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Space-time supersymmetry in WZW-like open superstring field theory

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We investigate space-time supersymmetry in the WZW-like open superstring field theory, whose complete action was recently constructed. Starting from a natural space-time supersymmetry transformation at the linearized level, we construct a nonlinear transformation so as to keep the complete action invariant. Then we show that the transformation satisfies the supersymmetry algebra up to an extra transformation, unphysical on the asymptotic string fields. This guarantees that the constructed transformation in fact acts as space-time supersymmetry on the physical S-matrix.

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1. Introduction

Construction of a complete action including both the Neveu–Schwarz (NS) sector representing space-time bosons and the Ramond sector representing space-time fermions are a long-standing problem in superstring field theory. While the action for the NS sector was constructed based on two different formulations, the WZW-like formulation (Ref. [1]) and the homotopy-algebra-based formulation (Ref. [2]), it had been difficult to incorporate the Ramond sector in a Lorentz-covariant way. Only recently, however, a complete action has been constructed for the WZW-like formulation (Ref. [3]), and soon afterwards for the homotopy-algebra-based formulation (Ref. [4]). Interestingly enough, in these complete actions, the string field in each sector appears quite asymmetrically. In the WZW-like formulation, e.g., the string field Φ in the NS sector is in the large Hilbert space, characterizing the WZW-like formulation, but the string field Ψ in the Ramond sector is in the restricted small Hilbert space defined using the picture-changing operators. Then the question is how space-time supersymmetry is realized between these two apparently asymmetric sectors. The purpose of this paper is to answer this question by explicitly constructing the space-time supersymmetry transformation in the WZW-like formulation.[1]

In the first quantized formulation, space-time supersymmetry is generated by the supercharge obtained by using the covariant fermion emission vertex (Ref. [6]), which interchanges each physical state in the NS sector with that in the Ramond sector. Therefore, it is natural to expect first that the

[1] Space-time supersymmetry in the homotopy-algebra-based formulation has recently been studied by Erler (Ref. [5]).

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space-time supersymmetry transformation in superstring field theory is realized as a linear transformation using this first-quantized supercharge (Ref. [7]). We will see, however, that this expectation is true only for the free theory, while the action including the interaction terms is not invariant under this linear transformation. We modify it so as to be a symmetry of the complete action, and then verify whether the constructed nonlinear transformation satisfies the supersymmetry algebra. We find that the supersymmetry algebra holds, up to the equations of motion and gauge transformation, only except for a nonlinear transformation. It is shown, however, that this extra transformation can also be absorbed into the gauge transformation up to the equations of motion at the linearized level. Under the assumption that the asymptotic condition holds also for the string field theory, this implies, at least perturbatively, that the constructed transformation acts as space-time supersymmetry on the physical states defined by the asymptotic string fields. This guarantees that supersymmetry is realized on the physical S-matrix.[2]

The rest of the paper is organized as follows. In Sect. 2, we summarize the known results on the complete action for the WZW-like open superstring field theory. In addition, restricting the background to the flat space-time, we introduce the GSO projection operator, which is essential to make the physical spectrum supersymmetric. For later use, some basic ingredients, such as the Maurer–Cartan equations and the covariant derivatives, are extended to those based on general derivations of the string product, which can be noncommutative. After this preparation, the space-time supersymmetry transformation is constructed in Sect. 3. Using the first-quantized supercharge, a linear transformation is first defined so as to be consistent with the restriction in the Ramond sector. Since this transformation is only a symmetry of the free theory, we first construct the nonlinear transformation perturbatively by requiring it to keep the complete action invariant. Based on some lower-order results, we suppose the full nonlinear transformation $\delta_{\mathcal{S}}$ in a closed form, and prove that it is actually a symmetry of the action. In Sect. 4, the commutator of two transformations is calculated explicitly. We show that it provides the space-time translation δ_p, up to the equations of motion and gauge transformation, except for a nonlinear transformation $\delta_{\tilde{p}}$ that can be absorbed into the gauge transformation only at the linearized level. Thus the supersymmetry algebra holds only on the physical states, and hence the physical S-matrix, defined by the asymptotic string fields under appropriate assumptions on asymptotic properties of the string fields. Although this extra symmetry is unphysical in this sense, it is nontrivial in the total Hilbert space including unphysical degrees of freedom. It produces further unphysical symmetries by taking commutators with supersymmetries, or themselves, successively. We have a sequence of unphysical symmetries corresponding to the first-quantized charges obtained by taking successive commutators of the supercharge and the unconventional translation charge with picture number $p = -1$. Section 5 is devoted to summary and discussion, and two appendices are added. In Appendix A, we summarize the conventions for the SO(1,9) spinor and the Ramond ground states, which are needed to identify the physical spectrum although they do not appear in this paper explicitly. The triviality of the extra transformation in the Ramond sector, which remains to be shown, is given in Appendix B. Further nonlinear transformations obtained by taking the commutator of two unphysical transformations $[\delta_{\tilde{p}_1}, \delta_{\tilde{p}_2}]$ are also discussed. All the extra symmetries obtained by taking commutators with $\delta_{\mathcal{S}}$ or $\delta_{\tilde{p}}$ repeatedly are shown to be unphysical.

[2] We further assume asymptotic completeness in this paper.

2. Complete gauge-invariant action

On the basis of the Ramond–Neveu–Schwarz (RNS) formulation of superstring theory, consisting of the matter sector, the reparametrization ghost sector, and the superconformal ghost sector. We assume in this paper that the background space-time is 10-dimensional Minkowski space, for which the matter sector is described by string coordinates $X^\mu(z)$ and their partners $\psi^\mu(z)$ $(\mu = 0, 1, \ldots, 9)$. The reparametrization ghost sector and superconformal ghost sector are described by a fermion pair $(b(z), c(z))$ and a boson pair $(\beta(z), \gamma(z))$, respectively. The superconformal ghost sector has another description by a fermion pair $(\xi(z), \eta(z))$ and a chiral boson $\phi(z)$ (Ref. [6]). The two descriptions are related through the bosonization relation:

$$\beta(z) = \partial\xi(z)e^{-\phi(z)}, \qquad \gamma(z) = e^{\phi(z)}\eta(z). \tag{2.1}$$

The Hilbert space for the $\beta\gamma$ system is called the small Hilbert space and that for the $\xi\eta\phi$ system is called the large Hilbert space.

The theory has two sectors depending on the boundary condition on the world-sheet fermions ψ^μ, β, and γ. The sector in which the world-sheet fermion obeys an antiperiodic boundary condition is known as the Neveu–Schwarz (NS) sector, and describes the space-time bosons. The other sector in which the world-sheet fermion obeys a periodic boundary condition is known as the Ramond (R) sector, and describes the space-time fermions. We can obtain the space-time supersymmetric theory by suitably combining two sectors (Ref. [9]).

2.1. *String fields and constraints*

In the WZW-like open superstring field theory, we use the string field Φ in the large Hilbert space for the NS sector. It is Grassmann even, and has ghost number 0 and picture number 0. Here we further impose the BRST-invariant GSO projection[3]

$$\Phi = \frac{1}{2}(1 + (-1)^{G_{\rm NS}})\Phi, \tag{2.2}$$

where $G_{\rm NS}$ is defined by

$$\begin{aligned} G_{\rm NS} &= \sum_{r>0}(\psi^\mu_{-r}\psi_{r\mu} - \gamma_{-r}\beta_r + \beta_{-r}\gamma_r) - 1 \\ &\equiv \sum_{r>0}\psi^\mu_{-r}\psi_{r\mu} + p_\phi \quad (\text{mod } 2), \end{aligned} \tag{2.3}$$

with $p_\phi = -\oint \frac{dz}{2\pi i}\partial\phi(z)$. This is necessary to remove the tachyon and makes the spectrum supersymmetric (Ref. [9]).

For the Ramond sector, we use the string field Ψ constrained on the restricted small Hilbert space satisfying the conditions (Ref. [3])

$$\eta\Psi = 0, \qquad XY\Psi = \Psi, \tag{2.4}$$

[3] This BRST-invariant GSO projection and that for the Ramond sector to be introduced shortly were first given in Ref. [8]. The operators $G_{\rm NS}$ and $G_{\rm R}$ are none other than world-sheet fermion number operators in the total Hilbert space including the ghost sectors.

where X and Y are the picture-changing operator and its inverse acting on the states in the small Hilbert space with picture numbers $-3/2$ and $-1/2$, respectively. They are defined by

$$X = -\delta(\beta_0)G_0 + \delta'(\beta_0)b_0, \qquad Y = -c_0\delta'(\gamma_0), \tag{2.5}$$

and satisfy

$$XYX = X, \qquad YXY = Y, \qquad [Q, X] = 0. \tag{2.6}$$

The string field Ψ is Grassmann odd, and has ghost number 1 and picture number $-1/2$. The picture-changing operator X is BRST exact in the large Hilbert space, and can be written using the Heaviside step function as $X = \{Q, \Theta(\beta_0)\}$. Here, instead of $\Theta(\beta_0)$, we introduce

$$\Xi = \xi_0 + (\Theta(\beta_0)\eta\xi_0 - \xi_0)P_{-3/2} + (\xi_0\eta\Theta(\beta_0) - \xi_0)P_{-1/2}, \tag{2.7}$$

and anew define

$$X = \{Q, \Xi\}. \tag{2.8}$$

This is identical to the one defined in Eq. (2.5) when it acts on the states in the small Hilbert space with picture number $-3/2$, but can act on the states in the large Hilbert space without the restriction on the picture number (Ref. [4]). The operator Ξ is nilpotent ($\Xi^2 = 0$) and satisfies $\{\eta, \Xi\} = 1$ (Ref. [4]), from which, with $\{Q, \eta\} = 0$, we can conclude

$$\begin{aligned}[\eta, X] &= [\eta, \{Q, \Xi\}] \\ &= -[Q, \{\Xi, \eta\}] - [\Xi, \{\eta, Q\}] = 0.\end{aligned} \tag{2.9}$$

We impose the BRST-invariant GSO projection as

$$\Psi = \tfrac{1}{2}(1 + \hat{\Gamma}_{11}(-1)^{G_{\mathrm{R}}})\Psi, \tag{2.10}$$

where G_{R} is given by

$$\begin{aligned}G_{\mathrm{R}} &= \sum_{n>0}(\psi^{\mu}_{-n}\psi_{n\mu} - \gamma_{-n}\beta_n + \beta_{-n}\gamma_n) - \gamma_0\beta_0 \\ &\equiv \sum_{n>0}\psi^{\mu}_{-n}\psi_{n\mu} + p_\phi + \frac{1}{2} \pmod{2}.\end{aligned} \tag{2.11}$$

The gamma matrix $\hat{\Gamma}_{11}$ is defined by using the zero-modes of the world-sheet fermion $\psi^{\mu}(z)$ as

$$\hat{\Gamma}_{11} = 2^5\psi_0^0\psi_0^1\cdots\psi_0^9. \tag{2.12}$$

We summarize the convention on how the zero modes ψ_0^{μ} act on the Ramond ground states in Appendix A.[4]

[4] In the context of string field theory, the GSO projections are also needed to make the Grassmann properties of string fields Φ and Ψ consistent with those of the coefficient space-time fields.

