Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 886 (2014) 328–363

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Quantum spin models for the SU(n)$_1$ Wess–Zumino–Witten model

Hong-Hao Tu [a,*], Anne E.B. Nielsen [a], Germán Sierra [b]

[a] *Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Str. 1, D-85748 Garching, Germany*
[b] *Instituto de Física Teórica, UAM-CSIC, Madrid, Spain*

Received 13 May 2014; accepted 30 June 2014

Available online 5 July 2014

Editor: Hubert Saleur

Abstract

We propose 1D and 2D lattice wave functions constructed from the SU(n)$_1$ Wess–Zumino–Witten (WZW) model and derive their parent Hamiltonians. When all spins in the lattice transform under SU(n) fundamental representations, we obtain a two-body Hamiltonian in 1D, including the SU(n) Haldane–Shastry model as a special case. In 2D, we show that the wave function converges to a class of Halperin's multilayer fractional quantum Hall states and belongs to chiral spin liquids. Our result reveals a hidden SU(n) symmetry for this class of Halperin states. When the spins sit on bipartite lattices with alternating fundamental and conjugate representations, we provide numerical evidence that the state in 1D exhibits quantum criticality deviating from the expected behaviors of the SU(n)$_1$ WZW model, while in 2D they are chiral spin liquids being consistent with the prediction of the SU(n)$_1$ WZW model.

1. Introduction

For decades, SU(n) quantum antiferromagnets have been an extensively studied class of strongly correlated systems in condensed matter. Initially, an important motivation of studying these models is that they may shed light on the properties of the spin-1/2 antiferromagnetic

* Corresponding author.
E-mail address: hong-hao.tu@mpq.mpg.de (H.-H. Tu).

http://dx.doi.org/10.1016/j.nuclphysb.2014.06.027

Heisenberg models with SU(2) symmetry [1–4], which are relevant for many strongly correlated electronic materials, including undoped high-T_c superconductors. Similar to the large-n expansion used in quantum chromodynamics, generalizing SU(2)-symmetric models to SU(n)-symmetric models allows stable mean-field solutions in the large-n limit [5,6], and furthermore, systematic calculations of corrections (organized in powers of $1/n$) can be carried out. Later on, the proposal [7,8] that the SU(4) Heisenberg model might describe certain materials with coupled spin–orbital degrees of freedom [9] brings SU(n) models closer to physical reality. By now, more and more evidences show that, depending on the magnitude of n, spatial dimensionality, lattice geometry, and form of couplings, the SU(n) models can support a zoo of exotic quantum states of matter [10–19]. Recently, considerable progress has been achieved in the experimental study of multi-flavor cold atoms in optical lattices [20–23]. With these experimental setups, atom species, lattice geometries, and interaction strengths can be manipulated and engineered in a highly controllable way [24,25]. The experimental advance spurs further theoretical investigations [26–34] of the SU(n) physics in the context of cold atomic setups. One may expect that, in the near future, the rich SU(n) physics might be experimentally explored in an unprecedented depth.

From the theoretical point of view, the SU(n) models are notoriously hard to tackle. Needless to say, the validity of the large-n solutions is questionable for physically relevant small n cases. Moreover, the SU(n) models usually suffer from the sign problem in quantum Monte Carlo simulations (except for special cases [35–38]), making them very difficult even for numerical study. For these models, important insights are gained from very few exactly solvable models, including integrable models and AKLT-type models. The former ones are restricted to 1D, including e.g. the SU(n) Uimin–Lai–Sutherland (ULS) model [39–41] and the SU(n) generalization [42–44] of the spin-1/2 Haldane–Shastry (HS) model [45,46], both of which exhibit Tomonaga–Luttinger liquid behaviors. The SU(n) AKLT-type models [47–52] generalize the SU(2) AKLT models [53,54], by extending the SU(n) singlets over multiple sites, and can be defined in one and higher dimensions.

Recently, a new approach of proposing strongly correlated wave functions has been suggested in Refs. [55–57]. This approach generalizes Moore and Read's construction [58] of fractional quantum Hall (FQH) wave functions in the continuum, by expressing both 1D and 2D *lattice* wave functions as chiral correlators of conformal field theories (CFTs). Apart from that, for rational CFTs, the existence of null fields allows to derive a (long-range) parent Hamiltonian [56]. Following this approach, wave functions have been constructed for the $\mathrm{SU}(2)_k$ and $\mathrm{SO}(n)_1$ WZW models [56,59], as well as $c = 1$ free boson CFTs at particular rational radii [60]. These simple wave functions, together with their parent Hamiltonians, provide important insight into the properties of their corresponding short-range realistic Hamiltonians [61], which are hard to solve directly.

In this work, we construct spin wave functions using the $\mathrm{SU}(n)_1$ WZW model and derive parent Hamiltonians of these states in 1D and 2D. In particular, we focus on two cases: (1) lattices with all spins transforming under SU(n) *fundamental* representations, and (2) lattices with a mixture of SU(n) *fundamental* and *conjugate* representations. In the former case, when the lattice sites are sitting on a unit circle in the complex plane, we derive a two-body parent Hamiltonian. This Hamiltonian can be viewed as an inhomogeneous extension of the SU(n) HS model. It recovers the SU(n) HS model when the lattice sites are *uniformly* distributed on the unit circle, which we call 1D uniform case. In 2D, we find that, on an infinite plane, the wave function converges to a special class of Halperin states that appeared in the context of the multilayer FQH effect. Interestingly, this reveals a hidden SU(n) symmetry for this class of Halperin states. Further numerical calculations based on topological entanglement entropy (TEE) [62,63] agree with

the prediction from the $\mathrm{SU}(n)_1$ WZW model and confirm that these 2D states are chiral spin liquids [64–66]. For the more general case of wave functions with both fundamental and conjugate representations, we concentrate on *bipartite* lattices with alternating fundamental and conjugate representations. In 1D uniform case, the wave function exhibits logarithmically increasing entanglement entropy and powerlaw decaying correlation functions, indicating quantum critical behaviors. Surprisingly, the estimated central charges for $n = 3$ and 4 show clear deviations from the expected values for the $\mathrm{SU}(n)_1$ WZW model. In 2D, we find that the states are again chiral spin liquids, with TEE being consistent with the prediction of the $\mathrm{SU}(n)_1$ WZW model.

