Scheme-independent calculations of anomalous dimensions of baryon operators in conformal field theories

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We present the first analytic scheme-independent series calculations of anomalous dimensions of several types of baryon operators at an infrared fixed point (IRFP) in an asymptotically free SU(3) gauge theory with N_f fermions. Separately, for an asymptotically free gauge theory with a gauge group G and N_f fermions in a representation R of G, we consider physical quantities at an IRFP, including the anomalous dimension of gauge-invariant fermion bilinears and the derivative of the beta function. These quantities have been calculated in series expansions whose coefficients have been proved to be scheme-independent at each order. We illustrate the scheme independence using a variety of schemes, including the RI′ scheme and several types of momentum subtraction schemes.

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I. INTRODUCTION

In conformal field theories, quantities of particular interest are the scaling dimensions, $D_{\mathcal{O}}$, of gauge-invariant operators, $\mathcal{O}$. In general, we write

$$D_{\mathcal{O}} = D_{\mathcal{O},cl.} - \gamma_{\mathcal{O}}, \tag{1.1}$$

where $D_{\mathcal{O},cl.}$ is the classical (free-field) dimension of $\mathcal{O}$, and $\gamma_{\mathcal{O}}$ is the anomalous dimension of $\mathcal{O}$ due to interactions. We shall focus on the determination of $\gamma_{\mathcal{O}}$ in perturbation theory at a fixed point of the renormalization group (RG). An important example of such a fixed point is encountered in the case of an asymptotically free non-Abelian gauge theory with gauge group G and sufficiently many massless fermions in a representation R of G. We denote the running gauge coupling at a Euclidean scale μ as $g = g(\mu)$ and denote $\alpha = g^2/(4\pi)$ and $a = g^2/(16\pi^2)$. In this theory, the gauge coupling evolves from small values in the ultraviolet (UV) at large μ to an infrared fixed point (IRFP) at a value denoted $\alpha_{\rm IR}$ as $\mu \to 0$. At this value, the theory is scale-invariant and is inferred to be conformally invariant [1,2]. This infrared behavior is commonly denoted the non-Abelian Coulomb phase (NACP) or conformal window. The RG evolution of the gauge coupling is described by the beta function,

$$\beta_\alpha = \frac{d\alpha}{d\ln\mu} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell a^\ell, \tag{1.2}$$

where b_ℓ is the ℓ-loop coefficient. At the two-loop (2ℓ) level [3–7], the IR zero of β_α function occurs at

$$\alpha_{{\rm IR},2\ell} = -\frac{4\pi b_1}{b_2}. \tag{1.3}$$

If N_f is only slightly smaller than the upper limit,

$$N_u = \frac{11 C_A}{4 T_f} \tag{1.4}$$

implied by the property of asymptotic freedom [5,6], then $\alpha_{{\rm IR},2\ell}$ is small and can be analyzed perturbatively [4,7]. As N_f decreases, the value of the coupling at the infrared zero of the beta function increases, motivating calculation of this IRFP value of α to higher-loop order. This was carried out to the four-loop level for general gauge group G and fermion representation R in [8–10], using b_3 [11] and b_4 [12] computed in the modified minimal subtraction scheme [13] for regularization and renormalization, denoted $\overline{\rm MS}$. (The minimal subtraction scheme was originally presented in [14].) Subsequently, the IRFP was calculated to the five-loop level [15], using b_5 in the $\overline{\text{MS}}$ scheme [16,17]. Effects of scheme dependence were studied in [18–25].

The anomalous dimension of a gauge-invariant operator $\mathcal{O}$, evaluated at a zero of the beta function (hence an RG fixed point), is, in principle, measurable, and hence cannot depend on the scheme used for regularization and renormalization. However, this property is not maintained in a conventional finite-order perturbative calculation of the anomalous dimension of such an operator as a power series in the coupling α,

$$\gamma_{\mathcal{O}} = \sum_{\ell=1}^{\infty} c_{\mathcal{O},\ell} a^{\ell}. \tag{1.5}$$

Once the perturbative expansion for $\gamma_{\mathcal{O}}$ is truncated at a finite order, scheme dependence is induced in the result for $\gamma_{\mathcal{O}}$. Only if one had the entire perturbative series available would the final result be guaranteed to be scheme-independent. Explicitly, to evaluate $\gamma_{\mathcal{O}}$ to finite order at an IRFP using Eqs. (1.2) and (1.5), one solves for the relevant zero of the n-loop beta function to obtain the n-loop value of α at this IRFP, denoted $\alpha_{\mathrm{IR},n\ell}$ and then substitutes this into Eq. (1.5) to obtain the value at the IRFP, $\gamma_{\mathcal{O},\mathrm{IR}}$. However, beyond the lowest orders, the result is scheme-dependent, because of scheme dependence in both the higher-order b_ℓ and the $c_{\mathcal{O},\ell}$ coefficients. The calculations of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ to four-loop order in [8,9] and to five-loop order in [15] used the four-loop and five-loop coefficients $c_{\bar{\psi}\psi,4}$ [26] and $c_{\bar{\psi}\psi,5}$ [27], respectively, calculated in the $\overline{\text{MS}}$ scheme. This scheme dependence of higher-order perturbative calculations is, of course, not limited to these quantities, but is a generic property of higher-order calculations. For example, it is well known that higher-order calculations of differential and total cross sections in quantum chromodynamics (QCD) are also scheme-dependent.

Intuitively, one expects that as one increases the order of the perturbative computation, there is more scheme-independent information contained in $\gamma_{\mathcal{O}}$. This expectation is justified by the fact that higher-order QCD calculations used, e.g., to analyze data from the Fermilab Tevatron and CERN Large Hadron Collider showed less dependence on the scheme/scale than lower-order calculations [28]. Indeed, for many years there has been work on the construction and application of schemes in QCD designed to reduce the scheme and scale dependence in higher-order QCD calculations (e.g., [29–31]).

Ideally, one would use a method of perturbative calculation of physical quantities that manifestly preserves the scheme independence at each finite order in the series expansion. That is, one would like to extract the scheme-independent information that is contained in the scheme-dependent higher-order coefficients b_ℓ and $c_{\mathcal{O},\ell}$. A key property of the IRFP in an asymptotically free gauge theory is that $\alpha_{\mathrm{IR}} \to 0$ as N_f (considered to be generalized to real numbers [6]) approaches the upper limit, N_u, allowed by asymptotic freedom. It follows that one can reexpress a physical quantity such as $\gamma_{\mathcal{O}}$ at the IRFP as a series expansion in powers of the difference

$$\Delta_f = N_u - N_f, \tag{1.6}$$

i.e.,

$$\gamma_{\mathcal{O}} = \sum_{n=1}^{\infty} \kappa_{\mathcal{O},n} \Delta_f^n. \tag{1.7}$$

Since Δ_f is obviously scheme-independent and so is $\gamma_{\mathcal{O}}$, each coefficient $\kappa_{\mathcal{O},n}$ is also scheme-independent. Some early work based on this was in [7,32].

Recently, extensive scheme-independent expansions for anomalous dimensions of a number of physical quantities have been calculated and analyzed in [33–42]. For asymptotically free vectorial gauge theories with gauge group G and N_f fermions transforming according to a representation R of G, physical quantities of interest include the fermion bilinears $\bar{\psi}\psi$ and $\bar{\psi}\mathcal{T}_j\psi$, where we suppress the sum over fermion flavor indices and $\mathcal{T}_j$ denotes a generator of the Lie algebra of $\mathrm{SU}(N_f)$. These have the same anomalous dimension [43]. We denote this anomalous dimension as $\gamma_{\bar{\psi}\psi}$ and its evaluation at the IRFP as $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$. The scheme-independent series expansion of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ is written as

$$\gamma_{\bar{\psi}\psi,\mathrm{IR}} = \sum_{n=1}^{\infty} \kappa_n \Delta_f^n. \tag{1.8}$$

In general, the calculation of the coefficient κ_n in Eq. (1.8) requires, as inputs, the values of the b_ℓ for $1 \le \ell \le n+1$ and the c_ℓ for $1 \le \ell \le n$.

