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Gauge boson mass dependence and chiral anomalies in generalized massless Schwinger models

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Abstract: I bosonize the position-space correlators of flavor-diagonal scalar fermion bilinears in arbitrary generalizations of the Schwinger model with n_F massless fermions coupled to n_A gauge bosons for $n_F \geq n_A$. For $n_A = n_F$, the fermion bilinears can be bosonized in terms of n_F scalars with masses proportional to the gauge couplings. As in the Schwinger model, bosonization can be used to find all correlators, including those that are forbidden in perturbation theory by anomalous chiral symmetries, but there are subtleties when there is more than one gauge boson. The new result here is the general treatment of the dependence on gauge boson masses in models with more than one gauge symmetry. For $n_A < n_F$, there are fermion bilinears with nontrivial anomalous dimensions and there are unbroken chiral symmetries so some correlators vanish while others are non-zero due to chiral anomlies. Taking careful account of the dependence on the masses, I show how the $n_A < n_F$ models emerge from $n_A = n_F$ as gauge couplings (and thus gauge boson masses) go to zero. When this is done properly, the limit of zero gauge coupling is smooth. Our consistent treatment of gauge boson masses guarantees that anomalous symmetries are broken while unbroken chiral symmetries are preserved because correlators that break the non-anomalous symmetries go to zero in the limit of zero gauge coupling.

Keywords: Field Theories in Lower Dimensions, Anomalies in Field and String Theories, Nonperturbative Effects

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1 Introduction

In this note, I analyze the position-space correlators of flavor-diagonal scalar fermion bilinears in arbitrary generalizations of the massless Schwinger model and focus on the behavior of gauge anomalies in the limit in which some gauge couplings go to zero. We will see that when the dimensional gauge couplings are properly accounted for (which I believe is done here for the first time) the limit is smooth.

We consider a general Schwinger model in 1+1 dimensions with n_F massless Dirac fermions, ψ_α for $\alpha = 1$ to n_F, and $n_A \leq n_F$ vector bosons, A_j^μ for $j = 1$ to n_A.

The Lagrangian is (using summation convention where it does not cause confusion)

$$\mathcal{L}_{Af} = \left(\sum_{\alpha=1}^{n_F} \overline{\psi}_\alpha \left(i \, \partial\!\!\!/ - e_{\alpha j} \, A\!\!\!/_j \right) \psi_\alpha \right) - \frac{1}{4} F_j^{\mu\nu} F_{j\mu\nu} \tag{1.1}$$

Note that we have assumed that the gauge couplings are diagonal in the fermion space. This is important to ensure that the model is exactly solvable. We could have non-diagonal couplings as long as the couplings to different gauge bosons commute with one another. But then we can simultaneously diagonalize them with a unitary transformation on the fermion fields, so we will assume that the gauge bosons only couple to diagonal fermion currents. The gauge couplings are an $n_F \times n_A$ matrix, e in which

$$e_{\alpha j} \text{ is the coupling of the } j\text{th vector to the } \alpha\text{th diagonal fermion current.} \tag{1.2}$$

If $n_F < n_A$, there are linear combinations of the gauge bosons that do not couple to anything and can be safely ignored, so we will always assume that $n_F \geq n_A$.

Each of the massless fermions generates a contribution to the vector boson mass matrix [1] and we will use our freedom to redefine the vector fields to diagonalize the physical vector boson mass so we can write[1]

$$\sum_\alpha e_{\alpha j} e_{\alpha j'} = \pi \, m_j^2 \, \delta_{jj'} \qquad m_j^2 = \frac{1}{\pi} \sum_\alpha e_{\alpha j}^2 \tag{1.3}$$

[1]This transformation may obscure charge quantization. See the discussion at the end section 4.

Then the m_j are the physical gauge boson masses. The basis of the fermions is fixed up to unbroken flavor symmetries because the currents must be diagonal. And unless some gauge boson masses are equal, the basis of the gauge boson fields is also fixed by (1.3).

We work in Lorenz gauge,

$$\partial_\mu A_j^\mu = 0 \tag{1.4}$$

Then a generalization of the arguments of [1][2] shows that the gauge invariant correlation functions to all orders in perturbation theory can be found using the free-field Lagrangian describing massless fermions, Ψ_α, massive bosons, $\mathcal{B}_j$, and massless scalar ghosts, $\mathcal{C}_j$

$$\mathcal{L} = \left(\sum_{\alpha=1}^{n_F} i\overline{\Psi}_\alpha \,\partial\!\!\!/\, \Psi_\alpha \right) - \frac{m_j^2}{2}\mathcal{B}_j^2 + \frac{1}{2}\partial_\mu \mathcal{B}_j \partial^\mu \mathcal{B}_j - \frac{1}{2}\partial_\mu \mathcal{C}_j \partial^\mu \mathcal{C}_j \tag{1.5}$$

using the replacements

$$A_j^\mu = \epsilon^{\mu\nu}\partial_\nu \left(\mathcal{B}_j - \mathcal{C}_j\right)/m_j \tag{1.6}$$

$$\psi_\alpha = e^{-i(\pi)^{1/2}\,(e_{\alpha j}/m_j)(\mathcal{C}_j-\mathcal{B}_j)\gamma^5}\,\Psi_\alpha \tag{1.7}$$

