Phys. Lett. B 853 (2024) 138693

Contents lists available at ScienceDirect

Physics Letters B

journal homepage: www.elsevier.com/locate/physletb

Letter

Nonrelativistic limits of the Klein-Gordon and Dirac equations in the Amelino-Camelia DSR

Nosratollah Jafari [a,b,c,*], Bekdaulet Shukirgaliyev [d,e,f]

[a] Fesenkov Astrophysical Institute, 050020, Almaty, Kazakhstan
[b] Al-Farabi Kazakh National University, Al-Farabi av. 71, 050040 Almaty, Kazakhstan
[c] Center for Theoretical Physics, Khazar University, 41 Mehseti Street, Baku, AZ1096, Azerbaijan
[d] Heriot-Watt International Faculty, Zhubanov University, 030000 Aktobe, Kazakhstan
[e] Energetic Cosmos Laboratory, Nazarbayev University, 010000 Astana, Kazakhstan
[f] Department of Computation and Data Science, Astana IT University, 010000 Astana, Kazakhstan

ARTICLE INFO

Editor: J. Hisano

Keywords:
Doubly special relativity
Non-relativistic limit
Relativistic quantum mechanics

ABSTRACT

This paper is devoted to the study of the nonrelativistic limit of Amelino-Camelia doubly special relativity and the corresponding modified Klein-Gordon and Dirac equations. We show that these equations reduce to the Schrödinger equations for the particle and the antiparticle with different inertial masses, however, their rest masses are the same. M. Coraddu and S. Mignemi have studied the nonrelativistic limit of the Magueijo-Smolin doubly special relativity. We compare their results with our study and show that these two models are reciprocal to each other in the nonrelativistic limit. We find that particle and antiparticle masses can be different. These different masses lead to the CPT violation. Also, we will find an upper limit on the photon mass.

Contents

1. Introduction

Doubly special relativity (DSR) theories have been proposed two decades ago for the nonlinear modification of special relativity. These

* Corresponding author.
E-mail address: nosrat.jafari@fai.kz (N. Jafari).

https://doi.org/10.1016/j.physletb.2024.138693
Received 9 February 2024; Received in revised form 29 April 2024; Accepted 30 April 2024
Available online 6 May 2024

theories have two invariant scales, the speed of light c and the Planck energy $E_p = \sqrt{\hbar c^5/G} \simeq 10^{19}$ GeV. Magueijo-Smolin (MS) DSR [1] and Amelino-Camelia DSR [2,3] are the two main examples of these theories. Although these models have similar structures, they belong to different realizations of kappa-Pioncare algebras [4].

According to the special relativity and relativistic quantum mechanics, the Schrödinger equation is the nonrelativistic limit of the Klein-Gordon and Dirac equations [5]. Also, we have the same mass for the particle and the antiparticle. This equality is the straightforward result of the dispersion relation $E = \pm\sqrt{m^2 + p^2}$.

The dispersion relation has been modified in the DSR theories, which leads to the modified Klein-Gordon and Dirac equations. It will be interesting to study the nonrelativistic limit of these modified equations. The nonlinearity of the dispersion relation and the fact that the dispersion relation is not invariant under space inversion and time reversal also imply the violation of CPT invariance.

M. Coraddu and S. Mignemi have studied the nonrelativistic limits of the MS DSR and the corresponding modified Klein-Gordon and Dirac equations [6]. They illustrated that the particle and the antiparticle rest masses are different, however their inertial masses are the same. Besides, they have showed that the modified Klein-Gordon and Dirac equations reproduce nonrelativistic quantum mechanics. To continue their proposal, we want to study the nonrelativistic limits of Amelino-Camelia doubly special relativity and the corresponding modified Klein-Gordon and Dirac equations.

We show that the corresponding modified Klein-Gordon and Dirac equations reduce to the Schrödinger equations for the particle and the antiparticle with different inertial masses. The difference between these two masses is proportional to mc^2/E_p in the first order of approximation. However, their rest masses are the same, but the different inertial masses lead to the violation of CPT invariance.

We compare the nonrelativistic limit of MS and Amelino-Camelia DSR. Our results are reciprocal to the M. Coraddu and S. Mignemi results. We can interpret MS DSR as modifying rest mass and Amelino-Camelia DSR as modifying momentum [7].

