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Metric for two unequal extreme Kerr-Newman black holes

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ABSTRACT

In the present paper, within the framework of stationary axisymmetric spacetimes, binary systems composed of two unequal co- and counter-rotating extreme Kerr-Newman black holes separated by a massless strut are reported. The metric describing both configurations is introduced in a closed analytical form in terms of five arbitrary parameters: the masses M_i, electric charges Q_i, and a coordinate distance R. We obtain novel results from these configurations; in particular, those related to the merging process.

1. Introduction

Binary black hole (BH) systems have attracted our attention since the early days of general relativity. The recent detection of gravitational waves [1] produced by binary BH mergers permits us to reconsider exact binary models to complement the vast amount of numerical results in the literature. However, from a technical point of view, it is quite complicated to take into account all the dynamical interactions in a binary setup, and for such a reason the stationary scenario seems to be a good candidate to develop analytical results. In static charged systems, the Majumdar-Papetrou metric [2–4] describes the simplest model of two extreme BHs, which remain in neutral equilibrium due to the balance of their electric charges and masses according to the relation $Q_i = \pm M_i$, regardless of the separation distance among sources. Moreover, in vacuum systems, the Kinnersley-Chitre (KCH) exact solution [5] allows us the description of rotating binary BHs, after solving appositely the axis conditions [6,7]. In this type of binary vacuum systems, the Kerr BHs are apart by a conical singularity [8,9], which can give us information on their gravitational attraction and spin-spin interaction.

In contrast, the treatment of unequal binary configurations of extreme Kerr-Newman (KN) BHs [10] has been a fairly complicated problem beyond our possibilities, due mainly to the fact that the axis conditions are not enough to define properly KN BHs, therefore, it is necessary to impose an extra condition in order to kill both magnetic charges, otherwise, Dirac strings linked to the KN BHs will appear [11–13]. The main purpose of this paper is to derive a five-parametric exact solution that completely describes binary co- and counter-rotating extreme KN BH separated by a massless strut in a unified manner. To accomplish such a goal, we are going to take into account the recent results of [14] where a complete derivation of the metric and thermodynamical properties for non-extreme KN BHs has been succeeded. Hence, the Ernst potentials and metric functions will be depicted in terms of physical Komar parameters [15]: the masses M_i, electric charges Q_i, and a coordinate distance R as well. In this scheme, the five arbitrary parameters compose an algebraic equation thus defining a dynamical law for interacting BHs with struts, which is reduced to some previous studied cases [7,16]. At the same time, the metric is concisely given in terms of Perjes' factor structure [17]. Since the physical limits in both rotating charged models are well identified, we derive quite simple formulas for the area of the horizon and the interaction force during the merger limit of BHs, where a deformed metric for a near horizon extreme binary KN BH is also given.

2. The charged Kinnersley-Chitre exact solution

Ernst's formalism [18] allows the description of Einstein-Maxwell equations in stationary axisymmetric spacetimes, in terms of a pair of complex functions $(\mathcal{E}, \Phi)$ satisfying

$$\begin{aligned} \left(\mathrm{Re}\mathcal{E} + |\Phi|^2\right)\Delta\mathcal{E} &= (\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi)\cdot\nabla\mathcal{E}, \\ \left(\mathrm{Re}\mathcal{E} + |\Phi|^2\right)\Delta\Phi &= (\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi)\cdot\nabla\Phi, \end{aligned} \tag{1}$$

where any exact solution of Eq. (1) can be derived via Sibgatullin's method (SM) [19,20], which is also useful to obtain the metric functions $f(\rho, z)$, $\omega(\rho, z)$ and $\gamma(\rho, z)$ of the line element [3]

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$$ds^2 = f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right] - f(dt - \omega d\varphi)^2. \qquad (2)$$

Due to the fact that SM needs a particular form of the Ernst potentials on the upper part of the symmetry axis, let us begin with a more suitable physical representation, namely

$$\begin{aligned}
&\mathcal{E}(0,z) = \frac{\mathfrak{e}_1}{\mathfrak{e}_2}, \qquad \Phi(0,z) = \frac{\mathcal{Q}z + \mathfrak{q}_o}{\mathfrak{e}_2},\\
&\quad \mathfrak{e}_1 = z^2 - [M + i(\mathfrak{q} + 2J_0)]z + P_+ + iP_1 - 2iJ_0(d - i\mathfrak{q}),\\
&\quad \mathfrak{e}_2 = z^2 + (M - i\mathfrak{q})z + P_- + iP_2,\\
&\quad P_\pm = \frac{M(2\Delta_o - R^2) \pm 2\left[\mathfrak{q}\mathfrak{s}_1 - 2(q_o Q + b_o B)\right]}{4M},\\
&\mathfrak{s}_1 = P_1 + P_2, \qquad d = M + \frac{P_2}{\mathfrak{q}}, \qquad \mathcal{Q} = Q + iB,\\
&\mathfrak{q}_o = q_o + ib_o, \qquad \Delta_o = M^2 - |\mathcal{Q}|^2 - \mathfrak{q}^2,
\end{aligned} \qquad (3)$$

where the aforementioned Ernst potentials Eq. (3) are the extreme case of that one considered in Ref. [14], from which the first Simon's multipole moments [21] can be explicitly calculated by means of the Hoenselaers-Perjés procedure [22,23]. In this sense, M plays the role of the total mass of the system and $Q + iB$ defines the total electromagnetic charge, while the total electric and magnetic dipole moment are $q_o - B(\mathfrak{q} + J_0)$ and $b_o + Q(\mathfrak{q} + J_0)$, respectively. Besides, R represents a separation distance between both sources. At the same time, the NUT charge J_0 [24] and total angular momentum of the system J are given by

$$\begin{aligned}
&J_0 = \frac{N}{8M^2(\mathfrak{q}P_- + P_2 d)}, \qquad J = M\mathfrak{q} - \frac{\mathfrak{s}_2}{2} + (M + d)J_0,\\
&N = M^2\left\{4(P_1P_2 + |\mathfrak{q}_o|^2) - \Delta_o(R^2 - \Delta_o)\right\}\\
&\quad - \left[\mathfrak{q}\mathfrak{s}_1 - 2(Qq_o + Bb_o)\right]^2, \qquad \mathfrak{s}_2 = P_1 - P_2.
\end{aligned} \qquad (4)$$

