On the free energy of solvable lattice models

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Abstract

We conjecture the inversion relations for thermalized solvable interaction round the face (IRF) two dimensional lattice models. We base ourselves on an ansatz for the Baxterization described in the 90's. We solve these inversion relations in the four main regimes of the models, to give the free energy of the models, in these regimes. We use the method of Baxter in the calculation of the free energy of the hard hexagon model. We believe these results to be quite general, shared by most of the known IRF models. Our results apply equally well to solvable vertex models. Using the expression for the free energy we calculate the critical exponent α, and from it the dimension of the perturbing (thermal) operator in the fixed point conformal field theory (CFT). We show that it matches either the coset $\mathcal{O}/\mathcal{G}$ or $\mathcal{G}/\mathcal{O}$, where $\mathcal{O}$ is the original CFT used to define the model and $\mathcal{G}$ is some unknown CFT, depending on the regime. This agrees with known examples of such models by Huse and Jimbo et al.

1. Introduction

Two dimensional solvable lattice models offer a rich ground to study such phenomena as phase transitions, universality, integrability and mathematical applications such as knot theory. For reviews see [1,2]. These models also enjoy a strong connection with two dimensional conformal quantum field theory (CFT). See, e.g., the reviews [3,4].

Some time ago we introduced a method to construct solvable interaction round the face (IRF) from the data of an arbitrary CFT [5]. We call such models IRF($\mathcal{O}, h, v$) where $\mathcal{O}$ is the defining CFT and h and v are two primary fields in the theory.

A long standing question is what is the fixed point CFT of the so defined models and how it is related to the original CFT $\mathcal{O}$. We solve this problem here by calculating the free energy of the thermalized models. From this we deduce the critical exponent α and the dimension of the perturbing field in the fixed point CFT.

To compute the free energy we first need to thermalize the trigonometric ansatz of [5]. This we do by calculating the two inversion relations for the general IRF models. Then we thermalize the models by replacing the $\sin(u)$ function in the inversion relations with the function $\theta_1(u, q)$, where θ_1 is the standard elliptic function. This agrees with all the models where the off–critical Boltzmann weights are known and we conjecture that it is true in general. Thus we are in a position to solve exactly models for which the Boltzmann weights are not explicitly known.

We find that in the four main regimes of the IRF model the fixed point CFT is given by a coset of the original theory. Namely, in regimes III and IV the fixed point CFT is consistent with the coset model $\mathcal{G}/\mathcal{O}$ where $\mathcal{G}$ is some unknown CFT. In regimes I and II, the fixed point CFT is $\mathcal{O}/\mathcal{G}$.

This fixed point RCFT is known exactly, in some cases. For example, in the Andrews–Baxter–Forrester model (ABF) [6], which is IRF($SU(2)_k, [1], [1]$), the fixed point field theory was determined to be the unitary minimal models, which are the coset $SU(2)_{k-1} \times SU(2)_1/SU(2)_k$, in regimes III and IV [7]. In regimes I and II the critical CFT was identified as the parafermionic field theory $SU(2)_k/U(1)$, which are the Fateev–Fateev model [8], by Jimbo et al. [9]. Indeed, this agrees with our general results. For the case of $\mathcal{O} = SU(N)_k$, and $h = v =$ fundamental, the fixed point field theory in regime III was shown to be $SU(N)_{k-1} \times SU(N)_1/SU(N)_k$ by Jimbo et al. [10]. Again agreeing with our result for the fixed point CFT.

To compute the free energy, in all the four regimes, we follow the method used by Baxter [1] in the hard hexagon model. Our results for the free energies agree with the hard hexagon case for $\mathcal{O} = SU(2)_3$, in the four regimes. In the last section, we discuss how our results for the free energy also agree with the eight vertex model and the ABF model.

2. The inversion relations

We wish to study IRF lattice models based on the braiding matrix of a rational conformal field theory (CFT). We fix a conformal field theory $\mathcal{O}$ and fixed primary fields in this theory h and v. The IRF model is denoted as IRF($\mathcal{O}, h, v$) following ref. [5], defined on a square lattice. The field h governs the horizontal admissibility condition and the field v the vertical one. We assume that the boundary conditions are periodic. Let B_i be the braiding matrix in the RCFT which exchanges the field h with the field v [11]. We define the operator

$$< a_1, a_2, \ldots, a_n | B_i | a_1', a_2', \ldots, a_n' > = B \begin{pmatrix} a_{i-1} & a_i \\ a_i' & a_{i+1} \end{pmatrix} \prod_{\substack{m=1 \\ m \neq i}}^{n} \delta_{a_m, a_m'}, \tag{2.1}$$

where the matrix B is the braiding matrix for the fields h and v, and it obeys the braiding relations, which are for $h = v$,

$$B_i B_{i+1} B_i = B_{i+1} B_i B_{i+1}, \qquad B_i B_j = B_j B_i \text{ if } |i - j| > 1. \tag{2.2}$$

The variables on the lattice a_m and a_m' are some primary fields in the RCFT $\mathcal{O}$.

From the braiding matrix one can define the projectors,

$$P_i^b = \prod_{\substack{a=1 \\ a \neq b}}^{n} \frac{B_i - \lambda_a}{\lambda_b - \lambda_a},$$ (2.3)

where n is the number of eigenvalues of B_i (called the number of blocks) and λ_a are the eigenvalues. The projection operators obey the relations,

$$P_i^a P_i^b = \delta_{a,b} P_i^a, \qquad \sum_{a=1}^{n} P_i^a = 1_i, \qquad \sum_{a=1}^{n} \lambda_a P_i^a = B_i,$$ (2.4)

where 1_i denotes the unit matrix.

