Dual properties of dense quark matter with color superconductivity phenomenon

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In this paper, the massless Nambu–Jona-Lasinio model extended by the diquark interaction channel is considered. We study its phase structure at zero temperature and in the presence of baryon μ_B, isospin μ_I, chiral μ_5, and chiral isospin μ_{I5} chemical potentials in the mean-field approximation. It is shown that the model thermodynamic potential, which depends on three order parameters, M, π_1, and Δ (where M, π_1, and Δ are, respectively, the chiral, charged pion, and diquark condensates of the model), is symmetric with respect to the (dual) transformation when $M \leftrightarrow \pi_1$ and simultaneously $\mu_I \leftrightarrow \mu_{I5}$. As a result, on the mean-field phase portrait of the model, the chiral symmetry breaking (CSB) and charged pion condensation (PC) phases turn out to be dually conjugate with each other, which greatly simplifies the study of the phase portrait of the model. In particular, the duality between CSB and charged PC phases means that in the (μ_I, μ_{I5})-phase portrait these phases are mirror symmetrical with respect to the line $\mu_I = \mu_{I5}$, which at the same time is the symmetry axis of the color superconducting (CSC) phase. Moreover, it follows from our analysis that chiral μ_5 chemical potential promotes the formation of CSC phase in dense quark matter. And together with μ_{I5}, it can generate the charged PC phase even at $\mu_I = 0$.

I. INTRODUCTION

It is well known that QCD is the theoretical basis for the study of strongly interacting matter. However, to consider the properties of dense quark (baryonic) matter, which can exist in the cores of compact stars or arise in collisions of heavy ions, perturbative QCD methods are inapplicable, since the coupling constant of strong interactions is quite large. In this case, effective QCD-like models, including the simplest Nambu–Jona-Lasinio (NJL) model which describes four-fermion interaction of u and d quarks [1–4], are often used to study dense baryonic medium composed of u and d quarks.

Extending this model by the terms with baryon μ_B and isospin μ_I chemical potentials, one can investigate the properties of quark matter with nonzero baryon n_B and nonzero isospin n_I densities (which is typical for neutron stars) [5–10]. More recently, it has become clear that chiral asymmetry (or chiral imbalance), i.e., unequal densities n_L and n_R of all left- and all right-handed quarks, is also one

of the properties of dense quark matter. Usually, the chiral asymmetry is characterized by the quantity n_5, called the chiral density, $n_5 \equiv n_R - n_L$. It can be generated dynamically at high temperatures, such as in a fireball after a heavy ion collision, due to the Adler-Bell-Jackiw anomaly and the interaction of quarks with gauge (gluon) field configurations with nontrivial topology, called sphalerons. However, in the most general case, the chiral densities of u and d quarks, i.e., the values $n_{u5} \equiv n_{uR} - n_{uL}$ and $n_{d5} \equiv n_{dR} - n_{dL}$, are not equal, and the quantity $n_{I5} \equiv n_{u5} - n_{d5}$ is also nonzero (it is the so-called chiral isospin density of quark matter). As it was discussed in Ref. [11], the nonzero n_{I5} can be created in dense quark matter in the presence of rather strong magnetic fields and even not so large values of temperature; i.e., chiral isospin density can be observed in magnetars. As a result, we see that baryon medium composed of u and d quarks can be characterized, in addition to temperature, by four more physical parameters, i.e., by densities n_B, n_I, n_5, and n_{I5}. Its thermodynamics can be studied, in particular, on the basis of the simplest NJL model extended with four chemical potentials μ_B, μ_I, μ_5, and μ_{I5}, which are thermodynamically conjugated to the corresponding densities.

In the mean-field approximation, the $(\mu_B, \mu_I, \mu_5, \mu_{I5})$-phase portrait of this *massless* NJL model has been investigated in the papers [12,13], where it was shown that

only two nontrivial phases are allowed for dense quark matter: (i) the chiral symmetry breaking (CSB) phase and (ii) the charged pion condensation (PC) one. Moreover, it was established that CSB and charged PC phases are dually conjugated to each other. It means that at fixed μ_B and μ_5 these phases are arranged mirror-symmetrically with respect to the line $\mu_I = \mu_{I5}$ on the mean-field (μ_I, μ_{I5})-phase portrait of the model, and so on. A more detailed influence of this dual symmetry on the phase structure of quark matter considered in the framework of the simplest massless NJL model has been investigated in Refs. [4,12–15]. As a result, it became clear that duality greatly simplifies the study of the phase structure of quark matter, especially when it is under the influence of several external factors (chemical potentials). However, it must be borne in mind that the duality between CSB and charged PC is exact (in the large-N_c or mean-field approximations) only in the chiral limit, i.e., when the bare quark mass m is zero. At $m \neq 0$, the duality between CSB and charged PC is only approximative [15]. Nevertheless, the NJL analysis at the physical point and observed approximative duality between these phases are in fairly good agreement with lattice QCD. In addition, it is important to be aware that duality has been proven only within the framework of the simplest NJL model, which is equivalent effectively to QCD only in the region of rather small baryon densities, i.e., at $\mu_B < 1$ GeV. At higher energies or values of chemical potentials, a more complicated NJL model, whose Lagrangians contain other quark interaction channels, should be taken into account. As a result, the set of possible ground states of dense quark matter can be expanded, which (in the absence of dual symmetry between some phases of matter) leads to significant difficulties in studying the phase structure of real quark matter, especially in the presence of four chemical potentials. The solution of the task could be greatly simplified if we had confidence in the existence of duality between some phases of matter even at rather large values of μ_B, i.e., in the framework of a more complex NJL models, which have a rather diverse structure of four-quark interactions. The proposed work is devoted to the consideration of this problem.

An illustration of the above can be the situation with quark matter inside neutron stars, whose baryon density is several times greater than the density of ordinary nuclear matter, i.e., in this case $\mu_B \gtrsim 1$ GeV. Moreover, there exist an evident isospin and chiral asymmetries. Under such extreme conditions, not only the quark-antiquark pairing, but also the formation of diquark pairs, as well as their condensation, is possible. It means that for large values of μ_B the phenomenon of color superconductivity (CSC) can be observed in the system. Therefore, to make it a more adequate investigation of quark matter properties, an extension of the standard NJL model with the help of four-fermion terms responsible for the diquark interaction

channel is usually used [see, e.g., the reviews [3,16] or Eq. (1) below]. The main goal of the present paper is to show that in this case, despite the complication of the model, its thermodynamic potential as a function of order parameters and the four above-mentioned chemical potentials has in the mean-field approximation a dual symmetry between the CSB and the charged PC phases. Thereby, the study of the phase structure of the model is simplified, and it is possible to shed new light on the role of both isospin and chiral (isospin) asymmetries in the formation of diquark condensation. Moreover, our results can serve as an argument in favor of the fact that duality is a characteristic feature of the QCD itself, and not only of its different effective models.

The paper is organized as follows. In Sec. II, a $(3 + 1)$-dimensional NJL model with two massless quark flavors (u and d quarks) that includes four kinds of chemical potentials, $\mu_B, \mu_I, \mu_{I5}, \mu_5$, is introduced. In addition to usual quark-antiquark channels, the model contains the diquark interaction one and is intended to describe the phenomenon of color superconductivity in a dense quark medium. Furthermore, the symmetries of the model are discussed, and its thermodynamic potential (TDP) is presented in the mean-field approximation. In Sec. III, the definition of the dual symmetry of the TDP is given. It means that TDP is invariant under some interchange of chemical potentials as well as, in some cases, simultaneous interchange of condensates. In particular, it was established, and this is one of the main results of the paper, that in the mean-field approximation the TDP of the model is invariant under the following two simultaneous transformations: $\mu_I \leftrightarrow \mu_{I5}$ and the chiral condensate $\leftrightarrow$ charged PC one. As a result, in the different (μ_I, μ_{I5})-phase portraits of the model, CSB and charged PC phases are dually conjugated to each other; i.e., they locate mirror symmetrically with respect to the line $\mu_I = \mu_{I5}$. In Sec. IV, a summary and conclusions are given. Some technical details are relegated to Appendixes A and B.

