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# Birdtracks of exotic $\mathrm{SU}(N)$ color structures

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Abstract: I introduce a systematic procedure for constructing complete and linearly independent sets of color structures for interactions of fields transforming under exotic representations of $\mathrm{SU}(N)$, in particular the $\mathrm{SU}(3)$ gauge group of QCD. It uncovers errors in previous results, starting with interactions of four fields including a single sextet.

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## Contents



# 1 Introduction

Physics Beyond the Standard Model (BSM) can be searched for in nonrenormalizable interactions of Standard Model (SM) particles and in the effects of new elementary particles that are not part of the SM. Due to the current absence of evidence for the latter, most efforts have concentrated on the former, in particular on constraining the parameter space of the dimension six and higher operators in the SM Effective Field Theory (SMEFT). Nevertheless, UV complete embeddings of these nonrenormalizable interactions typically need to introduce new heavy particles. These particles can transform under more complicated irreducible representations (irreps) of the color group SU(3) than the singlet, triplet and octet irreps that exhaust the repertoire of the SM.

The construction of a basis for the higher dimensional operators in an EFT involves the construction of a basis for the invariant tensors of the symmetry groups. In the case of flavor SU(3), this problem has been studied already a long time ago. Classic references for invariant tensors in products of triplet and octet representations are [1–3]. Most of these results can be generalized to products of the fundamental and adjoint representations of SU($N$) [3, 4]. The major technical difficulty for the construction of such bases lies in the fact that these tensor algebras are not freely generated. For example, $[T^a]^i{}_j$ and $f^{abc}$ are invariant tensors in the products $\mathbf{8}\otimes\mathbf{3}\otimes\overline{\mathbf{3}}$ and $\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{8}$, respectively, but the sum

$$[T^a]^i{}_j[T^b]^j{}_k - [T^b]^i{}_j[T^a]^j{}_k - \mathrm{i}f^{abc}[T^c]^i{}_k \tag{1.1}$$

of their products obviously vanishes in $\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{3}\otimes\overline{\mathbf{3}}$. There are many more non-trivial relations among products of invariant tensors [1–4] and a naive approach risks producing overcomplete sets. In the following, independence will always refer to linear independence and not algebraic independence. Correspondingly, complete sets are to be understood as spanning sets of a vector space of tensors and not as generating sets of an algebra.

If we want to include new exotic fields that transform under irreps other than the fundamental and adjoint, these classic results are not sufficient, of course. The case of SU(3)-sextets has been studied using methods inspired by the investigations of dualities in supersymmetric field theories [5]. The next step towards a complete classification of effective interactions involving SU(3) exotica has been done in [6, 7]. Their approach is based on a recursive decomposition of tensor products into irreps and a selection of SU(3)-singlets in the final product. It corresponds to integrating out heavy fields in different irreps of SU(3) in a top-down construction of effective Lagrangians. While such decompositions can be performed reliably with the aid of computers (e.g. [8–11]), one must verify that the tensors constructed this way are linearly independent and that their set is complete, as required for a systematic bottom-up exploration of BSM physics, in particular if the renormalization of these interactions is taken into account (see, e.g., [12] for examples of nontrivial relations among SU(3) tensors in such calculations).

A different systematic approach is suggested by the fact that all irreps of SU($N$) can be constructed as subspaces of the tensor product of a suitable number of fundamental and conjugated fundamental representations $\mathbf{N}$ and $\overline{\mathbf{N}}$ by enforcing permutation symmetries in the factors. Obviously, one can represent an arbitrary tensor product of irreps in the same way. It has been known for a long time that every invariant tensor in a product of fundamental and conjugated fundamental representations can be expressed as a product of Kronecker symbols $\delta^i{}_j$ and, in the case of unimodular transformations, Levi-Civita symbols $\epsilon^{i_1 i_2\cdots i_N}$ or $\overline{\epsilon}_{j_1 j_2\cdots j_N}$ [13–15]. Comprehensive proofs of this fact for GL($N$), SL($N$) and SO($N$) can be found in [15]. The proof for GL($N$) is elementary und the one for SL($N$) is not much more complicated. However, in the case of proper subgroups of GL($N$) and SL($N$), the conditions on invariant tensors are weaker and care must be taken not to overlook additional solutions. The constraints on group elements from SO($N$) have been implemented in [15] by Lagrange multipliers. Fortunately, this proof translates directly to the cases of SU($N$) by complex conjugating the matrix elements of the adjoint transformations.

Unfortunately, this result also does not guarantee that the tensors constructed in this way are linearly independent. Indeed, as described in section 3.3, tensors containing $\epsilon^{i_1 i_2 \cdots i_N}$ and $\bar{\epsilon}_{j_1 j_2 \cdots j_N}$ simultaneously can be expressed as a sum of products of Kronecker symbols. Furthermore, there are many less obvious dependencies among tensors. This is complicated by the fact that some of them are only valid for $N$ smaller than some threshold. Examples for this will be presented in sections 3.2 and 5.

If we want to systematically construct complete and linearly independent sets of invariant tensors, we require a computational test for the linear independence of tensors. For this purpose, I define the natural sesquilinear form

$$\mu_N(A,B) = \sum_{\substack{i_1\cdots \\ j_1\cdots}} \overline{A^{i_1\cdots}_{j_1\cdots}} B^{j_1\cdots}_{i_1\cdots} = \overline{\mu_N(B,A)} \tag{1.2}$$

on the vector space of SU($N$) tensors of a given rank. Generalizing an observation for products of adjoint representations of SU($N$) [2], all dependencies among tensors can then be found by computing the radical of this bilinear form, i.e. the eigenvectors of a matrix representation of this sesquilinear form with vanishing eigenvalue, as explained in section 4.3. It turns out that the number of vanishing eigenvalues can depend on $N$.

In this paper, I propose a general algorithm for constructing bases of invariant tensors describing interactions involving particles transforming under higher dimensional irreps of SU($N$). This algorithm has been implemented in the computer program TANGARA.[1] using the O'Caml [16] birdtrack libraries developed for O'Mega [17, 18].

Section 2 briefly introduces the colorflow formalism in order to establish the notation and presents a non-trival example that motivated the investigation presented here. Section 3 continues with a discussion of the pecularities of SU($N$) colorflows that follow from the tracelessness of the generators and the invariance of Levi-Civita symbols.

Section 4 presents the novel algorithm for identifying complete and linearly independent sets of invariant tensors. I apply it in section 5 to answer the questions posed by the example studied in section 2.3. Section 6 presents a revised catalogue for the simplest cases in detail and discusses in which cases it confirms the results of [6] and in which cases these results must be amended.

Appendix A briefly describes the program TANGARA, which has been used to obtain the results presented here. I will also discuss how to make the results available to Monte Carlo event generators and other tools for elementary particle physics.

## 2 Colorflows

We are faced with the task of efficiently computing the matrix elements of the inner product $\mu_N$ defined in (1.2). Fortunately, these are nothing but the "color factors" familiar from squared QCD scattering amplitudes. An efficient algorithm for their computation that works directly

[1]Note on the name: *Tangara* is a genus of spectacularly colored birds in the family *Thraupidae* found in South America.

in a product of fundamental and conjugated fundamental representations has been advocated in [19]: normalize the generators as

$$\mathrm{tr}(T^a T^b) = \delta^{ab} \tag{2.1}$$

and replace all contractions of indices in the adjoint representation of SU($N$) by

$$[T^a]^i{}_j [T^a]^k{}_l = \delta^i{}_l \delta^k{}_j - \frac{1}{N} \delta^i{}_j \delta^k{}_l \,. \tag{2.2}$$

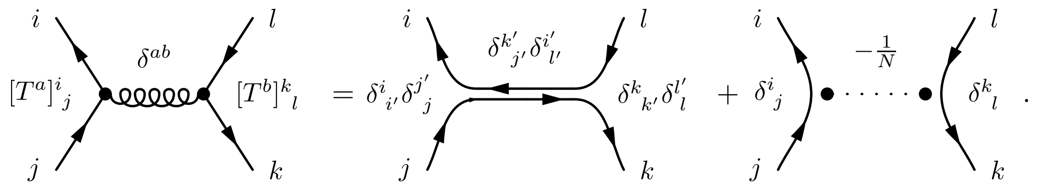
Then it only remains to keep track of the factors of $1/N$ and count the number of closed chains of Kronecker symbols

$$\delta^{i_1}{}_{i_2} \delta^{i_2}{}_{i_3} \cdots \delta^{i_n}{}_{i_1} = N \,, \tag{2.3}$$

each contributing a factor of $N$ to the color factor. Representing the Kronecker symbols by arrows leads to the colorflow representation where each closed loop corresponds to one factor of $N$ in the color factor.

This description has subsequently been developed into the comprehensive "birdtracks" approach to Lie algebras and their representations [20]. It has also been used to construct invariant tensors as building blocks for the color part of scattering amplitudes of SU(3)-triplets and octets [21–24], including implementations in computer programs [25, 26].

The identity (2.2) must be applied to two vertices in a Feynman diagram simultaneously when evaluating SU($N$) color factors. While this is not a problem for evaluating color sums for complete Feynman diagrams [27] or (1.2), it is an obstacle for the recursive algorithms that are the state of the art in perturbative calculations (see [18] and references cited therein). This can be avoided by reproducing the subtraction term in (2.2) via additional couplings to a fictitious particle, called a U(1)-ghost, whose sole purpose is to subtract the traces of the generators [28]



(2.4)

These U(1)-ghosts have to be included in the internal color exchanges and in the sums over external colors, of course. The resulting colorflow Feynman rules can automatically be derived from traditional Feynman rules as specified, e.g., in UFO [29, 30]. This has been implemented in the recursive matrix element generator O'Mega [17, 18] that is used in the general purpose Monte Carlo event generator Whizard [31].

In the colorflow Feynman rules, the couplings of the U(1)-ghost are fixed by a Ward identity to be the same as the couplings to the SU($N$)-gluons [28]. Therefore, they are not a new source of independent tensors and the formalism can be used in arbitrary orders of perturbation theory to construct complete and independent sets of interactions in color space.

### 2.1 A note on notation

In the calculations, I will keep $N \geq 2$ general as long as possible. This allows us to test the procedure by checking pecularities of SU(2) and to confirm simplifications in the limit $N \to \infty$.

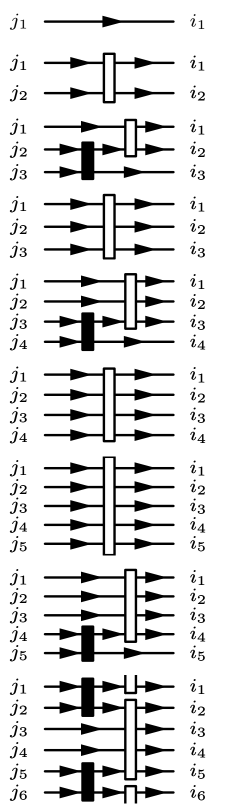
Results involving the invariance of the tensors $\epsilon^{ijk}$ and $\bar{\epsilon}_{ijk}$ apply only to SU(3), of course. Nevertheless, in applications I will only be interested in SU(3) and I shall engage in *abus de langage* throughout this paper when denoting the irreps of SU($N$). Instead of spelling out the Young tableaux, I will often use the familiar dimensions of the SU(3)-irreps, as in formula (2.5) below. With the exception of the **15** and **15′**, this is unambiguous for all small irreps and allows me to take advantage of an abbreviated notation for which much intuition as available among practitioners.