The paper is organized as follows. In Section 2, we explain how we construct wave functions of spin systems from primary fields of the $\mathrm{SU}(n)_1$ WZW models, and we derive decoupling equations that form the basis for obtaining parent Hamiltonians of the states. In Section 3, we consider the wave functions obtained from primary fields that transform under the fundamental representation of $\mathrm{SU}(n)$. We provide explicit analytical expressions for the wave functions and compute the TEE of the states in 2D numerically. In Section 4, we derive parent Hamiltonians of the states constructed from the fundamental representation. For a uniform lattice in 1D this Hamiltonian reduces effectively to the $\mathrm{SU}(n)$ HS model, and we also discuss CFT predictions for the spectrum of this model. In Section 5, we consider the more general case of wave functions constructed from primary fields transforming either under the fundamental or the conjugate representation of $\mathrm{SU}(n)$. The wave functions are expressed analytically, and we investigate their properties through Monte Carlo simulations. Parent Hamiltonians of the states are derived in Section 6, where we also discuss possibilities for obtaining a truncated short-range version of the Hamiltonian. Finally, Section 7 concludes the paper.

2. Constructing quantum spin models from the $\mathrm{SU}(n)_1$ WZW model

2.1. Wave functions

Before constructing the wave functions, let us briefly review the $\mathrm{SU}(n)_1$ WZW model [67]. This rational CFT has n primary fields, denoted by Λ_a, with $a = 0, 1, \ldots, n-1$, corresponding to particular $\mathrm{SU}(n)$ irreducible representations. The primary field Λ_0 is an $\mathrm{SU}(n)$ singlet, which is also the identity field with conformal weight $h(\Lambda_0) = 0$. The next primary field Λ_1 is the $\mathrm{SU}(n)$ fundamental representation, corresponding to a single box when the $\mathrm{SU}(n)$ irreducible representations are represented as the Young tableaux. In general, the primary field Λ_a corresponds to a Young tableau with a single column and a rows. Accordingly, Λ_a consists of $\dim \Lambda_a = \binom{n}{a}$ components, and we write these components as $\Lambda_{a,\alpha}$, where $\alpha \in \{1, 2, \ldots, \dim \Lambda_a\}$.

The central charge c, conformal weights $h(\Lambda_a)$, and fusion rules of the $\mathrm{SU}(n)_1$ WZW model are given by [68]

$$c = n - 1, \qquad h(\Lambda_a) = \frac{a(n-a)}{2n}, \qquad \Lambda_a \otimes \Lambda_b = \Lambda_{a+b \,(\mathrm{mod}\, n)}. \tag{1}$$

As we shall discuss further below, the $\mathrm{SU}(n)_1$ WZW model has a free-field representation with $n-1$ free bosons. In this representation, the primary fields are conveniently realized using vertex operators.

To build lattice wave functions, we consider N_T spins sitting at the fixed positions z_j ($j = 1, 2, \ldots, N_\mathrm{T}$) in the complex plane. Following Ref. [56], we define lattice wave functions

$$|\Psi\rangle = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{N_\mathrm{T}}} \langle 0|\Lambda_{a_1,\alpha_1}(z_1)\Lambda_{a_2,\alpha_2}(z_2)\ldots\Lambda_{a_{N_\mathrm{T}},\alpha_{N_\mathrm{T}}}(z_{N_\mathrm{T}})|0\rangle |\alpha_1, \alpha_2, \ldots, \alpha_{N_\mathrm{T}}\rangle \tag{2}$$

that are chiral correlators of primary fields. Here, $|0\rangle$ is the vacuum of the CFT and $|\alpha_j\rangle$ are the basis vectors of the internal state of spin number j. CFT states of the form (2) can be seen as a special type of matrix product states in which the finite-dimensional matrices have been replaced by infinite-dimensional conformal fields. They are therefore sometimes referred to as infinite-dimensional-matrix product states (IDMPS).

Regarding the wave function (2), there are several comments in order. First, choosing the primary field Λ_{a_j} at site j requires that the spin at this site also transforms under the SU(n) irreducible representation corresponding to a Young tableau with one column and a_j rows. Note that the SU(n)$_1$ WZW model does not have primary fields corresponding to a Young tableaux with more than one column. Secondly, the fusion rules in (1) always have a *unique* fusion outcome, which ensures that the wave function (2) is a unique function. Lastly, to have a nonvanishing wave function, the N_{T} primary fields in (2) must fuse into the identity Λ_0 (i.e. the SU(n) singlet),

$$\Lambda_{a_1} \otimes \Lambda_{a_2} \otimes \cdots \otimes \Lambda_{a_{N_{\mathrm{T}}}} = \Lambda_0. \tag{3}$$

In this work, we shall focus on the case, where each of the spins belong either to the SU(n) *fundamental* representation Λ_1 or to the SU(n) *conjugate* representation Λ_{n-1}. We shall denote the sublattice of spins transforming under the fundamental (conjugate) representation by A (B),

$$\begin{aligned} &A\text{:} \quad \text{Fundamental representation,} \\ &B\text{:} \quad \text{Conjugate representation,} \end{aligned} \tag{4}$$

and we shall let N ($\bar{N}$) denote the number of spins in A (B) such that $N + \bar{N} = N_{\mathrm{T}}$. The condition (3) then gives that $(N - \bar{N})/n$ must be an integer, and we shall assume this to be the case throughout. Note that the fundamental and conjugate representations are the same for $n = 2$, so that there is only one state in this particular case. For $n \geq 3$, however, they are different.

Before we continue with the above case, let us note that other choices for the primary fields are possible. For instance, for even n, one could use the primary field $\Lambda_{n/2}$ (self-conjugate representation) to build the wave function, according to the fusion rule $\Lambda_{n/2} \otimes \Lambda_{n/2} \otimes \cdots \otimes \Lambda_{n/2} = \Lambda_0$ (N_{T} even). For the SU(4) case, one has SU(4)$_1 \simeq$ SO(6)$_1$ and the SU(4) self-conjugate primary field Λ_2 becomes the vector representation of SO(6) with conformal weight $h(\Lambda_2) = 1/2$, which can be interpreted as a Majorana field and has been considered in Ref. [59]. Although we only consider states constructed from the fundamental and conjugate representations below, we note that the formalism we develop is general and that other cases can be treated in a similar way.

In the following, we shall find it convenient to use the notation

$$\varphi_{\alpha_j}(z_j) = \begin{cases} \Lambda_{1,\alpha_j}(z_j) & \text{for } j \in A \\ \Lambda_{n-1,\alpha_j}(z_j) & \text{for } j \in B. \end{cases} \tag{5}$$

We can then write the wave functions that we are interested in as

$$|\Psi\rangle = \sum_{\alpha_1,\ldots,\alpha_{N_{\mathrm{T}}}=1}^{n} \Psi(\alpha_1, \ldots, \alpha_{N_{\mathrm{T}}}) |\alpha_1, \ldots, \alpha_{N_{\mathrm{T}}}\rangle, \tag{6}$$

where

$$\Psi(\alpha_1, \ldots, \alpha_{N_{\mathrm{T}}}) = \langle 0 | \varphi_{\alpha_1}(z_1) \varphi_{\alpha_2}(z_2) \cdots \varphi_{\alpha_{N_{\mathrm{T}}}}(z_{N_{\mathrm{T}}}) | 0 \rangle. \tag{7}$$

Since we shall often refer to the wave function, for which all the primary fields belong to the fundamental representation, we shall give this wave function a particular name: $|\Psi_{\mathrm{F}}\rangle$. Explicit

representations of $|\Psi_{\mathrm{F}}\rangle$ and $|\Psi\rangle$ will be discussed in Sections 3 and 5, respectively. In the next two subsections, we shall use their abstract forms to derive relevant null fields and their corresponding decoupling equations, which are our starting point for deriving parent Hamiltonians.