Dirac equation in the MS DSR has been studied previously [8,9]. They solved the modified Dirac equation in a special case and obtained a deformed Berry phase. Also, in the κ-Poincare algebra the modifications of the Dirac equation have been studied [10]. A general formalism for finding the modified Klein-Gordon and Dirac equations in any given curved momentum space has been provided by S. A. Franchino-Vinas and J.J. Relancio [11]. For this purpose, we should obtain the modified Casimir invariant of the given curved momentum space. Note that these modified Klein-Gordon and Dirac equations are in the momentum space, and to find their corresponding in space-time, we should find their duals.

In the following section, we summarize some basics of Amelino-Camelia DSR. In section 3, we study the nonrelativistic limit of modified dispersion relation in Amelinio-Camelia DSR and make some comparisons with the nonrelativistic limit of MS DSR. The nonrelativistic limits of ordinary and modified Klein-Gordon and Dirac equations are presented in section 4. Then, in section 5 we have some conclusions.

2. Amelino-Camelia DSR

The essence of Amelino-Camelia DSR is the modification of ordinary special relativistic boosts

$$K_a = ip_a\frac{\partial}{\partial E} + iE\frac{\partial}{\partial p_a}, \tag{1}$$

to preserve the following nonlinear dispersion relation invariant [3], here a is an index from 1 to 3. In this DSR, the modified dispersion relation is

$$2k^2[\cosh(\frac{E}{k}) - \cosh(\frac{m}{k})] = \mathbf{p}^2 e^{E/k}, \tag{2}$$

in which k is the Planck energy. G. Amelino-Camelia DSR proposed the modified boosts

$$B_a = ip_a\frac{\partial}{\partial E} + i(\frac{1}{2k}\mathbf{p}^2 + k\frac{1-e^{-2E/k}}{2})\frac{\partial}{\partial p_a} - i\frac{p_a}{k}(p_b\frac{\partial}{\partial p_b}), \tag{3}$$

which leaves invariant the mentioned dispersion relation. Also, we can find new transformations for this doubly special relativity which is different from the Lorentz transformations [7].

The modified Klein-Gordon equation is obtained from equation (2) by substituting differential operators for $E = i\frac{\partial}{\partial t}$ and $\mathbf{p} = -i\vec{\nabla}$ in the standard fashion as in quantum mechanics

$$\left[\nabla^2 - 2k^2\exp(\frac{-i}{k}\frac{\partial}{\partial t})\left(\cosh(\frac{-i}{k}\frac{\partial}{\partial t}) - \cosh(\frac{m}{k})\right)\right]\Psi(\vec{x},t) = 0. \tag{4}$$

Also, we can construct a deformed Dirac equation

$$\left(\gamma^\mu D_\mu - I\right)\Psi(\mathbf{p}) = 0, \tag{5}$$

where D_μ is the modified Dirac operator

$$D_0 = \frac{e^{E/k} - \cosh(m/k)}{\sinh(m/k)},$$
$$D_a = \frac{p_a}{p}\frac{\left(2e^{E/k}[\cosh(E/k) - \cosh(m/k)]\right)^{1/2}}{\sinh(m/k)}, \tag{6}$$

and γ^μ are the familiar Dirac γ matrices [3].

3. Nonrelativistic limit

In ordinary special relativity, we go to the classical nonrelativistic limit for a free particle by expanding ordinary dispersion relation

$$E^2 - c^2\mathbf{p}^2 = m^2c^4. \tag{7}$$

In $(\mathbf{p}^2c^2/m^2c^4) \ll 1$ limit, we obtain

$$E = \pm\sqrt{\mathbf{p}^2c^2 + m^2c^4} \simeq \pm(mc^2 + \frac{\mathbf{p}^2}{2m} + ...). \tag{8}$$

The first term on the right-hand side is the rest mass, and the second term is the classical kinetic energy.

