



On the superiority of the $|V_{cb}| - \gamma$ plots over the unitarity triangle plots in the 2020s

Andrzej J. Buras^{1,2,a}

¹ TUM Institute for Advanced Study, Lichtenbergstr. 2a, 85747 Garching, Germany

² Physik Department, TU München, James-Frank-Straße, 85748 Garching, Germany

Received: 5 May 2022 / Accepted: 29 June 2022 / Published online: 13 July 2022

© The Author(s) 2022

Abstract The unitarity triangle (UT) plots played already for three decades an important role in the tests of the Standard Model (SM) and the determination of the CKM parameters. As of 2022, among the four CKM parameters, $|V_{us}|$ and β are already measured with respectable precision, while this is not the case of $|V_{cb}|$ and γ . In the case of $|V_{cb}|$ the main obstacle are the significant tensions between its inclusive and exclusive determinations from tree-level decays and it could still take some years before a unique value of this parameter will be known. The present uncertainty in γ of 4° from tree-level decays will be reduced to 1° by the LHCb and Belle II collaborations in the coming years. Unfortunately in the common UT plots $|V_{cb}|$ is not seen and the experimental improvements in the determination of γ from tree-level decays at the level of a few degrees are difficult to appreciate. In view of these deficiencies of the UT plots with respect to $|V_{cb}|$ and γ and the central role these two CKM parameters will play in this decade, the recently proposed plots of $|V_{cb}|$ versus γ extracted from various processes appear to be superior to the UT plots in the flavour phenomenology of the 2020s. We illustrate this idea with $\Delta F = 2$ observables ΔM_s , ΔM_d , ε_K and with rare decays $B_s \rightarrow \mu^+ \mu^-$, $B_d \rightarrow \mu^+ \mu^-$, $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$. In particular the power of ε_K , $\mathcal{B}(K^+ \rightarrow \pi^+ \nu \bar{\nu})$ and $\mathcal{B}(K_L \rightarrow \pi^0 \nu \bar{\nu})$ in the determination of $|V_{cb}|$, due to their strong dependence on $|V_{cb}|$, is transparently exhibited in this manner. Combined with future reduced errors on γ and $|V_{cb}|$ from tree-level decays such plots can better exhibit possible inconsistencies between various determinations of these two parameters, caused by new physics, than it is possible with the UT plots. This can already be illustrated on the example of the recently found 2.7σ anomaly in $B_s \rightarrow \mu^+ \mu^-$.

1 Introduction

The unitary CKM matrix [1, 2] can be conveniently parametrized by the four parameters

$$|V_{us}|, |V_{cb}|, \beta, \gamma \quad (1)$$

with β and γ being two angles in the UT, shown in Fig. 1.

By now $|V_{us}|$, β and γ , as measured in tree-level decays, are found to be [3].

$$|V_{us}| = 0.2253(8), \quad \beta = 22.2(7)^\circ, \quad \gamma = (65.4_{-4.2}^{+3.8})^\circ. \quad (2)$$

Moreover, in the coming years the determination of γ by the LHCb [4, 5] and Belle II [6] collaborations should be significantly improved with the uncertainty brought down to 1° . Also some reduction of the error on β from tree-level decays is expected. A review of various methods can be found in Chapter 8 in [7].

The situation with $|V_{cb}|$ is very different. There is a persistent tension between its inclusive and exclusive determinations [8, 9]¹

$$|V_{cb}|_{\text{incl}} = 42.16(50) \times 10^{-3}, \quad |V_{cb}|_{\text{excl}} = 39.21(62) \times 10^{-3}, \quad (3)$$

which is clearly disturbing. This is the case when one makes SM predictions for rare K and B decays branching ratios and for quark mixing observables like ΔM_s , ΔM_d and ε_K . In particular rare K decay branching ratios, like the ones for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$, exhibit $|V_{cb}|^{2.8}$ and $|V_{cb}|^4$ dependence, respectively, while $|\varepsilon_K|$ the $|V_{cb}|^{3.4}$ one [10]. B physics observables exhibit typically $|V_{cb}|^2$ dependence. This implies large modifications in the SM predictions for the observables in question when the inclusive values of $|V_{cb}|$ are replaced by the exclusive ones [11].

^a e-mail: aburas@ph.tum.de (corresponding author)

¹ The exclusive value for $|V_{cb}|$ should be considered as preliminary.

This problematic is not seen directly in the usual UT plots [12, 13], simply because $|V_{cb}|$ does not enter Fig. 1 explicitly. Moreover the bands resulting from rare K decays and ε_K are bound to be broad because these observables being very sensitive to $|V_{cb}|$ suffer from this large parametric uncertainty. However, already in 1994, it has been pointed out in [14] that the branching ratio for $K_L \rightarrow \pi^0 \nu \bar{\nu}$ growing like $|V_{cb}|^4$ is a powerful tool to determine $|V_{cb}|$ in the absence of NP provided its branching ratio could be measured precisely. But even a measurement of this branching ratio with 10% accuracy would determine $|V_{cb}|$ with an error of 2.5% because in this decay the hadronic uncertainties are negligible. A similar comment applies to $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ up to long distance charm quark contribution and to ε_K , for which the theoretical uncertainties have been recently reduced [15].

It is then evident that the usual UT plots are not the arena where this nice property of Kaon processes can be used properly and in fact it becomes their Achilles tendon there. Moreover, the $|V_{cb}|$ problematic cannot be properly monitored with the help of the UT plots as $|V_{cb}|$ is hidden in computer codes. In my view this is a deficiency which could be eliminated if the usual UT plots were accompanied by the $|V_{cb}| - \gamma$ plots proposed recently in collaboration with Elena Venturini in [10, 11, 16]. Indeed, as far as $|V_{cb}|$ and γ are concerned the $|V_{cb}| - \gamma$ plots are superior to the usual UT plots in three ways:

- They exhibit $|V_{cb}|$ and its correlation with γ determined through a given observable in the SM, allowing thereby monitoring the progress on both parameters expected in the coming years. Violation of this correlation in experiment will clearly indicate new physics (NP) at work.
- They utilize the strong sensitivity of rare K decay processes to $|V_{cb}|$ thereby providing precise determination of $|V_{cb}|$ even with modest experimental precision on their branching ratios.
- They exhibit the action of ΔM_s and of B_s decays together with other processes. In particular the recently found anomaly in $B_s \rightarrow \mu^+ \mu^-$ decay [11, 17] can be exhibited in this manner which is not possible in a UT-plot.

It appears then that the $|V_{cb}| - \gamma$ plots for ΔM_s , ΔM_d and ε_K presented in [10, 11] are more useful in this context than the usual UT plots that exhibit the impact of quark mixing and rare decays in the $(\bar{\varrho}, \bar{\eta})$ plane [18, 19]. The goal of the present paper is to demonstrate the usefulness of $|V_{cb}| - \gamma$ plots also for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$, $K_L \rightarrow \pi^0 \nu \bar{\nu}$, $B_s \rightarrow \mu^+ \mu^-$ and $B_d \rightarrow \mu^+ \mu^-$ decays. Clearly, other decays can be considered as well, in particular the short distance contribution to $K_S \rightarrow \mu^+ \mu^-$ which has the same CKM dependence as $K_L \rightarrow \pi^0 \nu \bar{\nu}$.