II. MODEL AND ITS THERMODYNAMIC POTENTIAL

Our investigation is based on the NJL type model with two quark flavors. Its Lagrangian describes the interaction in the quark-antiquark as well as scalar diquark channels,

$$L = \bar{q}[\gamma^\nu i\partial_\nu - m]q + G[(\bar{q}q)^2 + (\bar{q}i\gamma^5\vec{\tau}q)^2]$$
$$+ H\sum_{A=2,5,7}[\overline{q^c}i\gamma^5\tau_2\lambda_A q][\bar{q}i\gamma^5\tau_2\lambda_A q^c], \quad (1)$$

where the quark field $q \equiv q_{i\alpha}$ is a flavor doublet ($i = 1, 2$; alternatively, we use the notations $q_1 = u$ and $q_2 = d$) and color triplet ($\alpha = 1, 2, 3$ or $\alpha = r, g, b$) as well as a four-component Dirac spinor; $q^c = C\bar{q}^T$ and $\overline{q^c} = q^T C$ are charge-conjugated spinors, and $C = i\gamma^2\gamma^0$ is the charge conjugation matrix (the symbol T denotes the transposition

operation). It is supposed that u and d quarks have an equal current (bare) mass m. In Eq. (1) and below, τ_a stands for Pauli matrices, and λ_A for Gell-Mann matrices in flavor and color space, respectively. Clearly, the Lagrangian L is invariant under transformations from color $SU(3)_c$ as well as baryon $U(1)_B$ groups. In addition, at $m = 0$, this Lagrangian is invariant under the chiral $SU(2)_L \times SU(2)_R$ group. At $m \neq 0$, the chiral symmetry is broken to the diagonal isospin subgroup $SU(2)_I$ with the generators $I_k = \tau_k/2$ $(k = 1, 2, 3)$. Moreover, in our system, the electric and baryonic charges are conserved quantities, too, since $Q = I_3 + B/2$, where I_3 is the third generator of the isospin group $SU(2)_I$, Q is the electric charge generator, and B is the baryon charge generator (evidently, these quantities are unit matrices in color space, but in flavor space, they are $Q = \mathrm{diag}(2/3, -1/3)$, $I_3 = \mathrm{diag}(1/2, -1/2)$, and $B = \mathrm{diag}(1/3, 1/3)$). If the Lagrangian (1) is obtained from the QCD one-gluon exchange approximation, then the quantity H/G should not differ too much from the value of 0.75 [3,16].

Note that the Lagrangian (1) describes physical processes in vacuum. To use the model (1) to study properties of dense quark medium, it is necessary to modify Lagrangian (1) by adding terms with chemical potentials,

$$
\begin{aligned}
L_{\mathrm{dense}} &= L + \bar{q}\mathcal{M}\gamma^0 q \\
&\equiv L + \bar{q}\left[\frac{\mu_B}{3} + \frac{\mu_I}{2}\tau_3 + \frac{\mu_{I5}}{2}\gamma^5\tau_3 + \mu_5\gamma^5\right]\gamma^0 q, \quad (2)
\end{aligned}
$$

where the chemical potential matrix $\mathcal{M}$ contains in the most general case four different chemical potential terms responsible for the description of quark matter with nonzero baryon (μ_B), isospin (μ_I), chiral (μ_5), and chiral isospin (μ_{I5}) densities, respectively.

If all chemical potentials in (2) are nonzero quantities, then $SU(2)_I$ at $m \neq 0$ is not the symmetry group of this Lagrangian. Instead, due to the μ_I term, it is symmetric under the flavor $U(1)_{I_3}$ group, $q \to \exp(i\alpha\tau_3/2)q$. Note, however, that in the chiral limit $(m = 0)$ an additional symmetry of the Lagrangian (2) appears, $U(1)_{AI_3}$: $q \to \exp(i\alpha\gamma^5\tau_3)q$.

To study the phase diagram of the system (2), we need to get its thermodynamic potential (in the mean-field approximation). For this purpose, let us consider the linearized version $\mathcal{L}$ of Lagrangian (2) that contains auxiliary bosonic fields,

$$
\begin{aligned}
\mathcal{L} &= \bar{q}[\gamma^\nu i\partial_\nu + \mathcal{M}\gamma^0 - \sigma - m - i\gamma^5\vec{\pi}\vec{\tau}]q - \frac{1}{4G}[\sigma^2 + \pi_a^2] \\
&- \frac{1}{4H}\Delta_A^*\Delta_A - \frac{\Delta_A^*}{2}[\overline{q^c}i\gamma^5\tau_2\lambda_A q] - \frac{\Delta_{A'}}{2}[\bar{q}i\gamma^5\tau_2\lambda_{A'}q^c]. \quad (3)
\end{aligned}
$$

In Eq. (3) and later, a summation over repeated indices $a = 1, 2, 3$ and $A, A' = 2, 5, 7$ is implied. Clearly, the

Lagrangians (2) and (3) are equivalent, as can be seen by using the equations of motion for bosonic fields, which take the form

$$
\begin{aligned}
\sigma(x) &= -2G(\bar{q}q), \quad \pi_a(x) = -2G(\bar{q}i\gamma^5\tau_a q), \\
\Delta_A(x) &= -2H(\overline{q^c}i\gamma^5\tau_2\lambda_A q), \quad \Delta_A^*(x) = -2H(\bar{q}i\gamma^5\tau_2\lambda_A q^c).
\end{aligned}
$$
$$(4)$$

It follows from (4) that the mesonic fields $\sigma(x), \pi_a(x)$ are real quantities, i.e., $(\sigma(x))^\dagger = \sigma(x)$, $(\pi_a(x))^\dagger = \pi_a(x)$ (the superscript symbol † denotes the Hermitian conjugation), but all diquark fields $\Delta_A(x)$ are complex scalars, so $(\Delta_A(x))^\dagger = \Delta_A^*(x)$. Clearly, the real $\sigma(x)$ and $\pi_a(x)$ fields are color singlets, whereas scalar diquarks $\Delta_A(x)$ form a color antitriplet $\bar{3}_c$ of the $SU(3)_c$ group. Note that the auxiliary bosonic field $\pi_3(x)$ corresponds to real $\pi^0(x)$ meson, whereas the physical $\pi^\pm(x)$-meson fields are the following combinations of the composite fields (4), $\pi^\pm(x) = (\pi_1(x) \mp i\pi_2(x))/\sqrt{2}$. If some of the scalar diquark fields have a nonzero ground-state expectation value, i.e., $\langle\Delta_A(x)\rangle \neq 0$, the color symmetry of the model (2) is spontaneously broken down.

In the one fermion-loop approximation (or in the mean-field approximation) and in the presence of the dense quark medium, the effective action $\mathcal{S}_{\mathrm{eff}}(\sigma, \pi_a, \Delta_A, \Delta_{A'}^*)$ of the model (1) is expressed by means of the path integral over quark fields,

$$
\exp(i\mathcal{S}_{\mathrm{eff}}(\sigma, \pi_a, \Delta_A, \Delta_{A'}^*)) = N'\int[d\bar{q}][dq]\exp\left(i\int\mathcal{L}d^4x\right),
$$
$$(5)$$

where

$$
\mathcal{S}_{\mathrm{eff}}(\sigma, \pi_a, \Delta_A, \Delta_{A'}^*) = -\int d^4x\left[\frac{\sigma^2 + \pi_a^2}{4G} + \frac{\Delta_A\Delta_A^*}{4H}\right] + \tilde{\mathcal{S}}_{\mathrm{eff}}
$$
$$(6)$$

and N' is a normalization constant. The quark contribution to the effective action, i.e., the term $\tilde{\mathcal{S}}_{\mathrm{eff}}$ in (6), is

$$
\begin{aligned}
\exp(i\tilde{\mathcal{S}}_{\mathrm{eff}}) = N'\int[d\bar{q}][dq]\exp\Bigg(i\int\Bigg[\bar{q}Dq - \frac{\Delta_A^*}{2}[\overline{q^c}i\gamma^5\tau_2\lambda_A q] \\
- \frac{\Delta_{A'}}{2}[\bar{q}i\gamma^5\tau_2\lambda_{A'}q^c]\Bigg]d^4x\Bigg),
\end{aligned}
$$
$$(7)$$

where we have used the notation

$$
D = (\gamma^\nu i\partial_\nu + \mathcal{M}\gamma^0 - \sigma(x) - m - i\gamma^5\vec{\pi}(x)\vec{\tau})\cdot\mathbb{1}_{3_c}, \quad (8)
$$

where $\mathbb{1}_{3_c}$ is the unit operator in the tree-dimensional color space. Starting from Eqs. (6) and (7), one can define the

TDP $\Omega(\sigma, \pi_a, \Delta_A, \Delta_{A'}^*)$ of the model (1). Indeed, this quantity is defined by the following relation,

$$\mathcal{S}_{\mathrm{eff}}|_{\sigma, \pi_a, \Delta_A, \Delta_{A'}^* = \mathrm{const}} = -\Omega(\sigma, \pi_a, \Delta_A, \Delta_{A'}^*) \int d^4 x. \quad (9)$$

The ground-state expectation values (mean values) $\langle \sigma(x) \rangle, \langle \pi_a(x) \rangle, \langle \Delta_A(x) \rangle, \langle \Delta_{A'}^*(x) \rangle$ of the auxiliary bosonic fields (4) are solutions of the gap equations for the TDP $\Omega(\sigma, \pi_a, \Delta_A, \Delta_{A'}^*)$ (in our approach, all ground-state expectation values do not depend on coordinates x),[1]

$$\frac{\partial \Omega}{\partial \pi_a} = 0, \quad \frac{\partial \Omega}{\partial \sigma} = 0, \quad \frac{\partial \Omega}{\partial \Delta_A} = 0, \quad \frac{\partial \Omega}{\partial \Delta_{A'}^*} = 0, \quad (10)$$

and usually they are the coordinates of the global minimum point (GMP) of Ω vs $\sigma, \pi_a, \Delta_A, \Delta_{A'}^*$. Note that at nonzero bare quark mass $m \neq 0$, the quantity $\langle \sigma(x) \rangle$, called chiral condensate, is always nonzero. However, in the chiral limit, $m = 0$, there can exists a region of chemical potentials in which $\langle \sigma(x) \rangle = 0$. In this case, it is the chiral symmetric phase of the model. In contrast, at $m = 0$, the region with $\langle \sigma(x) \rangle \neq 0$ is called the chiral symmetry breaking phase. Moreover, at both $m = 0$ and $m \neq 0$, the GMP of the TDP with $\langle \pi^{\pm}(x) \rangle \neq 0$ corresponds to the so-called charged pion condensation phase. But in the case when one of the scalar diquark fields has a nonzero ground-state expectation value, $\langle \Delta_A(x) \rangle \neq 0$, we have a phase with spontaneously broken color $SU(3)_c$ symmetry. It is called the color superconducting one.