It is not difficult to show that once the SM [19,20] has been applied to the axis data Eq. (3), the Ernst potentials satisfying Eq. (1) acquire the final aspect

$$\begin{aligned}
&\mathcal{E} = \frac{\Lambda - 2\Gamma}{\Lambda + 2\Gamma}, \qquad \Phi = \frac{2\chi}{\Lambda + 2\Gamma},\\
&\Lambda = R^2\left[(R^2 - \delta)(x^2 - y^2)^2 + \delta(x^4 - 1)\right] + \left\{|\mathfrak{p}|^2 + (\mathfrak{q} + J_0)\mathfrak{r}\right.\\
&\quad \left. - R^2(R^2 - \delta)\right\}(y^4 - 1) + 2iR\left\{xy\left[\left[\mathfrak{r} + (\mathfrak{q} + J_0)R^2\right](y^2 - 1)\right.\right.\\
&\quad \left.\left. - (\mathfrak{q} + J_0)R^2(x^2 + y^2 - 2)\right] - RS_1\left(x^2 + y^2 - 2x^2y^2\right)\right\}\\
&\Gamma = (M + iJ_0)\mathbb{P}_1 - (\mathfrak{b} + iS_2)\mathbb{P}_2, \qquad \chi = \mathcal{Q}\mathbb{P}_1 + 2\mathfrak{q}_o\mathbb{P}_2\\
&\mathbb{P}_1 = R^3x(x^2 - 1) - (R\bar{\mathfrak{p}}x - i\mathfrak{r}y)(y^2 - 1),\\
&\mathbb{P}_2 = R^2y(x^2 - 1) - \left[\mathfrak{p}y - i(\mathfrak{q} + J_0)Rx\right](y^2 - 1),\\
&\quad \mathfrak{p} = R^2 - \delta + iS_1, \quad \mathfrak{r} = 2\mathfrak{a} - (\mathfrak{q} + J_0)(R^2 - 2\delta) - 2\mathfrak{b}J_0,\\
&\quad \mathfrak{a} = MS_2 + 2(b_oQ - q_oB), \quad \delta = \Delta_o - 2\mathfrak{q}J_0, \quad S_1 = \mathfrak{s}_1 - 2dJ_0,\\
&S_2 = \mathfrak{s}_2 - 2dJ_0, \quad \mathfrak{b} = \left[\mathfrak{q}\mathfrak{s}_1 - 2(q_oQ + b_oB)\right]/M - 2\mathfrak{q}J_0,
\end{aligned} \qquad (5)$$

where (x, y) are prolate spheroidal coordinates related to cylindrical coordinates (ρ, z) by means of

$$x = \frac{r_+ + r_-}{R}, \quad y = \frac{r_+ - r_-}{R}, \qquad r_\pm = \sqrt{\rho^2 + (z \pm R/2)^2}. \qquad (6)$$

Furthermore, the metric functions contained within the line element can be written down in a closed analytical form by using Perjes's factor structure [17], thus getting[1]

$$\begin{aligned}
&f = \frac{\mathcal{D}}{\mathcal{N}}, \qquad \omega = 2J_0(y + C) + \frac{R(y^2 - 1)\left[(x^2 - 1)\Sigma\Pi - \Theta\mathrm{T}\right]}{2\mathcal{D}},\\
&e^{2\gamma} = \frac{\mathcal{D}}{R^8(x^2 - y^2)^4}, \qquad \mathcal{N} = \mathcal{D} + \Theta\Pi - (1 - y^2)\Sigma\mathrm{T},\\
&\mathcal{D} = \Theta^2 + (x^2 - 1)(y^2 - 1)\Sigma^2,\\
&\Theta = R^2\left[(R^2 - \delta)(x^2 - y^2)^2 + \delta(x^2 - 1)^2\right]\\
&\quad + \left[|\mathfrak{p}|^2 + (\mathfrak{q} + J_0)\mathfrak{r} - R^2(R^2 - \delta)\right](y^2 - 1)^2,\\
&\Sigma = 2R\Big((\mathfrak{q} + J_0)R^2x^2 - \mathfrak{r}y^2 - 2RS_1xy\Big),\\
&\Pi = 4Rx\Big\{MR^2(x^2 - y^2) + \left[M\delta + (\mathfrak{q} + J_0)S_2 + 3J_0S_1\right](1 + y^2)\\
&\quad + \left[2\left(M^2 + J_0^2\right) - |\mathcal{Q}|^2\right]Rx - 2\left[J_0S_2 + (\mathfrak{q} + J_0)S_1\right]y - 4J_0S_1\Big\}\\
&\quad - 4y\Big\{\mathfrak{b}\left[R^2(x^2 - y^2) + \delta(1 + y^2) + 2MRx - 2\mathfrak{b}y\right] + S_1S_2\\
&\quad \times(1 + y^2) - 2\left(S_2^2 - 2|\mathfrak{q}_o|^2\right)y + J_0\left[\mathfrak{r}(1 - y^2) + 2(\mathfrak{q} + J_0)R^2x^2\right]\Big\},\\
&\mathrm{T} = \frac{2}{R}\Big\{2R^2\Big(\left[MS_1 - (\mathfrak{q} + J_0)\mathfrak{b} + J_0(R^2 - \delta)\right]y - S_2(Rx + M)\\
&\quad - \mathfrak{a} + 2\mathfrak{b}J_0\Big)(1 - x^2) + \Big(2\left[\mathfrak{b}S_1 + (R^2 - \delta)S_2 + M\mathfrak{r}\right](Rx + M)\\
&\quad + 2J_0\left[S_1S_2 - \mathfrak{b}(2R^2 - \delta) - \left[|\mathfrak{p}|^2 + (\mathfrak{q} + J_0)\mathfrak{r}\right]y\right] - (M^2 - \mathfrak{q}^2)\mathfrak{r}\\
&\quad + 2\mathfrak{a}R^2 + (\mathfrak{q} + J_0)\left[\delta R^2 + 2J_0\mathfrak{r} - 4|\mathfrak{q}_o|^2\right]\Big)(1 - y^2)\Big\}.
\end{aligned} \qquad (7)$$