The derivative of the beta function evaluated at the IRFP,

$$\beta'_{\mathrm{IR}} = \left.\frac{d\beta_\alpha}{d\alpha}\right|_{\alpha=\alpha_{\mathrm{IR}}}, \tag{1.9}$$

is also a physical quantity and hence is scheme-independent [44]. Indeed, from the trace anomaly [45] $T^\mu_\mu = [\beta_\alpha/(4\alpha)] F^a_{\mu\nu} F^{a\mu\nu}$, where $F^a_{\mu\nu}$ is the field-strength tensor, it follows that the full scaling dimension of $F^2 \equiv \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, satisfies the relation [46]

$$D_{F^2} = 4 + \frac{d\beta_\alpha}{d\alpha} - \frac{2}{\alpha}\beta_\alpha, \tag{1.10}$$

so that, at the IRFP, with $\beta_\alpha = 0$, $\gamma_{F^2,\mathrm{IR}} = -\beta'_{\mathrm{IR}}$, i.e., β'_{IR} is equivalent to the anomalous dimension of F^2 evaluated at the IRFP. The scheme-independent series expansion of β'_{IR} is written as

$$\beta'_{\mathrm{IR}} = \sum_{n=2}^{\infty} d_n \Delta_f^n. \tag{1.11}$$

(Note that $d_1 = 0$ for all G and R.) In general, the calculation of the coefficient d_j in Eq. (1.11) requires, as inputs, the values of the b_ℓ for $1 \le \ell \le j$. In addition to these calculations for vectorial gauge theories, Ref. [41] carried out scheme-independent calculations of $\beta'_{\rm IR}$ for chiral gauge theories.

The results of the scheme-independent series expansions in [33–40] are useful for several reasons, which are also motivations for the present study. First, they give new information about fundamental properties of conformal field theories, namely anomalous dimensions at an IRFP in the non-Abelian Coulomb phase of an asymptotically free gauge theory. A second important use of these calculations pertains to the determination of the size of the NACP. The upper end of the NACP, as a function of N_f, is known and is equal to N_u. However, for nonsupersymmetric theories, the lower end, at a value that we denote as $N_{f,cr}$, is not known, and there is an intensive ongoing effort to determine $N_{f,cr}$ by means of lattice simulations [47,48]. Applying scheme-independent calculations of $\gamma_{\bar\psi\psi,\rm IR}$, Refs. [34–38,40] obtained estimates of $N_{f,cr}$ in a manner complementary to lattice gauge simulations. This was done using the monotonic increase of $\gamma_{\bar\psi\psi,\rm IR}$ with decreasing N_f that was shown by the scheme-independent calculations, in conjunction with the rigorous upper limit on $\gamma_{\bar\psi\psi,\rm IR}$ from conformal invariance, namely $\gamma_{\bar\psi\psi,\rm IR} < 2$ [49]. A third application follows from the second, namely that a knowledge of $N_{f,cr}$ (for a given gauge group G and fermion representation R) is necessary for the construction and study of quasiconformal theories of physics beyond the Standard Model (BSM), since these require N_f to be slightly less than $N_{f,cr}$ in order to achieve the slow running of the gauge coupling and associated quasiconformal behavior. In turn, the dynamical breaking of the approximate dilatation invariance in these theories leads to a light approximate Nambu-Goldstone boson, the dilaton [47,48,50–52]. These vectorial BSM theories can naturally arise from the sequential breaking of asymptotically free chiral gauge theories [53]. This is relevant to the investigation of the Higgs boson; although its production and decay properties are consistent with the predictions of the Standard Model, there is the continuing question of whether it might be a composite, dilaton-like state resulting from a quasiconformal BSM theory [51].

The accuracy of the scheme-independent series expansions of $\gamma_{\bar\psi\psi,\rm IR}$ and $\beta'_{\rm IR}$ was studied in several ways in [33–42]. One way was to evaluate the stability of these quantities as higher-order terms in powers of Δ_f were added in the series. It was shown that the finite-order scheme-independent series calculations were most accurate at the upper end of the NACP, and remained reasonably accurate over a substantial portion of the NACP extending to lower values of N_f.

For the gauge group $G = {\rm SU}(3)$, a baryon operator has the form of a product of three fermion fields, each transforming as the fundamental (triplet) representation of G, with their gauge indices a, b, c contracted with the ϵ_{abc} tensor to form a color singlet. Relevant previous studies of anomalous dimensions of baryon operators in QCD include [54–61]. In particular, the anomalous dimensions of baryon operators have been calculated to one-loop [54], two-loop [55,57], and three-loop order [58,59] as powers series in α and related studies have been presented in [60,61].

In this paper we shall present, for the first time, analytic scheme-independent series calculations to order $O(\Delta_f^3)$ of anomalous dimensions of several types of baryon operators at an infrared fixed point of an asymptotically free SU(3) gauge theory with N_f fermions in the fundamental representation. An assessment of the accuracy of these calculations will also be given. As was discussed previously [33–35], the procedure for the calculation of scheme-independent series expansions requires that the IRFP be exact, and this is only the case in the non-Abelian Coulomb phase, in which the chiral flavor symmetry is exact [62]. Since we thus necessarily restrict our analysis to the NACP, where there is no confinement, we use the term "baryon" to refer only to the property that the baryon operators that we consider are singlets under the SU(3) gauge symmetry. We note that there is actually some irony in using the term "baryon" here, since it is derived from the Greek word $\beta\alpha\rho\upsilon\varsigma$, meaning "heavy." However, a gauge-singlet state produced by the operation of a baryon creation operator on the vacuum in the non-Abelian Coulomb phase is massless, as are all physical states in this phase.

As a second part of our paper, we shall present, for general gauge group G and fermion representation R, an explicit illustration of the scheme independence of the earlier calculations of Δ_f expansions of $\gamma_{\bar\psi\psi,\rm IR}$ and $\beta'_{\rm IR}$ [33–38,40]. These calculations naturally used the $\overline{\rm MS}$ scheme because the n-loop coefficients in the beta function and in $\gamma_{\bar\psi\psi}$ had been calculated to the highest loop order in this scheme, and these coefficients have the simplest form in this scheme. Since a rigorous proof was already given in these earlier works of the scheme independence of the coefficients in these Δ_f expansions, it is not necessary to carry out the calculations in schemes other than the simplest one. However, it is, nevertheless, quite instructive to see how the considerably more complicated higher-order coefficients in the beta function and anomalous dimensions in these more complicated schemes combine to reproduce exactly the results of the $\overline{\rm MS}$ scheme for the coefficients in the various Δ_f series expansions. For the purpose of these illustrations, we shall consider a variety of different schemes, including the RI′ scheme [63,64] and several varieties of momentum (MOM) subtraction schemes [29,65–70] (see also [71]).

It should be mentioned that this program of explicitly demonstrating scheme independence of the coefficients in the Δ_f expansions of anomalous dimensions of various operators was previously carried out for the $\mathcal{N} = 1$

supersymmetric gauge theories in [33,35,36,39], where it was shown that the use of two different schemes, namely the $\overline{\mathrm{DR}}$ scheme [72] and the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) scheme [73] yield the same scheme-independent results for the anomalous dimension of a holomorphic composite product of chiral superfields, $\gamma_{\Phi_{\mathrm{comp}},\mathrm{IR}}$, which, order by order are in precise agreement with the corresponding series expansion of the exactly known expression [74]. In addition to demonstrating explicitly that different schemes yield the same values of coefficients in the scheme-independent expansion of $\gamma_{\Phi_{\mathrm{comp}},\mathrm{IR}}$ of the form (1.7), this work showed that (i) the series (1.7) converges to the exact expression everywhere where the latter applies, i.e., in the NACP, (ii) for a fixed N_f in the NACP, a finite truncation of the series (1.7) to order $O(\Delta_f^p)$ approaches the exact expression exponentially rapidly, and (iii) throughout the entire NACP, one achieves excellent accuracy of a few percent even with a series calculated to a modest order of $n = 4$, i.e., $O(\Delta_f^4)$. These scheme-independent calculations of anomalous dimensions in an $\mathcal{N} = 1$ supersymmetric gauge theory thus improved upon conventional scheme-dependent series expansions in powers of α_{IR} [75–76] (see also [77]).