2.2. Complete gauge-invariant action

By use of the string fields introduced in the previous subsection, the complete action for the WZW-like open superstring field theory is given by (Ref. [3])

$$S = -\frac{1}{2}\langle\langle\Psi, YQ\Psi\rangle\rangle - \int_0^1 dt\langle A_t(t), QA_\eta(t) + (F(t)\Psi)^2\rangle, \tag{2.13}$$

and is invariant under the gauge transformations

$$A_{\delta_g} = D_\eta\Omega + Q\Lambda + \{F\Psi, F\Xi\{F\Psi, \Lambda\}\} - \{F\Psi, F\Xi\lambda\}, \tag{2.14a}$$

$$\delta_g\Psi = -X\eta F\Xi[F\Psi, D_\eta\Lambda] + Q\lambda + X\eta F\lambda, \tag{2.14b}$$

where we have introduced the one parameter extension $\Phi(t)$ of Φ ($t \in [0, 1]$) satisfying the boundary condition $\Phi(1) = \Phi$ and $\Phi(0) = 0$, and defined

$$A_{\mathcal{O}}(t) = (\mathcal{O}e^{\Phi(t)})e^{-\Phi(t)}, \tag{2.15}$$

with $\mathcal{O} = \partial_t$, η, or δ, which are analogs of (components) of the right-invariant one form, satisfying the Maurer–Cartan-like equation

$$\mathcal{O}_1 A_{\mathcal{O}_2}(t) - (-1)^{\mathcal{O}_1\mathcal{O}_2}\mathcal{O}_2 A_{\mathcal{O}_1}(t) - [\![A_{\mathcal{O}_1}(t), A_{\mathcal{O}_2}(t)]\!] = 0, \tag{2.16}$$

where $[\![A_1, A_2]\!]$ is the graded commutator of the two string field A_1 and A_2: $[\![A_1, A_2]\!] = A_1A_2 - (-1)^{A_1A_2}A_2A_1$. Using $A_\eta(t)$, the covariant derivative $D_\eta(t)$ is defined by the operator acting on the string field A as

$$D_\eta(t)A = \eta A - [\![A_\eta, A]\!], \tag{2.17}$$

which is nilpotent: $(D_\eta(t))^2 = 0$. Then the linear map $F(t)$ on a general string field Ψ in the Ramond sector is defined by

$$\begin{aligned} F(t)\Psi &= \frac{1}{1 + \Xi(D_\eta(t) - \eta)}\Psi \\ &= \Psi + \Xi[\![A_\eta(t), \Psi]\!] + \Xi[\![A_\eta(t), \Xi[\![A_\eta(t), \Psi]\!]]\!] + \cdots . \end{aligned} \tag{2.18}$$

The map $F(t)$ has a property that changes $D_\eta(t)$ into η:

$$D_\eta(t)F(t) = F(t)\eta. \tag{2.19}$$

Using $F(t)$, we can define a homotopy operator for $D_\eta(t)$ as $F(t)\Xi$ satisfying (Ref. [3])

$$\{D_\eta(t), F(t)\Xi\} = 1, \tag{2.20}$$

which trivializes the D_η-cohomology as well as the η-cohomology in the large Hilbert space. From the definition (2.18), we can show that the homotopy operator $F\Xi$ is BPZ even

$$\langle F\Xi\Psi_1, \Psi_2\rangle = (-1)^{\Psi_1}\langle\Psi_1, F\Xi\Psi_2\rangle, \tag{2.21}$$

and satisfies

$$\{Q, F\Xi\}A = FXF\Xi D_\eta A + FX\eta F\Xi A - F\Xi[QA_\eta, F\Xi A], \tag{2.22}$$

for a string field A. It is useful to note that we can define the projection operators

$$\mathcal{P}_{\mathrm{R}} = D_\eta F \Xi, \qquad \mathcal{P}_{\mathrm{R}}^{\perp} = F \Xi D_\eta, \tag{2.23}$$

onto the Ramond string field annihilated by D_η and its orthogonal complement, respectively.

The BPZ inner product in the small Hilbert space $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is related to that in the large Hilbert space $\langle \cdot, \cdot \rangle$ as

$$\begin{aligned}\langle\!\langle A, B\rangle\!\rangle &= \langle \Xi A, B\rangle = (-1)^{A}\langle A, \Xi B\rangle \\ &= \langle \xi_0 A, B\rangle = (-1)^{A}\langle A, \xi_0 B\rangle,\end{aligned} \tag{2.24}$$

where A and B are in the small Hilbert space, and also in the Ramond sector for the equations in the first line.

Using a general variation of the map $F(t)$ on a string field A,

$$(\delta F(t))A = -F(t)(\delta F^{-1}(t))F(t)A = F\Xi[\![\delta A_\eta(t), F(t)A]\!], \tag{2.25}$$

a general variation of the action (2.13) can be calculated as (Ref. [3])

$$\delta S = -\langle A_\delta, QA_\eta + (F\Psi)^2\rangle - \langle\!\langle \delta\Psi, Y(Q\Psi + X\eta F\Psi)\rangle\!\rangle, \tag{2.26}$$

from which we find the equations of motion

$$QA_\eta + (F\Psi)^2 = 0, \qquad Q\Psi + X\eta F\Psi = 0. \tag{2.27}$$

Before closing this section, we generalize several ingredients for later use. We can define $A_{\mathcal{O}}(t)$ not only for $\mathcal{O} = \partial_t, \eta$, or δ, but also for any other derivations of the string product. Although such general $\mathcal{O}$'s are not in general commutative, we assume that they satisfy a closed algebra with respect to the graded commutator of derivations, $\{\mathcal{O}_1, \mathcal{O}_2] = \mathcal{O}_1\mathcal{O}_2 - (-1)^{\mathcal{O}_1\mathcal{O}_2}\mathcal{O}_2\mathcal{O}_1$. The generalized $A_{\mathcal{O}}(t)$'s satisfy the equation

$$\mathcal{O}_1 A_{\mathcal{O}_2}(t) - (-1)^{\mathcal{O}_1\mathcal{O}_2}\mathcal{O}_2 A_{\mathcal{O}_1}(t) - [\![A_{\mathcal{O}_1}(t), A_{\mathcal{O}_2}(t)]\!] = A_{\{\mathcal{O}_1,\mathcal{O}_2]}(t), \tag{2.28}$$

which reduces to the Maurer–Cartan-like equation (2.16) when $\{\mathcal{O}_1, \mathcal{O}_2] = 0$. Using $A_{\mathcal{O}}(t)$, we can define the covariant derivative $D_{\mathcal{O}}(t)$ on a string field A by

$$D_{\mathcal{O}}(t)A = \mathcal{O}A - [\![A_{\mathcal{O}}(t), A]\!]. \tag{2.29}$$

From Eq. (2.28), we can show that

$$[\![D_{\mathcal{O}_1}(t), D_{\mathcal{O}_2}(t)]\!] = D_{\{\mathcal{O}_1,\mathcal{O}_2]}(t). \tag{2.30}$$

As an analog of the linear map $F(t)$ in the Ramond sector, we can also define the linear map $f(t)$ on a general string field Φ in the NS sector by

$$\begin{aligned}f(t)\Phi &= \frac{1}{1 + \xi_0(D_\eta(t) - \eta)}\Phi \\ &= \Phi + \xi_0[\![A_\eta(t), \Phi]\!] + \xi_0[\![A_\eta(t), \xi_0[\![A_\eta(t), \Phi]\!]]\!] + \cdots .\end{aligned} \tag{2.31}$$

A homotopy operator for $D_\eta(t)$ in the NS sector is given by the BPZ even operator $f(t)\xi_0$:

$$\{D_\eta(t), f(t)\xi_0\} = 1, \qquad \langle f\xi_0\Phi_1, \Phi_2\rangle = (-1)^{\Phi_1}\langle \Phi_1, f\xi_0\Phi_2\rangle. \tag{2.32}$$

We can define the projection operators

$$\mathcal{P}_{\mathrm{NS}} = D_\eta f \xi_0, \qquad \mathcal{P}_{\mathrm{NS}}^{\perp} = f \xi_0 D_\eta \tag{2.33}$$

onto the NS string field annihilated by D_η and its orthogonal complement, respectively.

3. Space-time supersymmetry

Now let us discuss how space-time supersymmetry is realized in the WZW-like formulation. Starting from a natural linearized transformation exchanging the NS string field Φ and the Ramond string field Ψ, we construct a nonlinear transformation that is a symmetry of the complete action (2.13). We show that the transformation satisfies the supersymmetry algebra, up to the equations of motion and gauge transformation, except for an unphysical symmetry.