### 2.2 Exotic birdtracks

In the colorflow representation, states in the reducible product of $n$ fundamental representations are described by $n$ parallel lines with arrows pointing into the diagram. The conjugated representation has the direction of the arrows reversed. As usual, the reducible representations are decomposed into irreps by imposing the permutation symmetries specified by the standard Young tableaux consisting of $n$ boxes [20]. For a given Young tableau, one first antisymmetrizes the lines in each column and subsequently symmetrizes the lines in each row. The normalizations are chosen such that the combined (anti)symmetrizations form a projection.

Instead of repeating the comprehensive account given in section 9.5 of [20] and in [21–23], I only list the simplest building blocks[2] in order to introduce the notation

| | | |
|---|---|---|
| $\mathbf{3}$ | $\begin{array}{\|c\|}\hline j_1\\ \hline\end{array}$ | (2.5a) |
| $\mathbf{6}$ | $\begin{array}{\|c\|c\|}\hline j_1 & j_2\\ \hline\end{array}$ | (2.5b) |
| $\mathbf{8}$ | $\begin{array}{\|c\|c\|}\hline j_2 & j_1\\ \hline j_3 \\ \cline{1-1}\end{array}$ | (2.5c) |
| $\mathbf{10}$ | $\begin{array}{\|c\|c\|c\|}\hline j_1 & j_2 & j_3\\ \hline\end{array}$ | (2.5d) |
| $\mathbf{15}$ | $\begin{array}{\|c\|c\|c\|}\hline j_3 & j_1 & j_2\\ \hline j_4 \\ \cline{1-1}\end{array}$ | (2.5e) |
| $\mathbf{15'}$ | $\begin{array}{\|c\|c\|c\|c\|}\hline j_1 & j_2 & j_3 & j_4\\ \hline\end{array}$ | (2.5f) |
| $\mathbf{21}$ | $\begin{array}{\|c\|c\|c\|c\|c\|}\hline j_1 & j_2 & j_3 & j_4 & j_5\\ \hline\end{array}$ | (2.5g) |
| $\mathbf{24}$ | $\begin{array}{\|c\|c\|c\|c\|}\hline j_4 & j_1 & j_2 & j_3\\ \hline j_5 \\ \cline{1-1}\end{array}$ | (2.5h) |
| $\mathbf{27}$ | $\begin{array}{\|c\|c\|c\|c\|}\hline j_2 & j_5 & j_3 & j_4\\ \hline j_1 & j_6 \\ \cline{1-2}\end{array}$ , | (2.5i) |



[2]Note that (2.5) depicts the Young projectors described in [20], which are readily available in the birdtracks library of O'MEGA [17, 18]. We can replace these projectors by the hermitian Young projectors advocated in [22, 32], without modifying the other parts of TANGARA. In the general case, this will change some matrix elements of the inner product $\mu_N$ (1.2), but the number of vanishing eigenvalues will remain the same. There will of course be no changes at all for totally symmetric or antisymmetric irreps.



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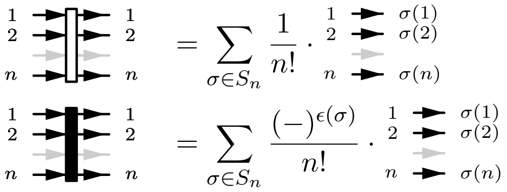
where the white boxes denote symmetrization and the black boxes antisymmetrization

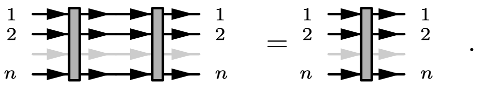


(2.6a)

(2.6b)

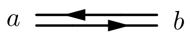
and the two parts of the symmetrizer for $i_1$ and $i_6$ in (2.5i) are to be understood as glued together at the open boundary.

Denoting the combination of all (anti)symmetrizations and the normalization factor corresponding to a Young tableau by a grey box, the projection property can be verified by connecting the arrows



(2.7)

When computing scattering amplitudes for a SU($N$) gauge theory, we use U(1)-ghosts [28] both in internal propagators and in external states when computing color sums

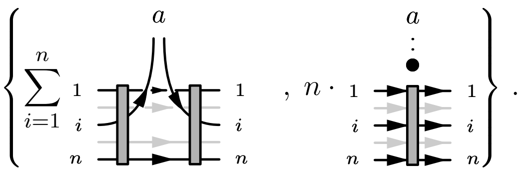


$$= \delta^{a_2}{}_{b_1} \delta^{b_2}{}_{a_1} \tag{2.8a}$$

$$a \cdots\cdots\cdots b \quad = -\frac{1}{N} \tag{2.8b}$$

as in (2.4). Indices in the adjoint representation will be written as single letters from the beginning of the latin alphabet, but they will appear in calculations as pairs of indices from the fundamental and conjugate representation (2.8a). In the case of the singlet ghosts, the indices are only written for illustration (2.8b). When constructing colorflows representing invariant tensors, we do not have to keep track of the U(1)-ghosts, because they can be added at the very end, as described in section 3.1. One could even ignore the ghosts altogether and use (2.2) for the evaluation of color summed scattering amplitudes.

The representations of the generators are invariant tensors in the product of the representation, its conjugate and the adjoint representation generators $[T^a]^{i_1 i_2 \cdots i_n}{}_{j_1 j_2 \cdots j_n}$. They are written as



(2.9)

The commutator relation

$$[T^a]^{i_1\cdots i_n}{}_{j_1\cdots j_n}[T^b]^{j_1\cdots j_n}{}_{k_1\cdots k_n} - [T^b]^{i_1\cdots i_n}{}_{j_1\cdots j_n}[T^a]^{j_1\cdots j_n}{}_{k_1\cdots k_n} = \mathrm{i} f^{abc}[T^c]^{i_1\cdots i_n}{}_{k_1\cdots k_n} \tag{2.10}$$

can be checked explicitely for any irrep considered here. The coupling to the U(1)-ghost generalizes (2.4). It drops out of (2.10), but is required to make the generator traceless and the coefficient $n$ is determined by

$$[T^a]^{i_1\cdots i_n}{}_{i_1\cdots i_n}[T^a]^{j_1\cdots j_n}{}_{j_1\cdots j_n} = 0\,. \tag{2.11}$$



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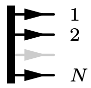
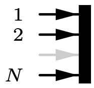
Note that, in the special cases of totally symmetric or antisymmetric states, the sum over $i$ in (2.9) is equivalent to diverting only a single line to the adjoint index and multiplying the result by $n$, but this shortcut is not available for mixed symmetries.

The totally antisymmetric rank-$n$ tensors $\epsilon^{i_1 i_2 \cdots i_n}$ and $\bar\epsilon_{j_1 j_2 \cdots j_n}$ are invariant in SU($N$) iff $n = N$. We can represent them as

$$\epsilon^{i_1 i_2 \cdots i_N} = \tag{2.12a}$$



$$\bar\epsilon_{i_1 i_2 \cdots i_N} = \quad . \tag{2.12b}$$



This has the consequence that the number of outgoing lines need not be the same as the number of incoming lines

$$\#_{\text{outgoing}} = \#_{\text{incoming}} \mod N\,. \tag{2.13}$$

In the following I will only consider the case of $\epsilon^{ijk}$ and $\bar\epsilon_{ijk}$ in SU(3).

For a systematic approach to the construction of a basis of operators involving exotic colorflows in the colorflow representation, we can start with double lines only for the adjoint representation. In a second step, we add systematically U(1)-ghosts [28] as in (2.9) to obtain the SU($N$) colorflows with traceless generators (see section 3.1 for non-trivial examples).

### 2.3 An example involving sextets and octets

The example that has motivated the present paper is the search for invariant tensors in the tensor product $\mathbf{8} \otimes \mathbf{8} \otimes \mathbf{6} \otimes \overline{\mathbf{6}}$ of SU(3) irreps. In order to make contact to the notation used in [6], I will use the correspondence

$$W^{ab\,s}{}_t = W^{a_1 b_1 s_1 s_2}_{a_2 b_2 t_1 t_2} \tag{2.14}$$

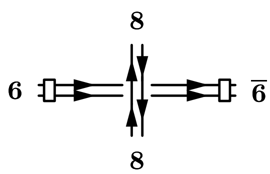
where $a, b = 1, \ldots, N^2{-}1$, $s, t = 1, \ldots, N(N{+}1)/2$, $a_i, b_i, s_i, t_i = 1, \ldots, N$ and $W$ is symmetric under the separate exchanges $s_1 \leftrightarrow s_2$ and $t_1 \leftrightarrow t_2$ for the tensors in this product. For example

$$\delta^s{}_t = \frac{1}{2!}\left(\delta^{s_1}_{t_1}\delta^{s_2}_{t_2} + +\delta^{s_1}_{t_2}\delta^{s_2}_{t_1}\right) \tag{2.15}$$

and analogously for the generators as tensors in $\mathbf{8} \otimes \mathbf{6} \otimes \overline{\mathbf{6}}$.

There are only four inequivalent ways to connect $\mathbf{8}$, $\mathbf{8}$, $\mathbf{6}$ and $\overline{\mathbf{6}}$ of SU(3), two of which are related be exchanging the factors of $\mathbf{8}$. Starting from the $\mathbf{6}$ we have the possibility to

1. connect both lines to the $\overline{\mathbf{6}}$

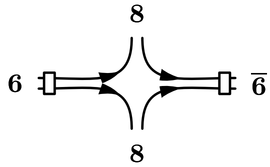
   $$\tag{2.16}$$

   producing the tensor

   $$[X^{ab}]^s{}_t = \delta^{ab}\delta^s{}_t \tag{2.17}$$

   after symmetrizing in the $\mathbf{6}$ and $\overline{\mathbf{6}}$ indices,

2. connect one line to one $\mathbf{8}$ and one to the other $\mathbf{8}$



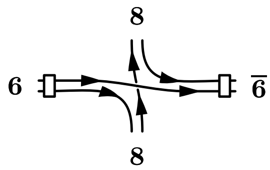
$$ \tag{2.18} $$

producing the tensor[3]

$$[Y^{ab}]^s{}_t = [\overline{K}^s_{\mathbf{6}}]_{ij}[T^a]^i{}_k[T^b]^j{}_l[K_{\mathbf{6}\,t}]^{kl} \tag{2.19}$$

which is symmetric in the exchange $a \leftrightarrow b$ since the tensors $[\overline{K}^s_{\mathbf{6}}]_{ij}$ and $[K_{\mathbf{6}\,t}]^{kl}$ are symmetric in both index pairs $(i,j)$ and $(k,l)$, or

3. connect one line to the $\overline{\mathbf{6}}$ and the other to one of the $\mathbf{8}$s



$$ \tag{2.20} $$

producing the tensor

$$[Z^{ab}]^s{}_t = [T^a_{\mathbf{6}}]^s{}_u[T^b_{\mathbf{6}}]^u{}_t - 2[Y^{ab}]^s{}_t \tag{2.21}$$

from which we can form two combinations,[4] symmetric and antisymmetric in the exchange $a \leftrightarrow b$ after symmetrizing in the $\mathbf{6}$ and $\overline{\mathbf{6}}$ indices. Note that the line connecting the two $\mathbf{8}$s is produced by the symmetrization between the factors in the products $T^a_{\mathbf{6}}T^b_{\mathbf{6}}$.