The massive and the massless free scalar propagators, respectively, are[3]

$$-i\left\langle 0 | T\mathcal{B}_j\left(x\right)\mathcal{B}_k\left(0\right) | 0 \right\rangle = \delta_{jk}\int \frac{d^2p}{(2\pi)^2}\frac{e^{-ipx}}{p^2-m_j^2+i\epsilon} = -\delta_{jk}\frac{i}{2\pi}K_0\left(m_j\sqrt{-x^2+i\epsilon}\right) \tag{1.8}$$

$$-i\left\langle 0 | T\mathcal{C}_j\left(x\right)\mathcal{C}_k\left(0\right) | 0 \right\rangle = -\delta_{jk}\int \frac{d^2p}{(2\pi)^2}\frac{e^{-ipx}}{p^2+i\epsilon} = -\delta_{jk}\frac{i}{2\pi}\ln\left(\xi m_j\sqrt{-x^2+i\epsilon}\right) \tag{1.9}$$

$$\xi \equiv e^{\gamma_E}/2 \quad \text{where } \gamma_E \text{ is Euler's constant} \tag{1.10}$$

The arbitrary dimensional constant in the logarithmic ghost propagators has been fixed so that the $\mathcal{B}_j$ and $\mathcal{C}_j$ propagators exactly cancel when the gauge couplings vanish, $m_j = 0$.

We will focus on the position-space correlators of the flavor-diagonal fermion-bilinear scalar operators

$$\begin{aligned} O_{\alpha+} &\equiv \psi_{2\alpha}^*\,\psi_{1\alpha} = e^{-2i\sum_j (e_{\alpha j}/m_j)(\mathcal{B}_j-\mathcal{C}_j)}\,\Psi_{2\alpha}^*\Psi_{1\alpha} \\ O_{\alpha-} &\equiv \psi_{1\alpha}^*\,\psi_{2\alpha} = e^{2i\sum_j (e_{\alpha j}/m_j)(\mathcal{B}_j-\mathcal{C}_j)}\,\Psi_{1\alpha}^*\Psi_{2\alpha} = O_{\alpha+}^* \end{aligned} \tag{1.11}$$

While (1.11) gives a complete description of the $O_{\alpha\pm}$ correlators to all orders in perturbation theory, the nonperturbative effects of gauge anomalies are much more transparent if we bosonize. For massless free fermions in 1+1, *any non-zero* correlator of diagonal fermion bilinears can be calculated by replacing the bilinears with exponentials of free massless scalar fields according to

$$\Psi_{2\alpha}^*\Psi_{1\alpha} \to \frac{\xi\mathcal{M}_\alpha}{2\pi}\,e^{-2i\pi^{1/2}\mathcal{D}^\alpha} \qquad \Psi_{1\alpha}^*\Psi_{2\alpha} \to \frac{\xi\mathcal{M}_\alpha}{2\pi}\,e^{2i\pi^{1/2}\mathcal{D}^\alpha} \tag{1.12}$$

[2] For details and definitions, see [2]. The conjecture in the paper was shown to be false in [3] but we can still use the calculational tools.

[3] K_0 is the modified Bessel function of the second kind.

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The $\mathcal{M}_\alpha$s are arbitrary and correlated with the propagators of the bosonization fields, $\mathcal{D}^\alpha$.

$$-i\left\langle 0\right|T\mathcal{D}^\alpha\left(x\right)\mathcal{D}^\beta\left(0\right)\left|0\right\rangle = \delta^{\alpha\beta}\int\frac{d^2p}{\left(2\pi\right)^2}\frac{e^{-ipx}}{p^2+i\epsilon} = \delta^{\alpha\beta}\frac{i}{2\pi}\ln\left(\xi\mathcal{M}_\alpha\sqrt{-x^2+i\epsilon}\right) \tag{1.13}$$

There is no mass in the position space correlators of the fermion bilinears and the mass $\mathcal{M}_\alpha$ is introduced by the bosonization procedure which requires a mass to get the dimensions right. Eq. (1.12) is a straightforward consequence of Fermi-Dirac statistics. Then

$$O_{\alpha\pm} = \frac{\xi\mathcal{M}_\alpha}{2\pi}\,e^{\mp 2i\sum_{j=1}^{n_A}(e_{\alpha j}/m_j)(\mathcal{B}_j-\mathcal{C}_j)}\,e^{\mp 2i\pi^{1/2}\mathcal{D}^\alpha} \tag{1.14}$$

The key to using (1.14) most effectively will be to cancel the ghost fields with linear combinations of the bosonization fields. This will be easier if we change notation slightly.

