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$\mathrm{SL}(2,R) \times \mathrm{U}(1)$ symmetry and quasinormal modes in the self-dual warped AdS black hole

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Abstract: The algebraic approach to the spectrum of quasinormal modes has been made as simple as possible for the BTZ black hole by the strategy developed in [16]. By working with the self-dual warped AdS black hole, we demonstrate in an explicit way that such a strategy can be well adapted to those warped AdS balck holes with the $\mathrm{SL}(2,R) \times \mathrm{U}(1)$ isometry. To this end, we first introduce two associated tensor fields with the quadratic Casimir of $\mathrm{SL}(2,R)\times\mathrm{U}(1)$ Lie algebra in the self-dual warped AdS black hole and show that they correspond essentially to the metric and volume element up to a constant prefactor, respectively. Then without appealing to any concrete coordinate system, we can further show that the solutions to the equations of motion for the scalar, vector, spinor fields all fall into the representations of the $\mathrm{SL}(2,R) \times \mathrm{U}(1)$ Lie algebra by a purely abstract tensor and spinor analysis. Accordingly, the corresponding spectrum of quasinormal modes for each fixed azimuthal quantum number can be derived algebraically as the infinite tower of descendants of the highest weight mode of the $\mathrm{SL}(2,R)$ Lie subalgebra.

Keywords: AdS-CFT Correspondence, Black Holes, Space-Time Symmetries

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1 Introduction

As we know, not only does the photon ring outside of the black hole play a vital role in predicting the intricate patterns of the black hole image, but also controls the photon sphere quasinormal ringdowns of massless fields such as the gravitational waves. Very recently, it has been discovered in [1] that the photon sphere quasinormal modes exhibit an emergent SL$(2,R)$ symmetry for the static spherically symmetry black hole, namely the spectrum of the photon sphere quasinormal modes forms the highest weight representation of the emergent SL$(2,R)$ symmetry, which is further found to persist also for the stationary rotating Kerr black hole [2]. Later on, to gain a deeper understanding of this emergent SL$(2,R)$ symmetry, the authors in [3] initiate the exploration of the self-dual warped AdS black hole, which arises as an approximation to the near-extremal Kerr black hole with the SL$(2,R)$ isometry. In particular, the resultant spectrum of quasinormal modes by explicit calculation can be identified as the highest weight representation of the SL$(2,R)$ isometry. It is further shown in [4] that the aforementioned emergent SL$(2,R)$ symmetry associated with the spectrum of the photon sphere quasinormal modes, if redefined appropriately by virtue of the residual degrees of freedom, can be regarded precisely as the eikonal limit of the SL$(2,R)$ isometry.

As in the context of AdS/CFT correspondence, the identification of such a symmetry is suspected to offer us a guiding principle to search for the holographic dual field theory description of the black hole under consideration through Kerr/CFT correspondence and warped AdS/CFT correspondence, respectively [5–8]. In particular, the SL$(2,R)$ symmetry dictated quasinormal modes correspond to the poles of the retarded Green function of the putative dual CFT. Such a specific correspondence has been confirmed in AdS_3/CFT_2, where analytic results can be readily obtained for both sides. Speaking specifically, the analytic result for the quasinormal modes of the scalar, vector, and spinor perturbations in the BTZ black hole is first obtained by directly solving the equations of motion [9, 10]. Later on, inspired by [11] and [12], the authors in [13] demonstrate that the quasinormal modes can be constructed as the left and right chiral highest weight representation of the $SL(2, R)$ isometry for the scalar and metric perturbations. This algebraic construction has been further generalized to other three dimensional black holes with the vector perturbation included [14, 15]. However, the involved calculation is somewhat complicated, which makes the $SL(2, R)$ isometry obscure albeit recovered in the final result as it should be the case. Such a technical deficiency is rescued in [16] by resorting to the two intrinsic tensor fields associated with the $SL(2, R)$ isometry of the BTZ black hole as well as the covariant derivative rather than the ordinary derivative in the course of analysis. Among others, one very advantage of this strategy is that the spinor perturbation can be incorporated and treated in a uniform manner as other tensor fields.