Recall that both the Lagrangian (2) and the effective action (6) are invariant under the color $SU(3)_c$ group. As a consequence of this fact, all Δ_A and $\Delta_{A'}^*$ dependence of the TDP $\Omega(\sigma, \pi_a, \Delta_A, \Delta_{A'}^*)$ (9) is shown in its form depending on the combination $\Delta_2 \Delta_2^* + \Delta_5 \Delta_5^* + \Delta_7 \Delta_7^* \equiv \Delta^2$, where Δ is a real quantity. Let us note also that in the chiral limit [due to a $U_{I_3}(1) \times U_{AI_3}(1)$ invariance of the model (2)] the TDP Ω (9) depends effectively only on the combinations $\sigma^2 + \pi_3^2$ and $\pi_1^2 + \pi_2^2$ (in addition to Δ). So, at $m = 0$, one can put $\pi_3 = 0$ and $\pi_2 = 0$ without loss of generality, whereas at the physical point (i.e., at $m \neq 0$), it depends effectively on the combination $\pi_1^2 + \pi_2^2$ as well as on σ and π_3. Since in this case the relations $\langle \sigma(x) \rangle \neq 0$ and $\langle \pi_3(x) \rangle = 0$ are always satisfied in the NJL model (2) at $H = 0$,[2] at $m \neq 0$, one can made a rather plausible assumption that in Eq. (9) $\pi_2 = \pi_3 = 0$ is also valid and study this TDP as a function of only three variables, σ, π_1, and Δ, i.e., $\Omega \equiv \Omega(\sigma, \pi_1, \Delta)$. It is clear that in order to calculate $\Omega(\sigma, \pi_1, \Delta)$ it is enough to suppose that in

Eqs. (6) and (7) $\Delta_2 = \Delta_2^* = \Delta$, $\Delta_5 = \Delta_7 = 0$, and $\pi_2 = \pi_3 = 0$ (note that in the following we also suppose that all auxiliary bosonic fields do not depend on space coordinate x). Since

$$\lambda_2 = \begin{pmatrix} 0, & -i, & 0 \\ i, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix} = \begin{pmatrix} \sigma_2, & 0 \\ 0, & 0 \end{pmatrix},$$

where σ_2 is the corresponding Pauli matrix acting in the two-dimensional fundamental representation of the $SU(2)_c$ subgroup of the $SU(3)_c$, it is clear that under this assumption the contribution of the blue q_b quarks in the expression (7) is factorized. Then,

$$\exp(i\tilde{\mathcal{S}}_{\mathrm{eff}}) = N' \int [d\bar{q}_b][dq_b] \exp\left(i \int [\bar{q}_b D^+ q_b] \right)$$
$$\times \int [d\bar{Q}][dQ] \exp\left(i \int [\bar{Q}(D^+ \cdot \mathbb{1}_{2_c})Q \right.$$
$$\left. - \frac{\Delta}{2}[\overline{Q^c} i\gamma^5 \tau_2 \sigma_2 Q] - \frac{\Delta}{2}[\bar{Q} i\gamma^5 \tau_2 \sigma_2 Q^c]] d^4 x \right),$$
$$(11)$$

where q_b is the flavor doublet of the blue quarks and Q is the flavor (u and d) and color (red and green) quark doublet. Moreover, here, we use the notation D^+ for the operator that is in the round brackets of Eq. (8) at $\pi_2 = \pi_3 = 0$ [see below in Eq. (14)] and $\mathbb{1}_{2_c}$ is the unit operator in the two-dimensional color space. Performing in Eq. (11) the functional integrations over q_b (which is a trivial one) and over Q (see Appendix A), we find

$$\exp(i\tilde{\mathcal{S}}_{\mathrm{eff}}) = N' \det D^+ \cdot \det^{1/2}(Z), \quad (12)$$

where

$$Z = \begin{pmatrix} D^+ \cdot \mathbb{1}_{2_c}, & -K \\ -K, & D^- \cdot \mathbb{1}_{2_c} \end{pmatrix}, \quad (13)$$

and

$$D^+ = i\gamma^\nu \partial_\nu - m + \mathcal{M}\gamma^0 - \Sigma, \quad \Sigma = \sigma + i\gamma^5 \pi_1 \tau_1,$$
$$D^- = i\gamma^\nu \partial_\nu - m - \gamma^0 \mathcal{M} - \Sigma, \quad K = i\Delta\gamma^5 \tau_2 \sigma_2. \quad (14)$$

Note that matrix elements of the matrix Z (13) are the operators in two-dimensional color and flavor spaces as well as in four-dimensional spinor and coordinate spaces. Then, it follows from Eqs. (6) and (12) that

$$\mathcal{S}_{\mathrm{eff}}(M, \pi_1, \Delta) = -\int d^4 x \left[\frac{(M-m)^2 + \pi_1^2}{4G} + \frac{\Delta^2}{4H} \right]$$
$$- \frac{i}{2} \ln \det(Z) - i \ln \det(D^+), \quad (15)$$

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[1] In thermodynamics, the quantities $\langle \sigma(x) \rangle, \ldots$ are usually called order parameters, which determine, in essence, the phase structure of the system.

[2] See, e.g., the gap equations (13) and (14) of Ref. [17], which at $m \neq 0$ have single solution $\langle \sigma(x) \rangle \neq 0$ and $\langle \pi_3(x) \rangle = 0$.

where we have introduced the gap $M \equiv \sigma + m$. The last term of Eq. (15), which does not depend on Δ, was calculated in our recent paper [12] [see there Eqs. (16)–(21)],

$$i \ln \det(D^+) = i \int \frac{d^4 p}{(2\pi)^4} \ln[(\eta^4 - 2a_+\eta^2 + b_+\eta + c_+)$$
$$\times (\eta^4 - 2a_-\eta^2 + b_-\eta + c_-)] \int d^4 x, \quad (16)$$

where $\eta = p_0 + \mu$, $|\vec{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2}$ and

$$a_\pm = M^2 + \pi_1^2 + (|\vec{p}| \pm \mu_5)^2 + \nu^2 + \nu_5^2;$$
$$b_\pm = \pm 8(|\vec{p}| \pm \mu_5)\nu\nu_5;$$
$$c_\pm = a_\pm^2 - 4\nu^2(M^2 + (|\vec{p}| \pm \mu_5)^2)$$
$$- 4\nu_5^2(\pi_1^2 + (|\vec{p}| \pm \mu_5)^2) - 4\nu^2\nu_5^2 \quad (17)$$

[we also use in Eq. (17) and below the notations $\mu = \mu_B/3$, $\nu = \mu_I/2$, and $\nu_5 = \mu_{I5}/2$]. The next undefined term of Eq. (15) is

$$\frac{i}{2} \ln \det(Z) = i \int \frac{d^4 p}{(2\pi)^4} \ln \det L(p) \int d^4 x, \quad (18)$$

where $L(p)$ is given by Eq. (A20); i.e., it is the 4×4 matrix in spinor space. Taking into account the relations (15)–(18) as well as the definition (9), it is easy to get the TDP of the model in the mean-field approximation,

$$\Omega(M, \pi_1, \Delta) = \left[\frac{(M - m)^2 + \pi_1^2}{4G} + \frac{\Delta^2}{4H}\right]$$
$$+ i \int \frac{d^4 p}{(2\pi)^4} \ln \det L(p)$$
$$+ i \int \frac{d^4 p}{(2\pi)^4} \ln[(\eta^4 - 2a_+\eta^2 + b_+\eta + c_+)$$
$$\times (\eta^4 - 2a_-\eta^2 + b_-\eta + c_-)]. \quad (19)$$

In what follows, when performing numerical calculations or finding symmetry properties of the TDP $\Omega(M, \pi_1, \Delta)$, it is very convenient to represent $\det L(p)$ in Eq. (19) as a production of four eigenvalues of the matrix $L(p)$,

$$\det L(p) = \tilde{\lambda}_1(p)\tilde{\lambda}_2(p)\tilde{\lambda}_3(p)\tilde{\lambda}_4(p). \quad (20)$$

