The classical Yang–Baxter equation and the associated Yangian symmetry of gauged WZW-type theories

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Dedicated to the memory of Sotirios Bonanos whose MATHEMATICA software has helped numerous researchers

Abstract

We construct the Lax-pair, the classical monodromy matrix and the corresponding solution of the Yang–Baxter equation, for a two-parameter deformation of the Principal chiral model for a simple group. This deformation includes as a one-parameter subset, a class of integrable gauged WZW-type theories interpolating between the WZW model and the non-Abelian T-dual of the principal chiral model. We derive in full detail the Yangian algebra using two independent methods: by computing the algebra of the non-local charges and alternatively through an expansion of the Maillet brackets for the monodromy matrix. As a byproduct, we also provide a detailed general proof of the Serre relations for the Yangian symmetry.
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1. Introduction and motivation

A class of σ-models was recently constructed via a gauging procedure involving the WZW action and the general Principal Chiral model (PCM) action for a group G [1]. The end result is the action

$$S_{k,\lambda}(g) = S_{\mathrm{WZW},k}(g) + \frac{k}{\pi} \int J_+^a (\lambda^{-1} - D^T)_{ab}^{-1} J_-^b, \tag{1.1}$$

where $S_{\mathrm{WZW},k}(g)$ is the WZW action at level k of a group element $g \in G$ and λ is a general $\dim(\mathbf{g})$ square real matrix. In addition, we have employed the standard definitions

$$J_+^a = \mathrm{Tr}(T_a \partial_+ g g^{-1}), \qquad J_-^a = \mathrm{Tr}(T_a g^{-1} \partial_- g), \qquad D_{ab} = \mathrm{Tr}(T_a g T_b g^{-1}), \tag{1.2}$$

with T_a, $a = 1, 2, \ldots, \dim(\mathbf{g})$ being the generators of the Lie algebra $\mathbf{g}$ satisfying the commutation rules, normalization and Killing form

$$[T_a, T_b] = f_{abc} T_c, \qquad \mathrm{Tr}(T_a T_b) = \delta_{ab}, \qquad K_b^a = \delta_b^a.$$

The key property of this action arises when λ is proportional to the identity, i.e. $\lambda_{ab} = \lambda \delta_{ab}$, since then it becomes integrable. This was shown in [1] by explicitly demonstrating that the current components $I_\pm = I_\pm^a T_a$ obey the standard integrability conditions

$$\partial_+ I_- + \partial_- I_+ = 0, \qquad \partial_+ I_- - \partial_- I_+ + [I_+, I_-] = 0. \tag{1.3}$$

The explicit realization in terms of the σ-model action variables is

$$H = \frac{1}{4e^2} \int\limits_{-\infty}^{+\infty} d\sigma \left(I_+^a I_+^a + I_-^a I_-^a \right),$$

$$I_+^a = \frac{2\lambda}{1+\lambda} (\mathbb{I} - \lambda D)_{ab}^{-1} J_+^b, \qquad I_-^a = -\frac{2\lambda}{1+\lambda} (\mathbb{I} - \lambda D^T)_{ab}^{-1} J_-^b, \tag{1.4}$$

where we have also included the expression for the Hamiltonian corresponding to (1.1). The general proof was done in [1] by explicitly demonstrating that certain integrability algebraic constraints provided in [2,3] were satisfied. A simpler way to prove the integrability of (1.1) has been given more recently in [4] by utilizing the fact that the construction involves, as mentioned, a gauging procedure reminiscent of the gauged WZW models.

As discussed in detail in [1] a motivation for studying this action relates to the global properties of the variables in σ-models arising via non-Abelian T-duality. The latter generalizes, in a certain sense, Abelian T-duality [5] and was initiated by [6–8]. It is easily seen that when the elements $\lambda_{ab} \to 0$ then (1.1) becomes the WZW $S_{\mathrm{WZW},k}(g)$. Also recall [1] that when λ approaches the identity matrix, $k \to \infty$ and $g \in G$ is appropriately expanded around the identity group element, then (1.1) becomes the non-Abelian T-dual for the general PCM.[1] Hence (1.1) interpolates between these two extreme cases and a way of thinking to the non-Abelian T-duality of the PCM is as a limiting case of (1.1). In the latter action the group element $g \in G$ is parametrized by compact variables. Hence the non-compactness displayed by the variables in the non-Abelian model is attributed to the zooming-limiting procedure we mentioned.

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[1] For the isotropic case see the derivation in [9,10] and for the general anisotropic case in [11]. Recent developments in non-Abelian T-duality in the presence of RR flux fields initiated with the work in [12]. For relations to the AdS/CFT correspondence, a discussion of global issues and further developments and references see [13–15].

The perturbation away from the WZW point is driven by the term $\lambda_{ab} J_+^a J_-^b$ which for generic λ_{ab} preserves no isometries (enhanced to $G_L \times G_R$ when $\lambda_{ab} = \lambda \delta_{ab}$). Based on that, on the matching of global symmetries and on the result of the computation of the renormalization group flow equations for the matrix λ in [16,17], one concludes that (1.1) provides the effective action for the bosonized anisotropic non-Abelian Thirring model valid to all orders in λ and to leading order in the $1/k$ expansion. In the same papers the following remarkable symmetry was also noticed

$$S_{-k,\lambda^{-1}}\big(g^{-1}\big) = S_{k,\lambda}(g), \tag{1.5}$$

which in fact mathematically dictates the form of all the aforementioned properties.

In this paper we will further investigate the integrable structure of a two-parameter deformation of the PCM, which includes as a one-parameter subset (1.1) for the prototypical isotropic case $\lambda_{ab} = \lambda \delta_{ab}$. In particular, based on the underlying algebraic structure, we will show the existence of a Yangian algebra [18] (for reviews see [19–21]) of classically conserved non-local charges in the spirit of a similar computation for the (generalized) Gross–Neveu and the isotropic PCM in [22]. In the isotropic case, the Yangian algebra corresponds to the adjoint action on g, i.e. $g \mapsto \Lambda_0^{-1} g \Lambda_0$, $\Lambda_0 \in G$. In addition, we will provide the Lax pair and we will compute the Poisson brackets of its spatial part which take the Maillet form [23,24]. This will provide an array of coefficients which, as required for consistency, solve a classical modified Yang–Baxter equation. This allows for the derivation of the Maillet brackets of the monodromy matrix [23,24]. An expansion of these brackets will provide an alternative derivation of the Yangian algebra.

This work is organized as follows: In Section 2 we review the derivation of the Lax pair and the corresponding (classical) monodromy matrix for a general class of two-dimensional systems. In Section 3 we compute the Maillet brackets of the spatial part of the Lax pair. Using this, we derive a class of solutions of the modified classical Yang–Baxter equation and the Maillet brackets of the monodromy matrix. In Section 4 we explore the realization of the Yangian algebra through the charge algebra and through an expansion of the Maillet brackets of the monodromy matrix. Details of the derivation are given in Appendices B–D respectively. In Section 5 we conclude with a discussion on possible future directions. Besides Appendices B–D we also include Appendix A where we revisit the proof of Drinfeld's relations.

2. Lax pair and the classical monodromy matrix

The purview of this section is to construct the Lax pair and the monodromy matrix of a class of integrable σ-models which were constructed in [1] and reviewed in Section 1. We will provide a rather general discussion by assuming that the equations of motion and the flat connection identities are given by[2]

$$(1 + \rho)\partial_+ I_- + (1 - \rho)\partial_- I_+ = 0, \quad \partial_+ I_- - \partial_- I_+ + [I_+, I_-] = 0, \tag{2.1}$$

---

[2] The world-sheet coordinates (σ^+, σ^-) and (τ, σ) are related by

$$\sigma^\pm := \tau \pm \sigma, \qquad \partial_0 := \partial_\tau = \partial_+ + \partial_-, \qquad \partial_1 := \partial_\sigma = \partial_+ - \partial_-,$$

so that $\star d\sigma^\pm = \pm d\sigma^\pm$ and $\star d\tau = d\sigma$, $\star d\sigma = d\tau$ in Lorentzian signature.

where I is a Lie algebra valued one-form

$$I = I^a T_a, \qquad I^a = I^a_+ \mathrm{d}\sigma^+ + I^a_- \mathrm{d}\sigma^-, \tag{2.2}$$

which for $\rho = 0$ describe the integrable (isotropic) σ-models reviewed in Section 1, whereas for $\rho \neq 0$ the form of the action is not known. We can rewrite (2.1) in a differential-form notation as

$$\mathcal{I} := I - \rho \star I, \qquad \mathrm{d}(\star\mathcal{I}) = 0, \qquad \mathrm{d}\mathcal{I} + \mathcal{I} \wedge \mathcal{I} = 0, \tag{2.3}$$

which makes manifest the classical integrability even when $\rho \neq 0$. Note that, even though the redefinition (2.3) has made the parameter ρ disappear from the integrability conditions, it may very well be present in the Poisson brackets for $\mathcal{I}^a_\pm$.

