_\alpha. \end{aligned} \tag{5.5}$$

The fermion critical points satisfy the equation

$$\partial_\tau (\partial_\tau + \mathrm{ad}_J)\dot{\psi} = 0, \tag{5.6}$$

where $J = J_{cl}$. We have

$$(\partial_\tau + \mathrm{ad}_J)\dot{\psi} = c,$$

and since $\psi \in \Pi\Omega\mathfrak{g}$, using Fourier series we have $c = 0$, so

$$(\partial_\tau + \mathrm{ad}_J)\dot{\psi} = 0. \tag{5.7}$$

It is sufficient to solve this equation for $v = 1$. It is convenient to rewrite the expansion (4.40) in the form

$$\psi = \sum_{n,j} \psi_{j,n} H_j e^{i\omega_n \tau} + \sum_{n,\alpha} \psi_{\alpha,n} E_\alpha e^{i\omega_n \tau} \in \Pi\Omega\mathfrak{g}, \tag{5.8}$$

where $\psi_{j,0} = \psi_{\alpha,0} = 0$ and $\overline{\psi}_{j,n} = -\psi_{j,-n}$, $\overline{\psi}_{\alpha,n} = -\psi_{-\alpha,-n}$. Then equation (5.7) gives

$$\sum_{n,j} i\omega_n \psi_{j,n} H_j e^{i\omega_n \tau} + \sum_{n,\alpha} \left(i\omega_n + 2\pi i \frac{\langle \alpha, \nu \rangle}{\beta} \right) \psi_{\alpha,n} E_\alpha e^{i\omega_n \tau} = 0,$$

so that $\psi_{n,j} = 0$ and $\psi_{\alpha,n} = 0$ except $\psi_{\alpha,-\langle \alpha, \nu \rangle}$ for $\langle \alpha, \nu \rangle \neq 0$, which are arbitrary. Thus the general solution of (5.7) is

$$\chi = \sum_{\langle \alpha, \nu \rangle \neq 0} \psi_{\alpha,-\langle \alpha, \nu \rangle} E_\alpha e^{i\omega_{-\langle \alpha, \nu \rangle}\tau} \tag{5.9}$$

and is parameterized by $\Pi(\mathfrak{g}/c_\gamma)$, where $c_\gamma = \{x \in \mathfrak{g} : [x, \gamma] = 0\}$ the centralizer of γ in $\mathfrak{g}$.

5.2 Quadratic form

It is easy to show that the map $J : \Omega G \to L\mathfrak{g}$ is injective and its image is a coadjoint orbit $\widehat{\mathcal{O}}_1$ of the element $0 \in L\mathfrak{g}$ (see appendix A.3). Correspondingly, a connected component of the critical locus of J_{cl} is a coadjoint orbit $\mathcal{O}_\gamma$ of G in $\mathfrak{g}$, and we have $\mathcal{O}_\gamma \subset \widehat{\mathcal{O}}_1$.

According to formula (4.36), the differential dJ of the map $J : \Omega G \to L\mathfrak{g}$ is the differential operator $D_J = \partial_\tau + \mathrm{ad}_J$. Let the vector $X \in \Omega\mathfrak{g}$, given by (5.3), be the tangent vector to ΩG at g_{cl}. Then for $g(\tau) = g_{cl}(\tau)e^{X(\tau)}$ we obtain

$$J = g^{-1}\dot{g} = J_{cl} + DX + \frac{1}{2}[DX, X] + \text{higher order terms}, \tag{5.10}$$

where we denote $D = D_{J_{cl}}$.

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[15]When G is simple and simply connected we have $\nu \in Q^\vee$.

It follows from (5.3) that[16]

$$DX = \sum_{n,j} i\omega_n z_{j,n} H_j e^{i\omega_n \tau} + \sum_{n,\alpha} i\omega_{n+\langle\alpha,\nu\rangle} z_{\alpha,n} E_\alpha e^{i\omega_n \tau}.$$

Using the Ad_G invariance of the localizing action we can reduce bosonic critical manifold M_γ to a representative γ and according to (5.5) wet decompose

$$X(\tau) = X^\perp(\tau) + u^\alpha(e^{-i\omega_{\langle\alpha,\nu\rangle}\tau} - 1)E_\alpha,$$

where the vector $X^\perp(\tau)$ is orthogonal to M_γ at γ and for its modes in addition to constraints above, we have $z_{\alpha,0} = z_{\alpha,-\langle\alpha,\nu\rangle}$ for $\langle\alpha,\nu\rangle \neq 0$. Similarly, we decompose

$$\psi(\tau) = \eta(\tau) + \chi(\tau),$$

where χ is given by (5.9) and η is orthogonal to χ, so it does have fermion modes $\psi_{\alpha,0}, \psi_{\alpha,-\langle\alpha,\nu\rangle}$ and $\psi_{j,0}$.