In the Amelino-Camelia DSR, we expand the modified dispersion relation (2) in $O(E^3/k^3)$ approximation and we obtain

$$E^2 - \mathbf{p}^2c^2 - \frac{\mathbf{p}^2c^2}{k}E = m^2c^4. \tag{9}$$

Solving this equation as a second-order equation for E, we have

$$E = \frac{\mathbf{p}^2c^2/k \pm \sqrt{\mathbf{p}^4c^4/k^2 + 4(\mathbf{p}^2c^2 + m^2c^4)}}{2}. \tag{10}$$

In the $(\mathbf{p}^2c^2/m^2c^4) \ll 1$ limit, we have

$$E \simeq \frac{\mathbf{p}^2c^2}{2k} \pm mc^2(1 + \frac{\mathbf{p}^2}{2m^2c^2}). \tag{11}$$

The positive sign corresponds to the classical nonrelativistic limit

$$E = mc^2 + \frac{\mathbf{p}^2}{2m^+}, \tag{12}$$

in which we have assumed

$$m^+ = \frac{m}{1 + \frac{mc^2}{k}}. \tag{13}$$

Thus, we find that the particle's inertial mass has been modified by the amount of mc^2/k, but the particle's rest mass remains unmodified. By inertial mass, we mean the mass which appears in the denominator of $\mathbf{p}^2/2m^+$. Also, by defining

$$m^- = \frac{m}{1 - \frac{mc^2}{k}} \tag{14}$$

we can interpret the negative sign solution as an antiparticle

$$E = -mc^2 - \frac{\mathbf{p}^2}{2m^-}, \tag{15}$$

moving in the opposite direction of time with a modified inertial mass m^-.

Difference between m^+ and m^- in the first order of approximation is

$$\frac{|m^+ - m^-|}{m^+} \simeq \frac{2m}{k}. \tag{16}$$

For the electron and positron this fraction will be

$$\frac{m_{e^+} - m_{e^-}}{m_{e^-}} \simeq \frac{2m_{e^-}}{k} \approx 10^{-22}. \tag{17}$$

Particle Data Group [12] has found

$$\frac{m_{e^+} - m_{e^-}}{m_{e^-}} < 8 \times 10^{-8}, \tag{18}$$

for this mass difference. The smallness of this number shows that quantum gravity effects are very tiny.

From the mass difference between electron and positron Eq. (17), we can obtain a bound on the photon mass as

$$m_\gamma^2 = D\frac{\alpha}{\pi}\Delta m^2 \Rightarrow m_\gamma = 2.5 \times 10^{-18} eV, \tag{19}$$

where α is the fine structure constant and D is a constant of the order of unity [13].

Also, by using of this difference Eq. (17) we can also put a lower bound [14] on the parameter k from Kaon physics

$$k > 1.1 \times 10^{18} \text{ GeV}. \tag{20}$$

The value of this parameter should be equal to the Planck energy, but it can, in principle be different from this value.

We can calculate the nonrelativistic limit of the velocity of a particle. The group velocity is

$$\mathbf{v}_g = \frac{\partial E}{\partial \mathbf{p}} \simeq \frac{\mathbf{p}}{m^+}, \tag{21}$$

and the particle velocity is

$$\mathbf{v}_{particle} = \frac{\mathbf{p}c^2}{E} \simeq \frac{\mathbf{p}}{m}. \tag{22}$$

In the ordinary special relativity we have the same value of $\mathbf{p}/m$ for the group velocity $\mathbf{v}_g$ and the particle velocity $\mathbf{v}_{particle}$. Here, we showed that the group velocity differ from the special relativistic value, but the particle velocity is the same as in the special relativity. In the nonrelativistic limit of MS DSR, the situation for velocities is inverse to our case and we have $\mathbf{v}_g \simeq \mathbf{p}/m$ and $\mathbf{v}_{particle} \simeq \mathbf{p}/m^+$ [6].

3.1. Comparison with the nonrelativistic limit of the MS DSR

We can compare the nonrelativistic limit of the Amelino-Camelia DSR with the nonrelativistic limit of the MS DSR. In MS DSR, the modified dispersion relation is

$$\frac{E^2 - \mathbf{p}^2 c^2}{1 - E^2/k^2} = m^2 c^4. \tag{23}$$

M. Coraddu and S. Mignemi [6] have reached to the modified relation

$$E = m^+ c^2 + \frac{\mathbf{p}^2}{2m}, \tag{24}$$

instead of Eq. (12) in which m^+ has the same value with Eq. (13). In the MS DSR case, the particle's rest mass is modified, and the inertial mass remains unmodified. Thus, we have an interesting result: the nonrelativistic limit of Magueijo-Smolin DSR is reciprocal to the nonrelativistic limit of Amelino-Camelia DSR.