2.2. Null vectors

As a rational CFT, the $\mathrm{SU}(n)_1$ WZW model has null vectors in its Verma modules of the Kac–Moody algebra. According to Ref. [56], identifying proper null vectors and deriving decoupling equations for the chiral correlators are the key for constructing parent Hamiltonians of the wave functions. In this subsection, we derive the null vectors relevant for (7).

The $\mathrm{SU}(n)_1$ Kac–Moody algebra is defined by

$$\left[J_m^a, J_{m'}^b\right] = i f_{abc} J_{m+m'}^c + \frac{m}{2}\delta_{ab}\delta_{m+m',0}, \quad m, m' \in \mathbb{Z}, \tag{8}$$

where $J_m^a = \oint_0 \frac{dz}{2\pi i} z^m J^a(z)$ is the mth mode of the Kac–Moody current $J^a(z)$ and f_{abc} are the structure constants of the $\mathrm{SU}(n)$ Lie algebra. Here and later on, we shall always assume that repeated indices are summed over. The operator product expansion (OPE) between the Kac–Moody currents and a primary field is [67]

$$J^a(z)\varphi_\alpha(w) \sim -\frac{1}{z-w}\sum_\beta \left(t^a\right)_{\alpha\beta}\varphi_\beta(w), \tag{9}$$

where the matrices t^a with elements $(t^a)_{\alpha\beta}$ are the generators of $\mathrm{SU}(n)$ in the representation of the primary field. Let us note here that the generators in the fundamental and conjugate representations are related though a complex conjugation and a multiplication by a minus sign, i.e.,

$$t^a = \begin{cases} \tau^a & \text{(fundamental representation)} \\ -(\tau^a)^* & \text{(conjugate representation)}, \end{cases} \tag{10}$$

where τ^a are the generators in the fundamental representation (see Appendix A).

To the primary field $\varphi_\alpha(z)$, one associates a primary state $|\varphi_\alpha\rangle$ satisfying the following properties [67]:

$$|\varphi_\alpha\rangle = \varphi_\alpha(0)|0\rangle, \qquad J_0^a|\varphi_\alpha\rangle = -\sum_\beta \left(t^a\right)_{\alpha\beta}|\varphi_\beta\rangle, \qquad J_n^a|\varphi_\alpha\rangle = 0, \quad n > 0, \tag{11}$$

and descendant states are obtained by multiplying $|\varphi_\alpha\rangle$ by any number of current operators J_n^a with $n < 0$. A null state is a state that is at the same time a descendant and a primary state. Since the wave function (7) only involves primary fields belonging to the fundamental or the conjugate representation, we shall here only need to deal with the two Verma modules formed by the corresponding primary states, as well as their descendants.

Let us first consider the primary field $\Lambda_{1,\alpha}(z)$ belonging to the *fundamental* representation. In Virasoro level $m = 1$, we look for null vectors with the following form:

$$\left|\chi^q\right\rangle = \sum_{a,\alpha} W_{q,a\alpha} J_{-1}^a |\Lambda_{1,\alpha}\rangle, \tag{12}$$

where $W_{q,a\alpha}$ can be interpreted as Clebsch–Gordan coefficients satisfying $\sum_{a\alpha} W^*_{q,a\alpha} W_{q',a\alpha} = \delta_{qq'}$. They come from the tensor product decomposition of the (n^2-1)-dimensional $\mathrm{SU}(n)$ adjoint representation (carried by J_{-1}^a) and the fundamental representation (carried by $|\Lambda_{1,\alpha}\rangle$),

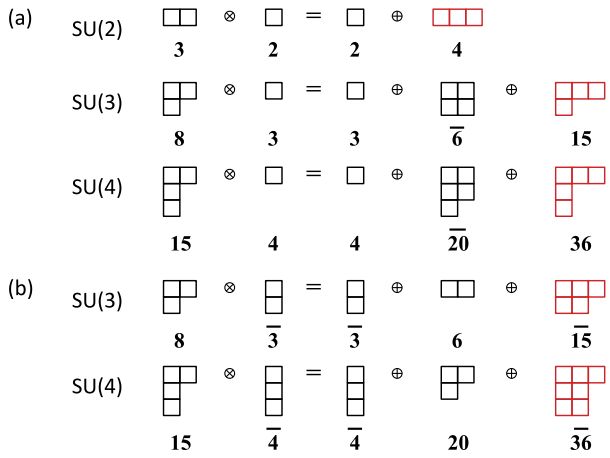

Fig. 1. Tensor product decompositions of the SU(n) adjoint representation and (a) the fundamental ((b) conjugate) representation with Young tableaux. The null vectors belong to the SU(n) high-dimensional representation (with red color). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$\left(n^2-1\right)\otimes n = n \oplus \frac{1}{2}n(n+1)(n-2) \oplus \frac{1}{2}n(n-1)(n+2), \tag{13}$$

where the irreducible representations are denoted by their dimensions (they are not distinguished with their complex conjugate representations). Fig. 1(a) shows the tensor product decomposition (13) for $n = 2, 3, 4$, using the Young tableaux. We have found that, for SU$(n)_1$ WZW model with all n, null vectors indeed exist in Virasoro level $m = 1$, and they belong to the SU(n) representation with dimension $\frac{1}{2}n(n-1)(n+2)$ in (13). In practice, the Clebsch–Gordan coefficients $W_{q,a\alpha}$ in (12) can be determined by requiring the null vector condition $\langle\chi^{b'}|\chi^{b}\rangle = 0$.

