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Lorentz-breaking weighted measures as quantum field theory regulators

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ABSTRACT

In this work we develop a re-formulation of quantum field theory through the more general weighted measures that the definition of quantum fields allows, and that violate explicitly the Lorentz symmetry; this approach provides finite answers for the long-live problems of the traditional formulations of quantum field theories, namely, smooth distributions for the field commutators that are finite a short distances, finite vacuum expectation values for the energy (without invoking normal ordering of operators), and finite fluctuations for the field operators. We shall show that the present scheme will allow us to construct an infinite family of noncommutative field theories that are compared with other formulations. Our regularization scheme breaks down for a massless field, which will be considered in future explorations.

1. Antecedents, motivations, and results

The origin of the divergences in QFT can be traced to the construction of the canonical commutation relations for the quantum fields and the requirement of Lorentz symmetry; hence, the creation of a finite quantum field theory would require the modification of the former and/or the abandonment of the later. Many different theoretical frameworks have been developed by exploring the Lorentz symmetry breaking as a quantum field theory regulator; additionally this idea is largely motivated by the theoretical predictions coming from quantum gravity and string theory frameworks, namely, that the Lorentz symmetry is not an exact symmetry at high energies (see for example [1], and [2]). Furthermore, the canonical commutation relations for quantum fields can be modified by the appearance of non-commutative spaces in string theory frameworks, and recent developments in non-commutative quantum mechanics [3–5]; the resulting theories violate relativistic invariance, but they suffer still of ultraviolet divergences (see for example [6]); hence the problem is far from to be solved.

In this manuscript we discuss whether actually the quantum gravity and/or string theory inspired frameworks are necessary for providing the clues for a finite quantum field theory; rather, we shall show that the conventional scheme may have the seeds for the possible creation of finite quantum field theories; the key observation is that the canonical hypothesis of promoting the commutation relations (of the dynamical variables) in quantum mechanics, to the commutation relations (for the dynamical variables) in quantum field theory, through the transition to the continuum through $\delta_{ij} \rightarrow \delta_{Dirac}(\vec{x}-\vec{y})$, allows to incorporate smooth distributions that will lead to smooth away the singularities that originate the divergences; this is achieved by introducing Lorentz-breaking weighted measures. As we shall see, the Dirac delta function is only one element (in fact the only divergent one in all dimensions that respects the Lorentz symmetry) of an infinite family of distributions available for describing finite quantum field theories, and thus we shall be able to construct the field commutators in terms of smooth distributions that will maintain the symmetries of the Dirac delta function, but will be also dependent on the background dimension, on the mass, and will be finite at short distances (section 2.1).

Once we have regularized the field commutators, the consequence direct is the regularization of the vacuum energy (section 3), one of the most famous long-live problems of quantum field theory; our regularization scheme does not require to invoke the normal ordering of operators for extracting the infinite vacuum energy, and thus it is consistent with general relativity. Additionally the vacuum expectation

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values for field operators, their fluctuations at fixed points, will be finite, as opposed to the traditional belief (section 4). We finalize the manuscript with some concluding remarks, on some extensions and future developments of our results.

As preliminary element, we recall that quantum field theories are constructed by imposing the canonical relations used in ordinary quantum mechanics, namely,

$$[x_i, p_j] = i\delta_{ij}, \tag{1}$$

$$\big[x_i, x_j\big] = 0, \quad [p_i, p_j] = 0, \tag{2}$$

on the field commutation relations, by considering the transition to the continuum $\delta_{ij} \to \delta(\vec{x} - \vec{y})$. For concreteness, for a complex scalar field we have

$$[\hat{\psi}(\vec{x}), \hat{\Pi}_\psi(\vec{y})] = i\delta(\vec{x} - \vec{y}); \tag{3}$$

$$\big[\hat{\psi}(\vec{x}), \hat{\psi}^\dagger(\vec{y})\big] = 0, \quad \big[\hat{\Pi}_\psi(\vec{x}), \hat{\Pi}^\dagger_\psi(\vec{y})\big] = 0; \tag{4}$$

$$\big[\hat{\psi}(\vec{x}), \hat{\Pi}^\dagger_\psi(\vec{y})\big] = 0; \tag{5}$$

where $\hat{\Pi}_\psi$ stands for the conjugate momentum for ψ; these relations are evaluated *at the same time*, at different spatial locations $\vec{x}$, and $\vec{y}$. These commutation relations described by Dirac delta functions are valid for arbitrary background dimensions, apply indistinctly for the massless or massive case, and are divergent at short distances $(\vec{x} - \vec{y}) \to 0$; by depending on the difference $(\vec{x} - \vec{y})$, they are translational invariant and symmetric under $(\vec{x} - \vec{y}) \to -(\vec{x} - \vec{y})$. In the next section, we smear the field commutator (3) over space, by relaxing the transition to the continuum through $\delta_{ij} \to D(\vec{x} - \vec{y})$, where $D(\vec{x} - \vec{y})$ is a smooth distribution that will be obtained by choosing all convergent Lorentz-breaking measures in the momenta space that the definition of field operators admits; this simple criterion will allow construct the infinite family of distributions commented above.