The above metric is the electromagnetically charged version of KCH's exact solution [5,6]. It contains nine physical parameters defined by the set $\{M, R, \mathfrak{q}, P_1, P_2, Q, B, q_o, b_o\}$. Naturally, in the absence of electromagnetic field (Q, B, q_o, and b_o set to zero) the KCH exact solution is recovered from it.

Taking into account Bonnor's description [25], the above metric is not asymptotically flat due to the presence of the NUT charge which represents a semi-infinite singular source located along the lower part of the symmetry axis at $y = -1$, for $C = -1$, thus providing additional rotation to the binary system. The last point can be better understood when analyzing the asymptotic behavior of the metric functions; i.e., $f \to 1$, $\gamma \to 0$, and $\omega \to 2J_0(y + C)$ at $x \to \infty$. In this regard, the condition $J_0 = 0$ is enough to ensure any asymptotically flat spacetime from Eq. (7). Such a task can be accomplished by means of

$$\begin{aligned}
&M^2\left[4(P_1P_2 + |\mathfrak{q}_o|^2) - \Delta(R^2 - \Delta)\right] - (\mathfrak{q}\mathfrak{s}_1 - 2Qq_o)^2 = 0,\\
&\Delta = M^2 - Q^2 - \mathfrak{q}^2,
\end{aligned} \qquad (8)$$

where we have imposed also the requirement $B = 0$ in order to describe afterward extreme KN binary BHs. In addition, the condition $\omega(x = 1, y = 2z/R) = 0$ permitting to disconnect the region among the sources is reduced to

[1] It is possible to locate up or down semi-infinite singularities along the axis depending on the values for $C = 0, \pm 1$ (see [6] and references therein).

$$2(R+M)\left\{\left[\mathfrak{q}s_1-2q_oQ\right]s_1+M(R^2-\Delta)s_2-\mathfrak{q}M^2R^2\right\}$$
$$+2MP_0\left[Ms_2+2b_oQ+\mathfrak{q}\Delta\right]+M\mathfrak{q}(Q^2R^2-4|\mathfrak{q}_o|^2)=0,$$
$$P_0=(R+M)^2+\mathfrak{q}^2. \quad (9)$$

If one is able to get an analytical solution from Eqs. (8) and (9), it will be possible to derive a binary model of rotating dyonic extreme BHs kept apart by a massless strut, where the sources are equipped with identical magnetic charges but endowed with opposite signs. Therefore, there exists a Dirac string joined to the BHs unless the magnetic charges are removed from the solution [11–13]. In this case, an absence of individual magnetic charges is accomplished with the following algebraic equation

$$\mathfrak{q}Q\left\{\left[MP_0-(R+M)(2M(R+M)-\mathfrak{q}^2)\right]s_1^2-4M^2(R+M)P_1P_2\right\}$$
$$+M^2\left[2\mathfrak{q}P_0b_o-Q\left(P_0-2\mathfrak{q}^2\right)(R^2-\Delta)\right]s_2-2q_o\left[M(\Delta+MR)P_0\right.$$
$$\left.+2\mathfrak{q}^2(R+2M)Q^2\right]s_1-4\mathfrak{q}q_o^2Q\left[MP_0-(R+M)Q^2\right]$$
$$-M^2(R+M)(R^2-\Delta)\left[2P_0b_o+\mathfrak{q}Q(R^2-\Delta)\right]=0, \quad (10)$$

that is achieved once the following condition on the real part of the potential Φ is imposed [11]

$$\lim_{\lambda\to 0}\left[\mathrm{Re}\Phi\big(x=1+\lambda,\,y=1\big)-\mathrm{Re}\Phi\big(x=1,\,y=1-\lambda\big)\right]=0, \quad (11)$$

and this condition only is established in the upper BH since the lower one contains the same magnetic charge with opposite sign as we have before mentioned. At this point, it is important to underline the fact that to study extreme KN BHs it is mandatory to analytically solve the set of Eqs. (8)-(10). Unfortunately, the most general exact solution cannot be derived directly from these entwined Eqs. (8)-(10) unless we get first a highly complicated fourth-degree algebraic equation in terms of any of the variables q_o, P_1, and P_2, consequently, one must circumvent this technical issue by adopting a different point of view. As we shall see next, we are going to derive the algebraic values for the set $\{q_o, b_o, P_1, P_2\}$ that solve exactly Eqs. (8)-(10) and define extreme KN binary BHs held apart by a massless strut in a physical representation.