Here,

$$\tilde{\lambda}_{1,2}(p) = \lambda_{1,2}(p)|_{|\vec{p}| \to |\vec{p}| - \mu_5}, \quad \tilde{\lambda}_{3,4}(p) = \lambda_{3,4}(p)|_{|\vec{p}| \to |\vec{p}| + \mu_5}, \quad (21)$$

and

$$\lambda_{1,2}(p) = N_1 \pm 4\sqrt{K_1}, \quad \lambda_{3,4}(p) = N_2 \pm 4\sqrt{K_2}, \quad (22)$$

where

$$N_2 = N_1 + 16\mu\nu\nu_5|\vec{p}|, \quad K_2 = K_1 + 8\mu\nu\nu_5|\vec{p}|p_0^4 - 8\mu\nu\nu_5|\vec{p}|p_0^2(M^2 + \pi_1^2 + |\Delta|^2 + |\vec{p}|^2 + \mu^2 + \nu^2 - \nu_5^2), \quad (23)$$

$$K_1 = \nu_5^2 p_0^6 - p_0^4[2\nu_5^2(|\Delta|^2 + \pi_1^2 + M^2 + |\vec{p}|^2 + \nu^2 + \mu^2 - \nu_5^2) + 4\mu\nu\nu_5|\vec{p}|] + p_0^2\{\nu_5^6 + 2\nu_5^4(M^2 - |\Delta|^2 - \pi_1^2$$
$$- \nu^2 - \mu^2 - |\vec{p}|^2) + 4\mu^2\nu^2(M^2 + |\vec{p}|^2) + 4|\vec{p}|\mu\nu\nu_5(|\Delta|^2 + \pi_1^2 + M^2 + |\vec{p}|^2 + \nu^2 + \mu^2 - \nu_5^2)$$
$$+ \nu_5^2[(|\Delta|^2 + \pi_1^2 + |\vec{p}|^2 + \nu^2 + \mu^2)^2 + 2|\vec{p}|^2 M^2 + M^4 + 2M^2(|\Delta|^2 - \nu^2 + \pi_1^2 - \mu^2)]\}, \quad (24)$$

$$N_1 = p_0^4 - 2p_0^2[|\Delta|^2 + \pi_1^2 + M^2 + |\vec{p}|^2 + \nu^2 + \mu^2 - 3\nu_5^2] + \nu_5^4 - 2\nu_5^2[|\Delta|^2 + \pi_1^2 + |\vec{p}|^2 + \nu^2 + \mu^2 - M^2]$$
$$- 8\mu\nu\nu_5|\vec{p}| + (|\vec{p}|^2 + M^2 + \pi_1^2 + |\Delta|^2 - \mu^2 - \nu^2)^2 - 4(\mu^2\nu^2 - \pi_1^2\nu^2 - |\Delta|^2\mu^2). \quad (25)$$

Exact expressions for the eigenvalues $\tilde{\lambda}_i(p)$ were obtained earlier in the study of the phase structure of the two-color QCD (see, e.g., Sec. III of Ref. [18]). We are now ready to discuss in the chiral limit, i.e., at $m = 0$, the dual properties of the NJL model (1) in the mean-field approximation.

Throughout the paper, we use in numerical investigations of the TDP (19) the soft cutoff regularization scheme when $d^4 p \equiv dp_0 d^3\vec{p} \to dp_0 d^3\vec{p} f_\Lambda(\vec{p})$. Here, the cutoff function is

$$f_\Lambda(\vec{p}) = \sqrt{\frac{\Lambda^{2N}}{\Lambda^{2N} + |\vec{p}|^{2N}}}, \quad (26)$$

and the parameter fit used is $G = 4.79$ GeV^{-2}, $\Lambda = 638.8$ MeV, and $N = 5$.

III. DUAL SYMMETRIES OF THE TDP (19) AND PHASE PORTRAITS OF THE MODEL

First of all, we note once again that in order to find the phase structure of the NJL model (2) it is necessary to study its TDP (19) as a function of three variables M, π_1, Δ for an absolute minimum and then see how the properties of its GMP change depending on the values of chemical potentials $\mu, \nu, \nu_{I5}, \mu_5$.

In this regard, we remark that in the chiral limit, $m = 0$, for sufficiently low values of the chemical potentials (say, at $\mu, \nu, \nu_5, \mu_5 < 1$ GeV) at the GMP (M, π_1, Δ) of the TDP (19) there can be no more than one nonzero coordinates. (This conclusion can be done based on the arguments presented earlier in our papers [12,18].) Therefore, with such a restriction on chemical potentials, in the chiral limit, only four different phases can be realized in the system:

(1) If GMP has the form $(M \neq 0, \pi_1 = 0, \Delta = 0)$, then the CSB phase appears in the model.

(2) If it has the form $(M = 0, \pi_1 \neq 0, \Delta = 0)$, the charged PC phase is realized.

(3) When the GMP looks like $(M = 0, \pi_1 = 0, \Delta \neq 0)$, it corresponds to the CSC or diquark condensation phase.

(4) The GMP of the form $(M = 0, \pi_1 = 0, \Delta = 0)$ corresponds to a symmetrical phase with all zero condensates.

Then, a phase portrait of the model is no more than a one-to-one correspondence between any point (μ, ν, ν_5, μ_5) of the four-dimensional space of chemical potentials and possible model phases (CSB, charged PC, CSC, and symmetric phase). However, it is clear that this four-dimensional phase portrait (diagram) is quite bulky, and it is rather hard to imagine it as a whole. So, to obtain a deeper understanding of the phase diagram as well as to get a greater visibility of it, it is very convenient to consider different low-dimensional cross sections of this general (μ, μ_5, ν, ν_5)-phase portrait, defined by the constraints of the form $\mu = $ const, $\mu_5 = $ const, etc.

In addition, note that the study of the phase structure of any model is greatly simplified if the so-called dual symmetries of its TDP is taken into account. Recall that by dual symmetry of the TDP we understand its symmetry (invariance) with respect to any discrete transformations as order parameters (in our case, these are M, π_1, and Δ condensates) and free external parameters of the system (these may be chemical potentials, coupling constants, etc.). The presence of the dual symmetry of the model TDP means that in its phase portrait there is some symmetry between phases with respect to the transformation of external parameters. And this circumstance can simplify the construction of the full phase diagram of the system.

For example, taking into account the Eqs. (17)–(25), it is possible to establish the invariance of the TDP (19) with respect to each of the following six transformations,

in each of which two chemical potentials change their sign simultaneously: (i) $\{\nu \to -\nu; \nu_5 \to -\nu_5\}$, (ii) $\{\nu \to -\nu; \mu_5 \to -\mu_5\}$, (iii) $\{\nu_5 \to -\nu_5; \mu_5 \to -\mu_5\}$, (iv) $\{\mu \to -\mu; \mu_5 \to -\mu_5\}$, (v) $\{\mu \to -\mu; \nu \to -\nu\}$, and (vi) $\{\mu \to -\mu; \nu_5 \to -\nu_5\}$. The invariance of the TDP (19) under the transformations (i)–(vi) is one of the simplest examples of its dual symmetries that can help us to simplify the analysis of the phase portrait of the model. In particular, due to the symmetry of the TDP (19) under transformations (i)–(vi), it is sufficient to study the phase structure of the NJL model (2) only, e.g., in the case when arbitrary three of the four chemical potentials have positive signs, whereas the sign of the rest chemical potential is not fixed. For example, let us assume that the phase portrait of the model is already established in the region R, in which $\mu \geq 0$, $\nu \geq 0$, $\nu_5 \geq 0$, but the sign of μ_5 is not fixed. In this case, one can be sure that the phase structure of the model is also established at arbitrary point $P(\mu, \nu, \nu_5, \mu_5)$ of the chemical potential space. Indeed, by applying one or more of the transformations (i)–(iv) to this point, one can transfer it to some point P_r of the above mentioned region R, at each point of which the structure of the ground state of the system is known. And due to the symmetry of the TDP (19) with respect to transformations (i)–(iv), the ground state of the model at the initial point $P(\mu, \nu, \nu_5, \mu_5)$ should have the same structure as at the point $P_r \in R$.

Another and more nontrivial example of the dual symmetry of the TDP $\Omega(M, \pi_1, \Delta)$ (19) is its invariance under the two discrete transformations $\mathcal{D}$,

$$\mathcal{D}: \nu \leftrightarrow \nu_5, M \leftrightarrow \pi_1, \qquad (27)$$

in the chiral limit. The invariance of the last term of Eq. (19) under the transformation $\mathcal{D}$ follows directly from Eqs. (16) and (17). However, checking the $\mathcal{D}$ invariance of its $\det L(p)$ term (11) is not such a simple task. However, this is true, since this fact is easy to establish taking into account the relations (21)–(25) and using any program of analytical calculations.