First, we define an orthogonal $n_F\times n_F$ matrix $\eta_{\alpha j}$ where the α index labels the fermion as usual but the j index is extended to make the matrix square and a subset b of the j indices are associated with the gauge fields.

$$\eta\,\eta^T = I \qquad \eta_{\alpha j} = \frac{e_{\alpha j}}{m_j\sqrt{\pi}} \quad \text{for} \quad j\in b \tag{1.15}$$

we can write

$$O_{\alpha\pm} = \frac{\xi\mathcal{M}_\alpha}{2\pi}\,e^{\mp 2i\pi^{1/2}\sum_{j\in b}\eta_{\alpha j}(\mathcal{B}_j-\mathcal{C}_j)}\,e^{\mp 2i\pi^{1/2}\mathcal{D}^\alpha} \tag{1.16}$$

Now we can define linear combinations of the bosonization fields

$$\mathcal{D}_j \equiv \eta_{\alpha j}\mathcal{D}^\alpha \tag{1.17}$$

and write

$$O_{\alpha\pm} = \frac{\xi\mathcal{M}_\alpha}{2\pi}\,e^{\mp 2i\pi^{1/2}\sum_{j\in b}\eta_{\alpha j}(\mathcal{B}_j-\mathcal{C}_j)}\,e^{\mp 2i\pi^{1/2}\sum_j\eta_{\alpha j}\mathcal{D}_j} \tag{1.18}$$

In this form, it looks like we can cancel the $\mathcal{D}_j$ fields for $j\in b$ with the ghosts. That is precisely what happens in the original Schwinger model and in the Schwinger model with flavors. But in general, this cancellation is not exact because of the different masses associated with different gauge couplings.

The $\mathcal{D}_j$ propagator is

$$-i\left\langle 0\right|T\mathcal{D}_j\left(x\right)\mathcal{D}_k\left(0\right)\left|0\right\rangle = \sum_\alpha\eta_{j\alpha}\eta_{k\alpha}\frac{i}{2\pi}\ln\left(\xi\mathcal{M}_\alpha\sqrt{-x^2+i\epsilon}\right) \tag{1.19}$$

Thus for the $\mathcal{D}_j$ for $j=1$ to n_A to exactly cancel the ghosts $\mathcal{C}_j$ we *must* have

$$\sum_\alpha\eta_{\alpha j}\eta_{\alpha k}\log\mathcal{M}_\alpha = \delta_{jk}\log m_j \quad \text{for } j \text{ or } k\in b \tag{1.20}$$

and (1.18) would be most useful if the $\mathcal{D}_j$ propagator were diagonal for all j. In general, this is impossible and the most useful thing we can do is to choose a common bosonization mass $\mathcal{M}_\alpha = m$ for the $\mathcal{D}^\alpha$s:

$$\mathcal{M}_\alpha = m \quad \text{for all } \alpha. \tag{1.21}$$

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so that

$$-i\left\langle 0\right|T\mathcal{D}_j\left(x\right)\mathcal{D}_k\left(0\right)\left|0\right\rangle=\delta_{jk}\frac{i}{2\pi}\ln\left(\xi m\sqrt{-x^2+i\epsilon}\right) \tag{1.22}$$

We could always cancel the ghost for a single j by choosing $m=m_j$. This is the way the issue is typically handled in models with a single gauge boson but it is not adequate for $n_A>1$.

If we adopt (1.22), we can *almost* cancel the ghosts in general.

$$O_{\alpha\pm}=\frac{\xi m}{2\pi}\,e^{\mp 2i\pi^{1/2}\sum_{j\in b}\eta_{\alpha j}(\mathcal{B}_j-\mathcal{E}_j)}\,e^{\mp 2i\pi^{1/2}\sum_{j\in b'}\eta_{\alpha j}\mathcal{D}_j} \tag{1.23}$$

where b' is the complement of b and the $\mathcal{E}_j$ are "constant" fields

$$\mathcal{E}_j=\mathcal{C}_j-\mathcal{D}_j\quad\text{for } j\in b \tag{1.24}$$

with propagators

$$-i\left\langle 0\right|T\mathcal{E}_j\left(x\right)\mathcal{E}_k\left(0\right)\left|0\right\rangle=\delta_{jk}\frac{i}{2\pi}\log\left(m-m_j\right) \tag{1.25}$$

so the Wick expansion gives

$$\langle 0|Te^{2i\pi^{1/2}\mathcal{E}_j(x)}\,e^{\mp 2i\pi^{1/2}\mathcal{E}_j(0)}|0\rangle=\left(m_j/m\right)^{\pm 2} \tag{1.26}$$