II. BARYON OPERATORS

In this section we consider a theory with gauge group $G = \mathrm{SU}(3)$ and N_f fermions in the fundamental (triplet) representation, $R = F$. Since the fermions are massless, the ultraviolet theory is invariant under the global flavor $(fl.)$ symmetry group

$$G_{fl.} = \mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R \otimes \mathrm{U}(1)_V. \tag{2.1}$$

This symmetry is unbroken in the non-Abelian Coulomb phase. Hence, the baryon operators that we consider transform according to definite representations of this group. Each fermion field can be decomposed into its left- and right-handed chiral components as $\psi = (P_L + P_R)\psi = \psi_L + \psi_R$, where $P_{R,L} = (1/2)(1 \pm \gamma_5)$ and we suppress color and flavor indices here. Showing these latter indices explicitly, each fermion field can thus be written formally as $\psi^a_{i,L} + \psi^a_{i,R}$, where a is an SU(3) color gauge index. Here, the flavor index i on $\psi^a_{i,L}$ refers to the fundamental representation of $\mathrm{SU}(N_f)_L$, while the flavor index i on $\psi^a_{i,R}$ refers to the fundamental representation of $\mathrm{SU}(N_f)_R$. This will be understood implicitly below. The chiral components $\psi^a_{i,L}$ and $\psi^a_{i,R}$ transform as $(N_f, 1)$ and $(1, N_f)$ under the chiral part of $G_{fl.}$, $\mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R$. The bilinear operator $\bar\psi\psi = \sum_{i=1}^{N_f}(\bar\psi_{i,L}\psi_{i,R} + \bar\psi_{i,R}\psi_{i,L})$ thus corresponds to what would be the flavor-singlet in the confined phase, where the chiral part of $G_{fl.}$. is broken to the diagonal $\mathrm{SU}(N_f)_V$ subgroup, while the operator $\bar\psi\mathcal{T}_j\psi$ corresponds to what would be the flavor-adjoint in the confined phase. In our present work we will use the symbols $S_{k,L}$ and $A_{k,L}$ to denote the k-fold symmetric and k-fold antisymmetric representations of $\mathrm{SU}(N_f)_L$, and similarly with $S_{k,R}$ and $A_{k,R}$ with $\mathrm{SU}(N_f)_R$.

Clearly, all of our baryon operators have unit baryonic charge under the $\mathrm{U}(1)_V$ factor group [which is equivalent to $\mathrm{U}(1)_B$ here] so we leave this implicit henceforth. Although we are in an NACP without any confinement of color, it is nonetheless convenient to deal with gauge-singlet operators, since they are gauge-invariant. The invariance of the baryon operator under the SU(3) gauge group is guaranteed by the contraction of the color indices a, b, c on the three fermion fields with the ϵ_{abc} tensor, so that the color part of the baryon wavefunction is totally antisymmetric. The other parts of the baryon operator depend on the chirality, spin contractions, and flavor structure of the three-fermion operator. These are constrained by the requirement that the full wavefunction must be totally antisymmetric under interchange of any two of the fermions.

As is well known, relevant representations of the Lorentz group SO(3,1) are specified by two spins, (j_1, j_2). It is convenient to construct a subset of baryon operators by combining two of the three fermions in a Majorana-type bilinear operator product, since this has spin 0 and is Lorentz-invariant. A Majorana-type bilinear links left-handed to left-handed chiral components of a fermion, and right-handed to right-handed chiral components. There are thus two of these, namely $\psi^{aT}_{i,L}C\psi^b_{j,L}$ and $\psi^{aT}_{i,R}\psi^b_{j,R}$. Here, C is the Dirac charge conjugation matrix defined by $C\gamma_\mu C^{-1} = -\gamma_\mu^T$ and satisfying the properties $C^T = -C$ and $C^{-1} = C^T$. The full baryon operator product is then obtained by combining each of these Majorana-type bilinears with the left-handed or right-handed chiral fermion. One thus has the operators

$$\mathcal{O}_{RLL} = \epsilon_{abc}\psi^a_{i,R}[\psi^{bT}_{j,L}C\psi^c_{k,L}] \tag{2.2}$$

$$\mathcal{O}_{LRR} = \epsilon_{abc}\psi^a_{i,L}[\psi^{bT}_{j,R}C\psi^c_{k,R}] \tag{2.3}$$

$$\mathcal{O}_{RRR} = \epsilon_{abc}\psi^a_{i,R}[\psi^{bT}_{j,R}C\psi^c_{k,R}] \tag{2.4}$$

and

$$\mathcal{O}_{LLL} = \epsilon_{abc}\psi^a_{i,L}[\psi^{bT}_{j,L}C\psi^c_{k,L}]. \tag{2.5}$$

To distinguish the chirality of the unpaired fermion, one could use a subscript L or R, but we shall follow the notational conventions of [55,58], according to which

$$\mathcal{O}^{(\frac{1}{2},0)}_{+} \equiv \mathcal{O}^{(\frac{1}{2},0)}_{LLL} \tag{2.6}$$

and

$$\mathcal{O}_{-}^{(\frac{1}{2},0)} \equiv \mathcal{O}_{RLL}^{(\frac{1}{2},0)}. \tag{2.7}$$

As is evident, in the Lorentz (j_1, j_2) labeling, the $j_1 = 1/2$ refers to the fermion field that is not a member of the Majorana fermion bilinear, and $j_2 = 0$ refers to the spin-0 transformation property of this Majorana fermion bilinear. These operators have anomalous dimensions denoted $\gamma_B^{(\frac{1}{2},0),+}$ and $\gamma_B^{(\frac{1}{2},0),-}$, respectively. Because the theory at the IRFP in the non-Abelian phase preserves the full flavor symmetry (2.1), the anomalous dimension $\gamma_B^{(\frac{1}{2},0),+}$ for $\mathcal{O}_{LLL}^{(\frac{1}{2},0)}$ is equal to the anomalous dimension for the corresponding operator with all L indices switched to R, namely $\mathcal{O}_{RRR}^{(\frac{1}{2},0)}$, and, separately, the anomalous dimension $\gamma_B^{(\frac{1}{2},0),-}$ for $\mathcal{O}_{RLL}^{(\frac{1}{2},0)}$ is equal to the anomalous dimension for the corresponding operator with L and R indices interchanged, namely $\mathcal{O}_{LRR}^{(\frac{1}{2},0)}$.

One part of the classification of baryon operators entails the analysis of the combination of the three spin 1/2 representations of angular momentum SU(2). In general, one has

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{3}{2} \tag{2.8}$$

(i.e., $2 \times 2 \times 2 = 2 + 2 + 4$ in terms of the dimensions $2s + 1$ of the representations). We have considered above the cases in which two of the spins are contracted to produce spin 0, corresponding to one of the two spin-1/2 terms on the right-hand side of Eq. (2.8). There are two remaining cases to consider, in which one combines two of the spins to produce a spin-1 state and then combines this with the third spin 1/2 to yield a net spin 1/2 or spin 3/2. We recall that the spin wavefunction in the case of spin 3/2 is totally symmetric, i.e., S_3 under the SU(2) of spin. In the analysis of baryon operators in QCD, it has proved useful to introduce a vector Δ_μ that is lightlike, i.e., has the property $\Delta^2 = 0$, and consider the operators (leaving the flavor indices implicit)

$$\mathcal{O}_{LLL}^{(\frac{3}{2},0)} = \epsilon_{abc} \not{\Delta}\psi_L^a \not{\Delta}\psi_L^b \not{\Delta}\psi_L^c \tag{2.9}$$

$$\mathcal{O}_{RRR}^{(\frac{3}{2},0)} = \epsilon_{abc} \not{\Delta}\psi_R^a \not{\Delta}\psi_R^b \not{\Delta}\psi_R^c \tag{2.10}$$

$$\mathcal{O}_{LLR}^{(1,\frac{1}{2})} = \epsilon_{abc} \not{\Delta}\psi_L^a \not{\Delta}\psi_L^b \not{\Delta}\psi_R^c \tag{2.11}$$

and

$$\mathcal{O}_{RRL}^{(1,\frac{1}{2})} = \epsilon_{abc} \not{\Delta}\psi_R^a \not{\Delta}\psi_R^b \not{\Delta}\psi_L^c. \tag{2.12}$$

In the notation of [55,58],

$$\mathcal{O}_{+}^{(\frac{3}{2},0)} \equiv \mathcal{O}_{LLL}^{(\frac{3}{2},0)} \tag{2.13}$$

and

$$\mathcal{O}_{-}^{(1,\frac{1}{2})} \equiv \mathcal{O}_{LLR}^{(1,\frac{1}{2})}. \tag{2.14}$$

The anomalous dimensions of these operators are denoted $\gamma^{(\frac{3}{2},0),+}$ and $\gamma^{(1,\frac{1}{2}),-}$, respectively. Again, owing to the exact chiral symmetry (2.1), the anomalous dimension $\gamma^{(\frac{3}{2},0),+}$ of $\mathcal{O}_{LLL}^{(\frac{3}{2},0)}$ is equal to the anomalous dimension of $\mathcal{O}_{RRR}^{(\frac{3}{2},0)}$, and the anomalous dimension $\gamma^{(1,\frac{1}{2}),-}$ of $\mathcal{O}_{LLR}^{(1,\frac{1}{2})}$ is equal to the anomalous dimension of $\mathcal{O}_{RRL}^{(1,\frac{1}{2})}$. The normalization of these anomalous dimensions is fixed by the basic relation (1.1).