We define the fusion products of the field h as,

$$h \cdot h = \sum_{a=0}^{n-1} \psi_a,$$ (2.5)

and

$$h \cdot \bar{h} = \sum_{a=0}^{n-1} \tilde{\psi}_a,$$ (2.6)

where $\bar{h}$ is the complex conjugate field of h, n is the number of blocks and the order of the fields is set in a certain way, which allows for the Yang–Baxter equation of the model. The order of the fields appears to be that ψ_{a+1} is contained in the fusion product of ψ_a with the adjoint representation, and similarly for $\tilde{\psi}_a$. The eigenvalues of B_i are [11]

$$\lambda_a = \epsilon_a e^{\pi i (2\Delta_h - \Delta_a)},$$ (2.7)

where $\epsilon_a = \pm 1$ according to whether the product is symmetric or anti–symmetric. Here Δ_a is the dimension of the field ψ_a [4]. The eigenvalues are all distinct in all the models that we discuss. (The fact that the number of blocks is the same in both eqs. (2.6), (2.7) is seen by computing the coefficient of the unit field in the fusion product $h \cdot h \cdot \bar{h} \cdot \bar{h}$ in two ways.) In particular, we set $\tilde{\psi}_0 = [1]$ (the unit primary field) and $\tilde{\psi}_1 = [\text{adjoint}]$ (the adjoint representation, assuming some quantum group structure). Namely, assuming that the CFT $\mathcal{O}$ is a WZW model defined with the group G [4] then the field $\tilde{\psi}_1$ is the primary field transforming in the adjoint representation of G. We denote the dimension of $\tilde{\psi}_a$ by $\tilde{\Delta}_a$. We define the crossing parameters as,

$$\zeta_a = \frac{\pi}{2}(\Delta_{a+1} - \Delta_a),$$ (2.8)

and

$$\tilde{\zeta}_a = \frac{\pi}{2}(\tilde{\Delta}_{a+1} - \tilde{\Delta}_a),$$ (2.9)

where $a = 0, 1, \ldots, n-2$. We note that $\zeta_a, \tilde{\zeta}_a < \pi/2$, which will be important later. This is since we assue that $\Delta_a, \tilde{\Delta}_a < 1$.

In ref. [5] an ansatz for the trigonometric solution of the Yang Baxter equation (YBE) was given. It is

$$R_i^{h,h}(u) = \sum_{a=0}^{n-1} f_a(u) P_i^a,$$ (2.10)

where

$$f_a(u) = \left[\prod_{j=1}^{a} \sin(\zeta_{j-1} - u) \right] \left[\prod_{j=a+1}^{n-1} \sin(\zeta_{j-1} + u) \right] \Big/ \left[\prod_{j=1}^{n-1} \sin(\zeta_{j-1}) \right], \tag{2.11}$$

where $a = 0, 1, \ldots, n - 1$. Our ansatz is that $R_i^{h,h}$ solves the Yang Baxter equation,

$$R_{i+1}(u) R_i(u + v) R_{i+1}(v) = R_i(v) R_{i+1}(u + v) R_i(u), \tag{2.12}$$

where we denoted by $R_i(u)$ instead of $R_i^{h,h}(u)$, for shortness. $R_i(u)$ is a trigonometric solution. Note that the YBE, eq. (2.12), holds only for one particular order of the fields in eqs. (2.6), (2.7), as was discussed earlier. Similarly, we define $R_i^{h,\bar{h}}(u)$ by substituting the crossing parameters $\tilde{\zeta}_a$ instead of ζ_a and substituting the projection operators of the braiding matrix of h with $\bar{h}$. $R_i^{h,\bar{h}}(u)$ also obeys the YBE,

$$R_i^{h,\bar{h}}(u) R_{i+1}^{h,\bar{h}}(u + v) R_i^{h,h}(v) = R_{i+1}^{h,h}(v) R_i^{h,\bar{h}}(u + v) R_{i+1}^{h,\bar{h}}(u). \tag{2.13}$$

The two YBE equations (2.12), (2.13) imply that the transfer matrices for $R_i^{h,h}(u)$ commute with each other for different spectral parameters u and the same for $R_i^{h,\bar{h}}(u)$ [1].

The R matrices obey the first inversion relation which follows from eqs. (2.10), (2.11),

$$R_i^{h,h}(u) R_i^{h,h}(-u) = \rho(u) \rho(-u) \, 1_i, \tag{2.14}$$

and

$$R_i^{h,\bar{h}}(u) R_i^{h,\bar{h}}(-u) = \tilde{\rho}(u) \tilde{\rho}(-u) \, 1_i, \tag{2.15}$$

where

$$\rho(u) = \prod_{r=0}^{n-2} \sin(\zeta_r - u) / \sin(\zeta_r), \tag{2.16}$$

and

$$\tilde{\rho}(u) = \prod_{r=0}^{n-2} \sin(\tilde{\zeta}_r - u) / \sin(\zeta_r). \tag{2.17}$$

The second inversion relation is crossing. We shall denote again $R_i(u)$ by its matrix form. Then the crossing relation is, (which is a conjecture),

$$R^{h,\bar{h}} \begin{pmatrix} d & c \\ a & b \end{pmatrix} (u) = \left(\frac{\psi_a \psi_c}{\psi_b \psi_d} \right)^{\frac{1}{2}} R^{h,h} \begin{pmatrix} a & d \\ b & c \end{pmatrix} (\lambda - u), \tag{2.18}$$

where λ is the crossing parameter,

$$\lambda = \tilde{\zeta}_0 = \pi \Delta_{\text{adjoint}}/2, \tag{2.19}$$

where we used eq. (2.8). The crossing multipliers are

$$\psi_a = S_{a0}/S_{00}, \tag{2.20}$$

where S is the modular matrix [4]. The fusion products, eqs. (2.6), (2.7) are related to the modular matrix by Verlinde formula. See e.g. [4].