For our purpose, we redefine the null vectors as [56]

$$\left|\chi^{a,\alpha}\right\rangle = \sum_{q} W^{*}_{q,a\alpha}\left|\chi^{q}\right\rangle = \sum_{b,\beta}(K_{\mathrm{F}})^{a\alpha}_{b\beta} J^{b}_{-1}|\Lambda_{1,\beta}\rangle, \tag{14}$$

where $(K_{\mathrm{F}})^{a\alpha}_{b\beta}$ is given by

$$(K_{\mathrm{F}})^{a\alpha}_{b\beta} = \sum_{q} W^{*}_{q,a\alpha} W_{q,b\beta}. \tag{15}$$

$\mathbf{K}_{\mathrm{F}}$ can be viewed as a matrix with its entries being $(\mathbf{K}_{\mathrm{F}})_{a\alpha,b\beta} = (K_{\mathrm{F}})^{a\alpha}_{b\beta}$, and it is a projector (i.e. $\mathbf{K}^2_{\mathrm{F}} = \mathbf{K}_{\mathrm{F}}$) onto the SU($n$) irreducible representation with dimension $\frac{1}{2}n(n-1)(n+2)$. This also lead to an additional equation, $\sum_a t^a (K_{\mathrm{F}})^a_b = 0$, where $(K_{\mathrm{F}})^a_b$ is a matrix with entries $[(K_{\mathrm{F}})^a_b]_{\alpha\beta} = (K_{\mathrm{F}})^{a\alpha}_{b\beta}$. These two equations are sufficient for determining the explicit form of $(K_{\mathrm{F}})^a_b$. For general n, we obtain

$$(K_{\mathrm{F}})^a_b = \frac{n+2}{2(n+1)}\delta_{ab} + \frac{1}{2(n+1)}\left[n d_{abc} - (n+2) i f_{abc}\right] t^c, \tag{16}$$

where d_{abc} is a totally symmetric tensor (see Appendix A).

If we build the null vector at Virasoro level $m = 1$ using the primary state $|\Lambda_{n-1,\alpha}\rangle$ in the conjugate representation, the representations appearing in (13) would be their complex conjugate representations. See Fig. 1(b) for this tensor product decomposition for $n = 3$ and 4. As a result,

the Clebsch–Gordan coefficients in (12) for obtaining the null vectors would be their complex conjugate. Then, the corresponding null vectors can be written as

$$\left|\chi^{a,\alpha}\right\rangle = \sum_{b,\beta} (K_{\mathrm{C}})^{a\alpha}_{b\beta} J^b_{-1} |\Lambda_{n-1,\beta}\rangle, \tag{17}$$

where $(K_{\mathrm{C}})^{a\alpha}_{b\beta} = (K^*_{\mathrm{F}})^{a\alpha}_{b\beta}$.

Utilizing (10), we can combine (14) and (17) into a single expression

$$\left|\chi^{a,\alpha}\right\rangle = \sum_{b,\beta} K^{a\alpha}_{b\beta} J^b_{-1} |\varphi_\beta\rangle, \tag{18}$$

where $K^{a\alpha}_{b\beta}$ are the matrix elements of

$$K^a_b = \frac{n+2}{2(n+1)}\delta_{ab} + \frac{1}{2(n+1)}\left[nr d_{abc} - (n+2) i f_{abc}\right] t^c. \tag{19}$$

Here, $r = +1$ for the fundamental representation, $r = -1$ for the conjugate representation, and t^c are the generators in the considered representation.

2.3. Decoupling equations

Following Ref. [56], a set of decoupling equations can be derived for the chiral correlator (7) using the null vectors (18). These decoupling equations provide operators annihilating the wave functions, which can be used to build parent Hamiltonians.

The null state (18) corresponds to the following null field:

$$\chi^{a,\alpha}(z_i) = \oint_{z_i} \frac{dz}{2\pi i} \frac{1}{z - z_i} \sum_{b,\beta} K^{a\alpha}_{b\beta} J^b(z) \varphi_\beta(z_i). \tag{20}$$

By definition of the null field, substituting it into the wave function (7), one obtains a vanishing expression

$$\begin{aligned} 0 &= \sum_{\alpha_1,\ldots,\alpha_N} \left\langle \varphi_{\alpha_1}(z_1) \cdots \chi^{a,\alpha_i}(z_i) \cdots \varphi_{\alpha_N}(z_N) \right\rangle |\alpha_1, \ldots, \alpha_N\rangle \quad \forall a \\ &= \sum_{\alpha_1,\ldots,\alpha_N} \sum_{b,\beta_i} K^{a,\alpha_i}_{b,\beta_i} \oint_{z_i} \frac{dz}{2\pi i} \frac{1}{z - z_i} \left\langle \varphi_{\alpha_1}(z_1) \cdots J^b(z) \varphi_{\beta_i}(z_i) \cdots \varphi_{\alpha_N}(z_N) \right\rangle |\alpha_1, \ldots, \alpha_N\rangle. \end{aligned} \tag{21}$$

After deforming the integral contour and using the OPE (9) between the Kac–Moody currents and primary fields, we arrive at

$$\begin{aligned} 0 &= \sum_{\alpha_1,\ldots,\alpha_N} \sum_{j(\neq i)} \sum_{\alpha'_j} \frac{(t^b_j)_{\alpha_j \alpha'_j}}{z_i - z_j} \\ &\quad \times \sum_{b,\beta_i} K^{a,\alpha_i}_{b,\beta_i} \left\langle \varphi_{\alpha_1}(z_1) \cdots \varphi_{\alpha'_j}(z_j) \cdots \varphi_{\beta_i}(z_i) \cdots \varphi_{\alpha_N}(z_N) \right\rangle |\alpha_1, \ldots, \alpha_j, \ldots, \alpha_i, \ldots, \alpha_N\rangle \\ &= \sum_{j(\neq i),b} \frac{t^b_j}{z_i - z_j} \left(K^{(i)}\right)^a_b |\Psi\rangle \quad \forall a, \end{aligned} \tag{22}$$

where $(K^{(i)})^a_b$ denotes the operator K^a_b in (19) acting on spin number i and $(t^b_j)_{\alpha_j\alpha'_j}$ denote the matrix elements of the operator t^b acting on spin number j. (Note that the representation chosen for t^b_j is the same as the representation of spin number j.) Thus, the resulting decoupling equation yields a set of operators

$$\mathcal{P}^a_i(z_1,\ldots,z_N) = \sum_{j(\neq i),b} \frac{t^b_j}{z_i - z_j}\big(K^{(i)}\big)^a_b, \tag{23}$$

which annihilate the wave function $|\Psi\rangle$, i.e. $\mathcal{P}^a_i(z_1,\ldots,z_N)|\Psi\rangle = 0\ \forall i,a$. Together with the fact that $|\Psi\rangle$ is a global SU(n) singlet, $T^a|\Psi\rangle = 0$ with $T^a = \sum_i t^a_i$, we obtain

$$\mathcal{C}^a_i(z_1,\ldots,z_N)|\Psi\rangle = 0, \tag{24}$$

where $\mathcal{C}^a_i(z_1,\ldots,z_N)$ is given by

$$\begin{aligned}\mathcal{C}^a_i(z_1,\ldots,z_N) &= \sum_{j(\neq i),b} w_{ij}\big(K^{(i)}\big)^a_b t^b_j \\ &= \frac{n+2}{2(n+1)} \sum_{j(\neq i)} w_{ij}\left[t^a_j + \left(\frac{n}{n+2} r_i d_{abc} + i f_{abc}\right) t^b_i t^c_j\right]\end{aligned} \tag{25}$$

and $w_{ij} = (z_i + z_j)/(z_i - z_j)$. For SU(2), we have $d_{abc} = 0$ and (25) recovers the result in Ref. [56]. Utilizing the formulas in Appendix A, we get

$$\mathcal{C}^a_i(z_1,\ldots,z_N) = \sum_{j(\neq i)} w_{ij}\left[\frac{1}{2}t^a_j - \frac{1}{n+1}t^a_i(\vec{t}_i\cdot\vec{t}_j) + (\vec{t}_i\cdot\vec{t}_j)t^a_i\right], \tag{26}$$

which is a convenient form for constructing parent Hamiltonians.