Using the above and assuming fields vanish at spatial infinity, we can construct the first two conserved charges [25]

$$Q_0 := \int\limits_{-\infty}^{+\infty} \mathrm{d}\sigma\, \mathcal{I}_0(\sigma),$$

$$\widehat{Q} := \int\limits_{-\infty}^{+\infty} \mathrm{d}\sigma\, \mathcal{I}_1(\sigma) + \int\limits_{-\infty}^{+\infty} \mathrm{d}\sigma\, \mathcal{I}_0(\sigma) \int\limits_{-\infty}^{\sigma} \mathrm{d}\sigma'\, \mathcal{I}_0(\sigma'), \tag{2.4}$$

and another (still conserved and, as we will see, particularly convenient) combination of them

$$Q_1 := \widehat{Q} - \frac{1}{2} Q_0^2$$

$$= \int\limits_{-\infty}^{+\infty} \mathrm{d}\sigma\, \mathcal{I}_1(\sigma) + \frac{1}{2} \int\limits_{-\infty}^{+\infty} \mathrm{d}\sigma \int\limits_{-\infty}^{\sigma} \mathrm{d}\sigma'\, \big[\mathcal{I}_0(\sigma), \mathcal{I}_0(\sigma')\big]. \tag{2.5}$$

In the above formula, we have rewritten the second term – corresponding to Q_0^2 – by splitting the double integral in the two domains $\sigma > \sigma'$ and $\sigma' > \sigma$, and changed variables $\sigma \leftrightarrow \sigma'$ in one of the pieces.

It is well known that the (infinite number) of conserved charges can be methodically constructed from the *Lax pair*

$$\partial_0 L_1 - \partial_1 L_0 = [L_0, L_1] \quad \text{or} \quad \mathrm{d}L = L \wedge L. \tag{2.6}$$

Using the latter we can show that the *monodromy matrix* (see, for instance, [26])[3]

$$M(v) := P \exp \int\limits_{-\infty}^{\infty} \mathrm{d}\sigma\, L_1(\sigma; v) \tag{2.7}$$

is conserved for all values of the complex *spectral parameter* v, namely $\partial_0 M(v) = 0$.

---

[3] The path ordered exponential reads

$$P \exp \int\limits_{-\infty}^{+\infty} \mathrm{d}\sigma f(\sigma) := 1 + \int\limits_{-\infty}^{+\infty} \mathrm{d}\sigma f(\sigma) + \int\limits_{-\infty}^{+\infty} \mathrm{d}\sigma \int\limits_{-\infty}^{\sigma} \mathrm{d}\sigma' f(\sigma) f(\sigma') + \cdots.$$

In our case the *Lax pair* reads

$$L_0(\sigma; \nu) = \frac{(\mathcal{I}_1 + \tilde{\nu}\mathcal{I}_0)\tilde{\nu}}{1 - \tilde{\nu}^2}, \qquad L_1(\sigma; \nu) = \frac{(\mathcal{I}_0 + \tilde{\nu}\mathcal{I}_1)\tilde{\nu}}{1 - \tilde{\nu}^2}, \quad \tilde{\nu} = \frac{\nu + \rho}{1 + \nu\rho}, \tag{2.8}$$

or, equivalently, $L_\pm(\sigma; \nu) = -\frac{\tilde{\nu}}{\tilde{\nu}\mp 1}\mathcal{I}_\pm$. By expanding M in powers of $\nu' := \nu + \rho$, we find an infinite set of classically conserved charges:

$$M(\nu') = 1 + \frac{\nu'}{1 - \rho^2}Q_0 + \frac{\nu'^2}{(1 - \rho^2)^2}(\widehat{Q} - \rho Q_0) + \mathcal{O}(\nu'^3). \tag{2.9}$$

One recognizes combinations of the charges (2.4) in the coefficients of the expansion.

3. The Maillet brackets and the Yang–Baxter equation

In this section we prove that when the above quantities are supplied by an appropriate algebraic structure, this allows to find explicit solutions to a modified classical Yang–Baxter equation. Consequently, the monodromy matrix obeys the associated Maillet brackets.

Following Sklyanin [27], we compute the Poisson brackets by first writing $L_1 = L_1^a T_a$ and then[4]

$$\big\{L_1^{(1)}(\sigma_1; \mu), L_1^{(2)}(\sigma_2; \nu)\big\} = \big\{L_1^a(\sigma_1; \mu), L_1^b(\sigma_2; \nu)\big\}T_a \otimes T_b. \tag{3.1}$$

The Poisson brackets assume the Maillet-type form [24]

$$\big([r_{-\mu\nu}, L_1^{(1)}(\sigma_1; \mu)] + [r_{+\mu\nu}, L_1^{(2)}(\sigma_1; \nu)]\big)\delta_{12} + \delta'_{12}(r_{-\mu\nu} - r_{+\mu\nu}), \tag{3.2}$$

where $r_{\pm\mu\nu}$ (as a shorthand notation for $r_\pm(\mu, \nu)$) are matrices in the basis $T_a \otimes T_b$. This is guaranteed to give a consistent Poisson structure, provided the Jacobi identities for these brackets are obeyed. This enforces $r_{\pm\mu\nu}$ to satisfy the modified classical Yang–Baxter relation

$$\big[r_{+\nu_1\nu_3}^{(13)}, r_{-\nu_1\nu_2}^{(12)}\big] + \big[r_{+\nu_2\nu_3}^{(23)}, r_{+\nu_1\nu_2}^{(12)}\big] + \big[r_{+\nu_2\nu_3}^{(23)}, r_{+\nu_1\nu_3}^{(13)}\big] = 0. \tag{3.3}$$

The non-vanishing coefficient of the δ' term in (3.2) is responsible for the above modification of the classical Yang–Baxter relation. Using (3.2), one can derive the Poisson brackets for the (classical) monodromy matrix [23]

$$\begin{aligned}\big\{M^{(1)}(\mu), M^{(2)}(\nu)\big\} = {}& \big[r_{\mu\nu}, M(\mu) \otimes M(\nu)\big] - M^{(2)}(\nu)s_{\mu\nu}M^{(1)}(\mu) \\ &+ M^{(1)}(\mu)s_{\mu\nu}M^{(2)}(\nu),\end{aligned} \tag{3.4}$$

which is consistent with the Jacobi identity, if we define the equal-point limits of the Poisson brackets through a generalized symmetric limit procedure [23].

Returning to the case at hand, it was pointed out in [28] that the Poisson structure of the isotropic PCM admits a one-parameter family of deformations (with parameter denoted by x).

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[4] The superscript in parenthesis stands for the notation of tensor products of spaces

$$M^{(1)} = M \otimes \mathbb{I}, \qquad M^{(2)} = \mathbb{I} \otimes M,$$

$$m^{(12)} = m_{ab}T_a \otimes T_b \otimes \mathbb{I}, \qquad m^{(13)} = m_{ab}T_a \otimes \mathbb{I} \otimes T_b, \qquad m^{(23)} = m_{ab}\mathbb{I} \otimes T_a \otimes T_b,$$

for an arbitrary matrix $m = m_{ab}T_a \otimes T_b$ in the tensor product algebra.

Subsequently, in [2] this deformation was further extended by introducing a second parameter ρ. In our conventions such two-parameter algebra reads[5]

$$
\begin{aligned}
\{\mathcal{I}_0^a, \mathcal{I}_0^b\} &= -2e^2 f_{abc}\big(1 + \rho^2 + (1 - \rho^2)x\big)\mathcal{I}_0^c \delta_{\sigma\sigma'} - 8e^2 \rho \delta_{ab}\delta_{\sigma\sigma'}', \\
\{\mathcal{I}_1^a, \mathcal{I}_1^b\} &= 2e^2 f_{abc}\big(4\rho\mathcal{I}_1^c + (1 + \rho^2 + x(\rho^2 - 1))\mathcal{I}_0^c\big)\delta_{\sigma\sigma'} - 8e^2 \rho \delta_{ab}\delta_{\sigma\sigma'}', \\
\{\mathcal{I}_0^a, \mathcal{I}_1^b\} &= \{\mathcal{I}_1^a, \mathcal{I}_0^b\} \\
&= -2e^2 f_{abc}\big(1 + \rho^2 + (1 - \rho^2)x\big)\mathcal{I}_1^c \delta_{\sigma\sigma'} + 4e^2\big(1 + \rho^2\big)\delta_{ab}\delta_{\sigma\sigma'}'.
\end{aligned}
\tag{3.5}
$$

When $\rho = 0$, the action (1.1) provides a realization of this algebra for a general group G [1] with $x = \frac{\lambda^2 + 1}{2\lambda}$ (for the $SU(2)$ case this realization was found by a brute force computation in [2]). There is no known action realizing the above algebra for $\rho \neq 0$.[6] Plugging (2.8) into (3.2) and using the algebra (3.5), we find that the matrix $r_{\pm\mu\nu}$ read

$$
r_{\pm\mu\nu} \mapsto r_{\pm\mu\nu}\Pi, \qquad \Pi := \sum_a T^a \otimes T^a,
$$

$$
r_{+\mu\nu} = 2e^2 \frac{(1 + \mu^2 + x(1 - \mu^2))(\mu + \rho)(\nu + \rho)}{(\nu - \mu)(1 - \mu^2)},
$$

$$
r_{-\mu\nu} = 2e^2 \frac{(1 + \nu^2 + x(1 - \nu^2))(\mu + \rho)(\nu + \rho)}{(\nu - \mu)(1 - \nu^2)} = -r_{+\nu\mu},
$$

and henceforth $r_{\pm\mu\nu}$ denotes the scalar. As for (3.3), it reduces to the single algebraic condition

$$
r_{+\nu_2\nu_3}r_{+\nu_1\nu_2} = r_{+\nu_1\nu_3}r_{-\nu_1\nu_2} + r_{+\nu_2\nu_3}r_{+\nu_1\nu_3},
\tag{3.6}
$$

extracted from the coefficient in front of the combination $f_{abc}T^a \otimes T^b \otimes T^c$. For completeness, we provide the values of $r_{\mu\nu}$ and $s_{\mu\nu}$, obtained by rewriting $r_{\pm\mu\nu} = r_{\mu\nu} \pm s_{\mu\nu}$:

$$
r_{\mu\nu} = -2e^2 \frac{(1 - \mu^2\nu^2 + x(1 - \mu^2)(1 - \nu^2))(\mu + \rho)(\nu + \rho)}{(\mu - \nu)(1 - \mu^2)(1 - \nu^2)},
$$

$$
s_{\mu\nu} = -2e^2 \frac{(\mu + \nu)(\mu + \rho)(\nu + \rho)}{(1 - \mu^2)(1 - \nu^2)},
\tag{3.7}
$$

which are generically non-vanishing.

