From Faddeev–Kulish to LSZ.
Towards a non-perturbative description of colliding electrons

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Dedicated to the memory of Wolfhart Zimmermann

Abstract

In a low energy approximation of the massless Yukawa theory (Nelson model) we derive a Faddeev–Kulish type formula for the scattering matrix of N electrons and reformulate it in LSZ terms. To this end, we perform a decomposition of the infrared finite Dollard modifier into clouds of real and virtual photons, whose infrared divergencies mutually cancel. We point out that in the original work of Faddeev and Kulish the clouds of real photons are omitted, and consequently their wave-operators are ill-defined on the Fock space of free electrons. To support our observations, we compare our final LSZ expression for $N = 1$ with a rigorous non-perturbative construction due to Pizzo. While our discussion contains some heuristic steps, they can be formulated as clear-cut mathematical conjectures.

1. Introduction

The seminal paper of Lehmann, Symanzik and Zimmermann [27] was among the first works on mathematical foundations of QFT. It starts from three general principles, namely Poincaré

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covariance, locality and an asymptotic condition, which specifies a certain form of convergence of the interacting fields to the asymptotic (free) fields at large times. Based on these assumptions, the paper provides the celebrated LSZ reduction formulae linking the scattering matrix elements to the amputated Green functions restricted to the mass-shells of the particles. The authors were aware of some limitations of their approach, in particular pertaining to bound states, but expected that the method could be applied to QED. Later studies showed that due to the infrared problems and local gauge invariance, all the LSZ assumptions have to be treated with care in this case: In fact, Lorentz symmetry is typically broken in charged sectors of QED [5,18]. Locality of the Dirac field requires the indefinite metric Gupta–Bleuler formulation [15]. Finally and most importantly, the simple-minded asymptotic condition, which presupposes ballistic motion of independent Wigner particles at large times, is undermined by the presence of long range forces [10] and the *infraparticle problem* [36,5,18]. Indeed, it had gradually become clear, that the Wigner concept of a particle is too restrictive to cover the electron, and has to be replaced with the notion of an infraparticle which is always accompanied by a cloud of *real* soft photons. In particular, an infraparticle does not have a sharp mass shell, to which the Green functions from the LSZ formula could be restricted. Consistently with the infraparticle picture of the electron, Yennie, Frautschi and Suura derived an algorithm for computation of inclusive collision cross-sections in QED, with a necessary summation over infinitely many real soft photons [38]. In contrast to the conceptually and mathematically clear LSZ paper, this derivation is heuristic and relies heavily on perturbation theory. In spite of several later refinements [26,39,37], a convincing derivation of the Yennie–Frautschi–Suura formula, starting from several physically clear assumptions is still an unsolved aspect of the infrared problem.

One line of developments towards a solution of this problem was initiated by Faddeev and Kulish. Building on the work of Dollard [10] on quantum-mechanical scattering in the presence of long-range potentials, these authors proposed a construction of a scattering matrix for QED [14]. However, due to a peculiar choice of a lower boundary of integration in the Dollard modifier, this S-matrix has no contributions from real soft photons, and thus it is difficult to link it to the Yennie–Frautschi–Suura prescription. For similar reasons, the relation between the Faddeev–Kulish approach and the LSZ asymptotic condition, widely discussed in various models and settings [4,33,29], has never been fully clarified. Given the recent revival of interest in the infrared problems triggered by works of Strominger et al. (see [35] for a review) and a prominent role of the Faddeev–Kulish approach for these developments [20,22,21,30,28,25], it is worthwhile to revisit these long-standing issues.

Another important line of developments towards a solution of the infrared problem was initiated by Fröhlich in [16,17]. These works concern a simplified, non-perturbative model of QED (the Nelson model) describing non-relativistic electrons coupled to photons. An LSZ-type asymptotic condition for one electron is formulated in [16], taking its infraparticle nature into account and carefully modeling the accompanying cloud of real photons. Taking the improvements due to Haag [23] and Ruelle [34] into account, the asymptotic condition is formulated in terms of strongly convergent asymptotic field approximants giving rise to scattering states, as opposed to the weakly convergent approximants from the original LSZ work. A rigorous proof of such an asymptotic condition was given relatively recently by Pizzo in the Nelson model [31] and by Chen–Fröhlich–Pizzo in a more realistic model of QED [7]. Given this mathematically satisfactory situation, the Nelson model is a suitable playground to bridge the gaps between the different approaches to the infrared problem mentioned above: the Yennie–Frautschi–Suura inclusive cross-sections, the Faddeev–Kulish scattering matrix and the LSZ asymptotic condition. In the present paper we focus on the step from the Faddeev–Kulish approach to the LSZ
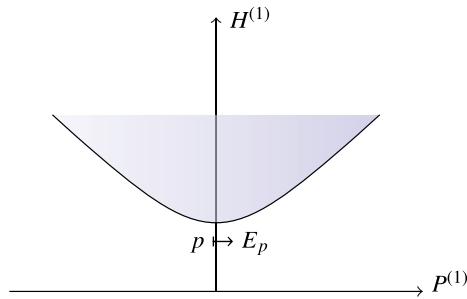
Fig. 1. The energy–momentum spectrum in the single-electron sector of the massless Nelson model.

asymptotic condition and leave a derivation of the Yennie–Frautschi–Suura algorithm via the corresponding LSZ reduction for future studies. (See, however, some remarks in this direction in Section 6.)

The Nelson model has been used for many decades for non-perturbative discussions of infrared problems (see e.g. [16,17,31,1,11]). Its Hamiltonian, stated in Section 2 below, can be obtained as a low energy approximation of the massless Yukawa theory with the interaction Lagrangian $\mathcal{L}_\text{I} = \lambda \overline{\psi} \phi \psi$. Here ψ is the massive Dirac field, whose excitations will be called electrons/positrons, and ϕ is the massless scalar field whose excitations will be called photons (although they are spinless). We fix an ultraviolet cut-off κ and approximate the dispersion relation of the massive particles by the non-relativistic formula $p \mapsto p^2/(2m)$, where $m = 1$ for simplicity. As the creation and annihilation processes of the electron–positron pairs can be neglected in the low-energy regime, we can restrict attention to the zero-positron sector and include only the electron–photon interactions in the Hamiltonian H of the Nelson model. This Hamiltonian commutes with the total number of electrons and we denote by $H^{(N)}$ the N-electron Hamiltonians. Furthermore, by the translation invariance of the model, $H^{(N)}$ commutes with the respective total momentum operator $P^{(N)}$ and thus this family of operators can be diagonalized simultaneously. For $N = 1$ the lower boundary of their joint spectrum is the physical (renormalized) energy–momentum relation of the electron which we denote $p \mapsto E_p$ (see Fig. 1). This dispersion relation has been a subject of study for many decades and it is relatively well understood [1,17,32,12]. Two comments about its properties are in order, since they anticipate our discussion in the later part of this paper:

(a) In the presence of interaction the physical dispersion relation $p \mapsto E_p$ differs from the bare one $p \mapsto \frac{p^2}{2}$ appearing in the free Hamiltonian (2.2). This is caused by certain photon degrees of freedom 'sitting' on the bare electron, which are responsible, in particular, for radiative corrections to its mass. We will refer to these photons as 'clouds of virtual photons', to distinguish them from 'clouds of real photons' described in (b) below.

