

Giant graviton expansion of Schur index and quasimodular forms

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ABSTRACT: The flavored superconformal Schur index of $\mathcal{N} = 4$ $U(N)$ SYM has finite N corrections encoded in its giant graviton expansion in terms of D3 branes wrapped in $AdS_5 \times S^5$. The key element of this decomposition is the non-trivial index of the theory living on the wrapped brane system. A remarkable feature of the Schur limit is that the brane index is an analytic continuation of the flavored index of $\mathcal{N} = 4$ $U(n)$ SYM, where n is the total brane wrapping number. We exploit recent exact results about the Schur index of $\mathcal{N} = 4$ $U(N)$ SYM to evaluate the closed form of the brane indices appearing in the giant graviton expansion. Away from the unflavored limit, they are characterized by quasimodular forms providing exact information at all orders in the index universal fugacity. As an application of these results, we present novel exact expressions for the giant graviton expansion of the unflavored Schur index in a class of four dimensional $\mathcal{N} = 2$ theories with equal central charges $a = c$, i.e. the non-Lagrangian theories $\widehat{\Gamma}(SU(N))$ with $\Gamma = E_6, E_7, E_8$.

KEYWORDS: AdS-CFT Correspondence, D-Branes

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1 Introduction

For a superconformal theory the index introduced in [1–3] encodes a large amount of information about the BPS spectrum. It may be regarded as the Witten index [4] in radial quantization and is invariant under smooth supersymmetric deformations. Indices are nowadays a standard and major tool for the investigation of superconformal theories [5]. Their applications range from the study of their dualities, strongly coupled fixed points, and even black hole physics in the context of AdS/CFT, see e.g. in AdS_4 [6–8] and in AdS_5 [9–11]. A review can be found in [12].

For $U(N)$ gauge invariant theories, the index admits a large N limit with finite N corrections coming from $U(N)$ trace relations that spoil orthogonality of single-trace operators. When the superconformal theory has a gravity dual, the interpretation of these corrections is far deeper. In the simplest example of four dimensional $\mathcal{N} = 4$ $U(N)$ super Yang-Mills (SYM) theory dual to type IIB superstring in $AdS_5 \times S^5$, finite N effects originate from wrapped D3 brane states maximally stretched in S^5 called giant gravitons [13] and having charge of order

N .¹ From the corresponding giant-graviton expansion of the index, it is possible to reproduce the entropy of dual black holes in $AdS_5 \times S^5$ as first shown in [15] for small black holes with charge $Q \ll N^2$ and later generalized to black holes with charge $Q \sim N^2$, first in [16], using a large-charge expansion, and later in [17], using a large N expansion. Recently, the giant graviton expansion has also been described by enumerating BPS geometries studying bubbling solutions in supergravity [18, 19].

The contribution to the index from a certain wrapped D3 brane state may be identified with the index of the theory living on the brane world-volume and having reduced unbroken superconformal invariance [20–22]. The computation of this “brane index” is a non-trivial issue. The superconformal index of 4d $\mathcal{N} = 4$ $U(N)$ SYM is defined as

$$I^{U(N)}(y, \mathbf{u}; q) = \text{Tr}_{\text{BPS}}[(-1)^F q^{H+\bar{J}} y^{2J} u_1^{R_1} u_2^{R_2} u_3^{R_3}], \quad u_1 u_2 u_3 = 1, \quad (1.1)$$

where the Hamiltonian H , the spins J, \bar{J} , and the R-charges R_1, R_2, R_3 are Cartan generators of the superconformal algebra $\mathfrak{psu}(2, 2|4)$ and the trace is restricted to states obeying certain BPS conditions.² The giant graviton expansion of the index reads [20, 21]

$$I^{U(N)}(y, \mathbf{u}; q) = I^{\text{KK}}(y, \mathbf{u}; q) \sum_{n_1, n_2, n_3=0}^{\infty} (qu_1)^{n_1 N} (qu_2)^{n_2 N} (qu_3)^{n_3 N} I_{n_1, n_2, n_3}^{\text{D}^3}(y, \mathbf{u}; q), \quad (1.2)$$

where each piece has a definite meaning on gravity dual side. The $N \rightarrow \infty$ factor $I^{\text{KK}}(y, \mathbf{u}; q)$ matches the index from IIB supergravity Kaluza-Klein states. The triple summation is over wrapped D3 branes with topology $S^1 \times S^3$ where S^1 is in AdS_5 and $S^3 \subset S^5$ with winding n_I along the 3-cycle obtained setting $z_I = 0$ in S^5 realized in \mathbb{C}^3 as $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. The quantities $I_{n_1, n_2, n_3}^{\text{D}^3}(y, \mathbf{u}; q)$ represent the superconformal index of the theory living on the wrapped branes world-volume, i.e. a $U(n_1) \times U(n_2) \times U(n_3)$ quiver gauge theory with three bi-fundamental hyper multiplets. The prefactor $\prod_{I=1}^3 (qu_I)^{n_I N}$ in (1.2) comes from the classical charges and energy of the brane system.

At generic values of fugacities y, \mathbf{u}, q , the brane indices $I_{n_1, n_2, n_3}^{\text{D}^3}$ do not depend on N and (1.2) organizes finite N corrections $\sim q^{nN}$ as a series of contributions from wrapped branes with increasing total winding number.³ At large N , the corrections are non-perturbative in the small q expansion. When algebraic constraints are imposed on fugacities to get simplified unrefined indices, extra polynomial contributions in N may appear. This is often referred to as a “wall-crossing” effect. It was discussed in [21, 22, 28] and explained in [29, 30] on gravity side in terms of zero modes of fluctuations on the wrapped branes, see also [31]. These zero modes are fully regularized when all fugacities are switched on and take generic values.

In this paper, we focus on some technical aspects of the determination of the brane indices $I_{n_1, n_2, n_3}^{\text{D}^3}$ in (1.2). To this aim, it will be convenient to consider the so-called Schur limit of the index [32, 33]. For a general $\mathcal{N} = 2$ superconformal theory, the Schur index gives the vacuum character of the associated chiral algebra [34]. Here, it corresponds to the

¹See [14] for a recent analysis of the precise correspondence between states of certain D3 giant gravitons branes in $AdS_5 \times S^5$ and auxiliary ghost states in $\mathcal{N} = 4$ $U(N)$ SYM.

²We follow the conventions in [23] where, in particular, details on the BPS condition can be found.

³It is possible to discuss (1.2) in the $U(N)$ SYM theory without reference to its string dual description [24–26], up to possible non-trivial reorderings of the sum [27].

specialization $y = q^{\frac{1}{2}}$, $\mathbf{u} = (u, u^{-1}, 1)$ of (1.1), i.e.⁴

$$I^{U(N)}(u; q) = \text{Tr}_{\text{BPS}}[(-1)^F q^{H+J+\bar{J}} u^{R_1-R_2}]. \quad (1.3)$$

The specific structure of the giant-graviton expansion of the Schur index of $\mathcal{N} = 4$ $U(N)$ SYM was first discussed in [35]. Quite remarkably, in this limit the right hand side of (1.2) can be written in terms of the SYM index itself, up to a tricky analytic continuation proposed in [23]. The explicit formula reads

$$I^{U(N)}(u; q) = I^{\text{KK}}(u; q) \sum_{n=0}^{\infty} \sum_{p=0}^n (uq)^{(n-p)N} I_{n-p}^{\text{D3}}(u; q) q^{2(n-p)p} (u^{-1}q)^{pN} I_p^{\text{D3}}(u^{-1}; q), \quad (1.4)$$

where the Kaluza-Klein factor is explicitly (see appendix A for special functions notation)

$$I^{\text{KK}}(u; q) = \frac{(q^2)_{\infty}}{(uq)_{\infty} (u^{-1}q)_{\infty}}, \quad (1.5)$$

and the brane indices are given by the key relation

$$I_n^{\text{D3}}(u; q) = I^{U(n)}(u^{-\frac{1}{2}}q^{-\frac{3}{2}}; u^{-\frac{1}{2}}q^{\frac{1}{2}}). \quad (1.6)$$

Thus, finite N corrections $\sim q^{nN}$ to the Schur index of $U(N)$ SYM are captured by the index of the $U(n)$ SYM theory by means of the above fugacity transformation. As a technical remark, the reduction of the triple sum in (1.2) to the single sum in (1.4) is a simplification due to vanishing of contribution from the cycle $z_3 = 0$ and a Weyl symmetry $u \rightarrow u^{-1}$ relating contributions from the other two cycles $z_1 = 0$ and $z_2 = 0$.

The analytic continuation rule (1.6) follows from the analysis of the unbroken superconformal symmetry on the wrapped brane system. This symmetry-based approach to the construction of brane indices was recently reconsidered in [30, 36] by a full calculation of fluctuation effects from all fields living on the brane world-volume. A similar treatment of finite N corrections to superconformal index was previously carried out in the case of the 6d $(2, 0)$ theory from semiclassical M2 brane wrapped on $S^1 \times S^2$ in the M-theory background $AdS_7 \times S^4$ [37], and for the 3d $\mathcal{N} = 8$ supersymmetric level-one $U(N) \times U(N)$ ABJM theory from semiclassical M5 brane wrapped on $S^1 \times S^5$ in $AdS_4 \times S^7$ [29]. Alternatively, localization on the wrapped brane was recently used in [38] to discuss the giant graviton expansion of the $\frac{1}{2}$ -BPS conformal index in $\mathcal{N} = 4$ $U(N)$ SYM.

Technically, a major problem in making use of (1.6) is that it cannot be applied when the $U(n)$ Schur index is only given as a (truncated) q -series. Reason is that the transformation (1.6) is actually an analytic continuation outside the convergence domain of the Schur index around $q = 0$. As a consequence, the strategy adopted in [35] to overcome this difficulty was to (i) apply the transformation $(u; q) \rightarrow (u^{-\frac{1}{2}}q^{-\frac{3}{2}}; u^{-\frac{1}{2}}q^{\frac{1}{2}})$ to the integrand of the holonomy integral representation of the $U(n)$ index and (ii) perform the contour integrations along suitable cycles by a prescription determining the set of relevant poles. The same technique was successfully applied in a variety of other models in [39–44] and further developed and clarified in [44, 45].

⁴To keep notation simple, the Schur index is denoted by the same symbol as the general index and only two arguments u, q instead of the five y, u_1, u_2, u_3, q .

The outcome of the symmetry-based approach were explicit expansions for the brane indices in (1.6). At leading and next-to-leading wrapping order, i.e. for $n = 1, 2$, they take the form

$$\begin{aligned} I_1^{\text{D3}}(u; q) &= \frac{u^3}{1-u^2} q + (1-u^2)q^2 + (u^{-3} - u^3)q^3 + \dots, \\ I_2^{\text{D3}}(u; q) &= \frac{u^{10}(2-u^2)}{(1-u^2)(1-u^4)} q^4 + u^5(2-u^4)q^5 + (2+2u^6-u^{12})q^6 + \dots, \end{aligned} \tag{1.7}$$

and were computed in [35] up to wrapping $n = 4$. Expressions (1.7) have a very rich dependence on u and q that can be (minimally) tested by matching the exact prediction for the index in the unflavored case $u = 1$. This was obtained in [46] and reads

$$\begin{aligned} I^{\text{U}(N)}(1; q) &= I^{\text{KK}}(1; q) \sum_{n=0}^{\infty} (-1)^n \left[\binom{N+n}{N} + \binom{N+n-1}{N} \right] q^{nN+n^2}, \\ &= I^{\text{KK}}(1; q) \left[1 - (N+2)q^{N+1} + \frac{1}{2}(1+N)(4+N)q^{2N+4} + \dots \right], \end{aligned} \tag{1.8}$$

where the Kaluza-Klein contribution is given by

$$I^{\text{KK}}(1; q) = \frac{1}{\vartheta_4(0)} = \frac{(q^2)_{\infty}}{(q)_{\infty}^2} = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n}. \tag{1.9}$$

Checking agreement between (1.4) and (1.8) turns out to be a non-trivial issue, because of the poles at $u = 1$ in the q -series of the brane indices (1.7). These have to cancel, but produce leftover powers of N , well visible in (1.8), due to the mentioned wall-crossing effect.

Besides, higher order terms of the expansion do not follow from an explicit closed formula and are not under full analytical control. Still, truncating the q -series at some high finite order, one empirically matches (1.8) [35].

Summary of new results. In this paper, we begin by considering the $u \rightarrow 1$ limit of (1.4) at all orders in q . To this aim, we exploit recent exact results about the flavored Schur index of $\mathcal{N} = 4$ $U(N)$ SYM [47, 48], see also [49] for other gauge groups. We show that its available explicit representations allow to evaluate the analytic continuation in (1.6) and to get closed form expressions for the brane indices $I_n^{\text{D3}}(u; q)$ in agreement with the expansions (1.7), and extending them at all orders in q . As a consequence, we prove that the $u \rightarrow 1$ limit of (1.4) reproduces the exact result (1.8). In more details, we find that the general structure of the brane index is

$$I_n^{\text{D3}}(u; q) = \frac{u^{n(2n+1)} A_n(u)}{\prod_{m=1}^n (1-u^{2m})} q^{n^2} + \frac{B_n(u; q)}{(1-u^2)^{n-2}} q^{n^2+1}, \tag{1.10}$$

where $A_n(u)$ are polynomials in u with degree $n(n-1)$ and integer coefficients that we compute explicitly for all n . The functions $B_n(u; q)$ have q -series with coefficients that are rational functions of u , smooth for $u \rightarrow 1$. They can be computed exactly near the point $u = 1$ and have the structure

$$B_n(u; q) = \frac{1}{q} \sum_{m=0}^{\infty} \frac{1}{m!} B_n^{(m)}(q) (u-1)^m, \tag{1.11}$$

where $B_n^{(m)}(q)$ are quasimodular forms, i.e. polynomials in the classical Eisenstein series $\mathbb{E}_2(q)$, $\mathbb{E}_4(q)$, $\mathbb{E}_6(q)$. Similar quasimodular properties were previously observed in the unflavored Schur indices of class \mathcal{S} theories [47, 50], see also [51–54].

From these expressions, one can extend systematically the unrefined expansion (1.8) by including near-unflavored corrections in powers of the deviation $u - 1$, and correct to all orders in q . To give an example, the leading term in the giant graviton expansion, i.e. the correction with weight $\sim q^N$, reads

$$\mathbb{I}^{\text{U}(N)}(u; q) = \mathbb{I}^{\text{KK}}(1; q) \left[1 + q^{N+1} \left(- (2 + N) + \sum_{p=2}^{\infty} \Delta_1^{(p)}(q; N) (u - 1)^p \right) + \mathcal{O}(q^{2N}) \right], \quad (1.12)$$

with

$$\Delta_1^{(2)}(q; N) = \frac{1}{72} [-12(2 + N)((2 + N)^2 - \mathbb{E}_2(q)) - \mathbb{E}_2^2(q) + \mathbb{E}_4(q)], \quad (1.13)$$

and similar expressions for the higher functions $\Delta_1^{(p)}(q; N)$ that we give in explicit form and involve higher powers of Eisenstein series, as p is increased.