The $\mathcal{E}_j$ fields keep the engineering dimensions right while eliminating the dependence on the arbitrary bosonization mass m. Eqs. (1.23)–(1.25) can be used to calculate any matrix element that is non-zero in perturbation theory and we can simplify (1.23) further by eliminating the $\mathcal{E}_j$ operators.[4]

Look at a general perturbatively non-zero correlator involving only bilinears (no higher powers). If there are n_α $O_{\alpha+}$s there must also be n_α $O_{\alpha-}$s or the correlator would vanish in perturbation theory. We can look separately at the contributions from each $\mathcal{E}_j$ because the propagators don't mix. So we are interested in

$$\langle 0|T\prod_{r=1}^{n_\alpha}e^{2i\pi^{1/2}\eta_{\alpha j}\mathcal{E}_j(x_r)}\,e^{-2i\pi^{1/2}\eta_{\alpha j}\mathcal{E}_j(y_r)}|0\rangle \tag{1.27}$$

We have put arguments (x_r) and (y_r) on the $\mathcal{E}_j$ from $O_{\alpha+}(x_r)$ and $O_{\alpha-}(y_r)$ for notational convenience, but the $\mathcal{E}_j$ are constant operators so the Wick expansion of (1.27) is independent of the coordinates. The contribution from contractions of the $\mathcal{E}_j(x_r)$s with the $\mathcal{E}_j(y_r)$s is

$$\left(\frac{m_j^2}{m^2}\right)^{n_\alpha^2\eta_{\alpha j}^2} \tag{1.28}$$

The contribution from contractions of the $\mathcal{E}_j(x_r)$s with $\mathcal{E}_j(x'_r)$s is

$$\left(\frac{m_j^2}{m^2}\right)^{-(n_\alpha^2-n_\alpha)\eta_{\alpha j}^2/2} \tag{1.29}$$

[4]Note that we will soon use (1.23) to calculate other matrix elements, but because these more general results follow from the perturbative calculation and cluster decomposition, the simplification will work in general.

as is the contribution from contractions of the $\mathcal{E}_j\,(y_r)$s with $\mathcal{E}_j\,(y_r')$s. So the total contribution is

$$\left(\frac{m_j^2}{m^2}\right)^{n_\alpha \eta_{\alpha j}^2} \tag{1.30}$$

This means that we can associate a factor

$$N_\alpha = \prod_{j\in b}\left(\frac{m_j}{m}\right)^{\eta_{\alpha j}^2} \tag{1.31}$$

with each operator in each $O_{\alpha+}$-$O_{\alpha-}$ pair and completely capture the effects of the $\mathcal{E}_j$s for perturbatively non-zero correlators.

Now we can eliminate the $\mathcal{E}_j$s and write

$$\begin{aligned} O_{\alpha\pm} = &\frac{\xi m}{2\pi}\, e^{2\pm i\pi^{1/2}\sum_{j\in b}\eta_{\alpha j}(\mathcal{E}_j-\mathcal{B}_j)}\, e^{2\mp i\pi^{1/2}\sum_{j\in b'}\eta_{\alpha j}\mathcal{D}_j} \to \\ &\frac{\xi m}{2\pi}\, N_\alpha\, e^{-2\pm i\pi^{1/2}\left(\sum_{j\in b}\eta_{\alpha j}\mathcal{B}_j+\sum_{j\in b'}\eta_{\alpha j}\mathcal{D}_j\right)} \end{aligned} \tag{1.32}$$

We can use (1.32) to calculate any correlator that is non-zero in perturbation theory. A particularly nice feature of (1.32) is the way it encodes the diagonal chiral symmetries. The theory is invariant under global translations of the massless $\mathcal{D}_j$ fields for $j \in b'$. These translations generate the $n_F - n_A$ unbroken chiral symmetries on the fermion bilinears.

$$O_{\alpha\pm} \text{ has } j \text{ chiral charge } q_j = \pm\eta_{\alpha j} \text{ for } \delta\mathcal{D}_j = \theta_j \text{ with } j \in b'. \tag{1.33}$$

There are no unbroken chiral transformations from translations of $\mathcal{D}_j$ for $j \in b$, because these $\mathcal{D}_j$s have been eaten by the ghosts. The corresponding chiral transformation from translations of the $\mathcal{B}_j$ for $j \in b$ are softly broken by the $\mathcal{B}_j$ mass terms generated by the gauge anomalies.