(b) It is also well established[1] that there are no normalizable states in the Hilbert space of the model, that would 'live' exactly at the lower boundary of the spectrum from Fig. 1. In other words, it is not possible to find normalizable states describing just the physical electron

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[1] Strictly speaking for the Nelson model only partial results in this direction have been published [17,32]. For a minimally coupled model of non-relativistic QED the absence of such states can be concluded by combining results from [24] and [8,19].

(including its cloud of virtual photons) and no other particles. Hence, the electron is always accompanied by some 'cloud of real photons', moving to lightlike infinity.

An early discussion of the Faddeev–Kulish formalism in the Nelson model is due to Fröhlich [16, Chapter 5], who was quite pessimistic about its rigorous mathematical justification. Our work still contains some heuristic steps, but they have the form of plausible, clear-cut conjectures (see Sections 5 and 6). In Section 3 below we start from the concept of the Dollard modifier which comes from quantum mechanical long-range scattering and encodes correlations between particles which persist at asymptotic times. It does not suffer from any infrared divergencies and thus does not require infrared regularization. Such divergencies appear only in Section 4 when we start rewriting the Faddeev–Kulish scattering states in the spirit of the LSZ asymptotic condition. This is completed in Section 5, where we express the Dollard modifier as a product of infrared divergent objects of two types: the clouds of real photons and the clouds of virtual photons, both of which are well-defined only in the presence of an infrared cut-off. From this perspective it is completely clear, that the two types of infrared divergencies, discussed in (a) and (b) above, must mutually cancel as the infrared cut-off is removed. In Section 6 we indicate that the resulting LSZ formula in the case $N = 1$ reproduces, up to minor technical differences, a rigorous formula for one-electron scattering states in the Nelson model due to Pizzo [31]. We conclude our discussion with several clear-cut mathematical conjectures concerning the convergence of N-electron scattering state approximants in the Nelson model, and with some comments on the LSZ reduction formulae for infraparticles.

Strangely, the original work of Faddeev and Kulish misses the central point above, namely the cancellation of the infrared divergences coming from the clouds of real and virtual photons. In fact, the omission of the lower boundary of integration in formula (9) of [14] (which corresponds to dropping term (4.2) below) ensures commutation of the S-matrix with the total momentum of charged particles. Consequently, there is no room for clouds of real photons and the S-matrix is ill-defined on the Fock space of free electron states. Faddeev and Kulish try to cure this problem by a contrived construction of the asymptotic Hilbert space, based on singular coherent states. While this strategy may work in some test-cases in perturbation theory, to our knowledge it has never matured into a general non-perturbative argument.

Some aspects of this problem have been discussed both in old [33] and very recent works [22], but our observation that the lower boundary of integration is responsible for clouds of real photons seems to be new. It leads to a very natural solution: we apply the Dollard formalism according to the rules of the art [10,9], without tampering with the lower boundary of integration. The resulting S-matrix may not commute with the total momentum of the electrons, but it acts on the usual Fock space. As mentioned above, the corresponding incoming and outgoing scattering states can be given a solid LSZ interpretation in terms of electrons dressed with clouds of virtual photons and accompanied by clouds of real photons. In the case of the outgoing states, these real photons can be interpreted as Bremsstrahlung, emitted by the colliding electrons. In the case of the incoming states one should rather think about the time-inverted process of photons absorbed in the collision. We note that only the former photons (i.e. Bremsstrahlung) have a counterpart in the Yennie–Frautschi–Suura algorithm for inclusive cross-sections [38]. This suggests that the latter (i.e. absorbed) photons, while indispensable for the existence of the incoming scattering states, may be superfluous at the level of collision cross-sections. Thus the step from the (time-reversal invariant) S-matrix to the Yennie–Frautschi–Suura algorithm will surely be more than a purely computational exercise. It will also involve the choice of the arrow of time, akin to

the choice of boundary conditions in classical electrodynamics. For a more extensive qualitative discussion of this aspect (which will not be further discussed here) we refer to [6,2].

2. The model

The Hilbert space of the Nelson model is given by $\mathcal{H} = \mathcal{F}_e \otimes \mathcal{F}_{ph}$, where $\mathcal{F}_e, \mathcal{F}_{ph}$ are the Fock spaces of the electrons and photons with the creation and annihilation operators denoted $b^{(*)}$, $a^{(*)}$, respectively. The Hamiltonian of this model is given by

$$H := H_0 + V, \tag{2.1}$$

$$H_0 := \int d^3p \, \frac{p^2}{2} b^*(p)b(p) + \int d^3k \, |k| a^*(k)a(k), \tag{2.2}$$

$$V := \int d^3p \, d^3k \, v(k)\big(b^*(p+k)a(k)b(p) + \text{h.c.}\big), \quad v(k) := \lambda \frac{\chi_{[0,\kappa]}(|k|)}{\sqrt{2|k|}}, \tag{2.3}$$

where H_0 involves the free evolution of the electrons and photons, V is the interaction, κ is a fixed ultraviolet cut-off and $\chi_{[0,\kappa]}(|k|) = 1$ for $0 \leq |k| \leq \kappa$ and $\chi_{[0,\kappa]}(|k|) = 0$ otherwise. As the Fermi statistics and the spin degrees of freedom of the electron will not play any role in the following discussion, we suppress the latter in the notation. Also, as we are primarily interested in electron collisions, we treat all photons in the model as 'soft' and do not introduce any division of the range of photon energies $[0, \kappa]$ into a soft and hard part.

Since this Hamiltonian commutes with the total number N of electrons, we can consider the Hamiltonians $H^{(N)}$ on the N-electron subspace $\mathcal{H}^{(N)} := \mathcal{F}_e^{(N)} \otimes \mathcal{F}_{ph}$, given by

$$H^{(N)} = \sum_{\ell=1}^{N} \frac{(-i\nabla_{x_\ell})^2}{2} + \int d^3k \, |k| a^*(k)a(k) + \sum_{\ell=1}^{N} \int d^3k \, v(k) \, (e^{ikx_\ell}a(k) + e^{-ikx_\ell}a^*(k)), \tag{2.4}$$

where x_ℓ is the position operator of the ℓ-th electron and $\mathcal{F}_e^{(N)}$ is the N-particle subspace of $\mathcal{F}_e$. This quantum-mechanical representation will facilitate the application of the Dollard prescription in Section 3.

3. The Dollard formalism

Let us recall very briefly the basics of the Dollard approach to long-range scattering [10, 9]. We consider for a moment a quantum-mechanical particle moving in an external potential. Its Hamiltonian, acting on a Hilbert space $\tilde{\mathcal{H}}$, is given by $\tilde{H} = \tilde{H}_0 + \tilde{V}(x)$, where $\tilde{H}_0$ is the free Hamiltonian and the potential $\tilde{V}$ depends on the position of the particle.[2] It is well known that e.g. for the Coulomb potential the conventional wave-operators $\tilde{\Omega}_{conv}^{out/in} = \lim_{t\to+/-\infty} e^{it\tilde{H}}e^{-it\tilde{H}_0}$ do not make sense, as they converge weakly to zero. Dollard proposed to replace the free dynamics $e^{-it\tilde{H}_0}$ with a modified dynamics $e^{-it\tilde{H}_0}\tilde{U}^D(t)$, where the Dollard modifier $\tilde{U}^D(t)$ is constructed as follows:

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[2] We use tilde here to distinguish the quantum mechanical objects from the corresponding objects in our discussion of the Nelson model.