We wish to thermalize now the IRF model. We do not know how to thermalize the Boltzmann weights. So instead we will thermalize the inversion relations. We define the theta function,

$$\theta_1(u, q^2) = \sin u \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2u + q^{4n})(1 - q^{2n}). \tag{2.21}$$

(This definition differs from the standard one by a factor of $2q^{1/4}$, which is irrelevant since we will only encounter ratios of theta functions.)

Now we conjecture that the thermalization of the first inversion relation, eq. (2.14), is given by replacing the sin by the theta function θ_1. We denote $\theta_1(u, q^2)$ by $\theta_1(u)$. Then, the thermalization of the R matrix is

$$R_i^{h,h}(u) R_i^{h,h}(-u) = \rho(u)\rho(-u) \, 1_i, \tag{2.22}$$

where

$$\rho(u) = \prod_{r=0}^{n-2} \theta_1(\zeta_r - u)/\theta_1(\zeta_r). \tag{2.23}$$

Similarly, we thermalize the second relation, eqs. (2.15), (2.18),

$$R_i^{h,\bar{h}}(u) R_i^{h,\bar{h}}(-u) = \tilde{\rho}(u)\tilde{\rho}(-u) \, 1_i, \tag{2.24}$$

where

$$\tilde{\rho}(u) = \prod_{r=0}^{n-2} \theta_1(\tilde{\zeta}_r - u)/\theta_1(\zeta_r). \tag{2.25}$$

Finally, the crossing relation eq. (2.18) remains the same for general q except for the crossing multiplier, eq. (2.20), whose explicit expression we will not need here. Note that for $q = 0$ (the critical limit), $\theta_1(u) = \sin u$ so we get the same inversion relations as before.

These conjectures can be indeed verified for many models for which we know the explicit Boltzmann weights, e.g., [1,2].

Next, we wish to define the free energy of the model. It is given by

$$\kappa = \lim_{N \to \infty} Z^{1/N}, \tag{2.26}$$

where N is the number of lattice sites and Z is the partition function calculated with $R^{h,h}$. The free energy is given as usual by,

$$F = -k_B T \log \kappa, \tag{2.27}$$

where k_B is Boltzmann constant and T is the temperature.

Now, since the transfer matrices commute for different spectral parameters u, the inversion relations translate to equations of $\kappa(u)$ (fixing some q),

$$\kappa(u)\kappa(-u) = \rho(u)\rho(-u), \tag{2.28}$$

and

$$\kappa(\lambda + u)\kappa(\lambda - u) = \tilde{\rho}(u)\tilde{\rho}(-u). \tag{2.29}$$

In deriving the last equation, we used the fact that the crossing multipliers cancel when calculating the partition function. We also used in an essential way the fact the boundary conditions are periodic [1].

Actually, the inversion relations, eqs. (2.28), (2.27), remain the same under the substitution $\zeta_i \to -\zeta_i$, or $\tilde{\zeta}_i \to -\tilde{\zeta}_i$. Thus, we take instead of the crossing relations their absolute values, $\zeta_i \to |\zeta_i|$ and $\tilde{\zeta}_i \to |\tilde{\zeta}_i|$.

We also find it convenient to change the second inversion relation by substituting $u \to \lambda - u$. The second inversion relation then becomes,

$$\kappa(u)\kappa(2\lambda - u) = \prod_{r=0}^{n-2} \left[\theta_1(\tilde{\zeta}_r - \lambda + u)\theta_1(\tilde{\zeta}_r + \lambda - u) \right]. \tag{2.30}$$

3. Regimes III

Our aim now is to solve the inversion relations eqs. (2.28), (2.30) and to calculate the free energy. We assume first that the model is in regime III. This is defined by

$$0 < q^2 < 1, \tag{3.1}$$
$$0 < u < d, \tag{3.2}$$

where $d = \min_i \zeta_i, \tilde{\zeta}_i$, and $q^2 = \exp(-\epsilon)$. Here ϵ is a parameter. It is convenient to use the modular transformation of the theta function. This is defined by [1]

$$\theta_1(u, e^{-\epsilon}) = \frac{1}{2} \left(\frac{2\pi}{\epsilon} \right)^{\frac{1}{2}} \exp\left[\frac{\epsilon}{8} - \frac{\pi^2}{2\epsilon} + \frac{2u(\pi - u)}{\epsilon} \right] f(e^{-4\pi u/\epsilon}, e^{-4\pi^2/\epsilon}), \tag{3.3}$$

where f is defined by,

$$f(w, q) = \prod_{n=1}^{\infty} (1 - q^{n-1}w)(1 - q^n w^{-1})(1 - q^n). \tag{3.4}$$