As a further simple application of our analysis, we present novel exact predictions for the giant graviton expansion of the Schur index of a class of non-Lagrangian 4d $\mathcal{N} = 2$ superconformal theories with equal central charges $a = c$, introduced in [55, 56]. They are denoted $\widehat{\Gamma}(\text{SU}(N))$ with $\Gamma = D_4, E_6, E_7, E_8$, and are built by gauging part of the flavor symmetry of a product of $D_p(\text{SU}(N))$ theories [57, 58]. In these models, the unflavored Schur index is equal to certain specializations of the flavored Schur index of $\mathcal{N} = 4$ $\text{SU}(N)$ SYM [55, 56].⁵ This identification come from an isomorphism between the chiral algebras of the respective theories and was explicitly checked for the Schur index of the $\widehat{E}_6(\text{SU}(2))$ theory [55, 60] showing that it is same as the vacuum character of the $\mathcal{A}(6)$ algebra [61, 62]. To exploit this relation to derive a giant graviton expansion requires more information than the unflavored limit in (1.8). From our results in flavored case we can work out the giant graviton expansion of the $\widehat{\Gamma}(\text{SU}(N))$ theories from the D3 brane representation (1.4). For instance, for the $\Gamma = E_6$ theory, the first two terms in its giant graviton expansion read

$$\begin{aligned} \mathbb{I}^{\widehat{E}_6(\text{SU}(N))}(q) &= \frac{(q; q^3)_{\infty} (q^2; q^3)_{\infty}}{(q)_{\infty} (q^2)_{\infty} (q^3)_{\infty}} \left[1 + q^{N+1} F_1(q) + q^{2N+3} \left(N + F_2(q) \right) + \mathcal{O}(q^{3N}) \right], \\ F_1(q) &= -\frac{(q^2)_{\infty}^2}{(q; q^2)_{\infty}^2} = -\frac{1}{4} q^{\frac{1}{4}} \vartheta_2(\sqrt{q})^2, \quad F_2(q) = 3 + \frac{1}{8} \frac{\partial_z^2 \vartheta_4(0; q)}{\vartheta_4(q)}. \end{aligned} \quad (1.14)$$

As we remarked, results like (1.14) originate from a fruitful combination of the D3 brane giant graviton expansion of the $\mathcal{N} = 4$ SYM $\text{U}(N)$ Schur index and the remarkable relation with the index of the $\widehat{\Gamma}(\text{SU}(N))$ theories. It would be interesting to give a gravity interpretation of these new brane-like expansions by exploring their large N limit and associating each q^{nN} term to suitable non-perturbative corrections. For instance, working with truncated expansions in q , this was shown to be possible in [43] for Argyres-Douglas [63, 64] and Minahan-Nemeschansky theories [65, 66] which are 4d $\mathcal{N} = 2$ theories on D3-branes in 7-brane backgrounds with constant axio-dilaton.

⁵See [59] for a recent discussion of modular properties in these models.

Plan of the paper. In section 2, we review explicit formulas for the exact Schur index in $\mathcal{N} = 4$ $U(N)$ SYM in terms of twisted Weierstrass functions. In section 3, we examine the corresponding expressions after application of the analytic continuation relation (1.6). We discuss the associated explicit brane indices I_n^{D3} and their general structure at increasing n . In section 4, we show how the exact unflavored index is recovered at all orders in q by presenting the quasimodular expressions of the coefficient functions in (1.11). Section 5 is devoted to the analysis of the corrections in the near-unflavored limit, i.e. the expansion of the index around in powers of $u - 1$. Finally, in section 6, we discuss the giant graviton expansion of the $\mathcal{N} = 2$ $\widehat{\Gamma}(SU(N))$ theories with $a = c$. Several appendices contain technical details and additional remarks.

2 Schur index of $\mathcal{N} = 4$ $U(N)$ SYM from Fermi gas

To simplify some of the following expressions, let us introduce a separate notation for the Schur index of $\mathcal{N} = 4$ $U(N)$ SYM in (1.3) with the replacement $q \rightarrow q^{1/2}$

$$\widetilde{\Gamma}^{U(N)}(u; q) = \Gamma^{U(N)}(u; q^{\frac{1}{2}}). \tag{2.1}$$

The index may be computed by the following holonomy multiple contour integral representation (see for instance eq. (2.6) of [67])

$$\widetilde{\Gamma}^{U(N)}(u; q) = \frac{1}{N!} \frac{(q)_\infty^{2N}}{(q^{\frac{1}{2}} u^{\pm 1}; q)_\infty^N} \oint_{|z_i|=1} \prod_{i=1}^N \frac{dz_i}{2\pi i z_i} \frac{\prod_{i \neq j} (\frac{z_i}{z_j}; q)_\infty (q \frac{z_i}{z_j}; q)_\infty}{\prod_{i \neq j} (q^{\frac{1}{2}} u^{-1} \frac{z_i}{z_j}; q)_\infty (q^{\frac{1}{2}} u \frac{z_i}{z_j}; q)_\infty}. \tag{2.2}$$

The multiple integrals (2.2) have been computed in [48] by Fermi gas methods. To present their result, we define ξ , ζ , and τ by relations

$$\xi = u q^{-\frac{1}{2}} = e^{2\pi i \zeta}, \quad q = e^{2\pi i \tau}, \tag{2.3}$$

and introduce an auxiliary fugacity w with associated chemical potential ν

$$w = e^{2\pi i \nu}. \tag{2.4}$$

Then, the function of ξ and q defined as⁶

$$\bar{\Gamma}^{U(N)}(\xi; q) \equiv \widetilde{\Gamma}^{U(N)}(\xi q^{\frac{1}{2}}; q), \tag{2.5}$$

is given by

$$\bar{\Gamma}^{U(N)}(\xi; q) = \frac{(-1)^N \xi^{\frac{N^2}{2}} \vartheta(w, q)}{\vartheta(w \xi^{-N}, q)} \mathcal{Z}_N(w, \xi, q). \tag{2.6}$$

In this expression, the q -theta function ϑ is defined in (A.7) and the functions $\mathcal{Z}_N(w, \xi, q)$ are given by

$$\mathcal{Z}_N(w, \xi, q) = \sum_{\Lambda \vdash N} (-1)^{N-|\Lambda|} \prod_{i=1}^r \frac{1}{\lambda_i^{m_i} m_i!} Z_{\lambda_i}(w, \xi, q)^{m_i}, \tag{2.7}$$

⁶This is simply $\widetilde{\Gamma}^{U(N)}(u; q)$ with u given in terms of ξ in (2.3). For later use, it will be convenient to avoid the common shortcut of using the same name for a function expressed in different variables.

where as usual $\Lambda \vdash N$ means that Λ is a partition of N represented as

$$\Lambda = (\lambda_1^{m_1} \cdots \lambda_r^{m_r}), \quad \sum_{i=1}^r m_i \lambda_i = N, \quad \lambda_1 > \lambda_2 > \cdots > \lambda_r > 0, \quad |\Lambda| = \sum_{i=1}^r m_i, \quad (2.8)$$

i.e. Λ corresponds to a Young tableau with m_1 rows with λ_1 columns, etc.. The quantities Z_ℓ in (2.7) are computed explicitly by the relations

$$\begin{aligned} Z_1(w, \xi, q) &= P_1 \begin{bmatrix} \xi \\ 1 \end{bmatrix} (\nu, \tau), \\ Z_\ell(w, \xi, q) &= \frac{w^{-(\ell-1)}}{(\ell-1)!} \sum_{k=1}^{\ell-1} k! |s_{\ell-1,k}| P_{k+1} \begin{bmatrix} q^{\ell-1} \xi^\ell \\ 1 \end{bmatrix} (\nu, \tau) \quad (\ell \geq 2), \end{aligned} \quad (2.9)$$

where $s_{a,b}$ are Stirling number of the first kind and P_k are twisted Weierstrass functions defined in (A.16). It can be shown that the index expression (2.6) does not depend on the choice of the auxiliary fugacity w and special choices of w may simplify expressions.

Let us give some explicit examples, following [48], for the purpose of illustration and to give an idea of the typical expressions. In the simplest U(1) case, the index expressed in terms of ξ and q reads

$$\bar{\Gamma}^{\text{U}(1)}(\xi; q) = \tilde{\Gamma}^{\text{U}(1)}(\xi q^{\frac{1}{2}}; q) = -\frac{\xi^{\frac{1}{2}} \vartheta(w, q)}{\vartheta(w \xi^{-1}, q)} P_1 \begin{bmatrix} \xi \\ 1 \end{bmatrix} (\nu, \tau). \quad (2.10)$$

Using (A.16), we get the simple result

$$\bar{\Gamma}^{\text{U}(1)}(\xi; q) = \frac{(q)_\infty^2}{(\xi^{-1}, q)_\infty (q\xi, q)_\infty} = -\sqrt{\xi} \frac{(q)_\infty^3}{\vartheta(\xi^{-1}; q)}, \quad (2.11)$$

that, in particular, shows explicitly the independence on the auxiliary fugacity w . Going back to $(u; q)$ variables, this is

$$\tilde{\Gamma}^{\text{U}(1)}(u; q) = \frac{(q)_\infty^2}{(q^{\frac{1}{2}} u^{\pm 1}, q)_\infty}, \quad (2.12)$$

whose q -series is

$$\begin{aligned} \tilde{\Gamma}^{\text{U}(1)}(u; q) &= 1 + (u + u^{-1}) q^{\frac{1}{2}} + (u^2 - 1 + u^{-1}) q + (u^3 + u^{-3}) q^{\frac{3}{2}} \\ &\quad + (u^4 + u^{-4}) q^2 + (u^5 - u - u^{-1} + u^{-5}) q^{\frac{5}{2}} + \cdots, \end{aligned} \quad (2.13)$$

and of course agrees with the explicit evaluation of (2.2). For the U(2) theory, we have

$$\bar{\Gamma}^{\text{U}(2)}(\xi, q) = \frac{\xi^2 \vartheta(w; q)}{2 \vartheta(w \xi^{-2}; q)} \left[P_1 \begin{bmatrix} \xi \\ 1 \end{bmatrix}^2 (\nu, \tau) - \frac{1}{w} P_2 \begin{bmatrix} q \xi^2 \\ 1 \end{bmatrix} (\nu, \tau) \right] \quad (2.14)$$

To derive a q -expansion, it is convenient to fix the freedom in the choice of w by taking $w = \xi$. This gives the simpler result⁷

$$\bar{\Gamma}^{\text{U}(2)}(\xi, q) = \frac{\xi}{2} P_2 \begin{bmatrix} q \xi^2 \\ 1 \end{bmatrix} (\zeta, \tau) \quad (2.15)$$

⁷From (A.16) we have $P_1 \begin{bmatrix} \xi \\ 1 \end{bmatrix} (\zeta, \tau) = \frac{(q)_\infty^3 \vartheta(1, q)}{\vartheta(\xi^{-1}, q) \vartheta(\xi, q)} = 0$ because $\vartheta(1, q) \sim (1, q)_\infty = 0$.

We have the explicit expression of the twisted Weierstrass function

$$P_2 \begin{bmatrix} x \\ 1 \end{bmatrix} (\nu, \tau) = \frac{(q)_\infty^3 \vartheta(x^{-1}w; q)}{\vartheta(x^{-1}; q) \vartheta(w; q)} \left[w \frac{\vartheta'(w; q)}{\vartheta(w; q)} - x^{-1} w \frac{\vartheta'(x^{-1}w; q)}{\vartheta(x^{-1}w; q)} \right], \quad (2.16)$$

and thus we can write

$$\bar{\Gamma}^{U(2)}(\xi, q) = \frac{\xi}{2} \frac{(q)_\infty^3 \vartheta(\frac{1}{q\xi}; q)}{\vartheta(\frac{1}{q\xi^2}; q) \vartheta(\xi; q)} \left[\xi \frac{\vartheta'(\xi; q)}{\vartheta(\xi; q)} - \frac{1}{q\xi} \frac{\vartheta'(\frac{1}{q\xi}; q)}{\vartheta(\frac{1}{q\xi}; q)} \right] \quad (2.17)$$

Replacing $\xi = uq^{-\frac{1}{2}}$ and expanding in small q gives same as evaluation of (2.2) as one can check, i.e. one obtains the q -series

$$\begin{aligned} \tilde{\Gamma}^{U(2)}(u; q) &= 1 + (u + u^{-1})q^{\frac{1}{2}} + 2(u^2 + u^{-2})q + 2(u^3 + u^{-3})q^{\frac{3}{2}} \\ &\quad + 3(u^4 + u^{-4})q^2 + (3u^5 + u + u^{-1} + 3u^{-5})q^{\frac{5}{2}} + \dots \end{aligned} \quad (2.18)$$

3 Wrapped D3 brane indices from analytic continuation

The giant-graviton expansion of the $\mathcal{N} = 4$ $U(N)$ SYM Schur index is given by the representation (1.4) and requires the brane indices defined by the analytic continuation (1.6). We have, cf. (2.1),

$$I_n^{D3}(u; q) = \Gamma^{U(n)}(q^{-\frac{3}{2}}u^{-\frac{1}{2}}; q^{\frac{1}{2}}u^{-\frac{1}{2}}) = \tilde{\Gamma}^{U(n)}\left(q^{-\frac{3}{2}}u^{-\frac{1}{2}}; \frac{q}{u}\right), \quad (3.1)$$

In the right hand side the combination ξ is $1/q^2$, cf. (2.3), and therefore relation (1.6) amounts to the simple correspondence

$$I_n^{D3}(u; q) = \bar{\Gamma}^{U(n)}\left(\frac{1}{q^2}, \frac{q}{u}\right). \quad (3.2)$$

Let us apply systematically this transformation to the $U(n)$ indices discussed in the previous section.

Single wrapping. At leading order in the giant graviton expansion, we apply (3.2) to (2.11) and get immediately the expression

$$I_1^{D3}(u; q) = -\frac{1}{q} \frac{(\frac{q}{u})_\infty^3}{\vartheta(q^2; \frac{q}{u})} = -qu^2 \frac{(\frac{q}{u})_\infty^3}{\vartheta(u^2; \frac{q}{u})}, \quad (3.3)$$

where we used (A.9). Expanding in small q gives

$$I_1^{D3}(u; q) = \frac{u^3}{1-u^2}q + (1-u^2)q^2 + (u^{-3} - u^3)q^3 + \dots \quad (3.4)$$

in agreement with the result (1.7) in [35]. Eq. (3.3) represents its closed expression at any q . The above derivation is straightforward. Other more compact representations of the $U(N)$ index turn out to be less convenient for analytic continuation, but will play a role in later section 3.1.