---

[5] Alternatively, in light-cone coordinates the algebra reads

$$
\begin{aligned}
\{\mathcal{I}_\pm^a, \mathcal{I}_\pm^b\} &= e^2 f_{abc}\big[(1 \mp \rho)^2 \mathcal{I}_\mp^c - ((1 \mp \rho)^2 + 2x(1 - \rho^2))\mathcal{I}_\pm^c\big]\delta_{\sigma\sigma'} \pm 2e^2(1 \mp \rho)^2 \delta_{ab}\delta_{\sigma\sigma'}', \\
\{\mathcal{I}_\pm^a, \mathcal{I}_\mp^b\} &= -e^2 f_{abc}\big[(1 - \rho)^2 \mathcal{I}_-^c + (1 + \rho)^2 \mathcal{I}_+^c\big]\delta_{\sigma\sigma'}.
\end{aligned}
$$

[6] The algebra for the PCM (pseudo-PCM) corresponds to choosing the parameters $x = 1$ ($x = -1$) and $\rho = 0$. The value $x = 1$ corresponds to taking the parameter λ in this paper to unity. The fact that then the action (1.1) does not become that for a PCM but rather for its non-Abelian T-dual is consistent since non-Abelian T-duality can be cast as a canonical transformation in phase space [9–11]. We also note that for $\rho = 0$ and $e \to 0$, $x \to \infty$, we obtain under an appropriate rescaling of the currents the algebra for the (generalized) Gross–Neveu model when $e^2 x$ is finite, and for the conformal case when ex is finite.

4. The realization of the Yangian algebra

The scope of this section is to explicitly realize the Yangian algebra by performing the mutual Poisson commutators between Q_0^a and Q_1^a and, alternatively, via an expansion of the Maillet brackets for the conserved monodromy matrix M.

The Yangian algebra $Y_C(\mathbf{g})$ is an associative Hopf algebra generated by the elements J_a and Q_a obeying [18]

$$[J_a, J_b] = F_{abc} J_c, \qquad [J_a, Q_b] = [Q_a, J_b] = F_{abc} Q_c. \tag{4.1}$$

In addition, the request that the co-product map (which we call f to avoid conflicts of notations) on J_a and Q_a, namely

$$f(J_a) = J_a \otimes \mathbb{I} + \mathbb{I} \otimes J_a, \qquad f(Q_a) = Q_a \otimes \mathbb{I} + \mathbb{I} \otimes Q_a + \frac{\alpha}{2} F_{abc} J_b \otimes J_c, \quad \alpha \in \mathbb{C}, \tag{4.2}$$

acts as a homomorphism,[7] implies the Serre relations – see Appendices A.1 and A.2 for details. The first Serre relation reads

$$[Q_a, [Q_b, J_c]] - [J_a, [Q_b, Q_c]] = \frac{\alpha^2}{24} a_{abcdef} J_{(d} J_e J_{f)},$$
$$a_{abcdef} = F_{adk} F_{bel} F_{cfm} F_{klm}, \tag{4.3}$$

where $J_{(a} J_b J_{c)}$ denotes the sum of all permutations of $J_a J_b J_c$,[8] which for (classical) commuting quantities simplifies to $6 J_a J_b J_c$. The first Serre relation is trivially satisfied for the $su(2)$ case, as it turns out that $a_{abcdef} = \varepsilon_{abe} \varepsilon_{cfd} - \varepsilon_{dbe} \varepsilon_{cfa}$. Using the Jacobi identity on the second term of the l.h.s. of (4.3), and using the second of the relations (4.1), we easily find that (4.3) can be written as

$$F_{dab}[Q_c, Q_d] + F_{dca}[Q_b, Q_d] + F_{dbc}[Q_a, Q_d] = \frac{\alpha^2}{24} a_{abcdef} J_{(d} J_e J_{f)}, \tag{4.4}$$

a form which is particularly convenient for our purposes. In addition, the second Serre relation reads

$$F_{kcd}\big[[Q_a, Q_b], Q_k\big] + F_{kab}\big[[Q_c, Q_d], Q_k\big]$$
$$= \frac{\alpha^2}{24} (a_{abkgef} F_{kcd} + a_{cdkgef} F_{kab}) J_{(g} J_e Q_{f)}. \tag{4.5}$$

---

[7] A homomorphism is a structure-preserving map between two algebraic structures (such as groups)

$$f : A \mapsto B \quad \text{with } f(a_1 + a_2) = f(a_1) + f(a_2), \quad f(a_1 a_2) = f(a_1) f(a_2), \ \forall a_1, a_2 \in A.$$

[8] This sum explicitly expands as

$$J_{(a} J_b J_{c)} = J_a J_b J_c + J_c J_a J_b + J_b J_c J_a + J_a J_c J_b + J_b J_a J_c + J_c J_b J_a.$$

The first Serre implies the second one (for details see Appendix A.2), except for the $su(2)$ case where it reads

$$
\begin{aligned}
&\big[[Q_a, Q_b], [J_c, Q_d]\big] + \big[[Q_c, Q_d], [J_a, Q_b]\big] \\
&= \frac{\alpha^2}{24}\big(\varepsilon_{eab}(\delta_{fc}\delta_{gd} - \delta_{fd}\delta_{cg}) + \varepsilon_{ecd}(\delta_{fa}\delta_{gb} - \delta_{fb}\delta_{ag})\big)J_{(d}J_eQ_{f)}.
\end{aligned}
\tag{4.6}
$$

Hence, for the $su(2)$ case only this relation is non-trivial.

4.1. Yangian algebra through the algebra of charges

Next we work out the algebra of the classical charges Q_0 and Q_1 defined in (2.4) and (2.5). Their components read[9]

$$
Q_0^a = \int_{-\infty}^{+\infty} d\sigma\, \mathcal{I}_0^a(\sigma),
$$

$$
Q_1^a = \int_{-\infty}^{+\infty} d\sigma\, \mathcal{I}_1^a(\sigma) + \frac{1}{4} f_{abc} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^2\sigma_{12}\, \varepsilon_{12} \mathcal{I}_0^b(\sigma_1)\mathcal{I}_0^c(\sigma_2).
\tag{4.7}
$$

A comment is in order regarding the form of the charges. This should be in agreement with the co-product (4.2) and realized as half-positive and half-negative axis splitting (see [22] for details), upon the identifications

$$
J_a \mapsto Q_0^a, \qquad Q_a \mapsto Q_1^a, \qquad F_{abc} \mapsto \frac{1}{\alpha} f_{abc}.
\tag{4.8}
$$

Using (3.5) we compute the Poisson brackets for the zeroth level charges[10]

$$
\{Q_0^a, Q_0^b\} = -2e^2\big(1 + \rho^2 + x(1 - \rho^2)\big) f_{abc} Q_0^c.
\tag{4.9}
$$

Using the Jacobi identity we find that

---

[9] We use the definition

$$
\varepsilon_{12} = \varepsilon(\sigma_1 - \sigma_2) = \begin{cases} 1 & \text{if } \sigma_1 > \sigma_2 \\ -1 & \text{if } \sigma_1 < \sigma_2 \end{cases}, \quad \text{and} \quad \varepsilon'(x) = 2\delta(x).
$$

[10] To avoid ambiguities arising from the non-utralocal terms, like

$$
\int d\sigma_1 d\sigma_2 \partial_1 \delta_{12} \neq \int d\sigma_2 d\sigma_1 \partial_1 \delta_{12}
$$

we follow [25] and we define the Poisson bracket

$$
\{Q_{0,1}^a, Q_{0,1}^b\} = \lim_{L_2 \to \infty} \lim_{L_1 \to \infty} \{Q_{0,1}^{a,L_1}, Q_{0,1}^{b,L_2}\},
$$

where $Q_{0,1}^{a,L}$ are volume cutoff charges (the same as $Q_{0,1}^a$, with range from $-L$ to L).

$$\{Q_0^a, Q_1^b\} = 2\Delta f_{abc} \int\limits_{-\infty}^{+\infty} d\sigma \, \mathcal{I}_1^c(\sigma)$$

$$+ \Delta(f_{bef} f_{aed} + f_{bde} f_{aef}) \int\limits_{-\infty}^{+\infty} d\sigma_1 \, \mathcal{I}_0^d(\sigma_1) \int\limits_{-\infty}^{\sigma_1} d\sigma_2 \, \mathcal{I}_0^f(\sigma_2)$$

$$\implies \quad \{Q_0^a, Q_1^b\} = -2e^2(1 + \rho^2 + x(1 - \rho^2)) f_{abc} Q_1^c. \tag{4.10}$$