To understand the physical meaning of the dual symmetry (27), let us suppose that $m = 0$ and that at the point $(\mu = a, \nu = b, \nu_5 = c, \mu_5 = d)$ of the mean-field phase portrait the GMP of the TDP (19) lies, e.g., at the point of the condensate space of the form $(M = A, \pi_1 = B, \Delta = C)$. Then, due to a dual symmetry (27) of the TDP, at the dually $\mathcal{D}$-conjugated point of the phase portrait, i.e., at the point $(\mu = a, \nu = c, \nu_5 = b, \mu_5 = d)$, the dually $\mathcal{D}$-conjugated phase should be located. Its condensate structure has the form $(M = B, \pi_1 = A, \Delta = C)$. Hence, we see that if, e.g., $M = A = 0$, $\pi_1 = B \neq 0$, $\Delta = C = 0$, then the CSB phase corresponding to a GMP of the form $M = B \neq 0$, $\pi_1 = A = 0$, $\Delta = C = 0$ should be dually conjugated to the initial charged PC phase, and vice versa. But if at the original point there is symmetric or CSC
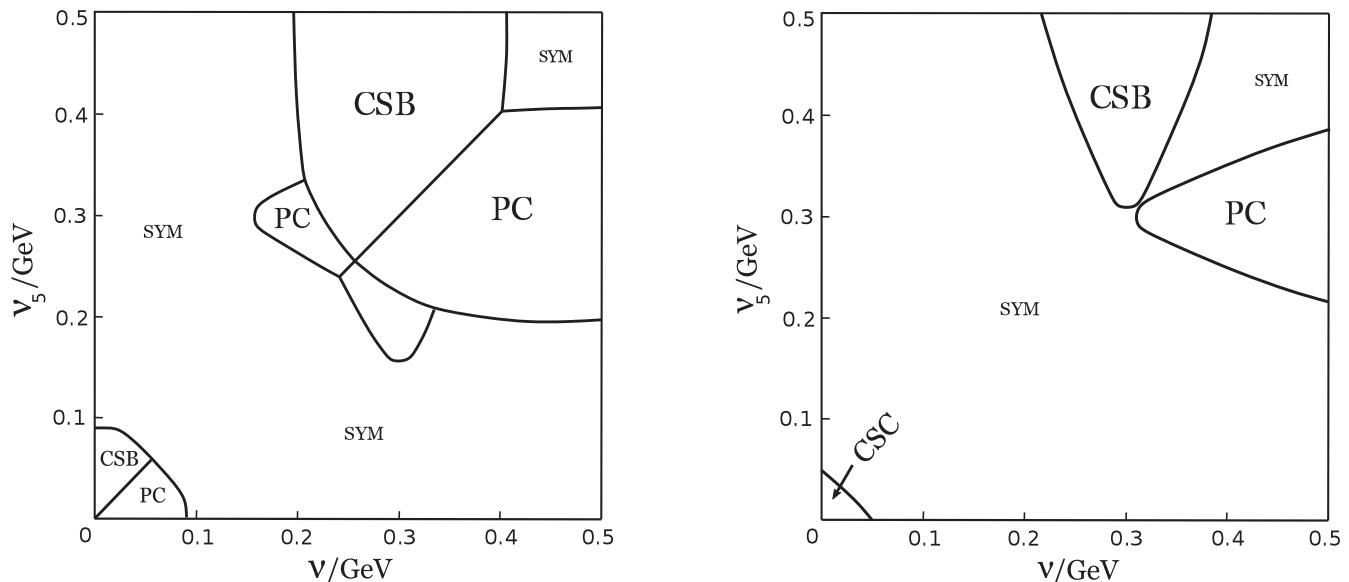
FIG. 1.   Some (ν, ν_5)-phase diagrams of the model at $H = 0.75G$. Left panel: the case of $\mu = 0.3$ GeV and $\mu_5 = 0$. Right panel: the case of $\mu = 0.3$ GeV and $\mu_5 = 0.15$ GeV. Here, PC denotes the charged pion condensation phase; CSB and CSC mean, respectively, the chiral symmetry breaking and color superconducting phases; and "sym" is the symmetric phase.

phase, then it is also realized at the dually conjugated point. Thus, knowing the phase of the model, which is realized at some point of its (μ, ν, ν_5, μ_5)-phase portrait, we can predict which phases are arranged at the dually conjugated points of this phase diagram. Moreover, the order parameter of the initial CSB phase of the point $(\mu = a, \nu = b, \nu_5 = c, \mu_5 = d)$, i.e., the quantity $M = A$, is equal to the order parameter $\pi_1 = A$ of the $\mathcal{D}$-dually conjugated charged PC phase of the point $(\mu = a, \nu = c, \nu_5 = b, \mu_5 = d)$ of the model phase portrait.

As a consequence, we see that at some fixed values of μ and μ_5, in the (ν, ν_5)-phase portrait of the model the CSB and charged PC phases should be mirror-symmetrical to each other with respect to the $\nu = \nu_5$ line. And each of the remaining phases, i.e., the symmetric or CSC one, must occupy an area symmetrical about this line. This fact is well illustrated by four (ν, ν_5)-phase portraits depicted in Figs. 1 and 2 at $\mu = 0.3$ GeV and different values of μ_5. Moreover, it is clear from Fig. 1 (left panel) that at $\mu_5 = 0$ and at rather small values of ν and ν_5 the CSC phase is absent. But when μ_5 is nonzero, this phase appears in a small neighborhood of the point $\nu = 0$, $\nu_5 = 0$ (see other phase diagrams of Figs. 1 and 2). Thus, chiral μ_5 chemical potential promotes the formation of the CSC phase in a dense baryonic medium. In addition, the phase diagram of Fig. 2 (right panel) confirms one more interesting property of μ_5 (as well as of ν_5 chemical potential) noted earlier in the framework of the ordinary NJL model [12] (see there Figs. 10 and 11): If μ_5 is nonzero, then the chiral isospin ν_5 chemical potential generates charged pion condensation in dense quark matter (it is the PC phase in the right panel of Fig. 2) even if isospin ν chemical potential equals zero (see also Figs. 10 and 11 of Ref. [12]). We emphasize once

again that, in order for this generation to take place, one needs to have nonzero chiral chemical potential μ_5. In contrast, as it was discussed in Ref. [13], this generation requires nonzero values of ν in the case of $\mu_5 = 0$. Hence, in the $\nu = 0$ case, chiral μ_5 chemical potential can take the role of ν and allow this generation to happen.

Finally, we would like to draw attention to the fact that at $m = 0$ the duality transformation $\mathcal{D}$ (27) of the TDP (19) can also be applied to a mean-field phase portrait of the model as a whole. Namely, in this case, i.e., when acting on the phase diagram, it is necessary to rename both the diagram axes and phases in such a way that $\nu \leftrightarrow \nu_5$ and CSB $\leftrightarrow$ charged PC. At the same time, the μ and μ_5 axes and CSC and symmetrical phases should not change their names and positions. It is evident that after such $\mathcal{D}$ transformation the full (μ, ν, ν_5, μ_5)-phase diagram is mapped to itself; i.e., the most general (μ, ν, ν_5, μ_5)-phase portrait of the model is self-$\mathcal{D}$-dual. In a similar way, it is clear that various (ν, ν_5)-phase diagrams at fixed μ and μ_5 values are transformed into themselves after applying to them the dual operation (27); i.e., they are also self-$\mathcal{D}$-dual (see, e.g., Figs. 1 and 2). But other cross sections of the full mean-field (μ, ν, ν_5, μ_5)-phase diagram, e.g., the (μ, ν)-phase portrait at some fixed values of ν_5 and μ_5, are not invariant, in general, under the action of dual transformation $\mathcal{D}$. As a result, a completely different phase portrait can be obtained. Hence, based on this mechanism, it is possible, having a well-known cross section of the full phase diagram of the model, to obtain its phase portrait in a less studied range of values of chemical potentials. For example, in Fig. 3 (left panel), one can see the (μ, ν)-phase portrait at fixed $\nu_5 = 0.05$ GeV and $\mu_5 = -0.05$ GeV. Applying to it the dual operation $\mathcal{D}$ according to the rule
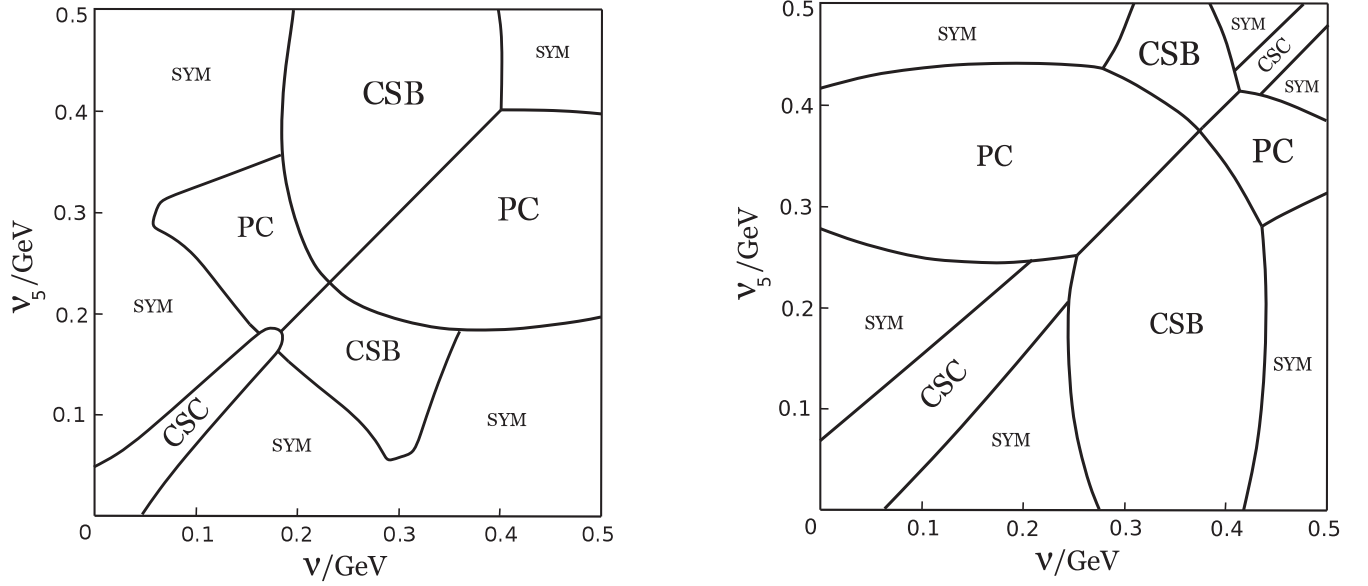
FIG. 2. Some (ν, ν_5)-phase diagrams of the model at $H = 0.75G$. Left panel: the case of $\mu = 0.3$ GeV and $\mu_5 = -0.1$ GeV. Right panel: the case of $\mu = 0.3$ GeV and $\mu_5 = -0.3$ GeV. All the notations are the same as in Fig. 1.