2 Cluster decomposition

While we only calculate to all orders in perturbation theory, in some situations, cluster decomposition gives us nonperturbative information. For any combination of n_o fermion bilinears in some region of space-time, we can look at a correlator that also includes all the conjugate fields in a region far away in space as in

$$\left\langle 0 \left| T \prod_{u=1}^{n_o} O_{\alpha_u s_u}(x_u)\, O^*_{\alpha_u s_u}(z+y_u) \right| 0 \right\rangle \tag{2.1}$$

where the s_u are ± 1, as in (1.11) and the x_a and y_a are clustered in some region of size ℓ $(-(x_a - x_b)^2 < \ell^2)$ around the (arbitrary) origin and z is a large space-like 2-vector. This correlator is calculable in perturbation theory. Using (1.32), we can write it as

$$\left(\frac{\xi m}{2\pi}\right)^{2n_o}\left(\prod_u N_{\alpha_u}\right)^2 (X\,Y\,Z) \tag{2.2}$$

where

$$
\begin{aligned}
X &= \prod_{u\neq u'} \exp\left[-2s_u s_{u'} \sum_j \eta_{\alpha_u j}\eta_{\alpha_{u'} j}\Delta_j(x_u - x_{u'})\right] \\
Y &= \prod_{u\neq u'} \exp\left[-2s_u s_{u'} \sum_j \eta_{\alpha_u j}\eta_{\alpha_{u'} j}\Delta_j(y_u - y_{u'})\right] \\
Z &= \prod_{u,u'} \exp\left[2s_u s_{u'} \sum_j \eta_{\alpha_u j}\eta_{\alpha_{u'} j}\Delta_j(z + y_{u'} - x_u)\right]
\end{aligned}
\tag{2.3}
$$

where

$$
\Delta_j(x) = \begin{cases} K_0\left(m_j\sqrt{-x^2+i\epsilon}\right) & \text{for } j \in b \\ -\log\left(m\sqrt{-x^2+i\epsilon}\right) & \text{for } j \in b' \end{cases}
\tag{2.4}
$$

We can now study the correlator as $-z^2$ goes to infinity.

If $b' \neq \emptyset$, that is for $n_F > n_A$ the correlator will in general go to zero, with the Z factor falling off like

$$
\frac{1}{\left(m\sqrt{-x^2+i\epsilon}\right)^{2\sum_{j\in b'} Q_j^2}}
\tag{2.5}
$$

where Q_j is the total j chiral charge of the operators in the X region,

$$
Q_j = \sum_u s_u \eta_{\alpha_u j}
\tag{2.6}
$$

Thus $\sum_{j\in b'} Q_j^2$ is the anomalous dimension of the leading operators in the operator product expansion of the set $\{O_{\alpha_u s_u}(x_u)\}$ and I will refer to this as the anomalous dimension of the set and we will have more to say about it below.

But we will begin by assuming that $n_A = n_F$ so there are no massless fields ($b' = \emptyset$) and

$$
\Delta_j(x) = K_0\left(m_j\sqrt{-x^2+i\epsilon}\right) \ \forall\, j
\tag{2.7}
$$

Then for sufficiently large z, all the Bessel functions go to 0 and $Z \to 1$. If the correlator is non-zero as the regions move infinitely far apart, it must factor into a product of contributions in the two regions.

$$
\left\langle 0 \left| T \prod_{u=1}^{n_o} O_{\alpha_u s_u}(x_u)\, O^*_{\alpha_u s_u}(z+y_u) \right| 0 \right\rangle \underset{-z^2\to\infty}{\longrightarrow} \left(\frac{\xi m}{2\pi}\right)^{2n_o} \left(\prod_u N_{\alpha_u}\right)^2 X\,Y
\tag{2.8}
$$

This gives non-perturbative information about anomalies. Conversely, if the correlators in the two regions are forbidden by unbroken symmetries, the correlator (2.1) must vanish as $z \to \infty$.