We find it convenient to redefine,

$$\tilde{\kappa}(u) = e^{2\delta u/\epsilon} e^{2(n-1)u^2/\epsilon} \kappa(u), \tag{3.5}$$

where

$$\delta = \frac{1}{\lambda} \sum_{r=0}^{n-2} [\zeta_r(\pi - \zeta_r) - \pi \tilde{\zeta}_r]. \tag{3.6}$$

The dependence on n is essential to simplify the inversion relations. Then, the first inversion relation, eq. (2.28), (2.23), becomes,

$$\tilde{\kappa}(w)\tilde{\kappa}(w^{-1}) = \eta(w), \tag{3.7}$$

where

$$\eta(w) = \prod_{r=0}^{n-2} \frac{f(z_r w) f(z_r w^{-1})}{f(z_r)^2}, \tag{3.8}$$

where

$$w = \exp(-4\pi u/\epsilon), \tag{3.9}$$

$$z_i = \exp(-4\pi \zeta_i/\epsilon), \tag{3.10}$$

$$\tilde{z}_i = \exp(-4\pi \tilde{\zeta}_i/\epsilon), \tag{3.11}$$

$$\tilde{q} = e^{-4\pi^2/\epsilon}, \tag{3.12}$$

and we denoted for brevity $f(w)$ for $f(w, \tilde{q})$. The second inversion relation, eq. (2.30), becomes,

$$\tilde{\kappa}(w)\tilde{\kappa}(x^2 w^{-1}) = \tilde{\eta}(w), \tag{3.13}$$

where

$$x = \exp(-4\pi \lambda/\epsilon) = \tilde{z}_0, \tag{3.14}$$

and

$$\tilde{\eta}(w) = \prod_{r=0}^{n-2} \frac{f(\tilde{z}_r w/x) f(\tilde{z}_r x w^{-1})}{f(z_r)^2}. \tag{3.15}$$

We wish to solve now for the free energy, using the two inversion relations. For this purpose we assume that $\log(\tilde{k}(w))$ is analytic in an annulus containing the point $w = 1$ and the point $w = x$. We analytically continue $k(u)$ to $-\lambda < u < 0$ and to $0 < u < \lambda$. This assumption is justified by considering explicit models, e.g., the hard hexagon model [1].

So we expand,

$$\log[\tilde{\kappa}(w)] = \sum_{m=-\infty}^{\infty} c_m w^m, \tag{3.16}$$

where the summation is convergent in an annulus containing 1 and x. In the neighborhood of 1, eqs. (3.7), (3.8), gives,

$$\log \eta = d_0 + \sum_{m=1}^{\infty} d_m (w^m + w^{-m}), \tag{3.17}$$

where $m > 0$,

$$d_0 = \sum_{r=0}^{n-2} -2\log[f(z_r)/\phi(\tilde{q})], \tag{3.18}$$

where

$$\phi(\tilde{q}) = \prod_{m=1}^{\infty} (1 - \tilde{q}^m), \tag{3.19}$$

and

$$d_m = -\sum_{r=0}^{n-2} (z_r^m + \tilde{q}^m z_r^{-m})/[m(1 - \tilde{q}^m)]. \tag{3.20}$$

We define $d_{-m} = d_m$ for $m > 0$. Similarly, the second inversion relation, eqs. (3.13)–(3.15), becomes, for $m \neq 0$,

$$\log \tilde{\eta} = \sum_{m=-\infty}^{\infty} d'_m w^m, \tag{3.21}$$

where

$$d'_0 = d_0, \tag{3.22}$$

$$d'_m = -\sum_{r=0}^{n-2} (\tilde{z}_r^m / x^m + \tilde{q}^m \tilde{z}_r^{-m} x^{-m}) / [m(1 - \tilde{q}^m)]. \tag{3.23}$$

Taking the logarithms of eqs. (3.13), (3.7) and equating coefficients, we find,

$$c_m + c_{-m} = d_m, \tag{3.24}$$

$$c_m + x^{-2m} c_{-m} = d'_m, \tag{3.25}$$

for $m \geq 0$. The solution of these two equations is,

$$c_0 = \tfrac{1}{2} d_0, \tag{3.26}$$

$$c_m = (-x^{2m} d'_m + d_m) / (1 - x^{2m}), \tag{3.27}$$

for $m \neq 0$. This completes the calculation of the free energy in regime III. Indeed

$$\log \tilde{\kappa}(w) = \sum_{m=-\infty}^{\infty} c_m w^m,$$

converges in an annulus containing the points 1 and x, as we assumed. We assume that $|\tilde{\zeta}_r| \leq \lambda$ always, for all r, which is needed to show convergence. We checked this in many models, but we do not have a general proof for this fact.