Double wrapping. At wrapping 2, i.e. at subleading order in the giant graviton expansion, we apply (3.2) to (2.17) and get the expression

$$I_2^{\text{D}3}(u; q) = \frac{1}{2q^2} \frac{\left(\frac{q}{u}\right)_\infty^3 \vartheta(qu; \frac{q}{u})}{\vartheta(q^3u; \frac{q}{u})\vartheta\left(\frac{1}{q^2}; \frac{q}{u}\right)} \left[\frac{1}{q^2} \frac{\vartheta'\left(\frac{1}{q^2}; \frac{q}{u}\right)}{\vartheta\left(\frac{1}{q^2}; \frac{q}{u}\right)} - qu \frac{\vartheta'(qu; \frac{q}{u})}{\vartheta(qu; \frac{q}{u})} \right]. \quad (3.5)$$

We can use (from (A.8)) the relations

$$\vartheta(x; q) = -\vartheta(x^{-1}; q), \quad \frac{\vartheta'(x; q)}{\vartheta(x; q)} = -\frac{1}{x^2} \frac{\vartheta'(x^{-1}; q)}{\vartheta(x^{-1}; q)}, \quad (3.6)$$

to simplify it into

$$I_2^{\text{D}3}(u; q) = \frac{1}{2q} \frac{\left(\frac{q}{u}\right)_\infty^3 \vartheta(qu; \frac{q}{u})}{\vartheta(q^3u; \frac{q}{u})\vartheta(q^2; \frac{q}{u})} \left[q \frac{\vartheta'(q^2; \frac{q}{u})}{\vartheta(q^2; \frac{q}{u})} + u \frac{\vartheta'(qu; \frac{q}{u})}{\vartheta(qu; \frac{q}{u})} \right]. \quad (3.7)$$

We cannot naively expand the ratios $\vartheta'/\vartheta = \partial_x \log \vartheta$ inside bracket by first expanding ϑ when the first argument of the ϑ function has an explicit integer power of q . However, using systematically (A.9) and (A.11), we get bring (3.7) to the form

$$I_2^{\text{D}3}(u; q) = \frac{q^4 u^8}{2} \frac{\left(\frac{q}{u}\right)_\infty^3}{\vartheta(u^4; \frac{q}{u})} \left[2u^2 \frac{\vartheta'(u^2, \frac{q}{u})}{\vartheta(u^2, \frac{q}{u})} - 3 \right]. \quad (3.8)$$

Expanding this expression at small q gives the series

$$I_2^{\text{D}3}(u; q) = \frac{u^{10}(2-u^2)q^4}{(1-u^2)(1-u^4)} + u^5(2-u^4)q^5 + (2+2u^6-u^{12})q^6 + (2u^{-5}+3u^7-u^{15})q^7 \\ + (3+2u^{-10}+3u^8-u^{18})q^8 + (2u^{-15}+u^{-3}+4u^9-u^{21})q^9 + \dots, \quad (3.9)$$

which is again in agreement with (1.7) and extends it to all orders in q .

Higher wrapping. The same procedure can be repeated at higher wrapping without difficulty, because the steps we illustrated for the U(1) and U(2) cases are fully algorithmic and easily coded. To present the results in compact form it is convenient to define the quantities

$$Q = \left(\frac{q}{u}\right)_\infty, \quad R_{k,p} = R_k\left(u^p, \frac{q}{u}\right), \quad \Theta_p = \vartheta\left(u^p, \frac{q}{u}\right), \quad (3.10)$$

where the function $R_k(x; q)$ is defined in (A.10). At wrapping 3,4,5, we obtain the brane indices

$$I_3^{\text{D}3}(u; q) = -\frac{q^9 u^{14} Q^3}{6\Theta_6} [20u^4 + 2R_{1,-2}^2 - 8u^8 R_{1,4} + R_{1,-2}(8u^2 - 2u^6 R_{1,4}) - R_{2,-2} + u^{12} R_{2,4}], \quad (3.11)$$

$$I_4^{\text{D}3}(u; q) = \frac{1}{24} q^{16} Q^3 u^{26} \left[\frac{3Q^3 u^2 \Theta_2^2 (3u^2 + R_{1,-2} - u^4 R_{1,2})^2}{\Theta_{-2} \Theta_4^2 \Theta_6} + \frac{1}{\Theta_8} [-210u^6 - 6R_{1,-2}^3 \right. \\ \left. + 6u^2 R_{1,-2}^2 (-5 + u^6 R_{1,6}) + 15u^2 R_{2,-2} + R_{1,6}(90u^{12} - 3u^8 R_{2,-2}) - 15u^{18} R_{2,6} \right. \\ \left. + R_{1,-2}(-90u^4 + 30u^{10} R_{1,6} + 6R_{2,-2} - 3u^{16} R_{2,6}) - R_{3,-2} + u^{24} R_{3,6} \right], \quad (3.12)$$

$$I_5^{\text{D}3}(u; q) = \frac{1}{120} q^{25} Q^3 u^{42} \left[-\frac{10Q^3 u^2 \Theta_2}{\Theta_{-2} \Theta_6 \Theta_8} [(-3u^2 - R_{1,-2} + u^4 R_{1,2})(20u^4 + 2R_{1,-2}^2 - 8u^8 R_{1,4} \right. \\ \left. + R_{1,-2}(8u^2 - 2u^6 R_{1,4}) - R_{2,-2} + u^{12} R_{2,4})] + \frac{1}{\Theta_{10}} [-3024u^8 - 24R_{1,-2}^4 \right.$$

$$\begin{aligned}
 &+ 24u^2 R_{1,-2}^3 (-6 + u^8 R_{1,8}) + 252u^4 R_{2,-2} - 6R_{2,-2}^2 - 252u^{24} R_{2,8} + 6u^{20} R_{2,-2} R_{2,8} \\
 &- 12R_{1,-2}^2 (42u^4 - 12u^{12} R_{1,8} - 3R_{2,-2} + u^{20} R_{2,8}) - 24u^2 R_{3,-2} \\
 &+ 4u^{10} R_{1,8} (336u^6 - 18u^2 R_{2,-2} + R_{3,-2}) + 24u^{32} R_{3,8} \\
 &+ 4R_{1,-2} (-336u^6 + 6u^{10} R_{1,8} (21u^4 - R_{2,-2}) + 36u^2 R_{2,-2} - 18u^{22} R_{2,8} - 2R_{3,-2} + u^{30} R_{3,8}) \\
 &+ R_{4,-2} - u^{40} R_{4,8} \Big]. \tag{3.13}
 \end{aligned}$$

Although the expressions have increasing complexity, their expansion in powers of q is straightforward and one finds (we write only the first terms for brevity, but it is clear that the expansions can be lengthen at will)

$$\begin{aligned}
 I_3^{\text{D}3}(u; q) &= \frac{u^{21}(5 - 3u^2 - 3u^4 + 2u^6)}{(1 - u^2)(1 - u^4)(1 - u^6)} q^9 + \frac{u^{14}(5 - 3u^4 - 3u^6 + 2u^{10})}{1 - u^4} q^{10} \\
 &+ \frac{u^7(5 - 5u^4 + 2u^6 + 4u^8 - 2u^{14} - u^{16} - 2u^{18} + 2u^{22})}{1 - u^4} q^{11} + \dots, \tag{3.14}
 \end{aligned}$$

$$\begin{aligned}
 I_4^{\text{D}3}(u; q) &= -\frac{u^{36}(-14 + 9u^2 + 10u^4 + 2u^6 - 6u^8 - 7u^{10} + 5u^{12})}{(1 - u^2)(1 - u^4)(1 - u^6)(1 - u^8)} q^{16} \\
 &- \frac{u^{27}(-14 + 9u^4 + 9u^6 + 10u^8 - 6u^{10} - 7u^{12} - 7u^{14} + 5u^{18})}{(1 - u^4)(1 - u^6)} q^{17} \\
 &- \frac{u^{18}(-14 + 14u^2 + 14u^4 - 19u^6 + 2u^{12} + u^{14} + u^{16} + 2u^{18} + 3u^{24} - 5u^{26} - 5u^{28} + 5u^{30})}{(1 - u^2)(1 - u^4)} q^{18} + \dots, \tag{3.15}
 \end{aligned}$$

$$\begin{aligned}
 I_5^{\text{D}3}(u; q) &= \frac{u^{55}(42 - 28u^2 - 32u^4 - 9u^6 - 6u^8 + 45u^{10} + 3u^{12} + 7u^{14} - 16u^{16} - 19u^{18} + 14u^{20})}{(1 - u^2)(1 - u^4)(1 - u^6)(1 - u^8)(1 - u^{10})} q^{25} \\
 &+ \frac{u^{44}}{(1 - u^4)(1 - u^6)(1 - u^8)} (42 - 28u^4 - 28u^6 - 32u^8 - 9u^{10} + 15u^{12} + 45u^{14} + 29u^{16} + 7u^{18} \\
 &- 16u^{20} - 19u^{22} - 19u^{24} + 14u^{28}) q^{26} + \frac{u^{33}}{(1 - u^4)(1 - u^6)(1 - u^8)} (42 - 42u^4 - 28u^6 - 28u^8 \\
 &+ 42u^{10} + 52u^{12} + 15u^{14} - 13u^{16} - 38u^{18} - 8u^{20} - 9u^{22} - 3u^{24} + 2u^{26} + 3u^{28} - 11u^{30} \\
 &+ 10u^{32} + 29u^{34} + 12u^{36} + 14u^{38} - 19u^{40} - 19u^{42} - 14u^{44} + 14u^{48}) q^{27} + \dots. \tag{3.16}
 \end{aligned}$$

Continuing this way, the general structure of $I_n^{\text{D}3}(u; q)$ turns out to be

$$I_n^{\text{D}3}(u; q) = \frac{u^{n(2n+1)} A_n(u)}{\prod_{k=1}^n (1 - u^{2k})} q^{n^2} + \frac{B_n(u; q)}{(1 - u^2)^{n-2}} q^{n^2+1}, \tag{3.17}$$

where $A_n(u)$ are polynomial in u with degree $n(n - 1)$ and $B_n(u; q)$ has a q -series with coefficients that are rational functions of u , smooth for $u \rightarrow 1$. The first cases of the polynomials $A_n(u)$ are

$$\begin{aligned}
 A_1(u) &= 1, & A_2(u) &= 2 - u^2, & A_3(u) &= 5 - 3u^2 - 3u^4 + 2u^6, \\
 A_4(u) &= 14 - 9u^2 - 10u^4 - 2u^6 + 6u^8 + 7u^{10} - 5u^{12}, \\
 A_5(u) &= 42 - 28u^2 - 32u^4 - 9u^6 - 6u^8 + 45u^{10} + 3u^{12} + 7u^{14} - 16u^{16} - 19u^{18} + 14u^{20}, \\
 A_6(u) &= 132 - 90u^2 - 104u^4 - 31u^6 - 29u^8 + 71u^{10} + 80u^{12} + 99u^{14} - 41u^{16} - 52u^{18} \\
 &\quad - 60u^{20} - 15u^{22} - 20u^{24} + 47u^{26} + 56u^{28} - 42u^{30}, \\
 A_7(u) &= 429 - 297u^2 - 345u^4 - 104u^6 - 104u^8 + 210u^{10} + 12u^{12} + 582u^{14} + 102u^{16} - 103u^{18} \\
 &\quad - 188u^{20} - 201u^{22} - 309u^{24} - 83u^{26} + 328u^{28} - 50u^{30} + 203u^{32} + 47u^{34} + 60u^{36} - 146u^{38} \\
 &\quad - 174u^{40} + 132u^{42}, \tag{3.18}
 \end{aligned}$$

$$\begin{aligned}
 A_8(u) = & 1430 - 1001u^2 - 1166u^4 - 351u^6 - 359u^8 + 695u^{10} - 64u^{12} + 1149u^{14} + 1177u^{16} \\
 & + 497u^{18} - 427u^{20} - 524u^{22} - 1715u^{24} - 344u^{26} - 318u^{28} - 507u^{30} + 1076u^{32} + 682u^{34} \\
 & + 826u^{36} + 304u^{38} - 398u^{40} - 361u^{42} + 86u^{44} - 652u^{46} - 149u^{48} - 188u^{50} \\
 & + 471u^{52} + 561u^{54} - 429u^{56},
 \end{aligned}$$

and one can check that in all cases $A_n(1) = 1$ and coefficients are integer. An explicit formula for them is illustrated in next section, see in particular formula (3.39). The functions $B_n(u; q)$ are non-trivial, even at a fixed value of u , having a full expansion in powers of q , and will be discussed later on.

3.1 Other representations of the $U(N)$ index and polynomials $A_N(u)$

We now show how to get an explicit formula for the polynomials $A_N(u)$ in (3.17). We start from an alternative compact representation of the indices $\bar{I}^{U(N)}(\xi; q)$ derived in [48]

$$\bar{I}^{U(N)}(\xi; q) = - \sum_{\substack{p_1, \dots, p_N \in \mathbb{Z} \\ p_1 < \dots < p_N}} \xi^{\frac{N^2}{2}} \prod_{n=1}^N \frac{\xi^{-p_n}}{1 - \xi^{\frac{N}{2}} q^{p_n}}. \tag{3.19}$$

In the simplest $U(1)$ case, it is

$$\bar{I}^{U(1)}(\xi; q) = - \sum_{p \in \mathbb{Z}} \frac{\xi^{-p+\frac{1}{2}}}{1 - \xi^{\frac{1}{2}} q^p}, \tag{3.20}$$

and the associated brane index is obtained from relation (3.2). Applying it to the summand, we get

$$I_1^{D3}(u; q) = - \sum_{p \in \mathbb{Z}} \frac{q^{2p-1}}{1 - u^{-p} q^{p-1}}. \tag{3.21}$$

We now split the sum with positive and negative indices and write

$$I_1^{D3}(u; q) = - \sum_{p=0}^{\infty} \frac{q^{2p-1}}{1 - u^{-p} q^{p-1}} - \sum_{p=1}^{\infty} \frac{q^{-2p-1}}{1 - u^p q^{-p-1}} = - \sum_{p=0}^{\infty} \frac{q^{2p-1}}{1 - u^{-p} q^{p-1}} + \sum_{p=1}^{\infty} \frac{u^{-p} q^{-p}}{1 - u^{-p} q^{p+1}}. \tag{3.22}$$

In the last term, we expand the denominator and exchange the order of the two sums

$$\sum_{p=1}^{\infty} \frac{u^{-p} q^{-p}}{1 - u^{-p} q^{p+1}} = \sum_{p=1}^{\infty} \sum_{n=0}^{\infty} u^{-p} q^{-p} u^{-pn} q^{n(p+1)} = - \sum_{n=0}^{\infty} \frac{q^{2n-1}}{q^{n-1} - u^{n+1}}. \tag{3.23}$$

Thus, we can write

$$I_1^{D3}(u; q) = \sum_{p=0}^{\infty} \frac{1}{u^2} \left(\frac{q}{u} \right)^{2p-1} \frac{u^p(1 - u^{p+1}) - q^{p-1}(1 - u^p)}{(1 - u^{-p} q^{p-1})(1 - u^{-p-1} q^{p-1})} \tag{3.24}$$

This expression can be expanded at small q with the result

$$F(u; q) = \frac{u^3 q}{1 - u^2} + (1 - u^2)q^2 + (u^{-3} - u^3)q^3 + (1 + u^{-6} - u^{-2} - u^4)q^4 + (u^{-9} - u^5)q^5 + \dots, \tag{3.25}$$

in agreement with (3.4). Extending this procedure to higher N by the same methods provides a way to compute the leading term $\sim q^{n^2}$ in $I_n^{\text{D}^3}(u; q)$, i.e. the polynomials $A_n(u)$ in (3.18). To show this, we begin with the algebraic identity, following from Polya theory [68], (cf. (2.8) for notation of partitions)

$$\sum_{0 \leq i_1 < \dots < i_k} x^{i_1 + \dots + i_k} = \frac{x^{k(k-1)/2}}{(1-x)(1-x^2) \dots (1-x^k)} = \sum_{\Lambda \vdash k} (-1)^{k+|\Lambda|} \prod_{i=1}^p \frac{1}{\lambda_i^{m_i} (m_i!)} \frac{1}{(1-x^{\lambda_i})^{m_i}}. \tag{3.26}$$