3.1. Space-time supersymmetry transformation

At the linearized level, a natural space-time supersymmetry transformation of string fields in the small Hilbert space, $\eta\Phi$ and Ψ, is given by

$$\delta^{(0)}_{\mathcal{S}(\epsilon)} \eta\Phi = \mathcal{S}(\epsilon)\Psi, \qquad \delta^{(0)}_{\mathcal{S}(\epsilon)} \Psi = X\mathcal{S}(\epsilon)\eta\Phi, \tag{3.1}$$

where

$$\mathcal{S}(\epsilon) = \epsilon_\alpha q^\alpha = \epsilon_\alpha \oint \frac{dz}{2\pi i} S^\alpha(z) e^{-\phi(z)/2} \tag{3.2}$$

is the first-quantized space-time supersymmetry charge with the parameter ϵ_α. The spin operator $S^\alpha(z)$ in the matter sector can be constructed from $\psi^\mu(z)$ using the bosonization technique (Ref. [6]). This $\mathcal{S}(\epsilon)$ is a (Grassmann-even) derivation of the string product, and is commutative with Q, η and ξ_0: $[Q, \mathcal{S}(\epsilon)] = [\eta, \mathcal{S}(\epsilon)] = [\xi_0, \mathcal{S}(\epsilon)] = 0$. It satisfies the algebra

$$[\mathcal{S}(\epsilon_1), \mathcal{S}(\epsilon_2)] = \tilde{p}(v_{12}), \tag{3.3}$$

with $v_{12}^\mu = (\epsilon_1 C \bar{\gamma}^\mu \epsilon_2)/\sqrt{2}$, where $\tilde{p}(v)$ is the operator with picture number $p = -1$ defined by

$$\tilde{p}(v) = v_\mu \tilde{p}^\mu = -v_\mu \oint \frac{dz}{2\pi i} \psi^\mu(z) e^{-\phi(z)}. \tag{3.4}$$

This is equivalent to the space-time translation operator $p(v) = v_\mu \oint \frac{dz}{2\pi i} i\partial X^\mu(z)$ (center of mass momentum of the string) in the sense that (Ref. [7]), e.g.,

$$(p(v) - X_0 \tilde{p}(v)) = \{Q, M(v)\}, \tag{3.5}$$

with

$$M(v) = v^\mu \oint \frac{dz}{2\pi i} (\xi(z) - \xi_0) \psi_\mu(z) e^{-\phi(z)}. \tag{3.6}$$

Note that $M(v)$ does not include ξ_0, and so is in the small Hilbert space: $\{\eta, M(v)\} = 0$. The algebra (3.3) and the Jacobi identity imply that $[Q, \tilde{p}(v)] = [\eta, \tilde{p}(v)] = [\xi_0, \tilde{p}(v)] = 0$.

We frequently omit specifying the parameters explicitly and denote, e.g., $\mathcal{S}(\epsilon_1)$ by $\mathcal{S}_1$. Since $\eta\Phi$ and Ψ are in the small Hilbert space containing the physical spectrum, Eq. (3.1) is the transformation law given in Ref. [7] except that the local picture-changing operator at the midpoint is replaced by the X in Eq. (2.5) so that the transformation is closed in the restricted space. As a transformation of Φ in the large Hilbert space, we adopt here that

$$\delta^{(0)}_{\mathcal{S}(\epsilon)}\Phi = \mathcal{S}(\epsilon)\Xi\Psi. \tag{3.7}$$

This is consistent with Eq. (3.1) but is not unique. A different choice, however, can be obtained by combining Eq. (3.7) and an Ω-gauge transformation, e.g.,

$$\begin{aligned}\tilde{\delta}^{(0)}_{\mathcal{S}(\epsilon)}\Phi &= \xi_0\mathcal{S}(\epsilon)\Psi\\ &= \delta^{(0)}_{\mathcal{S}(\epsilon)}\Phi - \eta(\xi_0\mathcal{S}(\epsilon)\Xi\Psi).\end{aligned} \tag{3.8}$$

Using the fact that $\mathcal{S}$ is BPZ odd,

$$\langle\mathcal{S}A, B\rangle = -\langle A, \mathcal{S}B\rangle, \tag{3.9}$$

it is easy to see that the quadratic terms of the action (2.13),

$$S^{(0)} = -\frac{1}{2}\langle\Phi, Q\eta\Phi\rangle - \frac{1}{2}\langle\langle\Psi, YQ\Psi\rangle\rangle, \tag{3.10}$$

are invariant under the transformation

$$\delta^{(0)}_{\mathcal{S}}\Phi = \mathcal{S}\Xi\Psi, \qquad \delta^{(0)}_{\mathcal{S}}\Psi = X\mathcal{S}\eta\Phi. \tag{3.11}$$

However, the action at the next order,

$$S^{(1)} = -\frac{1}{6}\langle\Phi, Q[\Phi, \eta\Phi]\rangle - \langle\Phi, \Psi^2\rangle, \tag{3.12}$$

is not invariant under $\delta^{(0)}_{\mathcal{S}}$ but is transformed as

$$\begin{aligned}\delta^{(0)}_{\mathcal{S}}S^{(1)} = &\ \langle\left(\tfrac{1}{2}[\Phi, \mathcal{S}\Xi\Psi] - \mathcal{S}\Xi[\Phi, \Psi] + \{\Psi, \Xi\mathcal{S}\Phi\}\right), Q\eta\Phi\rangle\\ &+ \langle\langle\left(-\tfrac{1}{2}X\eta[\Phi, \mathcal{S}\Phi] + X\eta[\Phi, \Xi\mathcal{S}\eta\Phi]\right), YQ\Psi\rangle\rangle.\end{aligned} \tag{3.13}$$

We have thus to modify the transformation by adding

$$\delta^{(1)}_{\mathcal{S}}\Phi = \frac{1}{2}[\Phi, \mathcal{S}\Xi\Psi] - \mathcal{S}\Xi[\Phi, \Psi] + \{\Psi, \Xi\mathcal{S}\Phi\}, \tag{3.14}$$

$$\delta^{(1)}_{\mathcal{S}}\Psi = -\frac{1}{2}X\eta[\Phi, \mathcal{S}\Phi] + X\eta[\Phi, \Xi\mathcal{S}\eta\Phi], \tag{3.15}$$

under which the kinetic terms (3.10) are transformed so as to cancel the contribution (3.13): $\delta^{(1)}_{\mathcal{S}}S^{(0)} + \delta^{(0)}_{\mathcal{S}}S^{(1)} = 0$. Then at the next order we have two contributions, $\delta^{(1)}_{\mathcal{S}}S^{(1)}$ and $\delta^{(0)}_{\mathcal{S}}S^{(2)}$, which are again nonzero and require to add

$$\begin{aligned}\delta^{(2)}_{\mathcal{S}}\Phi = &\ \frac{1}{12}[\Phi, [\Phi, \mathcal{S}\Xi\Psi]] + \frac{1}{2}\{[\Phi, \Psi], \Xi\mathcal{S}\Phi\} + \frac{1}{2}[\Xi[\Phi, \Psi], \mathcal{S}\Phi]\\ &+ \frac{1}{2}\{\Psi, \Xi\{\eta\Phi, \Xi\mathcal{S}\Phi\}\} + \frac{1}{2}\{\Psi, \Xi[\Phi, \Xi\mathcal{S}\eta\Phi]\} - [\Xi[\Phi, \Psi], \Xi\mathcal{S}\eta\Phi]\\ &- \frac{1}{2}\mathcal{S}\Xi[\Phi, \Xi\{\eta\Phi, \Psi\}] - \frac{1}{2}\mathcal{S}\Xi[\eta\Phi, \Xi[\Phi, \Psi]],\end{aligned} \tag{3.16}$$

$$\begin{aligned}\delta^{(2)}_{\mathcal{S}}\Psi = &\ \frac{1}{6}X\eta[\Phi, [\Phi, \mathcal{S}\Phi]] + \frac{1}{2}X\eta[\Phi, \Xi[\mathcal{S}\Phi, \eta\Phi]] + \frac{1}{2}X\eta\{\eta\Phi, \Xi[\Phi, \Xi\mathcal{S}\eta\Phi]\}\\ &+ \frac{1}{2}X\eta[\Phi, \Xi[\eta\Phi, \Xi\mathcal{S}\eta\Phi]],\end{aligned} \tag{3.17}$$

to cancel them by $\delta_{\mathcal{S}}^{(2)}S^{(0)}$: $\delta_{\mathcal{S}}^{(2)}S^{(0)}+\delta_{\mathcal{S}}^{(1)}S^{(1)}+\delta_{\mathcal{S}}^{(0)}S^{(2)}=0$. The procedure is not terminated, so we suppose a full transformation consistent with these results, and then show that it is in fact a symmetry of the complete action.

3.2. *Complete space-time supersymmetry transformation*

Here we suppose that the complete transformation is given by

$$A_{\delta_{\mathcal{S}}} = e^{\Phi}(\mathcal{S}\Xi(e^{-\Phi}F\Psi e^{\Phi}))e^{-\Phi}+\{F\Psi,F\Xi A_{\mathcal{S}}\}, \tag{3.18a}$$

$$\delta_{\mathcal{S}}\Psi = X\eta F\Xi D_{\eta}A_{\mathcal{S}} = X\eta F\Xi\mathcal{S}A_{\eta}, \tag{3.18b}$$

and show that the complete action (2.13) is invariant under this transformation. From the formula of the general variation of the action (2.26), we have

$$\delta_{\mathcal{S}}S = -\langle e^{\Phi}(\mathcal{S}\Xi(e^{-\Phi}F\Psi e^{\Phi}))e^{-\Phi},QA_{\eta}+(F\Psi)^2\rangle-\langle\{F\Psi,F\Xi A_{\mathcal{S}}\},QA_{\eta}+(F\Psi)^2\rangle \\ -\langle\!\langle X\eta F\Xi D_{\eta}A_{\mathcal{S}},Y(Q\Psi+X\eta F\Psi)\rangle\!\rangle. \tag{3.19}$$

We calculate each of these three terms, which we denote (I), (II), and (III), separately. First, using Eq. (2.21) and the cyclicity of the inner product, the second term is calculated as

$$(\mathrm{II}) = \langle A_{\mathcal{S}},F\Xi[QA_{\eta}+(F\Psi)^2,F\Psi]\rangle. \tag{3.20}$$