III. SCHEME-INDEPENDENT SERIES EXPANSION FOR ANOMALOUS DIMENSION OF GENERAL BARYON OPERATOR

A general expression, calculated to the two-loop level, was given for the anomalous dimension of a general baryon operator $\mathcal{O}_B$ in [57] and extended to the three-loop level in [58,59]. This depends on certain coefficients $\mathbb{C}_k$, which are listed in Table I. With the definition (1.1) (which sets the absolute normalization of the anomalous dimension), and noting that the sign convention in (1.1) is opposite to that in [58], we have

$$\begin{aligned}
\gamma_B = {} & \frac{1}{3}\mathbb{C}_2 a + \left[(-72 + 4N_f)\mathbb{C}_0 + \left(\frac{47}{18} - \frac{1}{27}N_f\right)\mathbb{C}_2 + \frac{1}{36}\mathbb{C}_2^2 - \frac{5}{36}\mathbb{C}_4\right] a^2 \\
& + \left[\left(-\frac{16094}{9} - 34\zeta_3 + \frac{1706}{9}N_f - \frac{20}{9}N_f^2\right)\mathbb{C}_0 + \left(\frac{5873}{108} - \frac{433}{18}\zeta_3 - \left(\frac{71}{27} + \frac{40}{9}\zeta_3\right)N_f - \frac{13}{81}N_f^2\right)\mathbb{C}_2\right. \\
& + \left(-\frac{209}{324} + \frac{71}{27}\zeta_3 + \frac{1}{324}N_f\right)\mathbb{C}_2^2 + \left(\frac{5}{648} - \frac{1}{27}\zeta_3\right)\mathbb{C}_2^3 + \left(\frac{91}{72} - \frac{29}{12}\zeta_3 + \frac{7}{324}N_f\right)\mathbb{C}_4 \\
& \left. + \left(-\frac{37}{432} + \frac{25}{144}\zeta_3\right)\mathbb{C}_2\mathbb{C}_4 + \left(-\frac{1}{8} + \frac{2}{9}\zeta_3\right)\mathbb{C}_{444}\right] a^3 + O(a^4).
\end{aligned} \tag{3.1}$$

TABLE I. Values of $\mathbb{C}_k$ coefficients.

(j_1, j_2)	chirality	$\mathbb{C}_0$	$\mathbb{C}_2$	$\mathbb{C}_4$	$\mathbb{C}_{444}$
$(\frac{1}{2}, 0)$	+	1	12	72	0
$(\frac{1}{2}, 0)$	−	1	12	−24	0
$(\frac{3}{2}, 0)$	+	1	−12	72	0
$(1, \frac{1}{2})$	−	1	−4	−24	0

We list the values of the $\mathbb{C}_k$ coefficients for various specific baryon operators in Table I.

We denote the anomalous dimension of the general baryon operator $\mathcal{O}_B$ as $\gamma_{\mathcal{O}_B}$ and write the scheme-independent series expansion for this as

$$\gamma_B = \sum_{n=1}^{\infty} \kappa_{B,n} \Delta_f^n. \tag{3.2}$$

For this SU(3) gauge theory with N_f fermions in the fundamental representation, $N_u = 33/2$, so the general expression for Δ_f in Eq. (1.6) yields $\Delta_f = (33/2) - N_f$.

We calculate the following coefficients in this scheme-independent series expansion for the general baryon operator:

$$\kappa_{B,1} = \frac{2}{3^2 \cdot (107)} \mathbb{C}_2, \tag{3.3}$$

$$\kappa_{B,2} = -\frac{8}{3 \cdot (107)^2} \mathbb{C}_0 + \frac{27083}{2 \cdot 3^4 \cdot (107)^3} \mathbb{C}_2 + \frac{1}{3^4 \cdot (107)^2} (\mathbb{C}_2^2 - 5\mathbb{C}_4), \tag{3.4}$$

and

$$\begin{aligned}\kappa_{B,3} = {} & \left(\frac{291892}{3^5 \cdot (107)^4} - \frac{272}{3^3 \cdot (107)^3} \zeta_3\right) \mathbb{C}_0 + \left(\frac{352124197}{2^2 \cdot 3^6 \cdot (107)^5} - \frac{238124}{3^5 \cdot (107)^4} \zeta_3\right) \mathbb{C}_2 \\ & + \left(-\frac{47365}{2 \cdot 3^7 \cdot (107)^4} + \frac{568}{3^6 \cdot (107)^3} \zeta_3\right) \mathbb{C}_2^2 + \left(\frac{16525}{2 \cdot 3^6 \cdot (107)^4} - \frac{58}{3^4 \cdot (107)^3} \zeta_3\right) \mathbb{C}_4 \\ & + \left(\frac{5}{3^7 \cdot (107)^3} - \frac{8}{3^6 \cdot (107)^3} \zeta_3\right) \mathbb{C}_2^3 + \left(-\frac{37}{2 \cdot 3^6 \cdot (107)^3} + \frac{25}{2 \cdot 3^5 \cdot (107)^3} \zeta_3\right) \mathbb{C}_2 \mathbb{C}_4 \\ & + \left(-\frac{1}{3^3 \cdot (107)^3} + \frac{16}{3^5 \cdot (107)^3} \zeta_3\right) \mathbb{C}_{444},\end{aligned} \tag{3.5}$$

where $\zeta_s = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. In Eqs. (3.3)–(3.5) we have indicated the simple factorizations of the denominators. The numerators do not, in general, have such simple factorizations.

In floating-point format, to the indicated precision,

$$\kappa_{B,1} = (2.076843 \times 10^{-3}) \mathbb{C}_2, \tag{3.6}$$

$$\kappa_{B,2} = -(2.329170 \times 10^{-4}) \mathbb{C}_0 + (1.364679 \times 10^{-4}) \mathbb{C}_2 + (1.078319 \times 10^{-6}) \mathbb{C}_2^2 - (5.391597 \times 10^{-6}) \mathbb{C}_4, \tag{3.7}$$

and

$$\begin{aligned}\kappa_{B,3} = {} & -(0.721139 \times 10^{-6}) \mathbb{C}_0 - (0.376693 \times 10^{-6}) \mathbb{C}_2 \\ & + (0.681918 \times 10^{-6}) \mathbb{C}_2^2 - (0.616147 \times 10^{-6}) \mathbb{C}_4 \\ & - (0.890178 \times 10^{-8}) \mathbb{C}_2^3 + (2.975975 \times 10^{-8}) \mathbb{C}_2 \mathbb{C}_4 \\ & + (0.343749 \times 10^{-7}) \mathbb{C}_{444}.\end{aligned} \tag{3.8}$$

IV. SCHEME-INDEPENDENT SERIES EXPANSIONS FOR ANOMALOUS DIMENSIONS OF SPECIFIC BARYON OPERATORS

In this section we present results for coefficients in scheme-independent series expansions for the anomalous dimensions of specific baryon operators. These analytic results are new here. The anomalous dimension of the baryon operator $\mathcal{O}_{\pm}^{(j_1,j_2)}$ is denoted $\gamma_B^{(j_1,j_2),\pm}$. We express the scheme-independent series expansion for this anomalous dimension as

$$\gamma_B^{(j_1,j_2),\pm} = \sum_{n=1}^{\infty} \kappa_n^{(j_1,j_2),\pm} \Delta_f^n \tag{4.1}$$

The truncation of this infinite series to maximal power (order) Δ_f^p is denoted $\gamma_{B,\Delta_f^p}^{(j_1,j_2),\pm}$. We note that numerical results for the Δ_f series expansions for two of the four specific operators, namely, $\mathcal{O}_{\pm}^{(\frac{1}{2},0)}$, were given previously in [60]. Since they were based on the results of [58], they should be multiplied by a factor of 2 [59].