described above, one can obtain (without any numerical calculations) its dual conjugation, i.e. the (μ, ν_5)-phase portrait at fixed $\nu = 0.05$ GeV and $\mu_5 = -0.05$ GeV (see the right panel of Fig. 3).

IV. SUMMARY AND CONCLUSIONS

In this paper, the phase structure of the generalized massless NJL model (1), which describes interactions both in quark-antiquark and diquark channels, is discussed at zero temperature and in the presence of four chemical potentials, baryon $\mu_B = 3\mu$, isospin $\mu_I = 2\nu$, chiral isospin $\mu_{I5} = 2\nu_5$, and chiral μ_5 chemical potentials, in the mean-field approximation. The model is intended to be a theoretical basis for considering the properties of dense quark matter, the ground state of which can be realized as one of the following phases: CSB, charged PC, CSC, and symmetric phase. To find out which of the phases is

implemented in the model, we have considered its thermodynamic potential in the mean field approximation (19). But it is obvious that even in this approximation the study of this TDP to an absolute minimum is a rather difficult problem, especially in the presence of the four above-mentioned chemical potentials. [By the way, we note that these chemical potentials are thermodynamically conjugated with the real physical characteristics of dense quark matter that can exist in the cores of neutron stars (see the Introduction).]

Previously, a similar problem arose when studying the phase structure of the simplest NJL model (without taking into account the diquark interaction channel) with the same four chemical potentials [12,13]. However, it turns out that in the latter case the TDP of the model in the mean-field approximation has (i) a more simple form since it is a function of only two order parameters and (ii) it is invariant
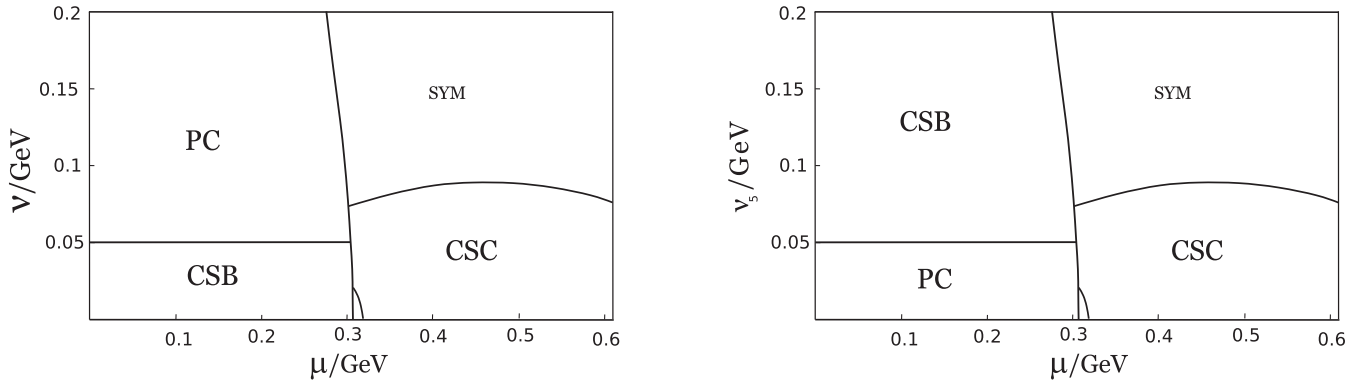

FIG. 3. Left panel: (μ, ν)-phase portrait at $H = 0.75G$. The case $\mu_5 = -0.05$ GeV and $\nu_5 = 0.05$ GeV. Right panel: (μ, ν_5)-phase portrait at $H = 0.75G$. The case $\mu_5 = -0.05$ GeV and $\nu = 0.05$ GeV. All the notations are the same as in Fig. 1.

under the so-called dual transformation $\mathcal{D}$ [its form is given by Eq. (27)]. Because of these circumstances, and in particular the property (ii), the solution of the problem is greatly simplified.

In the present work, we show that in the more complex four-fermion NJL model (1) the thermodynamic potential in the chiral limit and in the mean-field approximation also has dual discrete symmetry $\mathcal{D}$ (27), which greatly simplifies the study of the phase structure of the model. And it is the main result of the paper. This dual symmetry between the CSB and the charged PC phases is especially clearly manifested in the so-called (ν, ν_5)-phase diagrams by the mirror-symmetric arrangement of these phases relative to the line $\nu = \nu_5$. At the same time, the symmetric and CSC phases in these diagrams are symmetrical with respect to the same line (see Figs. 1 and 2). In addition, acting by the dual transformation $\mathcal{D}$ on more well-known phase portraits, one can get an idea of the phase structure of the model in the region of less-studied values of chemical potentials (compare the two diagrams in Fig. 3).

Hence, we see that in two qualitatively different QCD-like NJL models there is a duality between CSB and charged PC phases in the mean-field approximation. The conclusion suggests itself that the dual symmetry $\mathcal{D}$ is not only inherent in these models in the mean-field approximation but is also a characteristic property of both their microscopic Lagrangians and their full thermodynamic potentials. Moreover, we believe that the full massless two-flavor QCD Lagrangian also has a dual symmetry

between CSB and charged PC phenomena. In the future, these issues will be discussed in more detail.

We hope that our results might shed some new light on phase structure of dense quark matter with isospin and chiral (isospin) imbalances and hence could be important for describing physics, for example, in an interior of the compact stars.

APPENDIX A: DERIVATION OF EXPRESSIONS (12) AND (18)

Here, we perform a functional integration over flavor and color quark-doublet fields Q in Eq. (11). First, let us consider the half of the quantity $\bar{Q}(D^+ \cdot \mathbb{1}_{2_c})Q$,

$$\frac{1}{2}\bar{Q}(D^+ \cdot \mathbb{1}_{2_c})Q = \frac{1}{2}\bar{Q}[(i\widehat{\partial} - m + \mathcal{M}\gamma^0 - \sigma \\ - i\gamma^5\tau_1\pi_1) \cdot \mathbb{1}_{2_c}]Q, \tag{A1}$$

and try to rewrite it in terms of charge conjugated quark fields Q^c and $\overline{Q^c}$ using the well-known relations, $\overline{Q} = (Q^c)^T C$ and $Q = C(\overline{Q^c})^T$. Now, let us apply to the expression (A1) the well-known linear algebra relation, $x_\alpha O_{\alpha\beta} y_\beta = \pm y_\beta O_{\beta\alpha}^T x_\alpha$, where the sign $+(-)$ corresponds to the case when the quantities x_α and y_β commute (anticommute). Therefore, since Q_c and $\overline{Q^c}$ are anticommuting fields, it can be transformed to the following one:

$$\frac{1}{2}\overline{Q}(D^+ \cdot \mathbb{1}_{2_c})Q = -\frac{1}{2}\overline{Q^c}\{CD^+C\}^T \cdot \mathbb{1}_{2_c}Q^c = \frac{1}{2}\overline{Q^c}\{CD^+C^{-1}\}^T \cdot \mathbb{1}_{2_c}Q^c$$

$$= \frac{1}{2}\overline{Q^c}\{-i(\gamma^\mu)^T\partial_\mu - m - C\mathcal{M}C^{-1}(\gamma^0)^T - \sigma - i\gamma^5\vec{\tau}_1\pi_1\}^T \cdot \mathbb{1}_{2_c}Q^c$$

$$= \frac{1}{2}\overline{Q^c}\{i\gamma^\mu\partial_\mu - m - \gamma^0\mathcal{M} - \sigma - i\gamma^5\tau_1\pi_1\} \cdot \mathbb{1}_{2_c}Q^c \equiv \frac{1}{2}\overline{Q^c}(D^- \cdot \mathbb{1}_{2_c})Q^c. \tag{A2}$$

It is clear from Eq. (A2) that

$$D^- \equiv i\gamma^\mu\partial_\mu - m - \gamma^0\mathcal{M} - \sigma - i\gamma^5\tau_1\pi_1 = \{CD^+C^{-1}\}^T. \tag{A3}$$