2. Weighted measures with broken Lorentz symmetry

We study for simplicity a complex scalar field in $D+1$ dimensions with Lagrangian $\mathcal{L} = \partial_\mu\varphi\partial^\mu\overline{\varphi} - m^2\varphi\overline{\varphi}$, with equations of motion given by $(\square + m^2)\varphi = 0$; our results with global $U(1)$ symmetry can be generalized to local $U(1)$ gauge symmetry in a direct way. With the decompositions for the quantum field and its conjugate momentum,

$$\hat{\varphi}(\vec{x}, t) = \frac{1}{\sqrt{2}\sqrt{(2\pi)^D}} \int \frac{d\vec{k}}{w_k^{\frac{s}{2}}} \big[\hat{a}_k e^{-iw_k t} e^{i\vec{k}\cdot\vec{x}} + \hat{b}^\dagger_k e^{iw_k t} e^{-i\vec{k}\cdot\vec{x}}\big]; \tag{6}$$

$$\hat{\pi}(\vec{x}, t) = \frac{i}{\sqrt{2}\sqrt{(2\pi)^D}} \int \frac{d\vec{k}}{w_k^{\frac{s}{2}-1}} \big[\hat{a}^\dagger_k e^{iw_k t} e^{-i\vec{k}\cdot\vec{x}} - \hat{b}_k e^{-iw_k t} e^{i\vec{k}\cdot\vec{x}}\big]; \tag{7}$$

the equations of motion are satisfied provided that the (Lorentz invariant) dispersion relation $w_k^2 = m^2 + \vec{k}^2$ holds; the weight s of the measure $\int \frac{d\vec{p}}{w_k^{\frac{s}{2}}}$ is in general a real quantity. Typically with the choice $s = 1$, one is enforcing the transition to the continuum with $[\hat{\varphi}, \hat{\pi}] \to \delta(\vec{x} - \vec{y})$, respecting the Lorentz symmetry. Now, the expression $(w_k^2)^s = (m^2 + \vec{k}^2)^s$ breaks explicitly the Lorentz symmetry; as a function on s can be expanded around $s = 1$,

$$(w_k^2)^s = (m^2 + \vec{k}^2)^s = (m^2)^s(1 + \frac{\vec{k}^2}{m^2}) \sum_{n\geq 0}^{+\infty} \frac{(s-1)^n \ln^n(1 + \frac{\vec{k}^2}{m^2})}{n!}; \tag{8}$$

as we shall see, by sacrificing the Lorentz invariance for arbitrary s, we shall be able to construct finite quantum field theories. In order to gain insight into the physical content of the above expression, we consider as an example the case with $s = 2$; with the IR restriction $\frac{\vec{k}^2}{m^2} < 1$, one has the approximation $\ln\Big(1 + \frac{\vec{k}^2}{m^2}\Big) \approx \frac{\vec{k}^2}{m^2}$, and the sum reduces to an exponential function on $\frac{\vec{k}^2}{m^2}$, and thus, $(w_k^2)^2 \approx (m^4 + m^2\vec{k}^2)e^{\frac{\vec{k}^2}{m^2}}$. Moreover, for the same IR approximation, but for s arbitrary, the expression will have the form $(w_k^2)^s \approx (m^2)^s + s(m^2)^{s-1}\vec{k}^2 + (s-1)(m^2)^{s-2}\vec{k}^4 + \cdots$.

Furthermore, the (nonvanishing) commutation relations for the annihilation/creation operators read

$$[\hat{a}_k, \hat{a}^\dagger_{k'}] = \alpha\delta(k - k'), \quad [\hat{b}_k, \hat{b}^\dagger_{k'}] = \beta\delta(k - k'), \tag{9}$$

where α and β are real parameters, and will play a nontrivial role in the approach at hand; the traditional choice $\alpha = 1 = \beta$ is only a possible election. In fact our present criticism on the textbook statements includes the commutation relations between annihilation/creation operators, but we restricted ourselves to the deformation described in (9), and we shall develop a more general deformation scheme elsewhere.