2.4. Vertex operator representation

After working out the decoupling equations for (7) using an abstract form of the primary fields, we now turn to an explicit representation of these primary fields, using chiral vertex operators. This is possible, since SU(n)$_1$ WZW model is equivalent to a free theory of $n-1$ massless bosons.

For our purpose, it is convenient to label the spin states in each site by their weights (eigenvalues of the Cartan generators). The state $|\alpha\rangle$, $\alpha \in \{1,2,\ldots,n\}$, in the fundamental representation is therefore characterized by $n-1$ quantum numbers, which we collect into the vector $\vec{m}_\alpha$ given explicitly by

$$\begin{aligned}\vec{m}_1 &= \big(\tfrac{1}{2},\ \tfrac{1}{2\sqrt{3}},\ \ldots,\ \tfrac{1}{\sqrt{2n(n-1)}}\big),\\ \vec{m}_2 &= \big(-\tfrac{1}{2},\ \tfrac{1}{2\sqrt{3}},\ \ldots,\ \tfrac{1}{\sqrt{2n(n-1)}}\big),\\ \vec{m}_3 &= \big(0,\ -\tfrac{1}{\sqrt{3}},\ \ldots,\ \tfrac{1}{\sqrt{2n(n-1)}}\big),\\ &\vdots\\ \vec{m}_n &= \big(0,\ 0,\ \ldots,\ -\tfrac{n-1}{\sqrt{2n(n-1)}}\big).\end{aligned} \tag{27}$$

In the conjugate representation, the state $|\alpha\rangle$, $\alpha \in \{1,2,\ldots,n\}$, is characterized by the quantum numbers $-\vec{m}_\alpha$. The SU(3) and SU(4) weight diagrams are shown in Fig. 2 as examples.

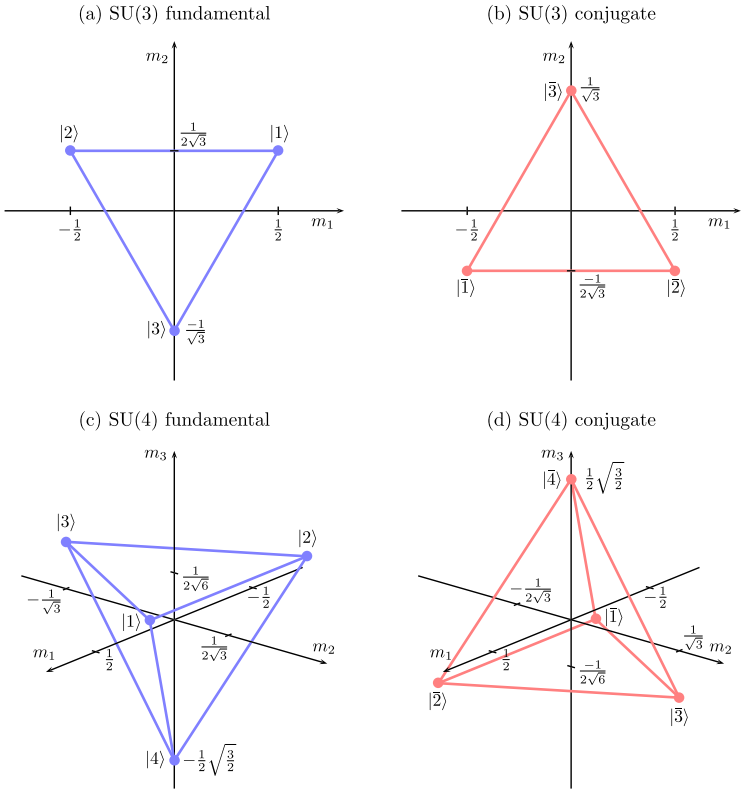

Fig. 2. Weight diagrams of the fundamental and conjugate representations for SU(3) and SU(4). Here, (m_1, m_2, m_3) is shorthand notation for the components of the vectors $\vec{m}_\alpha$ and $-\vec{m}_\alpha$, respectively.

Using the weights, the primary field $\varphi_\alpha(z)$ can be expressed as

$$\varphi_\alpha(z) = \kappa_\alpha : \exp\big(i\sqrt{2} r \vec{m}_\alpha \cdot \vec{\phi}(z)\big) : , \tag{28}$$

where $r = +1$ for the fundamental representation and $r = -1$ for the conjugate representation as above. The colons denote normal ordering and $\vec{\phi}(z)$ is a vector of $n-1$ independent fields of free, massless bosons. The factor κ_α is a Klein factor, commuting with the vertex operators and satisfying Majorana-like anticommutation relations

$$\{\kappa_\alpha, \kappa_{\alpha'}\} = 2\delta_{\alpha\alpha'}. \tag{29}$$

Note that κ_α is the same in the fundamental and in the conjugate representation. At this moment, the meaning of these Klein factors is not clear. In fact, their role is to ensure that the wave function (7) is an SU(n) singlet. We will go back to this point when discussing the wave functions in Section 3 and Section 5.

Let us note that the vertex operators in (28) have the anticipated conformal weights, since

$$\vec{m}_\alpha \cdot \vec{m}_\alpha = h(\Lambda_1) = h(\Lambda_{n-1}) = \frac{n-1}{2n}. \tag{30}$$

Another quantity, which will be used in later sections, is $\vec{m}_\alpha \cdot \vec{m}_{\alpha'}$ with $\alpha \neq \alpha'$. It is easy to convince ourselves that this value does not depend on the states we choose. For $\alpha \neq \alpha'$, we find

$$\vec{m}_\alpha \cdot \vec{m}_{\alpha'} = -\frac{1}{2n}. \tag{31}$$

Altogether, we thus conclude

$$\vec{m}_\alpha \cdot \vec{m}_{\alpha'} = \frac{1}{2}\delta_{\alpha\alpha'} - \frac{1}{2n}. \tag{32}$$