As alluded to in the very beginning, the $SL(2, R)$ isometry of the self-dual warped AdS black hole is supposed to play a similar role in controlling the spectrum of quasinormal modes algebraically. However, such a suspicion has so far been confirmed solely for the (massless) scalar perturbation due to its obvious simplicity. So it is tempting to check this suspicion for other perturbations explicitly by utilizing the strategy developed in [16]. The purpose of this paper is to show this is also the case for both vector and spinor perturbations. To make the whole analysis as simple as possible, we are required to introduce the two intrinsic tensor fields associated with the Casimir of the full $SL(2, R) \times U(1)$ isometry of the self-dual warped AdS black hole. This is the main difference from the BTZ black hole, where the full isometry is given by $SL(2, R)_L \times SL(2, R)_R$ but the two intrinsic tensor fields are associated with the Casimir of each $SL(2, R)$ sector. It is noteworthy that the warp factor makes the additional $U(1)$ isometry also display itself in Riemann curvature, which turns out to have an extra effect on the conformal weight of the spinor perturbation. Except for these nuances, the whole procedure devised in [16] proves to be applicable to the self-dual warped AdS black hole with the $SL(2, R) \times U(1)$ symmetry manifest throughout the whole analysis.

The rest of paper is organized as follows. In the subsequent section, we provide a brief review of the self-dual warped AdS black hole as a solution to the topological massive gravity, where the $SL(2, R) \times U(1)$ isometry is identified and relevant geometric quantities are presented in an explicit way for our later usage. In section 3, with the quadratic Casimir of $SL(2, R) \times U(1)$ Lie algebra, we introduce its Lie derivative representation and two associated tensor fields, which turn out to be proportional to the metric and the volume element, respectively. In section 4, we first derive how the solutions to the equations of motion fall in a uniform manner into the representations of the $SL(2, R) \times U(1)$ Lie algebra for the scalar, vector, and spinor field in the self-dual warped AdS black hole and then present the algebraic approach to the quasinormal modes as the highest weight representation of the $SL(2, R)$ Lie subalgebra for each fixed azimuthal quantum number, where the main bulk of tensor and spinor analysis is performed in a purely abstract way without resorting to any specific coordinate system.[1] We conclude our paper in the last section.

Notation and conventions follow [17] unless specified otherwise.

2 Self-dual warped AdS black hole, SL(2, R)×U(1) isometry and relevant geometric quantities

Let us start with the self-dual warped AdS black hole

$$ds^2 = \ell^2\left[\frac{-dT^2+dx^2}{\sinh^2 x}+\Lambda^2\left(d\phi+\frac{dT}{\tanh x}\right)^2\right], \tag{2.1}$$

where ℓ is the warped AdS radius and the warp factor Λ is so defined that the unwarped self-dual AdS black hole is given by $\Lambda = 1$. Whence the inverse metric can be obtained as

$$\begin{aligned} g^{ab} =& \frac{1}{\ell^2}\left\{-\sinh^2 x\left(\frac{\partial}{\partial T}\right)^a\left(\frac{\partial}{\partial T}\right)^b+\cosh x\sinh x\left[\left(\frac{\partial}{\partial T}\right)^a\left(\frac{\partial}{\partial \phi}\right)^b+\left(\frac{\partial}{\partial \phi}\right)^b\left(\frac{\partial}{\partial T}\right)^a\right]\right.\\ &\left.+\sinh^2 x\left(\frac{\partial}{\partial x}\right)^a\left(\frac{\partial}{\partial x}\right)^a+\frac{1-\Lambda^2\cosh^2 x}{\Lambda^2}\left(\frac{\partial}{\partial \phi}\right)^a\left(\frac{\partial}{\partial \phi}\right)^b\right\}, \end{aligned} \tag{2.2}$$

and the associated volume element reads

$$\epsilon = \Lambda\ell^3 \ \mathrm{csch}^2 x \ dT\wedge dx\wedge d\phi. \tag{2.3}$$

Such a black hole has the SL(2, R) × U(1) isometry with the corresponding Killing fields defined as follows

$$\begin{aligned} L_0^a &= -\left(\frac{\partial}{\partial T}\right)^a,\\ L_{-1}^a &= e^{-T}\left[-\cosh x\left(\frac{\partial}{\partial T}\right)^a+\sinh x\left(\frac{\partial}{\partial x}\right)^a+\sinh x\left(\frac{\partial}{\partial \phi}\right)^a\right],\\ L_{+1}^a &= e^{T}\left[-\cosh x\left(\frac{\partial}{\partial T}\right)^a-\sinh x\left(\frac{\partial}{\partial x}\right)^a+\sinh x\left(\frac{\partial}{\partial \phi}\right)^a\right],\\ W_0^a &= \left(\frac{\partial}{\partial \phi}\right)^a, \end{aligned} \tag{2.4}$$

whereby the commutators satisfy the following SL(2, R) × U(1) Lie algebra

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \quad [L_{+1}, L_{-1}] = 2L_0, \quad [W_0, L_m] = 0. \tag{2.5}$$

with $m = 0, \pm 1$. Note that our self-dual warped AdS black hole is not locally maximally symmetric. Instead, one can show that the corresponding Riemann tensor, Ricci tensors,

[1] To our best knowledge, the equation of motion for the metric perturbation on top of the warped AdS is essentially third order in the topological massive gravity, and has not been tamed into so well understood a form as in AdS. So we leave it to future work.