With the commutators (9), the general transition will be achieved through $[\hat{\varphi}, \hat{\pi}] \to D_s(\vec{x} - \vec{y})$, where D_s are smeared versions of the Dirac delta, that will depend on the mass, on the spatial interval, and are finite to short distances; for arbitrary dimension, the nontrivial commutators read

$$\left[\hat{\varphi}(\vec{x},t),\hat{\pi}(\vec{x}',t)\right]=\frac{i(\alpha+\beta)}{2(2\pi)^D}\int_{-\infty}^{\infty}\frac{d\vec{k}}{\omega_k^{s-1}}\cos\left(\vec{k}\cdot(\vec{x}-\vec{x}')\right); \tag{10}$$

$$\left[\hat{\varphi}(\vec{x},t),\hat{\varphi}^{\dagger}(\vec{x}',t)\right]=\frac{(\alpha-\beta)}{2(2\pi)^D}\int_{-\infty}^{\infty}\frac{d\vec{k}}{\omega_k^{s}}\cos\left(\vec{k}\cdot(\vec{x}-\vec{x}')\right); \tag{11}$$

$$\left[\hat{\pi}(\vec{x},t),\hat{\pi}^{\dagger}(\vec{x}',t)\right]=-\frac{(\alpha-\beta)}{2(2\pi)^D}\int_{-\infty}^{\infty}\frac{d\vec{k}}{\omega_k^{s-2}}\cos\left(\vec{k}\cdot(\vec{x}-\vec{x}')\right); \tag{12}$$

$$\left[\hat{\varphi}(\vec{x},t),\hat{\pi}^{\dagger}(\vec{x}',t)\right]=0; \tag{13}$$

the last vanishing commutator is due to the trivial commutators $[\hat{a},\hat{a}]=0=[\hat{b},\hat{b}]$, and it corresponds to the commutator (5); furthermore, with the choice $s=1$, and the election $\alpha=1=\beta$, one defines the traditional scheme with $\left[\hat{\varphi},\hat{\pi}\right]=i\delta_{Dirac}$, and $\left[\hat{\varphi},\hat{\varphi}^{\dagger}\right]=0=\left[\hat{\pi},\hat{\pi}^{\dagger}\right]$; note that with this choice of parameters, one decides which commutator is nontrivial, and then the commutation relations (3)-(5) are reproduced. However, this is only an election, since one can for example to choice $\alpha=1=-\beta$, and thus $\left[\hat{\varphi},\hat{\pi}\right]=0$, and $\left[\hat{\varphi},\hat{\varphi}^{\dagger}\right]\neq 0\neq\left[\hat{\pi},\hat{\pi}^{\dagger}\right]$, which corresponds certainly to an atypical version for quantum field theory.

Therefore, if in general $\alpha\neq\pm\beta$, we have the more stringent version of the commutation relations (3)-(5), which will be nonvanishing, close in spirit to noncommutative quantum mechanics, in which the commutation relations (2) are precisely nonvanishing [3–5]. Hence, one can start with the initial idea of a transition to the continuum from the relations (1), and (2), to the relations (3)-(4), but one obtains at the end a more general version for quantum field theory, without invoking any breakthrough of the modern physics. In the same sense, if we look back, in the converse direction of that transition, then one can infer that a more stringent version for quantum mechanics there exists, in which the coordinates and momenta do not commute to each other.

We discuss in detail only the fundamental commutator (10); independently on the choice of the parameters α, and β, the integrals for the relations (11), and (12) can be developed along the same lines. Thus, in the expression (10), the value $s=1$ corresponds to the usual Dirac delta in D dimensions; this general integral for s arbitrary will define the distributions D_s, that we shall determine explicitly in different dimensions. Note that the cases for $s<1$ are evidently divergent, and we work with the restriction $s\geq 1$, with the purpose of including the standard case $s=1$ with Lorentz symmetry; as we shall see, the conventional description in terms of the Dirac delta distribution separates the infinite family of divergent theories, from the infinite family of (Lorentz-breaking) convergent ones constructed here. The k-integration in the general expression (10) must be determined explicitly for each dimension, and in principle it exists in arbitrary spacetime dimensions.

2.1. 3+1 QFT

On the other hand, in three spatial dimensions the commutator in spherical coordinates is written as

$$\left[\hat{\varphi}(\vec{x},t),\hat{\pi}(\vec{x}',t)\right]=\frac{(\alpha+\beta)}{2}\frac{i}{2\pi^2|\Delta\vec{x}|}\int_0^{\infty}dk\frac{k\sin\left(k|\Delta\vec{x}|\right)}{\left(\sqrt{k^2+m^2}\right)^{s-1}}, \tag{14}$$