3. Extreme KN binary BHs

Let us begin the section by considering first the variables $\{q_o, b_o, P_1, P_2\}$ earlier derived in Ref. [14] for non-extreme KN BHS. These parameters containing a physical representation are given by

$$q_o=\frac{2\mathfrak{q}(Q_1J_2-Q_2J_1)}{P_0}+\frac{Q_1}{2}(R-2M_2)-\frac{Q_2}{2}(R-2M_1),$$
$$b_o=\left[(Q_1C_2-Q_2C_1)\left(\frac{J_1}{\mathcal{P}_1}-\frac{J_2}{\mathcal{P}_2}\right)-\frac{\mathfrak{q}}{P_0}\left(Q\right.\right.$$
$$-\frac{Q_1H_{1+}\left[Q_1(Q_1-Q_2)P_0-2(R+M)H_{1-}\right]}{\mathcal{P}_1}$$
$$\left.\left.-\frac{Q_2H_{2+}\left[Q_2(Q_2-Q_1)P_0-2(R+M)H_{2-}\right]}{\mathcal{P}_2}\right)\right]\frac{(R^2-\Delta)}{2},$$
$$P_{1,2}=\frac{(2H_2A_2-RP_0\mathcal{P}_2)J_1-(2H_1A_1-RP_0\mathcal{P}_1)J_2}{2P_0\mathcal{P}_0}\pm(M\mathfrak{q}-J), \quad (12)$$

with the following elements:

$$A_i=\mathcal{P}_i-(R^2-\Delta)H_{i+}P_0,\quad H_i=M_iP_0-Q_iQ(R+M),$$
$$\mathcal{P}_i=H_{i-}C_i-(-1)^i(M_1-M_2)Q_i^2P_0^2,$$
$$\mathcal{P}_0=M\mathcal{P}_1-(R^2-\Delta)H_{1+}H_{1-}\equiv M\mathcal{P}_2-(R^2-\Delta)H_{2+}H_{2-},$$
$$C_i=P_0^2-2M_i(R+M)P_0+2\mathfrak{q}^2Q_1Q_2,$$
$$H_{i\pm}=M_iP_0\pm Q_1Q_2(R+M),\qquad i=1,2. \quad (13)$$

Also, in Weyl's coordinates, the BH horizons in the binary setup are represented on the symmetry axis as thin rods of length $2\sigma_i$ whose centers are separated by a distance R. For unequal KN BHs, σ_i turns out to be [14]

$$\sigma_i=\sqrt{D_i-J_i\left(\frac{J_iG_i-2\mathfrak{q}A_iB_i}{P_0^2\mathcal{P}_i^2}\right)},$$
$$D_i=M_i^2-Q_i^2F_i-2(-1)^iQ_iF_0,$$
$$G_i=\left[2(R+M)\mathcal{P}_i+P_0(R^2-\Delta)C_i\right]^2-4P_0\mathcal{P}_1\mathcal{P}_2,$$
$$F_0=\frac{M_2Q_1-M_1Q_2}{R+M}\left(1-\frac{\mathfrak{q}^2}{P_0}\right),$$
$$F_i=1-\frac{Q_i^2\mathfrak{q}^2}{P_0^2}\left(1-\frac{A_i^2}{\mathcal{P}_i^2}\right)+\frac{Q^2\mathfrak{q}^2}{P_0^2},$$
$$B_i=Q_i^2P_0(R^2-\Delta)C_i-2H_i\mathcal{P}_i,\qquad i=1,2, \quad (14)$$

where the set of parameters $\{M_1, M_2, Q_1, Q_2, J_1, J_2\}$ are the physical Komar parameters [15] for each source. It should be pointed out, that the total mass, total electric charge, and total angular momentum are $M=M_1+M_2$, $Q=Q_1+Q_2$, and $J=J_1+J_2$, respectively. Furthermore, $G_1\equiv G_2$ and $A_1\equiv A_2$, so these terms have a symmetrical character under the change of constituents; i.e., $1\leftrightarrow 2$. A peculiar feature of the binary system is that the seven physical parameters satisfy a dynamical law for interacting KN sources (BHs $\sigma_i^2\geq 0$ or naked singularities $\sigma_i^2<0$) with struts defined by

$$\mathfrak{q}\mathcal{P}_0-J_1\mathcal{P}_2-J_2\mathcal{P}_1=0. \quad (15)$$

The extreme limit solution is achieved by setting $\sigma_1=\sigma_2=0$ in Eq. (14), where this condition enables one to get

$$J_1=\frac{\mathfrak{q}A_1B_1+\varepsilon_1P_0\mathcal{P}_1\sqrt{P_0(R^2-\Delta)E_0d_1}}{G_1},\quad \varepsilon_1=\pm1,$$
$$J_2=\frac{\mathfrak{q}A_2B_2+\varepsilon_2P_0\mathcal{P}_2\sqrt{P_0(R^2-\Delta)E_0d_2}}{G_2},\quad \varepsilon_2=\pm1,$$
$$E_0=4R\mathcal{P}_1+(R^2-\Delta)\left[(P_0+2Q_1Q_2)C_1-2(R-\delta_2)P_0H_{1+}\right]$$
$$\equiv 4R\mathcal{P}_2+(R^2-\Delta)\left[(P_0+2Q_1Q_2)C_2-2(R+\delta_2)P_0H_{2+}\right],$$
$$d_i=P_0^2D_i+\left(\frac{\mathfrak{q}Q_i^2A_i}{\mathcal{P}_i}\right)^2,\qquad \delta_2=M_1-M_2. \quad (16)$$

On one hand, if one takes into account the case $\varepsilon_1=\varepsilon_2=\pm1$, it might be possible to study co-rotating KN BHs, while on the other hand, the choice $\varepsilon_1=-\varepsilon_2=\pm1$, permits the description of the corresponding counter-rotating scenario. The substitution of Eq. (16) into Eq. (15) guides us to the simple formula

$$P_0(R^2-\Delta)E_0\left(\sqrt{d_2}+\epsilon\sqrt{d_1}\right)^2=\mathfrak{q}^2(E_0-2RA_0)^2,\quad \epsilon=\pm1, \quad (17)$$

and thereby the sign $+/-$ determines co/counter-rotating KN binary BHs, where A_0 defines either A_1 or A_2 since these two terms are equivalent.