3. Quantum states from the fundamental representation of SU(n)

In this section, we analyze the wave function (7) in detail, both theoretically and numerically, for the case where all spins transform under the fundamental representation. First, the chiral correlator can be evaluated and expressed in terms of a product of Jastrow factors [67]

$$\Psi_{\mathrm{F}}(\alpha_1, \alpha_2, \ldots, \alpha_N) = \chi(\alpha_1, \alpha_2, \ldots, \alpha_N)\delta_{\sum_i \vec{m}_{\alpha_i}=0} \prod_{i<j} (z_i - z_j)^{2\vec{m}_{\alpha_i}\cdot\vec{m}_{\alpha_j}}, \tag{33}$$

where $\chi(\alpha_1, \alpha_2, \ldots, \alpha_N) = \kappa_{\alpha_1}\kappa_{\alpha_2}\cdots\kappa_{\alpha_N}$ is a z_j-independent phase factor to be determined below and the Kronecker delta function $\delta_{\sum_i \vec{m}_{\alpha_i}=0}$, which is 1 for $\sum_i \vec{m}_{\alpha_i} = 0$ and zero otherwise, ensures charge neutrality. Referring to Eq. (27), we observe that the charge neutrality forces the number of spins N_α in the state $|\alpha\rangle$ to fulfill $N_1 = N_2 = \cdots = N_n$. This gives $N_\alpha = N/n$ for all α, and we shall therefore assume N/n to be an integer whenever we consider states constructed from only the fundamental representation of SU(n). Utilizing (32), we note that (33) simplifies to

$$\Psi_{\mathrm{F}}(\alpha_1, \alpha_2, \ldots, \alpha_N) \propto \chi(\alpha_1, \alpha_2, \ldots, \alpha_N)\delta_{\sum_i \vec{m}_{\alpha_i}=0} \prod_{i<j} (z_i - z_j)^{\delta_{\alpha_i\alpha_j}}. \tag{34}$$

We shall also find it useful to express the state $|\Psi_{\mathrm{F}}\rangle$ in another notation. For a given spin configuration $|\alpha_1, \alpha_2, \ldots, \alpha_N\rangle$, let $x_j^{(\alpha)}$, where $j = 1, 2, \ldots, N/n$, be the position within the ket of the jth spin in the state $|\alpha\rangle$. For example, if we choose $n = 3$ and $N = 9$ and consider the state ket $|1, 2, 1, 3, 2, 3, 3, 2, 1\rangle$, we would have $x_1^{(1)} = 1$, $x_2^{(1)} = 3$, $x_3^{(1)} = 9$, $x_1^{(2)} = 2$, $x_2^{(2)} = 5$, $x_3^{(2)} = 8$, $x_1^{(3)} = 4$, $x_2^{(3)} = 6$, and $x_3^{(3)} = 7$. We shall also write $\{x^{(\alpha)}_{1\to\frac{N}{n}}\}$ or simply $\{x^{(\alpha)}\}$ as shorthand notation for $x_1^{(\alpha)}, x_2^{(\alpha)}, \ldots, x^{(\alpha)}_{\frac{N}{n}}$. We can then express $|\Psi_{\mathrm{F}}\rangle$ as

$$|\Psi_{\mathrm{F}}\rangle = \sum_{\{x^{(1)}\},\{x^{(2)}\},\ldots,\{x^{(n)}\}\in S_N} \Psi_{\mathrm{F}}\big(\{x^{(1)}\}, \{x^{(2)}\}, \ldots, \{x^{(n)}\}\big)\big|\{x^{(1)}\}, \{x^{(2)}\}, \ldots, \{x^{(n)}\}\big\rangle, \tag{35}$$

where S_N is the symmetric group over the elements $\{1, 2, \ldots, N\}$ and

$$\Psi_{\mathrm{F}}\big(\{x^{(1)}\}, \{x^{(2)}\}, \ldots, \{x^{(n)}\}\big) \propto \chi\big(\{x^{(1)}\}, \{x^{(2)}\}, \ldots, \{x^{(n)}\}\big) \prod_{\alpha=1}^{n} \prod_{1\le i<j\le \frac{N}{n}} (z_{x_i^{(\alpha)}} - z_{x_j^{(\alpha)}}). \tag{36}$$

Let us next determine χ from the condition that $|\Psi_{\mathrm{F}}\rangle$ should be an SU(n) singlet. We shall find below that the wave function $|\Psi_{\mathrm{F}}\rangle$ is proportional to the ground state of the SU(n) HS model if we choose $z_j = e^{2\pi i j/N}$ and

$$\chi = \mathrm{sgn}\big(x_1^{(1)}, \ldots, x_{N/n}^{(1)}, x_1^{(2)}, \ldots, x_{N/n}^{(2)}, \ldots, x_1^{(n)}, \ldots, x_{N/n}^{(n)}\big), \tag{37}$$

where the right-hand side of (37) is the sign of the permutation needed to transform $x_1^{(1)},\dots,x_{N/n}^{(1)},x_1^{(2)},\dots,x_{N/n}^{(2)},\dots,x_1^{(n)},\dots,x_{N/n}^{(n)}$ into $1,2,\dots,N$. Since the ground state of the SU(n) HS model is an SU(n) singlet, it follows that (37) is the correct choice of χ for all choices of z_j. The result (37) can be obtained from $\chi=\kappa_{\alpha_1}\kappa_{\alpha_2}\cdots\kappa_{\alpha_N}$ by choosing the factors κ_α to be Klein factors, which satisfy the Majorana-like anticommutation relation (29), and choosing to work in a sector, in which $\kappa_1\kappa_2\cdots\kappa_n=1$. This follows from

$$\begin{aligned}
&\kappa_{\alpha_1}\kappa_{\alpha_2}\cdots\kappa_{\alpha_N}\\
&\quad=\operatorname{sgn}\bigl(x_1^{(1)},\dots,x_{N/n}^{(1)},x_1^{(2)},\dots,x_{N/n}^{(2)},\dots,x_1^{(n)},\dots,x_{N/n}^{(n)}\bigr)\kappa_1^{N/n}\kappa_2^{N/n}\cdots\kappa_n^{N/n}\\
&\quad=\begin{cases}\operatorname{sgn}(x_1^{(1)},\dots,x_{N/n}^{(1)},x_1^{(2)},\dots,x_{N/n}^{(2)},\dots,x_1^{(n)},\dots,x_{N/n}^{(n)}) & \text{for } N/n \text{ even}\\ \operatorname{sgn}(x_1^{(1)},\dots,x_{N/n}^{(1)},x_1^{(2)},\dots,x_{N/n}^{(2)},\dots,x_1^{(n)},\dots,x_{N/n}^{(n)})\kappa_1\kappa_2\cdots\kappa_n & \text{for } N/n \text{ odd}\end{cases}\\
&\quad=\operatorname{sgn}\bigl(x_1^{(1)},\dots,x_{N/n}^{(1)},x_1^{(2)},\dots,x_{N/n}^{(2)},\dots,x_1^{(n)},\dots,x_{N/n}^{(n)}\bigr).
\end{aligned}\tag{38}$$

The proof given in Appendix B shows directly that the state (34) with χ given by (37) and z_j arbitrary is an SU(n) singlet without referring to the SU(n) HS model.