Note also that the transpose operation presented in Eq. (A2) means both transposition of matrices in color, flavor, and spinor spaces and transposition of the differentiation operator, $\partial_\nu^T = -\partial_\nu$. Moreover, we also have used there the following relations: $C^{-1} = C^T = -C$, $C\gamma^\nu C^{-1} = -(\gamma^\nu)^T$, $C\gamma^5 C^{-1} = (\gamma^5)^T = \gamma^5$, $C\mathcal{M}C^{-1} = \mathcal{M}$, and $\mathcal{M}^T = \mathcal{M}$. Another half of the quantity $\bar{Q}(D^+ \cdot \mathbb{1}_{2_c})Q$ we still consider unchanged. So,

$$\bar{Q}(D^+ \cdot \mathbb{1}_{2_c})Q = \frac{1}{2}\bar{Q}(D^+ \cdot \mathbb{1}_{2_c})Q + \frac{1}{2}\overline{Q^c}(D^- \cdot \mathbb{1}_{2_c})Q^c. \tag{A4}$$

In the following, it is very convenient to use the Nambu-Gorkov formalism, in which Q quarks are composed into a bispinor Ψ such that

$$\Psi = \begin{pmatrix} Q \\ Q^c \end{pmatrix}, \quad \Psi^T = (Q^T, \overline{Q}C^T);$$

$$\bar{\Psi} = (\bar{Q}, \overline{Q^c}) = (\overline{Q}, Q^T C) = \Psi^T\begin{pmatrix} 0, & C \\ C, & 0 \end{pmatrix} \equiv \Psi^T Y. \tag{A5}$$

Now, taking into account the Eq. (A4) and introducing the matrix-valued operator Z [see in Eq. (13)], one can rewrite

the functional Gaussian integral over Q and $\bar{Q}$ in (11) in terms of Ψ and Z and then evaluate it as (clearly, in this case $[d\bar{Q}][dQ] = [dQ^c][dQ] = [d\Psi]$)

$$\int [d\bar{Q}][dQ] \exp\left(i \int [\bar{Q}(D^+ \cdot \mathbb{1}_{2_c})Q - \frac{\Delta}{2}[\overline{Q^c}i\gamma^5\tau_2\sigma_2 Q] - \frac{\Delta}{2}[\bar{Q}i\gamma^5\tau_2\sigma_2 Q^c]]d^4x\right)$$
$$= \int [d\Psi] \exp\left\{\frac{i}{2} \int \bar{\Psi}Z\Psi d^4x\right\} = \int [d\Psi] \exp\left\{\frac{i}{2} \int \Psi^T(YZ)\Psi d^4x\right\} = \det^{1/2}(YZ) = \det^{1/2}(Z), \quad (A6)$$

where the last equality is valid due to the evident relation $\det Y = 1$. Now, using a general formula

$$\det\begin{pmatrix} A, & B \\ C, & D \end{pmatrix} = \det[-CB + CAC^{-1}D] = \det[DA - DBD^{-1}C], \quad (A7)$$

it is possible to find that [the notations used below are introduced in Eqs. (13) and (14)]

$$\det(Z) = \det\begin{pmatrix} D^+ \cdot \mathbb{1}_{2_c}, & -K \\ -K, & D^- \cdot \mathbb{1}_{2_c} \end{pmatrix} = \det[(-KK + KD^+K^{-1}D^-) \cdot \mathbb{1}_{2_c}] \quad (A8)$$

$$= \det[\Delta^2 \cdot \mathbb{1}_{2_c} + (-i\hat{\partial} - m - \widetilde{\mathcal{M}}\gamma^0 - \sigma + i\gamma^5\tau_1\pi_1)(i\hat{\partial} - m - \gamma^0\mathcal{M} - \sigma - i\gamma^5\tau_1\pi_1) \cdot \mathbb{1}_{2_c}], \quad (A9)$$

where we use the notations $\hat{\partial} = \gamma^\alpha\partial_\alpha$, $\mu = \mu_B/3$, $\nu = \mu_I/2$, $\nu_5 = \mu_{I5}/2$, and

$$\widetilde{\mathcal{M}} = \mu + \mu_5\gamma^5 - \nu\tau_3 - \nu_5\gamma^5\tau_3. \quad (A10)$$

In the particular case when $\Delta = 0$ [in this case, the diquark channel of the NJL model (1) is ignored], it is clear from Eq. (A8) that

$$\det(Z) = \det[\gamma^5\tau_2 D^+ \gamma^5\tau_2 D^- \cdot \mathbb{1}_{2_c}] = \det^2[D^+D^-]$$
$$= \det^4(D^+). \quad (A11)$$

The last equality in this expression follows from Eq. (A3), which means that $\det(D^-) = \det(D^+)$. Taking into account in Eq. (15) the relation (A11), we see that at $\Delta = 0$ the quark contribution to the effective action (15) is equal to $-3i \ln \det(D^+)$, i.e., coincides with a corresponding expression for the effective action of the NJL model (1) with $N_c = 3$ and $H = 0$ (see Ref. [12]).

Since $\hat{\partial}\hat{\partial} = \partial^2$, we have from Eq. (A9)

$$\det(Z) = \det[(\Delta^2 + \partial^2 + i\hat{\partial}\mathcal{A} + \mathcal{B}i\hat{\partial} - \mathcal{B}\mathcal{A}) \cdot \mathbb{1}_{2_c}], \quad (A12)$$

where

$$\mathcal{A} = m + \sigma + \gamma^0\mathcal{M} + i\gamma^5\tau_1\pi_1,$$
$$\mathcal{B} = -m - \sigma - \widetilde{\mathcal{M}}\gamma^0 + i\gamma^5\tau_1\pi_1. \quad (A13)$$

Note that the expression in the square brackets of Eq. (A12) is proportional to a unit operator in the two-dimentional (i.e., $N_c = 2$ below) color space, so it follows from Eq. (A12) that

$$\det(Z) \equiv \det^{N_c}\mathcal{D} = \det^{N_c}\begin{pmatrix} D_{11}, & D_{12} \\ D_{21}, & D_{22} \end{pmatrix}, \quad (A14)$$

where $\mathcal{D}$ is the 2×2 matrix in the two-dimensional flavor space [its matrix elements D_{kl} are the nontrivial operators in the four-dimensional spinor and in the $(3 + 1)$-dimensional coordinate spaces]. To get the exact expressions of the matrix elements D_{kl}, we need to bear in mind that

$$i\hat{\partial}\mathcal{A} + \mathcal{B}i\hat{\partial} = i\hat{\partial}\gamma^0\mathcal{M} - \widetilde{\mathcal{M}}\gamma^0 i\hat{\partial} = (\mu + \mu_5\gamma^5)[i\hat{\partial}\gamma^0 - \gamma^0 i\hat{\partial}] + (\nu + \nu_5\gamma^5)[i\hat{\partial}\gamma^0 + \gamma^0 i\hat{\partial}]\tau_3,$$
$$-\mathcal{B}\mathcal{A} = \pi_1^2 + (m + \sigma + \widetilde{\mathcal{M}}\gamma^0)(m + \sigma + \gamma^0\mathcal{M}) - i\gamma^5\tau_1\pi_1\gamma^0\mathcal{M} + \widetilde{\mathcal{M}}\gamma^0 i\gamma^5\tau_1\pi_1$$
$$= \pi_1^2 + (m + \sigma)^2 + 2(m + \sigma)\gamma^0(\mu + \nu_5\gamma^5\tau_3) + \widetilde{\mathcal{M}}\mathcal{M} - i\gamma^5\tau_1\pi_1\gamma^0\mathcal{M} + \widetilde{\mathcal{M}}\gamma^0 i\gamma^5\tau_1\pi_1. \quad (A15)$$

Note also that

$$\widetilde{\mathcal{M}}\mathcal{M} = (\mu + \mu_5\gamma^5)^2 - (\nu + \nu_5\gamma^5)^2,$$

$$-i\gamma^5\tau_1\pi_1\gamma^0\mathcal{M} = i(\mu - \mu_5\gamma^5 - \nu\tau_3 + \nu_5\gamma^5\tau_3)\gamma^0\gamma^5\tau_1\pi_1,$$

$$i\widetilde{\mathcal{M}}\gamma^0\gamma^5\tau_1\pi_1 = i(\mu + \mu_5\gamma^5 - \nu\tau_3 - \nu_5\gamma^5\tau_3)\gamma^0\gamma^5\tau_1\pi_1,$$

$$-i\gamma^5\tau_1\pi_1\gamma^0\mathcal{M} + i\widetilde{\mathcal{M}}\gamma^0\gamma^5\tau_1\pi_1 = 2i(\mu - \nu\tau_3)\gamma^0\gamma^5\tau_1\pi_1 = 2i\mu\gamma^0\gamma^5\tau_1\pi_1 + 2\nu\gamma^0\gamma^5\tau_2\pi_1. \tag{A16}$$