For the third term, we find

$$\begin{aligned}(\mathrm{III}) &= -\langle\!\langle \eta F\Xi D_{\eta}A_{\mathcal{S}},Q\Psi+X\eta F\Psi\rangle\!\rangle\\ &= -\langle A_{\mathcal{S}},D_{\eta}F\Xi(Q\Psi+X\eta F\Psi)\rangle\\ &= -\langle A_{\mathcal{S}},F(Q\Psi+X\eta F\Psi)\rangle,\end{aligned} \tag{3.21}$$

where we have used Eq. (2.21), Eq. (2.19), and the fact that X is BPZ even with respect to the inner product in the small Hilbert space, $\langle\!\langle XA,B\rangle\!\rangle=\langle\!\langle A,XB\rangle\!\rangle$, and $Q\Psi+X\eta F\Psi$ is in the restricted small Hilbert space. In order to calculate the first term (I), some consideration is necessary. In addition to the cyclicity, we need the following relation for two graded commutative derivations of the string product, $\mathcal{O}_1$ and $\mathcal{O}_2$ satisfying $\{\mathcal{O}_1,\mathcal{O}_2\}=0$:

$$\begin{aligned}e^{-\Phi}(\mathcal{O}_1A_{\mathcal{O}_2})e^{\Phi} &= \mathcal{O}_1\widetilde{A}_{\mathcal{O}_2}+\widetilde{A}_{\mathcal{O}_1}\widetilde{A}_{\mathcal{O}_2}-(-1)^{\mathcal{O}_1\mathcal{O}_2}\widetilde{A}_{\mathcal{O}_2}\widetilde{A}_{\mathcal{O}_1}\\ &= (-1)^{\mathcal{O}_1\mathcal{O}_2}\mathcal{O}_2\widetilde{A}_{\mathcal{O}_1},\end{aligned} \tag{3.22}$$

where $\widetilde{A}_{\mathcal{O}}$ is an analog of the left-invariant current: $\widetilde{A}_{\mathcal{O}}=e^{-\Phi}(\mathcal{O}e^{\Phi})$. If we use this relation for $(\mathcal{O}_1,\mathcal{O}_2)=(Q,\eta)$, we find

$$\begin{aligned}(\mathrm{I}) &= -\langle\mathcal{S}\Xi(e^{-\Phi}F\Psi e^{\Phi}),e^{-\Phi}(QA_{\eta}+(F\Psi)^2)e^{\Phi}\rangle\\ &= \langle\mathcal{S}\Xi(e^{-\Phi}F\Psi e^{\Phi}),\eta\widetilde{A}_{Q}\rangle-\langle\mathcal{S}\Xi(e^{-\Phi}F\Psi e^{\Phi}),(e^{-\Phi}F\Psi e^{\Phi})^2\rangle.\end{aligned} \tag{3.23}$$

Here the second term vanishes owing to Eqs. (3.9) and (2.24):

$$
\begin{aligned}
-\langle \mathcal{S}\Xi(e^{-\Phi}F\Psi e^{\Phi}),(e^{-\Phi}F\Psi e^{\Phi})^2\rangle &= \langle\!\langle (e^{-\Phi}F\Psi e^{\Phi}),\{(e^{-\Phi}F\Psi e^{\Phi}),\mathcal{S}(e^{-\Phi}F\Psi e^{\Phi})\}\rangle\!\rangle \\
&= \tfrac{2}{3}\Big(\langle\!\langle \mathcal{S}(e^{-\Phi}F\Psi e^{\Phi}),(e^{-\Phi}F\Psi e^{\Phi})^2\rangle\!\rangle \\
&\quad + \langle\!\langle (e^{-\Phi}F\Psi e^{\Phi}),\{(e^{-\Phi}F\Psi e^{\Phi}),\mathcal{S}(e^{-\Phi}F\Psi e^{\Phi})\}\rangle\!\rangle\Big) \\
&= 0.
\end{aligned} \tag{3.24}
$$

The first term in Eq. (3.23) can further be calculated as

$$
\begin{aligned}
(\mathrm{I}) &= -\langle \mathcal{S}(e^{-\Phi}F\Psi e^{\Phi}),\widetilde{A}_Q\rangle = \langle F\Psi, e^{\Phi}(\mathcal{S}\widetilde{A}_Q)e^{-\Phi}\rangle \\
&= \langle F\Psi, QA_{\mathcal{S}}\rangle = \langle A_{\mathcal{S}}, QF\Psi\rangle,
\end{aligned} \tag{3.25}
$$

where we have used the relation (3.22) with $(\mathcal{O}_1,\mathcal{O}_2)=(Q,\mathcal{S})$, and the identity

$$
\eta(e^{-\Phi}F\Psi e^{\Phi}) = e^{-\Phi}(D_\eta F\Psi)e^{\Phi} = 0. \tag{3.26}
$$

Summing Eqs. (3.20), (3.21), and (3.25), the variation of the action under the space-time supersymmetry transformation finally becomes

$$
\delta_{\mathcal{S}} S = \langle A_{\mathcal{S}}, \big(QF\Psi - F(Q\Psi + X\eta F\Psi) + F\Xi[QA_\eta + (F\Psi)^2, F\Psi]\big)\rangle, \tag{3.27}
$$

which vanishes due to the identity (4.89) in Ref. [3]: $\delta_{\mathcal{S}}S = 0$. Hence the complete action (2.13) is invariant under the transformation (3.18).

4. Algebra of transformation

Starting from a natural linear transformation (3.11), we have constructed the nonlinear transformation (3.18) as a symmetry of the complete action (2.13). If this is in fact space-time supersymmetry, the commutator of two transformations should satisfy the supersymmetry algebra

$$
[\delta_{\mathcal{S}_1},\delta_{\mathcal{S}_2}] \overset{?}{=} \delta_{p(v_{12})}, \tag{4.1}
$$

up to the equations of motion (2.27) and gauge transformation (2.14) generated by some field-dependent parameters, where $\delta_{p(v_{12})}$ is the space-time translation defined by

$$
\delta_{p(v)}A_\eta = -p(v)A_\eta, \qquad \delta_{p(v)}\Psi = -p(v)\Psi, \tag{4.2}
$$

with the parameter v_{12} in Eq. (3.3). In this section, we show that the algebra (4.1) is slightly modified, but still the transformation (3.18) can be identified with space-time supersymmetry.

4.1. Preparation

As preparation, note that the relations

$$
\delta A_\eta = D_\eta A_\delta, \tag{4.3a}
$$

$$
A_\delta = f\xi_0\delta A_\eta + D_\eta \Omega_\delta, \tag{4.3b}
$$

hold with $\Omega_\delta = f\xi_0 A_\delta$, for general variation of the NS string field A_δ. The former, Eq. (4.3a), is the case of $(\mathcal{O}_1,\mathcal{O}_2)=(\delta,\eta)$ in Eq. (2.16), and the latter, Eq. (4.3b), is obtained by decomposing

A_δ by the projection operators (2.33) and using Eq. (4.3a). These relations (4.3) show that two variations A_δ and δA_η are in one-to-one correspondence up to the Ω-gauge transformation. Since any transformation of the string field is a special case of the general variation, Eq. (4.3) holds for any symmetry transformation δ_I,

$$\delta_I A_\eta = D_\eta A_{\delta_I}, \tag{4.4a}$$

$$A_{\delta_I} = f\xi_0 \delta_I A_\eta + D_\eta \Omega_I. \tag{4.4b}$$

This is the case even for the commutator of the two transformations $[\delta_I, \delta_J]$,

$$[\delta_I, \delta_J] A_\eta = D_\eta A_{[\delta_I,\delta_J]}, \tag{4.5a}$$

$$A_{[\delta_I,\delta_J]} = f\xi_0 [\delta_I, \delta_J] A_\eta + D_\eta \Omega_{IJ}, \tag{4.5b}$$

with

$$\begin{aligned}\Omega_{IJ} = & -f\xi_0[f\xi_0\delta_I A_\eta, f\xi_0\delta_J A_\eta] \\ & + \delta_I\Omega_J - [f\xi_0\delta_I A_\eta, \Omega_J] - \delta_J\Omega_I + [f\xi_0\delta_J A_\eta, \Omega_I] - [\Omega_I, D_\eta\Omega_J],\end{aligned} \tag{4.6}$$

which can be shown by explicit calculation using Eqs. (2.28) and (2.31) if we assume Eq. (4.4) with some field-dependent Ω_I. Therefore if the algebra of the transformation is closed on A_η,

$$[\delta_I, \delta_J] A_\eta = \sum_{K\neq\Omega} \delta_K A_\eta, \tag{4.7}$$

we have

$$A_{[\delta_I,\delta_J]} = \sum_{K\neq\Omega} A_{\delta_K} + D_\eta \Omega_{IJ} = \sum_K A_{\delta_K}, \tag{4.8}$$

or equivalently, the algebra is also closed on e^Φ:

$$[\delta_I, \delta_J] e^\Phi = \sum_K \delta_K e^\Phi, \tag{4.9}$$

with some field-dependent Ω_{IJ}. Here in Eq. (4.7) we used that A_η is invariant under the Ω-gauge transformation, $A_{\delta_\Omega} = D_\eta \Omega$, as seen from Eq. (4.4a).