3.1. Wave functions in the hardcore boson basis

In order to compare the state (33) to known models in particular limits, we shall now express the state in a hardcore boson basis. In this picture, the coordinates z_j are lattice sites that can be empty or occupied by at most one hardcore boson. A spin in the state $|n\rangle$ is interpreted as an empty site, and a spin in the state $|\alpha\rangle$, with $\alpha\in\{1,2,\dots,n-1\}$, is interpreted as a site occupied by a hardcore boson with color α.

Referring to (27), we observe that the $(n-1)$th component $m_{\alpha_j,n-1}$ of the vector $\vec{m}_{\alpha_j}$ can be written as

$$m_{\alpha_j,n-1}=\frac{n}{\sqrt{2n(n-1)}}p_j-\frac{n-1}{\sqrt{2n(n-1)}},\tag{39}$$

where p_j is one if $\alpha_j\in\{1,2,\dots,n-1\}$ and zero if $\alpha_j=n$. In other words, we can use this component to distinguish between occupied sites and holes, and we shall use this observation to eliminate the coordinates of the unoccupied sites from the Jastrow factor in (33). The part of this factor that includes the contribution from $m_{\alpha_j,n-1}$ can be written as

$$\begin{aligned}
\prod_{i<j}(z_i-z_j)^{2m_{\alpha_i,n-1}m_{\alpha_j,n-1}}&=\prod_{i<j}(z_i-z_j)^{2[\frac{n}{\sqrt{2n(n-1)}}p_i-\frac{n-1}{\sqrt{2n(n-1)}}][\frac{n}{\sqrt{2n(n-1)}}p_j-\frac{n-1}{\sqrt{2n(n-1)}}]}\\
&\propto\prod_{i<j}(z_i-z_j)^{\frac{n}{n-1}p_ip_j}\prod_{i<j}(z_i-z_j)^{-(p_i+p_j)}.
\end{aligned}\tag{40}$$

The second factor in the above expression can be simplified as [57,60]

$$\prod_{i<j}(z_i-z_j)^{-(p_i+p_j)}=\prod_j(-1)^{(j-1)p_j}\prod_i\bigl[f_N(z_i)\bigr]^{p_i},\tag{41}$$

where

$$f_N(z_i)=\prod_{j(\neq i)}(z_i-z_j)^{-1}.\tag{42}$$

Let us next consider the part of the Jastrow factor that includes the contributions from $m_{\alpha_j,l}$ with $l = 1, 2, \ldots, n-2$. Utilizing (27) and (32), we find

$$\sum_{l=1}^{n-2} m_{\alpha,l} m_{\alpha',l} = \frac{1}{2}\delta_{\alpha\alpha'} - \frac{1}{2(n-1)} \quad \big(\alpha \neq n,\ \alpha' \neq n\big). \tag{43}$$

If α or α' is equal to n, we instead get $\sum_{l=1}^{n-2} m_{\alpha,l} m_{\alpha',l} = 0$ as follows immediately from (27). The part of the Jastrow factor that includes the contributions for $m_{\alpha_j,l}$ with $l = 1, 2, \ldots, n-2$ can therefore be written as

$$\prod_{i<j} (z_i - z_j)^{2\sum_{l=1}^{n-2} m_{\alpha_i,l} m_{\alpha_j,l}} = \prod_{i<j} (z_i - z_j)^{(\delta_{\alpha_i\alpha_j} - \frac{1}{n-1}) p_i p_j}. \tag{44}$$

Combining (40) and (44), we get the expression

$$\prod_{i<j} (z_i - z_j)^{2\vec{m}_{\alpha_i}\cdot\vec{m}_{\alpha_j}} \propto \prod_{i<j} (z_i - z_j)^{(\delta_{\alpha_i\alpha_j}+1) p_i p_j} \prod_j (-1)^{(j-1)p_j} \prod_j \big[f_N(z_j)\big]^{p_j} \tag{45}$$

for the Jastrow factor.

We would like to also remove the hole coordinates from the sign factor χ. Doing so gives rise to a sign factor that compensates the factor $\prod_j (-1)^{(j-1)p_j}$ in the wave function. The remaining sign factor is then $\mathrm{sgn}(x_1^{(1)}, \ldots, x_{N/n}^{(1)}, x_1^{(2)}, \ldots, x_{N/n}^{(2)}, \ldots, x_1^{(n-1)}, \ldots, x_{N/n}^{(n-1)})$. We note, however, that this factor can be absorbed by rearranging the ordering in the Jastrow factor. Putting everything together, we thus conclude that the state (35) can also be written as

$$\begin{aligned} |\Psi_{\mathrm{F}}\rangle = \sum_{\{x^{(1)}\},\{x^{(2)}\},\ldots,\{x^{(n-1)}\}} & \Psi_{\mathrm{F}}\big(\{x^{(1)}\}, \{x^{(2)}\}, \ldots, \{x^{(n-1)}\}\big) \\ & \times \big|\{x^{(1)}\}, \{x^{(2)}\}, \ldots, \{x^{(n-1)}\}\big\rangle, \end{aligned} \tag{46}$$

where the sum is over all possible distributions of the $(n-1)N/n$ colored bosons on the N lattice sites with at most one boson per site, and

$$\begin{aligned} &\Psi_{\mathrm{F}}\big(\{x^{(1)}\}, \{x^{(2)}\}, \ldots, \{x^{(n-1)}\}\big) \\ &\quad \propto \prod_{\alpha} \prod_{i<j} (z_{x_i^{(\alpha)}} - z_{x_j^{(\alpha)}})^2 \prod_{\alpha<\beta} \prod_{i,j} (z_{x_i^{(\alpha)}} - z_{x_j^{(\beta)}}) \prod_{\alpha} \prod_j f_N(z_{x_j^{(\alpha)}}) \end{aligned} \tag{47}$$

with $\alpha, \beta \in \{1, 2, \ldots, n-1\}$ and $i, j \in \{1, 2, \ldots, N/n\}$. We shall now comment further on (47) for particular choices of the lattice.