We calculate the following:

$$\kappa_1^{(\frac{1}{2},0),+} = \frac{8}{3\cdot(107)} = 2.492212\times10^{-2} \tag{4.2}$$

$$\kappa_2^{(\frac{1}{2},0),+} = \frac{38758}{3^3\cdot(107)^3} = 1.171780\times10^{-2} \tag{4.3}$$

$$\kappa_3^{(\frac{1}{2},0),+} = \frac{314021069}{3^5\cdot(107)^5} - \frac{97792}{3^3\cdot(107)^4}\zeta_3 = 5.892227\times10^{-5} \tag{4.4}$$

$$\kappa_1^{(\frac{1}{2},0),-} = \frac{8}{3\cdot(107)} = 2.492212\times10^{-2} \tag{4.5}$$

$$\kappa_2^{(\frac{1}{2},0),-} = \frac{18626}{3^2\cdot(107)^3} = 1.689374\times10^{-3} \tag{4.6}$$

$$\kappa_3^{(\frac{1}{2},0),-} = \frac{40784885}{3^3\cdot(107)^5} - \frac{70400}{3^3\cdot(107)^4}\zeta_3 = 0.837892\times10^{-4} \tag{4.7}$$

$$\kappa_1^{(\frac{3}{2},0),+} = -\frac{8}{3\cdot(107)} = -(2.492212\times10^{-2}) \tag{4.8}$$

$$\kappa_2^{(\frac{3}{2},0),+} = -\frac{69574}{3^3\cdot(107)^3} = -(2.103448\times10^{-3}) \tag{4.9}$$

$$\kappa_3^{(\frac{3}{2},0),+} = -\frac{32245429}{3^3\cdot(107)^5} + \frac{1169920}{3^4\cdot(107)^4}\zeta_3 = 4.730261\times10^{-5} \tag{4.10}$$

$$\kappa_1^{(1,\frac{1}{2}),-} = -\frac{8}{3^2\cdot(107)} = -(0.830737\times10^{-2}) \tag{4.11}$$

$$\kappa_2^{(1,\frac{1}{2}),-} = -\frac{62726}{3^4\cdot(107)^3} = -(6.321370\times10^{-4}) \tag{4.12}$$

$$\kappa_3^{(1,\frac{1}{2}),-} = -\frac{314714429}{3^6\cdot(107)^5} + \frac{178688}{3^3\cdot(107)^4}\zeta_3 = 2.991050\times10^{-5}. \tag{4.13}$$

As is evident from these results, all of the scheme-independent coefficients $\kappa_n^{(\frac{1}{2},0),+}$ and $\kappa_n^{(\frac{1}{2},0),-}$ that have been calculated, namely those for $n = 1, 2, 3$, are positive. In contrast, we find mixed signs for the scheme-independent coefficients $\kappa_n^{(\frac{3}{2},0),+}$; while $\kappa_1^{(\frac{3}{2},0),+}$ and $\kappa_2^{(\frac{3}{2},0),+}$ are negative, $\kappa_3^{(\frac{3}{2},0),+}$ is positive, and similarly with the $\kappa_n^{(1,\frac{1}{2}),-}$ for $n = 1$, 2, 3.

In Figs. 1–4 we show curves of these anomalous dimensions, and in Tables II–V we list values of these anomalous dimensions, as calculated to the various orders in Δ_f in our scheme-independent expansions.

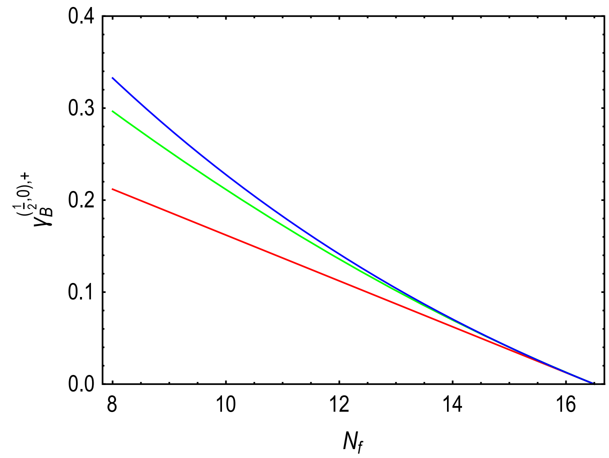

FIG. 1. Plot of $\gamma_B^{(\frac{1}{2},0),+}$, as calculated in the scheme-independent series expansion to $O(\Delta_f^p)$ with $1 \le p \le 3$, as a function of N_f. The curves refer to the calculation to $O(\Delta_f)$ (red); $O(\Delta_f^2)$ (green); and $O(\Delta_f^3)$ (blue), with colors online.

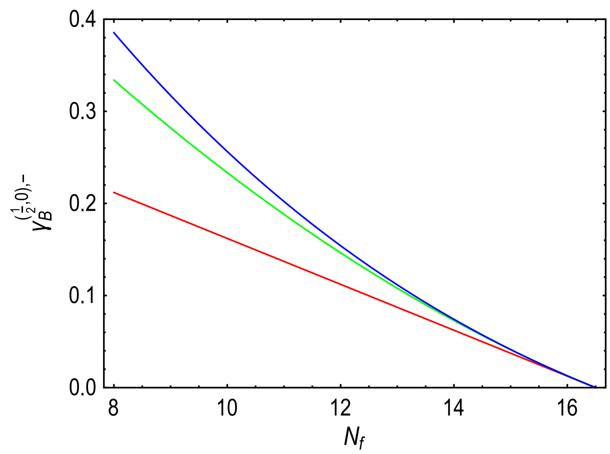

FIG. 2. Plot of $\gamma_B^{(\frac{1}{2},0),-}$, as calculated in the scheme-independent series expansion to $O(\Delta_f^p)$ with $1 \le p \le 3$, as a function of N_f. The curves refer to the calculation to $O(\Delta_f)$ (red); $O(\Delta_f^2)$ (green); and $O(\Delta_f^3)$ (blue), with colors online.

We comment further on the results for the coefficients $\kappa_n^{(\frac{1}{2},0),+}$ and $\kappa_n^{(\frac{1}{2},0),-}$ in the respective scheme-independent series expansions for $\gamma_B^{(\frac{1}{2},0),\pm}$. It will be recalled that an important property of the scheme-independent calculations of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ in [33–39] is that (a) the coefficients κ_1 and κ_2 are manifestly positive, and (b) for all groups and representations considered, κ_3 and κ_4 were also found to be positive. This result implied several monotonicity properties, namely that (i) for a fixed truncation order p, the scheme-independent series expansion for $\gamma_{\bar\psi\psi,\mathrm{IR}}$ is a monotonically increasing function of Δ_f, i.e., it increases monotonically with decreasing N_f, and (ii) for a fixed value of N_f, the

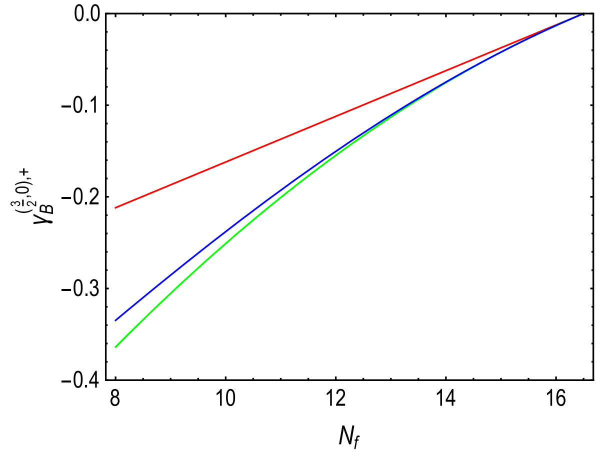

FIG. 3. Plot of $\gamma_B^{(\frac{3}{2},0),+}$, as calculated in the scheme-independent series expansion to $O(\Delta_f^p)$ with $1 \le p \le 3$, as a function of N_f. The curves refer to the calculation to $O(\Delta_f)$ (red); $O(\Delta_f^2)$ (green); and $O(\Delta_f^3)$ (blue), with colors online.