In the limit $s \to \infty$ the path integral $I(s)$ effectively reduces onto a small tubular neighborhood of $\mathcal{M}_\gamma$ in ΩG. Indeed, rescale $X^\perp$ and η such that $X^\perp \to \frac{X^\perp}{\sqrt{s}}$ and $\eta \to \frac{\eta}{\sqrt{s}}$, which according to Remark 3 does not change the integration measure. Next, consider the terms in the action left over in the $s \to \infty$ limit[17] and use the delta-function constraints $\delta(\sum_{n\in\mathbb{Z}} z_{j,n})$, $\delta(\sum_{n\in\mathbb{Z}} z_{\alpha,n})$. The integral over modes describing the orbit $\mathcal{O}_\gamma$ naturally gives the its volume $\text{vol}(\mathcal{O}_\gamma)$ with respect to the Cartan-Killing volume form on G and we obtain

$$\lim_{s\to\infty} I(s) = \sum_{\gamma \in \Gamma/W} \text{vol}(\mathcal{O}_\gamma) e^{-S_\gamma} \int e^{-S_\gamma^{loc}} \mathscr{D}Y \mathscr{D}\eta \, d\chi, \tag{5.11}$$

where S_γ is the classical action,

$$S_\gamma^{loc} =$$
$$\int \left(\frac{1}{2}\langle D\dot{Y}, D\dot{Y}\rangle + \frac{1}{2}\langle \chi, \dot{\chi}\rangle + \frac{1}{2}\langle \dot{\eta}, D\dot{\eta}\rangle + \frac{1}{2}\langle \dot{\eta}, [DY, \dot{\chi}]\rangle + \frac{1}{4}\langle \dot{\chi}, [[DY, Y], \dot{\chi}]\rangle \right) d\tau, \tag{5.12}$$

the tangent vector Y does not contain modes $z_{\alpha,0}, z_{\alpha,-\langle\alpha,\nu\rangle}$ and $z_{j,0}$ and the integration over χ is a finite-dimensional.

It is tempting to make a shift $\dot{\eta} \mapsto \dot{\eta} + [Y, \dot{\chi}]$ in the path integral (5.11). We write $[Y, \dot{\chi}]$ explicitly as

$$\left[\sum_{j,n\neq 0} z_{j,n} H_j e^{i\omega_n \tau} + \sum_{\alpha,n\neq 0,-\langle\alpha,\nu\rangle} z_{\alpha,n} E_\alpha e^{i\omega_n \tau}, -i \sum_\beta \omega_{\langle\beta,\nu\rangle} \psi_{\beta,-\langle\beta,\nu\rangle} E_\beta e^{-i\omega_{\langle\beta,\nu\rangle}\tau} \right] \tag{5.13}$$

The commutator does not contain modes H_j and $E_\alpha e^{-i\omega_{\langle\alpha,\nu\rangle}}$. First, there is no constant term H_j, since it could only comes from $[E_\alpha, E_{-\alpha}]e^{i\omega_{n+\langle\alpha,\nu\rangle}\tau}$, but $n + \langle\alpha,\nu\rangle \neq 0$. Similarly,

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[16]Here we are not assuming that constant terms of $X(\tau)$ vanish but rather imposing the constraints $X(0) = X(\tau) = 0$, which follows from $g \in \Omega G$.

[17]We could use this procedure for the derivation of the Eskin trace formula in section 4.2. Since the critical points in that case are isolated, we obtain the same path integral (4.35).

the term $E_\alpha e^{-i\omega\langle\alpha,\nu\rangle}$ cannot appear: it can come only from $[H_j, E_\alpha]e^{i\omega_n - \langle\alpha,\nu\rangle\tau}$ or from $[E_\alpha, E_\beta]e^{i\omega_n - \langle\beta,\nu\rangle\tau}$, but $n \neq 0$.

However, (5.13) contains modes $\psi_{\alpha,-\langle\alpha,\nu\rangle}$ with $\langle\alpha,\nu\rangle \neq 0$. Namely, it is

$$\zeta = \frac{1}{\beta} \int_0^\beta [Y, \dot\chi] d\tau$$

$$= \sum_j \sum_{\alpha\in R} \omega_{\langle\alpha,\nu\rangle}\alpha_j z_{j,\langle\alpha,\nu\rangle}\psi_{\alpha,-\langle\alpha,\nu\rangle}E_\alpha + \sum_{\alpha,\beta\in R,\alpha+\beta\neq 0} \omega_{\langle\beta,\nu\rangle}N(\alpha,\beta)z_{\alpha,\langle\beta,\nu\rangle}\psi_{\beta,-\langle\beta,\nu\rangle}E_{\alpha+\beta}.$$

Thus correct change of variables $\dot\eta \mapsto \dot\phi = \dot\eta + [Y,\dot\chi] - \zeta$. As a result, we get the action (renaming ϕ to η again)

$$S_\gamma^{loc} = \frac{1}{2}\int \Big(\langle D\dot Y, D\dot Y\rangle + \langle\chi, \dot\chi\rangle + \langle\dot\eta, D\dot\eta\rangle$$

$$+ \langle\zeta, \mathrm{ad}_{J_{cl}}\zeta\rangle - \langle[Y,\dot\chi],[DY,\dot\chi]\rangle + \frac{1}{2}\langle\dot\chi,[[DY,Y],\dot\chi]\rangle\Big)d\tau. \tag{5.14}$$

We have

$$[[DY,Y]\dot\chi] = -[[Y,\dot\chi],DY] - [[\dot\chi,DY],Y]$$

$$= [DY,[Y,\dot\chi]] + [Y,[\dot\chi,DY]],$$

so

$$\langle\dot\chi,[[DY,Y],\dot\chi]\rangle = \langle\dot\chi,[DY,[Y,\dot\chi]]\rangle + \langle\dot\chi,[Y,[\dot\chi,DY]]\rangle$$

$$= -\langle[DY,\dot\chi],[Y,\dot\chi]\rangle - \langle[Y,\dot\chi],[\dot\chi,DY]\rangle$$

$$= 2\langle[Y,\dot\chi],[DY,\dot\chi]\rangle$$

and we finally obtain

$$S_\gamma^{loc} = \frac{1}{2}\int \Big(\langle D\dot Y, D\dot Y\rangle + \langle\chi,\dot\chi\rangle + \langle\dot\eta, D\dot\eta\rangle + \langle\zeta, \mathrm{ad}_J\zeta\rangle\Big)d\tau. \tag{5.15}$$