Ricci scalar, and Einstein tensor are given by

$$\begin{aligned} R_{abcd} &= 2(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) - Rg_{a[c}g_{d]b}, \\ R_{ab} &= \frac{\Lambda^2 - 2}{2\ell^2} g_{ab} + \frac{1-\Lambda^2}{\Lambda^2\ell^4} W_{0a}W_{0b}, \\ R &= \frac{\Lambda^2 - 4}{2\ell^2}, \\ G_{ab} &= \frac{\Lambda^2}{4\ell^2} g_{ab} + \frac{1-\Lambda^2}{\Lambda^2\ell^4} W_{0a}W_{0b}. \end{aligned} \tag{2.6}$$

As one can see, the Riemann tensor receives an extra contribution from the U(1) generator. In particular, this U(1) generator will also leave its footprint in the following expression, i.e.,

$$R_{abcd}\epsilon^{ab}{}_e\epsilon^{cd}{}_f = \frac{\Lambda^2}{\ell^2} g_{ef} + \frac{4(1-\Lambda^2)}{\Lambda^2\ell^4} W_{0e}W_{0f}, \tag{2.7}$$

which, as mentioned before and shown later, will have an effect on the conformal weight of the spinor perturbation. On the other hand, although the Weyl tensor vanishes automatically, the self-dual warped AdS black hole is not conformally flat because the Cotton tensor defined as $C_{ab} = \epsilon_a{}^{cd}\nabla_c(R_{db} - \frac{1}{4}Rg_{db})$ is given by

$$C_{ab} = \frac{\Lambda(1-\Lambda^2)}{2\ell^3} g_{ab} + \frac{3(\Lambda^2-1)}{2\Lambda\ell^5} W_{0a}W_{0b}, \tag{2.8}$$

which does not vanish when $\Lambda \neq 1$. Accordingly, we have

$$G_{ab} - \frac{4-\Lambda^2}{12\ell^2} g_{ab} + \frac{2\ell}{3\Lambda} C_{ab} = 0, \tag{2.9}$$

which means that our self-dual warped AdS black hole can be supported by the gravitational Chern-Simons term as a solution to the topological massive gravity [3].

3 Quadratic Casimir, its Lie derivative representation and two associated tensor fields

The quadratic Casimir operator of $SL(2, R) \times U(1)$ Lie algebra is defined as

$$C^2 = L^2 + \left(1 - \frac{1}{\Lambda^2}\right) W_0^2 \tag{3.1}$$

with

$$L^2 = L_0^2 - \frac{1}{2}(L_{+1}L_{-1} + L_{-1}L_{+1}) \tag{3.2}$$

the Casimir operator of $SL(2, R)$ Lie subalgebra. Note that the Lie derivative of tensor and spinor fields obeys $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$ and $\mathcal{L}_{\alpha X} = \alpha\mathcal{L}_X$ for the arbitrary Killing vector fields X and Y with the arbitrary constant α. Therefore the above Lie algebra can be naturally represented by the Lie derivative. In particular, the quadratic Casimir operators can be realized by the Lie derivative as follows

$$\mathcal{C}^2 = \mathcal{L}^2 - \left(1 - \frac{1}{\Lambda^2}\right)\mathcal{L}_{W_0}\mathcal{L}_{W_0} = \mathcal{L}_{L_0}\mathcal{L}_{L_0} - \frac{1}{2}\left(\mathcal{L}_{L_{+1}}\mathcal{L}_{L_{-1}} + \mathcal{L}_{L_{-1}}\mathcal{L}_{L_{+1}}\right) - \left(1 - \frac{1}{\Lambda^2}\right)\mathcal{L}_{W_0}\mathcal{L}_{W_0}. \tag{3.3}$$