where $k=\sqrt{k_1^2+k_2^2+k_3^2}$; some solutions are [7],

s	$\left[\hat{\varphi},\hat{\pi}\right]$	$\lim_{\lvert\Delta\vec{x}\rvert\to 0}\left[\hat{\varphi},\hat{\pi}\right]$
1	$i\frac{1}{2\pi}\frac{\delta(\lvert\Delta\vec{x}\rvert)}{\lvert\Delta\vec{x}\rvert^2}$	$i\infty$
2	$i\frac{m}{2\pi^2\lvert\Delta\vec{x}\rvert}K_1(m\lvert\Delta\vec{x}\rvert)$	$\frac{i}{2\pi^2}\frac{1}{\lvert\Delta\vec{x}\rvert^2}$
3	$i\frac{1}{4\pi\lvert\Delta\vec{x}\rvert}e^{-m\lvert\Delta\vec{x}\rvert}$	$\frac{i}{4\pi}\frac{1}{\lvert\Delta\vec{x}\rvert}$
4	$i\frac{1}{2\pi^2}K_0(m\lvert\Delta\vec{x}\rvert)$	$-\frac{i}{2\pi^2}\ln\frac{m\lvert\Delta\vec{x}\rvert}{2}$
5	$i\frac{1}{8\pi m}e^{-m\lvert\Delta\vec{x}\rvert}$	$\frac{i}{8\pi}\frac{1}{m}$
6	$\frac{i}{6\pi^2}\frac{\lvert\Delta\vec{x}\rvert}{m}K_1(m\lvert\Delta\vec{x}\rvert)$	$\frac{i}{6\pi^2}\frac{1}{m^2}$

the convergence at short distances takes the form $\lim_{|\Delta\vec{x}|\to 0}\left[\hat{\varphi},\hat{\pi}\right]\sim\frac{\alpha+\beta}{m^{s-4}}$, with $s\geq 5$; in contrast with the previous cases such a convergence is achieved from $s\geq 4$, for two spatial dimension, and from $s\geq 3$ for one spatial dimension; in fact these results suggest that for a $D+1$ space-time dimension, the convergence is achieved for $s\geq D+2$; additionally the logarithmic divergence is achieved just when the weight coincides with the spacetime dimension, $s=D+1$. Such a coincidence allows, in particular, to choice the function K_0 as the common expression for the field commutator in all dimensions. General expressions for the integrals (14) for every s are available in the literature [7].

3. Finite observables

We construct now the Hamiltonian, momentum and charge operators; we start from their classical form

$$H = \int d^D x \left(\pi^*_\varphi \pi_\varphi + \nabla\varphi^* \cdot \nabla\varphi + m^2\varphi^*\varphi\right), \tag{15a}$$

$$Q = i\int d^D x\, (\varphi^\dagger\partial_0\varphi - \partial_0\varphi^\dagger\varphi), \tag{15b}$$

$$P_i = -i\int d^D x\, (\pi_\varphi\partial_i\varphi + \pi_{\varphi^\dagger}\partial_i\varphi^\dagger); \tag{15c}$$

as it is well known, the quantization ambiguities imply that one has that to choice the order in which the commuting classical quantities will be promoted to operators; for example, if one considers *symmetrized* classical expressions such as $\pi^*_\varphi\pi_\varphi = \frac{1}{2}\pi^*_\varphi\pi_\varphi + \frac{1}{2}\pi_\varphi\pi^*_\varphi$, then the operator versions of the observables read

$$\hat{H} = \int d^D k\, \omega_k^{2-s}\left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right] + \underbrace{\frac{\alpha+\beta}{2}\frac{L^D}{(2\pi)^D}\int d^D k\, \omega_k^{2-s}} \tag{16a}$$

$$\hat{Q} = \int d^D k\, \omega_k^{1-s}\left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) - \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right] + \underbrace{\frac{\alpha-\beta}{2}\frac{L^D}{(2\pi)^D}\int d^D k\, \omega_k^{1-s}}, \tag{16b}$$

$$\hat{P}_i = \int d^D k\, \frac{k_i}{\omega_k^{s-1}}\left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right] + \frac{\alpha+\beta}{2}\frac{L^D}{(2\pi)^D}\underbrace{\int d^D k\, k_i\, \omega_k^{1-s}}_{=0}, \tag{16c}$$

where we have confined the system in a box with sides of length L. The above expressions have been obtained by using the commutators (9), by locating the annihilation operators to the right hand side, without invoking normal ordering; in the approach at hand, the potentially divergent integrals can be controlled by choosing appropriately the weight s, and then they will be finite.

With the usual definition of the vacuum, $\hat{a}(k)\,|0\rangle = 0 = b(k)\,|0\rangle$, the first terms with the annihilation operators located to the right hand side vanish trivially. Therefore, the action of the observables on the vacuum state is determined by the terms in underbrace, which are potentially divergent, depending on the choice of the parameters α, β, and the weight s; for the momenta (16c) such an integral vanishes trivially for all dimensions, because the integrand is an odd function on k_i. Furthermore, the choice $\alpha = \beta$ will lead a neutral vacuum, and to a nontrivial vacuum energy; conversely, the choice $\alpha = -\beta$ will lead to a vanishing vacuum energy, and to a nontrivial vacuum charge. Hence, if one imposes the constraint $\alpha \neq \pm\beta$, then the vacuum will have both a nontrivial energy and a nontrivial charge; however in the present approach both quantities can be regularized, removing the UV divergences.