We summarize the comparison of the nonrelativistic limit of Amelino-Camelia and MS doubly special relativity in the following table.

Amelino-Camelia DSR	Magueijo-Smolin DSR
$E = mc^2 + \mathbf{p}^2/2m^+$	$E = m^+c^2 + \mathbf{p}^2/2m$
$\mathbf{v}_g = \mathbf{p}/m^+$	$\mathbf{v}_g = \mathbf{p}/m$
$\mathbf{v}_{particle} = \mathbf{p}/m$	$\mathbf{v}_{particle} = \mathbf{p}/m^+$

3.2. Amelino-Camelia and MS DSR theories in one DSR

A general DSR can contain the Amelino-Camelia and MS DSR theories in one DSR. This goal can be reachable at least in the first order of approximation of the Planck length. We have obtained a general finite boost for the DSR in the first order of the Planck length [15]. In this DSR, dispersion relation is

$$E^2 - \mathbf{p}^2 - \frac{2\alpha}{k}E^3 + \frac{2\beta}{k}E\mathbf{p}^2 = m^2, \tag{25}$$

in which α and β are free parameters, and in general are different.

The nonrelativistic limit of this dispersion relation for the particle will be

$$E = m^+_{rest}c^2 + \frac{\mathbf{p}^2}{2m^+_{inertial}}, \tag{26}$$

and the rest and inertial masses m^+_{rest} and $m^+_{inertial}$ are depended to the α and β parameters. In this DSR, the two rest and inertial masses have been modified.

3.3. CPT violation and mass splitting between particle and antiparticle

Different inertial masses m^+ and m^- in the equations (13) and (14) leads to the violation of the CPT invariance. Experimental tests of the CPT usually measure the mass differences of particles and antiparticles. However, interpreting the inertial mass as the mass of an elementary particle may have some difficulties, in principle, modification of mass-shell relation in Eq. (9) can be interpreted as CPT violation.

CPT violation is a necessary but not sufficient condition for the particle-antiparticle mass splitting. In a Lorentz invariant quantum field theory, CPT-violating interaction alone does not split the masses of a particle and antiparticle but breaks only the equality of lifetimes, magnetic moments, and cross sections [16].

Two necessary conditions to break the CPT theorem are [17]:

1. Non-local theory,
2. Lorentz symmetry breaking

The DSR theories are in some sense non-local theories, and we expect CPT violation in these theories. For clarifying effects of the discrete transformations of parity, charge conjugation and time reversal in the DSR theories we take for example a composition of two four-momenta as

$$(p \oplus q)_i = f p_i + g q_i + h\epsilon_{ijk} p_j q_k \tag{27}$$

where f, g, h are functions depending on the four moment and the Planck energy [18]. The parity symmetry is lost if h is nonzero, and the CPT invariance will be dependent on the f, g, h functions. Thus, the DSR theories involves a violation of CPT.

For investigating discrete and continuous aspects of the groups and their effects on the CPT violation we should re-analyse all of our discussions. However, we know that the Lorentz group is a continuous group, we can construct a discrete Lorentz group for which CPT symmetry can be broken differently [19].

4. Nonrelativistic limits of the modified Klein-Gordon and Dirac equations

We now consider the nonrelativistic limit of modified Klein-Gordon and Dirac equations. The nonrelativistic limit of the ordinary Klein-Gordon and Dirac equations is given in many relativistic quantum mechanics and quantum field theory books [5].