Despite these comments we do not claim by no means that during the RUN 3 of the LHC and the Belle II era the UT plots should be abandoned. Indeed with improved measurements

of various observables they will exhibit the CKM unitarity tests

$$\gamma \Leftrightarrow \frac{\Delta M_d}{\Delta M_s}, \quad \beta \Leftrightarrow \frac{|V_{ub}|}{|V_{cb}|}, \quad (4)$$

with γ and β measured in tree level non-leptonic B -decays. In this sense they offer complementary tests of the SM. Explicit relations like the ones in (4) are collected in Chapters 2 and 8 in [7] and some of them will be given below.

It should be remarked that the authors of [20] performed recently a determination of $|V_{cb}|$ and $|V_{ub}|$ from loop processes, rare decays and quark mixing, by assuming no NP contributions to these observables. To this end they used only well measured observables in the B system and ε_K . This strategy has already been explored in [21] where ε_K , ΔM_d and ΔM_s and $S_{\psi K_S}$ have been considered. This was also the case of the analyses in [10, 11].

Our present analysis extends the $|V_{cb}| - \gamma$ strategy, developed in [10, 11], to rare K decays, not considered in [20], resurrecting some old ideas from the 1990s [14, 22, 23] and improving significantly on them. In this context it should be noted that in [22] the possibility of the determination of β from $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$, basically independently of $|V_{cb}|$ and γ , has been pointed out. As pointed out recently in [10] β can also be determined practically independently of $|V_{cb}|$ and γ either from $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $|\varepsilon_K|$ or $K_L \rightarrow \pi^0 \nu \bar{\nu}$ and $|\varepsilon_K|$.

The determination of $|V_{cb}|$ from $K_L \rightarrow \pi^0 \nu \bar{\nu}$ alone proposed in [14] requires still the value of γ as we will see explicitly in Sect. 3. In [23] the determination of the full CKM matrix has been achieved with the help of the measurement of $K_L \rightarrow \pi^0 \nu \bar{\nu}$ and of the angle α in the UT in tree-level decays. However, from the present perspective, the use of γ is favoured over α because of smaller hadronic uncertainties in its tree-level determinations.

The outline of our paper is as follows. In Sect. 2 we recall some useful expressions related to the CKM matrix and the SM formulae for the seven observables in question that exhibit their dependence on the CKM parameters in (1). In Sect. 3 we list useful formulae for $|V_{cb}|$ as a function of γ and β resulting from the *magnificent seven* observables considered by us. We illustrate the application of these formulae by presenting a few examples of the $|V_{cb}| - \gamma$ plots. A brief summary and an outlook are given in Sect. 4.

2 Basic formulae

2.1 Elements of the CKM Matrix and the UT

It will be useful to recall first a number of very accurate expressions related to the UT shown in Fig. 1 that exhibit the

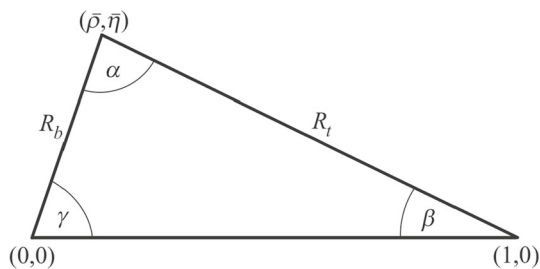


Fig. 1 The unitarity triangle

absence of $|V_{cb}|$ in the usual UT plots. With $\lambda = |V_{us}|$ we have

$$R_t = \sqrt{(1 - \bar{\rho})^2 + \bar{\eta}^2} = \frac{1}{\lambda} \left| \frac{V_{td}}{V_{cb}} \right|,$$

$$R_b = \sqrt{\bar{\rho}^2 + \bar{\eta}^2} = \left(1 - \frac{\lambda^2}{2}\right) \frac{1}{\lambda} \left| \frac{V_{ub}}{V_{cb}} \right|. \tag{5}$$

But

$$|V_{td}| = \lambda |V_{cb}| \sin \gamma, \quad |V_{ts}| = G(\beta, \gamma) |V_{cb}|,$$

$$|V_{ub}| = \lambda \sqrt{\sigma} |V_{cb}| \sin \beta, \tag{6}$$

where

$$G(\beta, \gamma) = 1 + \frac{\lambda^2}{2} (1 - 2 \sin \gamma \cos \beta), \quad \sigma = \left(\frac{1}{1 - \frac{\lambda^2}{2}} \right)^2. \tag{7}$$

Consequently

$$R_t = \sin \gamma, \quad R_b = \sin \beta \tag{8}$$

so that $|V_{cb}|$ disappears.

2.2 The magnificent seven

Explicit expressions for various observables in terms of the CKM parameters in (1), used in our paper, are the ones from [10] modified only by adjusting some reference input as stated below.

For ΔM_d and ΔM_s they are

$$\Delta M_d = 0.5065/\text{ps} \left[\frac{\sqrt{\hat{B}_{B_d}} F_{B_d}}{210.6 \text{MeV}} \right]^2 \left[\frac{S_0(x_t)}{2.307} \right]$$

$$\times \left[\frac{|V_{td}|}{8.67 \times 10^{-3}} \right]^2 \left[\frac{\eta_B}{0.5521} \right], \tag{9}$$

$$\Delta M_s = 17.749/\text{ps} \left[\frac{\sqrt{\hat{B}_{B_s}} F_{B_s}}{256.1 \text{MeV}} \right]^2 \left[\frac{S_0(x_t)}{2.307} \right]$$

$$\times \left[\frac{|V_{ts}|}{41.9 \times 10^{-3}} \right]^2 \left[\frac{\eta_B}{0.5521} \right]. \tag{10}$$

The value 2.307 in the normalization of $S_0(x_t)$ is its SM value for $m_t(m_t) = 162.83 \text{ GeV}$. The central values of $|V_{td}|$ and $|V_{ts}|$ exposed here are chosen to make the overall factors

in these formulae to be equal to the experimental values of the two observables. The reference values for $\sqrt{\hat{B}_{B_d}} F_{B_d}$ and $\sqrt{\hat{B}_{B_s}} F_{B_s}$ are those from the HPQCD collaboration [24] as used by us already in [11]. Correspondingly also the resulting reference values for $|V_{td}|$ and $|V_{ts}|$ agree perfectly with those quoted in [24]. Similar results for ΔM_d and ΔM_s hadronic matrix elements have been obtained within the HQET sum rules in [25, 26], respectively.

Of importance is also the mixing induced CP-asymmetry in the SM [3]

$$S_{\psi K_S} = \sin 2\beta = 0.699(17), \tag{11}$$

which implies the value of β in (2).

Next [10],

$$|\varepsilon_K| = 2.224 \times 10^{-3} \left[\frac{|V_{cb}|}{42.6 \times 10^{-3}} \right]^{3.4}$$

$$\times \left[\frac{\sin \gamma}{\sin 64.6^\circ} \right]^{1.67} \left[\frac{\sin \beta}{\sin 22.2^\circ} \right]^{0.87}. \tag{12}$$

The expression above provides an approximation of the exact formula of [15] with an accuracy of 1.5%, in the ranges $38 < |V_{cb}| \times 10^3 < 43$, $60^\circ < \gamma < 75^\circ$, $20^\circ < \beta < 24^\circ$.