Remark 8. The main reason that the shift $\dot\eta \to \dot\eta + [Y,\dot\chi]$ could be dangerous is because $[Y,\dot\chi]$ may contain a constant Fourier mode which cannot be integrated in $L\mathfrak{g}$. Namely, consider the following elementary example

$$I = \int_{\Pi\Omega\mathfrak{g}} e^{-\int_0^\beta \langle\dot\eta,[A,\dot\eta]\rangle d\tau}\mathscr{D}\eta = \mathrm{Pf}(\partial_\tau^2 \mathrm{ad}_A), \tag{5.16}$$

where $A \in \mathfrak{g}$ is a constant, and consider the shift $\dot\eta \to \dot\varphi = \dot\eta + \varepsilon$ where $\varepsilon \in \Pi\mathfrak{g}$ is a constant. Obviously, after integration φ is no longer in $L\mathfrak{g}$. Nevertheless, if we naively assume that the path integral measure does not change, $\mathscr{D}\eta = \mathscr{D}\varphi$, and integration is still over $\Pi\Omega\mathfrak{g}$, then we obtain

$$\int_0^\beta \langle\dot\eta,[A,\dot\eta]\rangle d\tau = \int_0^\beta \langle\dot\varphi,[A,\dot\varphi]\rangle d\tau - 2\int_0^\beta \langle\dot\varphi[A,\varepsilon]\rangle d\tau + \int_0^\beta \langle\varepsilon,[A,\varepsilon]\rangle d\tau$$

$$= \int_0^\beta \langle\dot\varphi,[A,\dot\varphi]\rangle d\tau + \beta\langle\varepsilon,[A,\varepsilon]\rangle, \tag{5.17}$$

where the middle term vanishes since it is a total derivative. Now since $\langle\varepsilon,[A,\varepsilon]\rangle$ is a non-zero constant, we get that under such change of variables the path integral gets multiplied by $e^{-\beta\langle\varepsilon,[A,\varepsilon]\rangle}$, which is a contradiction.

To summarize, the integral over $\mathscr{D}Y$ is Gaussian and everything reduces to evaluating the regularized determinant of the following non-local linear operator

$$D^2 \ddot{Y}(\tau) + \frac{1}{\beta}\left[\dot{\chi}(\tau), \int_0^\beta [J, [Y(\xi), \dot{\chi}(\xi)]]d\xi\right] = \lambda Y(\tau), \tag{5.18}$$

which will contain a finite product of eigenvalues that depend on the modes of $\chi(\tau)$. Note that according to the conditions on $Y(\tau)$, we need to evaluate in the Hilbert space $L^2_\gamma([0, \beta], \mathfrak{g})$ of the $\mathfrak{g}$-valued functions whose Fourier series do not contain coefficients $z_{\alpha,0}, z_{\alpha,-\langle\alpha,\nu\rangle}$ and $z_{j,0}$.

The second term in (5.18) can be written in a more suggestive form. Namely, let P be a projection on a constant term operator,

$$PY = \frac{1}{\beta}\int_0^\beta Y(\xi)d\xi.$$

Using equation $D\chi = 0$, we have

$$[J, [Y, \dot{\chi}]] = [Y, [J, \dot{\chi}]] - [\dot{\chi}, [J, Y]] = -[Y, \ddot{\chi}] - [\dot{\chi}, [J, Y]]$$

and integrating over ξ we get

$$\int_0^\beta [J, [Y(\xi), \dot{\chi}(\xi)]]d\xi = -\int_0^\beta [\dot{\chi}(\xi), DY(\xi)]d\xi,$$

so we can write our operator as

$$L = D^2\partial_\tau^2 - \mathrm{ad}_{\dot{\chi}}P\mathrm{ad}_{\dot{\chi}}D = (D\partial_\tau^2 - \mathrm{ad}_{\dot{\chi}}P\mathrm{ad}_{\dot{\chi}})D. \tag{5.19}$$

This means that one needs to use another basis in the Y-space, adapted to the commutativity of D and $\mathrm{ad}_{\dot{\chi}}$.

Therefore, after integrating over Y and η, we finally obtain the following result.

Theorem 2. *The following trace formula holds.*

$$e^{-\frac{1}{12}\beta R}\,\mathrm{Tr}\,e^{-\frac{1}{2}\beta\Delta} =$$

$$V_G \sum_{\gamma\in\Gamma/W} \frac{vol(\mathcal{O}_\gamma)}{(2\pi)^{(d_G+d_{\mathcal{O}_\gamma})/2}}e^{-S_\gamma} \int \frac{\mathrm{Pf}(D\partial_\tau^2)}{\sqrt{\det(D^2\partial_\tau^2 - \mathrm{ad}_{\dot{\chi}}P\mathrm{ad}_{\dot{\chi}}D)}}e^{-\frac{1}{2}\int\langle\chi,\dot{\chi}\rangle d\tau}d\chi,$$

where the Pfaffian and determinant are taken with respect to the real Hilbert space $L^2_\gamma([0, \beta], \mathfrak{g})$ and integration over χ is over the finite-dimensional space $\Pi\mathfrak{g}/c_\gamma$ defined in (5.9).

Remark 9. As an example, we apply Theorem 2 in the case of $G = \mathrm{SU}(2)$. For the spectral side, we have

$$e^{-\frac{1}{12}\beta R}\,\mathrm{Tr}\,e^{-\frac{1}{2}\beta\Delta} = \sum_{n=0}^\infty (n+1)^2 e^{-\frac{\beta(n+1)^2}{16}}$$

Examining the localization side, there are two distinct classes of critical orbits, distinguished by $\gamma = 0$ or $\gamma = 2\pi i n\sigma_3$ with $n \in \mathbb{N}$.