In order to illustrate how this dynamical law might be used to describe various scenarios among two interacting BHs, for instance, let us explore first a four parametric solution describing the case of a binary system of unequal counter-rotating KN BHs [16] that arises immediately when $\epsilon=-1$ and $\mathfrak{q}=0$, where it is pretty much obvious that Eq. (17) is satisfied with the condition

$$\sigma_{1E}^2=\sigma_{2E}^2,$$
$$\sigma_{iE}=\sqrt{M_i^2-Q_i^2-2(-1)^i Q_i\frac{M_2Q_1-M_1Q_2}{R+M}}, \quad i=1,2. \tag{18}$$

For such a case, both angular momenta displayed in Eq. (16) are reduced to

$$J_i=\varepsilon_i\frac{\sigma_{iE}P_{1i}(R+M)}{\sqrt{P_{00}(R^2-\delta_0)}}, \quad i=1,2, \quad \varepsilon_1=-\varepsilon_2=\pm 1,$$
$$P_{11}=M_1\big[(R+M_2)^2-M_1^2\big]-Q_1\big[Q_2R-\delta_2Q\big],$$
$$P_{00}=\big[(R+M_1)^2-M_2^2\big]\big[(R+M_2)^2-M_1^2\big]+(\delta_1R+\delta_2Q)^2,$$
$$P_{12}=P_{11(1\leftrightarrow 2)}, \qquad \delta_0=M^2-Q^2, \qquad \delta_1=Q_1-Q_2. \tag{19}$$

Better yet, the substitution of these formulas inside of Eq. (12) derives directly the simple results

$$q_o=\frac{Q_1}{2}(R-2M_2)-\frac{Q_2}{2}(R-2M_1), \quad b_o=\frac{Q_2P_2-Q_1P_1}{R+M},$$
$$P_1=-\frac{J_1(R+\delta_2)(R^2-\delta_0)}{P_{11}}, \quad P_2=\frac{J_2(R-\delta_2)(R^2-\delta_0)}{P_{12}}. \tag{20}$$

So, it is not quite difficult to show that Eqs. (8)-(10) are identically satisfied by the results given above in Eqs. (18)-(20), when $\mathfrak{q}=0$ is considered. The reader should notice from this counter-rotating example that if both angular momenta equal zero, then $\sigma_{iE}=0$ defines the extreme condition in electrostatic spacetimes. The function σ_{iE} and its thermodynamical properties have been derived first in [26].

On the other hand, the vacuum solution earlier studied in Ref. [7] is the second trivial scenario that appears in the absence of electric charges $Q_1=Q_2=0$ and it is well depicted by only three parameters, where after choosing $\epsilon=+1$, Eq. (17) provides us a bicubic equation for co-rotating Kerr BHs that is given by

$$(M^2-\mathfrak{q}^2)(4M_1M_2\mathfrak{q}^2-p_1p_2)-4M_1M_2\mathfrak{q}^2R^2=0,$$
$$p_1=(R+M_1)^2-M_2^2+\mathfrak{q}^2, \quad p_2=(R+M_2)^2-M_1^2+\mathfrak{q}^2. \tag{21}$$

A trivial combination of Eqs. (12), (16), and (21), will lead us to the following expressions in the co-rotating case

$$P_{1,2}=\frac{(M^2-\mathfrak{q}^2+MR)(M_2-M_1)}{2\mathfrak{q}}\pm(M\mathfrak{q}-J),$$
$$J_1=\frac{M_1^2\mathfrak{q}P_0}{Mp_1}, \qquad J_2=\frac{M_2^2\mathfrak{q}P_0}{Mp_2}. \tag{22}$$

In the meanwhile, whether $\epsilon=-1$ it is possible to derive another bicubic algebraic equation concerning counter-rotating Kerr BHs that now contains the form

$$(M^2-\mathfrak{q}^2)(4M_1M_2\mathfrak{q}^2-p_1p_2)+4M_1M_2R^2(R+M)^2=0, \tag{23}$$

where this binary configuration is entirely depicted by

$$P_{1,2}=\frac{M\mathfrak{q}(R^2+MR-M^2+\mathfrak{q}^2)}{2(M_2-M_1)(R+M)}\pm(M\mathfrak{q}-J),$$
$$J_1=\frac{M_1^2\mathfrak{q}}{M_1-M_2}\left(1-\frac{2M_2(R+M)p_2}{4M_1M_2P_0-p_1p_2}\right),$$
$$J_2=-\frac{M_2^2\mathfrak{q}}{M_1-M_2}\left(1-\frac{2M_1(R+M)p_1}{4M_1M_2P_0-p_1p_2}\right). \tag{24}$$

Regarding now the most general case, where after non-trivial algebraic manipulations on Eqs. (12), (16), and (17), eventually we will get the algebraic set of parameters $\{q_o, b_o, P_1, P_2\}$, which solve analytically Eqs. (8)-(10) and give us a complete description of extreme KN binary BHs in a physical representation; they read

$$q_o=\left(\frac{Q_1C_2-Q_2C_1}{2}+M\Big[(M_1Q_2-M_2Q_1)P_0\right.$$
$$\left.-Q_1Q_2(Q_1-Q_2)(R+M)\Big]\right)\frac{RP_0(R^2-\Delta)}{E_0-2RA_0},$$
$$b_o=\left[\frac{Q_1}{\mathfrak{q}}\left(RP_0\left(\frac{2H_{2+}(R^2-\Delta)+R(C_2+Q_2QP_0)}{E_0-2RA_0}\right)-1\right)\right.$$
$$\left.+\frac{Q_2}{\mathfrak{q}}\left(RP_0\left(\frac{2H_{1+}(R^2-\Delta)+R(C_1+Q_1QP_0)}{E_0-2RA_0}\right)-1\right)\right]$$
$$\times\frac{(R^2-\Delta)}{2},$$
$$P_{1,2}=\frac{RP_0(d_2-d_1)(R^2-\Delta)}{2\mathfrak{q}(E_0-2RA_0)}\pm(M\mathfrak{q}-J), \tag{25}$$

while the total angular momentum can be expressed as follows

$$J=M\mathfrak{q}-\left(\frac{R+M}{\mathfrak{q}}-RP_0^2\frac{P_0(R^2-\Delta+MR)-2Q_1Q_2(\Delta+MR)}{\mathfrak{q}(E_0-2RA_0)}\right)$$
$$\times\frac{(R^2-\Delta)}{2}. \tag{26}$$