Eq. (3.26) determines the coefficients in the decomposition of the total trace of a symmetric tensor with summation restricted to ordered indices in terms of unrestricted traces. For a symmetric tensor T_{i_1, \dots, i_k} we have from (3.26)

$$\sum_{i_1 < \dots < i_k} T_{i_1, \dots, i_k} = \sum_{\Lambda \vdash k} \frac{(-1)^{k+|\Lambda|}}{\prod_{q=1}^p \lambda_q^{m_q} (m_q!)} \sum_{j_1, \dots, j_{|\Lambda|}} T_J, \tag{3.27}$$

where J is a multi-index with, for all p , m_p groups of the same index repeated λ_p times. The number of free indices is $|\Lambda|$. For instance, for a symmetric tensor with rank 3 or 4, the expansion (3.27) gives the general identities

$$\sum_{i < j < k} T_{ijk} = \frac{1}{6} \sum_{i,j,k} T_{ijk} - \frac{1}{2} \sum_{i,j} T_{ijj} + \frac{1}{3} \sum_i T_{iii}, \tag{3.28}$$

$$\sum_{i < j < k < \ell} T_{ijkl} = \frac{1}{24} \sum_{i,j,k,\ell} T_{ijkl} - \frac{1}{4} \sum_{i,j,k} T_{ijkk} + \frac{1}{8} \sum_{i,j} T_{iijj} + \frac{1}{3} \sum_{i,j} T_{ijjj} - \frac{1}{4} \sum_i T_{iiii}. \tag{3.29}$$

Considering now the symmetric rank N tensor in (3.19)

$$T_{p_1, \dots, p_N} = - \prod_{n=1}^N \frac{\xi^{-p_n + \frac{N}{2}}}{1 - \xi^{\frac{N}{2}} q^{p_n}}, \tag{3.30}$$

we obtain from (3.27)

$$I_N^{\text{D}^3}(u; q) = \sum_{\lambda \vdash N} (-1)^{N+|\lambda|+1} \prod_{q=1}^p \frac{1}{\lambda_q^{m_q} (m_q!)} S_{\lambda_q}^{(N)}(u; q)^{m_q}. \tag{3.31}$$

The functions $S_r^{(N)}(u; q)$ are obtained by applying the procedure we adopted to treat the $U(1)$ case to the single index sums. Their explicit expression is

$$S_r^{(N)}(u; q) = \sum_{n=0}^{\infty} \left[\left(\frac{q^{2n-N}}{1 - u^{-n} q^{n-N}} \right)^r + (-1)^r \binom{n}{r-1} \frac{u^{-n-1} q^{(n+1)(N+1) - r(N+2)}}{1 - u^{-n-1} q^{n-2r+1}} \right], \tag{3.32}$$

where we used

$$\frac{1}{(1-x)^r} = \sum_{n=0}^{\infty} \binom{n}{r-1} x^{n-r+1}. \tag{3.33}$$

To give an example, for $N = 3$, we have

$$I_3^{\text{D}^3}(u; q) = -\frac{1}{6} (S_1^{(3)})^3 + \frac{1}{2} S_1^{(3)} S_2^{(3)} - \frac{1}{3} S_3^{(3)}, \tag{3.34}$$

with

$$\begin{aligned}
 S_1^{(3)}(u; q) &= \frac{u^8 q^3}{1+u-u^3-u^4} + (u^4-u^6)q^4 + (1-u^3+u^5-u^8)q^5 + \left(-1+\frac{1}{u^4}+u^6-u^{10}\right)q^6 \\
 &+ \left(1+\frac{1}{u^8}-\frac{1}{u^3}-u^4+u^7-u^{12}\right)q^7 + \left(\frac{1}{u^{12}}-\frac{1}{u^6}+u^8-u^{14}\right)q^8 \\
 &+ \left(\frac{1}{u^{16}}-\frac{1}{u^9}+\frac{1}{u^5}-u^5+u^9-u^{16}\right)q^9 + \dots, \tag{3.35}
 \end{aligned}$$

$$\begin{aligned}
 S_2^{(3)}(u; q) &= \frac{u^{15}(4+u+u^2-2u^3)q^6}{(-1+u^3)^2(1+u+u^2+u^3)} - 2(u^{10}(-2+u^2))q^7 - u^5(-4+u^3-5u^7+2u^{10})q^8 \\
 &+ (4+6u^{14}-2u^{18})q^9 + \dots, \tag{3.36}
 \end{aligned}$$

$$S_3^{(3)}(u; q) = \frac{u^{21}(15-19u^3+6u^6)q^9}{(-1+u^3)^3(1+u^3)} + \dots, \tag{3.37}$$

and from (3.34), we get

$$I_3^{D3}(u; q) = \frac{u^{21}(5-3u^2-3u^4+2u^6)}{(1-u^2)(1-u^4)(1-u^6)} q^9 + \dots, \tag{3.38}$$

in agreement with (3.14). The fact that $I_N^{D3}(u; q)$ starts at order q^{N^2} is consistent with $S_r^{(N)}(u; q) = C_r^{(N)}(u) q^{Nr} + \mathcal{O}(q^{Nr+1})$ and therefore $A_N(u)$ in (3.18) can be read from (3.31) that gives

$$\begin{aligned}
 A_N(u) &= u^{-N(2N+1)} \prod_{k=1}^N (1-u^{2k}) \sum_{\lambda \vdash N} (-1)^{N+|\lambda|+1} \prod_{q=1}^p \frac{1}{\lambda_q^{m_q} (m_q!)} [C_{\lambda_q}^{(N)}(u)]^{m_q}, \tag{3.39} \\
 C_r^{(N)}(u) &= (-1)^r u^{rN} \left[(1-u^N)^{-r} - \binom{2r-1}{r-1} \frac{u^{rN}}{1-u^{2r}} \right] - (-1)^r u^{rN} \sum_{n=0}^{2r-2} \binom{n}{r-1} u^{nN}.
 \end{aligned}$$

where the finite sum representation of $C_r^{(N)}(u)$ may be find after some manipulation of (3.32). Again, let us give an example by considering the case $N = 4$. From the formula in second line of (3.39), we get

$$\begin{aligned}
 C_1^{(4)}(u) &= \frac{u^{10}}{1-u^4}, & C_2^{(4)}(u) &= \frac{u^{20}}{(1-u^4)^2}, \tag{3.40} \\
 C_3^{(4)}(u) &= -\frac{u^{28}(15+5u^2-19u^4-4u^6+6u^8)}{(1+u^2)(1-u^4)^2(1-u^6)}, & C_4^{(4)}(u) &= -\frac{u^{36}(-56+119u^4-85u^8+20u^{12})}{(1-u^4)^3(1-u^8)},
 \end{aligned}$$

and the first line of (3.39) gives

$$A_4(u) = 14 - 9u^2 - 10u^4 - 2u^6 + 6u^8 + 7u^{10} - 5u^{12}, \tag{3.41}$$

in agreement with (3.18). The representation (3.39) is quite efficient and for instance we can get $A_N(u)$ for $N = 20$ in a few seconds.

4 Giant-graviton expansion of the index in unflavored limit

The giant graviton expansion of the Schur index in (1.4) and can be regrouped as⁸

$$\frac{\mathbb{I}^{\text{U}(N)}(u; q)}{\mathbb{I}^{\text{KK}}(u; q)} = \sum_{n=0}^{\infty} q^{nN} \widehat{\mathbb{I}}_n^{\text{D}3}(u; q), \quad (4.1)$$

$$\widehat{\mathbb{I}}_n^{\text{D}3}(u; q) = \sum_{p=0}^n q^{2(n-p)p} u^{(n-2p)N} \mathbb{I}_{n-p}^{\text{D}3}(u; q) \mathbb{I}_p^{\text{D}3}(u^{-1}; q). \quad (4.2)$$

The exact result in the unflavored limit $u \rightarrow 1$ was derived in [46] and already given in (1.8). Comparing it with (4.1) gives the prediction

$$\lim_{u \rightarrow 1} \widehat{\mathbb{I}}_n^{\text{D}3}(u; q) = (-1)^n \frac{(2n+N)(N+n-1)!}{n! N!} q^{n^2}, \quad (4.3)$$

i.e. the following polynomials in N times q^{n^2}

$$\begin{aligned} \widehat{\mathbb{I}}_1^{\text{D}3}(1; q) &= -(N+2)q, & \widehat{\mathbb{I}}_2^{\text{D}3}(1; q) &= \frac{1}{2}(1+N)(4+N)q^4, \\ \widehat{\mathbb{I}}_3^{\text{D}3}(1; q) &= -\frac{1}{6}(1+N)(2+N)(6+N)q^9, \\ \widehat{\mathbb{I}}_4^{\text{D}3}(1; q) &= \frac{1}{24}(1+N)(2+N)(3+N)(8+N)q^{16}, \dots \end{aligned} \quad (4.4)$$

As discussed in [29, 30], the N dependence of the giant graviton corrections is a familiar fact working in unrefined limits and comes from the zero modes of fluctuations of fields in the theory on the brane world-volume. In particular, the N term in the leading single wrapping contribution is associated with a total of two zero modes from both scalar fields and fermion fields fluctuations in a $\mathcal{N} = 4$ Maxwell multiplet [30].

4.1 Differential constraints

It is non-trivial to match the brane index prediction (4.3) starting from its structure (3.17). To see this in full details, let us begin with the case $n = 1$, i.e. at leading wrapping order. From (4.2) we get

$$\widehat{\mathbb{I}}_1^{\text{D}3}(u; q) = -[N(A_1(1) + A_1'(1)) + 2A_1(1)]q + \mathcal{O}(u-1). \quad (4.5)$$

This is in agreement with (4.4) if

$$A_1(1) = 1, \quad A_1'(1) = 0. \quad (4.6)$$

These conditions are trivially true since $A_1(u) = 1$, cf. (3.18). The next case is $n = 2$. Using (4.6), we get now

$$\begin{aligned} \widehat{\mathbb{I}}_2^{\text{D}3}(u; q) &= -\frac{q^4}{4}(1 - A_2(1)) \left[\frac{1}{(u-1)^2} + \frac{1}{u-1} \right] \\ &\quad - \frac{q^4}{16} \left[-1 - 95A_2(1) + 32qB_1(1, q) - 32qB_2(1, q) - 30A_2'(1) + 4A_1''(1) - 2A_2''(1) \right. \\ &\quad \left. - 8(7A_2(1) + A_2'(1))N - 8A_2(1)N^2 \right] + \mathcal{O}(u-1). \end{aligned} \quad (4.7)$$

⁸Notice that in the l.h.s. we have \mathbb{I} and not $\widetilde{\mathbb{I}}$.

Cancellation of the poles at $u = 1$ requires $A_2(1) = 1$ that holds, as we mentioned. We can use the known values of polynomials $A_k(u)$ and their derivatives at $u = 1$. In fact, as we have seen, the list of polynomials in (3.18) can be computed by (3.39) in a finite number of steps when focusing on a definite wrapping order. This simplifies (4.7) to

$$\widehat{\mathbb{I}}_2^{\text{D}^3}(u; q) = \frac{1}{2}q^4 \left[(1 + N)(4 + N) - 4q(B_1(1, q) - B_2(1, q)) \right] + \mathcal{O}(u - 1). \quad (4.8)$$

Comparing with (4.4) and denoting $B_k(q) \equiv B_k(1; q)$, we need the non-trivial condition

$$B_1(q) = B_2(q). \quad (4.9)$$

Similarly, considering higher wrapping contributions, we find additional differential conditions involving the $B_k(q)$ functions. Again, these constraints come from requiring cancellation of poles at $u = 1$ and matching powers of N in the finite part. At wrapping 3, we get two conditions (we omit the q argument)

$$\begin{aligned} B_1 + 2B_2 - 6B_3 &= 0, \\ 4B_1 - 4B_2 + 2B_3 - B_1^{(1)} + 2B_2^{(1)} - 2B_3^{(1)} &= 0, \end{aligned} \quad (4.10)$$

where a notation has been introduced for the partial derivatives with respect to u evaluated at the point $u = 1$ ⁹

$$B_k^{(p)}(q) = \partial_u^p B_k(u; q)|_{u=1}, \quad (4.11)$$

not to be confused with derivatives of $B_k(q)$ with respect to q . At wrapping 4, we get the four conditions

$$\begin{aligned} -B_1 + 3B_2 - 6B_3 + 6B_4 &= 0, \\ -B_1 - 6B_3 + 24B_4 &= 0, \\ -8B_1 + 18B_3 - 24B_4 + B_1^{(1)} - 6B_3^{(1)} + 12B_4^{(1)} &= 0, \\ -215B_1 + 279B_2 + 144qB_2^2 - 306B_3 - 288qB_1B_3 + 126B_4 + 90B_1^{(1)} - 162B_2^{(1)} \\ + 180B_3^{(1)} - 108B_4^{(1)} - 6B_1^{(2)} + 18B_2^{(2)} - 36B_3^{(2)} + 36B_4^{(2)} &= 0. \end{aligned} \quad (4.12)$$

Finally, at wrapping 5, there are six new constraints

$$\begin{aligned} 3B_1 - 4B_2 - 12B_3 + 72B_4 - 120B_5 &= 0, \\ 27B_1 - 4B_2 - 12B_3 + 648B_4 - 3000B_5 &= 0, \\ 13B_1 - 36B_2 + 72B_3 - 96B_4 + 72B_5 - B_1^{(1)} + 4B_2^{(1)} - 12B_3^{(1)} + 24B_4^{(1)} - 24B_5^{(1)} &= 0, \\ 117B_1 - 36B_2 + 72B_3 - 864B_4 + 1800B_5 - 9B_1^{(1)} + 4B_2^{(1)} - 12B_3^{(1)} + 216B_4^{(1)} - 600B_5^{(1)} &= 0, \\ 669B_1 - 420B_2 - 628B_3 - 384qB_2B_3 + 2184B_4 + 1152qB_1B_4 - 1960B_5 - 150B_1^{(1)} + 136B_2^{(1)} \\ + 264B_3^{(1)} - 1008B_4^{(1)} + 1200B_5^{(1)} + 6B_1^{(2)} - 8B_2^{(2)} - 24B_3^{(2)} + 144B_4^{(2)} - 240B_5^{(2)} &= 0, \\ 485B_1 + 324B_2 - 216B_3 + 1152qB_2B_3 - 2592B_4 - 3456qB_1B_4 + 936B_5 - 593B_1^{(1)} \\ - 1152qB_4B_1^{(1)} + 1052B_2^{(1)} + 1152qB_3B_2^{(1)} - 1476B_3^{(1)} - 1152qB_2B_3^{(1)} + 1656B_4^{(1)} \\ + 1152qB_1B_4^{(1)} - 792B_5^{(1)} + 72B_1^{(2)} - 192B_2^{(2)} + 360B_3^{(2)} - 432B_4^{(2)} + 288B_5^{(2)} \\ - 2B_1^{(3)} + 8B_2^{(3)} - 24B_3^{(3)} + 48B_4^{(3)} - 48B_5^{(3)} &= 0. \end{aligned} \quad (4.13)$$

⁹We remark that $B^{(1)}(q) \neq B'(q)$.

The same pattern continues at higher wrapping larger than 5. We will stop at this order for the sake of presentation.