For notational convenience we define

$$\Delta := -e^2(1 + \rho^2 + x(1 - \rho^2)). \tag{4.11}$$

Finally, we can compute $\{Q_1^a, Q_1^b\}$ as follows:

$$Q_1^a := x^a + y^a, \qquad \{Q_1^a, Q_1^b\} = \{x^a, x^b\} + \{x^a, y^b\} - \{x^b, y^a\} + \{y^a, y^b\},$$

$$\{x^a, x^b\} + \{x^a, y^b\} - \{x^b, y^a\} = f_{abc} Q_2^{(1)c},$$

$$Q_2^{(1)a} = 2e^2(1 + \rho^2 + x(\rho^2 - 1)) Q_0^a + 8e^2 \rho \int d\sigma \mathcal{I}_1^a$$

$$+ \Delta f_{abc} \int d^2\sigma_{12} \varepsilon_{12} \mathcal{I}_0^b(\sigma_1) \mathcal{I}_1^c(\sigma_2), \tag{4.12}$$

where x^a and y^a correspond to the first and second term in the expression of Q_1^a, respectively. Furthermore, we can show that

$$\{y^a, y^b\} = 2\Delta f_{acd} f_{bre} f_{dr\ell} \times \frac{1}{4} I_{ce\ell},$$

$$I_{ce\ell} = \int d^3\sigma_{123} \varepsilon_{13} \varepsilon_{32} \mathcal{I}_0^c(\sigma_1) \mathcal{I}_0^e(\sigma_2) \mathcal{I}_0^\ell(\sigma_3) = I_{ec\ell}, \tag{4.13}$$

which can be further simplified with the use of (B.8), proven in Appendix B and reported here for convenience:

$$f_{acd} f_{bre} f_{dr\ell} I_{ce\ell} = -\frac{1}{3} f_{acd} f_{bre} f_{dr\ell} Q_0^c Q_0^e Q_0^\ell - \frac{1}{3} f_{abr} f_{rcd} f_{de\ell} I_{ce\ell}.$$

Putting all together, the Poisson brackets of two Q_1^a's read

$$\{Q_1^a, Q_1^b\} = f_{abc}(Q_2^{(1)c} + Q_2^{(2)c}) - 2\Delta \times f_{acd} f_{bre} f_{dr\ell} \times \frac{1}{12} Q_0^c Q_0^e Q_0^\ell,$$

$$\text{where } Q_2^{(2)a} = -\frac{\Delta}{6} f_{acd} f_{de\ell} I_{ce\ell}. \tag{4.14}$$

By use of the Jacobi identity, we find the first Serre relation (4.4):

$$\frac{1}{2} f_{d[ab} \{Q_1^{c]}, Q_1^d\} = 2e^2(1 + \rho^2 + (1 - \rho^2)x) \times \frac{1}{24} f_{aip} f_{bjq} f_{ckr} f_{ijk} Q_0^{(p} Q_0^q Q_0^{r)}, \tag{4.15}$$

where we have used the identity (C.1) proven in Appendix C

$$\frac{1}{2} f_{d[ab} f_{c]pm} f_{dnq} f_{mnr} Q_0^p Q_0^q Q_0^r = 3 f_{aip} f_{bjq} f_{ckr} f_{ijk} Q_0^p Q_0^q Q_0^r. \tag{4.16}$$

In total, the charges (4.7) form a classical Yangian algebra in the sense of Poisson brackets, namely Eqs. (4.9), (4.10) and (4.15), under the correspondence (4.8) with

$$\alpha = \frac{1}{2\Delta} = -\frac{1}{2e^2(1+\rho^2+(1-\rho^2)x)}. \tag{4.17}$$

The above is a generalization of a proof originally given in [22] for the isotropic PCM and for the (generalized) Gross–Neveu model, whose corresponding algebras for the $I_{\pm}^a$'s are particular cases of (3.5) (see footnote 6).

The appearance of the classical Yangian algebra was guaranteed in the first place by the existence of the Yang–Baxter equation (3.3) and the realization of the co-product (4.2) by (4.7) and (4.8). The only additional step which needed to be made was to compute the value of α through the Poisson brackets of the level-zero charges (4.9).

4.1.1. The su(2) case

For the $su(2)$ case, the first Serre relation is trivially satisfied, therefore we only have to study the second one (4.6). Using (4.10) we can rewrite the l.h.s. of (4.6) (once again understood in its classical version of Poisson brackets) as

$$2\Delta\big(\varepsilon_{cde}\{\{Q_1^a, Q_1^b\}, Q_1^e\} + \varepsilon_{abe}\{\{Q_1^c, Q_1^d\}, Q_1^e\}\big). \tag{4.18}$$

Next we note that (4.14) trivializes to

$$\{Q_1^a, Q_1^b\} = \varepsilon_{abc}\mathcal{Q}^c,$$
$$\mathcal{Q}^a = Q_2^{(1)a} + \frac{\Delta}{4}\big(Q_0^a Q_0^d Q_0^d + I_{dda}\big), \tag{4.19}$$

where $Q_2^{(1)a}$ is given in (4.12) with f_{abc} replaced by ε_{abc}. Using these specialized expressions we find

$$\{\{Q_1^a, Q_1^b\}, \{Q_0^c, Q_1^d\}\} + \{\{Q_1^c, Q_1^d\}, \{Q_0^a, Q_1^b\}\}$$
$$= 2e^6\big(1+\rho^2+(1-\rho^2)x\big)^3 Q_0^e\big(\varepsilon_{abe}Q_0^{[c}Q_1^{d]} + \varepsilon_{cde}Q_0^{[a}Q_1^{b]}\big), \tag{4.20}$$

which is in agreement with (4.6), (4.8) and (4.17).

4.2. Yangian algebra through the Maillet brackets of the monodromy matrix

An alternative derivation of the Yangian algebra is obtained through the Maillet brackets (3.4) and the expansion of the monodromy matrix M (2.9).

Rewriting (2.9) in terms of $Q_{0,1}$ we find that

$$M\big(v'\big) = 1 + \frac{v'}{1-\rho^2}Q_0 + \frac{v'^2}{(1-\rho^2)^2}\Big(Q_1 - \rho Q_0 + \frac{1}{2}Q_0^2\Big) + \mathcal{O}\big(v'^3\big). \tag{4.21}$$

Plugging (4.21) into (3.4), expanding first in v' and only afterwards in μ', where $\mu' := \mu + \rho$, and keeping all the terms up to the order $\mathcal{O}(v'^2\mu')$, produces, after a good deal of algebra (and writing $Q = Q^a T_a$), (4.9) and (4.10), respectively

$$\{Q_0^a, Q_0^b\} = -2e^2\big(1+\rho^2+(1-\rho^2)x\big)f_{abc}Q_0^c,$$
$$\{Q_0^a, Q_1^b\} = -2e^2\big(1+\rho^2+(1-\rho^2)x\big)f_{abc}Q_1^c. \tag{4.22}$$

Next we consider the expansion of the Maillet brackets (3.4) up to $\mathcal{O}(\nu'^2\mu'^2)$. We will see that, in order to study this term, it is necessary to expand the monodromy matrix up to the order $\mathcal{O}(\nu'^3)$. By manipulating the $\mathcal{O}(\nu'^2\mu'^2)$ term in the brackets, after a rather tedious computation, we obtain the first Serre relation in (4.15). The related technical details are presented in Appendix D.

5. Conclusions and outlook

The purview of the present paper is the construction of the Lax pair $L_{0,1}$ for isotropic coupling matrices λ of the action (1.1) and the corresponding symmetry algebra (3.5) with $\rho = 0$. Using its spatial part L_1 we built the conserved classical monodromy matrix, derived the corresponding Poisson (*Maillet-type* [24,23]) brackets and the emerging modified Yang–Baxter equation as the Jacobi identity on these Poisson brackets. Employing the classical monodromy matrix we constructed the first two conserved charges and obtained their Yangian algebra, both through the charge algebra and also from an expansion of the Poisson brackets for the monodromy matrix. In addition, the renormalizability of this action at one-loop in the $1/k$ expansion [16] ensures that the above construction remains applicable at this order.

It would be interesting to study generalizations of the construction we have provided for vanishing ρ and anisotropic coupling matrices λ, whose action was given in (1.1). These σ-models generically interpolate from the WZW to the non-Abelian T-dual of the anisotropic PCM, and so a good place to start this study are cases which possess an integrable anisotropic PCM endpoint, like the $su(2)$ case [30–32]. Also note that the Yangian symmetries are preserved for the deformed WZW model on squashed spheres [33,34].

It is possible to replace the WZW term in (1.1) by a coset CFT with action realization in terms of a gauged WZW model. In these case the end point of the deformation corresponds to the non-Abelian T-dual of PCM for coset instead of group spaces [1]. In that respect, and for symmetric coset spaces, the deformation has been convincingly argued to correspond to a quantum deformation of the bosonic sector of the string theory, when the deformation parameter is a root of unity [4]. When instead it is real, achieved by analytic continuation, the models are those of [35–37], based on the construction of [38,39] and realized as σ-models in [40]. We believe that our treatment is generalizable to these cases as well.

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Appendix A. The Serre relations

The scope of this appendix is to provide an explicit proof of the Serre relations for pedagogical reasons.