On the other side, if the transition to operators is made without using the *symmetrization* of classical expressions, but we use for example the expressions such as Eqs. (15), then the coefficients in the integrals in underbrace are simply α, or β; however, the key observation in the approach at hand is that such integrals are finite, and the normal ordering of operators is not invoked.

3.1. The vacuum energy is finite

This case includes the traditional values $\alpha = 1 = \beta$, that together with the choice $s = 1$ lead to a neutral vacuum state $\hat{Q}\,|0\rangle = 0$, and with ultraviolet divergences for $\hat{H}$; this represents the first famous result of QFT, an infinite energy for the vacuum, which is removed by invoking normal ordering of operators; as we shall see, this procedure is not required in the present scheme.

In the approach at hand the zero-point energy reads,

$$\hat{H}\,|0\rangle = \frac{(\alpha+\beta)L^D}{2(2\pi)^D}\int \frac{d^D k}{\left(\sqrt{|\vec{k}|^2+m^2}\right)^{s-2}}\,|0\rangle \equiv \frac{(\alpha+\beta)L^D}{2(2\pi)^D}H_0\,|0\rangle\,. \tag{17}$$

In the next table we show some values for the integral H_0;

s	$H_{0,D=1}$	$H_{0,D=2}$	$H_{0,D=3}$
1	∞	∞	∞
2	∞	∞	∞
3	$ln\frac{k}{m}$	∞	∞
4	$\frac{\pi}{m}$	$ln\frac{\lvert\vec{k}\rvert}{m}$	∞
5	$\frac{2}{m^2}$	$\frac{2\pi}{m}$	$ln\frac{\lvert\vec{k}\rvert}{m}$
6	$\frac{\pi}{2}\frac{1}{m^3}$	$\frac{\pi}{m^2}$	$\frac{\pi^2}{m}$

the first row with $s = 1$ is justly the traditional divergent case, valid for all dimensions; this result is replicated for the case $s = 2$. In general for $D + 1$ space-time dimensions, the vacuum energy is finite from $s \geq D + 3$, and takes the form $\frac{\alpha+\beta}{m^{s-D-2}}$; therefore, one can extend the table to the right and below, since the diagonals have basically the same form. In particular the diagonal with logarithmic divergences is obtained with the approximation $\frac{m}{k} \ll 1$ as $k \to \infty$ after the integration. Therefore, the general conclusion is that one chooses whether the vacuum has or not an infinite energy; for a spacetime dimension given, there exist in fact an infinite number of values for s leading to finite values for the vacuum energy; see the Eqs. (34), and (35) below for some numerical examples.

Classically the potential for the complex field φ coming from the Lagrangian defines a paraboloid, namely $V(\varphi,\bar{\varphi}) = m^2\varphi\bar{\varphi} = m^2(\varphi_1^2 + \varphi_2^2)$, whose lowest energy level implies that $\varphi = 0$, *i.e.* the bottom of such a paraboloid. According to the table, the vev for the energy does not vanish (and in general does not diverge), rather it is finite and is defined in terms of the mass. This result can be considered as the analogous of the well known result for a quantum harmonic oscillator, whose potential energy is defined as $V(x) = \frac{1}{2}\omega^2 x^2$; classically the vacuum is the state in which the particle is motionless, with $x = 0$; however, quantum-mechanically the lowest energy state has an energy $E_0 = \frac{1}{2}\hbar\omega$. This zero-point energy was fundamental for recovering, at the Einstein time, the expected classical limit for the average energy of an oscillator in thermal equilibrium at temperature T; specifically such a limit is obtained by expanding the Planck formulae $E_W = \frac{\hbar\omega}{e^{\hbar\omega/kT}-1} + \frac{1}{2}\hbar\omega$, in the classical limit $kT \gg \hbar\omega$; for more details on this issue and its relationship with the cosmological constant problem see [8]. As well known, there no exists in the traditional scheme, an analogous result due to the divergence of the vev for the energy; furthermore, as we shall see in the section 4, the vev for the quantum field itself, will not diverge.

Furthermore, for the excited states of the quantum harmonic oscillator, obtained from the repeated action of the creation operator on the vacuum, $|n\rangle \equiv (\hat{a}^\dagger)^n |0\rangle$, we have the very known eigen-energy expression $\hat{H}|n\rangle = (n+\frac{1}{2})\hbar\omega|n\rangle$, and then the system has a ladder of energy states. In our case, we have a similar situation, by considering the excitations of the field; if $\hat{a}^\dagger|0\rangle \equiv |k_a\rangle$ and $\hat{b}^\dagger|0\rangle = |k_b\rangle$ are single excited states, then it is straightforward to construct the following energy eigenstates,