All other connections are obtained from even permutations inside the $\mathbf{6}$ and $\overline{\mathbf{6}}$.

Thus, there is a single colorflow $Z^{ab}_{\mathrm{A}} = Z^{ab} - Z^{ba}$ that is antisymmetric in the two $\mathbf{8}$s and there are three colorflows $X^{ab}$, $Y^{ab}$ and $Z^{ab}_{\mathrm{S}} = Z^{ab} + Z^{ba}$ that are symmetric in the two $\mathbf{8}$s. In section 5.1, we will see that one linear combination of the symmetric flows vanishes in the special case of SU(2) and that they remain independent for SU($N$) with $N \geq 3$.

## 3 Relations among colorflows

### 3.1 U(1)-ghosts

In the approach of [28], the identity (2.2) is replaced by the introduction of U(1)-ghosts, as in (2.4). This corresponds to including colorflows in which all possible subsets of the double

[3] In the notation of [6]

$$H^{n\;ab} = [T^a]^i{}_k[T^b]^j{}_l,\; F^{n\;s}{}_t = [\overline{K}^s_{\mathbf{6}}]_{kl}[K_{\mathbf{6}\,t}]^{ij}\,,$$

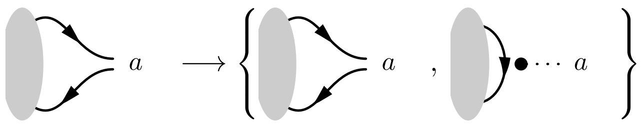
where $n$ combines the indices $i$, $j$, $k$, $l$, as in $\mathbf{27} \subset \overline{\mathbf{3}} \otimes \overline{\mathbf{3}} \otimes \mathbf{3} \otimes \mathbf{3}$.

[4] Here [6] lists only the antisymmetric commutator

$$\left[T^a_{\mathbf{6}}, T^b_{\mathbf{6}}\right] = \mathrm{i}f^{abc}T^c_{\mathbf{6}}\,,$$

using $f^{abc} = \mathrm{i}[T^a_{\mathbf{8}}]^b{}_c$, and not the symmetric anticommutator.

lines representing an index in the adjoint representation have been replaced by insertions of U(1)-ghosts

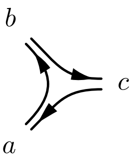


(3.1)

where I have represented the ghost by a dotted line and the rest of the diagram by a grey blob. A priori, this will replace each colorflow containing $n$ external double lines by $2^n$ colorflows, as in (3.3) below. Typically, some of these will cancel after antisymmetrization, but remain after symmetrization (see, e.g., (3.4) and (3.5), below). Note that, for the purpose of constructing inequivalent colorflows, the substitution (3.1) can be ignored until these colorflows are used in the computation of matrix elements or of the inner product $\mu_N$ (1.2) using the diagrammatical rule (2.4) instead of (2.2).

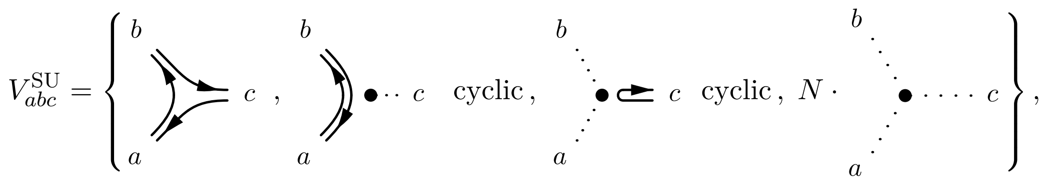
As a non-trivial example which has already been discussed in [28] in the context of the $H \to ggg$ coupling, consider the colorflow

$$V_{abc} =$$



(3.2)

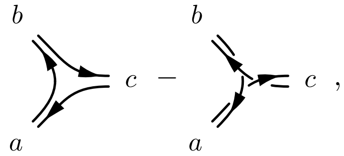
coupling three adjoint representations. Performing the substitutions (3.1) for the three external states results in $2^3 = 8$ colorflows



(3.3)

where the cyclic permutations of $(a, b, c)$ in the colorflows with one or two ghosts have not been drawn separately. The factor $N$ in front on the last colorflow arises from the closed loop remaining after replacing the double line by the U(1)-ghost on all external states. The antisymmetric combination of the $V^{\mathrm{SU}}_{abc}$ corresponds to the structure constants of the SU($N$) Lie algebra

$$\mathrm{i}f_{abc} = \mathrm{tr}\left(T_a\,[T_b, T_c]_-\right) = V^{\mathrm{SU}}_{abc} - V^{\mathrm{SU}}_{acb} =$$



(3.4)

and all U(1)-ghosts cancel, because they are symmetric. In the symmetric combination, on



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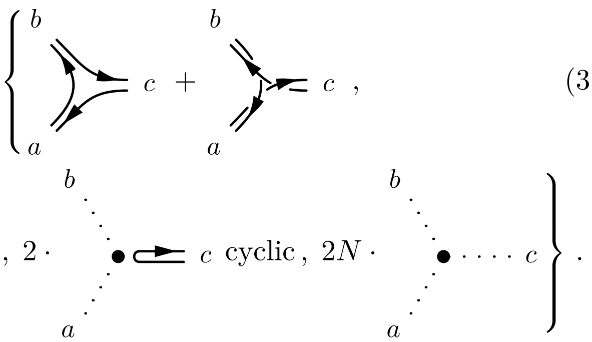
the other hand, the U(1)-ghosts add up

$$d_{abc} = \mathrm{tr}\left(T_a\,[T_b, T_c]_+\right) = V^{\mathrm{SU}}_{abc} + V^{\mathrm{SU}}_{acb} = \tag{3.5}$$




### 3.2 Spurious colorflows

The approach described in the previous subsection is straightforward for the evaluation of color factors [28], but in the present application, special care must be taken to avoid counting spurious colorflows. Indeed, the expression (3.5) does not appear to be correct for the special case of SU(2), where $d_{abc} = 0$, as can be checked directly using the Pauli matrices

$$d^{\mathrm{SU}(2)}_{ijk} = \frac{1}{\sqrt{8}}\,\mathrm{tr}\left(\sigma_i\,[\sigma_j, \sigma_k]_+\right) = \frac{1}{\sqrt{2}}\,\mathrm{tr}\left(\sigma_i\right)\delta_{jk} = 0\,. \tag{3.6}$$

Therefore, it appears that in the case of SU(2), the expression (3.5) does *not* represent an independent invariant tensor, but is a complicated way of writing 0 instead.

We can confirm this expectation by noticing that, up to permutations of the indices $a$, $b$ and $c$, $V_{abc}$ is the only possible colorflow for three adjoint representations. Thus, the symmetric and antisymmetric combinations $d_{abc}$ and $f_{abc}$ form a complete set. We can use the expressions (3.4) and (3.5) to compute an inner product in the vector space spanned by $d_{abc}$ and $f_{abc}$ by computing color sums as in [28]

$$d^{\mathrm{SU}}_{abc} d^{\mathrm{SU}}_{abc} = \frac{2(N^2-1)(N^2-4)}{N} \tag{3.7a}$$

$$d^{\mathrm{SU}}_{abc} f_{abc} = 0 \tag{3.7b}$$

$$f_{abc} f_{abc} = 2N(N^2-1)\,. \tag{3.7c}$$

This result is consistent with $d^{\mathrm{SU}(2)}_{ijk} = 0$ and $f^{\mathrm{SU}(2)}_{ijk} = \sqrt{2}\,\epsilon^{ijk}$.

Therefore, we must be aware of the fact that a naive application of the colorflow rules [28] for SU($N$) might produce sums of colorflows that are, for special values of $N$, just a complicated way of writing 0 and don't enlarge the basis. In section 4.3, I will describe a general algorithm for finding such redundancies.

Of course, the same results are obtained using (2.2) instead of the U(1)-ghosts (2.4).

### 3.3 Redundant $\epsilon$-tensors

In the case of matching dimension $N = \delta^m_m$ and rank $n$ of $\epsilon$ and $\bar\epsilon$, the tensor algebra of the $\delta^j_i$, $\epsilon^{i_1 i_2 \cdots i_n}$ and $\bar\epsilon_{j_1 j_2 \cdots j_n}$ is not freely generated. Indeed, introducing the *generalized*

*Kronecker* $\delta$ symbol

$$\delta^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_n} = \sum_{\sigma \in S_n} (-1)^{\varepsilon(\sigma)} \delta^{i_1}_{\sigma(j_1)} \delta^{i_2}_{\sigma(j_2)} \cdots \delta^{i_n}_{\sigma(j_n)}$$
$$= \sum_{\sigma \in S_n} (-1)^{\varepsilon(\sigma)} \delta^{\sigma(i_1)}_{j_1} \delta^{\sigma(i_2)}_{j_2} \cdots \delta^{\sigma(i_n)}_{j_n} = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_n} \\ \delta^{i_2}_{j_1} & \delta^{i_2}_{j_2} & \cdots & \delta^{i_2}_{j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{i_n}_{j_1} & \delta^{i_n}_{j_2} & \cdots & \delta^{i_n}_{j_n} \end{pmatrix}, \tag{3.8}$$

there is the relation $\forall n = N \in \mathbf{N}$ with $N \geq 2$:

$$\epsilon^{i_1 i_2 \cdots i_n} \bar{\epsilon}_{j_1 j_2 \cdots j_n} = \delta^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_n}, \tag{3.9}$$

which follows from antisymmetry and the choice of normalization $\epsilon^{12\cdots n} = 1 = \bar{\epsilon}_{12\cdots n}$ alone. Contracting $k$ indices in the relation (3.9), we find $\forall k, n, N \in \mathbf{N}$ with $0 \leq k \leq n = N \geq 2$:

$$\epsilon^{m_1 \cdots m_k i_{k+1} \cdots i_n} \bar{\epsilon}_{m_1 \cdots m_k j_{k+1} \cdots j_n} = k!\, \delta^{i_{k+1} i_{k+2} \cdots i_n}_{j_{k+1} j_{k+2} \cdots j_n}. \tag{3.10}$$

Because the left hand side of (3.9) is the most concise description of the $n!$ terms on the right hand side, it is tempting to keep it in the basis. On the other hand, replacing the left hand side immediately by the right hand side is the most symmetric evaluation rule possible and I will adopt it, including the rules (3.10) obtained by contracting pairs of indices.

## 4 Enumerating colorflows

Having identified all the dependencies, I can now describe the algorithm for constructing a basis for the invariant tensors in products of irreps of SU($N$).

### 4.1 Selection rules

Since all external states must be connected to the corresponding number of incoming or outgoing colorflow lines, not all products of irreps can contain invariant tensors. We start by summing the number of boxes in the Young diagrams corresponding to the irreps of particles and those of antiparticles. Each adjoint representation counts as one box for a particle and one box for an antiparticle. These sums correspond to the overall number of incoming and outgoing lines, respectively. They can only differ by $\nu N$ with $\nu \in \mathbf{Z}$ for SU($N$). Iff $\nu < 0$, the tensor contains exactly $|\nu|$ factors of $\epsilon^{i_1 i_2 \cdots i_N}$ and iff $\nu > 0$, there are $\nu$ factors of $\bar{\epsilon}_{i_1 i_2 \cdots i_N}$. According to the conventions described in section 3.3, $\epsilon^{i_1 i_2 \cdots i_N}$ and $\bar{\epsilon}_{i_1 i_2 \cdots i_N}$ must not appear together in the same tensor.