1. Define the time-dependent asymptotic potential $\tilde{V}^{\mathrm{as}}(t) := \tilde{V}(pt)$ by evaluating the potential $\tilde{V}(x)$ at the expected ballistic trajectory of the particle pt. Here p is the quantum-mechanical momentum operator, which can be treated as a number (i.e. a multiplication operator) if we work in the momentum representation of the particle. (We set the mass of the particle equal to one.)
2. Transform the asymptotic potential to the interaction picture $\tilde{V}^{\mathrm{as,I}}(t) := e^{it\tilde{H}_0}\tilde{V}(pt)e^{-it\tilde{H}_0}$.
3. Define the Dollard modifier as follows $\tilde{U}^{\mathrm{D}}(t) := \mathrm{T}\exp\left(-i\int_0^t d\tau\,\tilde{V}^{\mathrm{as,I}}(\tau)\right)$, where $\mathrm{T}\exp$ is the time-ordered exponential.

Now the Dollard wave-operators have the form $\tilde{\Omega}^{\mathrm{out/in}} = \lim_{t\to+/-\infty} e^{it\tilde{H}}e^{-it\tilde{H}_0}\tilde{U}^{\mathrm{D}}(t)$ and exist as strong limits of their approximating sequences for a large class of long-range potentials. For a physical justification of the above prescription one should note that for any $\tilde{\Psi}_0 \in \tilde{\mathcal{H}}$ the family of states $\tilde{\Psi}_t = e^{-it\tilde{H}_0}\tilde{U}^{\mathrm{D}}(t)\tilde{\Psi}_0$ satisfies

$$i\partial_t\tilde{\Psi}_t = (\tilde{H}_0 + \tilde{V}^{\mathrm{as}}(t))\tilde{\Psi}_t, \tag{3.1}$$

that is it evolves according to the asymptotic dynamics.

Let us now apply this formalism to the Nelson model defined in the previous section. Our starting point is the interaction V, which is given on $\mathcal{H}^{(N)}$ by

$$V = \sum_{\ell=1}^{N}\int d^3k\,v(k)\big(e^{-ikx_\ell}a^*(k) + e^{ikx_\ell}a(k)\big). \tag{3.2}$$

According to the Dollard prescription, we construct the asymptotic interaction as follows: We substitute $x_\ell \to \nabla E_{p_\ell}t$, where ∇E_{p_ℓ} is the velocity of the ℓ-th electron moving with momentum p_ℓ along the ballistic trajectory, as expected for asymptotic times. Thus we have

$$V_{\underline{p}}^{\mathrm{as}}(t) := \sum_{\ell=1}^{N}\int d^3k\,v(k)\left(e^{-ik\cdot\nabla E_{p_\ell}t}a^*(k) + e^{ik\cdot\nabla E_{p_\ell}t}a(k)\right), \tag{3.3}$$

where $\underline{p} := (p_1,\ldots,p_N)$ are momenta of the electrons, which are just numbers since we work in the momentum representation (cf. formula (3.7) below). As the physical dispersion relation of the electron is not $p \mapsto p^2/2$ appearing in H_0 but rather the lower boundary $p \mapsto E_p$ of the energy–momentum spectrum, we need to renormalize the free Hamiltonian and thus diverge slightly from the Dollard prescription outlined above. We define

$$H_0^{\mathrm{ren}} := \int d^3p\,(E_p - C_p)b^*(p)b(p) + \int d^3k\,|k|a^*(k)a(k), \quad C_p := \int d^3k\,\frac{v(k)^2}{\Omega_p(k)}. \tag{3.4}$$

Here $\Omega_p(k) := |k| - k\cdot\nabla E_p$ and the choice of the normalization constant C_p will be justified a posteriori in Section 5. (The need to renormalize the free Hamiltonian was noted already in [16].) Thus the asymptotic interaction in the interaction picture is

$$V_{\underline{p}}^{\mathrm{as,I}}(t) = e^{iH_0^{\mathrm{ren}}t}V_{\underline{p}}^{\mathrm{as}}(t)e^{-iH_0^{\mathrm{ren}}t}$$

$$= \sum_{\ell=1}^{N}\int d^3k\,v(k)\left(e^{i(|k|-k\cdot\nabla E_{p_\ell})t}a^*(k) + e^{-i(|k|-k\cdot\nabla E_{p_\ell})t}a(k)\right)$$

$$= \sum_{\ell=1}^{N} \int d^3k \, v(k) \left(e^{i\Omega_{p_\ell}(k)t} a^*(k) + e^{-i\Omega_{p_\ell}(k)t} a(k) \right). \tag{3.5}$$

Now we define the Dollard modifier

$$U_{\underline{p}}^{D}(t) := \mathrm{T} \exp \left(-i \int_0^t d\tau \, V_{\underline{p}}^{\mathrm{as,I}}(\tau) \right) = e^{-i \int_0^t d\tau \, V_{\underline{p}}^{\mathrm{as,I}}(\tau) - \frac{1}{2} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \, [V_{\underline{p}}^{\mathrm{as,I}}(\tau_1), V_{\underline{p}}^{\mathrm{as,I}}(\tau_2)]},$$
$$\tag{3.6}$$

where the second step above is standard [14]. For any family of functions $h_\ell \in C_0^\infty(\mathbb{R}^3)$, $\ell = 1, \ldots, N$, of the electron momenta we define the corresponding scattering state approximant as follows:

$$\Psi_{h,t} = e^{iHt} e^{-iH_0^{\mathrm{ren}}t} \int d^3 p_1 \ldots d^3 p_N \, U_{\underline{p}}^{D}(t) h_1(p_1) \ldots h_N(p_N) b^*(p_1) \ldots b^*(p_N) |0\rangle$$

$$= e^{iHt} e^{-iH_0^{\mathrm{ren}}t} \int d^{3N} \underline{p} \, U_{\underline{p}}^{D}(t) h(\underline{p}) b^*(\underline{p})^N |0\rangle, \tag{3.7}$$

where in the second step we introduced some obvious short-hand notation. We note that all the quantities above are well defined without infrared regularization. But a need for infrared regularization will arise in the next subsection, where we start reformulating states (3.7) in terms of the LSZ asymptotic creation operators of photons and electrons. Their approximating sequences are given schematically by

$$t \mapsto e^{iHt} \left(e^{-i|k|t} a^*(k) \right) e^{-iHt}, \quad t \mapsto e^{iHt} \left(e^{-iE_p t} b^*(p) \right) e^{-iHt}. \tag{3.8}$$

As we will see in (6.6)–(6.7) below, $b^*(p)$ will actually require renormalization.

To conclude this section, we define the wave-operators $\Omega^{\mathrm{in/out}} : \mathcal{F}_e \to \mathcal{H}$ for the electron scattering as follows

$$\Omega^{\mathrm{in/out}} \left(\int d^{3N} \underline{p} \, h(\underline{p}) b^*(\underline{p})^N |0\rangle \right) = \lim_{t \to -/+\infty} e^{iHt} e^{-iH_0^{\mathrm{ren}}t} \int d^{3N} \underline{p} \, U_{\underline{p}}^{D}(t) h(\underline{p}) b^*(\underline{p})^N |0\rangle \tag{3.9}$$

so that the corresponding scattering matrix $S := (\Omega^{\mathrm{out}})^* \Omega^{\mathrm{in}}$ is an operator on $\mathcal{F}_e$. The existence of the limit in (3.9) is not settled, but seems to be a feasible functional-analytic problem, as we discuss in Section 6.