$$\hat{H}|k_a\rangle = \Big[\frac{\alpha}{\omega_{k_a}^{s-2}} + \frac{\alpha+\beta}{2}\frac{L^D}{(2\pi)^D}H_0\Big]|k_a\rangle\,,$$
$$\hat{H}|k_b\rangle = \Big[\frac{\beta}{\omega_{k_b}^{s-2}} + \frac{\alpha+\beta}{2}\frac{L^D}{(2\pi)^D}H_0\Big]|k_b\rangle\,; \tag{18}$$

where the vacuum energy $H_0 = H_0(m;s)$ is described above in the table; thus, these one-particle states have excitation energies $\frac{\alpha}{\omega_{k_a}^{s-2}}$, and $\frac{\beta}{\omega_{k_b}^{s-2}}$ in relation to the vacuum energy. These single expressions can be generalized for multi-particle eigen-states with n a-bosons excited, and with m b-bosons excited, in any order,

$$\hat{H}\left|k_a^1,,,k_a^n;k_b^1,,,k_b^m\right\rangle = \Big[\alpha\sum_{i=1}^{n}(\frac{1}{\omega_{k_a^i}})^{s-2} + \beta\sum_{i=1}^{m}(\frac{1}{\omega_{k_b^i}})^{s-2} + \frac{\alpha+\beta}{2}\frac{L^D}{(2\pi)^D}H_0\Big]\left|k_a^1,,,k_a^n;k_b^1,,,k_b^m\right\rangle; \tag{19}$$

hence, the energy of this eigen-state is given by the sum of the energies of the various particles. Additionally these energy eigen-states are also eigen-states for the number operators, which are defined as usual,

$$\int d\vec{k}\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})\left|k_a^1,,,k_a^n;k_b^1,,,k_b^m\right\rangle = n\alpha\left|k_a^1,,,k_a^n;k_b^1,,,k_b^m\right\rangle,$$
$$\int d\vec{k}\hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\left|k_a^1,,,k_a^n;k_b^1,,,k_b^m\right\rangle = m\beta\left|k_a^1,,,k_a^n;k_b^1,,,k_b^m\right\rangle; \tag{20}$$

which count effectively the number of particles of each type in the multi-particle states.

In the traditional scheme the normal ordering implies to forget (for a moment) the fundamental commutators (9) for extracting an infinite vacuum energy; however, in order to construct a nontrivial quantum field theory, one must restore those commutators, and then one does not use normal ordering for constructing the eigenstates for the Hamiltonian and for the number operators. This ambiguity has darkened the traditional formulation of quantum field theory, as opposed to the approach at hand, in which the fundamental commutators (9) are maintained always switched on, namely, for constructing finite vev for the observables, and for the building of the multi-particle eigen-states described above.

4. The vev for the field operator is finite

One consequence of the usual choice $s = 1$ is the inexistence of normalizable states; with the conventional definition of the vacuum state $\hat{a}|0> = 0 = \hat{b}|0>$, one has that, $<0|\hat{\psi}\hat{\psi}^\dagger|0> \to \infty$; hence, the action of the field operators on the Hilbert space is not well defined. Since operators and expectation values normalize to delta functions in the usual formulation, one requires to construct well defined operators by smearing those singular distributions, which can be achieved by creating a wave-packet through $\int d\vec{p}e^{-ip\cdot x}f(p)\hat{a}^\dagger|0>$, where typically the smearing function f is chosen as the Gaussian $f = e^{-p^2/2m^2}$; this procedure is of course as arbitrary as the smearing functions chosen, and this kind of arbitrariness also has darkened the traditional formulation of quantum field theory.

However, as expected at this point, the fluctuation of the field operator at a fixed point is in general finite for arbitrary weight s; explicitly we have for arbitrary dimension,

$$<0|\hat{\varphi}\hat{\varphi}^\dagger|0> = \frac{<0|0>}{2(2\pi)^D}\alpha\int_{-\infty}^{+\infty}\frac{d\vec{k}}{w_k^s}; \quad <0|\hat{\varphi}^\dagger\hat{\varphi}|0> = \frac{<0|0>}{2(2\pi)^D}\beta\int_{-\infty}^{+\infty}\frac{d\vec{k}}{w_k^s}; \tag{21}$$

therefore,

$$\langle 0| \left(\hat{\varphi}(\mathbf{x})\hat{\varphi}^{\dagger}(\mathbf{x}) + \hat{\varphi}^{\dagger}(\mathbf{x})\hat{\varphi}(\mathbf{x})\right)|0\rangle = \frac{<0|0>}{2(2\pi)^D}(\alpha+\beta)\int_{-\infty}^{+\infty}\frac{d\vec{k}}{w_k^s}; \tag{22}$$

this expression will lead to a similar table to that constructed for the vacuum energy above, for different values of the weight s and for different background dimensions.