For the trivial orbit $\nu = 0$, there is no χ integral and therefore

$$\frac{\text{Pf}(D\partial_\tau^2)}{\sqrt{\det(D^2\partial_\tau^2 - \text{ad}_{\dot\chi}P\text{ad}_{\dot\chi}D)}} = \frac{1}{\text{Pf}(\partial_\tau)} = \frac{1}{\beta^{3/2}}$$

For the non-trivial critical orbit $\nu \neq 0$, the classical action for the critical orbit $\gamma = 2\pi i n \sigma_3$ is $S_\gamma = \frac{16\pi^2 n^2}{\beta}$ and the fermionic integral gives

$$\int \frac{\text{Pf}(D\partial_\tau^2)}{\sqrt{\det(D^2\partial_\tau^2 - \text{ad}_{\dot\chi}P\text{ad}_{\dot\chi}D)}} e^{-\frac{1}{2}\int\langle\chi,\dot\chi\rangle d\tau} d\chi = \frac{\beta - 32\pi^2 n^2}{2\beta^{5/2}}.$$

Together with the volume factor $V_G = 32\sqrt{2}\pi^2$ and $\text{vol}(\mathcal{O}_{\gamma\neq0}) = 8\pi$, we get the expression for the localization side

$$\frac{16\sqrt{\pi}}{\beta^{3/2}} + \sum_{n=1}^{\infty}\frac{32\sqrt{\pi}(\beta - 32n^2\pi^2)}{\beta^{5/2}}e^{-\frac{16\pi^2 n^2}{\beta}} = \sum_{n=-\infty}^{\infty}\frac{16\sqrt{\pi}(\beta - 32n^2\pi^2)}{\beta^{5/2}}e^{-\frac{16\pi^2 n^2}{\beta}}$$

This expression matches precisely the spectral side, which can be shown using the Poisson summation formula.

A Semi-simple Lie groups and algebras

Here we present, in a succinct form, basic necessary facts on semi-simple Lie groups and algebras (see [22, 25] for details and proofs).

A.1 Basic facts on $\mathfrak{g}$

Let $\mathfrak{g}$ be a semi-simple Lie algebra of dimension n, let $\mathfrak{g}^*$ be the dual vector space to $\mathfrak{g}$, and let B be the Killing form on $\mathfrak{g}$,

$$B(u,v) = \text{Tr}(\text{ad}_u \circ \text{ad}_v), \quad u,v \in \mathfrak{g}.$$

If the Killing form B is negative-definite, the semi-simple Lie algebra $\mathfrak{g}$ is called compact. Let $\mathfrak{g}$ be compact Lie algebra, $\mathfrak{t}$ be its Cartan subalgebra of dimension r, and let $\mathfrak{g}_\mathbb{C} = \mathfrak{g}\otimes_\mathbb{R}\mathbb{C}$ be it complexification, a complex semi-simple Lie algebra with the Cartan subalgebra $\mathfrak{t}_\mathbb{C} = \mathfrak{t}\otimes_\mathbb{R}\mathbb{C}$.

It will be convenient to denote by $\langle\,,\,\rangle$ the Euclidean inner product on real vector space $i\mathfrak{t}$, defined by the Killing form, and to use the same notation for the induce inner product the $\mathbb{R}$-dual vector space $i\mathfrak{t}^*$. Moreover, we will use $\langle\,,\,\rangle$ to denote the natural pairing between $i\mathfrak{t}^*$ and $i\mathfrak{t}$, $\langle\alpha,h\rangle = \alpha(h)$, where $\alpha \in i\mathfrak{t}^*$ and $h \in i\mathfrak{t}$. There will be no confusion since we will always specify the spaces to which arguments of $\langle\,,\,\rangle$ belong.

Let R be the root system for the pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, so that

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}_\mathbb{C} : [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{t}_\mathbb{C}\}$$

and

$$R = \{\alpha \in i\mathfrak{t}^* - 0 : \mathfrak{g}_\alpha \neq 0\}.$$

For every root $\alpha \in R$ denote by $\alpha^\vee \in i\mathfrak{t}$ the corresponding coroot, defined by

$$\langle \alpha^\vee, \beta \rangle = 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \quad \text{for all} \quad \beta \in i\mathfrak{t}^*.$$

A basis of $\mathfrak{g}_\mathbb{C}$ consisting of a basis of $\mathfrak{t}_\mathbb{C}$ and bases of $\mathfrak{g}_\alpha$ is called Cartan-Weyl basis. Orthonormal Cartan-Weyl basis is given by $H_j \in i\mathfrak{t}$ and bases E_α of $\mathfrak{g}_\alpha$, satisfying

$$\langle H_i, H_j \rangle = \delta_{ij}, \quad \langle E_\alpha, E_\beta \rangle = \delta_{\alpha,-\beta}$$

and having commutation relations

$$[H_i, E_\alpha] = \alpha(H_i)E_\alpha, \qquad [E_\alpha, E_{-\alpha}] = \sum_{i=1}^r \alpha(H_i)H_i,$$

and

$$[E_\alpha, E_\beta] = N_{\alpha,\beta}E_{\alpha+\beta} \quad \text{if} \quad 0 \neq \alpha + \beta \in R.$$

The compact Lie algebra $\mathfrak{g}$ is the following $\mathbb{R}$-linear subspace of $\mathfrak{g}_\mathbb{C}$

$$\mathfrak{g} = \bigoplus_{j=1}^r \mathbb{R} \cdot iH_j \oplus \bigoplus_{\alpha \in R} \mathbb{R} \cdot (E_\alpha - E_{-\alpha}) \oplus \bigoplus_{\alpha \in R} \mathbb{R} \cdot i(E_\alpha + E_{-\alpha}), \tag{A.1}$$

since restriction of the Cartan-Killing form to it is negative definite.