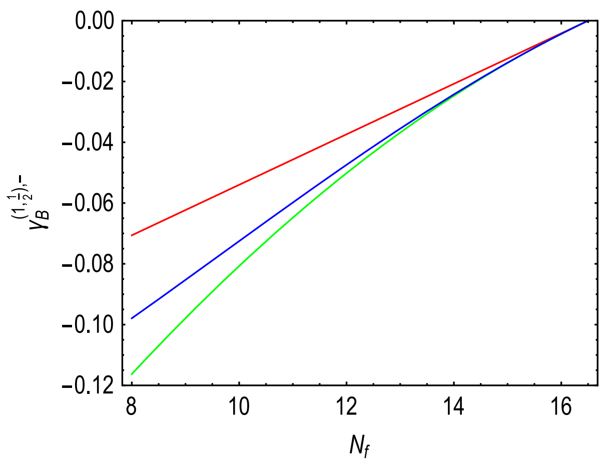

FIG. 4. Plot of $\gamma_B^{(1,\frac{1}{2}),-}$, as calculated in the scheme-independent series expansion to $O(\Delta_f^p)$ with $1 \le p \le 3$, as a function of N_f. The curves refer to the calculation to $O(\Delta_f)$ (red); $O(\Delta_f^2)$ (green); and $O(\Delta_f^3)$ (blue), with colors online.

series calculation to $O(\Delta_f^p)$ is a monotonically increasing function of p. Indeed, as was noted in several of these works, and was studied in detail in [39], the coefficients in the corresponding scheme-independent expansions of anomalous dimensions of composite holomorphic products of chiral superfields in $\mathcal{N}=1$ supersymmetric gauge theories are all positive.

In view of these previous positivity findings, it is of considerable interest that all of the $\kappa_n^{(\frac{1}{2},0),+}$ and $\kappa_n^{(\frac{1}{2},0),-}$ that have been calculated, namely those for $j = 1$, 2, 3, are positive, so the corresponding monotonicity results apply for $\gamma_B^{(\frac{1}{2},0),\pm}$. These calculations to finite order in $O(\Delta_f)$ are expected to be most accurate for small Δ_f, i.e., for N_f

TABLE II. Values of $\gamma_{B,\Delta^p}^{(\frac{1}{2},0),+}$ with $1 \le p \le 3$.

N_f	$\gamma_{B,\Delta^1}^{(\frac{1}{2},0),+}$	$\gamma_{B,\Delta^2}^{(\frac{1}{2},0),+}$	$\gamma_{B,\Delta^3}^{(\frac{1}{2},0),+}$
8	0.212	0.296	0.333
9	0.187	0.253	0.278
10	0.162	0.212	0.228
11	0.137	0.173	0.182
12	0.112	0.136	0.141
13	0.0872	0.102	0.104
14	0.0623	0.0696	0.0705
15	0.0374	0.0400	0.0402
16	0.0125	0.0128	0.0128

TABLE III. Values of $\gamma_{B,\Delta^p}^{(\frac{1}{2},0),-}$ with $1 \le p \le 3$.

N_f	$\gamma_{B,\Delta^1}^{(\frac{1}{2},0),-}$	$\gamma_{B,\Delta^2}^{(\frac{1}{2},0),-}$	$\gamma_{B,\Delta^3}^{(\frac{1}{2},0),-}$
8	0.212	0.334	0.385
9	0.187	0.282	0.317
10	0.162	0.233	0.256
11	0.137	0.188	0.202
12	0.112	0.146	0.154
13	0.0872	0.108	0.112
14	0.0623	0.0729	0.0742
15	0.0374	0.0412	0.0415
16	0.0125	0.0129	0.0129

TABLE IV. Values of $\gamma_{B,\Delta^p}^{(\frac{3}{2},0),+}$ with $1 \le p \le 3$.

N_f	$\gamma_{B,\Delta^1}^{(\frac{3}{2},0),+}$	$\gamma_{B,\Delta^2}^{(\frac{3}{2},0),+}$	$\gamma_{B,\Delta^3}^{(\frac{3}{2},0),+}$
8	−0.212	−0.364	−0.335
9	−0.187	−0.305	−0.285
10	−0.162	−0.251	−0.238
11	−0.137	−0.201	−0.193
12	−0.112	−0.155	−0.150
13	−0.0872	−0.113	−0.111
14	−0.0623	−0.0755	−0.0747
15	−0.0374	−0.0421	−0.0420
16	−0.0125	−0.0130	−0.0130

TABLE V. Values of $\gamma_{B,\Delta^p}^{(1,\frac{1}{2}),-}$ with $1 \le p \le 3$.

N_f	$\gamma_{B,\Delta^1}^{(1,\frac{1}{2}),-}$	$\gamma_{B,\Delta^2}^{(1,\frac{1}{2}),-}$	$\gamma_{B,\Delta^3}^{(1,\frac{1}{2}),-}$
8	−0.0706	−0.117	−0.0979
9	−0.0623	−0.0979	−0.0852
10	−0.0540	−0.0807	−0.0725
11	−0.0457	−0.0648	−0.0598
12	−0.0374	−0.0502	−0.0475
13	−0.0291	−0.0368	−0.0355
14	−0.0208	−0.0247	−0.0243
15	−0.0125	−0.0139	−0.0138
16	−0.00415	−0.00431	−0.00431

slightly below $N_u = 16.5$, while higher-order corrections become progressively larger as N_f decreases toward the lower end of the NACP. In [34–37] these scheme-independent calculations were used to derive estimates of the value of N_f at the lower end of the NACP. The method was to use the unitarity lower bound $D_{\mathcal{O}} \geq 1$ for a Lorentz-scalar operator $\mathcal{O}$ in a conformal field theory [49]. From the basic definition (1.1), taking into account that the free-field (classical) dimension of $\bar{\psi}\psi$ is $D_{\bar{\psi}\psi,cl.} = 3$, there follows the upper bound $\gamma_{\bar{\psi}\psi,\mathrm{IR}} \leq 2$. Combining this with the above-mentioned monotonicity results for the scheme-independent calculation of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ yielded the estimate [34–37] that the conformal non-Abelian Coulomb phase extends from $N_u = 16.5$ down to slightly above $N_f = 8$, so the maximal value of Δ_f in this NACP, is $(\Delta_f)_{\max} \simeq 8$.

As was done for $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ and β'_{IR} in previous works [33–35,37], we may estimate the accuracy of these $O(\Delta_f^3)$ series calculations of $\gamma_B^{(\frac{1}{2},0),+}$ and $\gamma_B^{(\frac{1}{2},0),-}$ in several ways. The first is to plot the various truncations to $O(\Delta_f^p)$ with $p = 1, 2, 3$ as functions of Δ_f, or equivalently, N_f in the conformal regime (non-Abelian Coulomb phase) and ascertain how close the curves are to each other. As expected, the curves of $\gamma_B^{(\frac{1}{2},0),+}$, calculated to the higher two orders, $O(\Delta_f^2)$ and $O(\Delta_f^3)$, remain close to each other over a larger range, extending to lower N_f, than the corresponding curves calculated to the lower two orders, $O(\Delta_f)$ and $O(\Delta_f^2)$. A similar comment applies to the corresponding curves of $\gamma_B^{(\frac{1}{2},0),-}$.

We recall that if a function $f(z)$ is analytic at $z = 0$ and thus has a Taylor series $f(z) = \sum_{n=1}^{\infty} s_n z^n$, then the ratio test states that the series converges to the function $f(z)$ if $|z| < z_0$, where

$$z_0 = \lim_{n\to\infty} \frac{|s_n|}{|s_{n+1}|}. \tag{4.14}$$

Of course, even if these series expansions in powers of Δ_f were Taylor series, it would not be possible to actually calculate the limit (4.14), since we have only the first few coefficients. Furthermore, the Δ_f expansion is not generically expected to be a Taylor series, because the properties of the theory change qualitatively as N_f increases through N_u and the theory becomes IR-free instead of UV-free. Nevertheless, a calculation of the first few ratios can give a rough idea of the accuracy of a truncation of the series to a given order. Accordingly, this was carried out for $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ and β'_{IR} in [33–38]. It was found that the series expansions for $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ to $O(\Delta_f^4)$ and β'_{IR} to $O(\Delta_f^5)$ were reasonably accurate over a substantial portion of the NACP.

It is thus worthwhile to carry out the analogous calculation of ratios here for $\gamma_B^{(\frac{1}{2},0),\pm}$. We find

$$\frac{\kappa_1^{(\frac{1}{2},0),+}}{\kappa_2^{(\frac{1}{2},0),+}} = 21.27 \tag{4.15}$$

$$\frac{\kappa_2^{(\frac{1}{2},0),+}}{\kappa_3^{(\frac{1}{2},0),+}} = 19.89 \tag{4.16}$$

$$\frac{\kappa_1^{(\frac{1}{2},0),-}}{\kappa_2^{(\frac{1}{2},0),-}} = 14.75 \tag{4.17}$$

and

$$\frac{\kappa_2^{(\frac{1}{2},0),-}}{\kappa_3^{(\frac{1}{2},0),-}} = 20.16. \tag{4.18}$$

These ratios are all substantially larger than $(\Delta_f)_{\max} \simeq 8$, indicating that the scheme-independent series expansions for $\gamma_B^{(\frac{1}{2},0),\pm}$ to $O(\Delta_f^3)$ may be reasonably accurate over a substantial part of the NACP for this SU(3) theory.