4. Infrared regularization

Let us consider the exponential in the Dollard modifier (3.6) and perform the time integral

$$-i \int_0^t d\tau \, V_{\underline{p}}^{\mathrm{as,I}}(\tau) = (-i) \sum_{\ell=1}^{N} \int d^3k \, v(k) \left(\frac{(e^{i\Omega_{p_\ell}(k)t} - 1)}{i\Omega_{p_\ell}(k)} a^*(k) + \frac{(e^{-i\Omega_{p_\ell}(k)t} - 1)}{(-i)\Omega_{p_\ell}(k)} a(k) \right) \tag{4.1}$$

$$= \sum_{\ell=1}^{N} \int d^3k \, \frac{v(k)}{\Omega_{p_\ell}(k)} \left(a^*(k) - a(k) \right) \tag{4.2}$$

$$- \sum_{\ell'=1}^{N} \int d^3k \, \frac{v(k)}{\Omega_{p_{\ell'}}(k)} \left(e^{i\Omega_{p_{\ell'}}(k)t} a^*(k) - e^{-i\Omega_{p_{\ell'}}(k)t} a(k) \right). \tag{4.3}$$

Since the l.h.s. of (4.1) is manifestly infrared finite, the same is true for the r.h.s. of this expression. However, terms (4.2) and (4.3) considered separately, coming from the lower and upper boundary of the τ-integration, are infrared singular. Indeed, they involve $a^{(*)}(k)$ integrated with functions which have a non-square-integrable singularity at zero momentum. This division of a regular expression into two singular parts, which will be needed to express the approximating vector (3.7) in the LSZ fashion, is the source of infrared divergencies, which must mutually cancel. As we pointed out above, in the work of Faddeev and Kulish [14] the counterpart of (4.2) is omitted.

To make sense out of (4.2) and (4.3), we need some infrared regularization of (3.7). To this end, we introduce an infrared cut-off $\sigma > 0$ and define a regularized version of the form factor from (2.3)

$$v^\sigma(k) := \lambda \frac{\chi_{[\sigma,\kappa]}(|k|)}{\sqrt{2|k|}}, \tag{4.4}$$

where $\chi_{[\sigma,\kappa]}(|k|) = 1$ for $\sigma \le |k| \le \kappa$ and $\chi_{[\sigma,\kappa]}(|k|) = 0$ otherwise. The corresponding potential and Hamiltonians are denoted V_σ, H_σ, $H_\sigma^{(N)}$ and $p \mapsto E_{p,\sigma}$ is the resulting dispersion relation of the electron. Next, we define the regularized approximating sequence analogously as in the previous section

$$\Psi_{h,t}^\sigma = e^{iHt} e^{-iH_{0;\sigma}^{\mathrm{ren}}t} \int d^{3N} \underline{p} \, U_{\underline{p},\sigma}^{\mathrm{D}}(t) h(\underline{p}) b^*(\underline{p})^N |0\rangle, \tag{4.5}$$

with the help of the regularized quantities:

$$H_{0;\sigma}^{\mathrm{ren}} := \int d^3 p \, (E_{p,\sigma} - C_{p,\sigma}) b^*(p) b(p) + \int d^3 k \, |k| a^*(k) a(k), \tag{4.6}$$

$$U_{\underline{p},\sigma}^{\mathrm{D}}(t) := \mathrm{T} \exp\left(-i \int_0^t d\tau \, V_{\underline{p},\sigma}^{\mathrm{as,I}}(\tau)\right) = e^{-i \int_0^t d\tau V_{\underline{p},\sigma}^{\mathrm{as,I}}(\tau) - \frac{1}{2} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 [V_{\underline{p},\sigma}^{\mathrm{as,I}}(\tau_1), V_{\underline{p},\sigma}^{\mathrm{as,I}}(\tau_2)]}, \tag{4.7}$$

$$V_{\underline{p},\sigma}^{\mathrm{as,I}}(t) = \sum_{\ell=1}^N \int d^3 k \, v^\sigma(k) \left(e^{i\Omega_{p_\ell,\sigma}(k)t} a^*(k) + e^{-i\Omega_{p_\ell,\sigma}(k)t} a(k)\right), \tag{4.8}$$

where $C_{p,\sigma} := \int d^3 k \, \frac{v^\sigma(k)^2}{\Omega_{p,\sigma}(k)}$ and $\Omega_{p,\sigma}(k) := |k| - k \cdot \nabla E_{p,\sigma}$. In this situation we have, analogously as in (4.2)–(4.3),

$$-i \int_0^t d\tau \, V_{\underline{p},\sigma}^{\mathrm{as,I}}(\tau) = \sum_{\ell=1}^N \int d^3 k \, \frac{v^\sigma(k)}{\Omega_{p_\ell,\sigma}(k)} \left(a^*(k) - a(k)\right) \tag{4.9}$$

$$- \sum_{\ell'=1}^N \int d^3 k \, \frac{v^\sigma(k)}{\Omega_{p_{\ell'},\sigma}(k)} \left(e^{i\Omega_{p_{\ell'},\sigma}(k)t} a^*(k) - e^{-i\Omega_{p_{\ell'},\sigma}(k)t} a(k)\right), \tag{4.10}$$

but the two terms (4.9) and (4.10) above are now well defined and can be analyzed separately. By a straightforward computation using the Baker–Campbell–Hausdorff formula, we thus obtain from (4.5)

$$\Psi_{h,t}^\sigma = e^{iHt} e^{-iH_{0;\sigma}^{\mathrm{ren}}t} \int d^{3N} \underline{p} \, e^{i\gamma_{\underline{p},\sigma}(t)} e^{-\theta_{\underline{p},\sigma}(t)} \prod_{\ell=1}^N \left(e^{iC_{p_\ell,\sigma}t} e^{\int d^3 k \, \frac{v^\sigma(k)}{\Omega_{p_\ell,\sigma}(k)} (a^*(k) - a(k))}\right) \times$$

$$\times \prod_{\ell'=1}^{N} \left(e^{-D_{p_{\ell'},\sigma}} e^{-\int d^3k \frac{v^\sigma(k)}{\Omega_{p_{\ell'},\sigma}(k)} e^{i\Omega_{p_{\ell'},\sigma}(k)t} a^*(k)} \right) h(\underline{p}) b^*(\underline{p})^N |0\rangle, \tag{4.11}$$

where $C_{p,\sigma}$ appeared below (4.8) and $D_{p,\sigma} := \frac{1}{2}\int d^3k \frac{v^\sigma(k)^2}{\Omega_{p,\sigma}(k)^2}$. The real-valued numerical functions $\gamma_{\underline{p},\sigma}, \theta_{\underline{p},\sigma}$ are stated in (5.9)–(5.12) below and will be discussed later.