4.2. $[\delta_{\mathcal{S}_1}, \delta_{\mathcal{S}_2}]$

Now let us explicitly calculate the supersymmetry algebra on A_η and Ψ, which is easier to calculate than the algebra on the fundamental string fields Φ (or e^Φ) and Ψ due to their Ω-gauge invariance and enough to know that on the fundamental string fields as was shown in the previous subsection. From Eq. (3.18) we find

$$A_{\delta_\mathcal{S}} = f\xi_0 \delta_\mathcal{S} A_\eta + D_\eta \Omega_\mathcal{S}, \tag{4.10a}$$

$$\delta_\mathcal{S}\Psi = X\eta F \Xi \mathcal{S} A_\eta, \tag{4.10b}$$

with

$$\delta_\mathcal{S} A_\eta = \mathcal{S}F\Psi + [F\Psi, F\Xi\mathcal{S}A_\eta] = D_\mathcal{S} F\Psi - [F\Psi, D_\eta F\Xi A_\mathcal{S}], \tag{4.11a}$$

$$\Omega_\mathcal{S} = f\xi_0 \left(e^\Phi(\mathcal{S}\Xi(e^{-\Phi}F\Psi e^\Phi))e^{-\Phi} + \{F\Psi, F\Xi A_\mathcal{S}\}\right). \tag{4.11b}$$

Here we used the relations

$$D_\eta(e^{\Phi}Ae^{-\Phi}) = e^{\Phi}(\eta A)e^{-\Phi}, \qquad \eta(e^{-\Phi}Ae^{\Phi}) = e^{-\Phi}(D_\eta A)e^{\Phi}, \tag{4.12}$$

which hold for a general string field A. The commutator of two transformations on Ψ,

$$[\delta_{\mathcal{S}_1}, \delta_{\mathcal{S}_2}]\Psi = \delta_{\mathcal{S}_1}(X\eta F\Xi\mathcal{S}_2 A_\eta) - (1 \leftrightarrow 2), \tag{4.13}$$

which is easier and straightforward, can be calculated as follows. Using Eqs. (2.25), (2.20) and (2.16) with $(\mathcal{O}_1, \mathcal{O}_2) = (\mathcal{S}, \eta)$ and (δ, η), we can find

$$\begin{aligned}\delta_{\mathcal{S}_1}(X\eta F\Xi\mathcal{S}_2 A_\eta) &= X\eta F\Xi[\delta_{\mathcal{S}_1}A_\eta, F\Xi\mathcal{S}_2 A_\eta] + X\eta F\Xi\mathcal{S}_2(\delta_{\mathcal{S}_1}A_\eta)\\ &= X\eta F\Xi D_{\mathcal{S}_2}(\delta_{\mathcal{S}_1}A_\eta) + X\eta F\Xi[D_\eta F\Xi A_{\mathcal{S}_2}, \delta_{\mathcal{S}_1}A_\eta].\end{aligned} \tag{4.14}$$

Then, using $[D_\eta, D_{\mathcal{S}}] = 0$,

$$\begin{aligned}[\delta_{\mathcal{S}_1}, \delta_{\mathcal{S}_2}]\Psi &= \Big(X\eta F\Xi D_{\mathcal{S}_2}D_{\mathcal{S}_1}F\Psi - X\eta F\Xi[F\Psi, D_{\mathcal{S}_2}D_\eta F\Xi A_{\mathcal{S}_1}]\\ &\quad - X\eta F\Xi[D_\eta F\Xi A_{\mathcal{S}_2}, [F\Psi, D_\eta F\Xi A_{\mathcal{S}_1}]]\Big) - (1 \leftrightarrow 2)\\ &= -X\eta F\Xi D_{\tilde{p}_{12}}F\Psi\\ &\quad + X\eta F\Xi[F\Psi, D_\eta\big(D_{\mathcal{S}_1}F\Xi A_{\mathcal{S}_2} - D_{\mathcal{S}_2}F\Xi A_{\mathcal{S}_1} + [F\Xi A_{\mathcal{S}_1}, D_\eta F\Xi A_{\mathcal{S}_2}]\big)],\end{aligned} \tag{4.15}$$

where we have used Eqs. (2.30) and (3.3), and denoted $\tilde{p}(v_{12}) = \tilde{p}_{12}$. Comparing with Eq. (2.14b), we find that the second line has the form of the gauge transformation with the parameter

$$\begin{aligned}D_\eta\Lambda_{\mathcal{S}_1\mathcal{S}_2} &= -D_\eta\Big(D_{\mathcal{S}_1}F\Xi A_{\mathcal{S}_2} - D_{\mathcal{S}_2}F\Xi A_{\mathcal{S}_1} + [F\Xi A_{\mathcal{S}_1}, D_\eta F\Xi A_{\mathcal{S}_2}]\Big)\\ &= -A_{\tilde{p}_{12}} + (\mathcal{S}_1 F\Xi\mathcal{S}_2 - \mathcal{S}_1 F\Xi\mathcal{S}_1)A_\eta - [F\Xi\mathcal{S}_1 A_\eta, F\Xi\mathcal{S}_2 A_\eta].\end{aligned} \tag{4.16}$$

The second form can be obtained using Eq. (2.28), and will be used below.

In order to calculate the algebra on A_η, we first calculate the transformation of $F\Psi$ using Eq. (2.25):

$$\begin{aligned}\delta_{\mathcal{S}}F\Psi &= F\Xi\{\delta_{\mathcal{S}}A_\eta, F\Psi\} + F\delta_{\mathcal{S}}\Psi\\ &= FX\eta F\Xi\mathcal{S}A_\eta + F\Xi\mathcal{S}(F\Psi)^2 + F\Xi[(F\Psi)^2, F\Xi\mathcal{S}A_\eta]\\ &= QF\Xi\mathcal{S}A_\eta + F\Xi\mathcal{S}\left(QA_\eta + (F\Psi)^2\right) + F\Xi[QA_\eta + (F\Psi)^2, F\Xi\mathcal{S}A_\eta]\\ &\cong QF\Xi\mathcal{S}A_\eta,\end{aligned} \tag{4.17}$$

where the third equality follows from Eq. (2.22), and the symbol $\cong$ denotes an equation that holds up to the equations of motion. Then the commutator of two transformations on A_η,

$$[\delta_{\mathcal{S}_1}, \delta_{\mathcal{S}_2}]A_\eta = \delta_{\mathcal{S}_1}\big(\mathcal{S}_2 F\Psi + [F\Psi, F\Xi\mathcal{S}_2 A_\eta]\big) - (1 \leftrightarrow 2), \tag{4.18}$$

can be calculated similarly to that on Ψ. Since the first term can be calculated as

$$\begin{aligned}\delta_{\mathcal{S}_1}\big(\mathcal{S}_2F\Psi + [F\Psi, F\Xi\mathcal{S}_2A_\eta]\big) &= \mathcal{S}_2(\delta_{\mathcal{S}_1}F\Psi) + [(\delta_{\mathcal{S}_1}F\Psi), F\Xi\mathcal{S}_2A_\eta]\\ &\quad + [F\Psi, F\Xi D_{\mathcal{S}_2}(\delta_{\mathcal{S}_1}A_\eta)] + [F\Psi, F\Xi[D_\eta F\Xi A_{\mathcal{S}_2}, (\delta_{\mathcal{S}_1}A_\eta)]]\\ &\cong \mathcal{S}_2QF\Xi\mathcal{S}_1A_\eta + [QF\Xi\mathcal{S}_1A_\eta, F\Xi\mathcal{S}_2A_\eta]\\ &\quad + [F\Psi, F\Xi D_{\mathcal{S}_2}D_{\mathcal{S}_1}F\Psi] - [F\Psi, F\Xi D_{\mathcal{S}_2}[F\Psi, D_\eta F\Xi A_{\mathcal{S}_1}]]\\ &\quad + [F\Psi, F\Xi[D_\eta F\Xi A_{\mathcal{S}_2}, D_{\mathcal{S}_1}F\Psi]]\\ &\quad - [F\Psi, F\Xi[D_\eta F\Xi A_{\mathcal{S}_2}, [F\Psi, D_\eta F\Xi A_{\mathcal{S}_1}]]],\end{aligned} \tag{4.19}$$

we find

$$\begin{aligned}[\delta_{\mathcal{S}_1}, \delta_{\mathcal{S}_2}]A_\eta &\cong -Q\big((\mathcal{S}_1F\Xi\mathcal{S}_2 - \mathcal{S}_2F\Xi\mathcal{S}_1)A_\eta - [F\Xi\mathcal{S}_1A_\eta, F\Xi\mathcal{S}_2A_\eta]\big)\\ &\quad - [F\Psi, F\Xi[D_{\mathcal{S}_1}, D_{\mathcal{S}_2}]F\Psi] - [F\Psi, F\Xi[F\Psi, D_\eta\Lambda_{\mathcal{S}_1\mathcal{S}_2}]]\\ &= -QA_{\tilde{p}_{12}} - [F\Psi, F\Xi D_{\tilde{p}_{12}}F\Psi]\\ &\quad - QD_\eta\Lambda_{\mathcal{S}_1\mathcal{S}_2} - [F\Psi, F\Xi[F\Psi, D_\eta\Lambda_{\mathcal{S}_1\mathcal{S}_2}]],\end{aligned} \tag{4.20}$$

using two expressions in Eq. (4.16). From Eqs. (4.15), (4.20) and (4.8) we can conclude that the the commutator of two space-time supersymmetry transformations satisfies the algebra

$$[\delta_{\mathcal{S}_1}, \delta_{\mathcal{S}_2}] \cong \delta_{p(v_{12})} + \delta_{g(\Lambda_{\mathcal{S}_1\mathcal{S}_2}, \Omega_{\mathcal{S}_1\mathcal{S}_2})} + \delta_{\tilde{p}(v_{12})}, \tag{4.21}$$

with the gauge parameters given in Eqs. (4.16) and (4.6). The last term absent in Eq. (4.1) is a new symmetry defined by

$$A_{\delta_{\tilde{p}(v)}} = A_{p(v)} - f\xi_0\big(QA_{\tilde{p}(v)} + [F\Psi, F\Xi D_{\tilde{p}(v)}F\Psi]\big), \tag{4.22a}$$

$$\delta_{\tilde{p}(v)}\Psi = p(v)\Psi - X\eta F\Xi D_{\tilde{p}(v)}F\Psi, \tag{4.22b}$$

where the former is determined so as to induce

$$\begin{aligned}\delta_{\tilde{p}(v)}A_\eta &= D_\eta\Big(A_{p(v)} - f\xi_0\big(QA_{\tilde{p}(v)} + [F\Psi, F\Xi D_{\tilde{p}(v)}F\Psi]\big)\Big)\\ &\cong p(v)A_\eta - QA_{\tilde{p}(v)} - [F\Psi, F\Xi D_{\tilde{p}(v)}F\Psi].\end{aligned} \tag{4.23}$$

This extra contribution can be absorbed into the gauge transformation, up to the equations of motion, at the linearized level as we will see shortly.