A.1. The first Serre relation

The proof goes along the lines suggested in [20]. Let us define the co-products of J_a and Q_a as in (4.2), and the quantity

$$Z_{ab} := f([Q_a, Q_b]) - [Q_a, Q_b] \otimes \mathbb{I} - \mathbb{I} \otimes [Q_a, Q_b], \tag{A.1}$$

on which f acts as a homomorphism (see footnote 7). Next we introduce

$$u_{ab} := F_{cda}v_{cdb} - F_{cdb}v_{cda}, \qquad F_{abc}u_{ab} = 0, \tag{A.2}$$

where v_{abc} is totally antisymmetric. Contracting (A.1) with u_{ab}, using (4.1) and the Jacobi identity on the term proportional to α, we find that

$$u_{ab}Z_{ab} = \frac{\alpha}{2} u_{ab} F_{abe} F_{cde} (Q_c \otimes J_d - J_c \otimes Q_d)$$
$$+ \frac{\alpha^2}{4} u_{ab} F_{acd} F_{bmn} F_{cmr} (J_r \otimes J_d J_n + J_d J_n \otimes J_r) \tag{A.3}$$

with the first term vanishing due to (A.2). Substituting the value of u_{ab} we find

$$\frac{\alpha^2}{4} (F_{ija}v_{ijb} - F_{ijb}v_{ija}) F_{acd} F_{bmn} F_{cmr} (J_r \otimes J_d J_n + J_d J_n \otimes J_r)$$
$$= (A - B) \times \frac{\alpha^2}{4} (J_r \otimes J_d J_n + J_d J_n \otimes J_r), \tag{A.4}$$

where

$$A = v_{ijb} F_{ija} F_{acd} F_{bmn} F_{cmr} = 2v_{ijb} (F_{ajd} F_{bmn} F_{cmi} F_{car} + F_{ajd} F_{bmn} F_{cam} F_{cir}),$$
$$B = v_{ija} F_{ijb} F_{acd} F_{bmn} F_{cmr} = 2v_{ijb} (F_{ajn} F_{bcd} F_{mci} F_{amr} + F_{ajn} F_{bcd} F_{mac} F_{imr}), \tag{A.5}$$

and we employed the Jacobi identity twice for each term. We then rewrite $A - B$ as

$$A - B = C + D,$$
$$C = 2v_{ijb} (F_{ajd} F_{bmn} F_{cmi} F_{car} - F_{ajn} F_{bcd} F_{mci} F_{amr}),$$
$$D = 2v_{ijb} (F_{ajd} F_{bmn} F_{cam} F_{cir} - F_{ajn} F_{bcd} F_{mac} F_{imr}). \tag{A.6}$$

Applying the Jacobi identity on C and relabelling the indices of D, we find

$$D = 4v_{ijb} F_{ajd} F_{bmn} F_{cam} F_{cir}, \qquad C = -\frac{D}{2}. \tag{A.7}$$

Thus we proved that

$$u_{ab} F_{acd} F_{bmn} F_{cmr} = 2v_{ijb} F_{ajd} F_{bmn} F_{cam} F_{cir}. \tag{A.8}$$

Using (A.8), we can rewrite the r.h.s. of (A.3) as

$$-\frac{\alpha^2}{2}v_{ija}a_{ijardn}(J_r \otimes J_d J_n + J_d J_n \otimes J_r), \qquad a_{abcdef} = F_{adk}F_{bel}F_{cfm}F_{klm}. \tag{A.9}$$

Thus (A.3) reads

$$2v_{ija}F_{ijb}Z_{ab} = \frac{\alpha^2}{2}v_{ija}a_{ijardn}(J_r \otimes J_d J_n + J_d J_n \otimes J_r). \tag{A.10}$$

Due to the contraction with a totally antisymmetric tensor we can rewrite (A.10) as

$$2v_{ija}F_{b[ij}Z_{a]b} = \frac{\alpha^2}{2}v_{ija}a_{[ija]rdn}(J_r \otimes J_d J_n + J_d J_n \otimes J_r). \tag{A.11}$$

Next, we note that

$$a_{[ija]rdn} = a_{ija(rdn)},$$
$$a_{ija(rdn)}(J_r \otimes J_d J_n + J_d J_n \otimes J_r) = a_{ijardn}(J_{(r} \otimes J_d J_{n)} + J_{(d} J_n \otimes J_{r)}), \tag{A.12}$$

where (.) denotes the sum of all permutations (see footnote 8). Using (A.11) and (A.12) we find

$$v_{ija}F_{b[ij}Z_{a]b} = \frac{\alpha^2}{4}v_{ija}a_{ijardn}(J_{(r} \otimes J_d J_{n)} + J_{(d} J_n \otimes J_{r)}). \tag{A.13}$$

In addition, using (4.2) we can easily prove that

$$J_{(r} \otimes J_d J_{n)} + J_{(d} J_n \otimes J_{r)} = \frac{1}{3}\big(f(J_{(r} J_d J_{n)}) - J_{(r} J_d J_{n)} \otimes \mathbb{I} - \mathbb{I} \otimes J_{(r} J_d J_{n)}\big). \tag{A.14}$$

Using (A.13) and (A.14) we find that

$$v_{ija}F_{b[ij}Z_{a]b} = \frac{\alpha^2}{12}v_{ija}a_{ijardn}\big(f(J_{(r} J_d J_{n)}) - J_{(r} J_d J_{n)} \otimes \mathbb{I} - \mathbb{I} \otimes J_{(r} J_d J_{n)}\big). \tag{A.15}$$

Finally, we make use of the properties (A.12) to manipulate the r.h.s. of (A.15) into

$$v_{ija}a_{ijardn}W_{(rdn)} = v_{ija}a_{ija(rdn)}W_{rdn} = v_{ija}a_{[ija]rdn}W_{rdn},$$
$$W_{rdn} = f(J_r J_d J_n) - J_r J_d J_n \otimes \mathbb{I} - \mathbb{I} \otimes J_r J_d J_n. \tag{A.16}$$

Using (A.16) we can write (A.15) as

$$v_{ija}F_{b[ij}Z_{a]b} = v_{ija} \times \frac{\alpha^2}{12}a_{[ija]rdn}W_{rdn}. \tag{A.17}$$

Since the latter holds for every antisymmetric tensor v_{ija} we conclude that

$$F_{b[ij}Z_{a]b} = \frac{\alpha^2}{12}a_{[ija]rdn}W_{rdn}$$
$$\implies \quad F_{b[ij}[Q_{a]}, Q_b] = \frac{\alpha^2}{12}a_{[ija]rdn}J_r J_d J_n = \frac{\alpha^2}{12}a_{ijardn}J_{(r}J_d J_{n)}. \tag{A.18}$$

Expanding the antisymmetric part on the l.h.s. of (A.18) we find the first Serre relation (4.4), namely

$$F_{mij}[Q_k, Q_m] + F_{mki}[Q_j, Q_m] + F_{mjk}[Q_i, Q_m] = \frac{\alpha^2}{24}a_{ijkrdn}J_{(r}J_d J_{n)}, \tag{A.19}$$

where we used that $F_{a[bc]} = 2F_{abc}$.

A.2. The second Serre relation

Applying the Jacobi identity we can easily prove that

$$[[Q_c, Q_d], Q_e] = [[Q_c, Q_e], Q_d] - [[Q_d, Q_e], Q_c],$$
$$F_{crk}F_{rde} - F_{drk}F_{rce} = F_{cdr}F_{rek}. \tag{A.20}$$

Using the first of (A.20) we can prove that

$$2(L_{ab|cd} + L_{bc|ad} + L_{ca|bd})$$
$$= F_{e[ab}[[Q_{c]}, Q_e], Q_d] - F_{e[ab}[[Q_{d]}, Q_e], Q_c]$$
$$+ F_{e[cd}[[Q_{a]}, Q_e], Q_b] - F_{e[cd}[[Q_{b]}, Q_e], Q_a], \tag{A.21}$$

where $L_{ab|cd}$ denotes the l.h.s. of (4.5). Using (4.4), the second relation in (4.1) and the second equation in (A.20), we can rewrite the r.h.s. of (A.21) as

$$R_{ab|cd} + R_{bc|ad} + R_{ca|bd}, \tag{A.22}$$

where $R_{ab|cd}$ denotes the r.h.s. of (4.5). Combining (A.21) and (A.22) we find

$$L_{ab|cd} + L_{bc|ad} + L_{ca|bd} = R_{ab|cd} + R_{bc|ad} + R_{ca|bd}. \tag{A.23}$$

The solution to this equation is the second Serre relation (4.5). In fact, we may suppose it is not, by assuming that there exists another choice of $X_{ab|cd}$ such that

$$L_{ab|cd} = R_{ab|cd} + X_{ab|cd}, \tag{A.24}$$

where $X_{ab|cd}$ is such that

$$X_{ab|cd} = -X_{ba|cd} = -X_{ab|dc} = X_{cd|ab} \tag{A.25}$$

and

$$X_{[ab|c]d} = 0, \qquad X_{a[b|cd]} = 0. \tag{A.26}$$

Applying the Jacobi identity on (A.24) and using (A.25) we find

$$F_{b[mn}X_{q]b|[r\underline{d}}F_{st]d} = 0 \quad \implies \quad X_{ab|cd} = F_{abe}Y_{e|cd} + F_{cde}Y_{e|ab},$$
$$Y_{a|bc} = -Y_{a|cb}, \tag{A.27}$$

where $\underline{d}$ is excluded from the anti-symmetrization. Using these relations and (A.26), we find

$$Y_{a|bc} \sim F_{abc} \quad \implies \quad X_{ab|cd} = \varepsilon F_{abe}F_{cde}, \tag{A.28}$$

where ε is an arbitrary constant. Thus (A.24) reads

$$L_{ab|cd} = R_{ab|cd} + \varepsilon F_{abe}F_{cde}. \tag{A.29}$$

Contracting the latter with $F_{ab\ell}F_{cd\ell}$ and using the Jacobi identity on commutators, we find

$$0 = 0 + \varepsilon c_G^2 \dim(\mathbf{g}) \quad \implies \quad \varepsilon = 0, \tag{A.30}$$

where $F_{acd}F_{bcd} = c_{\mathbf{g}}\delta_{ab}$, $a = 1, 2, \ldots, \dim(\mathbf{g})$. This completes the proof of the redundancy of the second Serre relation (4.5).