4.2 Quasimodular expansions and solution of the constraints

To prove that the differential constraints (4.9), (4.10), (4.12), (4.13) are satisfied at all orders in q we need explicit closed expressions for the functions $B_n^{(p)}(q)$. Let us begin with $n = 1$. The brane index was given in (3.3) and we can expose its singularity at $u \rightarrow 1$ by writing

$$\begin{aligned} I_1^{\text{D}3}(u; q) &= qu^3 \frac{\left(\frac{q}{u}\right)_\infty^2}{\left(\frac{q}{u^3}; \frac{q}{u}\right)_\infty (u^2; \frac{q}{u})_\infty} = qu^3 \prod_{k=0}^{\infty} \frac{[1 - (\frac{q}{u})^{k+1}]^2}{\left(1 - \frac{q^{k+1}}{u^{k+3}}\right)\left(1 - \frac{q^k}{u^{k-2}}\right)} \\ &= \frac{u^3}{1 - u^2} q \prod_{k=0}^{\infty} \frac{[1 - (\frac{q}{u})^{k+1}]^2}{\left(1 - \frac{q^{k+1}}{u^{k+3}}\right)\left(1 - \frac{q^{k+1}}{u^{k-1}}\right)}. \end{aligned} \tag{4.14}$$

We now expand the logarithm of the second factor around $u = 1$ obtaining

$$\log \prod_{k=1}^{\infty} \frac{(1 - \frac{q^k}{u^k})^2}{(1 - \frac{q^k}{u^{k+2}})(1 - \frac{q^k}{u^{k-2}})} = S_2(q) (u - 1)^2 + S_3(q) (u - 1)^3 + S_4(q) (u - 1)^4 + \dots, \tag{4.15}$$

with the following infinite sums

$$S_2(q) = \sum_{k=1}^{\infty} \frac{4q^k}{(1 - q^k)^2}, \tag{4.16}$$

$$S_3(q) = - \sum_{k=1}^{\infty} \frac{4(1 + k + (k - 1)q^k)q^k}{(1 - q^k)^3}, \tag{4.17}$$

$$S_4(q) = \sum_{k=1}^{\infty} \frac{[5 + 6k + 2k^2 - 2(1 - 4k^2)q^k + (5 - 6k + 2k^2)q^{2k}]q^k}{(1 - q^k)^4}. \tag{4.18}$$

These sums can be expressed in terms of the Eisenstein series, cf. (A.12), as illustrated in details in appendix B.1. One obtains ($\mathbb{E}_{2n} \equiv \mathbb{E}_{2n}(q)$)

$$S_2(q) = \frac{1}{6}(1 - \mathbb{E}_2), \tag{4.19}$$

$$S_3(q) = -\frac{1}{6}(1 - \mathbb{E}_2) + \frac{1}{72}(\mathbb{E}_2^2 - \mathbb{E}_4), \tag{4.20}$$

$$S_4(q) = \frac{53}{360} - \frac{11}{72}\mathbb{E}_2 - \frac{1}{48}\mathbb{E}_2^2 + \frac{19}{720}\mathbb{E}_4 - \frac{1}{864}\mathbb{E}_2^3 + \frac{1}{288}\mathbb{E}_2\mathbb{E}_4 - \frac{1}{432}\mathbb{E}_6, \tag{4.21}$$

and so on. At each order we need to include a generic combination of Eisenstein series with increased total degree. In other words, the sums $S_n(q)$ are quasimodular forms, i.e. polynomials in the Eisenstein series $\mathbb{E}_2, \mathbb{E}_4, \mathbb{E}_6$. Exponentiating (4.15) and replacing into (4.14)

gives the functions $B_1^{(p)}(q)$ that have the explicit quasimodular expressions

$$\begin{aligned}
 q B_1(q) &= \frac{1}{24}(1 - \mathbb{E}_2), \\
 q B_1^{(1)}(q) &= \frac{1}{24}(1 - \mathbb{E}_2) + \frac{1}{288}(\mathbb{E}_2^2 - \mathbb{E}_4), \\
 q B_1^{(2)}(q) &= -\frac{17}{720} + \frac{1}{72}\mathbb{E}_2 + \frac{1}{96}\mathbb{E}_2^2 - \frac{1}{1440}\mathbb{E}_4 - \frac{1}{1728}\mathbb{E}_2^3 + \frac{1}{576}\mathbb{E}_2\mathbb{E}_4 - \frac{1}{864}\mathbb{E}_6, \\
 q B_1^{(3)}(q) &= -\frac{175}{512} + \frac{3229}{8064}\mathbb{E}_2 - \frac{251}{2304}\mathbb{E}_2^2 + \frac{55}{1152}\mathbb{E}_4 + \frac{23}{3456}\mathbb{E}_2^3 - \frac{11}{960}\mathbb{E}_2\mathbb{E}_4 + \frac{1}{135}\mathbb{E}_6 \\
 &\quad - \frac{1}{4608}\mathbb{E}_2^4 + \frac{1}{2016}\mathbb{E}_2\mathbb{E}_6 - \frac{5}{16128}\mathbb{E}_4^2.
 \end{aligned} \tag{4.22}$$

We remark that the expansion of $B_1^{(p)}$ for any p can be obtained systematically from a differential equation discussed in appendix B.2. This proves that the same structure is found at each p . Still, for practical evaluation, it is more efficient to assume a quasimodular Ansatz and fix coefficients from the first terms of its q -series.

For the functions $B_2^{(p)}(q)$, $B_3^{(p)}(q)$, $B_4^{(p)}(q)$, and $B_5^{(p)}(q)$, we find similar expansions with different coefficients collected in appendix B.3. In particular, we see that all functions B_k are proportional to B_1 according to

$$B_2 = B_1, \quad B_3 = \frac{1}{2!}B_1, \quad B_4 = \frac{1}{3!}B_1, \quad B_5 = \frac{1}{4!}B_1, \quad \dots \tag{4.23}$$

Using the expressions in appendix B.3, the differential constraints (4.9)–(4.13) are readily checked to be satisfied. This proves that the unflavored index is reproduced in the limit $u \rightarrow 1$ at all orders in q , up to wrapping 5.

5 Finite N corrections in near-unflavored regime

We can go easily beyond the unflavored limit and give the exact q dependence of the giant graviton expansion coefficients in the near unflavored regime, i.e. at first non-trivial order in $u - 1$. From the expressions in appendix B.3, the expansion of $\widehat{\Gamma}_n^{\text{D}3}(u; q)$ around $u = 1$ gives the following corrections to the unflavored brane indices in (4.4)

$$\begin{aligned}
 \widehat{\Gamma}_1^{\text{D}3}(u; q) &= q \left[-(2 + N) + \Delta_1^{(2)}(q; N) (u - 1)^2 + \mathcal{O}((u - 1)^3) \right], \\
 \widehat{\Gamma}_2^{\text{D}3}(u; q) &= q^4 \left[\frac{1}{2}(1 + N)(4 + N) + \Delta_2^{(2)}(q; N) (u - 1)^2 + \mathcal{O}((u - 1)^3) \right], \\
 \widehat{\Gamma}_3^{\text{D}3}(u; q) &= q^9 \left[-\frac{1}{6}(1 + N)(2 + N)(6 + N) + \Delta_3^{(2)}(q; N)(u - 1)^2 + \mathcal{O}((u - 1)^3) \right],
 \end{aligned} \tag{5.1}$$

where $\Delta_n^{(2)}(q)$ admit the exact quasimodular representations

$$\Delta_1^{(2)}(q; N) = \frac{1}{72} \left[-12(2 + N)((2 + N)^2 - \mathbb{E}_2) - \mathbb{E}_2^2 + \mathbb{E}_4 \right], \tag{5.2}$$

$$\Delta_2^{(2)}(q; N) = \frac{1}{72} \left[12(-2 + N)(4 + N)((4 + N)^2 - \mathbb{E}_2) + N(\mathbb{E}_2^2 - \mathbb{E}_4) \right], \tag{5.3}$$

$$\Delta_3^{(2)}(q; N) = \frac{1}{144} \left[-12(6 - 3N + N^2)(6 + N)((6 + N)^2 - \mathbb{E}_2) - (2 + N + N^2)(\mathbb{E}_2^2 - \mathbb{E}_4) \right]. \tag{5.4}$$

Just to give an example, we consider the $U(2)$ index and denote $\delta u = u - 1$. We have

$$\frac{I^{U(2)}(u; q)}{I^{KK}(u; q)} = 1 - 4q^3 + 9q^8 - 16q^{15} + \dots + (-10q^3 - 12q^4 - 24q^5 - 16q^6 + 24q^8) \quad (5.5)$$

$$+ 88q^9 + 48q^{10} + 144q^{11} + 36q^{12} + 192q^{13} - 240q^{14} + 280q^{15} - 504q^{16} + \dots) \delta u^2 + \dots,$$

and, setting $N = 2$ in (5.1) and (5.2),

$$q^2 \widehat{I}_1^{D3}(u; q) = -4q^3 + (-10q^3 - 12q^4 - 24q^5 - 16q^6 + 24q^8 + 96q^9 + 96q^{10}$$

$$+ 240q^{11} + 260q^{12} + 432q^{13} + 336q^{14} + 896q^{15} + 504q^{16} + \dots) \delta u^2 + \dots,$$

$$q^{2 \times 2} \widehat{I}_2^{D3}(u; q) = 9q^8 + (-8q^9 - 48q^{10} - 96q^{11} - 224q^{12} - 240q^{13} - 576q^{14}$$

$$- 448q^{15} - 960q^{16} + \dots) \delta u^2 + \dots,$$

$$q^{3 \times 2} \widehat{I}_2^{D3}(u; q) = -16q^{15} + (-168q^{15} - 48q^{16} + \dots) \delta u^2 + \dots.$$
(5.6)

One can check that the three terms $1 + q^2 \widehat{I}_1^{D3} + q^4 \widehat{I}_2^{D3} + q^6 \widehat{I}_3^{D3}$ are enough to reproduce all terms in the r.h.s. of (5.5).

Similar expressions may be computed at higher order in $u - 1$. This requires some additional computational effort, but no conceptual difficulty. For instance, for the leading wrapping correction $n = 1$ we can write

$$\widehat{I}_1^{D3}(u; q) = q \left[-(2 + N) + \sum_{p=2}^{\infty} \Delta_1^{(p)}(q; N) (u - 1)^p \right], \quad (5.7)$$

where the $p = 2$ term was in the first line of (5.2) and the next ones are

$$\Delta_1^{(3)}(q; N) = -\Delta_1^{(2)}(q), \quad (5.8)$$

$$\Delta_1^{(4)}(q; N) = -\frac{1}{725760} \left[-63(14527 + 28800N + 18240N^2 + 5600N^3 + 960N^4 + 96N^5) \right.$$

$$- 210(1 + 144N + 24N^2) \mathbb{E}_2^2 + 140(-11 + 6N) \mathbb{E}_2^3 + 105 \mathbb{E}_2^4$$

$$+ 84(19 + 192N + 60N^2) \mathbb{E}_4 + 150 \mathbb{E}_4^2 + \mathbb{E}_2 \left(60(3155 + 5880N + 2016N^2 + 336N^3) \right.$$

$$\left. - 504(-1 + 5N) \mathbb{E}_4 - 240 \mathbb{E}_6 \right) + 112(-2 + 15N) \mathbb{E}_6 \left. \right], \quad (5.9)$$

$$\Delta_1^{(5)}(q; N) = \frac{1}{362880} \left[63(6847 + 17280N + 12480N^2 + 4640N^3 + 960N^4 + 96N^5) \right.$$

$$+ 210(-23 + 144N + 24N^2) \mathbb{E}_2^2 - 140(-11 + 6N) \mathbb{E}_2^3 - 105 \mathbb{E}_2^4$$

$$- 84(-41 + 192N + 60N^2) \mathbb{E}_4 - 150 \mathbb{E}_4^2 + \mathbb{E}_2 \left(-60(1139 + 4872N + 2016N^2 + 336N^3) \right.$$

$$\left. + 504(-1 + 5N) \mathbb{E}_4 + 240 \mathbb{E}_6 \right) - 112(-2 + 15N) \mathbb{E}_6 \left. \right]. \quad (5.10)$$

6 Giant graviton expansion of $\mathcal{N} = 2 \widehat{\Gamma}(SU(N))$ theories

As a further application of our analysis, we consider in this section the 4d $\mathcal{N} = 2$ superconformal theories proposed in [55, 56]. These models are generically non-Lagrangian and have equal

$\widehat{\Gamma}(G)$	$\widehat{D}_4(\text{SU}(2n+1))$	$\widehat{E}_6(\text{SU}(3n\pm 1))$	$\widehat{E}_7(\text{SU}(4n\pm 1))$	$\widehat{E}_8(\text{SU}(6n\pm 1))$
$a = c$	$2n(n+1)$	$2n(3n\pm 2)$	$6n(2n\pm 1)$	$10n(3n\pm 1)$

Table 1. $\widehat{\Gamma}(\text{SU}(N))$ four-dimensional $\mathcal{N} = 2$ superconformal theories with $\Gamma = D_4, E_6, E_7, E_8$ and $a = c$. The index n is a positive integer. Notice that three cases admits the description in [69] as Argyres-Douglas theories engineered from M5 brane, $\widehat{E}_6(\text{SU}(2)) = (A_2, D_4)$, $\widehat{E}_7(\text{SU}(3)) = (A_3, E_6)$, $\widehat{E}_8(\text{SU}(5)) = (A_5, E_8)$.

conformal anomaly coefficients $a = c$. They are denoted $\widehat{\Gamma}(G)$, where Γ and G are ADE simply laced Lie groups, and are built by gauging part of the flavor symmetry of a product of the $D_p(G)$ superconformal theories studied in [57, 58]. Here, we focus on the cases $\Gamma = D_4, E_6, E_7, E_8$ and $G = \text{SU}(N)$, listed in table 1. In all of these theories, the unflavored Schur index is same as the following specialization of the flavored Schur index of $\mathcal{N} = 4$ $\text{SU}(N)$ SYM theory [55, 56]

$$\begin{aligned} \widehat{\text{I}}^{\widehat{\Gamma}(\text{SU}(N))}(q) &= \bar{\text{I}}^{\text{SU}(N)}(q^{-1}; q^{m_\Gamma}), \\ m_{D_4} &= 2, \quad m_{E_6} = 3, \quad m_{E_7} = 4, \quad m_{E_8} = 6, \end{aligned} \tag{6.1}$$

where m_Γ is the largest comark of the affine Dynkin diagram $\widehat{\Gamma}$. We now show how the remarkable relation (6.1) may be combined with the closed formulas presented for the flavored Schur index of $\mathcal{N} = 4$ $\text{U}(N)$ SYM in order to derive the giant graviton expansion of the unflavored Schur index of the $\widehat{\Gamma}(\text{SU}(N))$ theories.