Now, taking into account the relations (A12)–(A16), we are ready to present the expressions for the matrix elements D_{kl} of the 2×2 matrix $\mathcal{D}$ from Eq. (A14),

$$D_{11} = \Delta^2 + \partial^2 + (\mu + \mu_5\gamma^5)[i\widehat{\partial}\gamma^0 - \gamma^0 i\widehat{\partial}] + (\nu + \nu_5\gamma^5)[i\widehat{\partial}\gamma^0 + \gamma^0 i\widehat{\partial}] + \pi_1^2 + (m + \sigma)^2$$

$$+ 2(m + \sigma)\gamma^0(\mu + \nu_5\gamma^5) + (\mu + \mu_5\gamma^5)^2 - (\nu + \nu_5\gamma^5)^2,$$

$$D_{22} = \Delta^2 + \partial^2 + (\mu + \mu_5\gamma^5)[i\widehat{\partial}\gamma^0 - \gamma^0 i\widehat{\partial}] - (\nu + \nu_5\gamma^5)[i\widehat{\partial}\gamma^0 + \gamma^0 i\widehat{\partial}] + \pi_1^2 + (m + \sigma)^2$$

$$+ 2(m + \sigma)\gamma^0(\mu - \nu_5\gamma^5) + (\mu + \mu_5\gamma^5)^2 - (\nu + \nu_5\gamma^5)^2,$$

$$D_{12} = 2i\gamma^0\gamma^5(\mu - \nu)\pi_1, \qquad D_{21} = 2i\gamma^0\gamma^5(\nu + \mu)\pi_1. \tag{A17}$$

Because of a rather general formula $\det O = \exp \mathrm{Tr}\ln O$, we see from Eq. (15) that indeed, in order to calculate an effective action, we should evaluate only the quantity $\mathrm{Tr}\ln Z$. Then, taking into account the general relations (B4) and (B5) as well as Eq. (A14), one can obtain

$$\mathrm{Tr}\ln Z = N_c \ln\det\mathcal{D} = N_c\mathrm{Tr}\ln\mathcal{D} = N_c \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\ln\bar{\mathcal{D}}(p) \int d^4x$$

$$= N_c \int \frac{d^4p}{(2\pi)^4} \ln\det\bar{\mathcal{D}}(p) \int d^4x, \tag{A18}$$

where Tr means the trace of an operator both in the coordinate and internal spaces, whereas tr is the trace only in internal space. Moreover, $\bar{\mathcal{D}}(p)$ is the 2×2 matrix, which is the momentum space representation of the matrix $\mathcal{D}$ from Eq. (A14). Its matrix elements $\bar{\mathcal{D}}_{kl}(p)$ can be obtained from the relations (A17) by simple replacements, $i\widehat{\partial} \to \hat{p}$ and $\partial^2 \to -p^2$. So, we have from (A17)

$$\bar{D}_{11}(p) = \Delta^2 - p^2 + (\mu + \mu_5\gamma^5)[\hat{p}\gamma^0 - \gamma^0\hat{p}] + (\nu + \nu_5\gamma^5)[\hat{p}\gamma^0 + \gamma^0\hat{p}] + \pi_1^2 + (m + \sigma)^2$$

$$+ 2(m + \sigma)\gamma^0(\mu + \nu_5\gamma^5) + (\mu + \mu_5\gamma^5)^2 - (\nu + \nu_5\gamma^5)^2,$$

$$\bar{D}_{22}(p) = \Delta^2 - p^2 + (\mu + \mu_5\gamma^5)[\hat{p}\gamma^0 - \gamma^0\hat{p}] - (\nu + \nu_5\gamma^5)[\hat{p}\gamma^0 + \gamma^0\hat{p}] + \pi_1^2 + (m + \sigma)^2$$

$$+ 2(m + \sigma)\gamma^0(\mu - \nu_5\gamma^5) + (\mu + \mu_5\gamma^5)^2 - (\nu + \nu_5\gamma^5)^2,$$

$$\bar{D}_{12}(p) = 2i\gamma^0\gamma^5(\mu - \nu)\pi_1, \qquad \bar{D}_{21}(p) = 2i\gamma^0\gamma^5(\nu + \mu)\pi_1. \tag{A19}$$

In Eq. (A18), we should evaluate the determinant of the operator $\bar{\mathcal{D}}(p)$, which is a 2×2 matrix in flavor space and 4×4 matrix in spinor space. It is a rather difficult task. But due to a general relation (A7), we have

$$\det\bar{D}(p) \equiv \det\begin{pmatrix} \bar{D}_{11}(p), & \bar{D}_{12}(p) \\ \bar{D}_{21}(p), & \bar{D}_{22}(p) \end{pmatrix}$$

$$= \det[-\bar{D}_{21}(p)\bar{D}_{12}(p) + \bar{D}_{21}(p)\bar{D}_{11}(p)(\bar{D}_{21}(p))^{-1}\bar{D}_{22}(p)] \equiv \det L(p), \tag{A20}$$

where the matrix $L(p)$, i.e., the expression/matrix in square brackets of Eq. (A20), is indeed a 4×4 matrix in four-dimensional spinor space only, which is composed of 4×4 matrices $\bar{D}_{ij}(p)$ [see Eq. (A19)]. So, since indeed $N_c = 2$,

$$\ln\det Z = \mathrm{Tr}\ln Z = 2 \int \frac{d^4p}{(2\pi)^4} \ln\det L(p) \int d^4x. \tag{A21}$$

APPENDIX B: TRACES OF OPERATORS AND THEIR PRODUCTS

Let $\hat{A}, \hat{B}, \ldots$ be some operators in the Hilbert space **H** of functions $f(x)$ depending on four real variables, $x \equiv (x^0, x^1, x^2, x^3)$. In the coordinate representation, their matrix elements are $A(x, y), B(x, y), \ldots$, correspondingly, so that

$$(\hat{A}f)(x) \equiv \int d^4 y A(x, y) f(y),$$

$$(\hat{A} \cdot \hat{B})(x, y) \equiv \int d^4 z A(x, z) B(z, y), \quad \text{etc.}$$

By definition,

$$\text{Tr}\hat{A} \equiv \int d^4 x A(x, x),$$

$$\text{Tr}(\hat{A} \cdot \hat{B}) \equiv \int d^4 x d^4 y A(x, y) B(y, x), \quad \text{etc.} \quad \text{(B1)}$$

Now, suppose that $A(x, y) \equiv A(x - y)$, $B(x, y) \equiv B(x - y)$, i.e., that $\hat{A}, \hat{B}$ are translationally invariant operators. Then, introducing the Fourier transformations of their matrix elements, i.e.

$$\bar{A}(p) = \int d^4 z A(z) e^{ipz}, \quad A(z) = \int \frac{d^4 p}{(2\pi)^4} \bar{A}(p) e^{-ipz}, \quad \text{etc.,}$$
$$\text{(B2)}$$

where $z = x - y$, it is possible to obtain from the above formulas

$$\text{Tr}\hat{A} = A(0) \int d^4 x = \int \frac{d^4 p}{(2\pi)^4} \bar{A}(p) \int d^4 x. \quad \text{(B3)}$$

If there is an operator function $F(\hat{A})$, where $\hat{A}$ is a translationally invariant operator, then in the coordinate representation, its matrix elements depend on the difference $(x - y)$. Obviously, it is possible to define the Fourier transformations $\overline{F(A)}(p)$ of its matrix elements, and the following relations are valid [$\bar{A}(p)$ is the Fourier transformation (B2) for the matrix element $A(x - y)$]:

$$\overline{F(A)}(p) = F(\bar{A}(p));$$

$$\text{Tr}F(\hat{A}) = \int \frac{d^4 p}{(2\pi)^4} F(\bar{A}(p)) \int d^4 x. \quad \text{(B4)}$$

Finally, suppose that $\hat{A}$ is an operator in some internal n-dimensional vector space, in addition. Evidently, the same is valid for the Fourier transformation $\bar{A}(p)$, which is now some $n \times n$ matrix. Let $\lambda_i(p)$ be eigenvalues of the $n \times n$ matrix $\bar{A}(p)$, where $i = 1, 2, \ldots, n$. Then,

$$\text{Tr}F(\hat{A}) = \int \frac{d^4 p}{(2\pi)^4} \text{tr}F(\bar{A}(p)) \int d^4 x$$

$$= \sum_{i=1}^{n} \int \frac{d^4 p}{(2\pi)^4} F(\lambda_i(p)) \int d^4 x. \quad \text{(B5)}$$

In this formula, we use the notation tr for the trace of any operator in the internal n-dimensional vector space only, whereas the symbol Tr means the trace of an operator in both the coordinate and internal spaces. In particular, if $F(\hat{A}) = \ln(\hat{A})$, then it follows from (B5) that [here, we use a well-known relation $\ln \det(\hat{A}) = \text{Tr} \ln(\hat{A})$]

$$\ln \det(\hat{A}) = \text{Tr} \ln(\hat{A}) = \sum_{i=1}^{n} \int \frac{d^4 p}{(2\pi)^4} \ln(\lambda_i(p)) \int d^4 x$$

$$= \int \frac{d^4 p}{(2\pi)^4} \ln(\det \bar{A}(p)) \int d^4 x. \quad \text{(B6)}$$

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