Appendix B. The triple integral

The scope of this appendix is to simplify (4.13). Let us first define the triple integral as

$$I_{ce\ell} = I_{ec\ell} = \int d^3\sigma_{123} \varepsilon_{13}\varepsilon_{32} \mathcal{I}_0^c(\sigma_1)\mathcal{I}_0^e(\sigma_2)\mathcal{I}_0^\ell(\sigma_3), \tag{B.1}$$

where $d^3\sigma_{123}$ stands for $d\sigma_1 d\sigma_2 d\sigma_3$. This can be rewritten as follows[11]:

$$I_{ce\ell} = -\frac{1}{2} \int d^3\sigma_{123} dw \varepsilon_{31}\mathcal{I}_0^c(\sigma_1)\varepsilon_{32}\mathcal{I}_0^e(\sigma_2)\partial_{\sigma_3}\left(\varepsilon_{3w}\mathcal{I}_0^\ell(w)\right). \tag{B.2}$$

Integrating by parts we can easily prove that

$$\begin{aligned}
I_{ce\ell} &+ I_{\ell ce} + I_{e\ell c} = -Q_0^c Q_0^e Q_0^\ell \\
&\implies (f_{acd}f_{bre}f_{dr\ell} + f_{a\ell d}f_{brc}f_{dre} + f_{aed}f_{br\ell}f_{drc})I_{ce\ell} \\
&\qquad = -f_{acd}f_{bre}f_{dr\ell}Q_0^c Q_0^e Q_0^\ell.
\end{aligned} \tag{B.3}$$

This formula could equivalently be found through the identity

$$\varepsilon_{13}\varepsilon_{32} + \varepsilon_{21}\varepsilon_{13} + \varepsilon_{32}\varepsilon_{21} = -1. \tag{B.4}$$

Using the Jacobi identity we can prove that

$$\begin{aligned}
f_{a\ell d}f_{dre}f_{brc} &= f_{acd}f_{bre}f_{dr\ell} + f_{aed}f_{brc}f_{dr\ell} - f_{acd}f_{br\ell}f_{dre} - f_{abr}f_{rcd}f_{d\ell e}, \\
f_{aed}f_{br\ell}f_{drc} &= f_{acd}f_{bre}f_{dr\ell} + f_{aed}f_{brc}f_{dr\ell} - f_{a\ell d}f_{bre}f_{drc} - f_{abr}f_{red}f_{d\ell c}.
\end{aligned} \tag{B.5}$$

Using the latter we can rewrite (B.3) as

$$\begin{aligned}
\big(3 f_{acd}f_{bre}f_{dr\ell} &+ 2 f_{aed}f_{brc}f_{dr\ell} - f_{acd}f_{br\ell}f_{dre} - f_{a\ell d}f_{bre}f_{drc} \\
&+ f_{abr}(f_{drc}f_{de\ell} - f_{dre}f_{d\ell c})\big)I_{ce\ell} = -f_{acd}f_{bre}f_{dr\ell}Q_0^c Q_0^e Q_0^\ell.
\end{aligned} \tag{B.6}$$

Using (B.4) we can prove that

$$\begin{aligned}
(2 f_{aed}f_{brc}f_{dr\ell} &- f_{acd}f_{br\ell}f_{dre} - f_{a\ell d}f_{bre}f_{drc})I_{ce\ell} \\
&= f_{acd}f_{bre}f_{dr\ell}\big(3 I_{ce\ell} + Q_0^c Q_0^e Q_0^\ell\big).
\end{aligned} \tag{B.7}$$

Combining (B.6) and (B.7) we find

$$f_{acd}f_{bre}f_{dr\ell}I_{ce\ell} = -\frac{1}{3} f_{acd}f_{bre}f_{dr\ell}Q_0^c Q_0^e Q_0^\ell - \frac{1}{3} f_{abr}f_{rcd}f_{de\ell}I_{ce\ell}. \tag{B.8}$$

Appendix C. Serre structure constants

In this appendix we prove (4.16). In order to do so, we use the equivalent rewriting

$$\frac{1}{2} f_{d[ab}f_{c]d}^3 = 3 f_{ai}f_{bj}f_{ck}f_{ijk}, \qquad f_{ij} := f_{ijk}Q_0^k. \tag{C.1}$$

We start from the r.h.s. and use the Jacobi identity to rewrite $f_{ck}f_{ijk}$ as

$$f_{ck}f_{ijk} = -f_{cik}f_{jk} - f_{jck}f_{ik}. \tag{C.2}$$

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[11] For manipulations of similar integrals, see [29].

Then we make use again of the Jacobi identity to deduce the following rewritings:

$$f_{ai} f_{cik} = f_{cai} f_{ik} - f_{iak} f_{ci}, \qquad f_{bj} f_{jck} = f_{bcj} f_{jk} + f_{jbk} f_{cj}. \tag{C.3}$$

Using (C.2) and (C.3), we can rewrite the r.h.s. of (C.1) as

$$f_{ai} f_{bj} f_{ck} f_{ijk} = f_{aj}^3 f_{bcj} + f_{bj}^3 f_{caj} + f_{bj}^2 f_{ci} f_{iaj} - f_{aj}^2 f_{ci} f_{ibj}, \tag{C.4}$$

or, equivalently, as

$$f_{ai} f_{bj} f_{ck} f_{ijk} = f_{bj}^3 f_{caj} + f_{cj}^3 f_{abj} + f_{cj}^2 f_{ai} f_{ibj} - f_{bj}^2 f_{ai} f_{icj}, \tag{C.5}$$

$$f_{ai} f_{bj} f_{ck} f_{ijk} = f_{cj}^3 f_{abj} + f_{aj}^3 f_{bcj} + f_{aj}^2 f_{bi} f_{icj} - f_{cj}^2 f_{bi} f_{iaj}. \tag{C.6}$$

Adding (C.4), (C.5) and (C.6) together we find

$$3 f_{ai} f_{bj} f_{ck} f_{ijk} = f_{d[ab} f_{c]d}^3 - f_{aj}^2 (f_{ci} f_{ibj} - f_{bi} f_{icj}) - f_{bj}^2 (f_{ai} f_{icj} - f_{ci} f_{iaj})$$
$$- f_{cj}^2 (f_{bi} f_{iaj} - f_{ai} f_{ibj}). \tag{C.7}$$

Using the Jacobi identity we can rewrite the terms in parentheses as

$$f_{ci} f_{ibj} - f_{bi} f_{icj} = f_{jd} f_{dbc}, \qquad f_{ai} f_{icj} - f_{ci} f_{iaj} = f_{jd} f_{dca},$$
$$f_{bi} f_{iaj} - f_{ai} f_{ibj} = f_{jd} f_{dab}. \tag{C.8}$$

Using (C.8), we can rewrite (C.7) as

$$3 f_{ai} f_{bj} f_{ck} f_{ijk} = f_{d[ab} f_{c]d}^3 - \frac{1}{2} f_{d[ab} f_{c]d}^3 = \frac{1}{2} f_{d[ab} f_{c]d}^3, \tag{C.9}$$

which completes the proof of (C.1) or equivalently (4.16).

Appendix D. Maillet brackets and the first Serre relation

Let us first consider the l.h.s. of (3.4) at the order $\mathcal{O}(\nu'^2 \mu'^2)$ of the expansion of the monodromy matrix (4.21). Specifically, if we define

$$g_1 := \frac{\nu'}{1 - \rho^2}, \qquad g_2 := \frac{\mu'}{1 - \rho^2}, \qquad q = \frac{Q_0}{2}, \qquad \widehat{q} = \frac{Q_1}{2}, \tag{D.1}$$

we get

$$4 g_1^2 g_2^2 \left\{ \left(\widehat{q} - \rho q + q^2 \right), \left(\widehat{q} - \rho q + q^2 \right) \right\}. \tag{D.2}$$

Expanding the Poisson brackets and using (4.22) written in terms of $q, \widehat{q}$, i.e.

$$\{q_a, q_b\} = \Delta f_{abc} q_c, \quad \{q_a, \widehat{q}_b\} = \Delta f_{abc} \widehat{q}_c, \quad \Delta = -e^2 \left(\rho^2 + 1 + x \left(1 - \rho^2 \right) \right), \tag{D.3}$$

we obtain

$$4 g_1^2 g_2^2 \big[\{\widehat{q}_a, \widehat{q}_b\} T_a \otimes T_b + 2\rho \Delta f_{bac} \widehat{q}_c T_a \otimes T_b - \Delta f_{bad} \widehat{q}_d q_c T_a \otimes \{T_b, T_c\}$$
$$+ \rho^2 \Delta f_{abc} q_c T_a \otimes T_b - \rho \Delta f_{abd} q_d q_c T_a \otimes \{T_b, T_c\} + \Delta f_{acd} \widehat{q}_d q_b \{T_a, T_b\} \otimes T_c$$
$$- \rho \Delta f_{bcd} q_a q_d \{T_a, T_b\} \otimes T_c + \Delta f_{bce} q_a q_e q_d \{T_a, T_b\} \otimes \{T_c, T_d\} \big], \tag{D.4}$$

where $\{T_a, T_b\} := T_a T_b + T_b T_a$.