A choice of element $h \in \mathfrak{t}$ with $\alpha(h) \neq 0$ for all $\alpha \in R$ determines the set R_+ of positive roots. The set of simple roots $S \subset R_+$ generates the root lattice Q in $i\mathfrak{t}^*$ and the corresponding set of coroots generates the coroot lattice $Q^\vee$ in $i\mathfrak{t}$. The weight lattice P in $i\mathfrak{t}^*$ is the dual lattice to $Q^\vee$, and a weight $\lambda \in P$ is called dominant if $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in R_+$. The set of dominant weights P_+ is isomorphic to $\operatorname{Irrep} G$, and for $\lambda \in P_+$ we have

$$C_2(\lambda) = \langle \lambda + \rho, \lambda + \rho \rangle - \langle \rho, \rho \rangle, \quad \text{where} \quad \rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha \in P_+.$$

Finally, an element $h \in \mathfrak{t}$ is called regular, if $\alpha(h) \notin 2\pi i\mathbb{Z}$ for all $\alpha \in R$. The following statement is crucial for the derivation of the Eskin trace formula on G.

Lemma 3. *If $X \in \mathfrak{t}$ is a regular element and $Y \in \mathfrak{g}$ satisfies $e^X = e^Y$, then $Y \in \mathfrak{t}$.*

Proof. First, $e^X = e^Y$ implies $e^X = e^{e^{sY}Xe^{-sY}}$ for all $s \in \mathbb{R}$. Now suppose that $[X, Y] \neq 0$, so $\operatorname{Ad}_{e^{sY}} X \neq X$, at least for some sufficiently small neighborhood of $s = 0$. This means that the exponential map is singular at X. Now the differential of the exponential map at X is given by

$$d\exp_X = e^X \frac{1 - e^{\operatorname{ad}_X}}{\operatorname{ad}_X}, \tag{A.2}$$

and it is singular iff ad_X has an eigenvalue $2\pi i\mathbb{Z}_{\neq 0}$. This is a contradiction since X is regular. This proves that $[X, Y] = 0$. Since $X \in \mathfrak{t}$ is regular, this implies that $Y \in \mathfrak{t}$. $\square$

A.2   Basic facts on G

Let G be connected semi-simple Lie group of dimension n, $\mathfrak{g}$ be its Lie algebra, and let B be the Killing form on $\mathfrak{g}$. Denote by $L_g : G \to G$ and $R_g : G \to G$ corresponding left and right translations, $L_g h = gh$ and $R_g h = hg$, $h \in G$, and denote by θ and $\tilde{\theta}$ corresponding left and right invariant Maurer-Cartan forms on G,

$$\theta_g(v) = (L_{g^{-1}})_* v \quad \text{and} \quad \tilde{\theta}_g(v) = (R_{g^{-1}})_* v, \quad v \in T_g G.$$

They satisfy the Maurer-Cartan equations

$$d\theta + \frac{1}{2}[\theta, \theta] = 0 \quad \text{and} \quad d\tilde{\theta} - \frac{1}{2}[\tilde{\theta}, \tilde{\theta}] = 0. \tag{A.3}$$

It is convenient to use Cartan method of moving frames (also called a tetrad formalism in general relativity). Namely, let $T_1, \ldots, T_n$ be a basis of $\mathfrak{g}$,

$$[T_a, T_b] = f_{ab}^c T_c,$$

where f_{ab}^c are the structure constants, and let $\boldsymbol{x} = (x^1, \ldots, x^n)$ be local coordinates in the neighborhood of $g \in G$. We have

$$\theta = \theta_\mu^a(\boldsymbol{x}) T_a dx^\mu \quad \text{and} \quad \tilde{\theta} = \tilde{\theta}_\mu^a(\boldsymbol{x}) T_a dx^\mu,$$

so corresponding bases of left-invariant and right-invariant vector fields on G are given by

$$e_a = \theta_a^\mu(\boldsymbol{x}) \partial_\mu \quad \text{and} \quad f_a = \tilde{\theta}_a^\mu(\boldsymbol{x}) \partial_\mu, \quad \text{where} \quad \partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{A.4}$$

Here $\theta_a^\mu(\boldsymbol{x})$ and $\tilde{\theta}_a^\mu(\boldsymbol{x})$ are inverse matrices to $\theta_\mu^a(\boldsymbol{x})$ and $\tilde{\theta}_\mu^a(\boldsymbol{x})$. For $v = v^\mu \partial_\mu \in T_g G$ we put $v_L = \theta_g(v) \in \mathfrak{g}$ and $v_R = \tilde{\theta}_g(v) \in \mathfrak{g}$, so

$$v_L = v_L^a T_a \quad \text{and} \quad v_R = v_R^a T_a, \quad \text{where} \quad v_L^a = \theta_\mu^a(\boldsymbol{x}) v^\mu \quad \text{and} \quad v_R^a = \tilde{\theta}_\mu^a(\boldsymbol{x}) v^\mu.$$

Equivalently, $v = v_L^a e_a = v_R^a f_a$.