For the rare decays $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$ we have [10]

$$\mathcal{B}(K^+ \rightarrow \pi^+ \nu \bar{\nu}) = (8.59 \pm 0.30) \times 10^{-11}$$

$$\times \left[\frac{|V_{cb}|}{42.6 \times 10^{-3}} \right]^{2.8} \left[\frac{\sin \gamma}{\sin 64.6^\circ} \right]^{1.39}, \tag{13}$$

$$\mathcal{B}(K_L \rightarrow \pi^0 \nu \bar{\nu}) = (2.93 \pm 0.04) \times 10^{-11}$$

$$\times \left[\frac{|V_{cb}|}{42.6 \times 10^{-3}} \right]^4 \left[\frac{\sin \gamma}{\sin(64.6^\circ)} \right]^2 \left[\frac{\sin \beta}{\sin(22.2^\circ)} \right]^2, \tag{14}$$

where we do not show explicitly the parametric dependence on $\lambda = |V_{us}|$ and set $\lambda = 0.225$. The 3.5% uncertainty in $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ is dominated by the long distance effects in the charm contribution [27], fully negligible in $K_L \rightarrow \pi^0 \nu \bar{\nu}$.

Similarly for $B_{s,d} \rightarrow \mu^+ \mu^-$ we have [28, 29]

$$\bar{\mathcal{B}}(B_s \rightarrow \mu^+ \mu^-)_{\text{SM}} = (3.77 \pm 0.06) \times 10^{-9}$$

$$\times \left[\frac{F_{B_s}}{230.3 \text{MeV}} \right]^2 \left| \frac{V_{ts}}{41.9 \times 10^{-3}} \right|^2 \bar{R}_s \tag{15}$$

where

$$\bar{R}_s = \left(\frac{\tau_{B_s}}{1.515 \text{ps}} \right) \left(\frac{0.935}{r(y_s)} \right)$$

$$\times \left(\frac{m_t(m_t)}{162.83 \text{ GeV}} \right)^{3.02} \left(\frac{\alpha_s(M_Z)}{0.1184} \right)^{0.032}. \tag{16}$$

Here $r(y_s)$ summarizes $\Delta \Gamma_s$ effects with $r(y_s) = 0.935 \pm 0.007$ within the SM [30–32].

For $B_d \rightarrow \mu^+ \mu^-$ we have

$$\begin{aligned}
\mathcal{B}(B_d \rightarrow \mu^+ \mu^-)_{\text{SM}} &= (1.02 \pm 0.02) 10^{-10} \\
&\times \left[\frac{F_{B_d}}{190.0 \text{ MeV}} \right]^2 \left| \frac{V_{td}}{8.67 \times 10^{-3}} \right|^2 \bar{R}_d. \quad (17)
\end{aligned}$$

As to an excellent accuracy $r(y_d) = 1$, one has this time

$$\bar{R}_d = \left(\frac{\tau_{B_d}}{1.519 \text{ ps}} \right) \left(\frac{m_t(m_t)}{162.83 \text{ GeV}} \right)^{3.02} \left(\frac{\alpha_s(M_Z)}{0.1184} \right)^{0.032}. \quad (18)$$

These expressions exhibit the following problem. Even if β can be determined precisely by measuring $S_{\psi K_S}$, the strong dependence of all rare decay branching ratios on $|V_{cb}|$ precludes in the presence of the tensions mentioned above, a useful determination of γ with the help of a given rare decay within the SM and its confrontation with its tree-level measurements.

In the usual UT analyses [12, 13] this problematic can be solved by considering ratios like the ones in (4) so that $|V_{cb}|$ drops out. But then no explicit information on $|V_{cb}|$, beyond the one resulting from computer codes, is provided by the UT plots. This deficiency is removed by accompanying the usual UT plots by the $|V_{cb}| - \gamma$ plots. They illustrate on the one hand that considering simultaneously $|\varepsilon_K|$, ΔM_d and ΔM_s and imposing the constraint on β in (11) one is able already now to obtain respectable determination on $|V_{cb}|$ and γ within the SM [11]. On the other hand they make clear that the future precise measurements of γ in tree-level decays and also experimental improvements on rare decay branching ratios that are very sensitive to $|V_{cb}|$ will offer very powerful tests of the SM. In particular when the experts also agree on the unique value of $|V_{cb}|$. Let us then have a closer look at the $|V_{cb}| - \gamma$ plots.

3 $|V_{cb}| - \gamma$ plots

Using the formulae just listed we can now find $|V_{cb}|$ as a function of γ and β resulting separately from each of the *magnificent seven* observables considered by us.

ΔM_d

$$\begin{aligned}
|V_{cb}| &= 42.6 \times 10^{-3} \left[\frac{\sin(64.6^\circ)}{\sin \gamma} \right] \left[\frac{210.6 \text{ MeV}}{\sqrt{\hat{B}_{B_d} F_{B_d}}} \right] \\
&\times \left[\frac{2.307}{S_0(x_t)} \right]^{0.5} \left[\frac{0.5521}{\eta_B} \right]^{0.5} \left[\frac{\Delta M_d}{0.5065/\text{ps}} \right]^{0.5}. \quad (19)
\end{aligned}$$

ΔM_s

$$|V_{cb}| = \left[\frac{41.9 \times 10^{-3}}{G(\beta, \gamma)} \right] \left[\frac{256.1 \text{ MeV}}{\sqrt{\hat{B}_{B_s} F_{B_s}}} \right] \left[\frac{2.307}{S_0(x_t)} \right]^{0.5}$$

$$\begin{aligned}
&\times \left[\frac{0.5521}{\eta_B} \right]^{0.5} \left[\frac{\Delta M_s}{17.749/\text{ps}} \right]^{0.5}. \quad (20)
\end{aligned}$$

$|\varepsilon_K|$

$$\begin{aligned}
|V_{cb}| &= 42.6 \times 10^{-3} \left[\frac{\sin(64.6^\circ)}{\sin \gamma} \right]^{0.491} \left[\frac{\sin(\beta)}{\sin(22.2^\circ)} \right]^{0.256} \\
&\times \left[\frac{|\varepsilon_K|}{2.224 \times 10^{-3}} \right]^{0.294}. \quad (21)
\end{aligned}$$

$K^+ \rightarrow \pi^+ \nu \bar{\nu}$

$$\begin{aligned}
|V_{cb}| &= 42.6 \times 10^{-3} \left[\frac{\sin(64.6^\circ)}{\sin \gamma} \right]^{0.496} \\
&\times \left[\frac{\mathcal{B}(K^+ \rightarrow \pi^+ \nu \bar{\nu})}{(8.59 \pm 0.30) \times 10^{-11}} \right]^{0.357}. \quad (22)
\end{aligned}$$

$K_L \rightarrow \pi^0 \nu \bar{\nu}$

$$\begin{aligned}
|V_{cb}| &= 42.6 \times 10^{-3} \left[\frac{\sin(64.6^\circ)}{\sin \gamma} \right]^{0.50} \\
&\times \left[\frac{\sin(22.2^\circ)}{\sin(\beta)} \right]^{0.50} \left[\frac{\mathcal{B}(K_L \rightarrow \pi^0 \nu \bar{\nu})}{(2.93 \pm 0.04) \times 10^{-11}} \right]^{0.25}. \quad (23)
\end{aligned}$$