Let us consider the transformation (4.22) at the linearized level:

$$\delta^{(0)}_{\tilde{p}}\Phi = p(v)\Phi - \xi_0Q\tilde{p}(v)\Phi = \big(p(v) - X_0\tilde{p}(v)\big)\Phi + Q(\xi_0\tilde{p}(v)\Phi), \tag{4.24a}$$

$$\delta^{(0)}_{\tilde{p}}\Psi = \big(p(v) - X\tilde{p}(v)\big)\Psi. \tag{4.24b}$$

Thanks to Eq. (3.5), the transformation of Φ in Eq. (4.24a) becomes the form of the gauge transformation up to the equation of motion at the linearized level:

$$\delta^{(0)}_{\tilde{p}}\Phi = Q\big((M(v) + \xi_0\tilde{p}(v))\Phi\big) + \eta\big(\xi_0M(v)Q\Phi\big) + \xi_0M(v)Q\eta\Phi. \tag{4.25}$$

We can similarly show that the transformation of Ψ in Eq. (4.24b) can also be written as a gauge transformation up to the equation of motion at the linearized level, as shown in Appendix B. Here we assume that the asymptotic condition (Ref. [10]) holds for string field theory as well as the conventional (particle) field theory. Then, at least perturbatively, we can identify that the transformation (4.24), or (4.25) and (B.5) can be interpreted, with appropriate (finite) renormalization, as that of asymptotic string fields. If we further assume asymptotic completeness, this implies that the extra transformation (4.24) acts trivially on the on-shell physical states defined by these asymptotic string fields, and thus the physical S-matrix. Thus the supersymmetry algebra is realized on the physical S-matrix, and we can identify the transformation (3.18) with space-time supersymmetry.

4.3. *Extra unphysical symmetries*

We have shown that the supersymmetry algebra is realized on the physical S-matrix but this is not the end of the story. The extra transformation $\delta_{\tilde{p}}$ produces another extra transformation if we consider the nested commutator $[\delta_{\mathcal{S}_1},[\delta_{\mathcal{S}_2},\delta_{\mathcal{S}_3}]]$. The extra contribution comes from the commutator $[\delta_{\mathcal{S}},\delta_{\tilde{p}}]$, which is nontrivial because the first-quantized charges $\mathcal{S}$ and $\tilde{p}$ are not commutative: $[\mathcal{S},\tilde{p}]\neq 0$. In fact, we can show that the algebra

$$[\delta_{\mathcal{S}},\delta_{\tilde{p}}]\cong\delta_g+\delta_{[\mathcal{S},\tilde{p}]} \tag{4.26}$$

holds with the gauge parameters

$$\begin{aligned}\Lambda_{\mathcal{S}\tilde{p}}=&\,f\xi_0\big(D_{\tilde{p}}f\xi_0D_{\mathcal{S}}-D_{\mathcal{S}}F\Xi D_{\tilde{p}}\big)F\Psi-[F\Psi,F\Xi D_{\tilde{p}}F\Xi A_{\mathcal{S}}]\\&-[F\Xi A_{\mathcal{S}},F\Xi D_{\tilde{p}}F\Psi]-D_{\tilde{p}}f\xi_0\{F\Psi,F\Xi A_{\mathcal{S}}\},\end{aligned} \tag{4.27a}$$

$$\lambda_{\mathcal{S}\tilde{p}}=X\eta F\Xi D_\eta D_{\tilde{p}}F\Xi A_{\mathcal{S}}, \tag{4.27b}$$

and $\Omega_{\mathcal{S}\tilde{p}}$ in Eq. (4.6). The new transformation $\delta_{[\mathcal{S},\tilde{p}]}$ is defined by

$$A_{\delta_{[\mathcal{S},\tilde{p}]}}=f\xi_0\Big(Qf\xi_0D_{[\mathcal{S},\tilde{p}]}F\Psi[F\Psi,F\Xi\big(QA_{[\mathcal{S},\tilde{p}]}+[F\Psi,f\xi_0D_{[\mathcal{S},\tilde{p}]}F\Psi]\big)]\Big), \tag{4.28a}$$

$$\delta_{[\mathcal{S},\tilde{p}]}\Psi=X\eta F\Xi\big(QA_{[\mathcal{S},\tilde{p}]}+[F\Psi,f\xi_0D_{[\mathcal{S},\tilde{p}]}F\Psi]\big), \tag{4.28b}$$

where $[\mathcal{S},\tilde{p}]$ denotes the first-quantized charge defined by the commutator $[q^\alpha,\tilde{p}^\mu]$ with the parameter $\zeta_{\mu\alpha}$,

$$[\mathcal{S},\tilde{p}]=\zeta_{\mu\alpha}[q^\alpha,\tilde{p}^\mu], \tag{4.29}$$

and in particular, $\zeta_{\mu\alpha}=\epsilon_\alpha v_\mu$ on the right-hand side of Eq. (4.26). This new symmetry is also unphysical in a similar sense to $\delta_{\tilde{p}}$. At the linearized level, the transformation (4.28) becomes[5]

$$\delta_{[\mathcal{S},\tilde{p}]}\Phi=\xi_0Q\xi_0[\mathcal{S},\tilde{p}]\Psi=\xi_0X_0[\mathcal{S},\tilde{p}]\Psi, \tag{4.30a}$$

$$\delta_{[\mathcal{S},\tilde{p}]}\Psi=X\eta\Xi Q[\mathcal{S},\tilde{p}]\Phi\cong XQ[\mathcal{S},\tilde{p}]\Phi, \tag{4.30b}$$

where we have used the fact that $\mathcal{S},\tilde{p}$, and thus $[\mathcal{S},\tilde{p}]$ are commutative with Q and η. If we note that $[\mathcal{S},p]=0$ and

$$[\mathcal{S},X_0]=[\mathcal{S},\{Q,\xi_0\}]=\{Q,[\mathcal{S},\xi_0]\}+\{\xi_0,[\mathcal{S},Q]\}=0, \tag{4.31}$$

[5] In this subsection, the symbol $\cong$ denotes an equation that holds up to the linearized equations of motion, $Q\eta\Phi=Q\Psi=0$.

the transformation of Φ, Eq. (4.28a), can further be rewritten in the form of a linearized gauge transformation:

$$\begin{aligned}\delta_{[\mathcal{S},\tilde{p}]}\Phi &= -\xi_0[\mathcal{S},(p-X_0\tilde{p})]\Psi = -\xi_0[\mathcal{S},\{Q,M\}]\Psi\\ &\cong -\xi_0 Q[\mathcal{S},M]\Psi\\ &= Q(\xi_0[\mathcal{S},M]\Psi) - \eta(\xi_0 X_0[\mathcal{S},M]\Psi).\end{aligned}\tag{4.32}$$

Similarly, the transformation of Ψ, Eq. (4.28b), can also be written as

$$\begin{aligned}\delta_{[\mathcal{S},\tilde{p}]}\Psi &\cong X\eta\xi_0 Q[\mathcal{S},\tilde{p}]\Phi\\ &= Q(X\eta\xi_0[\mathcal{S},\tilde{p}]\Phi) + X\eta X_0[\mathcal{S},\tilde{p}]\Phi\\ &\cong Q(X\eta\xi_0[\mathcal{S},\tilde{p}]\Phi + X\eta[\mathcal{S},M]\Phi).\end{aligned}\tag{4.33}$$

It should be noted that the gauge parameter in this form, $\lambda_{\mathcal{S}\tilde{p}} = X\eta\xi_0[\mathcal{S},\tilde{p}]\Phi + X\eta[\mathcal{S},M]\Phi$, is in the restricted small Hilbert space: $\eta\lambda_{\mathcal{S}\tilde{p}} = 0$ and $XY\lambda_{\mathcal{S}\tilde{p}} = \lambda_{\mathcal{S}\tilde{p}}$.

In addition, a further extra transformation is produced by considering the commutator between $\delta_{\tilde{p}_1}$ and $\delta_{\tilde{p}_2}$, and this sequence of extra transformations does not terminate as long as the nested commutators, $[\mathcal{O},[\mathcal{O},\mathcal{O}]]$, $[\mathcal{O},[\mathcal{O},[\mathcal{O},\mathcal{O}]]]$, $\ldots$, with $\mathcal{O}=\mathcal{S}$ or $\tilde{p}$, do not vanish. This complicates the structure of the algebra, but we can similarly show that all of these extra transformations act trivially on the physical S-matrix, as shown in Appendix B.

5. Summary and discussion

In this paper, we have explicitly constructed a space-time supersymmetry transformation of the WZW-like open superstring field theory in flat 10-dimensional space-time. Under the GSO projections, we have extended a linear transformation expected from space-time supersymmetry in the first-quantized theory to a nonlinear transformation so as to be a symmetry of the complete action (2.13). We have also shown that the transformation satisfies the supersymmetry algebra up to gauge transformation, the equations of motion, and a transformation $\delta_{\tilde{p}}$ acting trivially on the asymptotic physical states defined by the asymptotic string fields. This unphysical transformation produces a series of transformations $\delta_{[\mathcal{S},\tilde{p}]}, \delta_{[\tilde{p}\tilde{p}]}, \ldots$ by taking commutators with $\delta_{\mathcal{S}}$ or $\delta_{\tilde{p}}$ repeatedly. All of these symmetries also act trivially on the asymptotic physical states, and thus are unphysical, but it is interesting to clarify their complete structure, which is nontrivial in the total Hilbert space including unphysical degrees of freedom.

In any case, except for such an unphysical complexity, we have now understood how space-time supersymmetry is realized in superstring field theory, and therefore are ready to study various consequences of space-time supersymmetry (Refs. [11]–[14]) on a firm basis. We have to (re)analyze them precisely using the techniques developed in conventional quantum field theory.[6] We hope to report on them in the near future.

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[6] For such analyses of superstring field theory, see, e.g., Refs. [15]–[21].