Now inspired by the strategy developed in [16], we construct the following two tensor fields associated with the quadratic Casimir of $\mathrm{SL}(2,R)\times\mathrm{U}(1)$, i.e.,

$$H^{ab}=L_0^aL_0^b-\frac{1}{2}(L_{+1}^aL_{-1}^b+L_{-1}^aL_{+1}^b)-\left(1-\frac{1}{\Lambda^2}\right)W_0^aW_0^b \tag{3.4}$$

and

$$Z_{abc}=L_{0a}\nabla_bL_{0c}-\frac{1}{2}(L_{+1a}\nabla_bL_{-1c}+L_{-1a}\nabla_bL_{+1c})-\left(1-\frac{1}{\Lambda^2}\right)W_{0a}\nabla_bW_{0c}.$$

Obviously, H is a symmetric tensor field. By a straightforward calculation, one can show that H is actually proportional to our metric, i.e.,

$$H^{ab}=\ell^2g^{ab}. \tag{3.5}$$

Regarding Z, we first notice that it is antisymmetric between the last two indices due to the Killing equation $\nabla_a\xi_b=\nabla_{[a}\xi_{b]}$. Second, $\nabla_bH_{ac}=0$ tells us that it is also antisymmetric between the first and third indices. So Z is virtually a 3-form, which implies that it should be proportional to the volume element. In addition, we have

$$\begin{aligned}\nabla^aZ_{abc}&=L_{0a}\nabla^a\nabla_bL_{0c}-\frac{1}{2}(L_{+1a}\nabla^a\nabla_bL_{-1c}+L_{-1a}\nabla^a\nabla_bL_{+1c})-\left(1-\frac{1}{\Lambda^2}\right)W_{0a}\nabla^a\nabla_bW_{0c}\\&=R_{cb}{}^{ad}[L_{0a}L_{0d}-\frac{1}{2}(L_{+1a}L_{-1d}+L_{-1a}L_{+1d})-\left(1-\frac{1}{\Lambda^2}\right)W_{0a}W_{0c}]=0,\end{aligned} \tag{3.6}$$

where $R_{abcd}=R_{ab[cd]}$ is used in the last step while the Killing equation and the identity

$$\nabla_a\nabla_b\xi_c=R_{cba}{}^d\xi_d \tag{3.7}$$

for any Killing field ξ are used in the first and second steps, respectively. Therefore, the proportional coefficients in front of the volume element should be constant. In particular, an explicit computation leads to

$$Z_{abc}=-\frac{\Lambda\ell}{2}\epsilon_{abc}. \tag{3.8}$$

It is noteworthy that the nice properties exhibited in eq. (3.5) and eq. (3.8) respectively for H and Z fields will be lost if one naively constructs them simply out of the quadratic Casimir of $\mathrm{SL}(2,R)$ Lie subalgebra.

4 $\mathbf{SL(2,\mathit{R})\times U(1)}$ symmetry and quasinormal modes in the self-dual warped AdS black hole

As a warm-up, let us rework with the scalar field ϕ, whose equation of motion is given by

$$(\nabla_a\nabla^a-\mu^2)\phi=0. \tag{4.1}$$

The Lie derivative acting on the scalar field gives rise to

$$\mathcal{L}_X\mathcal{L}_Y\phi=X^a\nabla_a(Y^b\nabla_b\phi)=(X^a\nabla_aY^b)\nabla_b\phi+X^aY^b\nabla_a\nabla_b\phi, \tag{4.2}$$

whereby we have

$$\mathcal{C}^2\phi = Z^a{}_a{}^b\nabla_b\phi + H^{ab}\nabla_a\nabla_b\phi = \ell^2 g^{ab}\nabla_a\nabla_b\phi = (\mu\ell)^2\phi. \tag{4.3}$$

Now let us proceed with the massive vector field A, whose equation of motion is given by

$$\epsilon_a{}^{bc}\nabla_b A_c = -\mu A_a. \tag{4.4}$$

Whence we can see that the Lorenz condition is satisfied automatically as follows

$$\nabla_a A^a = -\frac{1}{\mu}\epsilon^{abc}\nabla_a\nabla_b A_c = -\frac{1}{2\mu}\epsilon^{abc}R_{abc}{}^d A_d = -\frac{1}{2\mu}\epsilon^{abc}R_{[abc]}{}^d A_d = 0 \tag{4.5}$$