5. On noncommutative field theory

The noncommutative version for the quantum mechanics of a harmonic oscillator, can be defined by the following commutation relations for the dynamical variables,

$$[q_i, p_j] = i\delta_{ij}; \quad [q_1, q_2] = i\theta, \quad [p_1, p_2] = iB; \tag{23}$$

inspired by this noncommutative scheme, in [6] a noncommutative field theory is introduced by considering the transition to the continuum through the following commutation relations for a complex scalar field,

$$[\hat{\psi}(\vec{x}), \hat{\Pi}_{\psi}(\vec{y})] = i\delta(\vec{x}-\vec{y}); \tag{24}$$

$$\left[\hat{\psi}(\vec{x}), \hat{\psi}^{\dagger}(\vec{y})\right] = \theta\delta(\vec{x}-\vec{y}), \quad \left[\hat{\Pi}_{\psi}(\vec{x}), \hat{\Pi}^{\dagger}_{\psi}(\vec{y})\right] = B\delta(\vec{x}-\vec{y}); \tag{25}$$

$$\left[\hat{\psi}(\vec{x}), \hat{\Pi}^{\dagger}_{\psi}(\vec{y})\right] = 0; \tag{26}$$

which correspond to a generalization of the commutation relations (3)-(5). There is no a comment on the commutator (26) in [6], and we assume here that it vanishes in that approach; this vanishing commutator coincides with our commutator (13), which is a consequence of the trivial commutators $[\hat{a}, \hat{a}] = 0 = [\hat{b}, \hat{b}]$. In the approach [6], the parameters θ, and B measure the noncommutativity, and their phenomenological bounds are established; such bounds do not contradict the low-energy phenomena, and will have observable effects at high energies.

According to the Eqs. (10)–(13) of the present approach, the more stringent version of the noncommutative quantum field theory can be obtained with the restriction $\alpha \neq \pm\beta$. Since each commutator is defined with a different weight s, one can not to identify all commutators with the Dirac delta, such as in the above proposal; one can identify only one commutator with that distribution.

According to the previous results, in a four dimensional space-time, the first value for the weight $s = 6$ guarantees the convergence for the vacuum energy and for the vev's of the field operators, which are of the form $\frac{\alpha+\beta}{m}$, and $\frac{\alpha+\beta}{m^3}$ respectively; moreover, for the field commutators we have,

$$\left[\hat{\varphi}(\vec{x}, t), \hat{\pi}(\vec{x'}, t)\right] = i\frac{\alpha+\beta}{2}\frac{|\Delta\vec{x}|}{6\pi^2 m}K_1(m|\Delta\vec{x}|); \tag{27}$$

$$\left[\hat{\varphi}(\vec{x}, t), \hat{\varphi}^{\dagger}(\vec{x'}, t)\right] = \frac{\alpha-\beta}{2}\frac{m|\Delta\vec{x}|+1}{32\pi m^3}e^{-m|\Delta\vec{x}|}; \tag{28}$$

$$\left[\hat{\pi}(\vec{x}, t), \hat{\pi}^{\dagger}(\vec{x'}, t)\right] = -\frac{\alpha-\beta}{2}\frac{e^{-m|\Delta\vec{x}|}}{8\pi m}; \tag{29}$$

$$\left[\hat{\varphi}(\vec{x}, t), \hat{\pi}^{\dagger}(\vec{x'}, t)\right] = 0. \tag{30}$$

In fact, from $s \geq 6$ there exists an infinite family of (finite) noncommutative quantum field theories in the more stringent version that can represent an alternative for the proposal given in [6]; we shall study in forthcoming works the phenomenological implications of our approach along the lines developed in that reference.

6. Particle physics numerology

In quantum field theory the energy scale is introduced through the commutators (9), and in the standard scheme, with the choice $\alpha = \hbar = \beta$, the fundamental commutator reads $\left[\hat{\varphi}, \hat{\pi}\right] \sim i\hbar\delta_{Dirac}$. In the approach at hand, the parameters α and β can be chosen independently, and in principle they define different energy scales; for simplicity we consider here that they define the same energy scale. In the system of units in which all fundamental quantities appear explicitly, we have that $[\alpha] = [\beta] = [\hbar]/[T]^{s-1}$, and hence these parameters will be adjusted according to the value of s; the choice $s = 1$ reproduces the traditional case. Therefore, we have a new timelike parameter encodes in α and β; this parameter can be identified with the corresponding time scale associated with the energy scale, and particularly it can be identified with the Planck time. If we call T such a parameter, then the vacuum energy can be rewritten as (neglecting factors like $2\pi, \pi^2$, etc., that appear in the tables),

$$H_0(T, s) = \frac{mc^2}{(\frac{T}{\hbar}mc^2)^{s-2}}; \tag{31}$$

where we have considered that in the new system of units $m \to \frac{mc}{\hbar}$ and $w_k \to cw_k$; note that the quantity $\frac{T}{\hbar}mc^2$ is dimensionless, and thus this expression is easily interpretable, since the vacuum energy turns out to be proportional to the rest energy of the field. Moreover, note that at quantum level, the particle has an *effective* mass $m_{effec} \equiv (\frac{T}{\hbar}mc^2)^{2-s}m$, which is different to that initially postulated at Lagrangian level.