5. Clouds of real and virtual photons, phases

Now we rewrite formula (4.11) in the LSZ fashion to facilitate its interpretation in terms of real and virtual photon clouds. By shifting the term $e^{-iH_{0;\sigma}^{\text{ren}}t}$ to the right and noting the cancellation of the constants $C_{p_\ell,\sigma}$ (cf. (4.6)) we get

$$\Psi_{h,t}^\sigma = e^{iHt} \int d^{3N}\underline{p}\, e^{i\gamma_{\underline{p},\sigma}(t)} e^{-\theta_{\underline{p},\sigma}(t)} \prod_{\ell=1}^{N} \left(e^{\int d^3k \frac{v^\sigma(k)}{\Omega_{p_\ell,\sigma}(k)} (e^{-i|k|t}a^*(k) - e^{i|k|t}a(k))} \right) \times \tag{5.1}$$

$$\times \prod_{\ell'=1}^{N} \left(e^{-D_{p_{\ell'},\sigma}} e^{-\int d^3k \frac{v^\sigma(k)}{\Omega_{p_{\ell'},\sigma}(k)} e^{-ik\cdot\nabla E_{p_{\ell'},\sigma}t} a^*(k)} \right) h_t(\underline{p}) b^*(\underline{p})^N |0\rangle, \tag{5.2}$$

where $h_t(\underline{p}) := \prod_{\ell=1}^{N} \left(e^{-iE_{p_\ell,\sigma}t} h_\ell(p_\ell) \right)$ is the (renormalized) free evolution of h.

In the bracket in (5.1) we recognize the LSZ approximants of the clouds of real photons. For future reference we set

$$\mathcal{W}_{p,\sigma}(t) := e^{\int d^3k \frac{v^\sigma(k)}{\Omega_{p,\sigma}(k)} (e^{-i|k|t}a^*(k) - e^{i|k|t}a(k))}. \tag{5.3}$$

It is more difficult to recast the expression in (5.2) as LSZ approximants pertaining to the electrons. For this purpose *we reverse the Dollard prescription* in the expression $e^{-ik\cdot\nabla E_{p_{\ell'},\sigma}t}$ in (5.2) that is we make a substitution $e^{-ik\cdot\nabla E_{p_{\ell'},\sigma}t} \to e^{-ik\cdot x_\ell}$. This leads us to a new family of approximating vectors

$$\tilde{\Psi}_{h,t}^\sigma = e^{iHt} \int d^{3N}\underline{p}\, e^{i\gamma_{\underline{p},\sigma}(t)} e^{-\theta_{\underline{p},\sigma}(t)} \left(\prod_{\ell=1}^{N} \mathcal{W}_{p_\ell,\sigma}(t) \right) \times$$

$$\times \left(\prod_{\ell'=1}^{N} e^{-D_{p_{\ell'},\sigma}} e^{-\int d^3k \frac{v^\sigma(k)}{\Omega_{p_{\ell'},\sigma}(k)} e^{-ik\cdot x_{\ell'}t} a^*(k)} \right) h_t(\underline{p}) b^*(\underline{p})^N |0\rangle. \tag{5.4}$$

Although we do not have a rigorous proof that $\lim_{t\to\infty} \|\Psi_{h,t}^\sigma - \tilde{\Psi}_{h,t}^\sigma\| = 0$, it is intuitively clear, that the position x of the freely evolving electron behaves asymptotically as $\nabla E_{p,\sigma}t$. (As a matter of fact, a similar substitution is used in the rigorous work of Pizzo, which we discuss in Section 6.) To simplify (5.4), we define the following (tentative) renormalized creation operator of the electron

$$\tilde{b}_\sigma^*(p) := \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \int d^{3m}k\, \tilde{f}_{p,\sigma}^m(k_1,\ldots,k_m) a^*(k_1)\ldots a^*(k_m) b^*(p - \underline{k}^{(m)}), \tag{5.5}$$

$$\tilde{f}_{p,\sigma}^m(k_1,\ldots,k_m) := (-1)^m e^{-D_{p,\sigma}} \frac{v^\sigma(k_1)}{\Omega_{p,\sigma}(k_1)} \cdots \frac{v^\sigma(k_m)}{\Omega_{p,\sigma}(k_m)}, \tag{5.6}$$

where $\underline{k}^{(m)} = k_1 + \cdots + k_m$. Using $e^{-ik\cdot x}b^*(p)|0\rangle = b^*(p-k)|0\rangle$, it is then easy to show that

$$e^{-D_{p,\sigma}}\left(e^{-\int d^3k \frac{v^\sigma(k)}{\Omega_{p,\sigma}(k)} e^{-ik\cdot x} a^*(k)} \right) b^*(p)|0\rangle = \tilde{b}^*_\sigma(p)|0\rangle. \tag{5.7}$$

Thus, intuitively, $\tilde{b}^*_\sigma(p)$ creates from the vacuum the electron with its cloud of virtual photons.

Consequently, we can rewrite (5.4) in the LSZ form:

$$\tilde{\Psi}^\sigma_{h,t} = e^{iHt}\int d^{3N}\underline{p}\, e^{i\gamma_{\underline{p},\sigma}(t)} e^{-\theta_{\underline{p},\sigma}(t)}\left(\prod_{\ell=1}^N \mathcal{W}_{p_\ell,\sigma}(t) \right)\left(\prod_{\ell'=1}^N e^{-iE_{p_{\ell'},\sigma}t} h_{\ell'}(p_{\ell'})\tilde{b}^*_\sigma(p_{\ell'}) \right)|0\rangle. \tag{5.8}$$

The real-valued functions $\gamma_{\underline{p},\sigma}$ and $\theta_{\underline{p},\sigma}$, appearing above, have the following explicit form

$$\gamma_{\underline{p},\sigma}(t) := \gamma_{1;\underline{p},\sigma}(t) + \gamma_{2;\underline{p},\sigma}(t),$$

$$\gamma_{1;\underline{p},\sigma}(t) := -2\sum_{\ell=1}^N \int d^3k\, v^\sigma(k)^2 \frac{\sin\Omega_{p_\ell,\sigma}(k)t}{\Omega^2_{p_\ell,\sigma}(k)}, \tag{5.9}$$

$$\gamma_{2;\underline{p},\sigma}(t) := -2\sum_{\ell<\ell'} \int d^3k\, v^\sigma(k)^2 \frac{(\sin\Omega_{p_{\ell'},\sigma}(k)t + \sin\Omega_{p_\ell,\sigma}(k)t)}{\Omega_{p_\ell,\sigma}(k)\Omega_{p_{\ell'},\sigma}(k)} \tag{5.10}$$

$$+ \sum_{\ell<\ell'} \int d^3k\, v^\sigma(k)^2 \left(\frac{1}{\Omega_{p_\ell,\sigma}(k)} + \frac{1}{\Omega_{p_{\ell'},\sigma}(k)} \right) \frac{\sin(\Omega_{p_\ell,\sigma}(k) - \Omega_{p_{\ell'},\sigma}(k))t}{(\Omega_{p_\ell,\sigma}(k) - \Omega_{p_{\ell'},\sigma}(k))}, \tag{5.11}$$

$$\theta_{\underline{p},\sigma}(t) := \sum_{\ell<\ell'} \int d^3k\, v^\sigma(k)^2 \frac{\cos(\Omega_{p_{\ell'},\sigma}(k) - \Omega_{p_\ell,\sigma}(k))t}{\Omega_{p_\ell,\sigma}(k)\Omega_{p_{\ell'},\sigma}(k)}. \tag{5.12}$$

Recalling that $\Omega_{p,\sigma}(k) = |k| - \nabla E_{p,\sigma}\cdot k$ and therefore $\Omega_{p_\ell,\sigma}(k) - \Omega_{p_{\ell'},\sigma}(k) = (\nabla E_{p_{\ell'},\sigma} - \nabla E_{p_\ell,\sigma})\cdot k$ we expect that the above contributions facilitate the asymptotic decoupling between the following particles:

- (5.9): the ℓ-th electron and a photon from the ℓ-th cloud.
- (5.10): the ℓ-th electron and a photon from the ℓ'-th cloud (and vice versa).
- (5.11), (5.12): the ℓ-th electron and the ℓ'-th electron.