$B_d \rightarrow \mu^+ \mu^-$

$$\begin{aligned}
|V_{cb}| &= 42.6 \times 10^{-3} \left[\frac{\sin(64.6^\circ)}{\sin \gamma} \right] \\
&\times \left[\frac{190.0 \text{ MeV}}{F_{B_d}} \right] \frac{1}{\sqrt{\bar{R}_d}} \left[\frac{\mathcal{B}(B_d \rightarrow \mu^+ \mu^-)}{(1.02 \pm 0.02) \times 10^{-10}} \right]^{0.50}. \quad (24)
\end{aligned}$$

$B_s \rightarrow \mu^+ \mu^-$

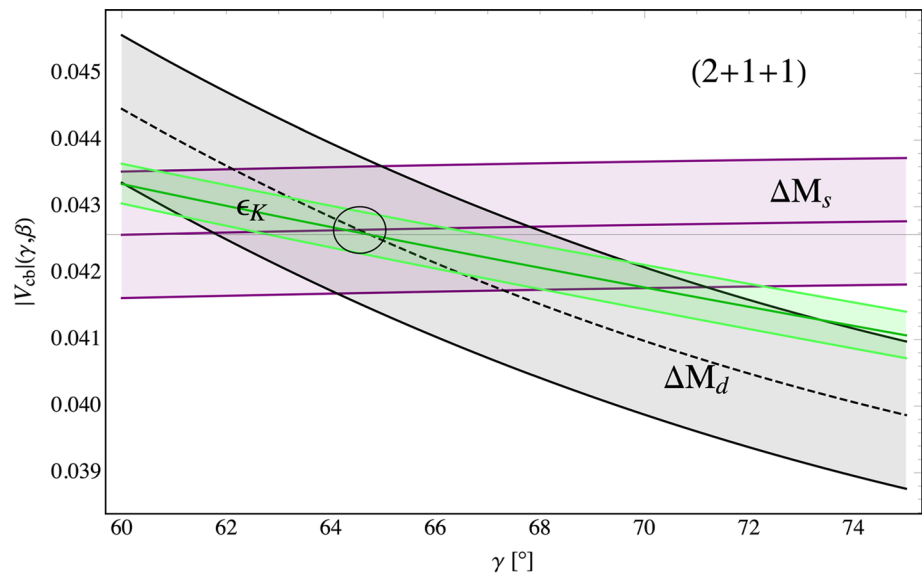
$$\begin{aligned}
|V_{cb}| &= \left[\frac{41.9 \times 10^{-3}}{G(\beta, \gamma)} \right] \left[\frac{230.3 \text{ MeV}}{F_{B_s}} \right] \frac{1}{\sqrt{\bar{R}_s}} \\
&\times \left[\frac{\mathcal{B}(B_s \rightarrow \mu^+ \mu^-)}{(3.77 \pm 0.06) \times 10^{-9}} \right]^{0.50}. \quad (25)
\end{aligned}$$

In [10, 11] only $|V_{cb}| - \gamma$ plots for ΔM_s , ΔM_d and ε_K have been presented. One of such plots from [11] is shown in Fig. 2.

An important observation should be made in this plot. The ε_K -band is thinner than the one coming from ΔM_d . This is dominantly related to the stronger dependence of ε_K on $|V_{cb}|$. This fact makes the action of ε_K less useful in the $(\bar{q}, \bar{\eta})$ plane than that of ΔM_d while in the $|V_{cb}| - \gamma$ plane it is reversed. The formulae above demonstrate this in explicit terms. Moreover, while the action of ΔM_s is invisible in a UT-plot it is clearly exhibited in Fig. 2.

Before presenting similar plots for rare decays $K^+ \rightarrow \pi^+ \nu \bar{\nu}$, $K_L \rightarrow \pi^0 \nu \bar{\nu}$, $B_s \rightarrow \mu^+ \mu^-$ and $B_d \rightarrow \mu^+ \mu^-$, let us recall the SM predictions for them [10, 11]. These are [10]

Fig. 2 The values of $|V_{cb}|$ extracted from ϵ_K , ΔM_d and ΔM_s as functions of γ with the hadronic matrix elements for $\Delta M_{s,d}$ obtained with 2 + 1 + 1 flavours [24]. The green band represents experimental $S_{\psi K_S}$ constraint on β . From [11]



$$\mathcal{B}(K^+ \rightarrow \pi^+ \nu \bar{\nu})_{\text{SM}} = (8.60 \pm 0.42) \times 10^{-11}, \quad \mathcal{B}(K_L \rightarrow \pi^0 \nu \bar{\nu})_{\text{SM}} = (2.94 \pm 0.15) \times 10^{-11}, \quad (26)$$

and [11]

$$\overline{\mathcal{B}}(B_s \rightarrow \mu^+ \mu^-)_{\text{SM}} = (3.78_{-0.10}^{+0.15}) \times 10^{-9}, \quad \mathcal{B}(B_d \rightarrow \mu^+ \mu^-)_{\text{SM}} = (1.02_{-0.03}^{+0.05}) \times 10^{-10}. \quad (27)$$

The most interesting at present is the SM prediction for the $B_s \rightarrow \mu^+ \mu^-$ branching ratio that exhibits a 2.7σ anomaly when confronted with its experimental value [33–35]

$$\overline{\mathcal{B}}(B_s \rightarrow \mu^+ \mu^-)_{\text{EXP}} = 2.86(33) \times 10^{-9}. \quad (28)$$

It updates the previous prediction of [17], based on different hadronic matrix elements, that exhibited a 2.1σ anomaly.

These are the most precise SM predictions for these decays to date. For $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$ they were obtained by using the experimental values of ϵ_K and $S_{\psi K_S}$. The ones for $B_{s,d} \rightarrow \mu^+ \mu^-$ using the strategy of [36] and experimental values of $\Delta M_{s,d}$.² No information on $|V_{cb}|$ was required to obtain these results and the left-over γ dependence in rare K decay branching ratios, once ϵ_K constraint was imposed, turned out to be negligible in the full range $60^\circ \leq \gamma \leq 75^\circ$ investigated by us. One can verify all these nice properties using the formulae above or inspect the plots in Fig. 15 of [10].

² This is the reason why the central values in (27) agree perfectly with those obtained in [24] where this strategy has been used as well.

The goal of this strategy is *not* to make an overall SM fit but to predict the SM branching ratios in question.