We now need to consider the r.h.s. of (3.4) at the same order. Let us define

$$q_1 := q \otimes \mathbb{I}, \qquad q_2 := \mathbb{I} \otimes q, \tag{D.5}$$

as in footnote 4. There are several contributions, which we list here below:

- both $r(\nu', \mu')$ and $s(\nu', \mu')$ are taken at the order $\mathcal{O}(\nu'\mu')$, which means that both $M(\nu')$ and $M(\mu')$ are taken at the linear order. This term contributes

$$16e^2 \rho g_1^2 g_2^2 \big(-[\Pi, q_1 q_2] + q_1 \Pi q_2 - q_2 \Pi q_1\big); \tag{D.6}$$

- $r(\nu', \mu')$ is taken at the order $\mathcal{O}(\nu'\mu'^0)$, which means that $M(\nu')$ is taken at the linear and $M(\mu')$ at the quadratic order. No $s(\nu', \mu')$ contribution is present at this order. We get

$$-8\Delta g_1^2 g_2^2 \big[\Pi, q_1 \big(\widehat{q}_2 - \rho q_2 + q_2^2\big)\big]; \tag{D.7}$$

- $r_-(\nu', \mu')$ is taken at the order $\mathcal{O}(\nu'\mu'^2)$, which means that $M(\nu')$ is taken at the linear and $M(\mu')$ at the zeroth order (making r_+ immaterial). We get

$$8e^2\big(1 + 3\rho^2\big) g_1^2 g_2^2 [\Pi, q_1]; \tag{D.8}$$

- $r_+(\nu', \mu')$ is taken at the order $\mathcal{O}(\nu'^2 \mu'^0)$, which means that $M(\nu')$ is taken at the zeroth and $M(\mu')$ at the quadratic order (making r_- irrelevant). We get

$$-16e^2 \rho g_1^2 g_2^2 \big[\Pi, \big(\widehat{q}_2 - \rho q_2 + q_2^2\big)\big]; \tag{D.9}$$

- $r_+(\nu', \mu')$ is taken at the order $\mathcal{O}(\nu'^2 \mu')$, which means that $M(\nu')$ is taken at the zeroth and $M(\mu')$ at the linear order. We will show that this term does not contribute to the final result, upon applying the procedure (D.12) we will introduce shortly.
- $r_+(\nu', \mu')$ is taken at the order $\mathcal{O}(\nu'^2 \mu'^{-1})$, which means that $M(\nu')$ is taken at the zeroth and $M(\mu')$ at the cubic order. The presence of this negative power in the expansion of r_+ forces us to go to the third order in the expansion of the monodromy matrix, which we will perform later on – see the discussion around (D.12).

  Putting all the terms together and performing a few manipulations, we get for the r.h.s. of the Poisson relations

$$
\begin{aligned}
16e^2 \rho g_1^2 g_2^2 q_a q_b \bigg(& f_{acd} f_{bce} T_d \otimes T_e + \frac{\Delta}{2e^2} f_{cbd} T_a T_c \otimes T_d - f_{cbe} T_c \otimes T_a T_e \\
& + \left[\frac{\Delta}{2e^2} - 1\right] f_{acd} T_c \otimes T_d T_b \bigg) - 8\Delta g_1^2 g_2^2 q_a \widehat{q}_b (f_{cbd} T_a T_c \otimes T_d \\
& + f_{cad} T_d \otimes T_c T_b) - 8\Delta g_1^2 g_2^2 q_a q_b q_d (f_{cbe} T_c T_a \otimes T_e T_d + f_{cde} T_c T_a \otimes T_b T_e \\
& + f_{ace} T_e \otimes T_b T_d T_c) - 16e^2 \rho g_1^2 g_2^2 \widehat{q}_b f_{cba} T_c \otimes T_a \\
& + 8e^2 \big(1 + 3\rho^2\big) g_1^2 g_2^2 q_a \left[f_{abc} T_b \otimes T_c - \frac{2\rho^2}{(1 + 3\rho^2)} f_{abc} T_b \otimes T_c\right].
\end{aligned} \tag{D.10}
$$

The strategy we will now follow is to bring everything on one side of the equation, namely to calculate

$$\frac{\text{l.h.s.–r.h.s.}}{g_1^2 g_2^2}, \tag{D.11}$$

and to act upon it with the following operation:

$$\frac{\Delta}{2} f_{\delta[\alpha\beta} \text{tr}(T_{\gamma]} \otimes T_\delta \circ), \tag{D.12}$$

where the three indices α, β and γ are totally antisymmetrized (without the $\frac{1}{6}$ factor). Upon performing the operation (D.12), and by using the Jacobi identity, the very first term of the l.h.s. as contributing to (D.11), namely $4\{\widehat{q}_a, \widehat{q}_b\} T_a \otimes T_b$, can be seen to coincide with

$$4\{\widehat{q}_\alpha, \{\widehat{q}_\beta, q_\gamma\}\} - 4\{q_\alpha, \{\widehat{q}_\beta, \widehat{q}_\gamma\}\}, \tag{D.13}$$

which is the desired combination appearing in the Serre relations. It is therefore a matter of analyzing all the other terms after this operation is performed. One thing to notice is that anything looking like

$$f_{\gamma\delta b}\Omega^b \tag{D.14}$$

will vanish upon this operation, as can be seen by using the Jacobi identity. This is the reason why the contribution of $r_+(\nu', \mu')$ taken at the order $\mathcal{O}(\nu'^2\mu')$ is absent, as we commented earlier, since it is precisely of the form (D.14). Disregarding this type of terms as irrelevant to the final result, we combine the remaining terms in (D.11) and perform quite extensive manipulations and simplifications. Performing then the operation (D.12) on the result of this simplification produces some terms that we will call *unwanted*, since they do not look like the standard terms appearing in the Serre relations, and some that we call *wanted*, since they have the desired form.

D.1. Unwanted terms

Let us begin with the unwanted terms. They come in two fashions:

- We get a quadratic contribution with level zero charges, specifically

$$-8\Delta\rho q_a q_b f_{\gamma eb} \mathrm{tr}(T_\delta\{T_e, T_a\}) + \text{contrib. } r_+(\nu', \mu') \text{ at } \mathcal{O}(\nu'^2\mu'^{-1}); \tag{D.15}$$

- We get a quadratic term with level zero and one charges, specifically

$$4\Delta q_a \widehat{q}_b\left[-f_{e\gamma b}\mathrm{tr}(T_\delta\{T_e, T_a\}) - f_{e\gamma a}\mathrm{tr}(T_\delta\{T_e, T_b\})\right]$$
$$+ \text{contrib. from } r_+(\nu', \mu') \text{ at } \mathcal{O}(\nu'^2\mu'^{-1}). \tag{D.16}$$

The respective first terms in (D.15) and (D.16) vanish upon the operation (D.12). In order to see this, one needs to proceed in steps. Let us consider the first unwanted term. The first step consists of repeatedly using

$$f_{abc} = \mathrm{tr}(T_a[T_b, T_c]), \tag{D.17}$$

to re-write the first term in (D.15), after the action of (D.12), as (indicating only the matrix part)

$$-\frac{1}{2} f_{\delta[\alpha\beta} f_{\gamma]eb}\mathrm{tr}(T_\delta\{T_e, T_a\}) = \mathrm{tr}(\{T_e, T_a\}[T_\alpha, T_\beta])\mathrm{tr}(T_e[T_\gamma, T_b]) + \text{``2''}, \tag{D.18}$$

where "2" means that we have to add the other two cyclic permutations $\beta\gamma\alpha$ and $\gamma\alpha\beta$ of the same structure on the l.h.s. of (D.18). At this point, it is convenient to open up the anti-commutator, involving the generator T_e, and move T_e close to the other trace by using cyclicity of the trace. When this is done, it produces two terms, in each of which one recognizes a structure of the type

$$\mathrm{tr}(x T_e)\mathrm{tr}(T_e y), \quad y \in \text{Lie algebra}. \tag{D.19}$$

The fact that y is Lie-algebra valued makes it possible to fuse the traces producing $\mathrm{tr}(xy)$. For the matrix part of the first unwanted term, we are therefore left with

$$\text{tr}\big(T_a[T_\alpha, T_\beta][T_\gamma, T_b]\big) + \text{tr}\big([T_\alpha, T_\beta]T_a[T_\gamma, T_b]\big) + \text{``2''}.\tag{D.20}$$

Adding the "2" explicitly to this term, symmetrizing $a \leftrightarrow b$ given the $q_a q_b$ in front of the first term in (D.15) and eventually expanding all the (anti-)commutators explicitly, one sees that all terms cancel and the total contribution vanishes. We will calculate the contribution from $r_+(\nu', \mu')$ at $\mathcal{O}(\nu'^2 \mu'^{-1})$ later on.