The above argument means that for $n_A = n_F$, up to phases [4–6] that are arbitrary and that we will set to zero, we can use (2.1)–(2.4) to calculate any correlator using (1.32) whether or not it is non-zero in perturbation theory and the result is

$$
\begin{aligned}
\left\langle 0 \left| T \prod_{u=1}^{n_o} O_{\alpha_u s_u}(x_u) \right| 0 \right\rangle = \left(\frac{\xi m}{2\pi}\right)^{n_o} \left(\prod_u N_{\alpha_u}\right) \\
\prod_{u\neq u'} \exp\left[-2s_u s_{u'} \sum_j \eta_{\alpha_u j}\eta_{\alpha_{u'} j} K_0\left(m_j\sqrt{-(x_u - x_{u'})^2+i\epsilon}\right)\right]
\end{aligned}
\tag{2.9}
$$

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This makes sense because when $b' = \emptyset$ with no $\mathcal{D}$ fields, there are no global chiral symmetries that are not broken by the gauge boson mass terms. So there are no constraints on the use of (2.1)–(2.4). Because the bosonization mass m is arbitrary, it will cancel explicitly in (2.9), so we can also write (2.9) entirely in terms of the m_js as

$$\left\langle 0 \left| T \prod_{u=1}^{n_o} O_{\alpha_u s_u}(x_u) \right| 0 \right\rangle = \left(\frac{\xi}{2\pi}\right)^{n_o} \left(\prod_u M_{\alpha_u}\right) \prod_{u\neq u'} \exp\left[-2 s_u s_{u'} \sum_j \eta_{\alpha_u j}\eta_{\alpha_{u'} j} K_0\left(m_j\sqrt{-(x_u-x_{u'})^2+i\epsilon}\right)\right] \tag{2.10}$$

where

$$M_\alpha = \prod_{j\in b} m_j^{\eta^2_{\alpha j}} \tag{2.11}$$

Equations (1.31), (2.9), (2.11), and (2.10) are the basic technical results of this note. Eqs. (1.31), (2.9) and (2.11), (2.10) are equivalent, but (1.31), (2.9) is more convenient for taking m_js to zero which we will do in the next section. The m_j dependence in (1.31) and (2.11) is useful in understanding the structure of the general theory.

Now we can use (1.31), (2.9) to analyze the general theory for $b' \neq \emptyset$. While we could do the general analysis directly, constructing the general theory as the limit of (2.9) as some gauge couplings go to zero automatically includes the nonperturbative effects of the gauge anomalies and will give us insights into their structure. It may seem surprising that this works at all because the corresponding limit in 3+1 dimensions is quite dangerous. But here everything goes smoothly because we have properly included the important dimensional parameters in (1.31).

In particular, look at the limit $m_j \to 0$ for some j in (2.9). Because of the product structure of the correlator, we can focus just on the factors that depend on j:

$$\left(\prod_u \left(\frac{m_j}{m}\right)^{\eta^2_{\alpha_u j}}\right) \prod_{u\neq u'} \exp\left[-2 s_u s_{u'} \eta_{\alpha_u j}\eta_{\alpha_{u'} j} K_0\left(m_j\sqrt{-(x_u-x_{u'})^2+i\epsilon}\right)\right] \tag{2.12}$$

If $m_j\sqrt{-(x_u-x_{u'})^2}$ is much less than 1 for all the coordinate pairs, we can approximate the Bessel functions by logs:

$$\left(\prod_u \left(\frac{m_j}{m}\right)^{\eta^2_{\alpha_u j}}\right) \prod_{u\neq u'} \exp\left[2 s_u s_{u'} \eta_{\alpha_u j}\eta_{\alpha_{u'} j} \log\left(m_j\sqrt{-(x_u-x_{u'})^2+i\epsilon}\right)\right] \tag{2.13}$$

This can be written

$$\left(\frac{m_j}{m}\right)^{\left(\sum_u \eta^2_{\alpha_u j}\right)+\left(\sum_{u\neq u'} 2 s_u s_{u'}\eta_{\alpha_u j}\eta_{\alpha_{u'} j}\right)} \prod_{u\neq u'} \exp\left[2 s_u s_{u'} \eta_{\alpha_u j}\eta_{\alpha_{u'} j} \log\left(m\sqrt{-(x_u-x_{u'})^2+i\epsilon}\right)\right] \tag{2.14}$$

$$= \left(\frac{m_j}{m}\right)^{\left(\sum_u s_u\eta_{\alpha_u j}\right)^2} \prod_{u\neq u'} \exp\left[2 s_u s_{u'} \eta_{\alpha_u j}\eta_{\alpha_{u'} j} \log\left(m\sqrt{-(x_u-x_{u'})^2+i\epsilon}\right)\right] \tag{2.15}$$

Now we can take the limit $m_j \to 0$. The limit vanishes unless

$$Q_j = \sum_u s_u \eta_{\alpha_u j} = 0 \tag{2.16}$$

This is expected from (1.33). In (2.16), Q_j is the total j chiral charge of the product of operators in the correlator. The limit is just the statement that the correlator vanishes if the product of operators carries an unbroken chiral symmetry. But (2.15) shows us exactly how the limit is approached.