6.1 $\Gamma = D_4$

Let us begin with the simplest case $\Gamma = D_4$. We have from (6.1), cf. (2.5),

$$\widehat{\text{I}}^{\widehat{D}_4(\text{SU}(N))}(q) = \bar{\text{I}}^{\text{SU}(N)}(q^{-1}; q^2) = \widetilde{\text{I}}^{\text{SU}(N)}(q^2), \tag{6.2}$$

where the index in the r.h.s. is the unflavored one, i.e. at $u = 1$. Also, we recall that

$$\bar{\text{I}}^{\text{SU}(N)}(\xi; q) = \frac{\bar{\text{I}}^{\text{U}(N)}(\xi; q)}{\bar{\text{I}}^{\text{U}(1)}(\xi; q)}. \tag{6.3}$$

Taking this relation in unflavored limit and using (2.12) we get

$$\widehat{\text{I}}^{\widehat{D}_4(\text{SU}(N))}(q) = \frac{(q; q^2)_\infty^2}{(q^2)_\infty^2} \widetilde{\text{I}}^{\text{U}(N)}(q^2) = \frac{(q; q^2)_\infty^2}{(q^2)_\infty^2} \text{I}^{\text{U}(N)}(q), \tag{6.4}$$

as argued in [56]. This case is trivial in the sense that its giant graviton expansion follows directly from the unflavored index expansion (1.8). The presence of factors of N in the giant graviton expansion of the $\Gamma = D_4$ index comes thus from the fact that relation (6.1) involves the unrefined index in the right hand side.

6.2 $\Gamma = E_6$

A more interesting case is $\Gamma = E_6$. Relation (6.1) gives now

$$\widehat{\mathbb{I}}^{E_6(\text{SU}(N))}(q) = \overline{\mathbb{I}}^{\text{SU}(N)}(q^{-1}; q^3) = \widetilde{\mathbb{I}}^{\text{SU}(N)}(q^{\frac{1}{2}}; q^3) = \frac{(q; q^3)_\infty (q^2; q^3)_\infty}{(q^3)_\infty^2} \widetilde{\mathbb{I}}^{\text{U}(N)}(q^{\frac{1}{2}}, q^3). \quad (6.5)$$

From (1.5), we have¹⁰

$$\widetilde{\mathbb{I}}^{\text{U}(\infty)}(u; q) \equiv \widetilde{\mathbb{I}}^{\text{KK}}(u; q) = \frac{(q)_\infty}{(uq^{\frac{1}{2}})_\infty (u^{-1}q^{\frac{1}{2}})_\infty}, \quad (6.6)$$

and in particular

$$\widetilde{\mathbb{I}}^{\text{U}(\infty)}(q^{\frac{1}{2}}; q^3) = \frac{(q^3)_\infty}{(q)_\infty (q^2)_\infty}. \quad (6.7)$$

Thus, we can write

$$\widehat{\mathbb{I}}^{E_6(\text{SU}(N))}(q) = \frac{(q; q^3)_\infty (q^2; q^3)_\infty}{(q)_\infty (q^2)_\infty (q^3)_\infty} \frac{\widetilde{\mathbb{I}}^{\text{U}(N)}(q^{\frac{1}{2}}; q^3)}{\widetilde{\mathbb{I}}^{\text{U}(\infty)}(q^{\frac{1}{2}}; q^3)}. \quad (6.8)$$

The ratio can be evaluated by specializing (4.1) at $u = q^{\frac{1}{2}}$

$$\frac{\widetilde{\mathbb{I}}^{\text{U}(N)}(q^{\frac{1}{2}}; q^3)}{\widetilde{\mathbb{I}}^{\text{U}(\infty)}(q^{\frac{1}{2}}; q^3)} = \frac{\mathbb{I}^{\text{U}(N)}(q^{\frac{1}{2}}; q^{\frac{3}{2}})}{\mathbb{I}^{\text{U}(\infty)}(q^{\frac{1}{2}}; q^{\frac{3}{2}})} = \sum_{n=0}^{\infty} \sum_{p=0}^n q^{(n+p)N+3(n-p)p} \mathbb{I}_{n-p}^{\text{D}3}(q^{\frac{1}{2}}; q^{\frac{3}{2}}) \mathbb{I}_p^{\text{D}3}(q^{-\frac{1}{2}}; q^{\frac{3}{2}}). \quad (6.9)$$

The next-to-leading giant graviton expansion is thus

$$\widehat{\mathbb{I}}^{E_6(\text{SU}(N))}(q) = \frac{(q; q^3)_\infty (q^2; q^3)_\infty}{(q)_\infty (q^2)_\infty (q^3)_\infty} \left[1 + q^N W_1(q) + q^{2N} W_2(q) + \mathcal{O}(q^{3N}) \right], \quad (6.10)$$

with the explicit functions

$$W_1(q) = \mathbb{I}_1^{\text{D}3}(q^{-\frac{1}{2}}, q^{\frac{3}{2}}), \quad W_2(q) = \mathbb{I}_1^{\text{D}3}(q^{\frac{1}{2}}; q^{\frac{3}{2}}) + \mathbb{I}_2^{\text{D}3}(q^{-\frac{1}{2}}; q^{\frac{3}{2}}). \quad (6.11)$$

From (3.3), the leading term $W_1(q)$ is provided by the exact expression

$$W_1(q) = -q^{-\frac{3}{2}} \frac{(q^2)_\infty^3}{\vartheta(q^3; q^2)} = \frac{(q^2)_\infty^2}{(q^{-1}; q^2)_\infty (q^3; q^2)_\infty} = -q \frac{(q^2)_\infty^2}{(q; q^2)_\infty^2}, \quad (6.12)$$

and is remarkably N independent. Notice that in terms of Jacobi elliptic theta functions, cf. (A.1), we can also write W_1 as

$$W_1(q) = -\frac{1}{4} q^{\frac{3}{4}} \vartheta_2(\sqrt{q})^2. \quad (6.13)$$

The next-to-leading term $W_2(q)$ is the sum of two contributions that are separately singular, suggesting some extra factor of N appearing at double wrapping order. Again, this is not surprising since the key relation (6.1) involves the flavored Schur index evaluated with a

¹⁰By analogy with the $\mathcal{N} = 4$ SYM case, we use the label KK for the $N = \infty$ limit of the index, although we don't have here a clear Kaluza-Klein interpretation of this contribution.

special algebraic relation between fugacities, as it happened in the $\Gamma = D_4$ case. Singularities cancel in the sum and to get the finite correction we have to regularize the flavor fugacity by scaling $q^{\frac{1}{2}} \rightarrow \eta q^{\frac{1}{2}}$ in the u argument in (6.9) and taking $\eta \rightarrow 1$ in the end. This gives

$$W_2(q) = \lim_{\eta \rightarrow 1} W_2(\eta; q), \quad W_2(\eta; q) = \eta^N \mathbb{I}_1^{\text{D}3}(\eta q^{\frac{1}{2}}; q^{\frac{3}{2}}) + \eta^{-2N} \mathbb{I}_2^{\text{D}3}(\eta^{-1} q^{-\frac{1}{2}}; q^{\frac{3}{2}}) \quad (6.14)$$

The derivation of the limit $W_2(q)$ is a little involved and is discussed in appendix B.4. The result is

$$W_2(q) = \left(N + \frac{23}{8}\right) q^3 - \frac{q}{4} R_2(q^{-1}; q^2) + \frac{1}{6} R_3(q^{-1}; q^2), \quad (6.15)$$

where we remind that the functions R_k are related to derivatives of $\log \theta$ according to (A.10). A more explicit expression is, cf. definitions in (A.1),

$$W_2(q) = q^3 \left[N + 3 + \frac{1}{8} \frac{\partial_z^2 \vartheta_4(0; q)}{\vartheta_4(q)} \right]. \quad (6.16)$$

The expansion of $W_2(q)$ in powers of q reads

$$W_2(q) = (N + 3)q^3 + q^4 + 2q^5 + 4q^6 + 4q^7 + 6q^8 + 8q^9 + \dots, \quad (6.17)$$

with a peculiar N dependence in the q^3 term only. In summary, the explicit giant graviton expansion of the $\widehat{E}_6(\text{SU}(N))$ Schur index is

$$\begin{aligned} \mathbb{I}^{\widehat{E}_6(\text{SU}(N))}(q) &= \frac{(q; q^3)_\infty (q^2; q^3)_\infty}{(q)_\infty (q^2)_\infty (q^3)_\infty} \left[1 + q^N (-q - 2q^2 - q^3 - 2q^4 - 2q^5 - 3q^7 - 2q^8 + \mathcal{O}(q^{10}) \dots) \right. \\ &\quad \left. + q^{2N} \left((N + 3)q^3 + q^4 + 2q^5 + 4q^6 + 4q^7 + 6q^8 + 8q^9 + \mathcal{O}(q^{10}) \right) + \mathcal{O}(q^{3N+7}) \right]. \end{aligned} \quad (6.18)$$

Notice that the first omitted terms are not just $\sim q^{3N}$ since one finds an leading term $-5q^7$ in the expression for W_3 , which is the first contribution omitted in (6.10).

Comparison with available data. As a check, the final expansion (6.18) may be compared with the explicit results computed in [56]. They read¹¹

$$\begin{aligned} \mathbb{I}^{\widehat{E}_6(\text{SU}(2))} &= 1 + q^2 + q^3 + 2q^6 + q^8 + q^{11} + 2q^{12} + q^{15} + 2q^{18} + \dots, \\ \mathbb{I}^{\widehat{E}_6(\text{SU}(4))} &= 1 + q^2 + 2q^3 + 2q^4 + q^5 + 6q^6 + 2q^7 + 4q^8 + 7q^9 + 7q^{10} + 4q^{11} + \dots, \\ \mathbb{I}^{\widehat{E}_6(\text{SU}(5))} &= 1 + q^2 + 2q^3 + 2q^4 + 2q^5 + 7q^6 + 2q^7 + 8q^8 + 10q^9 + 8q^{10} + \dots. \end{aligned} \quad (6.19)$$

Dividing by the Kaluza-Klein factor in (6.10), we find

$$\begin{aligned} \mathbb{I}^{\widehat{E}_6(\text{SU}(2))} / \mathbb{I}^{\text{KK}} &= 1 - q^3 - 2q^4 - q^5 - 2q^6 + 3q^7 + q^8 - q^9 + 2q^{10} + \dots, \\ \mathbb{I}^{\widehat{E}_6(\text{SU}(4))} / \mathbb{I}^{\text{KK}} &= 1 - q^5 - 2q^6 - q^7 - 2q^8 - 2q^9 + 4q^{11} + \dots, \\ \mathbb{I}^{\widehat{E}_6(\text{SU}(5))} / \mathbb{I}^{\text{KK}} &= 1 - q^6 - 2q^7 - q^8 - 2q^9 - 2q^{10} + \dots. \end{aligned} \quad (6.20)$$

¹¹The $\widehat{E}_6(\text{SU}(2))$ case is known in closed form as $\mathbb{I}^{\widehat{E}_6(\text{SU}(2))}(q) = \sum_{n=1}^{\infty} \frac{q^{n-1}}{1+q^n+q^{2n}}$ [48].

To reproduce these series, it is enough to use the two terms we computed, i.e.

$$1 + q^N W_1 + q^{2N} W_2 + \dots \tag{6.21}$$

For instance for $N = 2$ one has, up to terms of order q^{10} included

$$\begin{aligned} & 1 + q^2(-q - 2q^2 - q^3 - 2q^4 - 2q^5 - 3q^7 - 2q^8 + \dots) + q^4(5q^3 + 1q^4 + 2q^5 + 4q^6 + \dots) + \mathcal{O}(q^{11}) \\ & = 1 - q^3 - 2q^4 - q^5 - 2q^6 + (-2 + 5)q^7 + (0 + 1)q^8 + (-3 + 2)q^9 + (-2 + 4)q^{10} + \mathcal{O}(q^{11}), \end{aligned} \tag{6.22}$$

in agreement with the first line in (6.20). The other cases can be checked similarly.

6.3 $\Gamma = E_7, E_8$

For a generic m_Γ , we can repeat the same steps. Let us discuss in particular the leading wrapping correction. Relation (6.1) gives

$$\begin{aligned} \widehat{\Gamma}^{\text{SU}(N)}(q) &= \overline{\Gamma}^{\text{SU}(N)}(q^{-1}; q^{m_\Gamma}) = \widetilde{\Gamma}^{\text{SU}(N)}(q^{\frac{m_\Gamma-2}{2}}; q^{m_\Gamma}) \\ &= \frac{(q; q^{m_\Gamma})_\infty (q^{m_\Gamma-1}; q^{m_\Gamma})_\infty}{(q^{m_\Gamma})_\infty^2} \widetilde{\Gamma}^{\text{U}(N)}(q^{\frac{m_\Gamma-2}{2}}; q^{m_\Gamma}). \end{aligned} \tag{6.23}$$

From (6.6) we have the Kaluza-Klein factor

$$\widetilde{\Gamma}^{\text{U}(\infty)}(q^{\frac{m_\Gamma-2}{2}}; q^{m_\Gamma}) = \frac{(q^{m_\Gamma})_\infty}{(q^{m_\Gamma-1})_\infty (q)_\infty}, \tag{6.24}$$

and thus

$$\widehat{\Gamma}^{\text{SU}(N)}(q) = \frac{(q; q^{m_\Gamma})_\infty (q^{m_\Gamma-1}; q^{m_\Gamma})_\infty}{(q)_\infty (q^{m_\Gamma-1})_\infty (q^{m_\Gamma})_\infty} \frac{\widetilde{\Gamma}^{\text{U}(N)}(q^{\frac{m_\Gamma-2}{2}}; q^{m_\Gamma})}{\widetilde{\Gamma}^{\text{U}(\infty)}(q^{\frac{m_\Gamma-2}{2}}; q^{m_\Gamma})}. \tag{6.25}$$

The ratio can be evaluated by specializing (4.1) at $u = q^{\frac{m_\Gamma-2}{2}}$

$$\frac{\widetilde{\Gamma}^{\text{U}(N)}(q^{\frac{m_\Gamma-2}{2}}; q^{m_\Gamma})}{\widetilde{\Gamma}^{\text{U}(\infty)}(q^{\frac{m_\Gamma-2}{2}}; q^{m_\Gamma})} = \frac{\Gamma^{\text{U}(N)}(q^{\frac{m_\Gamma-2}{2}}; q^{\frac{m_\Gamma}{2}})}{\Gamma^{\text{U}(\infty)}(q^{\frac{m_\Gamma-2}{2}}; q^{\frac{m_\Gamma}{2}})} = \left[1 + q^N W_1^\Gamma(q) + \mathcal{O}(q^{2N}) \right], \tag{6.26}$$

where the leading order wrapping correction reads

$$W_1^\Gamma(q) = I_1^{\text{D}3}(q^{-\frac{m_\Gamma-2}{2}}, q^{\frac{m_\Gamma}{2}}) = -q \frac{(q^{m_\Gamma-1})_\infty^2}{(q; q^{m_\Gamma-1})_\infty (q^{m_\Gamma-2}, q^{m_\Gamma-1})_\infty}. \tag{6.27}$$

For $m_\Gamma = 3$ it reproduces (6.12). In the E_7, E_8 cases, i.e. for $m_\Gamma = 4, 6$, its series expansion reads

$$\begin{aligned} W_1^{E_7}(q) &= -q - q^2 - 2q^3 - 2q^5 - q^6 - 2q^7 - q^9 + \mathcal{O}(q^{10}), \\ W_1^{E_8}(q) &= -q - q^2 - q^3 - q^4 - 2q^5 - q^7 - q^8 - 2q^9 + \mathcal{O}(q^{10}). \end{aligned} \tag{6.28}$$

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A Special functions

We collect in this appendix the definition of special functions appearing in the text and some useful identities.