4.1. Ordinary Klein-Gordon equation

The ordinary Klein-Gordon equation reads

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2c^2)\Psi(\vec{x},t) = 0. \tag{28}$$

We define new operator $M = \sqrt{m^2c^4 - c^2\vec{\nabla}^2}$ and rewrite the Klein-Gordon equation as

$$-\frac{\partial^2}{\partial t^2}\Psi = M^2\Psi. \tag{29}$$

By introducing new fields

$$\phi^\pm = \Psi \pm iM^{-1}\frac{\partial\Psi}{\partial t}, \tag{30}$$

we can convert the Klein-Gordon equation to the first-order equations

$$i\frac{\partial\phi^\pm}{\partial t} = \pm M\phi^\pm. \tag{31}$$

In the non-relativistic limit $(\mathbf{p}^2c^2/m^2c^4) \ll 1$ or large c we have

$$M \simeq mc^2 - \frac{1}{2m}\vec{\nabla}^2. \tag{32}$$

For subtracting the rest of the mass-energy, we transform to the fields $\tilde{\phi}^\pm = \exp(\pm imc^2t)\phi^\pm$ and we reach to the Schrödinger equations

$$i\frac{\partial\tilde{\phi}^\pm}{\partial t} = \mp\frac{1}{2m}\vec{\nabla}^2\tilde{\phi}^\pm. \tag{33}$$

The positive sign in the $\tilde{\phi}^\pm$ corresponds to the particle solution and the negative sign to the antiparticle case [6].

4.2. Modified Klein-Gordon equation

Modified Klein-Gordon equation (4) in $O(E^3/k^3)$ approximation is

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi + \frac{i}{k}\frac{\partial}{\partial t}\vec{\nabla}^2\Phi - (m^2c^2 - \vec{\nabla}^2)\Phi = 0. \tag{34}$$

We can rewrite this equation as

$$-\frac{\partial^2}{\partial t^2}\Phi = \left[m^2c^4 - \left(1 + \frac{i}{k}\frac{\partial}{\partial t}\right)\vec{\nabla}^2\right]\Phi. \tag{35}$$

By defining a new operator

$$\tilde{M} = \sqrt{m^2c^4 - c^2\left(1 + \frac{i}{k}\frac{\partial}{\partial t}\right)\vec{\nabla}^2}, \tag{36}$$

and introducing new fields

$$\psi^\pm = \Phi \pm i\tilde{M}^{-1}\frac{\partial\Phi}{\partial t}, \tag{37}$$

we can convert the modified Klein-Gordon equation to the first-order equations

$$i\frac{\partial\psi^\pm}{\partial t} = \pm\tilde{M}\psi^\pm. \tag{38}$$

In the non-relativistic limit $(\mathbf{p}^2c^2/m^2c^4) \ll 1$ or large c we have

$$\tilde{M} \simeq mc^2 - \frac{1}{2m}\left(1 + \frac{i}{k}\frac{\partial}{\partial t}\right)\vec{\nabla}^2. \tag{39}$$

By transforming to the fields $\tilde{\psi}^\pm = e^{\pm imc^2t}\psi^\pm$, we reach to the Schrödinger equations

$$i\frac{\partial\tilde{\psi}^\pm}{\partial t} = \mp\frac{1}{2m^\pm}\vec{\nabla}^2\tilde{\psi}^\pm. \tag{40}$$

As expected, the particle and the antiparticle states are satisfied in the Schrödinger equations with the modified inertial masses m^+ and m^-.

4.3. Dirac equation

Ordinary Dirac Equation is

$$(i\gamma^\mu\frac{\partial}{\partial x^\mu} - m)\psi = 0. \tag{41}$$

Defining two-component form as

$$\psi(\vec{x},t) = \begin{pmatrix}\chi\\ \eta\end{pmatrix}, \tag{42}$$

and using the standard form of γ matrices

$$\gamma^0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \gamma^i = \begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}, \tag{43}$$

we reach to the coupled equations

$$\begin{cases}(E-m)\,\chi = \vec{\sigma}.\vec{p}\,\eta\\ (E+m)\,\eta = \vec{\sigma}.\vec{p}\,\chi\end{cases} \tag{44}$$

Note that $p^\mu = (E,\mathbf{p})$, $p_\mu = (E,-\mathbf{p})$ and $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