due to the cyclic identity $R_{[abc]d} = 0$. Furthermore, we have

$$\begin{aligned}\mu^2 A^d &= -\mu\epsilon^{dea}\nabla_e A_a = \epsilon^{dea}\nabla_e(\epsilon_a{}^{bc}\nabla_b A_c) = \epsilon^{ade}\epsilon_a{}^{bc}\nabla_e\nabla_b A_c\\ &= (g^{dc}g^{eb} - g^{db}g^{ec})\nabla_e\nabla_b A_c = \nabla_a\nabla^a A^d - \nabla_a\nabla^d A^a\\ &= \nabla_a\nabla^a A^d + \nabla^d\nabla_a A^a - \nabla_a\nabla^d A^a = \nabla_a\nabla^a A^d + R^{dabc}A_c g_{ab}\\ &= \nabla_a\nabla^a A^d - R^{dc}A_c.\end{aligned} \tag{4.6}$$

On the other hand, by the Lie derivative acting on this vector field, we have

$$\begin{aligned}\mathcal{L}_X\mathcal{L}_Y A_a &= X^b\nabla_b(\mathcal{L}_Y A_a) + \mathcal{L}_Y A_b\nabla_a X^b\\ &= X^b\nabla_b(Y^c\nabla_c A_a + A_c\nabla_a Y^c) + (Y^c\nabla_c A_b + A_c\nabla_b Y^c)\nabla_a X^b\\ &= (X^b\nabla_b Y^c)\nabla_c A_a + X^b Y^c\nabla_b\nabla_c A_a + (X^b\nabla_a Y^c)\nabla_b A_c + A_c X^b\nabla_b\nabla_a Y^c\\ &\quad +(Y^c\nabla_a X^b)\nabla_c A_b + A_c\nabla_b(Y^c\nabla_a X^b) - A_c Y^c\nabla_b\nabla_a X^b.\end{aligned} \tag{4.7}$$

Whence we can further obtain

$$\begin{aligned}\mathcal{C}^2 A_a &= Z^b{}_b{}^c\nabla_c A_a + H^{bc}\nabla_b\nabla_c A_a + 2Z^c{}_a{}^b\nabla_c A_b + A_c\nabla_b Z^c{}_a{}^b + A^c R_{cabd}H^{bd} - A_c R_{ad}H^{dc}\\ &= \ell^2 g^{bc}\nabla_b\nabla_c A_a + \ell\Lambda\epsilon_a{}^{cb}\nabla_c A_b - \ell^2 R_{ac}A^c\\ &= [(\mu\ell)^2 - \Lambda\mu\ell]A_a,\end{aligned} \tag{4.8}$$

where the identity (3.7) is used in the first step.

With the above experience, let us manipulate the spinor field, which is more involved. To this end, we start with the Dirac equation

$$(\gamma^a\nabla_a + \mu)\psi = 0. \tag{4.9}$$

Here $\gamma^a = e^a_I\gamma^I$ and the covariant derivative acting on the spinor field is given by $\nabla_a = \partial_a + \frac{1}{4}\omega_{IJa}\gamma^{IJ}$, where e^a_I constitute a set of orthogonal normal vector bases, and Gamma matrices satisfy $\{\gamma^I, \gamma^J\} = 2\eta^{IJ}$ with the spin connection $\omega_{IJa} = e_{Ib}\nabla_a e^b_J$ and $\gamma^{IJ} = \frac{1}{2}[\gamma^I, \gamma^J]$. Next acting on the Dirac equation with $\gamma^b\nabla_b - \mu$, we obtain

$$\begin{aligned}0 &= (\gamma^b\nabla_b - \mu)(\gamma^a\nabla_a + \mu)\psi = (\gamma^a\gamma^b\nabla_a\nabla_b - \mu^2)\psi\\ &= (g^{ab}\nabla_a\nabla_b + \gamma^{ab}\nabla_a\nabla_b - \mu^2)\psi = (\nabla_a\nabla^a - \mu^2)\psi + \gamma^{ab}\nabla_{[a}\nabla_{b]}\psi\\ &= \left(\nabla_a\nabla^a - \mu^2 + \frac{1}{8}R_{abcd}\gamma^{ab}\gamma^{cd}\right)\psi,\end{aligned} \tag{4.10}$$