If Q_j vanishes, the first factor in (2.15) is 1, the m_j dependence disappears, and the bosonization mass m shows up in the second factor in the expected way for a bosonization field. Notice that it was critical here to include the $N(\alpha)$ factor.

We can now find the correlators for a general model by starting with all gauge couplings non-zero and then taking $g_j \to 0$ for $j \in b'$, so in general the collections of n_o operators that have non-zero correlators will satisfy

$$Q_j = \sum_u s_u \eta_{\alpha_u j} = 0 \quad \forall\, j \in b' \tag{2.17}$$

The correlators are then simply

$$\begin{aligned} \left\langle 0 \left| T \prod_{u=1}^{n_o} O_{\alpha_u s_u}(x_u) \right| 0 \right\rangle = \left(\frac{\xi m}{2\pi}\right)^{n_o} \left(\prod_u N_{\alpha_u}\right) \\ \prod_{u \neq u'} \exp\left[-2 s_u s_{u'} \sum_j \eta_{\alpha_u j} \eta_{\alpha_{u'} j} \Delta_j\left(x_u - x_{u'}\right)\right] \end{aligned} \tag{2.18}$$

where as in (2.4)

$$\Delta_j(x) = \begin{cases} K_0\left(m_j \sqrt{-x^2 + i\epsilon}\,\right) & \text{for } j \in b \\ -\log\left(m\sqrt{-x^2 + i\epsilon}\,\right) & \text{for } j \in b' \end{cases} \tag{2.19}$$

3 Anomalies

Obviously, as $m_j \propto g_j \to 0$, the gauge anomalies associated with A_j^μ disappear. But the requirement of (2.17) puts interesting constraints on the form of the remaining anomalies for $m_j = 0$. In general a set of operators $O_{\alpha_u s_u}$ satisfying (2.17) can be decomposed into irreducible sets which cannot be split into subsets satisfying (2.17). These always include any single operators $\{O_{\alpha+}\}$ and $\{O_{\alpha-}\}$ for which $\eta_{\alpha j} = 0\, \forall j \in b'$ and the $\pm$ pairs, $\{O_{\alpha+}, O_{\alpha-}\}$ for each α for which $\eta_{\alpha j} \neq 0$ for any $j \in b'$. I will refer to any other irreducible sets as "anomaly sets" because they are related to the remaining gauge anomalies. They are characterized by a set of integers n_α:

$$S(n_\alpha) \text{ consists of } |n_\alpha| \text{ copies of } O_{\alpha\, \mathrm{sgn}(n_\alpha)} \text{ for each } \alpha \tag{3.1}$$

In terms of the n_α, (2.17) is

$$\sum_\alpha |n_\alpha|\, \mathrm{sgn}(n_\alpha) \eta_{\alpha j} = \sum_\alpha n_\alpha \eta_{\alpha j} = 0 \quad \forall\, j \in b' \tag{3.2}$$

Then (1.15) implies that n_α is a linear combination of $\eta_{\alpha j}$ for $j \in b$,

$$n_\alpha = \sum_{j\in b} \mathcal{N}_j \eta_{\alpha j} \tag{3.3}$$

for some $\mathcal{N}_j$. It makes sense that the irreducible anomaly sets depend only on the remaining gauge couplings.

4 Examples

If there is only one gauge boson, there is only one $\eta_{\alpha j}$ for $j \in b$ — call it $\eta_{\alpha 1}$. Then (3.3) is

$$n_\alpha \propto \eta_{\alpha 1} \tag{4.1}$$

which fixes the irreducible anomaly sets (if they exist) up to a sign (which is just associated with complex conjugation of all the bilinears). But n_α exists if an only if the components of $\eta_{\alpha 1}$ are commensurate.

Here is the 2-flavor Schwinger model with $b = \{1\}$:

$$\eta_{\alpha 1} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad \eta_{\alpha 2} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad n_\alpha = \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix} \tag{4.2}$$

The irreducible anomaly sets are $\{O_{1\pm}, O_{2\pm}\}$.

Here is a similar model again with $b = \{1\}$ but gauge couplings that differ by a factor of 2.

$$\eta_{\alpha 1} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \quad \eta_{\alpha 2} = \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \quad n_\alpha = \begin{pmatrix} \pm 2 \\ \pm 1 \end{pmatrix} \tag{4.3}$$

The irreducible anomaly sets are $\{2 \times O_{1\pm}, O_{2\pm}\}$.