For compact G the bilinear form $-B$ is positive-definite and determines a bi-invariant Riemannian metric $(\,,\,)$ on G, the Cartan-Killing metric

$$(u, v)_g = -B(\theta_g(u), \theta_g(v)), \quad u, v \in T_g G. \tag{A.5}$$

In local coordinates we have

$$(u, v)_g = g_{\mu\nu}(\boldsymbol{x}) u^\mu v^\nu, \quad \text{where } v = v^\mu \partial_\mu, \quad u = u^\mu \partial_\mu,$$

and it follows from (A.5) that

$$g_{\mu\nu}(\boldsymbol{x}) = g_{ab} \theta_\mu^a(\boldsymbol{x}) \theta_\nu^b(\boldsymbol{x}), \quad \text{where} \quad g_{ab} = -B(T_a, T_b). \tag{A.6}$$

A.3 Coadjoint orbits of *LG*

Let LG be the loop group of G and $\widehat{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{R}$ be the central extension of the Lie algebra $L\mathfrak{g}$ by the 2-cocycle

$$c(X, Y) = \frac{1}{\beta} \int_0^\beta \langle \dot{X}(\tau), Y(\tau) \rangle d\tau.$$

The adjoint action of LG on $\widehat{L\mathfrak{g}}$ is given by

$$\mathrm{Ad}_g(X, \alpha) = \left(\mathrm{Ad}_g(X), \alpha + \frac{1}{\beta} \int_0^\beta \langle J(\tau), X(\tau) \rangle d\tau \right), \quad J = J(g) = g^{-1}\dot{g}.$$

The non-degenerate bilinear form on $\widehat{L\mathfrak{g}}$,

$$\langle (X_1, \alpha_1), (X_2, \alpha_2) \rangle = \frac{1}{\beta} \int_0^\beta \langle X_1(\tau), X_2(\tau) \rangle d\tau + \alpha_1 \alpha_2,$$

allows to identify $\widehat{L\mathfrak{g}}^*$ with $\widehat{L\mathfrak{g}}$. Under this identification, the coadjoint action of LG on $\widehat{L\mathfrak{g}}^*$ takes the form

$$\mathrm{Ad}_g^*(X, \alpha) = (\mathrm{Ad}_g(X) - \alpha \, \dot{g}g^{-1}, \alpha), \quad (X, \alpha) \in \widehat{L\mathfrak{g}}. \tag{A.7}$$

The Lie algebra $L\mathfrak{g}$ is naturally identified with the hyperplane $\alpha = 1$ in $\widehat{L\mathfrak{g}}$, and since $-\dot{g}g^{-1} = J(g^{-1})$, it follows from (A.7) that the image of the map $J : LG \to L\mathfrak{g}$ is a coadjoint orbit $\widehat{\mathcal{O}}_1$ of the element $(0, 1) \in \widehat{L\mathfrak{g}}$.

B Symplectic geometry of T^*G

Here G is a Lie group, $\mathfrak{g}$ is its Lie algebra, and $\mathfrak{g}^*$ is the dual vector space to $\mathfrak{g}$ with the natural pairing which is this section we denote by $(\, , \,) : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$.

B.1 The symplectic form

- The canonical 1-form ϑ on T^*G (the Liouville 1-form on a cotangent bundle) is defined as follows. Let $\pi : T^*G \to G$ be the natural projection and $\pi_* : T(T^*G) \to TG$ be its differential. Then for $\xi \in T_{(g,p)}T^*G$, where $g \in G$ and $p \in T_g^*G$, we have

$$\vartheta(\xi) = p(v), \quad \text{where} \quad v = \pi_*(\xi) \in T_gG. \tag{B.1}$$

- The canonical symplectic form on T^*G is defined by $\omega = d\vartheta$.

- The left translations (and also the right translations) trivialize the tangent and cotangent bundles to G,

$$TG \simeq G \times \mathfrak{g} \quad \text{and} \quad T^*G \simeq G \times \mathfrak{g}^*,$$

so putting in (B.1) $p = (L_g^*)^{-1}\alpha$, where $\alpha \in \mathfrak{g}^*$, we obtain

$$\vartheta(\xi) = (\alpha, (L_{g^{-1}})_* v). \tag{B.2}$$

Using Maurer-Cartan form θ, (B.2) can be written as

$$\vartheta_{(g,\alpha)} = (\alpha, \theta_g),$$

so using the Maurer-Cartan equation, we obtain a simple formula for the symplectic form ω

$$\omega = (d\alpha, \theta) + (\alpha, d\theta) = (d\alpha, \theta) - \frac{1}{2}(\alpha, [\theta, \theta]). \tag{B.3}$$

Explicitly, for $X_1 = (v_1, \alpha_1), X_2 = (v_2, \alpha_2) \in T_{(g,\alpha)}T^*G \simeq T_gG \times \mathfrak{g}^*$ we have[18]

$$\omega(X_1, X_2) = (\alpha_1, \theta(v_2)) - (\alpha_2, \theta(v_1)) - (\alpha, [\theta(v_1), \theta(v_2)]). \tag{B.4}$$

B.2 Poisson brackets

The Poisson bracket of smooth functions on T^*G is defined by

$$\{f_1, f_2\} = \omega(X_{f_1}, X_{f_2}).$$

Here X_f is the Hamiltonian vector field for a function f, defined by

$$i_{X_f}\omega = -df.$$

- For every $u \in \mathfrak{g}$ consider the function l_u on T^*G, defined by

$$l_u(g, \alpha) = (\alpha, u).$$

The corresponding Hamiltonian vector field X_{l_u} at a point $(g, \alpha) \in T^*G$ is

$$X_{l_u} = ((L_g)_*u, \mathrm{ad}_u^*\alpha) \in T_{(g,\alpha)}T^*G,$$

where ad^* is the coadjoint action of $\mathfrak{g}$,

$$(\mathrm{ad}_u^*\alpha, v) = (\alpha, \mathrm{ad}_u v) = (\alpha, [u, v]).$$

Indeed, $dl_u = u$ and for a vector field $Y = (v, \beta)$ on T^*G we have

$$\begin{aligned}
\omega(X_{l_u}, Y) &= (\mathrm{ad}_u^*\alpha), \theta(v)) - (\beta, \theta((L_g)_*u)) - (\alpha, [\theta((L_g)_*u), \theta(v)]) \\
&= (\mathrm{ad}_u^*\alpha), \theta(v)) - (\beta, u) - (\alpha, [u, \theta(v)]) \\
&= -(\beta, u) = -dl_u(Y).
\end{aligned}$$