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where $\gamma^{ab} = e^a_I e^b_J \gamma^{IJ}$. On the other hand, the Lie derivative of spinor fields with respect to Killing fields is given by [18]

$$\mathcal{L}_X\psi = X^a\nabla_a\psi - \frac{1}{4}\gamma^{ab}\psi\nabla_b X_a, \tag{4.11}$$

whereby we have

$$\begin{aligned}
\mathcal{L}_X\mathcal{L}_Y\psi &= X^a\nabla_a\mathcal{L}_Y\psi - \frac{1}{4}\gamma^{ab}\mathcal{L}_Y\psi\nabla_b X_a \\
&= X^a\nabla_a\left(Y^c\nabla_c\psi - \frac{1}{4}\gamma^{cd}\psi\nabla_d Y_c\right) - \frac{1}{4}\gamma^{ab}\left(Y^c\nabla_c\psi - \frac{1}{4}\gamma^{cd}\psi\nabla_d Y_c\right)\nabla_b X_a \\
&= (X^a\nabla_a Y^c)\nabla_c\psi + X^a Y^c\nabla_a\nabla_c\psi - \frac{1}{4}\gamma^{cd}\psi X^a\nabla_a\nabla_d Y_c - \frac{1}{4}(X^a\nabla_d Y_c)\gamma^{cd}\nabla_a\psi \\
&\quad -\frac{1}{4}(Y^c\nabla_b X_a)\gamma^{ab}\nabla_c\psi + \frac{1}{16}\gamma^{ab}\gamma^{cd}\psi\nabla_d(Y_c\nabla_b X_a) - \frac{1}{16}\gamma^{ab}\gamma^{cd}\psi Y_c\nabla_d\nabla_b X_a.
\end{aligned} \tag{4.12}$$

Then it is not hard to show

$$\begin{aligned}
\mathcal{C}^2\psi &= Z^a{}_a{}^c\nabla_c\psi + H^{ac}\nabla_a\nabla_c\psi - \frac{1}{4}\gamma^{cd}\psi R_{cdae}H^{ae} - \frac{1}{2}Z^a{}_{dc}\gamma^{cd}\nabla_a\psi \\
&\quad + \frac{1}{16}\gamma^{ab}\gamma^{cd}\psi\nabla_d Z_{cba} - \frac{1}{16}\gamma^{ab}\gamma^{cd}\psi R_{abde}H^e{}_c \\
&= \ell^2\nabla_a\nabla^a\psi + \frac{\Lambda\ell}{4}\epsilon^a{}_{bc}\gamma^{cb}\nabla_a\psi + \frac{\ell^2}{16}R_{abcd}\gamma^{ab}\gamma^{cd}\psi \\
&= \ell^2(\mu^2 - \frac{1}{16}R_{abcd}\gamma^{ab}\gamma^{cd})\psi - \frac{\Lambda\ell}{4}\epsilon^a{}_{bc}\gamma^{bc}\nabla_a\psi \\
&= \ell^2(\mu^2 - \frac{1}{16}R_{abcd}\epsilon^{abe}\epsilon^{cdf}\gamma_e\gamma_f)\psi + \frac{\Lambda\ell}{2}\gamma^a\nabla_a\psi \\
&= \left[(\mu\ell)^2 - \frac{\Lambda\mu\ell}{2}\right]\psi - \left(\frac{\Lambda^2}{16}g_{ef} + \frac{1-\Lambda^2}{4\Lambda^2\ell^2}W_eW_f\right)\gamma^{(e}\gamma^{f)}\psi \\
&= \left[(\mu\ell)^2 - \frac{\Lambda\mu\ell}{2}\right]\psi - \left(\frac{3\Lambda^2}{16} + \frac{1-\Lambda^2}{4}\right)\psi \\
&= \left[(\mu\ell)^2 - \frac{\Lambda\mu\ell}{2} + \frac{\Lambda^2-4}{16}\right]\psi,
\end{aligned} \tag{4.13}$$

where the identity $\gamma^{ab} = \epsilon^{abc}\gamma_c$ and eq. (2.7) have been used in the fourth and fifth steps, respectively.

So the upshot of the whole bulk of tensor and spinor analysis presented above is that the solutions to the equations of motion for various fields all turn out to fall into the representations of the $\mathrm{SL}(2,R)\times\mathrm{U}(1)$ Lie algebra characterized by the value of the Casimir, i.e.,

$$\mathcal{C}^2\Phi = \lambda\Phi \tag{4.14}$$

with $\lambda = (u\ell)^2$ for the scalar field, $\lambda = (u\ell)^2 - \Lambda\mu\ell$ for the vector field, and $\lambda = (u\ell)^2 - \frac{\Lambda\mu\ell}{2} + \frac{\Lambda^2-4}{16}$ for the spinor field.