V. UNITARITY BOUNDS ON ANOMALOUS DIMENSIONS OF BARYONIC OPERATORS

Since our scheme-independent series expansions for baryon operators apply at an infrared fixed point in the non-Abelian Coulomb phase, where the theory is conformally invariant, it is of interest to study how the resultant anomalous dimensions compare with the unitarity bounds on a conformal field theory. In general [49], for an operator $\mathcal{O}$ characterized by Lorentz spins (j_1, j_2), unitarity in a conformal field theory requires that the full scaling dimension $D_{\mathcal{O}}$ is bounded below according to

$$D_{\mathcal{O}} \geq j_1 + j_2 + 1. \tag{5.1}$$

For our case of SU(3), the free-field dimension of a baryon operator is $D_{B,\mathrm{free}} = 3(3/2) = 9/2$, so, with Eq. (1.1), the lower bound (5.1) implies the upper bound on the anomalous dimension

$$\mathrm{SU}(3)\colon\ \gamma_B^{(j_1,j_2)} \leq \frac{7}{2} - (j_1 + j_2). \tag{5.2}$$

Specifically, for the various operators considered here (suppressing $\pm$),

$$\gamma_B^{(\frac{1}{2},0)} \leq 3 \tag{5.3}$$

$$\gamma_B^{(\frac{3}{2},0)} \leq 2 \tag{5.4}$$

and

$$\gamma_B^{(1,\frac{1}{2})} \le 2. \tag{5.5}$$

For the present theory with gauge group SU(3) and N_f fermions in the fundamental representation, the previous work in [34–37] led to the inference that the lower end of the NACP occurs at $N_{f,cr}$ around 8-9. In Fig. 1 and Fig. 2, one can see that our scheme-independent calculations of $\gamma_B^{(\frac{1}{2},0)+}$ and $\gamma_B^{(\frac{1}{2},0)-}$ to $O(\Delta_f^3)$ are well below the upper bound of 3 in (5.3). Our results for $\gamma_B^{(\frac{3}{2},0)+}$ and $\gamma_B^{(1,\frac{1}{2})-}$ are negative, so they obviously also satisfy the respective upper bounds (5.4) and (5.5).

The fact that these baryon anomalous dimensions, as calculated to $O(\Delta_f^3)$, do not saturate their respective unitarity upper bounds as N_f decreases toward the lower end of the non-Abelian Coulomb phase is reminiscent of the situation for an $\mathcal{N}=1$ supersymmetric gauge theory with gauge group SU(N_c) and N_f pairs of chiral superfields, transforming respectively as the representations R and $\bar{R}$ of SU(N_c), as studied in [39]. For this supersymmetric gauge theory, the only composite chiral superfield for which the anomalous dimension saturates its unitarity upper bound from conformal invariance as N_f approaches the lower end of the NACP from above is the gauge-invariant quadratic chiral superfield, which contains the $\bar{\psi}\psi$ component field product. In contrast [aside from the pseudoreal case of SU(2)], a baryonic chiral superfield does not saturate its unitarity upper bound from conformal invariance at the lower end of the NACP [39].

VI. SCHEMES FOR ILLUSTRATIVE CALCULATIONS

In this section we review some background and methods relevant for our calculations illustrating the scheme independence of the Δ_f series expansions for $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ and β'_{IR}. We consider several schemes for regularization and renormalization. We first discuss these schemes. Recall that a common expression that one obtains from loop integrals performed in d-dimensional spacetime is

$$\frac{\Gamma(2-(d/2))}{(4\pi)^{d/2}}\frac{1}{A^{(d/2)-2}}, \tag{6.1}$$

where $\Gamma(z)$ is the Euler gamma function, and A is a denominator depending on some external momenta. Defining $\epsilon = 4-d$ and expanding about $\epsilon=0$, using the Taylor-Laurent expansion of $\Gamma(z)$ about a pole at $z=0$,

$$\Gamma(z) = \frac{1}{z} - \gamma_E + O(z), \tag{6.2}$$

Eq. (6.1) becomes

$$\frac{1}{(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) - \ln A + O(\epsilon)\right], \tag{6.3}$$

where

$$\gamma_E = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) \simeq 0.5772157. \tag{6.4}$$

In the minimal subtraction scheme MS [14], one subtracts the pole term, $2/\epsilon$. In the modified minimal subtraction scheme $\overline{\mathrm{MS}}$ [13], one subtracts the pole term and also the two following terms, namely the combination $2/\epsilon - \gamma_E + \ln(4\pi)$. Both the MS and $\overline{\mathrm{MS}}$ schemes are mass-independent and have the appeal that the beta function and anomalous dimensions of gauge-invariant operators are gauge-invariant. As was noted above, the calculations of [33–38,40,41] used this scheme, although the resulting Δ_f expansions were proved to be scheme-independent.

In addition to the $\overline{\mathrm{MS}}$ scheme used in the previous work [34–38,40], the schemes that we use for our present illustrative demonstrations of scheme independence of Δ_f expansions are the following:

(1) the modified renormalization-invariant scheme (RI′) [63,64],
(2) the momentum subtraction scheme MOMggg defined by focusing on the triple-gluon vertex [29,70],
(3) the momentum subtraction scheme MOMh defined by focusing on the gluon-ghost-ghost vertex [29,70],
(4) the momentum subtraction scheme MOMq defined by focusing on the gluon-fermion-fermion vertex [29,70] (indicated with the subscript q for "quark"), and
(5) the minimal momentum subtraction (mMOM) scheme [65,68].

We write the conventional expansion of $\gamma_{\bar{\psi}\psi}$ as

$$\gamma_{\bar{\psi}\psi} = \sum_{\ell=1}^{\infty} c_\ell a^\ell, \tag{6.5}$$

where the c_ℓ are the ℓ-loop coefficients and, where no confusion will result, we set $c_\ell \equiv c_{\bar{\psi}\psi,\ell}$. The one-loop coefficient, $c_1 = 6C_f$, is scheme-independent, while the c_ℓ with $\ell \ge 2$ are scheme-dependent [44]. The evaluation of the n-loop truncation of (6.5) at the IRFP is obtained by substituting $\alpha = \alpha_{\mathrm{IR},n\ell}$ and is denoted $\gamma_{\mathrm{IR},n\ell}$.

Concerning the beta function (1.2), the one-loop coefficient, b_1 [3], is scheme-independent. In mass-independent schemes, the two-loop coefficient, b_2 [4], is also independent of the specific scheme [44]. We have mentioned above the calculations of b_3 [11], b_4 [12], and b_5 [16,17] in the $\overline{\mathrm{MS}}$ scheme. As noted, the c_ℓ were calculated to four-loop order [26] and to five-loop order in [27], in the $\overline{\mathrm{MS}}$ scheme [78].

The b_ℓ and c_ℓ have been calculated to four-loop order in the RI′ scheme [64] and the minimial MOM (mMOM) scheme [68]. Additional calculations in generalized MOM schemes were presented in [70]. A comparison of conventional calculations of $\alpha_{\mathrm{IR},n\ell}$ and $\gamma_{\mathrm{IR},n\ell}$ was given up to the four-loop order in [20,21,23,25]. An important aspect in which the RI′ and MOM schemes differ with the $\overline{\mathrm{MS}}$ scheme is that beyond the lowest orders, the b_ℓ and c_ℓ are gauge-dependent. We consider a covariant gauge-fixing term so that the gauge part of the Lagrangian is [with our $(+---)$ metric]

$$\mathcal{L}_{\mathrm{gauge}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \mathrm{F.P.}, \tag{6.6}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{6.7}$$

is the field-strength tensor, with $a = 1, .., o(G)$ is the group index, $o(G)$ is the order of the gauge group, f^{abc} are the structure constants of the Lie algebra of G, and F.P. denote Faddeev-Popov terms. The gauge field propagator is thus

$$\Delta^{ab}_{\mu\nu}(k) = -\frac{\delta^{ab}[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}]}{k^2}. \tag{6.8}$$