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Appendix A. Spinor conventions and Ramond ground states

In this paper, although it is mostly implicit, we adopt the chiral representation for SO(1,9) gamma matrices Γ^{μ}, in which Γ^{μ} is given by

$$\Gamma^{\mu} = \begin{pmatrix} 0 & (\gamma^{\mu})_{\alpha\dot{\beta}} \\ (\bar{\gamma}^{\mu})^{\dot{\alpha}\beta} & 0 \end{pmatrix}, \tag{A.1}$$

where γ^{μ} and $\bar{\gamma}^{\mu}$ satisfy

$$(\gamma^{\mu}\bar{\gamma}^{\nu} + \gamma^{\nu}\bar{\gamma}^{\mu})_{\alpha}{}^{\beta} = 2\eta^{\mu\nu}\delta_{\alpha}{}^{\beta}, \qquad (\bar{\gamma}^{\mu}\gamma^{\nu} + \bar{\gamma}^{\nu}\gamma^{\mu})^{\dot{\alpha}}{}_{\dot{\beta}} = 2\eta^{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{A.2}$$

The charge conjugation matrix $\mathcal{C}$ satisfies the relations

$$(\Gamma^{\mu})^{T} = -\mathcal{C}\Gamma^{\mu}\mathcal{C}^{-1}, \qquad \mathcal{C}^{T} = -\mathcal{C}, \tag{A.3}$$

and is given in the chiral representation by

$$\mathcal{C} = \begin{pmatrix} 0 & C^{\alpha}{}_{\dot{\beta}} \\ -(C^{T})_{\dot{\alpha}}{}^{\beta} & 0 \end{pmatrix}. \tag{A.4}$$

The matrices $\mathcal{C}\Gamma^{\mu}$ are symmetric, or equivalently,

$$(C\bar{\gamma}^{\mu})^{\alpha\beta} = (C\bar{\gamma}^{\mu})^{\beta\alpha}, \qquad (C^{T}\gamma^{\mu})_{\dot{\alpha}\dot{\beta}} = (C^{T}\gamma^{\mu})_{\dot{\beta}\dot{\alpha}}. \tag{A.5}$$

The world-sheet fermion $\psi^{\mu}(z)$ in the Ramond sector has zero-modes that satisfy the SO(1,9) Clifford algebra

$$\{\psi_0^{\mu}, \psi_0^{\nu}\} = 0. \tag{A.6}$$

The degenerate ground states therefore become the space-time spinor, on which ψ_0^{μ} act as space-time gamma matrices. We summarize here the related convention. We denote the ground state spinor as $\begin{pmatrix} |^{\alpha}\rangle \\ |_{\dot{\alpha}}\rangle \end{pmatrix}$, on which ψ_0^{μ} acts as

$$\psi_0^{\mu}|^{\alpha}\rangle = |_{\dot{\alpha}}\rangle\frac{1}{\sqrt{2}}(\bar{\gamma}^{\mu})^{\dot{\alpha}\alpha}, \qquad \psi_0^{\mu}|_{\dot{\alpha}}\rangle = |^{\alpha}\rangle\frac{1}{\sqrt{2}}(\gamma^{\mu})_{\alpha\dot{\alpha}}. \tag{A.7}$$

Then $\hat{\Gamma}_{11}$ defined by Eq. (2.12) acts on the ground states as

$$\hat{\Gamma}_{11}|^{\alpha}\rangle = |^{\alpha}\rangle, \qquad \hat{\Gamma}_{11}|_{\dot{\alpha}}\rangle = -|_{\dot{\alpha}}\rangle, \tag{A.8}$$

by which the definition of the GSO projection (2.10) is supplemented. Similarly, the BPZ conjugate of the ground state spinor $(\langle^{\alpha}|, \langle_{\dot{\alpha}}|)$ satisfies

$$\langle^{\alpha}|\psi_0^{\mu} = \frac{i}{\sqrt{2}}(\bar{\gamma}^{\mu})^{\dot{\alpha}\alpha}\langle_{\dot{\alpha}}|, \qquad \langle_{\dot{\alpha}}|\psi_0^{\mu} = -\frac{i}{\sqrt{2}}(\gamma^{\mu})_{\alpha\dot{\alpha}}\langle^{\alpha}|, \tag{A.9}$$

with the normalization

$$\langle{}^{\alpha}|_{\dot{\alpha}}\rangle = C^{\alpha}{}_{\dot{\alpha}}, \qquad \langle{}_{\dot{\alpha}}|^{\alpha}\rangle = i(C^{T})_{\dot{\alpha}}{}^{\alpha} = -iC^{\alpha}{}_{\dot{\alpha}}. \tag{A.10}$$

The nontrivial matrix elements of ψ_0^{μ} are then given by

$$\langle{}^{\alpha}|\psi_0^{\mu}|^{\beta}\rangle = \frac{1}{\sqrt{2}}(C\bar{\gamma}^{\mu})^{\alpha\beta}, \qquad \langle{}_{\dot{\alpha}}|\psi_0^{\mu}|_{\dot{\beta}}\rangle = -\frac{i}{\sqrt{2}}(C^{T}\gamma^{\mu})_{\dot{\alpha}\dot{\beta}}. \tag{A.11}$$

Appendix B. Triviality of the extra unphysical symmetries at the linearized level

First, in order to show the triviality of Eq. (4.24b), it is useful to introduce the local inverse picture-changing operator

$$Y(z_0) = -c(z_0)\delta'(\gamma(z_0)), \tag{B.1}$$

which also satisfies

$$XY(z_0)X = X, \tag{B.2}$$

and in addition is commutative with Q: $[Q, Y(z_0)] = 0$. The point z_0 can be chosen to be any point on the string, e.g., the midpoint $z_0 = i$. Due to Eq. (B.2), we can define another projection operator $XY(z_0)$ that is commutative with Q, and acts identically with XY in the restricted small Hilbert space:

$$[Q, XY(z_0)] = 0, \tag{B.3}$$

and if $XY\Psi = \Psi$ then

$$XY(z_0)\Psi = XY(z_0)XY\Psi = XY\Psi. \tag{B.4}$$

Using this projection operator, the linearized transformation (4.24b) can be written as the a linearized gauge transformation,

$$\begin{aligned}\delta_{\tilde{p}}^{(0)}\Psi &= XY(z_0)\,(p(v) - X\tilde{p}(v))\,\Psi = XY(z_0)\{Q, \tilde{M}(v)\}\Psi\\ &\cong Q(XY(z_0)\tilde{M}(v)\Psi),\end{aligned} \tag{B.5}$$

up to the linearized equation of motion, $Q\Psi = 0$, with

$$\tilde{M}(v) = v^{\mu}\oint\frac{dz}{2\pi i}(\xi(z) - \Xi)\psi_{\mu}(z)e^{-\phi(z)}. \tag{B.6}$$

We can see that the gauge parameter in Eq. (B.5),

$$\lambda_{\tilde{p}} = XY(z_0)\tilde{M}(v)\Psi, \tag{B.7}$$

is in the restricted small Hilbert space,

$$\eta\lambda_{\tilde{p}} = 0, \qquad XY\lambda_{\tilde{p}} = \lambda_{\tilde{p}}, \tag{B.8}$$

if we note that $\{\eta, \tilde{M}\} = 0$.

As was mentioned in Sect. 4.3, the commutator $[\delta_{\tilde{p}_1}, \delta_{\tilde{p}_2}]$ produces another unphysical transformation $\delta_{[\tilde{p},\tilde{p}]}$:

$$[\delta_{\tilde{p}_1}, \delta_{\tilde{p}_2}] \cong \delta_g + \delta_{[\tilde{p},\tilde{p}]_{12}}, \tag{B.9}$$

where the field-dependent parameters are given by

$$\begin{aligned}\Lambda_{\tilde{p}_1\tilde{p}_2} =& f\xi_0\Big((D_{\tilde{p}_1}f\xi_0 D_{\tilde{p}_2} - D_{\tilde{p}_2}f\xi_0 D_{\tilde{p}_1})A_Q + D_{\tilde{p}_1}f\xi_0[F\Psi, F\Xi D_{\tilde{p}_2}F\Psi] \\ &- D_{\tilde{p}_2}f\xi_0[F\Psi, F\Xi D_{\tilde{p}_1}F\Psi] + \{F\Psi, F\Xi(D_{\tilde{p}_1}F\Xi D_{\tilde{p}_2} - D_{\tilde{p}_2}F\Xi D_{\tilde{p}_1})F\Psi\} \\ &- [F\Xi D_{\tilde{p}_2}F\Psi, F\Xi D_{\tilde{p}_2}F\Psi]\Big),\end{aligned} \tag{B.10}$$

$$\lambda_{\tilde{p}_1\tilde{p}_2} = -X\eta F\Xi(D_{\tilde{p}_1}F\Xi D_{\tilde{p}_2} - D_{\tilde{p}_2}F\Xi D_{\tilde{p}_1})F\Psi, \tag{B.11}$$

and $\Omega_{\tilde{p}_1\tilde{p}_2}$ in Eq. (4.6). The unphysical transformation $\delta_{[\tilde{p},\tilde{p}]}$ is defined by

$$\begin{aligned}A_{\delta_{[\tilde{p},\tilde{p}]}} =& -f\xi_0\bigg(Qf\xi_0\big(QA_{[\tilde{p},\tilde{p}]} + [F\Psi, F\Xi D_{[\tilde{p},\tilde{p}]}F\Psi]\big) \\ &+ \Big[F\Psi, F\Xi\Big(QF\Xi D_{[\tilde{p},\tilde{p}]}F\Psi + [F\Psi, f\xi_0\big(QA_{[\tilde{p},\tilde{p}]} + [F\Psi, F\Xi D_{[\tilde{p},\tilde{p}]}F\Psi]\big)]\Big)\Big]\bigg),\end{aligned} \tag{B.12a}$$

$$\delta_{[\tilde{p},\tilde{p}]}\Psi = -X\eta F\Xi\Big(QF\Xi D_{[\tilde{p},\tilde{p}]}F\Psi + [F\Psi, f\xi_0\big(QA_{[\tilde{p},\tilde{p}]} + [F\Psi, F\Xi D_{[\tilde{p},\tilde{p}]}F\Psi]\big)]\Big). \tag{B.12b}$$