For the example from section 2.3, we have $1 + 1 + 2$ incoming lines from **8**, **8** and **6**, the same number of outgoing lines from **8**, **8** and $\overline{\mathbf{6}}$. Therefore, no $\epsilon$ or $\bar{\epsilon}$ appear in this example.

### 4.2 Combinatorics

Having established the number of $\epsilon$s or $\bar{\epsilon}$s required, we can proceed by drawing *all* combinations of arrows starting at a particle or at an $\epsilon$ and ending at an antiparticle or at an $\bar{\epsilon}$. The lines starting at the same particle or at the same $\epsilon$ obey symmetrization and antisymmetrization conditions specified by the Young tableau describing the irrep. Therefore there will be equivalent colorflows that should not be counted more than once. The same applies to lines ending at the same antiparticle or $\bar{\epsilon}$.

In principle, the procedure described in section 4.3 will weed out all double counting. In the worst case, the size of the matrices to be diagonalized in that step can grow with a factorial of the number of all arrows. Thus it is worthwhile to reduce the size of these matrices by keeping only one representative of obviously equivalent color flows. Therefore, we proceed as follows:

1. Create a list $S$ of starting points of lines (adjoints, products of fundamental representations and $\epsilon$s). Adjoints and fundamental representations are represented by a single integer $n$ identifying the external state. The factors in products of fundamental representations are represented by the integer $n$ denoting the external state combined with a second integer $i$ identifying the factor, i.e. $(n, i)$. Analogously for each $\epsilon$, but we must treat them as indistinguishable. In the example of section 2.3, we have $S = \{1, 2, (3, 1), (3, 2)\}$ if the four external states in $\mathbf{8} \otimes \mathbf{8} \otimes \mathbf{6} \otimes \overline{\mathbf{6}}$ are enumerated from 1 to 4.

2. Create a corresponding list $E$ of endpoints of lines (adjoints, products of conjugate representations and $\bar{\epsilon}$s). In the example of section 2.3, we have $E = \{1, 2, (4, 1), (4, 2)\}$.

3. Generate all the permutations of $E$, i.e. all one-to-one maps $S \to E$. In the example of section 2.3, there are 4! maps:

$$\begin{bmatrix} 1 & \to & 2 \\ 2 & \to & 1 \\ (3,1) & \to & (4,1) \\ (3,2) & \to & (4,2) \end{bmatrix}, \begin{bmatrix} 1 & \to & (4,1) \\ 2 & \to & (4,2) \\ (3,1) & \to & 1 \\ (3,2) & \to & 2 \end{bmatrix}, \begin{bmatrix} 1 & \to & 2 \\ 2 & \to & (4,1) \\ (3,1) & \to & 1 \\ (3,2) & \to & (4,2) \end{bmatrix}, \begin{bmatrix} 1 & \to & (4,1) \\ 2 & \to & 1 \\ (3,1) & \to & 2 \\ (3,2) & \to & (4,2) \end{bmatrix}, \ldots \tag{4.1}$$

   where I have spelled out four representatives for (2.16), (2.18) and the two permutations of (2.20).

4. Drop all maps $S \to E$ with at least one line looping back to the same state, e.g.

$$\begin{bmatrix} 1 & \to & 1 \\ 2 & \to & (4,1) \\ (3,1) & \to & (4,2) \\ (3,2) & \to & 2 \end{bmatrix}, \tag{4.2}$$

   because they do not correspond to valid SU$(N)$ colorflows. In the example there are 10 of those and 14 remain.

5. Keep one representative of the equivalence classes under the permutations according to the Young tableaux describing the irreps, i.e. according to permutations of the

subsets $\{(n,i)\}_i$ of $S$ and $E$. One of these equivalence classes in the example is

$$\begin{bmatrix} 1 & \to & 2 \\ 2 & \to & (4,1) \\ (3,1) & \to & 1 \\ (3,2) & \to & (4,2) \end{bmatrix} \sim \begin{bmatrix} 1 & \to & 2 \\ 2 & \to & (4,2) \\ (3,1) & \to & 1 \\ (3,2) & \to & (4,1) \end{bmatrix} \sim \begin{bmatrix} 1 & \to & 2 \\ 2 & \to & (4,1) \\ (3,2) & \to & 1 \\ (3,1) & \to & (4,2) \end{bmatrix} \sim \begin{bmatrix} 1 & \to & 2 \\ 2 & \to & (4,2) \\ (3,2) & \to & 1 \\ (3,1) & \to & (4,1) \end{bmatrix} . \tag{4.3}$$

This can be done by computing the orbits of the permutations described by the Young tableau for the irrep of each external state. In the example, the orbits containing (2.16), (2.18) and one of (2.20) consist of 2, 4 and 4 maps respectively, adding up to $2+4+2\cdot 4 = 14$, as required. If necessary, this process can be sped up by restricting the permutations generated in step 3 to one representative of these orbits.

6. Optionally symmetrize and antisymmetrize with respect to permutations of external states transforming under the same irrep of $\mathrm{SU}(N)$.

7. Apply the Young projection operators for all the factors. In the example, all irreps are symmetric and the resulting colorflow $Z^{ab}$ (2.20) is just the sum of the four maps in (4.3). This does not determine the overall normalization, which can be chosen to ensure that only integers appear as coefficients and to minimize the number of minus signs in the case of antisymmetric and mixed irreps.

Due to the subsequent test for redundancy described in section 4.3, it is less important to avoid accidental double counting than it is to produce all colorflows. In particular, step 5 could be skipped without affecting the final result. It just speeds up the subsequent search for independent tensors, because it keeps the matrices used in section 4.3 substantially smaller. This implies that the implementation of any optimization in steps 3 and 5 can be checked for moderately sized irreps by verifying that the constructed sets of independent invariant tensors are the same with and without including the optimization.

### 4.3 Finding dependent tensors

Since all terms in the sum (1.2) for $\mu_N(A,A)$ are the squared modulus of a component of the tensor $A$, it is positive by construction. Therefore the sesquilinear form $\mu_N$ induces an inner product and a norm on the vector space $\mathcal{V}$ of invariant tensors of a given rank

$$\|A\|_N = \sqrt{\mu_N(A,A)} \geq 0 \tag{4.4a}$$

$$\|A\|_N = 0 \quad \Leftrightarrow \quad A = 0 \tag{4.4b}$$

and it is not degenerate

$$\forall B \in \mathcal{V} : \mu_N(B,A) = 0 \iff A = 0\,. \tag{4.5}$$

The form $\mu_N$ can be employed to generalize a calculation [2] for small products of adjoint representations of SU(3): given a complete, but not necessarily linearly independent, set of $n \geq \dim(\mathcal{V})$ tensors

$$\mathcal{T} = \{T^i\}_{i=1,\ldots,n} \subseteq \mathcal{V}\,, \tag{4.6}$$



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we can expand every tensor $A \in \mathcal{V}$ as

$$A = \sum_{i=1}^{n} a_i T^i \,, \tag{4.7}$$

although this expansion will not be unique, in general. The inner product

$$\mu_N(A, B) = \langle a, M(N, \mathcal{T}) b \rangle \tag{4.8}$$

can then be expressed by the natural sesquilinear form

$$\langle a, b \rangle = \sum_{i=1}^{n} \overline{a_i} b_i \tag{4.9}$$

on $\mathbf{C}^n$ and the self-adjoint matrix $M(N, \mathcal{T})$

$$M^{ij}(N, \mathcal{T}) = \mu_N(T^i, T^j) = \overline{M^{ji}(N, \mathcal{T})} \tag{4.10}$$

which depends on the number of colors $N$ and the set $\mathcal{T}$. It can be computed either by using the identity (2.2) or by adding U(1) ghosts as described in section 3.1. The condition (4.5) now reads

$$\forall i : \sum_{j=1}^{n} M^{ij}(N, \mathcal{T})\, a_j = 0 \iff A = 0 \tag{4.11}$$

and we find that the linear relations among elements of $\mathcal{T}$ are just the eigenvectors of the matrix $M(N, \mathcal{T})$ corresponding to vanishing eigenvalues. Conversely, the number of independent invariant tensors is given by the rank $r_N$ of the matrix $M(N, \mathcal{T})$. The rank $r_N$ is independent of the set $\mathcal{T}$ of invariant tensors used to compute $M(N, \mathcal{T})$, as long as it is complete. The orthogonal projector $\mathcal{P}_N(\mathcal{T})$ on the subspace of $\mathbf{C}^n$ spanned by the eigenvectors corresponding to positive eigenvalues depends on $\mathcal{T}$, but the orthogonal projector $P_N$ on the corresponding subspace of $\mathcal{V}$ does not.

Since $M(N, \mathcal{T})$ is a finite and self-adjoint $n \times n$-matrix, it is always possible to compute $r_N$ and $P_N$ for any chosen value of $N$. This task is simplified by the observation that $\mu_N(A, B) = 0$, if $A$ and $B$ have different symmetries under permutations of the factors in the tensor product. Thus the matrix $M(N, \mathcal{T})$ assumes a block diagonal form, if the elements of $\mathcal{T}$ are chosen to be symmetric or antisymmetric under permutations of the factors. In the colorflow basis, the matrix elements of $M(N, \mathcal{T})$ will be polynomials in $N$ with real coefficients, possibly multiplied by a negative power of $N$. $M(N, \mathcal{T})$ will also be symmetric because transposition corresponds to reversing all colorflow lines.

There is the option to construct a basis of invariant tensors that are mutually orthogonal with respect to $\mu_N$. Unfortunately, except for the simplest cases, the real eigenvalues and eigenvectors can only be computed after fixing a value for $N$. The resulting real numbers are then not very illuminating. Therefore, one should rather use $P_N$ only to eliminate dependent tensors and to choose a linearly independent set $\{T^i\}_{i=1,\ldots,r_N}$ that is calculationally convenient, but not necessarily orthonormal with respect to $\mu_N$.



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### 4.3.1 Exceptional Values of $N$

The identity (2.2) or the rule (2.4) guarantee that all matrix elements $M^{ij}(N,\mathcal{T})$ are polynomials in $N$, possibly multiplied by $N^{-k}$ with $k$ a small natural number. Thus the characteristic polynomial has the form

$$\det\left(M(N,\mathcal{T})-\lambda\mathbf{1}\right)=N^{-k}\sum_{i=0}^{d}p_i(N)\,\lambda^i=N^{-k}\sum_{i=c_N}^{d}p_i(N)\,\lambda^i \tag{4.12}$$

with polynomials $\{p_i\}_{i=0,\dots,d}$ in $N$ as coefficients and $d$ the dimension of the matrix $M(N,\mathcal{T})$. The corank $c_N$ of the matrix $M(N,\mathcal{T})$, i.e. the number of eigenvectors with vanishing eigenvalue, is the multiplicity of the root of the characteristic polynomial at $\lambda=0$. For a given $N$, this is the number of consecutive $p_i(N)$ starting from $p_0(N)$ that vanish simultaneously

$$c_N=\max_i\left\{\forall j<i:p_j(N)=0\right\}\,. \tag{4.13}$$

As a polynomial in $N$, $p_i$ either vanishes for all $N$ or has at most $\deg(p_i)$ positive real roots, where $\deg(p)$ denotes the degree of $p$. Thus there can be at most a finite number of exceptional values of $N$, where the rank and corank of $M(N,\mathcal{T})$ are not constant and additional relations among invariant tensors appear. In particular

$$\exists r_\infty,\hat{N}:\forall N>\hat{N}:r_N=r_\infty\,, \tag{4.14}$$

i.e. there is a maximum $N$ above which the rank $r_N$ no longer changes.