Then we have that a whole description for co- and counter-rotating electrically charged BHs is made once the parameters contained in Eqs. (25) and (26) are inserted inside the asymptotically flat exact solution, which is obtainable from Eqs. (5) and (7) by making simply $J_0=0$ and $B=0$, where a physical representation in terms of the parameters $\{M_1, M_2, Q_1, Q_2, R\}$ will be achieved through the parameter $\mathfrak{q}$ once Eq. (17) is taken into account. Finally, to complete the full solution, we show also the Kinnersley potential [27]

$$\Phi_2=\frac{(4\mathfrak{q}+iRxy)\chi-i\mathcal{I}}{\Lambda+2\Gamma},$$
$$\mathcal{I}=R^2\Big\{2\mathfrak{q}_o\Big[Rx-4i\mathfrak{q}y-M(1-y^2)\Big]-$$
$$Q\Big[\bar{\varepsilon}(1+y^2)+\big(2MRx-4\mathfrak{b}_oy-2\bar{\mathfrak{p}}+R^2\big)y+2i\mathfrak{q}Rx\Big]\Big\}(x^2-1)$$
$$+\Big\{2\mathfrak{q}_o\Big[(Mp-i\mathfrak{q}\bar{\varepsilon})(1+y^2)-Rx\big(4MRx-2\bar{\varepsilon}y+\bar{\mathfrak{p}}+4\Delta\big)$$
$$+i(2\mathfrak{q}R^2-r)y\Big]+Q\Big[(\bar{\varepsilon}\bar{\mathfrak{p}}-iMr)(1+y^2)-Rx\big[2\varepsilon Rx+$$
$$i(2r-\mathfrak{q}R^2)\big]-\Big(R^2(\bar{\mathfrak{p}}+2R^2-2\Delta)-2(|p|^2+\mathfrak{q}r)\Big)y\Big]\Big\}(y^2-1),$$
$$p=R^2-\Delta+is_1, \quad r=2\mathfrak{a}_o-\mathfrak{q}(R^2-2\Delta), \quad \varepsilon=\mathfrak{b}_o+is_2,$$
$$\mathfrak{a}_o=Ms_2+2b_oQ, \quad \mathfrak{b}_o=\big(\mathfrak{q}s_1-2q_oQ\big)/M, \tag{27}$$

with the aim to derive the magnetic potential by considering its real part. This potential is useful to deal with the contribution of the Dirac string in the horizon mass [11–13].

3.1. Physical characteristics of extreme KN binary BHs

The thermodynamical properties of each extreme KN BH satisfy the well-known Smarr formula for the mass [11,28]

$$M_i = 2\Omega_i J_i + \Phi_i^H Q_i, \quad i=1,2, \tag{28}$$

where Ω_i and Φ_i define the angular velocity and electric potential in the corotating frame of each BH. Another important thermodynamical aspect to be considered in the binary system is the area of the horizon S_i. In the extreme limit case of BHs, their corresponding formulas are obtained by setting $\sigma_i = 0$ in those ones previously derived in Ref. [14]. The result is

$$\begin{aligned}
&\Omega_i = \frac{\mathfrak{q} A_i}{P_0 \mathcal{P}_i} + \frac{J_i P_0^3 \mathcal{P}_i R^2 (R^2-\Delta)}{\mathcal{P}_i^2 \mathcal{N}_i^2 + P_0^2 (R^2-\Delta)^2 \mathcal{M}_i^2},\\
&\Phi_i^H = \frac{M_i - 2\Omega_i J_i}{Q_i}, \qquad S_i = 4\pi \frac{\mathcal{P}_i^2 \mathcal{N}_i^2 + P_0^2 (R^2-\Delta)^2 \mathcal{M}_i^2}{R^2 P_0 \mathcal{P}_i^2},\\
&\mathcal{N}_i = M_i P_0 - 2\mathfrak{q} J_i - Q_i Q(R+M),\\
&\mathcal{M}_i = J_i C_i + \mathfrak{q} Q_i^2 \left[M_i P_0 + Q_1 Q_2 (R+M)\right], \quad i=1,2.
\end{aligned} \tag{29}$$