We wish to calculate now the critical exponent α. This is defined as the singularity of the free energy,

$$\kappa_{\text{singular}} \propto |T - T_c|^{2-\alpha}, \tag{3.28}$$

where T_c is the critical temperature. We have that $\tilde{q} = \exp(-4\pi^2/\epsilon)$. Since c_m is divided by $1 - x^{2m}$ this means that it becomes a theta function at the modulus x^2, when summed back. Since $\lambda = \pi \tilde{\Delta}_0 / 2$, eq. (2.9), we may write this modulus as

$$x^2 = \tilde{q}^{\tilde{\Delta}_0}. \tag{3.29}$$

Under the inverse modulus transformation, eq. (3.3), x^2 becomes,

$$(q^2)^{1/\tilde{\Delta}_0}. \tag{3.30}$$

Since $q^2 \propto |T - T_c|$ this implies that

$$\kappa_{\text{singular}} \propto |T - T_c|^{1/\tilde{\Delta}_0}, \tag{3.31}$$

and so,

$$2 - \alpha = \frac{1}{\tilde{\Delta}_0}. \tag{3.32}$$

The dimension of the perturbing operator at the critical conformal field theory is given by [3],

$$\Delta_p = \frac{1-\alpha}{2-\alpha} = 1 - \tilde{\Delta}_0. \tag{3.33}$$

Now, since we assume that $\tilde{\psi}_1$ is the adjoint operator, in quantum group models, the dimension of the perturbing field is seen to be,

$$\Delta_p = 1 - \Delta_{\text{adjoint}}, \tag{3.34}$$

where $\tilde{\Delta}_0 = \Delta_{\text{adjoint}}$.[1] Such a field appears in the coset theory,

$$\mathcal{C} = \frac{\mathcal{G}}{\mathcal{O}}, \tag{3.35}$$

where $\mathcal{O}$ is the original CFT used to define the model and $\mathcal{G}$ is some CFT model [4], where we take the currents in $\mathcal{G}$ and the adjoint representation in $\mathcal{O}$. Thus, we conjecture that the fixed point of Regime III is given by the coset $\mathcal{C}$. In many cases where the fixed point theory was calculated explicitly, this was indeed shown to be the case. For example, the RCFT $\mathcal{O}$ of the ABF model [6] is $SU(2)_k$ WZW model. The fixed point in Regime III is the $k+1$ minimal model [7], defined by the coset

$$\mathcal{C} = \frac{SU(2)_{k-1} \times SU(2)_1}{SU(2)_k}, \tag{3.36}$$

consistent with eq. (3.35).

4. Regime II

Let us consider regime II. This is defined by

$$0 < q < 1, \qquad 0 > u > -d. \tag{4.1}$$

The first inversion relation remains the same, eqs. (2.28), (2.23). The second inversion relation becomes,

$$\kappa(u)\kappa(2\lambda - \pi - u) = \prod_{r=0}^{n-2} \theta_1(\tilde{\zeta}_r - \lambda + u)\theta_1(\tilde{\zeta}_r + \lambda - \pi - u)/\theta_1(\zeta_r)^2, \tag{4.2}$$

where the theta function is invariant under a shift by π. We define,

$$\tilde{\kappa}(u) = e^{\delta u/\epsilon} e^{2(n-1)u^2/\epsilon} \kappa(u), \tag{4.3}$$

where

$$\delta = \sum_{r=0}^{n-2} \left[2\pi^2 - 4\pi \tilde{\zeta}_r + 4\zeta_r(\pi - \zeta_r) \right] \Big/ (2\lambda - \pi). \tag{4.4}$$

The dependence on n is essential to simplify the inversion relations. Then the second inversion relation, eq. (4.2), becomes,

$$\tilde{\kappa}(w)\tilde{\kappa}(w_0^2/w) = \tilde{\eta}(w) = \prod_{r=0}^{n-2} \left[f(\tilde{z}_r x^{-1} w) f(\tilde{z}_r x \tilde{q}^{-1} w^{-1})/f(z_r)^2 \right], \tag{4.5}$$

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[1] An exception to this is the hard hexagon model, where $\kappa(w)$ is not singular in regime III, as a result of cancellations [1].

where

$$w_0^2 = \tilde{q}^{-1} x^2, \tag{4.6}$$

where $\tilde{q}$, x, z_r and $\tilde{z}_r$ are given by eqs. (3.12), (3.14), (3.10), (3.11). We assume now that

$$\log[w^\mu \tilde{\kappa}(w)] \tag{4.7}$$

is analytic in an annulus $a < |w| < b$ containing the points $w = 1$ and $w = w_0$. This is an analytic continuation of w to $w < 1$ and $w > w_0$ even though the regime is defined for a subset of these values.

Then we can calculate

$$\log \tilde{\eta}(w) = \sum_{m=-\infty}^{\infty} d_m' w^m, \tag{4.8}$$

and

$$d_m' = \sum_{r=0}^{n-2} [\tilde{z}_r^m x^{-m} + \tilde{q}^{-m} (\tilde{z}_r x \tilde{q}^{-1})^{-m}] / [m(1 - \tilde{q}^{-m})], \tag{4.9}$$

$$d_0' = d_0 + \sum_{r=0}^{n-2} \log(\tilde{z}_r^2 \tilde{q}^{-1}). \tag{4.10}$$

Thus,

$$\mu = (d_0' - d_0)/(2 \log w_0) = \frac{\tilde{\Delta}_{n-1}}{\Delta_{\text{adjoint}}}, \tag{4.11}$$

and

$$\log[w^\mu \tilde{\kappa}(w)] = \sum_{m=-\infty}^{\infty} c_m w^m, \tag{4.12}$$

where

$$c_m = (d_m - w_0^{2m} d_m')/(1 - w_0^{2m}), \tag{4.13}$$

for $m \neq 0$, and

$$c_0 = \tfrac{1}{2} d_0. \tag{4.14}$$

Indeed the series, eq. (4.12), for $\tilde{\kappa}$ converges in an annulus containing $w = 1$ and $w = w_0$.

