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On elementary proof of AGT relations from six dimensions

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ABSTRACT

The actual definition of the Nekrasov functions participating in the AGT relations implies a peculiar choice of contours in the LMNS and Dotsenko–Fateev integrals. Once made explicit and applied to the original triply-deformed (6-dimensional) version of these integrals, this approach reduces the AGT relations to symmetry in $q_{1,2,3}$, which is just an elementary identity for an appropriate choice of the integration contour (which is, however, a little non-traditional). We illustrate this idea with the simplest example of $\mathcal{N} = (1,1)$ $U(1)$ SYM in six dimensions, however all other cases can be evidently considered in a completely similar way.

Seiberg–Witten theory and its quantization [1–5], provided by Nekrasov's evaluation [6] of the LMNS integrals [7], is one of the cornerstones of modern theoretical physics. In different ways this story is related to a majority of other important subjects. In particular, the AGT relations [8] provide a connection to 2d conformal theories [9,10] and, perhaps, further to the generic stringy AdS/CFT correspondence. Lifting the original four-dimensional story to five and six dimensions makes a contact with q- and elliptic Virasoro algebras [11], with (refined) topological string theories [12] and, finally, with still mysterious double-elliptic integrable systems [13]. As usual, things are rather obscure in low dimensions and get clarified when their number increases, at expense of an undeveloped language to describe these simple, but somewhat non-classical structures. In this letter, we provide a brief summary of the last years efforts to understand and prove refinement procedures on one side and the AGT relations on another, and emphasize that the choice of the right language is sufficient to convert the latter into an elementary identity. Below is a very brief, though exhaustive presentation. A less formal and more traditional version will be provided in a longer accompanying paper.

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We consider the integrals

$$Z_\gamma\{Q,q\} = \int_\gamma dx^N F\{x_i|Q,q\} \tag{1}$$

where F are basically the products and ratios (perhaps, infinite) of Van-der-Monde like quantities $\prod_{i\neq j}(x_i - cx_j)$ over some Q, q-dependent families C of parameters c:

$$F\{x_i|Q,q\} \sim \frac{\prod_{c\in C^+}\prod_{i\neq j}(x_i - cx_j)}{\prod_{c\in C^-}\prod_{i\neq j}(x_i - cx_j)} \tag{2}$$

Such F has a variety of poles. The integration contour γ can be chosen to pick up some of these poles, so that the integral becomes a sum of residues over them. In other words, γ defines a set Π_γ of poles,

$$x_i = x_i^\pi(Q,q) \quad \text{for} \quad \pi \in \Pi_\gamma \tag{3}$$

so that

$$\boxed{Z_\gamma\{C_\pm\} = \sum_{\pi\in\Pi_\gamma} F'_\pi\{Q,q\}} \tag{4}$$

where

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$$F'_\pi\{Q,q,u\} = F\{x_i|Q,q\}\prod_{i=1}^{N}\left(x_i - x_i^\pi(Q,q)\right)\Big|_{x_i=x_i^\pi(Q,q)} \tag{5}$$

and we switched from Q, q to notation with $C_\pm$, which is sometimes more adequate. It would be convenient for us to normalize the partition function so that the contribution of the first pole is one. This amounts to the following:

$$\widetilde{Z}_\gamma\{C_\pm\} = \sum_{\pi\in\Pi_\gamma}\frac{F'_\pi\{Q,q\}}{F'_\varnothing\{Q,q\}} \tag{6}$$

Since we now have the *ratio* of two residues in each term, we can simply replace it with the ratio of the integrands *evaluated* at the poles:

$$\widetilde{Z}_\gamma\{C_\pm\} = \sum_{\pi\in\Pi_\gamma}\frac{F\{x_i^\pi(Q,q)|Q,q\}}{F\{x_i^\varnothing(Q,q)|Q,q\}} \tag{7}$$

The ratio of integrands (2) can be rewritten in terms of time-variables $p_n = \sum_i x_i^n$,

$$\frac{F\{x_i^\pi(Q,q)|Q,q\}}{F\{x_i^\varnothing(Q,q)|Q,q\}} = \exp\left(\sum_{n\ge1}\left(\sum_{c\in C_-}c^n - \sum_{c\in C_+}c^n\right)\frac{(p_n^\pi p_{-n}^\pi - p_n^\varnothing p_{-n}^\varnothing)}{n}\right) \tag{8}$$

which can be further expanded into the Schur/Macdonald polynomials of various types [14,15], depending on particular sets $C_\pm$.