## 5 Revisiting the example

We can now return to the example of the four-fold product $\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{6}\otimes\overline{\mathbf{6}}$ from section 2.3 and compute the matrix $M(N,\mathcal{T})$ for the colorflows

$$\mathcal{T}=\{X,Y,Z_\mathrm{S},Z_\mathrm{A}\}\,. \tag{5.1}$$

### 5.1 SU($N$)

We obtain for the colorflows $\mathcal{T}_\mathrm{S}=\{X,Y,Z_\mathrm{S}\}$ that are symmetric in the two adjoint factors

$$\mu_N(X,X)=\frac{1}{2}N^2(N+1)C_\mathrm{F} \tag{5.2a}$$

$$\mu_N(Y,Y)=\frac{1}{4}(N^3+2N^2-2)C_\mathrm{F} \tag{5.2b}$$

$$\mu_N(Z_\mathrm{S},Z_\mathrm{S})=\frac{1}{2}(N^3+2N^2-4)C_\mathrm{F} \tag{5.2c}$$

$$\mu_N(X,Y)=\mu_N(Y,X)=\frac{1}{2}NC_\mathrm{F} \tag{5.2d}$$

$$\mu_N(X,Z_\mathrm{S})=\mu_N(Z_\mathrm{S},X)=N(N+1)C_\mathrm{F} \tag{5.2e}$$

$$\mu_N(Y,Z_\mathrm{S})=\mu_N(Z_\mathrm{S},Y)=\frac{1}{2}(N^2-2)C_\mathrm{F} \tag{5.2f}$$

and for the antisymmetric colorflow $\mathcal{T}_\mathrm{A}=\{Z_\mathrm{A}\}$

$$\mu_N(Z_\mathrm{A},Z_\mathrm{A})=\frac{1}{2}N^2(N+2)C_\mathrm{F}\,, \tag{5.3}$$

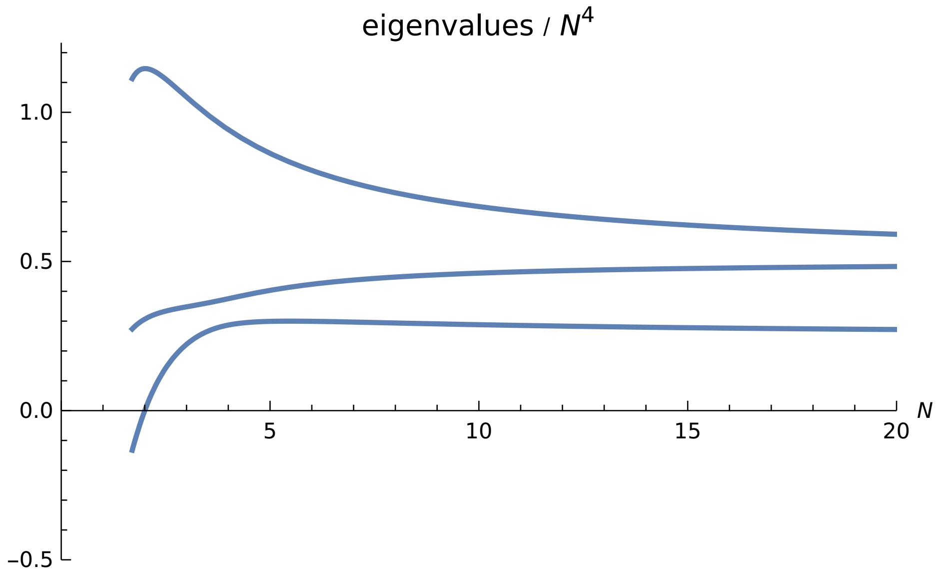


**Figure 1.** Eigenvalues of the matrix $M(N,\mathcal{T})$ in (5.5) for SU($N$) as a function of $N$, divided by the asymptotic scaling $N^4$.

where the quadratic Casimir operator in the fundamental representation appears as a common factor. It takes the value

$$C_2(\mathrm{F}) = C_\mathrm{F} = \frac{N^2-1}{N} \tag{5.4}$$

in the normalization (2.1).

All products of the symmetric and antisymmetric colorflows vanish, of course. The eigenvalues of the real symmetric matrix

$$M(N,\mathcal{T}_\mathrm{S}) = \begin{pmatrix} \mu_N(X,X) & \mu_N(X,Y) & \mu_N(X,Z_\mathrm{S}) \\ \mu_N(Y,X) & \mu_N(Y,Y) & \mu_N(Y,Z_\mathrm{S}) \\ \mu_N(Z_\mathrm{S},X) & \mu_N(Z_\mathrm{S},Y) & \mu_N(Z_\mathrm{S},Z_\mathrm{S}) \end{pmatrix} \tag{5.5}$$

can be computed numerically for arbitrary values of $N$. As illustrated in figure 1, they are all positive for $N > 2$, but one eigenvalue vanishes for SU(2). It corresponds to the relation

$$X = Z_\mathrm{S}\,. \tag{5.6}$$

Thus we have found an invariant tensor that vanishes for SU(2), but is independent for SU($N$) with $N \geq 3$, similarly to the $d_{ijk}^{\mathrm{SU}}$ discussed in section 3.2.

### 5.2 U($N$)

For illustration, we can compute the elements of the matrix $M(N,\mathcal{T})$ also in the case of U($N$). This can be done by dropping all contributions of U(1)-ghosts or by using (2.2) without the $1/N$-term. Thus neither negative coefficients nor negative powers of $N$ can appear in



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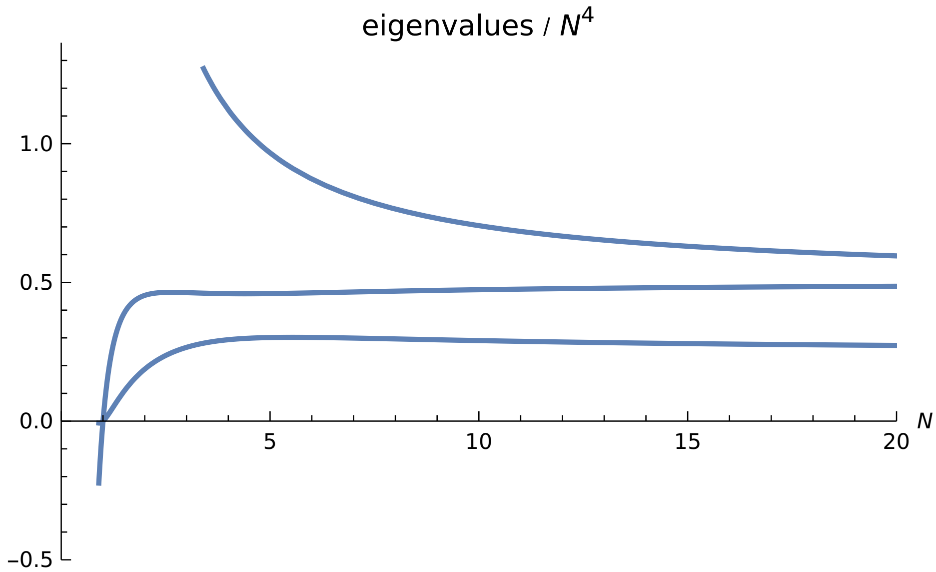



**Figure 2.** Eigenvalues of the matrix $M(N, \mathcal{T})$ in (5.5) for U($N$) as a function of $N$, divided by the asymptotic scaling $N^4$.

the results for U($N$). Indeed, we compute

$$\mu_N(X,X) = \frac{1}{4}N^2 C \tag{5.7a}$$

$$\mu_N(Y,Y) = \frac{1}{2}N(N+1)C \tag{5.7b}$$

$$\mu_N(Z_\mathrm{S},Z_\mathrm{S}) = \frac{1}{2}(N^2+N+2)C \tag{5.7c}$$

$$\mu_N(X,Y) = \mu_N(Y,X) = \frac{1}{2}C \tag{5.7d}$$

$$\mu_N(X,Z_\mathrm{S}) = \mu_N(Z_\mathrm{S},X) = NC \tag{5.7e}$$

$$\mu_N(Y,Z_\mathrm{S}) = \mu_N(Z_\mathrm{S},Y) = \frac{1}{2}(N+1)C\,, \tag{5.7f}$$

where the common factor is now $C = N(N+1)$, and

$$\mu_N(Z_\mathrm{A},Z_\mathrm{A}) = \frac{1}{2}N^2(N+2)C_\mathrm{F} \tag{5.8}$$

for $\mathcal{T}_\mathrm{S}$ and $\mathcal{T}_\mathrm{A}$ respectively. It is not surprising that the result for $\mu_N(Z_\mathrm{A},Z_\mathrm{A})$ is the same for U($N$) and SU($N$), because the U(1)-ghosts cancel in the antisymmetric case, but not in the symmetric case. As illustrated in figure 2, all eigenvalues are positive for $N \geq 2$, but only one non-vanishing eigenvalue survives in the abelian limit U(1). It can be written

$$X + Y + 2Z_\mathrm{S} \tag{5.9}$$

and the orthogonal combinations vanish.

### 5.3 $N \to \infty$

The coefficients of the leading powers of $N$ agree for U($N$) and SU($N$). This was to be expected, because the difference must contain powers of $1/N$.

It is easy to see that, unless two colorflows $A$ and $B$ are related by a permutation of the factors, their inner product $\mu_N(A,B)$ contains fewer closed chains (2.3) than $\mu_N(A,A)$ or $\mu_N(B,B)$. Therefore the off-diagonal elements of the matrix $M(N,\mathcal{T})$ will scale with a smaller power of $N$ for $N\to\infty$ and $M(N,\mathcal{T})$ will asymptotically become diagonal. This is indeed the case

$$\lim_{N\to\infty}\left(\frac{M(N,\mathcal{T})}{N^4}\right)=\frac{1}{4}\begin{pmatrix}2&0&0\\0&1&0\\0&0&2\end{pmatrix} \tag{5.10}$$

and the two larger eigenvalues will approach $N^4/2$, while the smaller will approach $N^4/4$. In the case of the two smaller eigenvalues, the asymptotic behaviour is already reached for small values of $N$, as illustrated in figures 1 and 2. This asymptotic behaviour is compatible with the observations made in section 4.3.1, of course.

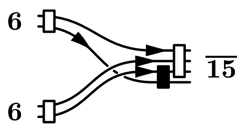
## 6 A catalogue of exotic birdtracks of SU($N$)

The method described in section 4 can be used to prepare a catalogue of bases of invariant tensors. In this section, I compare the results with the catalogue presented in [6].