The first-quantized charge $[\tilde{p},\tilde{p}]$ is defined by

$$[\tilde{p},\tilde{p}] = w_{\mu\nu}[\tilde{p}^\mu, \tilde{p}^\nu], \tag{B.13}$$

with the parameter $w_{\mu\nu}(= -w_{\nu\mu})$, and $[\tilde{p},\tilde{p}]_{12} = [\tilde{p},\tilde{p}](w_{12} = (v_1 v_2 - v_2 v_1)/2)$ in Eq. (B.9). At the linearized level, the transformation (B.12) becomes

$$\delta_{[\tilde{p},\tilde{p}]}\Phi = -\xi_0 Q\xi_0 Q[\tilde{p},\tilde{p}]\Phi = -\xi_0 QX_0[\tilde{p},\tilde{p}]\Phi, \tag{B.14}$$

$$\delta_{[\tilde{p},\tilde{p}]}\Psi = -X\eta\Xi Q\Xi[\tilde{p},\tilde{p}]\Psi = -X\eta\Xi X[\tilde{p},\tilde{p}]\Psi, \tag{B.15}$$

and can further be rewritten in the form of a linearized gauge transformation:

$$\begin{aligned}\delta_{[\tilde{p},\tilde{p}]}\Phi &= \xi_0 Q[\tilde{p}, \{Q, M\}]\Phi = \xi_0 Q[\tilde{p}, M]Q\Phi \\ &\cong -Q(\xi_0[\tilde{p}, M]Q\Phi) + \eta(\xi_0 X_0[\tilde{p}, M]Q\Phi),\end{aligned} \tag{B.16}$$

and

$$\begin{aligned}\delta_{[\tilde{p},\tilde{p}]}\Psi &= X\eta\Xi[\tilde{p}, \{Q, \tilde{M}\}]\Psi \cong X\eta\Xi Q[\tilde{p}, \tilde{M}]\Psi \\ &= Q(X\eta\Xi[\tilde{p}, \tilde{M}]\Psi),\end{aligned} \tag{B.17}$$

up to the linearized equations of motion. The parameter $\lambda_{\tilde{p}\tilde{p}} = X\eta\Xi[\tilde{p},\tilde{M}]\Psi$ is in the restricted small Hilbert space: $\eta\lambda_{\tilde{p}\tilde{p}} = 0$ and $XY\lambda_{\tilde{p}\tilde{p}} = \lambda_{\tilde{p}\tilde{p}}$.

Finally we show that all the extra symmetries obtained from the repeated commutators of $\delta_{\mathcal{S}}$'s and $\delta_{\tilde{p}}$'s act trivially on the physical states defined by the asymptotic string fields. For this purpose, it is enough to consider the transformations of $\eta\Phi$ and Ψ at the linearized level for a similar reason to that discussed in Sect. 4. Using the linearized form of Eq. (4.3) for general variation,

$$\delta\Phi = \xi_0\delta\eta\Phi + \eta(\xi_0\delta\Phi), \tag{B.18}$$

we can show that if the transformation of $\eta\Phi$ has the form of a gauge transformation, $\delta\eta\Phi = -Q\eta\Lambda$, with some field-dependent parameter Λ, then the transformation of Φ also has the form of a gauge transformation:

$$\begin{aligned}\delta\Phi &= -\xi_0 Q\eta\Lambda + \eta\Omega \\ &= Q\Lambda + \eta(\Omega - \xi_0 Q\Lambda),\end{aligned} \tag{B.19}$$

with some field-dependent Ω.

Starting from the linearized transformations

$$\delta_{\mathcal{S}}\eta\Phi = \mathcal{S}\Psi, \qquad \delta_{\mathcal{S}}\Psi = X\mathcal{S}\eta\Phi, \tag{B.20}$$

$$\delta_{\tilde{p}}\eta\Phi = (p - X_0\tilde{p})\eta\Phi, \qquad \delta_{\tilde{p}}\Psi = (p - X\tilde{p})\Psi, \tag{B.21}$$

extra symmetries can be read from repeated commutators, $[\delta_{\mathcal{O}_1}, [\delta_{\mathcal{O}_2}, \ldots, [\delta_{\mathcal{O}_n}, \delta_{\tilde{p}}], \ldots]]$, where $\mathcal{O}_i = \mathcal{S}$ or $\tilde{p}$. For example, we can read $\delta_{[\mathcal{S},\tilde{p}]}$ from $[\delta_{\mathcal{S}}, \delta_{\tilde{p}}]$,

$$\begin{aligned}[\delta_{\mathcal{S}}, \delta_{\tilde{p}}]\eta\Phi &= (p - X_0\tilde{p})\mathcal{S}\Psi - \mathcal{S}(p - X\tilde{p})\Psi \\ &= -X_0\tilde{p}\mathcal{S}\Psi + \mathcal{S}X\tilde{p}\Psi \\ &\cong -X_0\tilde{p}\mathcal{S}\Psi + Q\mathcal{S}\{\xi_0, \eta\}\Xi\tilde{p}\Psi \\ &= [\mathcal{S}, X_0\tilde{p}]\Psi + Q\eta(\mathcal{S}\xi_0\Xi\tilde{p}\Psi) \\ &= -[\mathcal{S}, (p - X_0\tilde{p})]\Psi + Q\eta(\mathcal{S}\xi_0\Xi\tilde{p}\Psi),\end{aligned} \tag{B.22}$$

and

$$\begin{aligned}[\delta_{\mathcal{S}}, \delta_{\tilde{p}}]\Psi &= (p - X\tilde{p})X\mathcal{S}\eta\Phi - X\mathcal{S}(p - X_0\tilde{p})\eta\Phi \\ &= -X\tilde{p}X\mathcal{S}\eta\Phi + X\mathcal{S}X_0\tilde{p}\eta\Phi \\ &\cong -QX\{\xi_0, \eta\}\tilde{p}\Xi\mathcal{S}\eta\Phi + X\mathcal{S}X_0\tilde{p}\eta\Phi \\ &= X[\mathcal{S}, X_0\tilde{p}]\eta\Phi - Q\eta(X\xi_0\tilde{p}\Xi\mathcal{S}\eta\Phi) \\ &= -X[\mathcal{S}, (p - X_0\tilde{p})]\eta\Phi - Q\eta(X\xi_0\tilde{p}\Xi\mathcal{S}\eta\Phi),\end{aligned} \tag{B.23}$$

as

$$\delta_{[\mathcal{S},\tilde{p}]}\eta\Phi = -[\mathcal{S}, (p - X_0\tilde{p})]\Psi, \tag{B.24}$$

$$\delta_{[\mathcal{S},\tilde{p}]}\Psi = -X[\mathcal{S}, (p - X_0\tilde{p})]\eta\Phi, \tag{B.25}$$

up to the equations of motion and gauge transformation. Similarly, we can find that general extra symmetries have the form

$$\delta_{[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l-1},\tilde{p}]]]}\eta\Phi = -(-1)^l (X_0)^{k+l-1}[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l-1},(p-X_0\tilde{p})]]]\Psi, \quad \text{(B.26a)}$$

$$\delta_{[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l-1},\tilde{p}]]]}\Psi = -(-1)^l (X)^{k+l}[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l-1},(p-X_0\tilde{p})]]]\eta\Phi, \quad \text{(B.26b)}$$

or

$$\delta_{[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l},\tilde{p}]]]}\eta\Phi = (-1)^l (X_0)^{k+l}[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l},(p-X_0\tilde{p})]]]\eta\Phi, \quad \text{(B.27a)}$$

$$\delta_{[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l},\tilde{p}]]]}\Psi = (-1)^l (X)^{k+l}[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l},(p-X_0\tilde{p})]]]\Psi, \quad \text{(B.27b)}$$

with $k = 1, 2, \ldots$ and $l = 0, 1, \ldots$, up to the equations of motion and gauge transformation. Here $2k-1$ (l) of the $\mathcal{O}$'s are $\mathcal{S}$ ($\tilde{p}$) in Eq. (B.26) and $2k$ (l) of the $\mathcal{O}$'s are $\mathcal{S}$ ($\tilde{p}$) in Eq. (B.27). All the picture-changing operators, except for the last one, can be put together in front of the right-hand side, aligning X_0 or X, which is always possible in a similar way to Eqs. (B.22) or (B.23). If an X is in front of some $\mathcal{O}_{i_0}$, we can move it to the top, e.g.,

$$\begin{aligned}
&(X_0)^p[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[X\mathcal{O}_{i_0},\ldots,[\mathcal{O}_n,(p-X_0\tilde{p})]]]]\eta\Phi \\
&\quad\cong Q\{\xi_0,\eta\}(X_0)^p[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\Xi\mathcal{O}_{i_0},\ldots,[\mathcal{O}_n,(p-X_0\tilde{p})]]]]\eta\Phi \\
&\quad= Q\xi_0(X_0)^p[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{i_0},\ldots,[\mathcal{O}_n,(p-X_0\tilde{p})]]]]\eta\Phi \\
&\qquad+ Q\eta(\xi_0(X_0)^p[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{i_0},\ldots,[\mathcal{O}_n,(p-X_0\tilde{p})]]]]\eta\Phi) \\
&\quad\cong (X_0)^{p+1}[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{i_0},\ldots,[\mathcal{O}_n,(p-X_0\tilde{p})]]]]\eta\Phi \\
&\qquad+ Q\eta(\xi_0(X_0)^p[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{i_0},\ldots,[\mathcal{O}_n,(p-X_0\tilde{p})]]]]\eta\Phi).
\end{aligned} \quad \text{(B.28)}$$

Using Eq. (3.5), it is easy to show that the transformations (B.26) or (B.27) can further be written in the form of a gauge transformation as

$$\delta_{[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l-1},\tilde{p}]]]}\eta\Phi \cong -(-1)^l Q\eta((X_0)^{k+l-1}\xi_0[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l-1},M],\ldots]]\Psi), \quad \text{(B.29a)}$$

$$\delta_{[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l-1},\tilde{p}]]]}\Psi \cong -(-1)^l Q((X)^{k+l}\eta\xi_0[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l-1},M],\ldots]]\eta\Phi), \quad \text{(B.29b)}$$

or

$$\delta_{[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l},\tilde{p}]]]}\eta\Phi \cong (-1)^l Q\eta((X_0)^{k+l}\xi_0[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l},M],\ldots]]\eta\Phi), \quad \text{(B.30a)}$$

$$\delta_{[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l},\tilde{p}]]]}\Psi \cong (-1)^l Q((X)^{k+l}\eta\xi_0[\mathcal{O}_1,[\mathcal{O}_2,\ldots,[\mathcal{O}_{2k+l},M],\ldots]]\Psi), \quad \text{(B.30b)}$$

respectively. Hence all the extra symmetries obtained as repeated commutators of $\delta_{\mathcal{S}}$'s and $\delta_{\tilde{p}}$'s act trivially on the on-shell physical states, and thus the physical S-matrix, defined by the asymptotic string fields.

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