Jacobi elliptic theta functions.

$$\begin{aligned}
 \vartheta_1(z; q) &= 2 q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n \sin[(2n+1)z] q^{n(n+1)}, \\
 \vartheta_2(z; q) &= 2 q^{\frac{1}{4}} \sum_{n=0}^{\infty} \cos[(2n+1)z] q^{n(n+1)}, \\
 \vartheta_3(z; q) &= 1 + 2 q^{\frac{1}{4}} \sum_{n=1}^{\infty} \cos(2nz) q^{n^2}, \\
 \vartheta_4(z; q) &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n \cos(2nz) q^{n^2}.
 \end{aligned}
 \tag{A.1}$$

Also, $\vartheta_n(q) \equiv \vartheta_n(0; q)$, $n = 2, 3, 4$.

Dedekind η function.

$$\eta(\tau) = q^{\frac{1}{12}} \prod_{k=1}^{\infty} (1 - q^{2k}), \quad q = e^{i\pi\tau}.
 \tag{A.2}$$

q -Pochhammer symbol.

$$(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - a q^k), \quad (a^{\pm}; q)_{\infty} = (a; q)_{\infty} (a^{-1}; q)_{\infty},
 \tag{A.3}$$

$$(q)_{\infty} \equiv (q; q)_{\infty} = \prod_{k=1}^{\infty} (1 - q^k).
 \tag{A.4}$$

Notice that we can write the Dedekind function in (A.2) as

$$\eta(\tau) = q^{\frac{1}{12}} (q^2)_{\infty}.
 \tag{A.5}$$

An elementary but useful relation is

$$(aq; q)_{\infty} = \frac{1}{1-a} (a; q)_{\infty}.
 \tag{A.6}$$

q -theta function. The q -theta function is defined as

$$\vartheta(x, q) = -x^{-\frac{1}{2}} (q)_{\infty} (x; q)_{\infty} (qx^{-1}; q)_{\infty},
 \tag{A.7}$$

with

$$\vartheta(x; q) = -\vartheta(x^{-1}; q), \quad \vartheta'(x; q) = \frac{1}{x^2} \vartheta'(x^{-1}; q), \quad \vartheta''(x; q) = -\frac{2}{x^3} \vartheta'(x^{-1}; q) - \frac{1}{x^4} \vartheta''(x^{-1}; q).
 \tag{A.8}$$

It obeys the very useful relation

$$\vartheta(q^m x; q) = (-1)^m q^{-\frac{m^2}{2}} x^{-m} \vartheta(x; q). \tag{A.9}$$

Introducing the ratios

$$R_k(x; q) = \frac{\vartheta^{(k)}(x; q)}{\vartheta(x; q)}, \quad \vartheta^{(k)}(x; q) = \partial_x^k \vartheta(x; q), \tag{A.10}$$

we obtain from (A.9) the relations

$$R_k(q^m x; q) = \sum_{p=0}^k (-1)^p \binom{k}{p} \frac{(m)_p}{x^p} R_{k-p}(x; q). \tag{A.11}$$

Eisenstein series. The classical Eisenstein series are defined as

$$\mathbb{E}_{2m}(q) = 1 - \frac{4m}{B_{2m}} \sum_{n=1}^{\infty} \frac{n^{2m-1} q^n}{1 - q^n}, \tag{A.12}$$

where B_{2m} are Bernoulli numbers. We will need the differential equations

$$q \frac{d}{dq} \mathbb{E}_2(q) = \frac{1}{12} (\mathbb{E}_2^2 - \mathbb{E}_4), \tag{A.13}$$

$$q \frac{d}{dq} \mathbb{E}_4(q) = \frac{1}{3} (\mathbb{E}_2 \mathbb{E}_4 - \mathbb{E}_6), \tag{A.14}$$

$$q \frac{d}{dq} \mathbb{E}_6(q) = \frac{1}{2} (\mathbb{E}_2 \mathbb{E}_6 - \mathbb{E}_4^2). \tag{A.15}$$

Twisted Weierstrass functions.

$$\begin{aligned} P_1 \begin{bmatrix} x \\ 1 \end{bmatrix} (\nu, \tau) &= \frac{(q)_\infty^3 \vartheta(x^{-1} w; q)}{\vartheta(x^{-1}; q) \vartheta(w; q)}, \quad w = e^{2\pi i \nu}, \\ P_k \begin{bmatrix} x \\ q \end{bmatrix} (\nu, \tau) &= \frac{(-1)^{k-1}}{(k-1)!} \frac{(q)_\infty^3}{\vartheta(x^{-1}; q)} \left(w \frac{\partial}{\partial w} \right)^{k-1} \frac{\vartheta(x^{-1} w; q)}{\vartheta(w; q)}. \end{aligned} \tag{A.16}$$

B Technical details

B.1 Generalized Lambert series and the sums S_3, S_4

Let us consider the generalized Lambert series

$$L_{a,b}(q) = \sum_{k=1}^{\infty} \frac{k^b q^k}{(1 - q^k)^a}. \tag{B.1}$$

The cases $L_{1,b}$ with odd b are directly related to Eisenstein series (A.12), but other cases are not. For instance the sum $L_{1,0}$ is the generating function of the number of positive divisors and can be written in terms of the q -polygamma function, but this is actually the definition of that function. We will adopt the following abbreviated notation for the sums $L_{1,b}$ with even index b

$$L_{2m}(q) \equiv L_{1,2m}(q) = \sum_{k=1}^{\infty} \frac{k^{2m} q^k}{1 - q^k}, \quad m = 0, 1, 2, \dots \tag{B.2}$$

We now show how to write $L_{a,b}$ in terms of Eisenstein series, sums of type L_{2m} , and their derivatives. To this aim, we start by remarking that a large set of identities among $L_{a,b}$ with different indices comes after summing over ℓ in the trivial identity

$$\sum_{k,\ell=1}^{\infty} (k - \ell) \times (\text{symmetric polynomial in } k, \ell) q^{k\ell} = 0. \tag{B.3}$$

For example,

$$\sum_{k,\ell=1}^{\infty} (k - \ell) q^{k\ell} = 0 \quad \rightarrow \quad \sum_{k=1}^{\infty} \left[\frac{k}{1 - q^k} - \frac{1}{(1 - q^k)^2} \right] q^k = 0, \tag{B.4}$$

gives the relation

$$L_{1,1} = L_{2,0}. \tag{B.5}$$

Another useful set of relations is obtained by differentiating with respect to q , for instance

$$q \frac{d}{dq} \mathbb{E}_2 = -24q \frac{d}{dq} \sum_{k=1}^{\infty} \frac{kq^k}{1 - q^k} = -24 \sum_{k=1}^{\infty} \frac{k^2 q^k}{(1 - q^k)^2} = -24 L_{2,2}. \tag{B.6}$$

Considering all instances of (B.3) with a generic symmetric polynomial of degree 4 and adding to (B.6) the analogous relation for $q^2 \mathbb{E}_2''$, one gets the following relations for $L_{a,b}$. With first index equal to 2:

$$\begin{aligned} L_{2,0} &= \frac{1}{24}(1 - \mathbb{E}_2), & L_{2,3} &= qL'_2, \\ L_{2,1} &= qL'_0, & L_{2,4} &= \frac{1}{240}q\mathbb{E}'_4 \\ L_{2,2} &= -\frac{1}{24}q\mathbb{E}'_2. \end{aligned} \tag{B.7}$$

With first index 3:

$$\begin{aligned} L_{3,0} &= \frac{1}{48}(1 - \mathbb{E}_2) + \frac{1}{2}L_2, & L_{3,2} &= -\frac{1}{48}q\mathbb{E}'_2 + \frac{1}{2}qL'_0 + \frac{1}{2}q^2L''_0, \\ L_{3,1} &= -\frac{1}{48}q\mathbb{E}'_2 + \frac{1}{2}qL'_0, & L_{3,3} &= -\frac{1}{48}q\mathbb{E}'_2 - \frac{1}{48}q^2\mathbb{E}''_2 + \frac{1}{2}qL'_2. \end{aligned} \tag{B.8}$$

With first index 4, 5, 6

$$\begin{aligned} L_{4,0} &= \frac{19}{1440} - \frac{1}{72}\mathbb{E}_2 + \frac{1}{1440}\mathbb{E}_4 + \frac{1}{2}L_2, \\ L_{4,1} &= -\frac{1}{48}q\mathbb{E}'_2 + \frac{1}{3}qL'_0 + \frac{1}{6}qL'_2, \\ L_{4,2} &= -\frac{1}{48}\mathbb{E}'_2 - \frac{1}{144}q^2\mathbb{E}''_2 + \frac{1}{2}qL'_0 + \frac{1}{2}q^2L''_0, \\ L_{5,0} &= \frac{3}{320} - \frac{1}{96}\mathbb{E}_2 + \frac{1}{960}\mathbb{E}_4 + \frac{11}{24}L_2 + \frac{1}{24}L_4, \\ L_{5,1} &= -\frac{11}{576}q\mathbb{E}'_2 + \frac{1}{5760}q\mathbb{E}'_4 + \frac{1}{4}qL'_0 + \frac{1}{4}qL'_2, \\ L_{6,0} &= \frac{863}{120960} - \frac{1}{120}\mathbb{E}_2 + \frac{7}{5760}\mathbb{E}_4 - \frac{1}{60480}\mathbb{E}_6 + \frac{5}{12}L_2 + \frac{1}{12}L_4. \end{aligned} \tag{B.9}$$

The sum $S_3(q)$. Let us now consider the sum in (4.17). Factoring q^k and splitting $q^k = 1 - (1 - q^k)$ in the rest, we get

$$S_3(q) = \sum_{k=1}^{\infty} \frac{4(1+k+(k-1)q^k)q^k}{(1-q^k)^3} = 4L_{2,0} - 4L_{2,1} + 8L_{3,1}. \quad (\text{B.10})$$

Using the previous relations, we obtain

$$S_3(q) = \frac{1}{6}(1 - \mathbb{E}_2 - q\mathbb{E}'_2), \quad (\text{B.11})$$

which is same as (4.20) using the differential equation (A.13).

The sum $S_4(q)$. The procedure for this sum is similar. First, we write

$$\begin{aligned} S_4(q) &= \sum_{k=1}^{\infty} \frac{[5 + 6k + 2k^2 - 2(1 - 4k^2)q^k + (5 - 6k + 2k^2)q^{2k}]q^k}{(1 - q^k)^4} \\ &= 5L_{2,0} - 6L_{2,1} + 2L_{2,2} - 8L_{3,0} + 12L_{3,1} - 12L_{3,2} + 8L_{4,0} + 12L_{4,2}. \end{aligned} \quad (\text{B.12})$$

Then, using the previous relations, we get

$$S_4(q) = \frac{53}{360} - \frac{11}{72}\mathbb{E}_2 + \frac{1}{180}\mathbb{E}_4 - \frac{1}{3}q\mathbb{E}'_2 - \frac{1}{12}q^2\mathbb{E}''_2. \quad (\text{B.13})$$

This is same as (4.21) after using (A.13) and (A.14) that give

$$q^2\mathbb{E}''_2 = \left(q \frac{d}{dq}\right)^2 \mathbb{E}_2 - q \frac{d}{dq} \mathbb{E}_2 = \frac{1}{72}\mathbb{E}_2^3 - \frac{1}{24}\mathbb{E}_2\mathbb{E}_4 + \frac{1}{36}\mathbb{E}_6 - \frac{1}{12}\mathbb{E}_2^2 + \frac{1}{12}\mathbb{E}_4. \quad (\text{B.14})$$

B.2 Differential equation for $B_1(u; q)$

Let us start from the single wrapping brane index

$$I_1^{\text{D3}}(u; q) = -qu^2 \frac{\left(\frac{q}{u}\right)_{\infty}^3}{\vartheta(u^2; \frac{q}{u})}, \quad (\text{B.15})$$

and define

$$f(x; q) = I_1^{\text{D3}}(\sqrt{x}; \sqrt{x}q) = -qx^{3/2} \frac{(q)_{\infty}^3}{\vartheta(x; q)}. \quad (\text{B.16})$$

From the differential equation for the ϑ function

$$q \frac{\partial}{\partial q} \vartheta(x; q) - \frac{1}{2} \left(x \frac{\partial}{\partial x}\right)^2 \vartheta(x; q) + \frac{1}{8} \vartheta(x; q) = 0, \quad (\text{B.17})$$

and the well known derivative

$$q \frac{d}{dq} \log(q)_{\infty} = \frac{1}{24}(\mathbb{E}_2 - 1), \quad (\text{B.18})$$

we obtain the non-linear differential equation for $f(x; q)$

$$\frac{9}{8} + \frac{1}{8}(\mathbb{E}_2 - 1) - \left(q\partial_q - \frac{1}{2}x\partial_x - \frac{1}{2}x^2\partial_x^2\right) \log f - \frac{1}{2}x^2 \left(\frac{3}{2x} - \partial_x \log f\right)^2 = 0. \quad (\text{B.19})$$

From (3.17), $f(x; q)$ has the structure

$$f(x, q) = \frac{x^2}{1-x} q + x(1-x)q^2 B_1(\sqrt{x}; q\sqrt{x}), \quad (\text{B.20})$$

and therefore, expanding around $x = 1$, we get

$$\begin{aligned} \log f(x, q) &= \log \frac{q}{1-x} + 2(x-1) + [-1 + q B_1(q)](x-1)^2 \\ &\quad + \frac{1}{6}(4 - 6q B_1(q) + 3q^2 B_1'(q) + 3q B_1^{(1)}(q))(x-1)^3 + \mathcal{O}((x-1)^4), \end{aligned} \quad (\text{B.21})$$

where notation is as in (4.11). Plugging this expansion in (B.19) gives

$$0 = \frac{1}{8}(-1 + 24q B_1(q) + \mathbb{E}_2) + 3q[q B_1'(q) + B_1^{(1)}(q)](x-1) + \mathcal{O}((x-1)^2). \quad (\text{B.22})$$

The leading order gives

$$B_1(q) \equiv B_1(1; q) = \frac{1}{24}(1 - \mathbb{E}_2(q)). \quad (\text{B.23})$$

At next orders in $x - 1$, using recursively results like (B.23) and its derivatives with respect to q , we get the following expressions for $\partial_u^p B_1(1; q)$ in terms of $\mathbb{E}_2(q)$ and its derivatives

$$\begin{aligned} q\partial_u B_1(1; q) &= \frac{1}{24}(1 - \mathbb{E}_2) + \frac{1}{24}\mathbb{E}_2', \\ q\partial_u^2 B_1(1; q) &= -\frac{17}{720} + \frac{1}{72}\mathbb{E}_2 + \frac{7}{720}\mathbb{E}_2^2 - \frac{1}{30}q\mathbb{E}_2' - \frac{1}{24}q^2\mathbb{E}_2'', \\ q\partial_u^3 B_1(1; q) &= \frac{7}{120}q\mathbb{E}_2' - \frac{7}{120}q\mathbb{E}_2\mathbb{E}_2' + \frac{9}{40}q^2\mathbb{E}_2'' + \frac{1}{24}q^3\mathbb{E}_2^{(3)}, \\ q\partial_u^4 B_1(1; q) &= \frac{377}{5040} - \frac{\mathbb{E}_2}{30} - \frac{7\mathbb{E}_2^2}{240} - \frac{31\mathbb{E}_2^3}{2520} - \frac{22}{105}q\mathbb{E}_2' + \frac{71}{210}q\mathbb{E}_2\mathbb{E}_2' + \frac{7}{60}q^2\mathbb{E}_2'' - \frac{181}{140}q^2\mathbb{E}_2'' \\ &\quad + \frac{7}{60}q^2\mathbb{E}_2\mathbb{E}_2'' - \frac{8}{15}q^3\mathbb{E}_2^{(3)} - \frac{1}{24}q^4\mathbb{E}_2^{(4)}, \end{aligned} \quad (\text{B.24})$$

and so on. These results are in agreement with (4.22), using the differential equations (A.13) and (A.14). This way, it is straightforward to compute any function $B_1^{(p)}(q)$ at arbitrarily large p .