The Landau gauge corresponds to $\xi = 0$, where this propagator is transverse, i.e., $k^\mu \Delta^{ab}_{\mu\nu}(k) = 0$. In these other schemes, the gauge parameter ξ also depends on the Euclidean scale μ, and so there is an associated function that measures this dependence, namely

$$\beta_\xi = \frac{d\xi}{d\ln\mu}. \tag{6.9}$$

We write the series expansion for this in powers of the coupling as

$$\beta_\xi = -2\xi \sum_{\ell=1}^{\infty} b_{\xi,\ell} a^\ell. \tag{6.10}$$

Evidently, the situation is the simplest in Landau gauge, since in this gauge, $\beta_\xi = 0$ and the gauge parameter is independent of the Euclidean scale. The value of α at the IR zero of β_α and the resultant value of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ were calculated in Landau gauge at the three-loop level in the RI′ scheme in [20] and in the mMOM scheme in [21], and at the four-loop level in [23]. We recall the procedure for this calculation. One looks for a physically acceptable simultaneous solution to the two coupled equations

$$\beta_\alpha(\alpha,\xi) = 0, \qquad \beta_\xi(\alpha,\xi) = 0, \tag{6.11}$$

where we have explicitly indicated the dependence of β_α and β_ξ on the variables α and ξ. Because β_ξ is proportional to ξ, one is always guaranteed to find a solution with $\xi = 0$. That is, if $\xi = 0$ at some value $\mu = \mu_0$, then $\xi = 0$ for all μ. This was the basis for the choice of Landau gauge in Refs. [20,21,23]. As was discussed in [23], there also exist fixed points for which $\xi \neq 0$, but these solutions are on a different footing from the $\xi = 0$ solution. As was noted in [20], at the two-loop level in the mMOM scheme, there is also an IRFP with $\xi_{2\ell} = -3$, and calculations at the three-loop level exhibit an IRFP with $\xi_{3\ell}$ near to this value (see also [79]). A list of the b_ℓ, $b_{\xi,\ell}$, and c_ℓ for general ξ, with $1 \le \ell \le 3$ in the mMOM scheme was given in [20] and a list of the b_ℓ, $b_{\xi,\ell}$, and c_ℓ for $\xi = 0$, i.e., Landau gauge, with $1 \le \ell \le 4$ was given in [23] for the RI′ and mMOM schemes. We will also remark on the general case in which ξ is not necessarily zero. The corresponding expressions for the b_ℓ, c_ℓ, and $b_{\xi,\ell}$ are too long and complicated to include here; they have been given, for example, as external files with the arXiv version of [70]. An important difference between the c_ℓ in the RI′ scheme and the b_ℓ and c_ℓ in the MOM schemes, as contrasted with the b_ℓ and c_ℓ in the $\overline{\mathrm{MS}}$ scheme is that in the non-MS schemes, these coefficients depend on a number of additional mathematical functions and constants. For example, as was discussed in [25], at the four-loop level, in addition to the dependence on the group invariants C_A, C_f, and T_f, the b_ℓ and c_ℓ in the $\overline{\mathrm{MS}}$, RI′, and mMOM schemes contain dependence on the quantities

$$\{\mathbb{Q}, \zeta_3, \zeta_5\}. \tag{6.12}$$

For the following, note that ζ_m with even $m = 2r$ are proportional to π^{2r}:

$$\zeta_{2r} = \frac{(-1)^{r+1} B_{2r} (2\pi)^{2r}}{2(2r)!}, \tag{6.13}$$

where the B_n are the Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \tag{6.14}$$

so listing π^2 is equivalent to listing ζ_2, etc. In contrast to Eq. (6.12), even at the lower, three-loop level, b_ℓ and c_ℓ in the other MOM schemes have a considerably more complicated form, since they depend on the following set of mathematical functions and constants:

$$\Big\{\mathbb{Q}, \pi^2, \zeta_3, \pi^4, \psi'(1/3), \psi'''(1/3), s_2(\pi/k), s_3(\pi/k), \frac{\pi\ln(3)}{\sqrt{3}}, \frac{\pi\ln(3)^2}{\sqrt{3}}, \frac{\pi^3}{\sqrt{3}}\Big\}, \tag{6.15}$$

where here k takes the values $k = 2$ and $k = 6$; $\psi(s)$ is the Euler ψ function, $\psi(s) = d\ln[\Gamma(s)]/ds$, $\psi'(s) = d\psi(s)/ds$; and $s_n(z)$ is defined as

$$s_n(z) = \frac{1}{\sqrt{3}} \mathrm{Im}\left[\mathrm{Li}_n\left(\frac{e^{iz}}{\sqrt{3}}\right)\right], \tag{6.16}$$

where $\mathrm{Li}_n(z)$ is the polylogarithm function,

$$\mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{Li}_{n-1}(z)}{t} dt \tag{6.17}$$

with $\mathrm{Li}_0(z) = z/(1-z)$ and $\mathrm{Li}_1(z) = -\ln(1-z)$. For $|z| \le 1$, this function has the series representation

$$\mathrm{Li}_n(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^n}, n = 2, 3, \ldots. \tag{6.18}$$

As noted above, the calculation of the coefficient d_n in Eq. (1.11) requires, as input, the ℓ-loop coefficients b_ℓ with $1 \le \ell \le n$. The calculation of the coefficient κ_n in Eq. (1.8) requires, as inputs, the values of the b_ℓ for $1 \le \ell \le n+1$, and the ℓ-loop coefficients c_ℓ in Eq. (6.5) with $1 \le \ell \le n$.

In addition to our explicit demonstration that different schemes yield the same values for the coefficients d_n and κ_n in the scheme-independent expansions (1.11) and (1.8), our work shows that the full physical content of these scheme-independent coefficients is derived from the use of the simplest scheme, namely $\overline{\mathrm{MS}}$. Thus, there is a huge cancellation of the additional mathematical functions and quantities in (6.15) in the scheme-independent coefficients d_n and κ_n. On the one hand, one may take the view that this had to be true, since a rigorous proof was given already that these coefficients are scheme-independent and their values were therefore already completely determined from the calculations in [33–37] in the $\overline{\mathrm{MS}}$ scheme. But nevertheless, our explicit demonstration of the cancellation is quite a striking result.

VII. SCHEME-INDEPENDENT EXPANSION OF $\gamma_{\bar\psi\psi,\mathrm{IR}}$

The coefficients κ_n in the scheme-independent expansion of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ in powers of Δ_f, Eq. (1.8), were calculated for a gauge group G with N_f fermions in a representation R up to $n=3$ in [33] and up to $n=4$ in [36,37]. [The coefficient κ_4 was calculated for $G = \mathrm{SU}(3)$ and $R = F$ in [34].] For example, the first two of these coefficients are

$$\kappa_1 = \frac{8C_f T_f}{C_A(7C_A + 11C_f)} \tag{7.1}$$

and

$$\kappa_2 = \frac{4C_f T_f^2(5C_A + 88C_f)(7C_A + 4C_f)}{3C_A^2(7C_A + 11C_f)^3}. \tag{7.2}$$

For the present work we have explicitly verified that we obtain the same results for these κ_n using the RI$'$, mMOM, and other MOM schemes. We have carried out this check to the highest order possible with existing inputs available in these schemes, i.e., to order $n=3$.

VIII. SCHEME-INDEPENDENT EXPANSION OF β'_{IR}

The derivative β'_{IR} is an important physical quantity characterizing the conformal field theory at α_{IR}. For general gauge group G with N_f fermions in a general representation R, the scheme-independent coefficients d_n were calculated up to $n=4$ in [35] and up to $n=5$ in [36,37]. The first two nonzero coefficients are

$$d_2 = \frac{2^5 T_f^2}{3^2 C_A(7C_A + 11C_f)} \tag{8.1}$$

and

$$d_3 = \frac{2^7 T_f^3(5C_A + 3C_f)}{3^3 C_A^2(7C_A + 11C_f)^2}. \tag{8.2}$$

We have explicitly verified that we obtain the same results for d_n with the RI$'$, mMOM, and other MOM schemes. We have carried out this check to the highest order possible with existing inputs available in these schemes, i.e., to order $n=4$.

IX. CONCLUSIONS

In conclusion, in this paper we have presented the first analytic scheme-independent expansions to $O(\Delta_f^3)$ for the anomalous dimensions of a variety of (gauge-invariant) baryon operators at an infrared fixed point of an asymptotically free SU(3) gauge theory with N_f fermions in the fundamental (triplet) representation. Furthermore, for an asymptotically free theory with a general gauge group G and N_f fermions in a general representation R of G, we have given explicit illustrative demonstrations of the scheme independence of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ and β'_{IR} at an IRFP. Although this scheme independence had been proved rigorously earlier, it is worthwhile to see how different schemes yield identical results for the coefficients in the scheme-independent expansions. We have carried out these calculations for the RI$'$ and several MOM schemes.

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