The 2.7σ anomaly in $B_s \rightarrow \mu^+ \mu^-$ can also be seen in the $|V_{cb}|$ -independent ratio [11]

$$\left[\frac{\overline{\mathcal{B}}(B_s \rightarrow \mu^+ \mu^-)}{\Delta M_s} \right]_{\text{SM}} = 2.13(7) \times 10^{-10} \text{ ps},$$

$$\left[\frac{\overline{\mathcal{B}}(B_s \rightarrow \mu^+ \mu^-)}{\Delta M_s} \right]_{\text{EXP}} = 1.61(18) \times 10^{-10} \text{ ps}. \quad (29)$$

Yet, it also turns out that the simultaneous description of the data for ΔM_d , ΔM_s , ϵ_K and $S_{\psi K_S}$ can be made without any participation of NP which gives an additional support for the SM predictions in (26) and (27). Indeed, as seen in Fig. 2, the SM predictions for ϵ_K , ΔM_d and ΔM_s turn out to be consistent with each other and with the data for the following values of the CKM parameters [11]

$$|V_{cb}| = 42.6(4) \times 10^{-3}, \quad \gamma = 64.6(16)^\circ, \quad \beta = 22.2(7)^\circ,$$

$$|V_{ub}| = 3.72(11) \times 10^{-3}. \quad (30)$$

The uncertainties shown here have been computed by propagating the non-parametric errors of the four $\Delta F = 2$ observables involved. More details can be found in [11]. In particular the determination of γ have been obtained by considering first the $|V_{cb}|$ -independent ratio $\Delta M_d/\Delta M_s$ from which one derives an accurate formula for $\sin \gamma$

$$\sin \gamma = \frac{0.983(1)}{\lambda} \sqrt{\frac{m_{B_s}}{m_{B_d}}} \xi \sqrt{\frac{\Delta M_d}{\Delta M_s}},$$

$$\xi = \frac{F_{B_s} \sqrt{\hat{B}_{B_s}}}{F_{B_d} \sqrt{\hat{B}_{B_d}}} = 1.216(16), \quad (31)$$

with the value for ξ from HPQCD [24]. The advantage of using this ratio for the determination of γ over studying ΔM_s and ΔM_d separately is its independence of $|V_{cb}|$ and the reduced error on ξ from LQCD relative to the individual errors of hadronic parameters in ΔM_s and ΔM_d . See Appendix B in [11].

As emphasized in [11] the very good agreement between ΔM_s , ΔM_d and ε_K is only obtained with the hadronic matrix elements with $2 + 1 + 1$ flavours from the lattice HPQCD collaboration [24]. For $2 + 1$ flavours significant inconsistencies within the SM were found. See Fig. 8 in [11]. All the input parameters used by us are collected in Table 1.

Our value for $|V_{cb}|$ is consistent with the inclusive one from [8] and $|V_{ub}|$ value with the exclusive one from FLAG [9]. It should be noted that the determination of γ in this manner is more accurate than its present determination from tree-level decays in (2).

In order to illustrate the action of the seven observables in the $|V_{cb}| - \gamma$ plane we show in Fig. 3 the results in the SM setting all uncertainties for transparency reasons to zero. We make the following observations.

- For fixed $\beta = 22.2^\circ$, ε_K , $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$ are represented to an excellent approximation by the same line which is already a very good test of the SM. This is simply because as seen in (21), (22) and (23) the γ dependence in the three observables is practically the same, the fact pointed out first in [22] and strongly emphasized in [10]. The dependence on β is different and

this allows to determine within the SM the angle β from any pair of these observables independently of the value of γ . For the pair of the rare K branching ratios this was pointed out in [22]. For the other two pairs in [10]. But the determination of β with the help of the plot in Fig. 3 is not useful and it is better to use the $|V_{cb}|$ -independent ratios R_0 , R_{11} , R_{12} of [10] with [16]

$$R_0 = \frac{\mathcal{B}(K^+ \rightarrow \pi^+ \nu \bar{\nu})}{\mathcal{B}(K_L \rightarrow \pi^0 \nu \bar{\nu})^{0.7}} = (2.03 \pm 0.08)$$

$$\times 10^{-3} \left[\frac{\sin 22.2^\circ}{\sin \beta} \right]^{1.4} = (2.03 \pm 0.11) \times 10^{-3}. \quad (32)$$

and

$$R_{11} = \frac{\mathcal{B}(K^+ \rightarrow \pi^+ \nu \bar{\nu})}{|\varepsilon_K|^{0.82}}, \quad R_{12} = \frac{\mathcal{B}(K_L \rightarrow \pi^0 \nu \bar{\nu})}{|\varepsilon_K|^{1.18}} \quad (33)$$

with explicit expressions for R_{11} and R_{12} given in (90) and (91) of [10], respectively. They all can be derived from (12), (13) and (14) given in the previous section.

- ΔM_d and $B_d \rightarrow \mu^+ \mu^-$ are represented by a single line and a different line represents ΔM_s and $B_s \rightarrow \mu^+ \mu^-$. This is precisely the illustration of the SM relations pointed out long time ago in [36].

While, as seen in Fig. 2, SM describes ε_K , ΔM_d , ΔM_s simultaneously very well, this not need to be the case for the four rare decays in question. This is illustrated in Fig. 4. To obtain these results we have set the branching ratio for $B_s \rightarrow \mu^+ \mu^-$ to the experimental world average from LHCb, CMS and ATLAS [33–35] but decreased its error from 11% down to 5%. For the remaining branching ratios we have chosen

Table 1 Values of the experimental and theoretical quantities used as input parameters. For future updates see FLAG [9], PDG [3] and HFLAV [38]

$m_{B_s} = 5366.8(2)\text{MeV}$ [3]	$m_{B_d} = 5279.58(17)\text{MeV}$ [3]
$\Delta M_s = 17.749(20)\text{ps}^{-1}$ [3]	$\Delta M_d = 0.5065(19)\text{ps}^{-1}$ [3]
$\Delta M_K = 0.005292(9)\text{ps}^{-1}$ [3]	$m_{K^0} = 497.61(1)\text{MeV}$ [3]
$S_{\psi K_S} = 0.699(17)$ [3]	$F_K = 155.7(3)\text{MeV}$ [38]
$ V_{us} = 0.2253(8)$ [3]	$ \varepsilon_K = 2.228(11) \cdot 10^{-3}$ [3]
$F_{B_s} = 230.3(1.3)\text{MeV}$ [9]	$F_{B_d} = 190.0(1.3)\text{MeV}$ [9]
$F_{B_s} \sqrt{\hat{B}_s} = 256.1(5.7)\text{MeV}$ [24]	$F_{B_d} \sqrt{\hat{B}_d} = 210.6(5.5)\text{MeV}$ [24]
$\hat{B}_s = 1.232(53)$ [24]	$\hat{B}_d = 1.222(61)$ [24]
$m_t(m_t) = 162.83(67)\text{GeV}$ [39]	$m_c(m_c) = 1.279(13)\text{GeV}$
$S_{tt}(x_t) = 2.303$	$S_{ut}(x_c, x_t) = -1.983 \times 10^{-3}$
$\eta_{tt} = 0.55(2)$ [15]	$\eta_{ut} = 0.402(5)$ [15]
$\kappa_\varepsilon = 0.94(2)$ [40]	$\eta_B = 0.55(1)$ [41, 42]
$\tau_{B_s} = 1.515(4)\text{ps}$ [43]	$\tau_{B_d} = 1.519(4)\text{ps}$ [43]

Fig. 3 Schematic illustration of the action of the seven observables in the $|V_{cb}| - \gamma$ plane in the context of the SM. We set $\beta = 22.2^\circ$ and all uncertainties in (19)–(25) to zero