With regards to the first term in (D.16), let us re-write it as

$$4\Delta(q_a \widehat{q}_b + q_b \widehat{q}_a) f_{\gamma e b} \text{tr}\big(T_\delta \{T_e, T_a\}\big).\tag{D.21}$$

From this we see that, due to the $a \leftrightarrow b$ symmetry of the pre-factor, perfectly analogous considerations apply as for the term we have just shown to vanish. Once again, we will study the contribution from $r_+(\nu', \mu')$ at $\mathcal{O}(\nu'^2 \mu'^{-1})$ later on.

D.2. Third order of the monodromy

We have seen that, to be able to calculate the two *unwanted* terms left-over from (D.15) and (D.16), and also the related contribution to the *wanted* terms, we need the monodromy matrix up to the third order in the spectral parameter. We will derive this term in the expansion in this section.

From Section 2 we have learned that the monodromy matrix M admits an expansion (adapted to the parameter g used in this section)

$$M(g) = P \exp g \int \frac{\mathcal{I}_0 - s g I_1}{(1 + \rho g)^2 - g^2},\tag{D.22}$$

where we note that both $\mathcal{I}_0$ and I_1 appear and

$$s = \rho^2 - 1.\tag{D.23}$$

We need to isolate the third order term in g. If we recall the density (2.4) of the level-zero charge

$$2j(\sigma) = \mathcal{I}_0(\sigma),\tag{D.24}$$

we see that the third-order term we are after reads, in compact notation,

$$2g^3\big(4\rho^2 - s\big) \int j + 2\rho s g^3 \int I_1 - 16\rho g^3 \int\int^\sigma j(\sigma)j(\sigma')$$

$$-2s g^3 \int\int^\sigma \big[j(\sigma)I_1(\sigma') + I_1(\sigma)j(\sigma')\big] + 8g^3 \int\int^\sigma\int^{\sigma'} j(\sigma)j(\sigma')j(\sigma'') := M_3.\tag{D.25}$$

We have to put this term into a form which is ready to be used for the Serre relations, therefore we parameterize

$$\frac{M_3}{g^3} = A Q_0^3 + \frac{B}{2} Q_0 \widehat{Q} + \frac{C}{2} \widehat{Q} Q_0 + D Q_0^2 + \frac{E}{2} \widehat{Q} + F Q_0 + \text{``Lie''},\tag{D.26}$$

where we recall that (in the notation of this section) $\widehat{Q}$ reads

$$\widehat{Q} = \int (-2\rho j - s I_1) + 4 \int\int^\sigma j(\sigma)j(\sigma').\tag{D.27}$$

"Lie" carries such a name because it is something which is not easily expressed in terms of Q or $\widehat{Q}$, nevertheless it is Lie-algebra valued, hence it will drop after operation (D.12). In particular, we choose

$$
\text{"Lie"} = H \int\!\!\int^{\sigma} \big[j(\sigma), j(\sigma')\big] + \frac{V}{2} \int\!\!\int^{\sigma} \big[j(\sigma), I_1(\sigma')\big] + \frac{U}{2} \int\!\!\int^{\sigma} \big[I_1(\sigma), j(\sigma')\big]
$$

$$
+ N \int\!\!\int\!\!\int^{\sigma}^{\sigma'} \big[j(\sigma), \big[j(\sigma'), j(\sigma'')\big]\big] + P \int\!\!\int\!\!\int^{\sigma}_{\sigma} \big[j(\sigma), \big[j(\sigma'), j(\sigma'')\big]\big]
$$

$$
+ R \int\!\!\int\!\!\int^{\sigma'}_{\sigma\ \sigma} \big[j(\sigma), \big[j(\sigma'), j(\sigma'')\big]\big]. \tag{D.28}
$$

By appropriately splitting the integration domains and taking into account the ordering of the generators, one can show that the terms we use in our parametrization of M_3 are enough to reconstruct the most general integral appearing at this order. In fact, we find in this way that they are more than sufficient, as we find a family of solutions when we try and match with (D.25):

$$
A = -\frac{8}{3}, \qquad F = \rho E + 2\big(4\rho^2 - s\big), \qquad G = s(E + 4\rho) \qquad V = s(B - 4),
$$

$$
U = -sB, \qquad C = 4 - B, \qquad R = N - \frac{8}{3}, \qquad P = \frac{16}{3} - 2B - N,
$$

$$
H = -E - 8\rho, \qquad D = -E - 4\rho, \tag{D.29}
$$

from which we see that we can set

$$
B = E = N = 0 \tag{D.30}
$$

as a convenient choice.

The only contribution from the third order of the monodromy that can survive the operation (D.12) is then

$$
g^3 \left[\frac{4}{3} q^3 + 4\widehat{q}q - 4\rho q^2 \right]. \tag{D.31}
$$

- The q^2 term in (D.31) represents the contribution to the unwanted term (D.15) from the third order expansion in the monodromy. After acting with (D.12) and performing a few manipulations on the indexes, one can see that this contribution reproduces the same structure as the first addendum in (D.15), hence it vanishes for the same reasons.
- The $\widehat{q}q$ term in (D.31) combines with the order $\mathcal{O}(\nu'^2 \mu'^{-1})$ of $r_+(\nu', \mu')$ to give

$$
-2\Delta g_1^2 g_2^2 \widehat{q}_b q_d [f_{cbe} T_c \otimes T_e T_d + f_{cde} T_c \otimes T_b T_e]. \tag{D.32}
$$

After acting with (D.12), one can see that this term as well reduces the same structure as the first addendum in (D.16), hence vanishing by the same token.

D.3. Wanted terms

We now need to calculate the contribution we do expect to appear in the first Serre relation, namely we evaluate the cubic term in the level-zero charges in the expression (D.11). This amounts to the following – after acting with (D.12):

$$2\Delta^2 q_a q_b q_d \, f_{\delta[\alpha\beta} \big[4\mathrm{tr}(T_{\gamma]} T_a T_e)\mathrm{tr}(T_\delta T_c T_d) f_{ecb} + 2 f_{\gamma]ea} f_{ecb}\mathrm{tr}(T_\delta T_c T_d)$$

$$+ 2\mathrm{tr}(T_{\gamma]} T_a T_e) f_{ecb} f_{dc\delta} + f_{\gamma]ea} f_{ecb} f_{dc\delta} + 2\mathrm{tr}(T_{\gamma]} T_c T_a)\mathrm{tr}(T_\delta T_e T_d) f_{cbe}$$

$$+ 2\mathrm{tr}(T_{\gamma]} T_c T_a)\mathrm{tr}(T_\delta T_b T_e) f_{cde} - 2 f_{\gamma]ac}\mathrm{tr}(T_\delta T_b T_d T_c)\big]$$

$$+ \text{ contribution from } r_+\big(\nu', \mu'\big) \text{ at } \mathcal{O}\big(\nu'^2 \mu'^{-1}\big). \tag{D.33}$$

Let us start with the part in (D.33) which is not coming from $r_+(\nu', \mu')$ at $\mathcal{O}(\nu'^2 \mu'^{-1})$. Exploiting the total symmetry of the pre-factor $q_a q_b q_c$, after repeated use of the Jacobi identity, and by use of reconstructing commutators inside the traces to reduce the length of the traces as much as possible, we can recast that contribution into

$$-4\Delta^2 q_a q_b q_c \, f_{\delta[\alpha\beta} f_{\gamma]be}\mathrm{tr}(T_\delta T_a T_e T_c) + 2\Delta^2 q_a q_b q_c \, f_{\delta[\alpha\beta} f_{\gamma]br} f_{rce} f_{ae\delta}. \tag{D.34}$$

Before proceeding with the calculation, let us compare with the contribution from $r_+(\nu', \mu')$ at $\mathcal{O}(\nu'^2 \mu'^{-1})$ and see whether the difficult-to-handle length-four trace cancels. It does indeed, since, by performing similar manipulations, the contribution from $r_+(\nu', \mu')$ at $\mathcal{O}(\nu'^2 \mu'^{-1})$ term results into

$$4\Delta^2 q_a q_b q_c \, f_{\delta[\alpha\beta} f_{\gamma]be}\mathrm{tr}(T_\delta T_a T_e T_c) - \frac{4}{3}\Delta^2 q_a q_b q_c \, f_{\delta[\alpha\beta} f_{\gamma]br} f_{rce} f_{ae\delta}. \tag{D.35}$$

We are then left with the two purely structure-constant contribution, which, by repeated use of the Jacobi identity, can be combined and manipulated into

$$-4\Delta^2 q_a q_b q_c \, f_{\alpha ae} f_{\beta bd} f_{\gamma cr} f_{edr}. \tag{D.36}$$

We now recall that this contributes to "l.h.s.–r.h.s." of the Poisson brackets, hence it changes sign when brought back to the r.h.s., giving

$$\big\{\widehat{q}_\alpha, \{\widehat{q}_\beta, q_\gamma\}\big\} - \big\{q_\alpha, \{\widehat{q}_\beta, \widehat{q}_\gamma\}\big\} = \frac{1}{6}\Delta^2 q_{(a} q_b q_{c)} f_{\alpha ae} f_{\beta bd} f_{\gamma cr} f_{edr}. \tag{D.37}$$

Using the Jacobi identity, Eqs. (D.1), (D.3) and the definition of Δ (4.11), we ultimately re-obtain the first Serre relation (4.15).

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