### 6.1 Three fields

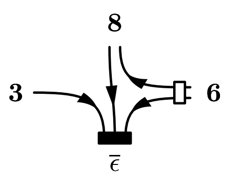
Table 1 lists the results for the three-fold products presented in table I of [6]. They confirm the latter. To illustrate the colorflow formalism, I nevertheless display the colorflows from table 1 involving a mixed symmetry **15** or one or two $\epsilon$s. Some results will be used in section 6.2.

- $\mathbf{6}\otimes\mathbf{6}\otimes\overline{\mathbf{15}}$: one line of both **6** must be connected to the antisymmetrizer and the other to the symmetrizer of the **15**

  
  
  (6.1)

  to obtain a non-vanishing colorflow. It is antisymmetric under the exchange of the two factors of **6**.

- $\mathbf{3}\otimes\mathbf{6}\otimes\mathbf{8}$: there is only one invariant tensor

  
  
  , (6.2)

  because the antisymmetric $\epsilon$ must not be connected twice to the symmetric **6**. This is the colorflow representation of the formula (A15) of [6].

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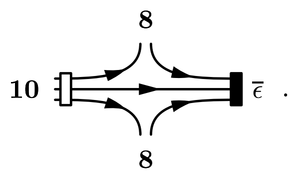
| | $n_\epsilon$ | $n_\uparrow$ | $r_3$ | remarks |
|---|---|---|---|---|
| $\mathbf{3}\otimes\mathbf{3}\otimes\overline{\mathbf{6}}$ | 0 | 2 | 1 | CG: $\mathbf{6}\subset\mathbf{3}\otimes\mathbf{3}$ |
| $\mathbf{3}\otimes\overline{\mathbf{3}}\otimes\mathbf{8}$ | 0 | 2 | 1 | $[T^a]^i{}_j$ |
| $\mathbf{6}\otimes\overline{\mathbf{6}}\otimes\mathbf{8}$ | 0 | 3 | 1 | $[T^a_{\mathbf{6}}]^s{}_u$ |
| $\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{8}$ | 0 | 3 | 2 | $f_{abc}$, $d_{abc}$ (but $r_2=1$) |
| $\mathbf{3}\otimes\mathbf{6}\otimes\overline{\mathbf{10}}$ | 0 | 3 | 1 | CG: $\mathbf{10}\subset\mathbf{3}\otimes\mathbf{6}$ |
| $\mathbf{6}\otimes\mathbf{6}\otimes\overline{\mathbf{15}}$ | 0 | 4 | 1 | CG: $\mathbf{15}\subset\mathbf{6}\otimes\mathbf{6}$ (6.1) |
| $\mathbf{6}\otimes\mathbf{6}\otimes\overline{\mathbf{15}'}$ | 0 | 4 | 1 | CG: $\mathbf{15}'\subset\mathbf{6}\otimes\mathbf{6}$ (6.9) |
| $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$ | 1 | 0 | 1 | totally antisymmetric |
| $\mathbf{3}\otimes\mathbf{6}\otimes\mathbf{8}$ | 1 | 1 | 1 | (6.2) |
| $\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{10}$ | 1 | 2 | 1 | antisymmetric (6.3) |
| $\mathbf{3}\otimes\mathbf{8}\otimes\overline{\mathbf{15}}$ | 1 | 2 | 1 | (6.4a) |
| $\mathbf{3}\otimes\overline{\mathbf{6}}\otimes\mathbf{15}$ | 1 | 2 | 1 | (6.5) |
| $\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}$ | 2 | 0 | 1 | totally symmetric |
| $\mathbf{6}\otimes\mathbf{8}\otimes\mathbf{15}$ | 2 | 1 | 1 | (6.6) |
| $\mathbf{6}\otimes\overline{\mathbf{6}}\otimes\mathbf{27}$ | 2 | 2 | 1 | (6.7) |
| $\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{27}$ | 2 | 2 | 1 | (6.8) |

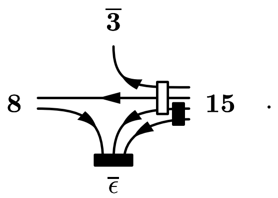
**Table 1.** Invariant tensors in three-fold products of irreps of SU(3), ordered in increasing numbers of epsilons $n_\epsilon$, arrows $n_\uparrow$ and rank $r_3$, the number of independent colorflows for $N=3$. This extends table I of [6].

- $\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{10}$: the only invariant tensor is antisymmetric in the two factors $\mathbf{8}$ due to the $\bar\epsilon$



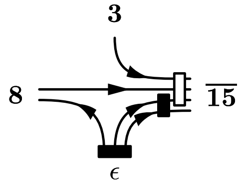

(6.3)

- $\mathbf{3}\otimes\mathbf{8}\otimes\overline{\mathbf{15}}$: it is easier to see that there is indeed only one inequivalent colorflow by looking at the conjugate $\overline{\mathbf{3}}\otimes\mathbf{8}\otimes\mathbf{15}$ instead: there must be exactly one $\bar\epsilon$ to saturate all lines and one of the lines entering this $\bar\epsilon$ must be connected to the only line of the $\mathbf{15}$ that is not symmetrized



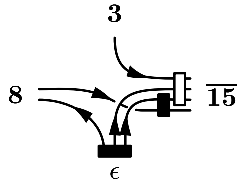

(6.4a)

All other contributions are then uniquely determined by symmetry. With the opposite order of symmetrization and antisymmetrization in the original $\mathbf{3} \otimes \mathbf{8} \otimes \overline{\mathbf{15}}$




(6.4b)

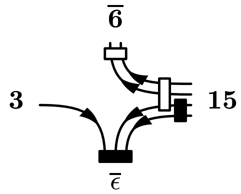
it is not immediately obvious that




(6.4c)

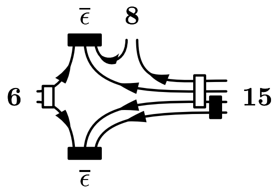
is equivalent after applying the Young projector[5] for the $\overline{\mathbf{15}}$. However, the method described in section 4.3 confirms that $r_N = 1$ in both cases.

- $\mathbf{3} \otimes \overline{\mathbf{6}} \otimes \mathbf{15}$: the single invariant tensor looks very similar to (6.4a)




. (6.5)

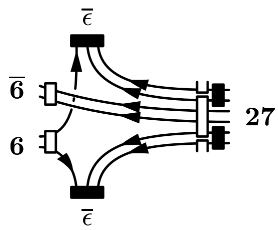
- $\mathbf{6} \otimes \mathbf{8} \otimes \mathbf{15}$: this needs two $\overline{\epsilon}$ and their lines must avoid the symmetrization of the **15**.




. (6.6)

Note that all other ways of inserting the **8** can be obtained by exchanging the $\overline{\epsilon}$s and the lines ending at them. Since the combinatorics is already not completely obvious, the method described in section 4.3 has been helpful for confirming the result $r_N = 1$.

- $\mathbf{6} \otimes \overline{\mathbf{6}} \otimes \mathbf{27}$: here is again only one way to saturate all antisymmetric lines ending in the $\overline{\epsilon}$s




, (6.7)

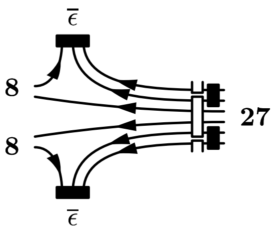
with the symmetrizer of the pair of outer lines wrapping around, as in (2.5i). En passant we note that this graphical representation makes it is obvious that there can be no invariant tensor in the product $\mathbf{3} \otimes \overline{\mathbf{3}} \otimes \mathbf{27}$.

[5] The hermitian Young projectors advocated in [21, 22] make both variants equally complicated.

| | $n_\epsilon$ | $n_\uparrow$ | $r_3$ | remarks |
|---|---|---|---|---|
| $\mathbf{3}\otimes\overline{\mathbf{3}}\otimes\mathbf{6}\otimes\overline{\mathbf{6}}$ | 0 | 3 | 2 | |
| $\mathbf{3}\otimes\mathbf{3}\otimes\overline{\mathbf{6}}\otimes\mathbf{8}$ | 0 | 3 | 2 | (6.10) |
| $\mathbf{6}\otimes\mathbf{6}\otimes\overline{\mathbf{6}}\otimes\overline{\mathbf{6}}$ | 0 | 4 | 3 | 1 anti-, 2 symmetric |
| $\mathbf{6}\otimes\overline{\mathbf{6}}\otimes\mathbf{8}\otimes\mathbf{8}$ | 0 | 4 | 4 | sections 2.3, 5.1 |
| $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\overline{\mathbf{10}}$ | 0 | 3 | 1 | CG: $\mathbf{10}\subset\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$ |
| $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{6}\otimes\overline{\mathbf{15}}$ | 0 | 4 | 2 | CG: $\mathbf{15}\subset\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{6}$ |
| $\mathbf{3}\otimes\overline{\mathbf{3}}\otimes\overline{\mathbf{3}}\otimes\overline{\mathbf{6}}$ | 1 | 1 | 1 | (6.11) |
| $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{8}$ | 1 | 1 | 2 | |
| $\mathbf{3}\otimes\mathbf{6}\otimes\mathbf{6}\otimes\overline{\mathbf{6}}$ | 1 | 2 | 1 | (6.12) |
| $\mathbf{3}\otimes\overline{\mathbf{6}}\otimes\overline{\mathbf{6}}\otimes\mathbf{8}$ | 1 | 2 | 2 | 1 anti-, 1 symmetric |
| $\mathbf{3}\otimes\mathbf{6}\otimes\mathbf{8}\otimes\mathbf{8}$ | 1 | 2 | 3 | (6.13) |
| $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{6}\otimes\mathbf{6}$ | 2 | 0 | 1 | symmetric |
| $\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{8}$ | 2 | 1 | 2 | (6.18) |

**Table 2.** Invariant tensors in four-fold products of irreps of SU(3), ordered in increasing numbers of epsilons $n_\epsilon$, arrows $n_\uparrow$ and rank $r_3$, the number of independent colorflows for $N=3$. This extends table II of [6].

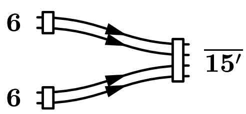
- $\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{27}$: the symmetry in the two adjoint factors is obvious



, (6.8)

  with the symmetrizer of the pair of outer lines wrapping around again.

- $\mathbf{6}\otimes\mathbf{6}\otimes\overline{\mathbf{15}'}$ The authors of [6] did not spell out the single invariant tensor



(6.9)

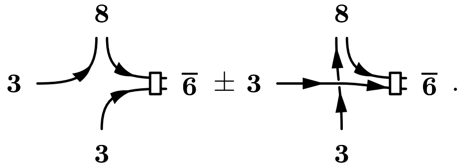
  in their catalogue, maybe because it is just a trivial symmetric Clebsch-Gordan coefficient.