With this in mind, we can construct the quasinormal modes by the standard algebraic approach. Namely, we start from the highest weight mode of the SL$(2,R)$ Lie subalgebra with a fixed azimuthal quantum number m as follows

$$\mathcal{L}_{W_0}\Phi^{(m0)} = im\Phi^{(m0)}, \quad \mathcal{L}^2\Phi^{(m0)} = \lambda_L\Phi^{(m0)}, \quad \mathcal{L}_{L_{+1}}\Phi^{(m0)} = 0, \quad \mathcal{L}_{L_0}\Phi^{(m0)} = h\Phi^{(m0)}, \tag{4.15}$$

where the component of Φ^{m0} associated with the orthogonal normal basis chosen in appendix B can be written formally as

$$\Phi^{(m0)} = e^{-hT+im\phi}\Psi^{(m0)}(x) \tag{4.16}$$

with the conformal weight h and the eigenvalue λ_L of the SL$(2,R)$ Casmir given respectively by

$$h = \frac{1 \pm \sqrt{1+4\lambda_L}}{2}, \quad \lambda_L = \lambda - \frac{(\Lambda^2-1)m^2}{\Lambda^2}. \tag{4.17}$$

Then the quasinormal modes can be obtained as the infinite tower of the descendent modes, i.e.,

$$\Phi^{(mn)} = \mathcal{L}^n_{L_{-1}}\Phi^{(m0)} \tag{4.18}$$

with $n = 0, 1, 2, \cdots$. Note that $\mathcal{L}_{L_0}\Phi^{(mn)} = (h+n)\Phi^{(mn)}$, so we have the following spectrum of quasinormal frequencies

$$\omega_{mn} = -i(h+n) \tag{4.19}$$

with the imaginary part of ω_{mn} required to be negative by definition.

We conclude this section by mentioning that in the eikonal regime $|m| \gg 1$, λ in eq. (4.17) can be neglected. This amounts to saying that the effects from the mass and spin are both subleading in the eikonal limit, which substantiates the claim made in [3].

5 Conclusion

Partially motivated by the emergent SL$(2,R)$ symmetry in the photon sphere quasinormal modes of the Kerr black hole, we have successfully obtained the analytic expression for the spectrum of quasinormal modes of the scalar, vector, and spinor fields in the exactly soluble self-dual warped AdS black hole in a uniform manner by fully exploiting its SL$(2,R)\times$U(1) isometry. To achieve this, we have introduced the two tensor fields associated with the Casimir of the full SL$(2,R)\times$U(1) Lie algebra and unveiled their pleasing relations to the metric and volume element respectively. Then we show that the solutions to the equations of motion of the scalar, vector, and spinor fields all fall into the representations of the SL$(2,R)\times$U(1) Lie algebra by our tensor and spinor analysis, where no specific coordinate system is used and the aforementioned two tensor fields make SL$(2,R)\times$U(1) symmetry transparent in the whole analysis. The resultant spectrum of quasinormal modes can be further constructed as the highest weight representation of the SL$(2,R)$ Lie subalgebra.

Although the self-dual warped AdS black hole is more involved than the simplest BTZ black hole, our work demonstrates that the strategy previously developed in [16] for the algebraic approach to the spectrum of quasinormal modes in the BTZ black hole turns

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out to be utterly applicable to the self-dual warped AdS black hole. As we know, there are other three dimensional warped AdS black hole solutions with the $SL(2,R)\times U(1)$ isometry [19–21] , where a variety of quasinormal modes have been calculated out mainly by solving the equations of motion analytically [22–26]. Note that the key to making our algebraic approach work lies in the three properties of our self-dual AdS black hole exhibited in eq. (2.7), eq. (3.5), and eq. (3.8), which are believed to hold also for other warped AdS black holes except that the prefactors in front of the metric, quadratic of the U(1) generator and volume element may be varied. Thus with our present work, we are convinced that the derivation of these quasinormal modes can also be made as simple as possible by our algebraic approach.