Expression (5.11) corresponds to the Coulomb phase and it is easy to show that it behaves as $\log t$ for large t and $\sigma = 0$. The remaining terms do not have counterparts in many-body quantum mechanical scattering.

6. Comparison with a rigorous LSZ approach

For $N = 1$ formula (5.8) is very similar to the single-electron state approximants obtained by Pizzo in [31]. To obtain these latter states from (5.8) one has to make the following modifications:

1. **Cell partition:** The region of p-integration in (5.8) has to be divided into time-dependent cubes. Suppose, for convenience, that this region is a cube of volume equal to one, centered at zero. At time $1 \le |t|$ the linear dimension of each cell is $1/2^{\bar{n}}$, where $\bar{n} \in \mathbb{N}$ is s.t.

$$(2^{\overline{n}})^{1/\overline{\varepsilon}} \leq |t| < (2^{\overline{n}+1})^{1/\overline{\varepsilon}} \tag{6.1}$$

for a small exponent $\overline{\varepsilon} > 0$. Thus there are $2^{3\overline{n}} \leq |t|^{3\overline{\varepsilon}}$ cells. Each such cell is denoted $\Gamma_j^{(t)}$ and the collection of all cells $\Gamma^{(t)}$.

2. **Photon clouds:** The photon cloud $\mathcal{W}_{p,\sigma}(t)$ from (5.8) should be replaced with the cloud $\mathcal{W}_\sigma(v_j, t)$, defined in (6.3) below, associated with the cube $\Gamma_j^{(t)}$ containing p and depending on the velocity $v_j := \nabla E_{p_j,\sigma}$ in the center of the cube $\Gamma_j^{(t)}$. Thus one makes the following substitution

$$\mathcal{W}_{p,\sigma}(t) := \exp\left\{ -\int d^3k \, v^\sigma(k) \frac{a(k)e^{i|k|t} - a^*(k)e^{-i|k|t}}{|k|(1 - \hat{k} \cdot \nabla E_{p,\sigma})} \right\} \tag{6.2}$$

$$\downarrow$$

$$\mathcal{W}_\sigma(v_j, t) := \exp\left\{ -\int d^3k \, v^\sigma(k) \frac{a(k)e^{i|k|t} - a^*(k)e^{-i|k|t}}{|k|(1 - \hat{k} \cdot v_j)} \right\}, \tag{6.3}$$

where $v_j := \nabla E_{p_j,\sigma}$ is the velocity in the center of the cube $\Gamma_j^{(t)}$ and $\hat{k} := k/|k|$. Clearly, the difference $|\nabla E_{p,\sigma} - v_j|$ tends to zero as $t \to \infty$ and the size of each cube $\Gamma^{(t)}$ shrinks to zero, so it should not be difficult to justify this substitution.

3. **Phases:** The phase $\gamma_{p,\sigma}(t)$ from (5.8) should be replaced with the phase defined in (6.5) below. Thus in view of (5.9) and the definition above $\Omega_{p,\sigma}(k) := |k| - k \cdot \nabla E_{p,\sigma}$, we make the substitution

$$\gamma_{p,\sigma}(t) = -\int_0^t d\tau \left\{ \int_{0 \leq |k|} d|k| d\omega(\hat{k}) \, v^\sigma(k)^2 (2|k|) \left(\frac{\cos(k \cdot \nabla E_{p,\sigma} \tau - |k|\tau)}{1 - \hat{k} \cdot \nabla E_{p,\sigma}} \right) \right\} \tag{6.4}$$

$$\downarrow$$

$$\gamma_\sigma(v_j, t)(p) = -\int_1^t d\tau \left\{ \int_{0 \leq |k| \leq \sigma_\tau^S} d|k| d\omega(\hat{k}) \, v^\sigma(k)^2 (2|k|) \left(\frac{\cos(k \cdot \nabla E_{p,\sigma} \tau - |k|\tau)}{1 - \hat{k} \cdot v_j} \right) \right\}, \tag{6.5}$$

where $d\omega(\hat{k}) := \sin\theta_{\hat{k}} d\theta_{\hat{k}} d\phi_{\hat{k}}$ is the measure on the unit sphere, and $\tau \mapsto \sigma_\tau^S = \kappa \tau^{-\alpha}$, $1/2 < \alpha < 1$, is the slow infrared cut-off. (As stated in 5. below, the cut-off σ will tend to zero with t much faster.) Since the region of momenta $|k| \geq \sigma_\tau^S$ affected by the above change is well separated from the infrared singularity, it is easy to justify the above step using stationary phase arguments.

4. **Renormalized creation operators:** The tentative renormalized creation operator of the electron (5.5)–(5.6) should be replaced with the actual renormalized creation operator, given by (6.7) below. That is, we make the following replacement:

$$\tilde{b}_\sigma^*(p) := \sum_{m=0}^\infty \frac{1}{\sqrt{m!}} \int d^{3m}k \, \tilde{f}_{p,\sigma}^m(k_1, \ldots, k_m) a^*(k_1) \ldots a^*(k_m) b^*(p - \underline{k}^{(m)}), \tag{6.6}$$

$$\downarrow$$

$$\hat{b}_\sigma^*(p) := \sum_{m=0}^\infty \frac{1}{\sqrt{m!}} \int d^{3m}k \, f_{p,\sigma}^m(k_1, \ldots, k_m) a^*(k_1) \ldots a^*(k_m) b^*(p - \underline{k}^{(m)}), \tag{6.7}$$

where the functions $\tilde{f}^m_{p,\sigma}$ are given by (5.6) and $f^m_{p,\sigma}$ are the wave-functions of the normalized ground states $\psi_{p,\sigma}$ of the fiber Hamiltonians $H_{p,\sigma}$. These latter Hamiltonians are defined via the direct integral decomposition

$$H^{(1)}_\sigma = \Pi^* \left(\int^\oplus d^3 p \, H_{p,\sigma} \right) \Pi, \tag{6.8}$$

where Π is a certain unitary identification of Hilbert spaces and $H_{p,\sigma}$ is a concrete operator on an auxiliary fiber Fock space $\mathcal{F}_{\text{fi}}$. The key property of the operator (6.7) is that it creates a freely-evolving physical electron from the vacuum (at fixed $\sigma > 0$), i.e.

$$e^{iH_\sigma t} \int d^3 p \, h(p) \hat{b}^*(p) |0\rangle = \Pi^* \int^\oplus d^3 p \, e^{-it E_{p,\sigma} t} h(p) \psi_{p,\sigma}. \tag{6.9}$$

Starting from [13, formula (4.43)], [12, formula (5.2)] and using methods from these references one can show that

$$f^m_{p,\sigma}(k_1, \ldots, k_m) = \tilde{f}^m_{p,\sigma}(k_1, \ldots, k_m) + \cdots, \tag{6.10}$$

where the omitted terms are either of order λ or more regular near zero than $\tilde{f}^m_{p,\sigma}$, at least in some variables k_i. Thus in the weak coupling regime $\tilde{f}^m_{p,\sigma}$ captures the leading part of the infrared singularity of $f^m_{p,\sigma}$. Further analysis in this direction is needed to justify the substitution (6.6) → (6.7), which takes correlations between the virtual photons dressing the electron into account.