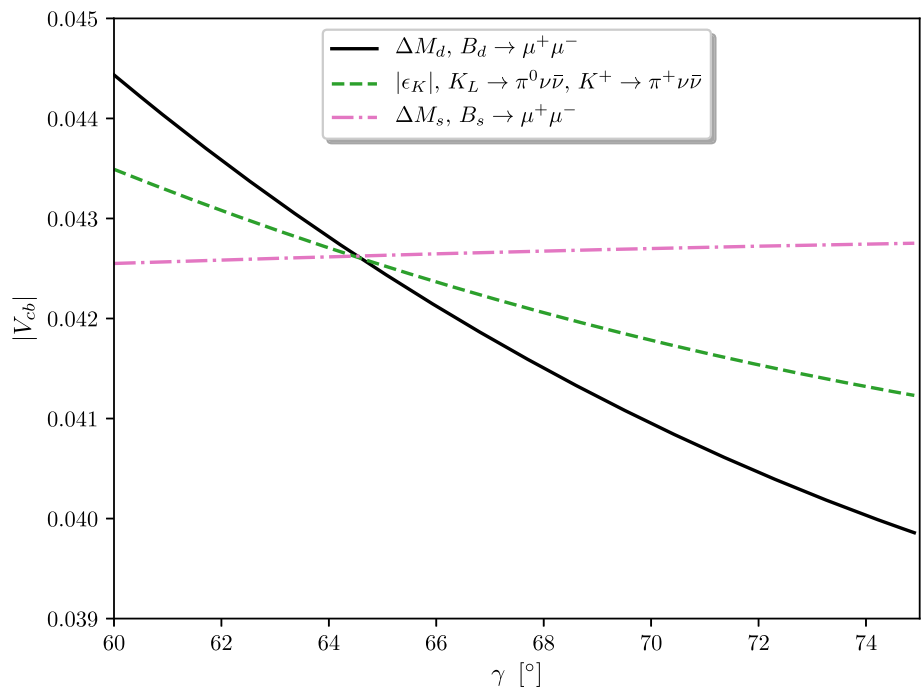
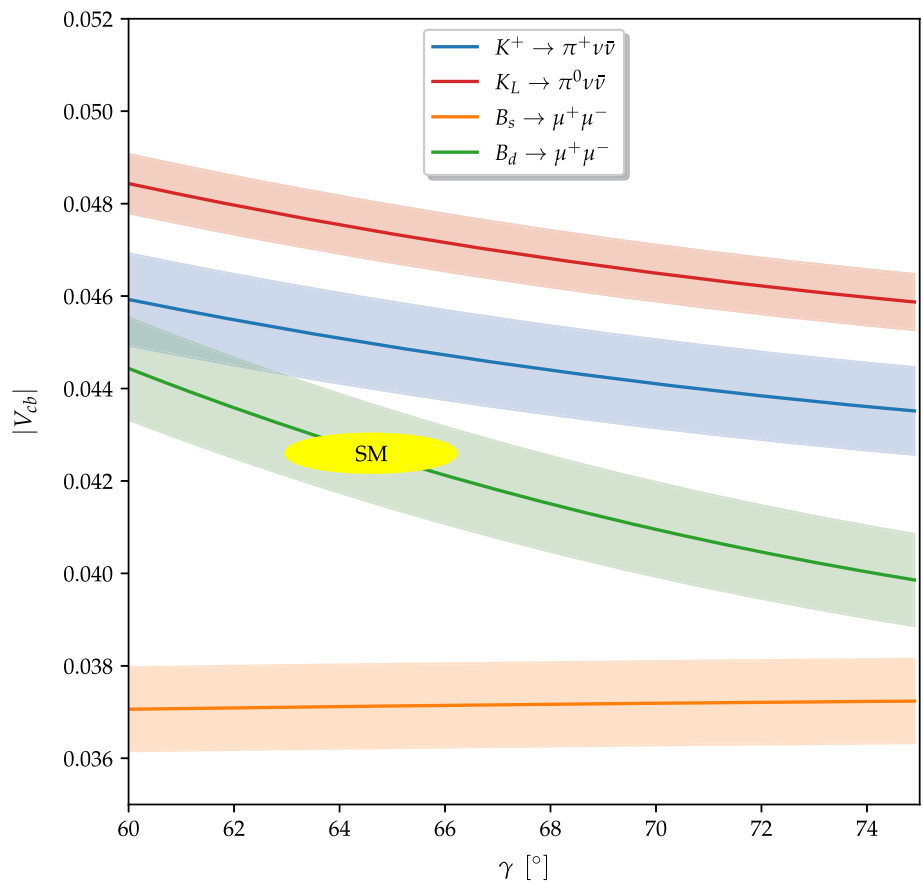


Fig. 4 The impact of hypothetical future measurements of the branching ratios for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$, $K_L \rightarrow \pi^0 \nu \bar{\nu}$, $B_d \rightarrow \mu^+ \mu^-$ and $B_s \rightarrow \mu^+ \mu^-$ as given in (34) and (35) on the $|V_{cb}| - \gamma$ plane. All uncertainties in (19)–(25) are included. The yellow disc represents the SM as obtained in (30)



values resulting from hypothetical future measurements that differ from the SM predictions in (26) and (27). We kept the errors at 5% as in the case of $B_s \rightarrow \mu^+\mu^-$ to exhibit the superiority of rare K decays over rare B decays as far as the determination of $|V_{cb}|$ is concerned. We use then

$$\mathcal{B}(K^+ \rightarrow \pi^+\nu\bar{\nu}) = (10.00 \pm 0.50) \times 10^{-11},$$

$$\mathcal{B}(K_L \rightarrow \pi^0\nu\bar{\nu}) = (4.50 \pm 0.23) \times 10^{-11}, \quad (34)$$

$$\overline{\mathcal{B}}(B_s \rightarrow \mu^+\mu^-) = (2.86 \pm 0.14) \times 10^{-9},$$

$$\mathcal{B}(B_d \rightarrow \mu^+\mu^-) = (1.00 \pm 0.05) \times 10^{-10}. \quad (35)$$

While the experimental errors are futuristic, we expect that the theoretical errors will go down with time so that the bands in Fig. 4 could apply one day with less accurate measurements.

This plot confirms all the statements made above. The superiority of $K_L \rightarrow \pi^0\nu\bar{\nu}$ over the remaining decays is clearly seen. The blue band will be narrowed once the long distance charm contributions to $K^+ \rightarrow \pi^+\nu\bar{\nu}$ will be known with higher precision from lattice QCD calculations [37] than they are known now [27].

4 Conclusions and outlook

In the present paper we have emphasized, resurrecting by now almost thirty years old ideas of [14,22,23], that the rare K and B decay branching ratios, being subject to small hadronic uncertainties, could soon give us a powerful tool to determine the CKM parameters, in particular the controversial parameter $|V_{cb}|$. They could also provide a useful insight in the value of γ beyond its tree-level determinations. In this context we have proposed to monitor future progress on the determination on $|V_{cb}|$ and γ in the $|V_{cb}| - \gamma$ plane rather than in the $(\bar{q}, \bar{\eta})$ plane used in the context of the common UT-fits. We also reemphasized in this context the important role of ΔM_s , ΔM_d and ε_K . To this end we derived seven expressions for $|V_{cb}|$ by means of which this CKM element can be determined. First in the case of

$$\Delta M_d, \quad \Delta M_s, \quad |\varepsilon_K|, \quad (36)$$

with the relevant expression for $|V_{cb}|$ as a function of γ , β and the observable involved given in (19), (20) and (21), respectively. The corresponding four formulae for $|V_{cb}|$ from rare decays

$$K^+ \rightarrow \pi^+\nu\bar{\nu}, \quad K_L \rightarrow \pi^0\nu\bar{\nu}, \quad B_d \rightarrow \mu^+\mu^-,$$

$$B_s \rightarrow \mu^+\mu^-, \quad (37)$$

are given in (22), (23), (24) and (25), respectively.