The free energy $\log[w^\mu \tilde{\kappa}(w)]$ is given by the series

$$c_m = a_m/(1 - w_0^{-2m}), \tag{4.15}$$

where a_m is some function, defied in eq. (4.13). Thus it is given by a theta function with the modulus w_0^{-2} which is

$$p = w_0^{-2} = e^{-4\pi^2(1-\tilde{\Delta}_0)/\epsilon} = \tilde{q}^{1-\tilde{\Delta}_0}, \tag{4.16}$$

under the inverse modulus transformation, as in eq. (3.3), we find

$$\kappa_{\text{singular}} = (q^2)^{1/(1-\tilde{\Delta}_0)}, \tag{4.17}$$

where q^2 is proportional to $|T - T_c|$. Thus we find for the exponent α,

$$2 - \alpha = 1/(1 - \tilde{\Delta}_0). \tag{4.18}$$

For a quantum group model, where $\mathcal{O}$ is a WZW model, $\tilde{\Delta}_0$ is the dimension of the adjoint representation, conjecturally. Thus the dimension of the perturbing thermal operator is

$$\Delta_p = \frac{1 - \alpha}{2 - \alpha} = \tilde{\Delta}_0 = \Delta_{\text{adjoint}}. \tag{4.19}$$

An operator with such a dimension appears in the Coset model,

$$\mathcal{C} = \frac{\mathcal{O}}{\mathcal{G}}, \tag{4.20}$$

where $\mathcal{O}$ is the RCFT used to define the model and $\mathcal{G}$ is some CFT model [4]. Thus we conjecture that $\mathcal{C}$ is the fixed point theory in regime II. The perturbing operator is the adjoint representation in $\mathcal{O}$ and the unit in $\mathcal{G}$. For example, for the ABF model in regime II, where $\mathcal{O}$ is $SU(2)_k$ WZW model, the fixed point theory is the parafermionic theory [9],

$$\mathcal{C} = \frac{SU(2)_k}{U(1)}. \tag{4.21}$$

5. Regime IV

Let us get now to regimes IV and I. Regime IV is defined by

$$-1 < q^2 < 0, \qquad 0 < u < d, \tag{5.1}$$

where $d = \min_i \zeta_i, \tilde{\zeta}_i$. We use the inverse modulus [1],

$$\theta_1(u, -e^{-\epsilon}) = -\frac{1}{2} \left(\frac{\pi}{\epsilon} \right)^{\frac{1}{2}} \exp\left[\frac{\epsilon}{8} - \frac{\pi^2}{8\epsilon} - \frac{u(2u + \pi)}{\epsilon} \right] f(e^{2\pi u/\epsilon}, -e^{-\pi^2/\epsilon}), \tag{5.2}$$

where, as before,

$$f(w, q) = \prod_{n=1}^{\infty} (1 - q^{n-1}w)(1 - q^n w^{-1})(1 - q^n). \tag{5.3}$$

We define

$$q^2 = -\exp(-\epsilon), \quad \tilde{q} = -e^{-\pi^2/\epsilon}, \tag{5.4}$$

$$x = -\exp(-\pi\lambda/\epsilon), \quad z_r = -\exp(-\pi\zeta_r/\epsilon), \quad \tilde{z}_r = -\exp(-\pi\tilde{\zeta}_r/\epsilon), \tag{5.5}$$

$$w = \exp(2\pi u/\epsilon), \tag{5.6}$$

where the crossing parameters were defined in eqs. (2.8), (2.9).

In regime IV we write the two inversion relations as

$$\kappa(u)\kappa(-u) = \prod_{r=0}^{n-2} \left[\theta_1(-\zeta_r - u)\theta_1(-\zeta_r + u)/\theta_1(-\zeta_r)^2 \right], \tag{5.7}$$

$$\kappa(u)\kappa(2\lambda - u) = \prod_{r=0}^{n-2} \left[\theta_1(\tilde{\zeta}_r - \lambda + u)\theta_1(\tilde{\zeta}_r + \lambda - u)/\theta_1(-\zeta_r)^2 \right], \tag{5.8}$$

which are eqs. (2.28), (2.30). We define

$$\tilde{\kappa}(u) = e^{\delta u/(2\lambda\epsilon)} e^{2(n-1)u^2/\epsilon} \kappa(u), \tag{5.9}$$

where

$$\delta = \sum_{r=0}^{n-2} 2\pi \tilde{\zeta}_r + 2\zeta_r(\pi - 2\zeta_r), \tag{5.10}$$

then, the first inversion relation for regime IV becomes,

$$\tilde{\kappa}(w)\tilde{\kappa}(w^{-1}) = \eta(w) = \prod_{r=0}^{n-2} \frac{f(z_r^2 w) f(z_r^2/w)}{f(z_r^2)^2}. \tag{5.11}$$

This also holds for regime I (see section 6). The second inversion relation, for regime IV, becomes,

$$\tilde{\kappa}(w)\tilde{\kappa}(x^{-4}/w) = \tilde{\eta}(w) = \prod_{r=0}^{n-2} \left[f(wx^2\tilde{z}_r^{-2}) f(w^{-1}\tilde{z}_r^{-2}x^{-2})/f(z_r^2)^2 \right], \tag{5.12}$$

where we define the crossing parameter,

$$w_0 = x^{-2}. \tag{5.13}$$

We denoted for brevity $f(w, \tilde{q})$ as $f(w)$.