Let $\frac{T}{\hbar}c^2 \equiv \frac{1}{M}$ be the inverse of the mass scale defined by T, then

$$H_0(M,s) = \frac{mc^2}{(\frac{m}{M})^{s-2}}; \tag{32}$$

therefore for masses above the mass scale $m > M$, the vacuum energy satisfies $H_0 < mc^2$, and conversely for the case $m < M$, one has that $H_0 > mc^2$; thus, in relation to the mass scale, the light masses will have a higher vacuum energy than those with heavy masses. In particular if $m = M$, then the vacuum energy will reduce to the energy that defines the scale, $H_0 = Mc^2$. This result is valid without reference to a mass scale, since we can compare the vacuum energy for two masses m_1, and m_2 (and for the same weight s),

$$\frac{H_0(m_2,s)}{H_0(m_1,s)} = \left(\frac{m_1}{m_2}\right)^{s-3}; \tag{33}$$

thus, if $m_1 < m_2$, then $H_0(m_1) > H_0(m_2)$.

The present scheme can be applied to estimate the vev for the energy of a neutral Higgs boson-like particle, which is realized by imposing that the field operator is Hermitian, thus $\hat{b} = \hat{a}$, and $\alpha = \beta$, and then we have a neutral field (see Eq. (16b)).

First we choice as scenario the energy scale for the strong interactions with $M \approx 1$ Gev, and for the mass of the Higgs-like field the value determined experimentally for the Higgs boson, $m \approx 125$ Gev; therefore, from Eq. (32),

$$\frac{H_0}{c^2} \approx (125)^{3-s}\ \text{GeV} \approx \begin{cases} 15\ \text{TeV}, & s=1, \\ 125\ \text{GeV}, & s=2, \\ 1\ \text{GeV}, & s=3, \\ 8\ \text{MeV}, & s=4, \\ 60\ \text{KeV}, & s=5, \\ 500\ \text{eV}, & s=6, \end{cases} \tag{34}$$

since that $m > M$, the energy is decreasing as $s \to \infty$; thus, if the value of this observable is determined experimentally, and such a value is below the mass of the Higgs-like particle, then one can to choose the appropriate value for s, and thus to set the corresponding quantum field theory. Note that, since there exist orders of magnitude between the values of the vev's for the different values of s, the adjustment of s may require a fine tuning by using fractional values.

Another scenario of interest is the electroweak energy scale with $M \approx 246$ Gev, and hence we have the following values for the vev's of the energy for same Higgs-like particle of mass $m \approx 125$ Gev, in order to compare with the previous table of values; in this case the vev's are above the mass m, and they are increasing as $s \to \infty$;

$$\frac{H_0}{c^2} \approx 125(2)^{s-2}\ \text{GeV} \approx \begin{cases} 62\ \text{GeV}, & s=1, \\ 125\ \text{GeV}, & s=2, \\ 250\ \text{GeV}, & s=3, \\ 500\ \text{GeV}, & s=4, \\ 1\ \text{TeV}, & s=5, \\ 2\ \text{TeV}, & s=6. \end{cases} \tag{35}$$

Note that between the first three values for the vev's in the table (34), there exist orders of magnitude of difference, as opposed to the four first values in the table (35), which are approximately of the same order; along the same lines, one can compare the set of the vev's of the last three values in both tables. Similarly one can compare the vev's in both tables for the value $s = 6$, which applies for the four-dimensional case as the first value for finite observables; we realize that there exist ten orders of magnitude of difference.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] M. Visser, Lorentz symmetry breaking as a quantum field theory regulator, Phys. Rev. D 80 (2009) 025001.
[2] D. Mattingly, Modern tests of Lorentz invariance, Living Rev. Relativ. 8 (2005) 5.
[3] H. Falomir, J. Gamboa, M. Loewe, F. Méndez, J.C. Rojas, Testing spatial noncommutativity via the Aharonov-Bohm effect, Phys. Rev. D 66 (2002) 045018.
[4] C. Acatrinei, Path integral formulation of noncommutative quantum mechanics, J. High Energy Phys. 09 (2001) 007.
[5] V.P. Nair, A.P. Polychronakos, Quantum mechanics on the noncommutative plane and the sphere, Phys. Lett. B 505 (2001) 267.
[6] J.M. Carmona, J.L. Cortes, J. Gamboa, F. Mendez, Quantum theory of noncommutative fields, J. High Energy Phys. 058 (2003).
[7] I.S. Gradshteyn, I.M. Ryzhik, Tables of Integrals, Series, and Products, 7th ed., Academic Press, 2007.
[8] J. Solá, Cosmological constant and the vacuum energy: old and new ideas, J. Phys. Conf. Ser. 453 (2013) 012015.