5. **Fast infrared cut-off:** The infrared cut-off σ appearing in (5.8) should be removed in the limit $t \to \infty$. More precisely, one sets

$$\sigma \to \sigma_t := 1/t^\beta, \tag{6.11}$$

for $\beta \geq 1$ sufficiently large.

After the above changes, we obtain from (5.8) the following approximating sequence

$$\hat{\Psi}_{h,t} := e^{iHt} \sum_{j \in \Gamma^{(t)}} \mathcal{W}_{\sigma_t}(v_j, t) \int_{\Gamma^{(t)}_j} d^3 p \, e^{-iE_{p,\sigma_t} t} e^{i\gamma_{\sigma_t}(v_j, t)(p)} h(p) \hat{b}^*_{\sigma_t}(p) |0\rangle. \tag{6.12}$$

It was rigorously proven by Pizzo in [31] that the outgoing and incoming single-electron states $\hat{\Psi}^{\text{in/out}}_h := \lim_{t \to -/+\infty} \hat{\Psi}_{h,t}$ exist and are non-zero.

Given the above considerations, there is hope for proving convergence of the Faddeev–Kulish type approximating sequence (3.7) in the single-electron case by estimating the norm distance to the Pizzo state (6.12). The most difficult parts will be the partial reversal of the Dollard prescription (5.2) → (5.4) and the step from the tentative to the actual renormalized creation operator of the electron (6.6) → (6.7). A more ambitious strategy consists in proving the existence of the limit of (3.6) directly, e.g. via an application of the Cook's method. Also here it seems necessary to make contact with the renormalized creation operator $\hat{b}^*(p)$, in order to exploit the key property (6.9). We hope to come back to these problems in future publications.

So far there is no counterpart of the result of Pizzo for two or more electrons. Actually, it is not even clear how the approximating sequence (6.12) should look like in this case. As scattering of two electrons in the Nelson model is currently under investigation [11–13], it is worth pointing

out that the Faddeev–Kulish type analysis from previous sections gives a reasonable candidate. In fact, let us simply apply the modifications 1.–5. listed above to the approximating vector (5.8) in the case $N = 2$. We obtain

$$\hat{\Psi}^{(2)}_{h,t} := e^{iHt} \sum_{j_1, j_2 \in \Gamma^{(t)}} \mathcal{W}_{\sigma_t}(v_{j_1}, t) \mathcal{W}_{\sigma_t}(v_{j_2}, t) \int_{\Gamma^{(t)}_{j_1} \times \Gamma^{(t)}_{j_2}} d^3 p_1 d^3 p_2 \, e^{i\gamma_{2;\underline{p},\sigma_t}(t)} e^{-\theta_{\underline{p},\sigma_t}(t)} \times$$

$$\times \left(e^{-i E_{p_1,\sigma_t} t} e^{i\gamma_{\sigma_t}(v_{j_1}, t)(p_1)} h_1(p_1) \hat{b}^*_{\sigma_t}(p_1) \right) \left(e^{-i E_{p_2,\sigma_t} t} e^{i\gamma_{\sigma_t}(v_{j_2}, t)(p_2)} h_2(p_2) \hat{b}^*_{\sigma_t}(p_2) \right) |0\rangle, \tag{6.13}$$

where $\gamma_{2;\underline{p},\sigma}$, $\theta_{\underline{p},\sigma}$ are given by (5.10)–(5.12) and may require some small modifications, akin to (6.4)→(6.5). We are confident that the above observations will facilitate mathematically rigorous research on scattering of two electrons in the Nelson model.

If the existence of the scattering states $\hat{\Psi}^{(2),\mathrm{out/in}}_h := \lim_{t \to +/-\infty} \hat{\Psi}^{(2)}_{h,t}$ can be established, it will constitute an LSZ-type asymptotic condition for the two-electron scattering in the Nelson model. It is a natural question, how to derive the LSZ reduction formula for the corresponding S-matrix elements $S_{h',h} := \langle \hat{\Psi}^{(2),\mathrm{out}}_{h'}, \hat{\Psi}^{(2),\mathrm{in}}_h \rangle$. Although this question, and a comparison of the result with the Yennie–Frautschi–Suura formula, is outside the scope of the present paper, let us provide some comments in this direction: We define the relevant LSZ approximants, similarly as in (3.8):

$$\hat{b}^*_j(t) := e^{itH} \left(\int d^3 p \, e^{-i E_p t} h_j(p) \hat{b}^*_{\sigma_t}(p) \right) e^{-itH}, \quad \mathcal{W}_j(t) := e^{itH} \mathcal{W}_{\sigma_t}(v_j, t) e^{-itH}, \tag{6.14}$$

where h_j is the restriction of h to the cube j and we disregard here the time-dependence of the cubes. The matrix element $S_{h',h}$ can be expressed in terms of the expectation values of the form

$$\langle 0| \hat{b}_{j'_2}(t) \hat{b}_{j'_1}(t) \mathcal{W}_{j'_2}(t)^* \mathcal{W}_{j'_1}(t)^* \mathcal{W}_{j_1}(-t) \mathcal{W}_{j_2}(-t) \hat{b}^*_{j_1}(-t) \hat{b}^*_{j_2}(-t) |0\rangle, \tag{6.15}$$

for large $t > 0$. Next, for any pair of operators C_1, C_2 one considers functions $\tau \mapsto F_{j,\tau}$ which satisfy (cf. [3, Lemma 5.7]

$$F_{j,\tau} = \begin{cases} C_1 \hat{b}^*_j(\tau) & \text{for } \tau \ll -1, \\ \hat{b}^*_j(\tau) C_2 & \text{for } t \gg 1. \end{cases} \tag{6.16}$$

Then one notes the identity

$$\int_{-t}^{t} d\tau \, \partial_\tau F_{j,\tau} = \hat{b}^*_j(t) C_2 - C_1 \hat{b}^*_j(-t). \tag{6.17}$$

(Of course an analogous relation can be written for the photon fields $\mathcal{W}_j(t)$.) Such identities allow us to replace the incoming LSZ approximants in (6.15) with the outgoing ones at a cost of time-ordered products of the photon and electron fields. The resulting scalar product of two outgoing scattering states corresponds to the trivial part of the scattering matrix and the expression involving time-ordered products encodes the interaction. Clearly, the above formulas differ in some respects from the original LSZ formalism, as the asymptotic condition is adapted to the non-relativistic model. In particular, we note the absence of the typical $\overset{\leftrightarrow}{\partial}_0$ expression in (6.14) and of the Klein–Gordon operator in (6.17). More importantly, the 'restriction to the electron's

mass-shell' is a subtle two-step procedure in the infraparticle case: First, at a non-zero infrared cut-off $\sigma_t > 0$, the renormalized creation operator $\hat{b}^*_{\sigma_t}(p)$ creates a physical single-electron state according to (6.9). Next, the infrared singularity appearing in the limit $\sigma_t \to 0$ (i.e. $t \to \infty$) must be cancelled by the clouds of real photons $\mathcal{W}_j(t)$. Surely, it will still require a considerable amount of work and insight to turn the above remarks into convincing computations, but there is little doubt that the LSZ approach can be adapted to scattering of several infraparticles.