Then

$$\{l_{u_1}, l_{u_2}\} = \omega(X_{l_{u_1}}, X_{l_{u_2}}) = -(\mathrm{ad}_{u_2}^*\alpha, u_1) = (\alpha, [u_1, u_2]),$$

so

$$\{l_{u_1}, l_{u_2}\} = l_{[u_1, u_2]}, \quad u_1, u_2 \in \mathfrak{g}. \tag{B.5}$$

---

[18]Note that due to the definition of the exterior product of two 1-forms, the factor $\frac{1}{2}$ in the Maurer-Cartan formula becomes 1.

- Trivialization of TG and T^*G by the right translations leads to the functions r_u on T^*G, defined by

$$r_u(g, \alpha) = (\alpha, \mathrm{Ad}_{g^{-1}} u).$$

The corresponding Hamiltonian vector field is $X_{r_u} = ((R_g)_* u, 0)$, so

$$\{r_{u_1}, r_{u_2}\} = -r_{[u_1, u_2]} \quad \text{and} \quad \{l_{u_1}, r_{u_2}\} = 0. \tag{B.6}$$

- Another class of functions on T^*G is given by smooth $f : G \to \mathbb{R}$. It is easy to see that

$$X_f = (0, -(L_g)^* df).$$

Indeed, for $Y = (v, \beta)$ be have

$$\omega(X_f, Y) = -((L_g)^* df, \theta(v)) = -(df, (L_g)_* \theta(v)) = -(df, v),$$

so such functions Poisson commute and

$$\{l_u, f\} = ((L_g)^* df, \theta((L_g)_* u)) = (df, (L_g)_* u) = e_u(f). \tag{B.7}$$

C Free particle on a Lie group

In local coordinates on G we have

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu}(\boldsymbol{x}) \dot{x}^\mu \dot{x}^\nu, \tag{C.1}$$

where $g_{\mu\nu}(\boldsymbol{x})$ is given by (A.6). Consider left-invaraint conserved current $J = g^{-1}\dot{g}$

$$J = J^a T_a, \quad J^a = \theta_\mu^a \dot{x}^\mu \quad \text{and} \quad \dot{x}^\mu = \theta_a^\mu J^a, \tag{C.2}$$

so the Lagrangian (C.1) can be rewritten in terms of currents

$$\mathcal{L} = \frac{1}{2} g_{ab} J^a J^b.$$

By the Legendre transform

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu} \dot{x}^\nu$$

so using $\dot{x}^\mu = g^{\mu\nu} p_\nu$ we have

$$H = p_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu. \tag{C.3}$$

Using (C.2), we can rewrite Legendre transform in terms of J

$$p_\mu = g_{ab} \theta_\mu^a J^b = \theta_\mu^a J_a, \quad J_a = g_{ab} J^b.$$

Note that J^a are functions on TG, so introducing the functions l_a on T^*G by

$$l_a = \theta_a^\mu p_\mu, \tag{C.4}$$

we have

$$p_\mu \dot{x}^\mu = g^{ab} l_a l_b \quad \text{and} \quad H = \frac{1}{2} g^{ab} l_a l_b.$$

We see that l_a are precisely the functions l_u on T^*G, defined in the appendix B, for $u = T_a$. According to (B.5), they have the following Poisson brackets

$$\{l_a, l_b\} = f^c_{ab} l_c. \tag{C.5}$$

One can do exactly same exercise with a right-invariant conserved current $\tilde{J} = \dot{g} g^{-1}$ with a similarly defined function $r_a = \tilde{\theta}^\mu_a p_\mu$ on T^*G. Namely, it follows from (B.6) that

$$\{r_a, r_b\} = -f^c_{ab} r_c \quad \text{and} \quad \{r_a, l_b\} = 0. \tag{C.6}$$

References

[1] A. Alekseev and E. Meinrenken, *The non-commutative Weil algebra*, *Invent. Math.* **139** (2000) 135 [math/9903052].

[2] L. Alvarez-Gaume, *Supersymmetry and the Atiyah-Singer Index Theorem*, *Commun. Math. Phys.* **90** (1983) 161 [INSPIRE].

[3] M.F. Atiyah, *Circular symmetry and stationary-phase approximation*, *Astérisque* **131** (1985) 43.

[4] F. Benini and S. Cremonesi, *Partition Functions of $\mathcal{N} = (2,2)$ Gauge Theories on S^2 and Vortices*, *Commun. Math. Phys.* **334** (2015) 1483 [arXiv:1206.2356] [INSPIRE].

[5] F. Benini, R. Eager, K. Hori and Y. Tachikawa, *Elliptic Genera of 2d $\mathcal{N} = 2$ Gauge Theories*, *Commun. Math. Phys.* **333** (2015) 1241 [arXiv:1308.4896] [INSPIRE].

[6] N. Berline, E. Getzler and M. Vergne, *Heat kernels and dirac operators*, Springer Science & Business Media (2003) [DOI:10.1007/978-3-642-58088-8].

[7] J.-M. Bismut, *A local index theorem for non Kähler manifolds*, *Math. Ann.* **284** (1989) 681 [INSPIRE].

[8] J.-M. Bismut, *The hypoelliptic Laplacian on the cotangent bundle*, *J. Am. Math. Soc.* **18** (2005) 379.