On the other hand, the interaction force related to the strut has the form [14]

$$\begin{aligned}
&\mathcal{F} = \frac{\mathcal{N}_0}{P_0^3 (R^2 - M^2 + Q^2 + \mathfrak{q}^2)},\\
&\mathcal{N}_0 = (M_1 M_2 P_0^2 - \mathfrak{q}^2 Q_1^2 Q_2^2)\left[(R+M)^2 - \mathfrak{q}^2\right]\\
&\quad - (Q_1 - F_0)(Q_2 + F_0) P_0^3 + \mathfrak{q}^2 \Big\{(M_1 Q_2 - M_2 Q_1)^2 P_0\\
&\quad + Q_1 Q_2 \left[2(R^2 + MR + \mathfrak{q}^2) P_0 + (P_0 + Q_1 Q_2) Q^2\right]\Big\},
\end{aligned} \tag{30}$$

and it acquires the same aspect regardless if the binary system is co/counter-rotating. The distinction between each configuration will be dictated by the choice of the sign +/− in the dynamical law shown in Eq. (17). The conical singularity in the middle region among the BHs can be removed if the force equals zero. However, a non vanishing force can give us more insight on how the sources are attracting or repelling each other via the gravitational, spin-spin, and electric interactions. Besides, the force provides the limits of the interaction distance, for instance, the merger limit of BHs is obtainable by equating the denominator of its formula to zero, thus getting the value $R \equiv R_0 = \sqrt{M^2 - Q^2 - \mathfrak{q}^2}$, where $\mathfrak{q} = J/M$ [see Eq. (26)]. Luckily, the merger limit of extreme BHs brings us rather simple expressions for Ω_i and Φ_i, which are given by

$$\begin{aligned}
&\Omega_1 = \Omega_2 = \frac{J/M}{d_0}, \qquad \Phi_i = \frac{Q(R_0+M)}{d_0} + \frac{R_0}{2Q_i}, \quad i=1,2,\\
&d_0 = (R_0+M)^2 + (J/M)^2,
\end{aligned} \tag{31}$$

while each angular momentum takes the final form

$$\begin{aligned}
&J_1 = M_1 \mathfrak{q} + \frac{Q\nu}{2\mathfrak{q}} + \frac{R_0\left[2\delta_2(R_0+M) - Q\delta_1\right]}{4\mathfrak{q}},\\
&J_2 = M_2 \mathfrak{q} - \frac{Q\nu}{2\mathfrak{q}} - \frac{R_0\left[2\delta_2(R_0+M) - Q\delta_1\right]}{4\mathfrak{q}},\\
&\nu = M_1 Q_2 - M_2 Q_1.
\end{aligned} \tag{32}$$

Table 1
Numerical values at the merging limit for the values $M_1 = 1.2$ and $M_2 = 0.8$.

Q_1	Q_2	R_0	$\mathfrak{q}$	J_1	J_2
2.5	-1.8	0.11772	1.86980	1.44435	2.29524
0.9	1.06	0.09520	0.38644	1.98612	-1.21324
0.3	1.65	0.34519	0.27990	7.78701	-7.22721
1.0	0.95455	0.0	0.42394	1.30509	-0.45720
2.5	-1.55364	0.0	0.38644	1.07652	2.44735

In this limit, Eq. (17) is simplified as follows

$$\begin{aligned}
&\mathfrak{q} R_0 \bigg[\Big(2M_1 M_2 (R_0+M) - Q(M_2 Q_1 + M_1 Q_2) - Q_1 Q_2 R_0\Big) d_0^2\\
&- 2\mathfrak{q}^2 Q_1^2 Q_2^2 (R_0+M)\bigg] = 0, \qquad \mathfrak{q} = J/M,
\end{aligned} \tag{33}$$

and this equation is quite practical to straightforwardly identify some non trivial situations. On one hand, exclusively the counter-rotating case establishes as $\mathfrak{q} = 0$ the lowest value for $\mathfrak{q}$ at the distance $R_0 = \sqrt{M^2 - Q^2}$. Naturally that, in this physical scenario, the total angular momentum of the system is $J = 0$, where both angular momenta fulfill the condition $J_1 = -J_2 = \infty$ [see Eqs. (19) and (32)]. On the other hand, if $R_0 = 0$, then it follows that $\mathfrak{q} = \sqrt{M^2 - Q^2}$, where it can be possible to recover the well-known expression for extreme KN BHs

$$J_1 + J_2 = (M_1 + M_2)\sqrt{(M_1+M_2)^2 - (Q_1+Q_2)^2}. \tag{34}$$

However, this second case may adopt co- and counter-rotating setups, depending on the chosen values for the masses and electric charges. Additionally, other examples during the merger limit can be explored when the polynomial enclosed in square brackets within Eq. (33) is solved. Regarding this point, Table 1 illustrates a set of numerical values carrying out this task.

Conversely, when the BHs are far away from each other, in the limit $R \to \infty$, the parameter $\mathfrak{q} \to J_1/M_1 + \epsilon J_2/M_2$, $\epsilon = \pm 1$, where now each angular momentum satisfies the relation $J_i = M_i \sqrt{M_i^2 - Q_i^2}$ for extreme KN BHs. Once again, it must be recalled that the sign +/− for ϵ is related to co/counter-rotating KN BHs, respectively.

Continuing with the analysis, but centering our attention in the case where $R = 0$, one should be aware that the interaction force and the horizon area become indeterminate at this value. To calculate their correct expressions, one must apply a Taylor expansion around $R = 0$ after using the term $\mathfrak{q} = \sqrt{M^2 - Q^2} + C_0 R$ in Eq. (17) with the objective to obtain the correct value for C_0, which is governed by the quadratic equation

$$\begin{aligned}
&4C_0^2 + 4\sqrt{(\alpha_+^2 - 1)(1 - \alpha_-^2)}\, C_0 + \alpha_-^2 - 1 = 0,\\
&\alpha_\pm = 2\sqrt{\frac{2M^2 - Q^2}{\alpha_0}}\left(\sqrt{4J_1^2 + Q_1^4} \pm \sqrt{4J_2^2 + Q_2^4}\right),\\
&\alpha_0 = \delta_0 (4M^2 - Q^2 - \delta_1^2)^2 + 4(2M^2 - Q^2)^2 \delta_2^2,
\end{aligned} \tag{35}$$

and, therefore, it can be shown that the final expressions for the area of the horizon and the force are given by