### 6.2 Four fields

Table 2 lists the results for the four-fold products presented in table II of [6]. In this case, we can not confirm them all:

1. $\mathbf{3}\otimes\mathbf{3}\otimes\overline{\mathbf{6}}\otimes\mathbf{8}$: here [6] reports two additional invariant tensors. However, there are only two ways to insert a gluon into the Clebsch-Gordan coefficient $\mathbf{3}\otimes\mathbf{3}\otimes\overline{\mathbf{6}}$. Thus

there is only one invariant tensor in $\mathbf{3} \otimes_{\mathrm{S}} \mathbf{3} \otimes \overline{\mathbf{6}} \otimes \mathbf{8}$ and one in $\mathbf{3} \otimes_{\mathrm{A}} \mathbf{3} \otimes \overline{\mathbf{6}} \otimes \mathbf{8}$

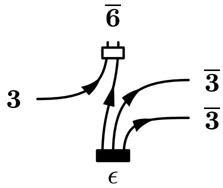



(6.10)

They can be expressed as combinations of $K_{\mathbf{6}}T_{\mathbf{3}}$ and $K_{\mathbf{6}}T_{\mathbf{6}}$. The other two tensors in table II of [6], $L\overline{J}$ and $QV$, both contain $\epsilon^{i_1 i_2 k}\overline{\epsilon}_{j_1 j_2 k} = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}$ and are therefore redundant, as described in section 3.3.

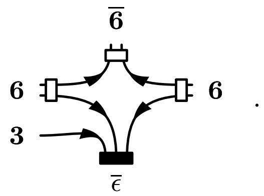
2. $\mathbf{6} \otimes \overline{\mathbf{6}} \otimes \mathbf{8} \otimes \mathbf{8}$: one symmetric tensor is missing in [6]. This has been discussed at length in sections 2.3 and 5.1.

3. $\mathbf{3} \otimes \overline{\mathbf{3}} \otimes \overline{\mathbf{3}} \otimes \overline{\mathbf{6}}$: the only independent invariant tensor is antisymmetric



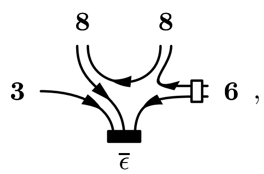

(6.11)

4. $\mathbf{3} \otimes \mathbf{6} \otimes \mathbf{6} \otimes \overline{\mathbf{6}}$: since each leg of the $\overline{\epsilon}$ must be connected to a different $\mathbf{3}$ or $\mathbf{6}$, there is again only one independent invariant tensor and it is antisymmetric



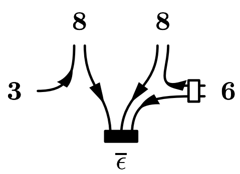

(6.12)

5. $\mathbf{3} \otimes \mathbf{6} \otimes \mathbf{8} \otimes \mathbf{8}$: there are two invariant tensors antisymmetric in the factors $\mathbf{8}$ and one symmetric. Up to permutations of the external $\mathbf{8}$s, there are three different ways to connect the $\overline{\epsilon}$ to the other external states: to both the $\mathbf{3}$ and the $\mathbf{6}$



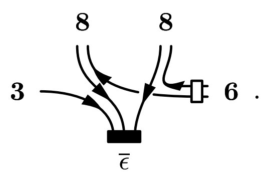

(6.13a)

to the $\mathbf{6}$ only




(6.13b)

and to the $\mathbf{3}$ only




(6.13c)

The colorflow (6.13c) is antisymmetric in the two **8**s, while (6.13a) and (6.13b) contain both symmetric and antisymmetric contributions. These three colorflows correspond to the invariant tensors

$$A^{ab}_{is} = \bar{\epsilon}_{ij_1k}[T^a]^k{}_l[T^b]^l{}_{j_2}[K_{\mathbf{6}\,s}]^{j_1j_2} \tag{6.14a}$$
$$B^{ab}_{is} = \bar{\epsilon}_{j_1kl}[T^a]^k{}_i[T^b]^l{}_{j_2}[K_{\mathbf{6}\,s}]^{j_1j_2} \tag{6.14b}$$
$$C^{ab}_{is} = \bar{\epsilon}_{ikl}[T^a]^k{}_{j_1}[T^b]^l{}_{j_2}[K_{\mathbf{6}\,s}]^{j_1j_2} \tag{6.14c}$$

Due to the presence of an $\bar{\epsilon}$, the matrix (4.10) can only be computed for $N = 3$

$$M\left(3, \{A, B, C\}\right) = \begin{pmatrix} 7 & 4 & -3 \\ 4 & 8 & 4 \\ -3 & 4 & 7 \end{pmatrix} \cdot 16 \tag{6.15}$$

and has the eigenvalues 0, 160 and 192. The eigenvector for the eigenvalue 0 is $(1, -1, 1)^T$ and corresponds to the relation

$$A - B + C = 0\,, \tag{6.16}$$

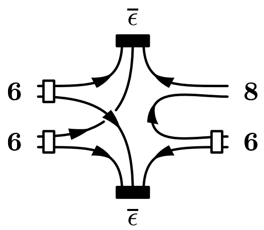
revealing that one symmetric and one antisymmetric tensor is redundant. It is easy to verify that the relation (6.16) is just the invariance of the tensor (6.2)

$$D^a_{is} = \bar{\epsilon}_{ij_1k}[T^a]^k{}_{j_2}[K_{\mathbf{6}\,s}]^{j_1j_2} \tag{6.17}$$

in the product $\mathbf{3}\otimes\mathbf{6}\otimes\mathbf{8}$.

The corresponding row in table II of [6] lists six invariant tensors: three of mixed symmetry, two antisymmetric and one symmetric. Therefore there are three non-trivial relation among them.

6. $\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{8}$: the only two independent invariant tensors are combinations of permutations of




. (6.18)

which can be viewed as insertions of a gluon into the only invariant tensor in $\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}$. The corresponding invariant tensors are

$$A^a_{s_1s_2s_3} = [T^a]^k{}_{i_1}\bar{\epsilon}_{i_2i_3k}\bar{\epsilon}_{j_1j_2j_3}[K_{\mathbf{6}\,s_1}]^{i_1j_1}[K_{\mathbf{6}\,s_2}]^{i_2j_2}[K_{\mathbf{6}\,s_3}]^{i_3j_3} \tag{6.19}$$

and its cyclic permutations in $\{s_1, s_2, s_3\}$, while the non-cyclic permutations are trivially related by the antisymmetry of the $\bar{\epsilon}$s. The eigenvector of the matrix $M$ corresponding to the eigenvalue 0 turns out to be the sum of the cyclic permutations. This can again be understood as the invariance of the invariant tensor in $\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}$. Therefore only two combinations of the $A$ are independent.

The tensors $A$ correspond to the $ST_\mathbf{6}$ tensors in table II of [6]. The $WX$ tensors are linear combinations of these, as can be seen by gluing the conjugate of (6.1) to (6.6) at the **15**.

There are three more products that have been left out of table II of [6]

1. $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\overline{\mathbf{10}}$: there is only one colorflow and it is totally symmetric.

2. $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{6}\otimes\overline{\mathbf{15}}$: there is one symmetric and one antisymmetric colorflow, corresponding to $\mathbf{15}\subset\mathbf{6}\otimes\mathbf{6}$ and $\mathbf{15}\subset\overline{\mathbf{3}}\otimes\mathbf{6}$.

3. $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{8}$: the two independent colorflows are just like (6.18), with one of the $\bar{\epsilon}$s removed and all **6**s replaced by **3**s. As in the case of $\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{8}$, only two of the

$$A^a_{i_1 i_2 i_3} = [T^a]^k{}_{i_1}\bar{\epsilon}_{i_2 i_3 k}\,, \tag{6.20}$$

are independent: the sum of the cyclic permutations vanishes because $\bar{\epsilon}_{i_1 i_2 i_3}$ is an invariant tensor in $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$.

I can confirm the remaining results of [6] for the four-fold products and only use this opportunity to clarify permutation symmetries in the factors:

- $\mathbf{3}\otimes\overline{\mathbf{3}}\otimes\mathbf{6}\otimes\overline{\mathbf{6}}$: there are two independent colorflows and they are linear combinations of the invariant tensors listed in [6].

- $\mathbf{6}\otimes\mathbf{6}\otimes\overline{\mathbf{6}}\otimes\overline{\mathbf{6}}$: there are one antisymmetric and two symmetric invariant tensors. This agrees with [6], since $\delta^{s_1}_{t_1}\delta^{s_2}_{t_2}$ contains both a symmetric and an antisymmetric contribution.

- $\mathbf{3}\otimes\overline{\mathbf{6}}\otimes\overline{\mathbf{6}}\otimes\mathbf{8}$: there is one symmetric and one antisymmetric invariant tensor. This is compatible with [6], except for obvious typos in the indices of the $K\overline{J}$ term.

- $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{6}\otimes\mathbf{6}$: since each $\bar{\epsilon}$ must be connected with both **6**s, the only colorflow is symmetric.

### 6.3 Five and more fields

Table III of [6] sketches a catalogue of invariant tensors in five-fold products of irreps of SU(3). Since a complete catalogue can easily be produced with the program TANGARA together with MATHEMATICA [33], I only count them in table 3 and refrain from presenting a graphical representation and a detailled discussion.

There are again products involving adjoint representations in table 3 for which the number of independent invariant tensors changes when going from SU(2) to SU(3). As a curiosity, table 4 displays the number of independent invariant tensors in products of $n$ adjoint representations of SU($N$) for different values of $N$. The products in this table contain no exotic irreps and the results for $r_3$ can already be found in [2, 3]. The values of $r_3$ and $r_\infty$ for $n=6$ are given in the caption of table 6 in [21], where they have been derived using purely combinatorial arguments.



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| | $n_\epsilon$ | $n_\uparrow$ | $r_3$ | remarks |
|---|---|---|---|---|
| $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{6}\otimes\overline{\mathbf{6}}\otimes\overline{\mathbf{6}}$ | 0 | 4 | 4 | |
| $\mathbf{3}\otimes\overline{\mathbf{3}}\otimes\mathbf{6}\otimes\overline{\mathbf{6}}\otimes\mathbf{8}$ | 0 | 4 | 5 | but $r_2 = 4$ |
| $\mathbf{6}\otimes\mathbf{6}\otimes\overline{\mathbf{6}}\otimes\overline{\mathbf{6}}\otimes\mathbf{8}$ | 0 | 5 | 8 | but $r_2 = 6$ |
| $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\overline{\mathbf{3}}\otimes\overline{\mathbf{6}}$ | 1 | 0 | 3 | |
| $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{6}$ | 2 | 0 | 2 | |
| $\mathbf{3}\otimes\mathbf{6}\otimes\overline{\mathbf{6}}\otimes\overline{\mathbf{6}}\otimes\overline{\mathbf{6}}$ | 2 | 0 | 3 | |
| $\mathbf{3}\otimes\overline{\mathbf{3}}\otimes\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}$ | 2 | 1 | 3 | |
| $\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{8}\otimes\mathbf{8}$ | 2 | 2 | 10 | 4 anti-, 6 symmetric |
| $\mathbf{3}\otimes\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}$ | 3 | 0 | 3 | |
| $\mathbf{6}\otimes\overline{\mathbf{6}}\otimes\overline{\mathbf{6}}\otimes\overline{\mathbf{6}}\otimes\overline{\mathbf{6}}$ | 3 | 0 | 6 | |

**Table 3.** Invariant tensors in five-fold products of irreps of SU(3), ordered in increasing numbers of epsilons $n_\epsilon$, arrows $n_\uparrow$ and rank $r_3$, the number of independent colorflows for $N = 3$, cf. table III of [6].

| n | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $\cdots$ | $r_\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | $\cdots$ | 2 |
| 4 | 3 | 8 | 9 | 9 | 9 | 9 | 9 | $\cdots$ | 9 |
| 5 | 6 | 32 | 43 | 44 | 44 | 44 | 44 | $\cdots$ | 44 |
| 6 | 15 | 145 | 245 | 264 | 265 | 265 | 265 | $\cdots$ | 265 |
| 7 | 36 | 702 | 1557 | 1824 | 1853 | 1854 | 1854 | $\cdots$ | 1854 |

**Table 4.** The rank $r_N$ of the matrix $M(N, \mathcal{T})$ of color factors (4.10), i.e. the number of independent invariant tensors, in the product of $n$ adjoint representations of SU($N$) for $2 \leq N \leq 8$.