[9] J.-M. Bismut, *The hypoelliptic Laplacian on a compact Lie group*, *J. Funct. Anal.* **255** (2008) 2190.

[10] J.-M. Bismut, *Hypoelliptic laplacian and orbital integrals (am-177)*, Princeton University Press (2011) [DOI:10.1515/9781400840571].

[11] H.W. Braden, *σ Models With Torsion*, *Annals Phys.* **171** (1986) 433 [INSPIRE].

[12] R. Camporesi, *Harmonic analysis and propagators on homogeneous spaces*, *Phys. Rept.* **196** (1990) 1 [INSPIRE].

[13] S. Cordes, G.W. Moore and S. Ramgoolam, *Lectures on 2-d Yang-Mills theory, equivariant cohomology and topological field theories*, *Nucl. Phys. B Proc. Suppl.* **41** (1995) 184 [hep-th/9411210] [INSPIRE].

[14] D. Delmastro, D. Gaiotto and J. Gomis, *Global anomalies on the Hilbert space*, *JHEP* **11** (2021) 142 [arXiv:2101.02218] [INSPIRE].

[15] B.S. DeWitt, *Dynamical theory in curved spaces. I. A review of the classical and quantum action principles*, *Rev. Mod. Phys.* **29** (1957) 377 [INSPIRE].

[16] N. Doroud, J. Gomis, B. Le Floch and S. Lee, *Exact Results in D = 2 Supersymmetric Gauge Theories*, *JHEP* **05** (2013) 093 [arXiv:1206.2606] [INSPIRE].

[17] J.S. Dowker, *When is the "sum over classical paths" exact?*, *J. Phys. A* **3** (1970) 451.

[18] J.S. Dowker, *Quantum mechanics on group space and Huygens' principle*, *Annals Phys.* **62** (1971) 361 [INSPIRE].

[19] L.D. Èskin, *Heat equation on Lie groups* (in Russian), in *Memoriam: N.G.Čebotarev*, Izdat. Kazan University, Kazan (1964), p. 113 [MR 0206535].

[20] I.B. Frenkel, *Orbital theory for affine Lie algebras*, *Invent. Math.* **77** (1984) 301.

[21] M.C. Gutzwiller, *Classical quantization of a Hamiltonian with ergodic behavior*, *Phys. Rev. Lett.* **45** (1980) 150.

[22] S. Helgason, *Differential geometry, lie groups, and symmetric spaces*, Academic press (1979) [ISBN: 9780080873961].

[23] C.G.J. Jacobi, *Suite des notices sur les fonctions elliptiques*, *J. Reine Angew. Math* **3** (1828) 403.

[24] A. Kapustin, B. Willett and I. Yaakov, *Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter*, *JHEP* **03** (2010) 089 [arXiv:0909.4559] [INSPIRE].

[25] A. Kirillov, *An introduction to lie groups and lie algebras*, Cambridge University Press (2008) [DOI:10.1017/cbo9780511755156].

[26] B. Kostant, *A cubic dirac operator and the emergence of euler number multiplets of representations for equal rank subgroups*, *Duke Math. J.* **100** (1999) 447.

[27] A. Losev, N. Nekrasov and S.L. Shatashvili, *Issues in topological gauge theory*, *Nucl. Phys. B* **534** (1998) 549 [hep-th/9711108] [INSPIRE].

[28] M.S. Marinov and M.V. Terentev, *Dynamics on the group manifold and path integral*, *Fortsch. Phys.* **27** (1979) 511 [INSPIRE].

[29] G.W. Moore, N. Nekrasov and S. Shatashvili, *Integrating over Higgs branches*, *Commun. Math. Phys.* **209** (2000) 97 [hep-th/9712241] [INSPIRE].

[30] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2003) 831 [hep-th/0206161] [INSPIRE].

[31] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[32] V. Pestun et al., *Localization techniques in quantum field theories*, *J. Phys. A* **50** (2017) 440301 [arXiv:1608.02952] [INSPIRE].

[33] R.F. Picken, *The Propagator for Quantum Mechanics on a Group Manifold From an Infinite Dimensional Analog of the Duistermaat-heckman Integration Formula*, *J. Phys. A* **22** (1989) 2285 [INSPIRE].

[34] S.D. Poisson, *Mémoires de l'Académie (royale) des sciences de l'Institut (imperial) de France*, Académie royale des sciences (France) (1827), p. 592.

[35] H. Rumpf, *Supersymmetric Dirac Particles in Riemann-Cartan Space-time*, *Gen. Rel. Grav.* **14** (1982) 773 [InSPIRE].

[36] L. Schulman, *A path integral for spin*, *Phys. Rev.* **176** (1968) 1558 [InSPIRE].

[37] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric riemannian spaces with applications to dirichlet series*, *J. Indian Math. Soc.* **20** (1956) 47.

[38] D. Stanford and E. Witten, *Fermionic Localization of the Schwarzian Theory*, *JHEP* **10** (2017) 008 [arXiv:1703.04612] [InSPIRE].

[39] E. Witten, *Constraints on Supersymmetry Breaking*, *Nucl. Phys. B* **202** (1982) 253 [InSPIRE].

[40] E. Witten, *Supersymmetry and Morse theory*, *J. Diff. Geom.* **17** (1982) 661 [InSPIRE].

[41] E. Witten, *Topological Quantum Field Theory*, *Commun. Math. Phys.* **117** (1988) 353 [InSPIRE].

[42] E. Witten, *Two-dimensional gauge theories revisited*, *J. Geom. Phys.* **9** (1992) 303 [hep-th/9204083] [InSPIRE].