As before, we assume that $\log(w^\mu \tilde{\kappa}(w))$ is analytic in an annulus $a < |w| < b$ containing $w = 1$ and $w = w_0$. Thus we expand

$$\log(\eta(w)) = \sum_{m=-\infty}^{\infty} d_m w^m, \tag{5.14}$$

and we find in regimes IV and I (see section 6)

$$d_m = -\sum_{r=0}^{n-2} \frac{z_r^{2m} + z_r^{-2m}\tilde{q}^m}{m(1-\tilde{q}^m)}, \tag{5.15}$$

$$d_0 = -2\sum_{r=0}^{n-2} \log[f(z_r^2)/\phi(\tilde{q})], \tag{5.16}$$

where $\phi(\tilde{q})$ was defined in eq. (3.19). Similarly we expand

$$\log(\tilde{\eta}(w)) = \sum_{m=-\infty}^{\infty} d'_m w^m, \tag{5.17}$$

where

$$d'_m = \sum_{r=0}^{n-2} \frac{(x^2\tilde{z}_r^{-2})^m + (\tilde{z}_r^{-2}x^{-2})^{-m}\tilde{q}^{-m}}{m(1-\tilde{q}^{-m})}, \tag{5.18}$$

for $m \neq 0$, and

$$d'_0 = d_0 + 2\mu \log w_0, \tag{5.19}$$

where

$$\mu = \sum_{r=0}^{n-2} \log(\tilde{z}_r^{-4})/\log(x^{-4}) = \frac{\tilde{\Delta}_{n-1}}{\Delta_{\text{adjoint}}}. \tag{5.20}$$

Thus, as before, eq. (3.27), we find

$$\log(w^{\mu} \tilde{\kappa}(w)) = \sum_{m=-\infty}^{\infty} c_m w^m, \tag{5.21}$$

where

$$c_m = \frac{(d_m - w_0^{2m} d'_m)}{(1 - w_0^{2m})}, \tag{5.22}$$

for $m \neq 0$ and

$$c_0 = \tfrac{1}{2} d_0. \tag{5.23}$$

We note that indeed this series converges in an annulus containing $w = 1$ and $w = w_0$.

Let us compute now the critical exponent α. The series for c_m, eq. (5.22), is an expression for a theta function with the modulus,

$$w_0^{-2} = x^4 = e^{-2\pi^2 \Delta_{\text{adjoint}}/\epsilon} = \tilde{q}^{2\Delta_{\text{adjoint}}}, \tag{5.24}$$

(we ignore the overall sign). The expression for $\tilde{\kappa}$ includes factors of $\theta_4(u, -\tilde{q}^2)$. The function θ_4 is defined by

$$\theta_4(u, q^2) = \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2u + q^{4n-2})(1 - q^n),$$

and it satisfies the conjugate modulus relation [1]

$$\theta_4(u, e^{-\epsilon}) = \left(\frac{2\pi}{\epsilon}\right)^{\frac{1}{2}} \exp\left[\frac{-\pi^2}{2\epsilon} + \frac{2u(\pi - u)}{\epsilon}\right] f(-e^{-4\pi u/\epsilon}, e^{-4\pi^2/\epsilon}).$$

Denoting $p = -q^2$, we find that the inverse modulus transformation gives the modulus,

$$p^{2/\Delta_{\text{adjoint}}}. \tag{5.25}$$

(Note that we get a factor of 4 in the modulus as a result of the inverse modulus relation.) Now, since the function θ_4 includes powers of $q = p^{\frac{1}{2}}$ and defining $p = t$ where t is the temperature, we find that

$$\kappa_{\text{singular}} \propto t^{1/\Delta_{\text{adjoint}}}, \tag{5.26}$$

and

$$2 - \alpha = 1/\Delta_{\text{adjoint}}. \tag{5.27}$$

Thus the dimension of the perturbing (thermal) operator in the fixed point RCFT is

$$\Delta_p = \frac{1 - \alpha}{2 - \alpha} = 1 - \Delta_{\text{adjoint}}. \tag{5.28}$$

This is the same dimension as in Regime III and we conclude that it is the other side of the same fixed point. The RCFT in Regime IV is thus conjectured as the coset model

$$\mathcal{C} = \frac{\mathcal{G}}{\mathcal{O}}, \tag{5.29}$$

where $\mathcal{O}$ is the original CFT used to define the model and $\mathcal{G}$ is some unknown CFT.

6. Regime I

Let us turn now to regime I. It is defined by

$$-1 < q < 0, \qquad 0 > u > -d. \tag{6.1}$$

The first inversion relation is the same as in regime IV, eq. (5.7). For the second inversion relation we take as in regime II,

$$\kappa(u)\kappa(2\lambda - \pi - u) = \prod_{r=0}^{n-2} \theta_1(\tilde{\zeta}_r - \lambda + u)\theta_1(\tilde{\zeta}_r + \lambda - \pi - u)/\theta_1(-\zeta_r)^2. \tag{6.2}$$

We define as in regime IV, q, $\tilde{q}$, x, $\tilde{z}_r$ and w, eqs. (5.4)–(5.6). We further define

$$\tilde{\kappa}(u) = e^{2(n-1)u^2/\epsilon} e^{\delta u/[\epsilon(2\lambda - \pi)]} \kappa(u), \tag{6.3}$$

where

$$\delta = \sum_{r=0}^{n-2} 2\zeta_r(\pi - 2\zeta_r) - \pi^2 + 2\pi \tilde{\zeta}_r. \tag{6.4}$$

Then the second inversion relation, eq. (6.2), then becomes

$$\tilde{\kappa}(w)\tilde{\kappa}(w_0^2/w) = \prod_{r=0}^{n-2} f(wx^2\tilde{z}_r^{-2})f(w^{-1}\tilde{z}_r^{-2}x^{-2}\tilde{q}^2) = \tilde{\eta}(w), \tag{6.5}$$

where

$$w_0 = -x^{-2}\tilde{q}, \tag{6.6}$$

and $f(w, \tilde{q})$ was defined in eq. (5.3).