This technique was intensively used to demonstrate that the substitution (1) $\longrightarrow$ (6) is the outcome of more standard procedures like Selberg integration, and actually (6), perhaps, in the form

$$\boxed{Z_\gamma\{C_\pm\} = \exp\left(\sum_{n\ge1}\left(\sum_{c\in C_-}c^n - \sum_{c\in C_+}c^n\right)\frac{(p_n^\pi p_{-n}^\pi - p_n^\varnothing p_{-n}^\varnothing)}{n}\right)} \tag{9}$$

can be taken as a *definition* of relevant quantities in Seiberg–Witten–Nekrasov theory. Moreover, the sets Π_γ in this context are *postulated* to be some collections of ordinary or 3d Young diagrams. The role of parameters Q and q is different in these theories: Q's are *moduli* (dimensions, brane lengths, masses, couplings depending on the preferred language), while q's are theory parameters (like the compactification radii of the 5-th, 6-th and 11-th dimensions). Technically, Q and q enter in different ways into the sets $C_\pm$ and Π_γ.

The AGT relations are then identities between sums/integrals with different sets $\{C_\pm\}$, describing the LMNS and Dotsenko–Fateev integrals. The identities can be actually understood as a symmetry in parameters q, which is, however, obvious only when their number is at least three, $q=\{q_1,q_2,q_3\}$, while it gets obscure in the limits when some of these parameters turn to zero (which corresponds to reducing the dimension of associated Yang–Mills theory from six to five and four). Thus, understanding the "6-dimensional and M-theory" origin of the theory allows one to provide an **elementary proof of the AGT relations**. Now we provide a simple illustration of this thesis with quite a non-trivial (!) example of the AGT identity.

The LMNS integral in 5d $U(1)$ theory with fundamental matter hypermultiplets in a specific point of the moduli space (equal masses of the multiplets), which will be our basic example is (the integral appeared, e.g., in [16])

$$Z_{LMNS} = \int d^Nx\prod_{i\ne j}\frac{(x_i-x_j)(x_i-t\tilde{t}x_j)}{(x_i-tx_j)(x_i-\tilde{t}x_j)} \tag{10}$$

The Dotsenko–Fateev (DF) like integral [17,18] describing the AGT-related conformal block of the q-deformed Virasoro algebra is [19,20],

$$Z_{DF} = \int d^Nx\prod_{k\ge0}\prod_{i\ne j}\frac{x_i-q^kx_j}{x_i-tq^kx_j} \tag{11}$$

It is clear that these integrals are two different limits of the following "affine Selberg integral"

$$Z(q,t,\tilde{t}) = \int d^Nx\prod_{i\ne j}\prod_{k=0}^{\infty}\frac{(x_i-q^kx_j)(x_i-t\tilde{t}q^kx_j)}{(x_i-tq^kx_j)(x_i-\tilde{t}q^kx_j)} \tag{12}$$

namely

$$Z_{LMNS}(t,\tilde{t}) = Z(q=0,t,\tilde{t}) \tag{13}$$

and

$$Z_{DF}(q,t) = Z(q,t,\tilde{t}=0) \tag{14}$$

Our claim is that the AGT relation

$$\boxed{Z_{LMNS}(q,t) = Z_{DF}(q,t)} \tag{15}$$

is just a trivial corollary of the symmetry

$$\boxed{Z(q,t,\tilde{t}) = Z(\tilde{t},t,q) = \text{four other permutations}} \tag{16}$$