In the nonrelativistic limit for the particle solution, we take $E \simeq m$ and $(\mathbf{p}^2c^2/m^2c^4) \ll 1$ which leads to $\eta \simeq (\vec{\sigma}.\vec{p}/2m)\chi$. Substituting this in Eq. (44) and introducing $\chi' = e^{+imt}\chi$, we reach to the following Schrödinger equation

$$i\frac{\partial\chi'}{\partial t} \simeq -\frac{1}{2m}\vec{\nabla}\cdot 2\chi'. \tag{45}$$

For the antiparticle solution in the nonrelativistic limit we take $E \simeq -m$. Doing as in particle case and introducing $\eta' = e^{-imt}\eta$, we can also show that η' satisfies in a similar Schrödinger equation

$$i\frac{\partial\eta'}{\partial t} \simeq +\frac{1}{2m}\vec{\nabla}^2\eta'. \tag{46}$$

4.4. Modified Dirac equation

Modified Dirac equation Eq. (5) in the $O(E^2/k^2)$ approximation is

$$\left[i\gamma^0\frac{1}{c}\frac{\partial}{\partial t} + i\gamma^i\frac{\partial}{\partial x^i}\left(1 + \frac{i}{2k}\frac{\partial}{\partial t}\right) - m\right]\tilde{\psi} = 0. \tag{47}$$

In this case, we obtain the two-component by introducing

$$\tilde{\psi}(\vec{x},t) = \begin{pmatrix}\tilde{\chi}\\ \tilde{\eta}\end{pmatrix}. \tag{48}$$

Doing as in the unmodified case, we reach the following coupled equations

$$\begin{cases}[E-m]\;\tilde{\chi} = (1+\frac{E}{2k})\vec{\sigma}.\vec{p}\,\tilde{\eta},\\ [E+m]\;\tilde{\eta} = (1+\frac{E}{2k})\vec{\sigma}.\vec{p}\,\tilde{\chi}.\end{cases} \tag{49}$$

In the nonrelativistic limit $(\mathbf{p}^2c^2/m^2c^4) \ll 1$, we take $E \simeq m$ for the particle solution, and we have

$$\tilde{\eta} \simeq (1 + \frac{E}{2k})\frac{\vec{\sigma}.\vec{p}}{2m}\tilde{\chi}. \tag{50}$$

By introducing $\tilde{\chi}' = \exp(+imc^2t)\tilde{\chi}$, we reach to the following Schrödinger equation

$$i\frac{\partial\tilde{\chi}'}{\partial t} \simeq -\frac{1}{2m^+}\vec{\nabla}^2\tilde{\chi}'. \tag{51}$$

We note that $\tilde{\chi}'$ satisfies in the Schrödinger equation with the modified inertial mass m^+. Also, for the antiparticle we take $E \simeq -m$ and we have

$$\tilde{\chi} \simeq -(1 + \frac{E}{2k})\frac{\vec{\sigma}.\vec{p}}{2m}\tilde{\eta}. \tag{52}$$

If we define $\tilde{\eta}' = \exp(-imc^2 t)\tilde{\eta}$ we can show that $\tilde{\eta}'$ satisfies in a similar Schrödinger equation

$$i\frac{\partial \tilde{\eta}'}{\partial t} \simeq +\frac{1}{2m^-}\vec{\nabla}^2\tilde{\eta}', \tag{53}$$

in which the modified mass m^- is given by Eq. (14).

5. Conclusion and some remarks

In this paper, we showed that the nonrelativistic limit of Amelino-Camelia DSR leads to the particle and the antiparticle with different inertial masses m^+ and m^- but the same rest mass. We interpreted this difference as a sign of the CPT violation. Also, we studied the corresponding modified Klein-Gordon and Dirac equations, which reproduce the Schrödinger equations in the nonrelativistic limit with these modified masses for the particle and the antiparticle.

The ratio $|m^+ - m^-|/m$ is equal to $2mc^2/k$. We used this ratio to find an upper limit on the photon mass. Also, we have obtained a lower bound on the amount of k. In DSR theories k is a fundamental constant with dimension of energy and can be different form the Planck energy $E_p = \sqrt{\hbar c^5/G} \simeq 10^{19}$.