An inspection of table 4 suggests a curious pattern for the products of $n \geq 3$ adjoints of SU($N$)

$$r_n = r_{n-1} + 1 \tag{6.21a}$$

$$\forall N \geq n : r_N = r_n \,. \tag{6.21b}$$

The considerations in section 4.3.1 show that the limit $r_\infty$ in (4.14) exists, but they are not sufficient to show that $r_\infty = r_n$. I don't know if there is a deeper reason for, a general proof or even a practical application of (6.21).

## 7 Conclusions

I have presented a systematic construction of complete and linearly independent sets of invariant tensors in products of irreducible representations of SU($N$). This construction is algorithmic and has been implemented in the computer code TANGARA. There is no fundamental limit on the size of the irreps and the number of factors. However, there are



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practical limits since the computational complexity of the most straightforward unoptimized algorithms grows combinatorially.

In section 6, I have compared the results of the new algorithm to a catalogue of invariant tensors published previously [6]. There are several discrepancies and I explain for each case in detail why the new result is the correct one.

The study of invariant tensors in products of representations larger than the **10** appears at the moment to be more of mathematical than phenomenological interest. But section 6.2 also lists six examples of colorflows involving only four triplets, sextets or octets, where previous published results are wrong. Three of these contain only a single sextet and another one a single pair. These are of immediate phenomenological interest for the study of BSM models containing such particles.

Nothing precludes the application of the method to other Lie algebras that appear in more exotic BSM models, such as SO($N$). For an implementation in TANGARA, only the underlying birdtrack library must be extended to support undirected lines.


## Acknowledgments

I thank Manuel Kunkel for triggering this paper by asking me to count the invariant tensors in the $\mathbf{8} \otimes \mathbf{8} \otimes \mathbf{6} \otimes \overline{\mathbf{6}}$ representation of SU(3). I thank Wolfgang Kilian, Jürgen Reuter for many productive discussions on making O'MEGA more colorful. I also thank JR for useful comments on the manuscript. I thank Andreas Trautner (MPIK Heidelberg) for useful suggestions on the presentation and for pointing out errors in the original manuscript.

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## A Implementation and interoperation

### A.1 tangara

The algorithm of section 4 has been implemented in the computer program TANGARA. As illustrated in figures 3 and 5, the program is given a tensor product, optionally with a representation of the permutation symmetry groups of identical factors, and computes a list $\mathcal{T}$ of candidates for a complete and linearly independent set of invariant tensors together with the matrix $M(N, \mathcal{T})$ of color factors (4.10).

Note that colorflows representing the invariant tensors do *not* contain the Young projectors nor the ghosts, because these can be added trivially by other programs using this output as input. As can be seen in figure 3, TANGARA lists the colorflows from section 2.3 in the order $\{X, Z_{\mathrm{S}}, Y\}$ and normalizes them in such a way that the coefficients are integers with the smallest modulus possible. This normalization is fixed, but the order of the colorflows is not guaranteed to be the same for different versions of TANGARA. The chosen normalization is the most convenient one, since it directly corresponds to the graphical representation of colorflows, as in section 6. Conceptually, the normalization $\mu_N(T, T) = 1$ might appear to be more satisfactory, but it would require dividing by square roots of polynomials in $N$ and make the output much more complicated.

```
$ tangara_tool -s '8 *S 8 * 6 * ~6'

 0: [(1) * [1>2; 2>1; 3.0>4.0; 3.1>4.1]]
 1: [(1) * [1>2; 2>4.0; 3.0>1; 3.1>4.1];
     (1) * [1>4.0; 2>1; 3.0>2; 3.1>4.1]]
 2: [(1) * [1>4.0; 2>4.1; 3.0>1; 3.1>2]]

colorfactors[n_] :=
 { { (1/2)*n^4+(1/2)*n^3-(1/2)*n^2-(1/2)*n,
     n^3+n^2-n-1                          ,
     (1/2)*n^2-(1/2)                          },
   { n^3+n^2-n-1                                ,
     (1/2)*n^4+n^3-(1/2)*n^2-3*n+2/n,
     (1/2)*n^3-(3/2)*n+1/n                    },
   { (1/2)*n^2-(1/2)                            ,
     (1/2)*n^3-(3/2)*n+1/n                ,
     (1/4)*n^4+(1/2)*n^3-(1/4)*n^2-n+(1/2)/n } }
```

**Figure 3.** TANGARA command line parameters and output: colorflows of invariant tensors in the symmetric tensor product $\mathbf{8} \otimes_S \mathbf{8} \otimes \mathbf{6} \otimes \overline{\mathbf{6}}$ and their color factor matrix $M(N, \mathcal{T})$.

```
\SU(2): rank = 2

  eval(1) = (3*(31 + Sqrt[321]))/8 = 18.3
  eval(2) = (3*(31 - Sqrt[321]))/8 = 4.9
  eval(3) = 0 = 0.

\SU(3): rank = 3

  eval(1) = (10*(17 + Sqrt[73]))/3 = 85.2
  eval(2) = (10*(17 - Sqrt[73]))/3 = 28.2
  eval(3) = 18 = 18.
```

**Figure 4.** MATHEMATICA [33] output for the rank $r_N$ and the eigenvalues of the color factor matrix $M(N, \mathcal{T})$ in figure 3 for SU(2) and SU(3).

The matrix of color factors is accompanied by a short script that computes the rank $r_N$, the eigenvalues and the eigenvectors using MATHEMATICA [33]. The output is shown in figure 4 without the eigenvectors. The computed eigenvectors can then be used to eliminate dependent tensors from the set $\mathcal{T}$. The script does not try to make a recommendation for a canonical or "best" choice of linearly independent invariant tensors, since mutually excluding goals are bound to enter into this decision. Optionally, if the colorflows contain no $\epsilon$s, the $N$-dependence of the eigenvalues can be plotted for illustration, as shown in figures 1 and 2.

The complete source code of TANGARA will be made publicly available in the O'MEGA subdirectory of a forthcoming WHIZARD [17, 18, 31] release.

```
$ tangara_tool -s '8 *A 8 * 6 * ~6'

0: [ ( 1)*[1>2; 2>4.0; 3.0>1; 3.1>4.1];
     (-1)*[1>4.0; 2>1; 3.0>2; 3.1>4.1] ]

colorfactors[n_] :=
 { { (1/2)*n^4+n^3-(1/2)*n^2-n } }
```

**Figure 5.** TANGARA command line parameters and output: colorflows of invariant tensors in the antisymmetric tensor product $\mathbf{8} \otimes_{\mathrm{A}} \mathbf{8} \otimes \mathbf{6} \otimes \overline{\mathbf{6}}$ and their color factor matrix $M(N, \mathcal{T})$.

### A.2 UFO

Counting the number of linearly independent invariant tensors is an interesting exercise, but the ultimate goal is their application in the study of BSM physics. For this purpose, the results must be made available to other tools. The UFO format [29, 30] has established itself as the *lingua franca* for describing models of BSM physics to automatic computation systems that compute renormalization group running, decay rates and cross sections. The latter are subsequently used by Monte Carlo event generators to simulate scattering processes at colliders.

The building blocks for color structures specified in the current UFO format [30] are sufficient to express all possible invariant tensors describing interactions of particles transforming under the $\mathbf{3}$, $\overline{\mathbf{3}}$, $\mathbf{6}$, $\overline{\mathbf{6}}$, and $\mathbf{8}$ of SU(3), including baryon number violating terms containing $\epsilon^{ijk}$ or $\bar{\epsilon}_{ijk}$. The $\mathbf{6}$ and $\overline{\mathbf{6}}$ irreps are described in UFO by the Clebsch-Gordan coefficients $[K_{\mathbf{6}\,s}]^{ij}$ and $[\overline{K}^s_{\mathbf{6}}]_{ij}$ and generators $[T^a_{\mathbf{6}}]^s{}_t$, where the latter could have be expressed by the Clebsch-Gordan coefficients and the generators in the fundamental representation

$$[T^a_{\mathbf{6}}]^s{}_t = 2[\overline{K}^s_{\mathbf{6}}]_{ik}[T^a]^i{}_j[K_{\mathbf{6}\,t}]^{jk}\,. \tag{A.1}$$

All this can be translated automatically into a colorflow basis, as has been demonstrated by the implementation in O'MEGA [17, 18] and WHIZARD [31].

Replacing the arrows by pairs of summation indices and symmetrizers by Clebsch-Gordan coefficients, all colorflows connecting triplets, sextets and octets can be translated to UFO directly, using only these building blocks. For example, the colorflow (6.2) in $\mathbf{3} \otimes \mathbf{6} \otimes \mathbf{8}$ corresponds to

$$C^a_{ir} = \bar{\epsilon}_{ijk}[T^a]^j{}_l[K_{\mathbf{6}\,r}]^{kl},. \tag{A.2a}$$

Using the UFO notation [29, 30], this is written

$$C^a_{ir} = \epsilon_{ijk}[T^a]^{\bar{j}}{}_l[K_{\mathbf{6}\,r}]^{\bar{k}\bar{l}} \tag{A.2b}$$

and can be encoded as a UFO expression

```
Epsilon(1,-1,-2) * T(3,-3,-1) * K6(2,-2,-3),        (A.2c)
```

taking into account that T(a,i,j) is translated to $[T^a]^{\bar{j}}_{\ i}$. The conjugate

$$C^{a\bar{i}\bar{r}} = \epsilon^{\bar{i}\bar{j}\bar{k}}[T^a]^{\bar{l}}_{\ j}[\overline{K}_{\mathbf{6}}^{\bar{r}}]_{kl} \tag{A.3a}$$

is written

```
EpsilonBar(1,-1,-2) * T(3,-1,-3) * K6Bar(2,-2,-3).        (A.3b)
```

Such translations can be performed directly by TANGARA and similar programs. In principle, this approach can be continued by adding dedicated Clebsch-Gordan coefficients and generators for each higher representation, where the catalogue (2.5) should be more than enough for all practical purposes in the foreseeable future: `K10`, `K15`, `K15prime`, `K21`, `K24`, `K27`.

An even more flexible solution would be to extend the syntax of UFO by generic particle declarations, Clebsch-Gordan coefficients and generators that accept a Young tableau as an additional argument specifying the irrep: e.g. `K[[1,2,3],[4]]` instead of `K15`. At the same time, one could add the option to encode interactions more concisely in a colorflow basis using only Kronecker and Levi-Civita symbols of external indices instead of the current building blocks that force the user to introduce summation indices as in (A.2c). In order to avoid a fragmentation of the UFO format [29, 30], this should be decided as a community effort for a future iteration of the format, after some experience has been gained with example implementations of concrete syntax in WHIZARD [17, 18, 31] and other programs.



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