$$\begin{aligned}
&S_i = \frac{4\pi (2M^2 - Q^2)^{3/2}\sqrt{4J_i^2 + Q_i^4}}{\sqrt{\alpha_0}\,\alpha_+}\left[1 + \left(\sqrt{\alpha_+^2 - 1} - \epsilon_0 \alpha_+\right)^2\right],\\
&\mathcal{F} = \frac{\alpha_0 \left(\sqrt{\alpha_+^2 - 1} + \epsilon_0 \alpha_+\right)^2 - 4(2M^2 - Q^2)^3}{16(2M^2 - Q^2)^3}, \quad i=1,2,
\end{aligned} \tag{36}$$

where $\epsilon_0 = \pm 1$. In this occasion, the signs $+/-$ define two different scenarios during the merger limit where both sources attract/repel to each other. In order to verify the accuracy of our result, for $Q_1 = Q_2 = 0$, it is possible to attain directly from Eq. (36) the result [29],

$$S_i = 8\pi M_i (M_1+M_2)^2 \left(\frac{M_1+M_2-\epsilon_0\sqrt{2M_1M_2}}{M_1^2+M_2^2}\right),$$
$$\mathcal{F} = \frac{M_1M_2+\epsilon_0\sqrt{2M_1M_2}(M_1+M_2)}{2(M_1+M_2)^2}, \quad i=1,2, \tag{37}$$

where the attractive scenario has been deduced before in Ref. [30], but it was expressed in terms of dimensionless parameters. Finally, after performing a first order expansion around $R=0$ and the simple coordinate changes $\rho = (r-M)\sin\theta$, $z = (r-M)\cos\theta$ on Eqs. (5) and (7) with $J_0 = 0$, $B = 0$, and the formulas contained in Eqs. (25)-(26), it is possible to get

$$\mathcal{E} \simeq 1 - \frac{2M}{r - ia\cos\theta}\left(1 - \mathcal{K}\frac{(r-M)\cos\theta - ia\sin^2\theta}{r - ia\cos\theta}\frac{R}{r-M}\right),$$
$$\Phi \simeq \frac{Q}{r - ia\cos\theta}\left(1 - \mathcal{K}\frac{(r-M)\cos\theta - ia\sin^2\theta}{r - ia\cos\theta}\frac{R}{r-M}\right),$$
$$f \simeq f_0\left[1 + 2\mathcal{K}\frac{(Mr+a^2)\Xi - (2Mr-Q^2)(Mr+a^2\cos 2\theta)}{\Xi(\Xi - 2Mr + Q^2)}\right.$$
$$\left.\times\frac{R\cos\theta}{r-M}\right], \qquad e^{2\gamma} \simeq e^{2\gamma_0} + \frac{4a^2\mathcal{K}\sin^2\theta}{(r-M)^3}R\cos\theta,$$
$$\omega \simeq \omega_0\left[1 - 2\mathcal{K}\left(\frac{2(r-M)}{\Xi - 2Mr + Q^2} - \frac{M}{2Mr - Q^2}\right)R\cos\theta\right],$$
$$\mathcal{K} = \frac{(2M^2-Q^2)\left[Q\delta_1(4M^2-Q^2-\delta_1^2) - 4M(2M^2-Q^2)\delta_2\right]}{\alpha_0},$$
$$\times\left(1 - \frac{\epsilon_0\sqrt{\alpha_+^2-1}}{\alpha_+}\right), \qquad f_0 = 1 - \frac{2Mr - Q^2}{\Xi},$$
$$e^{2\gamma_0} = 1 - \frac{a^2\sin^2\theta}{(r-M)^2}, \qquad \omega_0 = -\frac{a(2Mr-Q^2)\sin^2\theta}{\Xi - 2Mr + Q^2},$$
$$\Xi = r^2 + a^2\cos^2\theta, \qquad a = J/M, \qquad \epsilon_0 = \pm 1, \tag{38}$$

being (r,θ) the Boyer-Lindquist coordinates. Eq. (38) defines a deformed metric for a near horizon extreme binary KN BH, where in the physical limit at $R=0$ it is possible to recover the metric for a single extreme KN BH of mass $M = M_1 + M_2$, electric charge $Q = Q_1 + Q_2$, and angular momentum $J = M\sqrt{M^2 - Q^2}$, in other words

$$ds^2 = f_0^{-1}\left[e^{2\gamma_0}\left[dr^2 + (r-M)^2 d\theta^2\right] + (r-M)^2\sin^2\theta d\varphi^2\right]$$
$$- f_0(dt - \omega_0 d\varphi)^2. \tag{39}$$

4. Conclusion

The derivation of the metric that completely characterizes unequal configurations of extreme KN BHs in a physical representation finally has been succeeded. The task of solving the conditions on the axis and the one eliminating the magnetic charges is accomplished by adopting a fitting parametrization that has been earlier introduced in [14]. It follows that the asymptotically flat metric has been written in a quite simple form by means of Perjes' approach [17], where it contains a physical representation in terms of the set $\{M_1, M_2, Q_1, Q_2, R\}$ that is very suitable to concrete some applications in rotating charged binary systems. Similarly to the non-extreme case [14], the physical parameters are related to each other through an algebraic equation that represents a dynamical law for interacting BHs with struts. Unfortunately, there is no chance to solve exactly this higher degree equation except for some special unequal cases [7,16].

Due to the fact that our solution reported in this work is presented with a more physical aspect, the physical limits of the interaction of BHs can be readily identified in both co- and counter-rotating cases. Even better, the thermodynamical characteristics of each BH during the merger limit have been also derived and concisely introduced, where it is possible to conceive an attractive or repulsive final state at this limit. In this respect, the deformed metric for a near horizon extreme binary KN BH is also obtained, from which we do not exclude that it might be helpful to develop some analytical studies related to the collision of two BHs like gravitational waves by assuming a quasi stationary process, in a similar way to that previously considered in [31].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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