The very good consistency of the three observables (36) with each other within the SM allowed, after the imposition of the $S_{\psi K_S}$ constraint, a satisfactory determination of the four

CKM parameters in [11] and given in (30). In the present paper this agreement is seen in the plot in Fig. 2.

The 2.7σ anomaly in $B_s \rightarrow \mu^+\mu^-$ found in [11] would increase with the improved measurement as assumed in (35) to 5.2σ . In the $|V_{cb}| - \gamma$ plane it will be signaled by the inconsistency with the SM yellow disc in Fig. 4 on which the SM prediction for this decay in (27) is based.

The data on the remaining rare decay branching ratios allow still for significant NP contributions and inconsistencies between various determinations of $|V_{cb}|$ as a function of γ from different decays could be found one day. We illustrated it in Fig. 4. In this context a precise measurement of γ by the LHCb and Belle II collaborations and the improvements on β will allow a very precise determination of $|V_{cb}|$ within the SM, first with the help of precisely measured $|\varepsilon_K|$ and later $K^+ \rightarrow \pi^+\nu\bar{\nu}$ and $K_L \rightarrow \pi^0\nu\bar{\nu}$.

Simultaneously the very accurate $|V_{cb}|$ independent SM predictions for rare decay branching ratios found in [10,11] and recalled here in (26) and (27) will play, in case of inconsistencies in the $|V_{cb}| - \gamma$ plane, an important role in the identification of a particular NP at work. The 16 $|V_{cb}|$ -independent ratios of various flavour observables derived in [10] will also be useful in this context.

We are looking forward to new data from LHCb, NA62, KOTO and Belle II collaborations as well as to improved hadronic matrix elements from LQCD which will allow one to use this strategy and the ones outlined in [10,11] more efficiently than it is possible now.

Acknowledgements I would like to thank Elena Venturini for the most pleasant and efficient collaboration leading to [10,11,16] that had clearly an important impact on the present paper. In particular I would also like to thank her for continuous discussions on classical music, first of all the one of Bach, Beethoven, Brahms, Mozart, Chopin, Rachmaninow, Schumann, Schubert, Vivaldi and Tschajkowski, but recently also of Alma Deutscher, the *Mozarta* of the 21st century. The invaluable help from Marcin Chrzaszcz in constructing the plots in Figs. 3 and 4 is highly appreciated. Financial support from the Excellence Cluster ORIGINS, funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), Excellence Strategy, EXC-2094, 390783311 is acknowledged.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and no experimental data.]

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Funded by SCOAP³. SCOAP³ supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