Now, we need to expand as before

$$\log \tilde{\eta}(w) = \sum_{m=-\infty}^{\infty} d'_m w^m, \tag{6.7}$$

and we find,

$$d'_0 = d_0, \tag{6.8}$$

where d_0 is given by eq. (5.16). Also,

$$d'_m = -\sum_{r=0}^{n-2} \frac{(x^2\tilde{z}_r^{-2})^m + (\tilde{z}_r^{-2}x^{-2}\tilde{q}^2)^{-m}\tilde{q}^m}{m(1 - \tilde{q}^m)}, \tag{6.9}$$

for $m \neq 0$. So, as before,

$$\log \tilde{\kappa}(w) = \sum_{m=-\infty}^{\infty} c_m w^m, \tag{6.10}$$

where

$$c_0 = \tfrac{1}{2} d_0, \tag{6.11}$$

and

$$c_m = -\frac{w_0^{2m} d_m' - d_m}{1 - w_0^{2m}}, \tag{6.12}$$

as before, for $m \neq 0$. The series, eq. (6.10), indeed converges in an annulus containing $w = 1$ and $w = w_0$.

Now, we wish to compute the exponent α. The expression for c_m is a theta function with the modulus

$$w_0^2 = \tilde{q}^2 x^{-4}, \tag{6.13}$$

which can be written as

$$w_0^2 = \tilde{q}^{2(1-\Delta_{\text{adjoint}})}, \tag{6.14}$$

where we used the definition of x and λ, eqs. (5.5), (2.9). Thus, the modulus of the theta function has the modulus

$$(\tilde{q}^2)^{2(1-\Delta_{\text{adjoint}})}. \tag{6.15}$$

Now, since the expression for the theta function contains θ_4, at positive modulus, the same discussion as in section 5 shows that

$$2 - \alpha = \frac{1}{1 - \Delta_{\text{adjoint}}}, \tag{6.16}$$

and the dimension of the perturbing field is

$$\Delta_p = \frac{1-\alpha}{2-\alpha} = \Delta_{\text{adjoint}}. \tag{6.17}$$

This is the same dimension as in regime II and we conclude that it is the other side of the same phase transition with the critical theory given as in regime II by the coset

$$\mathcal{C} = \frac{\mathcal{O}}{\mathcal{G}}, \tag{6.18}$$

where $\mathcal{O}$ is the original theory used to define the model and $\mathcal{G}$ is some unknown CFT.

This concludes the expression for the free energy in all the four regimes. We can check our results for the hard hexagon model which is IRF($SU(2)_3$, [1]). There $\lambda = \pi/5$, $n = 2$, $\zeta_0 = \tilde{\zeta}_0 = \lambda$. We find a complete agreement with the results of Baxter for the free energy [1], p. 425.

7. Discussion

Another construction of two dimensional solvable lattice models is the vertex models. These are defined by some CFT $\mathcal{O}$ and some representation h, where the fields are weights in this representation. The Boltzmann weights are elements of End($V \times V$) where V are the weights of

the representation h. As was discussed in refs. [12,13], the Baxterization of the models is exactly the same formula as the IRF models, eq. (2.10), (2.11). For the case of $SU(2)$ this was described in ref. [14]. Thus, our calculation of the free energy holds equally well for these models, with the parameters ζ_i and $\tilde{\zeta}_i$ given by eqs. (2.8), (2.9). We assume here periodic boundary conditions. The free energy then obeys the same inversion relations, eqs. (2.28), (2.30). Thus also the solution for the free energy is the same as described in this paper.

In ref. [15], the inversion relations for the eight vertex model are described by Baxter. They coincide precisely with our conjectured inversion relations eqs. (2.28), (2.29), where we denote the crossing parameter by λ, eqs. (6.6a), (6.6b) of [15], and the same analyticity requirements. The eight vertex model is a thermalization of the six vertex model [15], which is the vertex model of general level $SU(2)$ with the representation h being the fundamental representation [14]. Thus, the two results fit nicely.

In fact, as was discussed in ref. [6] the same inversion relations hold for the kth ABF model with $\lambda = \pi/(k + 2)$, for all natural integer k. In our language, this is the IRF with the CFT $\mathcal{O}$ which is $SU(2)_k$ WZW model with the fields $h = v =$ the fundamental representation. So, again we find a consistency with eq. (2.19). Thus also the solution we found for the free energy is exactly the same as the one found for the eight vertex model and for the ABF model, in all the four regimes. This supports important credence to our inversion relations and to our calculation of the free energy, in these particular cases.

Another interesting point is that in the theories $\mathcal{C}$ define integrable models when perturbed by the thermal operator. This may be interesting for the building of new integrable models of massive quantum field theories.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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