We compared the nonrelativistic limit of Amelino-Camelia and Magueijo-Smolin DSRs. In this limit, we reached the $E = mc^2 + \mathbf{p}^2/2m^+$ for Amelino-Camelia DSR, also we have $E = m^+c^2 + \mathbf{p}^2/2m$ for MS DSR [6]. These findings are natural results of the dispersion relations. We put the $O(E^3/k^3)$ approximation of Amelino-Camelia dispersion relation Eq. (2) and the MS dispersion relation [6] together:

$$\begin{cases} E^2 = \mathbf{p}^2c^2 + m^2c^4(1 - E/k)^2 & \text{Magueijo-Smolin} \\ E^2 \simeq \mathbf{p}^2c^2(1 + E/k) + m^2c^4 & \text{Amelino-Camelia} \end{cases} \tag{54}$$

Then, observing their similarities and differences is easy, and interpreting the nonrelativistic limit of MS DSR as modifying rest mass and Amelino-Camelia DSR as modifying momentum is not difficult. Thus, in the nonrelativistic limit, the MS and Amelino-Camelia DRS theories seem to be complementary to each other.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Nosratollah Jafari reports financial support was provided by Ministry of Science and Higher Education of the Republic of Kazakhstan. Bekdaulet Shukirgaliyev reports a relationship with Ministry of Science and Higher Education of the Republic of Kazakhstan that includes: funding grants. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgement

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19677351 and Program No. BR21881880).

Also, we would like to thank Dr. Ahmad Shariati for his kind and valuable discussions.

References

[1] J. Magueijo, L. Smolin, Phys. Rev. Lett. 88 (2002) 190403, arXiv:hep-th/0112090.
[2] G. Amelino-Camelia, Int. J. Mod. Phys. D 11 (2002) 35, arXiv:gr-qc/0012051.
[3] G. Amelino-Camelia, Int. J. Mod. Phys. D 11 (2002) 1643, arXiv:gr-qc/0210063.
[4] J. Kowalski-Glikman, Lect. Notes Phys. 669 (2005) 131, arXiv:hep-th/0405273.
[5] F. Gross, John Willey and Sons Publishing Inc., 1993.
[6] M. Coraddu, S. Mignemi, Europhys. Lett. 91 (2010) 51002, arXiv:0911.4241 [hep-th].
[7] N.R. Bruno, G. Amelino-Camelia, J. Kowalski-Glikman, Phys. Lett. B 522 (2001) 133, arXiv:hep-th/0107039.
[8] P. Gosselin, A. Berard, H. Mohrbach, S. Ghosh, Phys. Lett. B 660 (2008) 267, arXiv:0709.0579 [hep-th].
[9] Z. Belhadi, F. Menas, A. Berard, P. Gosselin, H. Mohrbach, Int. J. Mod. Phys. A 27 (2012) 1250031, arXiv:1106.1302 [hep-th].
[10] E. Harikumar, M. Sivakumar, Mod. Phys. Lett. A 26 (2011) 1103, arXiv:0910.5778 [hep-th].
[11] S.A. Franchino-Viñas, J.J. Relancio, Class. Quantum Gravity 40 (2023) 054001, arXiv:2203.12286 [hep-th].
[12] K.A. Olive, et al., Particle Data Group, Chin. Phys. C 38 (2014) 090001.
[13] A.D. Dolgov, EPJ Web Conf. 95 (2015) 03007, arXiv:1411.2280 [astro-ph.CO].
[14] B. Grinstein, PoS KAON13 (2013) 002.
[15] N. Jafari, M.R.R. Good, Phys. Lett. B 135735 (2020), arXiv:2009.06096 [gr-qc].
[16] L. Álvarez-Gaumé, M.M. Chaichian, M.A. Oksanen, A. Tureanu, Phys. Lett. B 850 (2024) 138483, arXiv:2310.16008 [hep-th].
[17] K. Fujikawa, A. Tureanu, Int. J. Mod. Phys. A 32 (2017) 1741014, arXiv:1607.01409 [hep-ph].
[18] J.M. Carmona, J.L. Cortés, J.J. Relancio, Symmetry 13 (2021) 1266, arXiv:2106.05904 [hep-ph].
[19] M. Lorente, P.D. Kramer, J. Phys. A 32 (1999) 2481, arXiv:hep-lat/0401019.