- N. Cabibbo, Unitary symmetry and leptonic decays. *Phys. Rev. Lett.* **10**, 531–533 (1963). [648 (1963)]
- M. Kobayashi, T. Maskawa, CP violation in the renormalizable theory of weak interaction. *Prog. Theor. Phys.* **49**, 652–657 (1973)
- Particle Data Group Collaboration, P.A. Zyla et al., Review of particle physics. *PTEP***2020**(8), 083C01 (2020)
- A. Cerri, V.V. Gligorov, S. Malvezzi, J. Martin Camalich, J. Zupan, Opportunities in flavour physics at the HL-LHC and HE-LHC. [arXiv:1812.07638](https://arxiv.org/abs/1812.07638)
- LHCb Collaboration, R. Aaij et al., Physics case for an LHCb Upgrade II—opportunities in flavour physics, and beyond, in the HL-LHC era. [arXiv:1808.08865](https://arxiv.org/abs/1808.08865)
- Belle-II Collaboration, W. Altmannshofer et al., The Belle II physics book. *PTEP***2019**(12), 123C01 (2019). [arXiv:1808.10567](https://arxiv.org/abs/1808.10567). [Erratum: *PTEP* **2020**, 029201 (2020)]
- A.J. Buras, *Gauge theory of weak decays*. (Cambridge University Press, 2020), p. 6
- M. Bordone, B. Capdevila, P. Gambino, Three loop calculations and inclusive V_{cb} . *Phys. Lett. B* **822**, 136679 (2021). [arXiv:2107.00604](https://arxiv.org/abs/2107.00604)
- Y. Aoki et al., FLAG Review 2021. [arXiv:2111.09849](https://arxiv.org/abs/2111.09849)
- A.J. Buras, E. Venturini, Searching for new physics in rare K and B decays without $|V_{cb}|$ and $|V_{ub}|$ uncertainties. [arXiv:2109.11032](https://arxiv.org/abs/2109.11032)
- A.J. Buras, E. Venturini, The exclusive vision of rare K and B decays and of the quark mixing in the standard model. [arXiv:2203.11960](https://arxiv.org/abs/2203.11960)
- UTfit Collaboration, M. Bona et al., Model-independent constraints on $\Delta F=2$ operators and the scale of new physics, *JHEP***0803** (2008) 049, [arXiv:0707.0636](https://arxiv.org/abs/0707.0636)
- CKMfitter Group Collaboration, J. Charles et al., CP violation and the CKM matrix: assessing the impact of the asymmetric B factories. *Eur. Phys. J. C* **41**, 1–131 (2005). [arXiv:hep-ph/0406184](https://arxiv.org/abs/hep-ph/0406184)
- A.J. Buras, Precise determinations of the CKM matrix from CP asymmetries in B decays and $K_L \rightarrow \pi^0 \nu \bar{\nu}$. *Phys. Lett. B* **333**, 476–483 (1994). [arXiv:hep-ph/9405368](https://arxiv.org/abs/hep-ph/9405368)
- J. Brod, M. Gorbahn, E. Stamou, Standard-model prediction of ϵ_K with manifest quark-mixing unitarity. *Phys. Rev. Lett.* **125**(17), 171803 (2020). [arXiv:1911.06822](https://arxiv.org/abs/1911.06822)
- A.J. Buras, E. Venturini, Standard model predictions for rare K and B decays without $|V_{cb}|$ and $|V_{ub}|$ uncertainties, p. 3 (2022). [arXiv:2203.10099](https://arxiv.org/abs/2203.10099)
- C. Bobeth, A.J. Buras, Searching for new physics with $\bar{B}(B_{s,d} \rightarrow \mu \bar{\mu})/\Delta M_{s,d}$. *Acta Phys. Polon. B* **52**, 1189 (2021). [arXiv:2104.09521](https://arxiv.org/abs/2104.09521)
- L. Wolfenstein, Parametrization of the Kobayashi–Maskawa matrix. *Phys. Rev. Lett.* **51**, 1945 (1983)
- A.J. Buras, M.E. Lautenbacher, G. Ostermaier, Waiting for the top quark mass, $K^+ \rightarrow \pi^+ \nu \bar{\nu}$, $B_s^0 - \bar{B}_s^0$ mixing and CP asymmetries in B decays. *Phys. Rev. D* **50**, 3433–3446 (1994). [arXiv:hep-ph/9403384](https://arxiv.org/abs/hep-ph/9403384)
- W. Altmannshofer, N. Lewis, Loop-induced determinations of V_{ub} and V_{cb} . *Phys. Rev. D* **105**(3), 033004 (2022). [arXiv:2112.03437](https://arxiv.org/abs/2112.03437)
- A.J. Buras, D. Buttazzo, J. Girschbach-Noe, R. Knegjens, $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$ in the Standard Model: status and perspectives. *JHEP* **11**, 033 (2015). [arXiv:1503.02693](https://arxiv.org/abs/1503.02693)
- G. Buchalla, A.J. Buras, $\sin 2\beta$ from $K \rightarrow \pi \nu \bar{\nu}$. *Phys. Lett. B* **333**, 221–227 (1994). [arXiv:hep-ph/9405259](https://arxiv.org/abs/hep-ph/9405259)
- G. Buchalla, A.J. Buras, $K \rightarrow \pi \nu \bar{\nu}$ and high precision determinations of the CKM matrix. *Phys. Rev. D* **54**, 6782–6789 (1996). [arXiv:hep-ph/9607447](https://arxiv.org/abs/hep-ph/9607447)
- R.J. Dowdall, C.T.H. Davies, R.R. Horgan, G.P. Lepage, C.J. Monahan, J. Shigemitsu, M. Wingate, Neutral B -meson mixing from full lattice QCD at the physical point. *Phys. Rev. D* **100**(9), 094508 (2019). [arXiv:1907.01025](https://arxiv.org/abs/1907.01025)
- M. Kirk, A. Lenz, T. Rauh, Dimension-six matrix elements for meson mixing and lifetimes from sum rules. *JHEP* **12**, 068 (2017). [arXiv:1711.02100](https://arxiv.org/abs/1711.02100). [Erratum: *JHEP* **06**, 162 (2020)]
- D. King, A. Lenz, T. Rauh, B_s mixing observables and $|V_{td}/V_{ts}|$ from sum rules. *JHEP* **05**, 034 (2019). [arXiv:1904.00940](https://arxiv.org/abs/1904.00940)
- G. Isidori, F. Mescia, C. Smith, Light-quark loops in $K \rightarrow \pi \nu \bar{\nu}$. *Nucl. Phys. B* **718**, 319–338 (2005). [arXiv:hep-ph/0503107](https://arxiv.org/abs/hep-ph/0503107)
- C. Bobeth, M. Gorbahn, T. Hermann, M. Misiak, E. Stamou et al., $B_{s,d} \rightarrow \ell^+ \ell^-$ in the standard model with reduced theoretical uncertainty. *Phys. Rev. Lett.* **112**, 101801 (2014). [arXiv:1311.0903](https://arxiv.org/abs/1311.0903)
- M. Beneke, C. Bobeth, R. Szafron, Power-enhanced leading-logarithmic QED corrections to $B_q \rightarrow \mu^+ \mu^-$. *JHEP* **10**, 232 (2019). [arXiv:1908.07011](https://arxiv.org/abs/1908.07011)
- S. Descotes-Genon, J. Matias, J. Virto, An analysis of $B_{d,s}$ mixing angles in presence of New Physics and an update of $B_s \rightarrow K^{0*} \bar{K}^{0*}$. *Phys. Rev. D* **85**, 034010 (2012). [arXiv:1111.4882](https://arxiv.org/abs/1111.4882)
- K. De Bruyn, R. Fleischer, R. Knegjens, P. Koppenburg, M. Merk et al., Branching ratio measurements of B_s decays. *Phys. Rev. D* **86**, 014027 (2012). [arXiv:1204.1735](https://arxiv.org/abs/1204.1735)
- K. De Bruyn, R. Fleischer, R. Knegjens, P. Koppenburg, M. Merk et al., Probing new physics via the $B_s^0 \rightarrow \mu^+ \mu^-$ effective lifetime. *Phys. Rev. Lett.* **109**, 041801 (2012). [arXiv:1204.1737](https://arxiv.org/abs/1204.1737)
- LHCb Collaboration, R. Aaij et al., Measurement of the $B_s^0 \rightarrow \mu^+ \mu^-$ decay properties and search for the $B^0 \rightarrow \mu^+ \mu^-$ and $B_s^0 \rightarrow \mu^+ \mu^- \gamma$ decays. [arXiv:2108.09283](https://arxiv.org/abs/2108.09283)
- CMS Collaboration, Combination of the ATLAS, CMS and LHCb results on the $B_{(s)}^0 \rightarrow \mu^+ \mu^-$ decays. CMS-PAS-BPH-20-003
- ATLAS Collaboration, Combination of the ATLAS, CMS and LHCb results on the $B_{(s)}^0 \rightarrow \mu^+ \mu^-$ decays. ATLAS-CONF-2020-049
- A.J. Buras, Relations between $\Delta M_{s,d}$ and $B_{s,d} \rightarrow \mu^+ \mu^-$ in models with minimal flavour violation. *Phys. Lett. B* **566**, 115–119 (2003). [arXiv:hep-ph/0303060](https://arxiv.org/abs/hep-ph/0303060)
- RBC, UKQCD Collaboration, N.H. Christ, X. Feng, A. Portelli, C.T. Sachrajda, Lattice QCD study of the rare kaon decay $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ at a near-physical pion mass. *Phys. Rev. D***100**(11), 114506 (2019). [arXiv:1910.10644](https://arxiv.org/abs/1910.10644)
- Flavour Lattice Averaging Group Collaboration, S. Aoki et al., FLAG Review 2019: Flavour Lattice Averaging Group (FLAG). *Eur. Phys. J. C* **80**(2), 113 (2020). [arXiv:1902.08191](https://arxiv.org/abs/1902.08191)
- J. Brod, M. Gorbahn, E. Stamou, Updated standard model prediction for $K \rightarrow \pi \nu \bar{\nu}$ and ϵ_K , in *19th international conference on B-physics at frontier machines*, p. 5 (2021). [arXiv:2105.02868](https://arxiv.org/abs/2105.02868)
- A.J. Buras, D. Guadagnoli, G. Isidori, On ϵ_K beyond lowest order in the operator product expansion. *Phys. Lett. B* **688**, 309–313 (2010). [arXiv:1002.3612](https://arxiv.org/abs/1002.3612)
- A.J. Buras, M. Jamin, P.H. Weisz, Leading and next-to-leading QCD corrections to ϵ parameter and $B^0 - \bar{B}^0$ mixing in the presence of a heavy top quark. *Nucl. Phys. B* **347**, 491–536 (1990)
- J. Urban, F. Krauss, U. Jentschura, G. Soff, Next-to-leading order QCD corrections for the $B^0 - \bar{B}^0$ mixing with an extended Higgs sector. *Nucl. Phys. B* **523**, 40–58 (1998). [arXiv:hep-ph/9710245](https://arxiv.org/abs/hep-ph/9710245)
- Heavy Flavor Averaging Group (HFAG) Collaboration, Y. Amhis et al., Averages of b -hadron, c -hadron, and τ -lepton properties as of summer 2016. [arXiv:1612.07233](https://arxiv.org/abs/1612.07233)