B.3 Quasimodular expressions for the functions $B_n^{(p)}(q)$

The functions $B_n^{(p)}(q)$ for $n = 3, 4, 5$ and $p = 1, 2, 3$ have a quasimodular structure completely similar to that in (4.22). The explicit expressions are

Functions $B_2^{(p)}$.

$$\begin{aligned} q B_2(q) &= \frac{1}{24}(1 - \mathbb{E}_2), \\ q B_2^{(1)}(q) &= \frac{1}{12}(1 - \mathbb{E}_2) + \frac{1}{288}(\mathbb{E}_2^2 - \mathbb{E}_4), \\ q B_2^{(2)}(q) &= -\frac{43}{30} + \frac{17}{12}\mathbb{E}_2 + \frac{1}{96}\mathbb{E}_2^2 + \frac{1}{160}\mathbb{E}_4 - \frac{1}{1728}\mathbb{E}_2^3 + \frac{1}{576}\mathbb{E}_2\mathbb{E}_4 - \frac{1}{864}\mathbb{E}_6, \\ q B_2^{(3)}(q) &= -\frac{443}{20} + \frac{45}{2}\mathbb{E}_2 - \frac{125}{144}\mathbb{E}_2^2 + \frac{373}{720}\mathbb{E}_4 - \frac{1}{576}\mathbb{E}_2^3 - \frac{11}{960}\mathbb{E}_2\mathbb{E}_4 + \frac{19}{1440}\mathbb{E}_6 + \frac{1}{6912}\mathbb{E}_2^4 \\ &\quad + \frac{1}{864}\mathbb{E}_2\mathbb{E}_6 - \frac{1}{2304}\mathbb{E}_4^2 - \frac{1}{1152}\mathbb{E}_2^2\mathbb{E}_4. \end{aligned} \quad (\text{B.25})$$

Functions $B_3^{(p)}$.

$$\begin{aligned}
 q B_3(q) &= \frac{1}{48}(1 - \mathbb{E}_2), \\
 q B_3^{(1)}(q) &= \frac{1}{12}(1 - \mathbb{E}_2) + \frac{1}{576}(\mathbb{E}_2^2 - \mathbb{E}_4), \\
 q B_3^{(2)}(q) &= -\frac{1597}{1440} + \frac{79}{72}\mathbb{E}_2 + \frac{5}{576}\mathbb{E}_2^2 + \frac{1}{320}\mathbb{E}_4 - \frac{1}{3456}\mathbb{E}_2^3 + \frac{1}{1152}\mathbb{E}_2\mathbb{E}_4 - \frac{1}{1728}\mathbb{E}_6, \\
 q B_3^{(3)}(q) &= -\frac{303}{160} + \frac{9}{4}\mathbb{E}_2 - \frac{73}{96}\mathbb{E}_2^2 + \frac{97}{240}\mathbb{E}_4 - \frac{1}{1152}\mathbb{E}_2^3 - \frac{53}{5760}\mathbb{E}_2\mathbb{E}_4 + \frac{29}{2880}\mathbb{E}_6 \\
 &\quad + \frac{1}{13824}\mathbb{E}_2^4 + \frac{1}{1728}\mathbb{E}_2\mathbb{E}_6 - \frac{1}{4608}\mathbb{E}_4^2 - \frac{1}{2304}\mathbb{E}_2^2\mathbb{E}_4.
 \end{aligned} \tag{B.26}$$

Functions $B_4^{(p)}$.

$$\begin{aligned}
 q B_4(q) &= \frac{1}{144}(1 - \mathbb{E}_2), \\
 q B_4^{(1)}(q) &= \frac{7}{144}(1 - \mathbb{E}_2) + \frac{1}{1728}(\mathbb{E}_2^2 - \mathbb{E}_4), \\
 q B_4^{(2)}(q) &= -\frac{343}{1080} + \frac{5}{16}\mathbb{E}_2 + \frac{1}{192}\mathbb{E}_2^2 - \frac{1}{8640}\mathbb{E}_4 - \frac{1}{10368}\mathbb{E}_2^3 + \frac{1}{3456}\mathbb{E}_2\mathbb{E}_4 - \frac{1}{5184}\mathbb{E}_6, \\
 q B_4^{(3)}(q) &= \frac{1951}{240} - \frac{191}{24}\mathbb{E}_2 - \frac{275}{864}\mathbb{E}_2^2 + \frac{637}{4320}\mathbb{E}_4 - \frac{1}{1728}\mathbb{E}_2^3 - \frac{29}{8640}\mathbb{E}_2\mathbb{E}_4 \\
 &\quad + \frac{17}{4320}\mathbb{E}_6 + \frac{1}{41472}\mathbb{E}_2^4 + \frac{1}{5184}\mathbb{E}_2\mathbb{E}_6 - \frac{1}{13824}\mathbb{E}_4^2 - \frac{1}{6912}\mathbb{E}_2^2\mathbb{E}_4,
 \end{aligned} \tag{B.27}$$

Functions $B_5^{(p)}$.

$$\begin{aligned}
 q B_5(q) &= \frac{1}{576}(1 - \mathbb{E}_2), \\
 q B_5^{(1)}(q) &= \frac{11}{576}(1 - \mathbb{E}_2) + \frac{1}{6912}(\mathbb{E}_2^2 - \mathbb{E}_4), \\
 q B_5^{(2)}(q) &= \frac{163}{17280} - \frac{19}{1728}\mathbb{E}_2 + \frac{5}{2304}\mathbb{E}_2^2 - \frac{7}{11520}\mathbb{E}_4 - \frac{1}{41472}\mathbb{E}_2^3 + \frac{1}{13824}\mathbb{E}_2\mathbb{E}_4 - \frac{1}{20736}\mathbb{E}_6, \\
 q B_5^{(3)}(q) &= \frac{17447}{2880} - \frac{865}{144}\mathbb{E}_2 - \frac{23}{288}\mathbb{E}_2^2 + \frac{83}{2880}\mathbb{E}_4 - \frac{1}{3456}\mathbb{E}_2^3 - \frac{1}{1440}\mathbb{E}_2\mathbb{E}_4 + \frac{17}{17280}\mathbb{E}_6 \\
 &\quad + \frac{1}{165888}\mathbb{E}_2^4 + \frac{1}{20736}\mathbb{E}_2\mathbb{E}_6 - \frac{1}{55296}\mathbb{E}_4^2 - \frac{1}{27648}\mathbb{E}_2^2\mathbb{E}_4.
 \end{aligned} \tag{B.28}$$

As an alternative basis, all functions $B_k^{(1)}$ can be expressed in terms of B_1 and $B_1^{(1)}$ according to

$$\begin{aligned}
 B_2^{(1)} &= B_1^{(1)} + B_1, & B_3^{(1)} &= \frac{1}{2}B_1^{(1)} + \frac{3}{2}B_1, \\
 B_4^{(1)} &= \frac{1}{6}B_1^{(1)} + B_1, & B_5^{(1)} &= \frac{1}{24}B_1^{(1)} + \frac{5}{12}B_1.
 \end{aligned} \tag{B.29}$$

Similarly, $B_k^{(2)}$ can be given in terms of $B_1, B_1^{(1)}, B_1^{(2)}$ with the addition of a term proportional to qB_1^2

$$\begin{aligned}
 B_2^{(2)} &= B_1^{(2)} - 2B_1^{(1)} - 32B_1 + 4qB_1^2, \\
 B_3^{(2)} &= \frac{1}{2}B_1^{(2)} - B_1^{(1)} - \frac{51}{2}B_1 + 4qB_1^2, \\
 B_4^{(2)} &= \frac{1}{6}B_1^{(2)} - \frac{137}{18}B_1 + 2qB_1^2, \\
 B_5^{(2)} &= \frac{1}{24}B_1^{(2)} + \frac{1}{6}B_1^{(1)} + \frac{1}{18}B_1 + \frac{2}{3}qB_1^2.
 \end{aligned}
 \tag{B.30}$$

Third derivatives require a further contribution proportional to $qB_1B_1^{(1)}$ and are given by

$$\begin{aligned}
 B_2^{(3)} &= B_1^{(3)} - 9B_1^{(2)} - \frac{732}{5}B_1^{(1)} - 384B_1 - \frac{876}{5}qB_1^2 + 24qB_1B_1^{(1)}, \\
 B_3^{(3)} &= \frac{1}{2}B_1^{(3)} - \frac{15}{2}B_1^{(2)} - \frac{1149}{10}B_1^{(1)} + 72B_1 - \frac{936}{5}qB_1^2 + 24qB_1B_1^{(1)}, \\
 B_4^{(3)} &= \frac{1}{6}B_1^{(3)} - 3B_1^{(2)} - \frac{1261}{30}B_1^{(1)} + \frac{1433}{6}B_1 - \frac{468}{5}qB_1^2 + 12qB_1B_1^{(1)}, \\
 B_5^{(3)} &= \frac{1}{24}B_1^{(3)} - \frac{3}{4}B_1^{(2)} - \frac{247}{30}B_1^{(1)} + \frac{617}{4}B_1 - \frac{146}{5}qB_1^2 + 4qB_1B_1^{(1)}.
 \end{aligned}
 \tag{B.31}$$

B.4 Evaluation of $W_2(q)$

In this appendix, we compute the function $W_2(q)$ from the limit (6.14), i.e.

$$W_2(q) = \lim_{\eta \rightarrow 1} \left[\eta^N I_1^{\text{D}3}(\eta q^{\frac{1}{2}}; q^{\frac{3}{2}}) + \eta^{-2N} I_2^{\text{D}3}(\eta^{-1} q^{-\frac{1}{2}}; q^{\frac{3}{2}}) \right].
 \tag{B.32}$$

The first term in (B.32) can be written using (A.6) in the form

$$I_1^{\text{D}3}(\eta q^{\frac{1}{2}}; q^{\frac{3}{2}}) = - \frac{q^3 \eta^6}{(1 - \eta q^2)(1 - \eta^2 q)(1 - \eta^3)} \frac{\left(\frac{q}{\eta}\right)_\infty^2}{(q^3; \frac{q}{\eta})_\infty \left(\frac{q}{\eta^4}; \frac{q}{\eta}\right)_\infty}.
 \tag{B.33}$$

We now observe that

$$\begin{aligned}
 \log \frac{\left(\frac{q}{\eta}\right)_\infty^2}{(q^3; \frac{q}{\eta})_\infty \left(\frac{q}{\eta^4}; \frac{q}{\eta}\right)_\infty} &= \sum_{n=0}^{\infty} \left[\log \frac{1 - q^{n+1}}{1 - q^{n+3}} + \left(\frac{(n-2)q^{n+1}}{1 - q^{n+1}} - \frac{nq^{n+3}}{1 - q^{n+3}} \right) (\eta - 1) + \mathcal{O}((\eta - 1)^2) \right] \\
 &= \log[(1 - q)(1 - q^2)] - \left(\frac{2q}{1 - q} + \frac{q^2}{1 - q^2} \right) (\eta - 1) + \mathcal{O}((\eta - 1)^2),
 \end{aligned}
 \tag{B.34}$$

and therefore

$$\frac{\left(\frac{q}{\eta}\right)_\infty^2}{(q^3; \frac{q}{\eta})_\infty \left(\frac{q}{\eta^4}; \frac{q}{\eta}\right)_\infty} = (1 - q)(1 - q^2) - q(1 - q)(2 + 3q)(\eta - 1) + \mathcal{O}((\eta - 1)^2).
 \tag{B.35}$$

Remarkably, the term linear in $\eta - 1$ is a simple polynomial in q , a property that does not hold for the next corrections quadratic in $\eta - 1$. Still, this is enough to give the following expansion of the first term in (B.32) around $\eta = 1$

$$\eta^N I_1^{\text{D}3}(\eta q^{\frac{1}{2}}; q^{\frac{3}{2}}) = \frac{q^3}{3(\eta - 1)} + \frac{5 + N}{3} q^3 + \mathcal{O}(\eta - 1).
 \tag{B.36}$$

The second piece in (B.32) is treated similarly. First, we expose the singularity for $\eta \rightarrow 1$ by writing

$$I_2^{D3}(\eta^{-1}q^{-\frac{1}{2}}; q^{\frac{3}{2}}) = -\frac{q^3}{2\eta^3(1-\eta^3)(1-\eta^4q^2)} \frac{(\eta q^2)_\infty^2}{\left(\frac{q^2}{\eta^2}; \eta q^2\right)_\infty (\eta^5 q^4; \eta q^2)_\infty} \left[-3 + \frac{2}{\eta^2 q} R_1\left(\frac{1}{\eta^2 q}; \eta q^2\right) \right]. \tag{B.37}$$

The combination of q-Pochhammer functions can be expanded around $\eta = 1$ using

$$\begin{aligned} & \log \frac{(\eta q^2)_\infty^2}{\left(\frac{q^2}{\eta^2}; \eta q^2\right)_\infty (\eta^5 q^4; \eta q^2)_\infty} \\ &= \sum_{n=0}^{\infty} \left[\log \frac{1-q^{2+2n}}{1-q^{4+2n}} + \left(-\frac{(n+4)q^{2+2n}}{1-q^{2+2n}} + \frac{(n+5)q^{4+2n}}{1-q^{4+2n}} \right) (\eta-1) + \mathcal{O}((\eta-1)^2) \right] \\ &= \log(1-q^2) - \frac{4q^2}{1-q^2} (\eta-1) + \mathcal{O}((\eta-1)^2). \end{aligned} \tag{B.38}$$

and therefore

$$\frac{(\eta q^2)_\infty^2}{\left(\frac{q^2}{\eta^2}; \eta q^2\right)_\infty (\eta^5 q^4; \eta q^2)_\infty} = 1 - q^2 - 4q^2(\eta-1) + \mathcal{O}((\eta-1)^2). \tag{B.39}$$

Hence,

$$\begin{aligned} \eta^{-2N} I_2^{D3}(\eta^{-1}q^{-\frac{1}{2}}; q^{\frac{3}{2}}) &= -\frac{q^3}{3(\eta-1)} \\ &+ \frac{1}{3}q \left[(2+3N)q^2 + q^3 R_1^{(0,1)}(q^{-1}; q^2) - 2R_1^{(1,0)}(q^{-1}; q^2) \right] + \mathcal{O}(\eta-1), \end{aligned} \tag{B.40}$$

where we used $R_1(q^{-1}; q^2) = q/2$ and $R_1^{(p,q)}$ are partial derivatives. This gives the finite expression

$$W_2(q) = \left(\frac{8}{3} + N\right)q^3 + \frac{q^4}{3} R_1^{(0,1)}(q^{-1}; q^2) - \frac{2q}{3} R_1^{(1,0)}(q^{-1}; q^2). \tag{B.41}$$

Using the differential equation (B.17) to work out $R_1^{(0,1)}$, we get the expression (6.15).

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