7. Conclusion

In this paper we revisited the Faddeev–Kulish approach to the electron scattering in the context of the massless Nelson model. In contrast to the original paper of Faddeev and Kulish, we applied the Dollard formalism according to the rules of the art, without dropping the lower boundary of integration. This led us to a scattering matrix which is meaningful on the usual Fock space of free electrons, but does not commute with the total electron momentum. This latter point was clarified in the later part of our analysis, where we reformulated this scattering matrix in the LSZ terms: The lower boundary of integration gives rise to clouds of real photons which always carry some momentum. Furthermore, we checked that the resulting LSZ formula at the one-electron level reproduces single-electron states constructed rigorously by Pizzo, up to minor technical differences. Our observations provide clear-cut mathematical conjectures, which will facilitate rigorous research of N-electron scattering in the massless Nelson model. Our findings may also provide a more solid basis for heuristic discussions of scattering theory in QED, which is a popular topic in current physics literature.

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References

[1] A. Abdesselam, D. Hasler, Analyticity of the ground state energy for massless Nelson models, Commun. Math. Phys. 310 (2012) 511–536.

[2] S. Alazzawi, W. Dybalski, Compton scattering in the Buchholz–Roberts framework of relativistic QED, Lett. Math. Phys. 107 (2017) 81–106.

[3] H. Araki, Mathematical Theory of Quantum Fields, Oxford University Press, 1999.

[4] E. Bagan, M. Lavelle, D. McMullan, Charges from dressed matter: construction, Ann. Phys. 282 (2000) 471–502.

[5] D. Buchholz, Gauss' law and the infraparticle problem, Phys. Lett. B 174 (1986) 331–334.

[6] D. Buchholz, J.E. Roberts, New light on infrared problems: sectors, statistics, symmetries and spectrum, Commun. Math. Phys. 330 (2014) 935–972.

[7] T. Chen, J. Fröhlich, A. Pizzo, Infraparticle scattering states in non-relativistic QED: I. The Bloch–Nordsieck paradigm, Commun. Math. Phys. 294 (2010) 761–825.

[8] T. Chen, J. Fröhlich, A. Pizzo, Infraparticle scattering states in non-relativistic QED: II. Mass shell properties, J. Math. Phys. 50 (2009) 012103.

[9] J. Dereziński, C. Gérard, Scattering Theory of Classical and Quantum N-Particle Systems, Springer, 1997.

[10] J.D. Dollard, Asymptotic convergence and the Coulomb interaction, J. Math. Phys. 5 (1964) 729–738.
[11] W. Dybalski, A. Pizzo, Coulomb scattering in the massless Nelson model I. Foundations of two-electron scattering, J. Stat. Phys. 154 (2014) 543–587.
[12] W. Dybalski, A. Pizzo, Coulomb scattering in the massless Nelson model II. Regularity of ground states, Preprint, arXiv:1302.5012.
[13] W. Dybalski, A. Pizzo, Coulomb scattering in the massless Nelson model III. Ground state wave-functions and non-commutative recurrence relations, Preprint, arXiv:1704.02924v2.
[14] L.D. Faddeev, P.P. Kulish, Asymptotic conditions and infrared divergencies in quantum electrodynamics, Theor. Math. Phys. 5 (1970) 153–166.
[15] R. Ferrari, L.E. Picasso, F. Strocchi, Some remarks on local operators in Quantum Electrodynamics, Commun. Math. Phys. 35 (1974) 25–38.
[16] J. Fröhlich, On the infrared problem in a model of scalar electrons and massless, scalar bosons, Ann. Inst. Henry Poincaré A (N.S.) 19 (1973) 1–103.
[17] J. Fröhlich, Existence of dressed one electron states in a class of persistent models, Fortschr. Phys. 22 (1974) 158–198.
[18] J. Fröhlich, G. Morchio, F. Strocchi, Charged sectors and scattering states in quantum electrodynamics, Ann. Phys. 119 (1979) 241–284.
[19] J. Fröhlich, A. Pizzo, Renormalized electron mass in non-relativistic QED, Commun. Math. Phys. 294 (2010) 439–470.
[20] B. Gabai, A. Sever, Large gauge symmetries and asymptotic states in QED, J. High Energy Phys. 12 (2016) 095.
[21] C. Gomez, R. Letschka, Memory of the infrared, Preprint, arXiv:1704.03395.
[22] C. Gomez, M. Panchenko, Asymptotic dynamics, large gauge transformations and infrared symmetries, Preprint, arXiv:1608.05630.
[23] R. Haag, Quantum field theories with composite particles and asymptotic conditions, Phys. Rev. 112 (1958) 669–675.
[24] D. Hasler, I. Herbst, Absence of ground states for a class of translation invariant models of non-relativistic QED, Commun. Math. Phys. 279 (2008) 769–787.
[25] D. Kapec, M. Perry, A.-M. Raclariu, A. Strominger, Infrared divergencies in QED, revisited, Preprint, arXiv: 1705.04311.
[26] T.W.B. Kibble, Coherent soft-photon states and infrared divergences. IV. The scattering operator, Phys. Rev. 175 (1968) 1624–1640.
[27] H. Lehmann, K. Symanzik, W. Zimmermann, Zur Formulierung quantisierter Feldtheorien, Nuovo Cimento 1 (1955) 205–225.
[28] M. Mirbabayi, M. Porrati, Shaving off black hole soft hair, Preprint, arXiv:1607.03120.
[29] G. Morchio, S. Strocchi, The infrared problem in QED: a lesson from a model with Coulomb interaction and realistic photon emission, Ann. Henri Poincaré 17 (2016) 2699–2739.
[30] M. Panchenko, The infrared triangle in the context of IR safe S matrices, Preprint, arXiv:1704.03739.
[31] A. Pizzo, Scattering of an infraparticle: the one particle sector in Nelson's massless models, Ann. Henri Poincaré 4 (2005) 553–606.
[32] A. Pizzo, One-particle (improper) states in Nelson's massless model, Ann. Henri Poincaré 4 (2003) 439–486.
[33] L. Polley, Modelle zum Infrarotproblem in der Quantenelektrodynamik, PhD thesis, University of Würzburg, 1980.
[34] D. Ruelle, On the asymptotic condition in quantum field theory, Helv. Phys. Acta 35 (1962) 147–163.
[35] A. Strominger, Lectures on the infrared structure of gravity and gauge theory, Preprint, arXiv:1703.05448.
[36] B. Schroer, Infrateilchen in der Quantenfeldtheorie, Fortschr. Phys. 11 (1963) 1–31.
[37] O. Steinmann, Perturbative Quantum Electrodynamics and Axiomatic Field Theory, Springer, 2000.
[38] D.R. Yennie, S.C. Frautschi, H. Suura, The infrared divergence phenomena and high-energy processes, Ann. Phys. 13 (1961) 379–452.
[39] D. Zwanziger, Scattering theory for quantum electrodynamics. II. Reduction and cross-section formulas, Phys. Rev. D 11 (1975) 3504–3530.