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A An explicit calculation of Z

By $\nabla_a\xi_b = \frac{1}{2}(d\xi)_{ab}$ for any Killing field ξ, we have

$$\begin{aligned} Z_{abc} = Z_{[abc]} &= \frac{1}{2}\left[L_{0[a}dL_{0bc]} - \frac{1}{2}(L_{+1[a}dL_{-1bc]} + L_{-1[a}dL_{+1bc]}) - \left(1-\frac{1}{\Lambda^2}\right)W_{0[a}dW_{0bc]}\right] \\ &= \frac{1}{6}\left[L_0\wedge dL_{0abc} - \frac{1}{2}(L_{+1}\wedge dL_{-1abc} + L_{-1}\wedge dL_{+1abc}) - \left(1-\frac{1}{\Lambda^2}\right)W_0\wedge dW_{0abc}\right], \end{aligned} \tag{A.1}$$

where

$$L_{0a} = g_{ab}L^b_0 = -\ell^2[(\Lambda^2\coth^2 x - \mathrm{csch}^2 x)(dT)_a + \Lambda^2\coth x(d\phi)_a], \tag{A.2}$$

$$\begin{aligned} L_{-1a} &= g_{ab}L^b_{-1} = -e^{-T}\ell^2\mathrm{csch}\ x[(\Lambda^2-1)\coth x(dT)_a - (dx)_a + \Lambda^2(d\phi)_a], \\ L_{+1a} &= g_{ab}L^b_{-1} = -e^{T}\ell^2\mathrm{csch}\ x[(\Lambda^2-1)\coth x(dT)_a + (dx)_a + \Lambda^2(d\phi)_a], \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} (dL_0)_{ab} &= \ell^2\mathrm{csch}^2 x[\Lambda^2(dx)_a\wedge(d\phi)_b - 2(\Lambda^2-1)\coth x(dT)_a\wedge(dx)_b], \\ (dL_{-1})_{ab} &= e^{-T}\ell^2\mathrm{csch}\ x[(2-\Lambda^2-\Lambda^2\cosh^2 x)\mathrm{csch}^2 x(dT)_a\wedge(dx)_b - \Lambda^2(d\phi)_a\wedge(dT)_b \\ &\quad +\Lambda^2\coth x(dx)_a\wedge(d\phi)_b], \\ (dL_{+1})_{ab} &= e^{T}\ell^2\mathrm{csch}\ x[(2-\Lambda^2-\Lambda^2\cosh^2 x)\mathrm{csch}^2\ x(dT)_a\wedge(dx)_b + \Lambda^2(d\phi)_a\wedge(dT)_b \\ &\quad +\Lambda^2\coth x(dx)_a\wedge(d\phi)_b], \\ (dW_0)_{ab} &= \ell^2\Lambda^2\mathrm{csch}^2 x(dT)_a\wedge(dx)_b. \end{aligned} \tag{A.4}$$

With this, we can finally obtain

$$Z = -\frac{\Lambda^2\ell^4\mathrm{csch}^2 x}{2}dT\wedge dx\wedge d\phi. \tag{A.5}$$

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B A little bit of spinor analysis

Associated with the choice of the orthogonal normal bases as $e_0^a = \frac{\sinh x}{\ell}(\frac{\partial}{\partial T})^a - \frac{\cosh x}{\ell}(\frac{\partial}{\partial \phi})^a$, $e_1^a = \frac{\sinh x}{\ell}(\frac{\partial}{\partial x})^a$, and $e_2^a = \frac{1}{\Lambda \ell}(\frac{\partial}{\partial \phi})^a$, the non-vanishing spin connections can be written as

$$
\begin{aligned}
\omega_{01a} = -\omega_{10a} &= \frac{(2-\Lambda^2)\coth x}{2}(dT)_a - \frac{\Lambda^2}{2}(d\phi)_a, \\
\omega_{02a} = -\omega_{20a} &= -\frac{\Lambda \text{csch } x}{2}(dx)_a, \\
\omega_{12a} = -\omega_{21a} &= \frac{\Lambda \text{csch } x}{2}(dT)_a.
\end{aligned} \tag{B.1}
$$

Thus we have

$$
\begin{aligned}
\mathcal{L}_{L_0}\Psi(x) &= L_0^a \nabla_a \Psi(x) - \frac{1}{4}\gamma^{ab}\Psi(x)\nabla_b L_{0a} \\
&= L_0^a \partial_a \Psi(x) + \frac{1}{4} L_0^a \omega_{aIJ}\gamma^{IJ}\Psi(x) - \frac{1}{4}\gamma^{IJ}\Psi(x) e_I^a e_J^b \nabla_b L_{0a} \\
&= \frac{1}{4}\gamma^{IJ}\Psi(x)(L_0^a \omega_{aIJ} - e_I^a e_J^b \nabla_b L_{0a}) = 0.
\end{aligned} \tag{B.2}
$$

Similarly, we can also obtain $\mathcal{L}_{W_0}\Psi(x) = 0$.

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