Again with $b = \{1\}$

$$\eta_{\alpha 1} = \begin{pmatrix} n_1/\sqrt{n_1^2+n_2^2} \\ n_2/\sqrt{n_1^2+n_2^2} \end{pmatrix} \quad \eta_{\alpha 2} = \begin{pmatrix} -n_2/\sqrt{n_1^2+n_2^2} \\ n_1/\sqrt{n_1^2+n_2^2} \end{pmatrix} \tag{4.4}$$

Now if n_1 and n_2 are relatively prime,

$$n_\alpha = \begin{pmatrix} \pm n_1 \\ \pm n_2 \end{pmatrix} \tag{4.5}$$

and the irreducible anomaly sets are $\{n_1 \times O_{1\pm}, n_2 \times O_{2\pm}\}$. If n_1 and n_2 have a common factor n_c the irreducible anomaly sets are $\{(n_1/n_c) \times O_{1\pm}, (n_2/n_c) \times O_{2\pm}\}$.

Finally for $b = \{1\}$, if the gauge couplings are not commensurate — as in.

$$\eta_{\alpha 1} = \begin{pmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{pmatrix} \quad \eta_{\alpha 2} = \begin{pmatrix} -\sqrt{1/3} \\ \sqrt{2/3} \end{pmatrix} \tag{4.6}$$

there are no anomaly sets. [7]

If there is more than gauge boson, more complicated scenarios are possible. If the components of $\eta_{\alpha j}$ are commensurate for each $j \in b$, there is an independent pair for anomaly sets for each $j \in b$ and they may be reducible. For example, with two gauge bosons, $b = 1, 2$, we could have a "diagonal color" model: [8–12]

$$\eta_{\alpha 1} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \quad \eta_{\alpha 2} = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix} \quad \eta_{\alpha 3} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \tag{4.7}$$

The irreducible anomaly sets are $\{O_{1\pm}, O_{2\mp}\}$, $\{O_{2\pm}, O_{3\mp}\}$, and $\{O_{3\pm}, O_{1\mp}\}$.

But incommensurate charges can further eliminate all the anomaly sets as in (4.6) or just limit them, as in this example (again with $b = 1, 2$):

$$\eta_{\alpha 1} = \begin{pmatrix} 1/2 \\ \left(2+\sqrt{2}\right)/4 \\ \left(2-\sqrt{2}\right)/4 \end{pmatrix} \quad \eta_{\alpha 2} = \begin{pmatrix} -1/2 \\ \left(2-\sqrt{2}\right)/4 \\ \left(2+\sqrt{2}\right)/4 \end{pmatrix} \quad \eta_{\alpha 3} = \begin{pmatrix} 1/\sqrt{2} \\ -1/2 \\ 1/2 \end{pmatrix} \tag{4.8}$$

Here because the only linear combination of $\eta_{\alpha 1}$ and $\eta_{\alpha 2}$ with commensurate components is $\eta_{\alpha 1} + \eta_{\alpha 2}$, the only irreducible anomaly sets are $\{O_{2\pm}, O_{3\pm}\}$.

One might ask[5] whether commensurate components of the $\eta_{\alpha j}$ vectors is associated with quantization of charge. That is obviously the case for $n_A = 1$, as shown in examples (4.2), (4.3), (4.6). For $n_A > 1$, the situation is more complicated because generically, the mass-eigenstates of the gauge fields will be linear combinations of the fields in the original Lagrangian. But the $\eta_{\alpha j}$ for $j \in b'$ are orthogonal to ALL of the $\eta_{\alpha j}$ for $j \in b$ and thus to any linear combination. So if there is any linear combination of the $\eta_{\alpha j}$ for $j \in b$ with commensurate coefficients, it will be associated with an anomaly set AND the corresponding combination of gauge fields will be coupled to a quantized charge. For example, in (4.8), the combination $\eta_{\alpha 1} + \eta_{\alpha 2}$ describes a coupling to a quantized charge.

In general, the independent anomaly sets will be associated with the independent linear combinations coupled to quantized charges. This is exactly what we would expect from Coleman's interpretation of a θ parameter as a background electric field [13]. Only the linear combinations corresponding to quantized charges will have θ parameters because a background field coupled to a non-quantized charge can be canceled by pair production of massless fermions. Thus the independent anomaly sets will be associated with the independent θ parameters. This connection can be obscured in models with more than one gauge boson by a nontrivial gauge boson mass matrix, but we have shown how a complete analysis of the mass dependence describes the general case.

5 Conclusions

While the physics is trivial, the correlators in generalizations of the massless Schwinger model depend in nontrivial ways on the gauge boson masses. I believe that the complete analysis of the mass dependence in this note clarifies the relationship between models with different numbers of gauge bosons and chiral anomalies.

[5] As a clever referee did.

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