As to (16), it is, indeed, an elementary identity, provided this integral is *defined* as (9),[1]

$$\widetilde{Z} = \sum_{\pi\in\Pi}\exp\left(-\sum_{n\ge1}\frac{(1-t^n)(1-\tilde{t}^n)}{1-q^n}\frac{(p_n^\pi p_{-n}^\pi - p_n^\varnothing p_{-n}^\varnothing)}{n}\right) \tag{17}$$

with $N=\infty$ and Π being the set of all 3d partitions, and

$$\{x_i^\pi\} = \{q^{\pi_{b,c}}\cdot t^{1-b}\cdot\tilde{t}^{1-c}\} \tag{18}$$

with $\pi_{b,c}$ being the height of partition $\pi\in\Pi$ at the point b,c (see for a similar calculation [21]). Indeed, then

$$\begin{aligned}
p_n^\pi &= \sum_{b,c\ge1}q^{n\pi_{b,c}}t^{n(1-b)}\tilde{t}^{n(1-c)}\\
&= \sum_{b,c\ge1}t^{n(1-b)}\tilde{t}^{n(1-c)}\left(1+(q^{n\pi_{b,c}}-1\right) =\\
&= \frac{1}{(1-t^{-n})(1-\tilde{t}^{-n})} - (1-q^n)\sum_{(a,b,c)\in\pi}q^{n(a-1)}t^{n(1-b)}\tilde{t}^{n(1-c)}\\
&= \frac{E_\pi(q^n,t^{-n},\tilde{t}^{-n})}{(1-t^{-n})(1-\tilde{t}^{-n})}
\end{aligned}$$

The function

[1] Since one cannot just evaluate the integrand of (12) at the poles, we actually compute the ratio of the integrands at the pole and at the pole corresponding to the empty diagram (this ratio is finite). The same trick was used in [20] and leads to an inessential normalization factor.

$$E_\pi(q_1,q_2,q_3) = 1-(1-q_1)(1-q_2)(1-q_3) \cdot \sum_{(a,b,c)\in\pi} q_1^{a-1}q_2^{b-1}q_3^{c-1} \tag{19}$$

itself is not symmetric in $q_{1,2,3}$, but permutations of variables transpose the 3d Young diagram π, so that the sum over all π in

$$\boxed{\widetilde{Z}(q,t,\tilde{t}) = \sum_{\text{all 3d partitions } \pi} \exp\left(-\sum_{n\geq 1}\frac{E_\pi(q^n,t^{-n},\tilde{t}^{-n})E_\pi(q^{-n},t^n,\tilde{t}^n)-1}{n(1-q^n)(1-t^{-n})(1-\tilde{t}^{-n})}\right)} \tag{20}$$

is symmetric, i.e. (16) is true. Notice also that this expression exactly reproduces the instanton partition function of the $\mathcal{N}=(1,1)$ $U(1)$ gauge theory in six dimensional Ω-background [22] with three equivariant parameters set to q^{-1}, t, $\tilde{t}$ respectively, if one inserts an additional product $\prod_i x_i^u$ inside the integral (12) and sets the instanton counting parameter $Q=q^u$.

One can check that the choice (18) is consistent with the usual definitions of Z_{LMNS} and Z_{AGT}: the residues of these two integrals are enumerated by ordinary partitions, which are just two different projections of a single 3d partition. The integral (18) easily generalizes to $U(N)$ theory, which basically amounts to considering a *set* of 3d Young diagrams instead of one. One can also introduce external parameters Q, corresponding to the massive matter of the gauge theory, and to the vertex operators in CFT respectively, but we postpone all these details, including a more accurate explanation of (12) to a longer paper, to avoid unnecessary complications in the presentation of a simple idea. Let us notice that the AGT relation, which we consider here is actually a combination of the standard AGT relation and the spectral duality of conformal block [23,15,20], so that e.g. the $U(N)$ gauge theory corresponds to the N-point conformal block.

To conclude, we claim that after an appropriate extension and with clever definitions like (9) the somewhat *transcendental* AGT relation (15) between the LMNS and DF-like integrals reduces to *elementary symmetry properties* (16) of sums over 3d partition like (20). This symmetry seems to be the true meaning of the Hubbard–Stratonovich duality of [19,24].

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