3.1.1. Jastrow wave functions for the uniform 1D lattice

We first consider a uniform lattice in 1D with periodic boundary conditions, which is achieved by choosing $z_j = e^{2\pi i j/N}$. For this particular case, we have the simple expression $f_N(z_j^{(\alpha)}) \propto z_j^{(\alpha)}$ [56]. Inserting this in (47), we see that the wave function for the particular case of a uniform 1D lattice reduces to the ground state of the SU(n) HS Hamiltonian [42–44]

$$\begin{aligned} &\Psi_{\mathrm{HS}}\big(\{x^{(1)}\}, \{x^{(2)}\}, \ldots, \{x^{(n-1)}\}\big) \\ &\quad = \prod_{\alpha} \prod_{i<j} (z_{x_i^{(\alpha)}} - z_{x_j^{(\alpha)}})^2 \prod_{\alpha<\beta} \prod_{i,j} (z_{x_i^{(\alpha)}} - z_{x_j^{(\beta)}}) \prod_{\alpha} \prod_j z_{x_j^{(\alpha)}}. \end{aligned} \tag{48}$$

3.1.2. 2D Halperin wave functions

Let us next consider a regular lattice in 2D. We shall assume that the area of each lattice site (defined as the area of the region consisting of all points that are closer to the given lattice site than to any other lattice site) is the same for all lattice sites. In this case, it has been shown in [60] that

$$\left|f_{N\to\infty}(z)\right| \propto e^{-|z|^2/4} \tag{49}$$

for N large. The state (47) can therefore be written as

$$\begin{aligned}\Psi_{\mathrm{F}}^{N\to\infty}\left(\{x^{(1)}\},\ldots,\{x^{(n-1)}\}\right) \propto \prod_{\alpha}\prod_{j} e^{-ig_{x_j^{(\alpha)}}} \prod_{\alpha}\prod_{i<j}\left(z_{x_i^{(\alpha)}}-z_{x_j^{(\alpha)}}\right)^2 \\ \times \prod_{\alpha<\beta}\prod_{i,j}\left(z_{x_i^{(\alpha)}}-z_{x_j^{(\beta)}}\right)\prod_{\alpha}\prod_{j} e^{-|z_{x_j^{(\alpha)}}|^2/4}\end{aligned} \tag{50}$$

in the thermodynamic limit, where

$$g_{x_j^{(\alpha)}} = \mathrm{Im}\left(\sum_{k(\neq x_j^{(\alpha)})} \ln\left(z_{x_j^{(\alpha)}}-z_k\right)\right). \tag{51}$$

Up to a local phase factor that can be removed with a simple transformation (of both the wave function and the parent Hamiltonian), we thus observe that the wave function (34) reduces to the lattice version of the Halperin state [69], which appeared in the context of the multilayer FQH effect. For example, the SU(3) state corresponds to Halperin's 221 double-layer spin-singlet state. One consequence of this interesting connection is that the wave function (34) describes an SU(n) chiral spin liquid state, supporting Abelian anyonic excitations (the same as those in Halperin states). Another consequence is that the particular series of Halperin FQH states in (50) have a hidden enhanced SU(n) symmetry. For instance, one may expect that the chiral gapless edge excitations of these states are described by the $\mathrm{SU}(n)_1$ WZW model.

3.2. Numerical results

Since the properties of the uniform 1D SU(n) HS state are already well-known, we shall here only investigate the states in 2D. We compute the TEE $-\gamma$ by considering the state on an $R\times L$ square lattice on the cylinder and using the formula [62,63,70]

$$S_L = \xi L - \gamma \tag{52}$$

for the entanglement entropy of half of the cylinder. In (52), we assume the cut to be perpendicular to the cylinder axis, L is the number of spins along the cut, and the formula is valid asymptotically for large L and R. The mapping of the IDMPS (36) to a cylinder is done through a conformal transformation, which amounts to choosing

$$z_j = \exp\left(2\pi(r_j + il_j)/L\right) \tag{53}$$

and considering r_j and l_j as the coordinates rather than $\mathrm{Re}(z_j)$ and $\mathrm{Im}(z_j)$. This will also change the chiral correlator by a constant factor, but we can ignore this, since the factor does not depend on the state of the spins. The square lattice is then obtained by choosing $r_j \in \{-R/2+1/2, -R/2+3/2,\ldots, R/2-1/2\}$ and $l_j \in \{1,2,\ldots,L\}$ and $N = RL$.

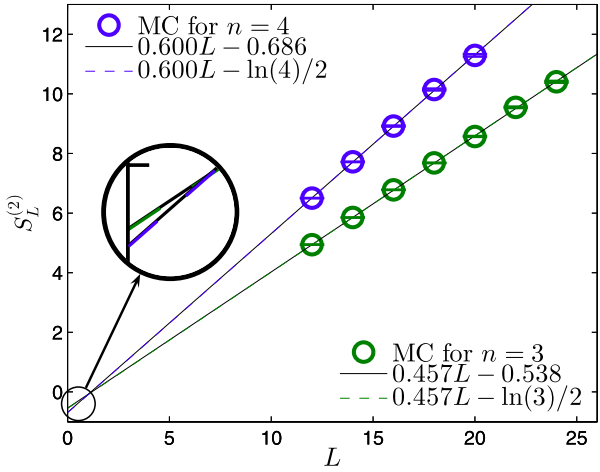

Fig. 3. Renyi entanglement entropy $S_L^{(2)}$ of the 2D IDMPSs (6) obtained from the fundamental representation of SU(n) for $n = 3$ and $n = 4$. The states are defined on an $R \times L$ square lattice on the cylinder, the cylinder is cut into two halves, and L is the number of spins along the cut. The length of the cylinder is $R = 12$ lattice sites in both cases. The intersection with the y-axis gives the TEE. The points are obtained from Monte Carlo simulations, and the lines are linear fits with the constant term being a free parameter (solid lines) or being fixed at $-\ln(n)/2$ (dashed lines). The inset is an enlarged view.

Since it is easier to compute numerically, we choose to consider the Renyi entropy with index 2, which is defined as $S_L^{(2)} = -\ln(\mathrm{Tr}(\rho_L^2))$, where ρ_L is the reduced density operator of half of the cylinder. Let us label the spins in the left half of the cylinder by the indices $1, 2, \ldots, N/2$. As observed in [55,71], one can use the Metropolis Monte Carlo algorithm to compute $\exp(-S_L^{(2)})$ by noting that

$$\begin{aligned}\exp\bigl(-S_L^{(2)}\bigr) &= \sum_{\alpha_1,\ldots,\alpha_N,\alpha'_1,\ldots,\alpha'_N} \frac{\Psi_{\mathrm{F}}(\alpha'_1,\ldots,\alpha'_{\frac{N}{2}},\alpha_{\frac{N}{2}+1},\ldots,\alpha_N)\Psi_{\mathrm{F}}(\alpha_1,\ldots,\alpha_{\frac{N}{2}},\alpha'_{\frac{N}{2}+1},\ldots,\alpha'_N)}{\Psi_{\mathrm{F}}(\alpha_1,\ldots,\alpha_N)\Psi_{\mathrm{F}}(\alpha'_1,\ldots,\alpha'_N)} \\ &\quad \times P\bigl(\alpha_1,\ldots,\alpha_N,\alpha'_1,\ldots,\alpha'_N\bigr) \end{aligned} \tag{54}$$

and interpreting

$$P\bigl(\alpha_1,\ldots,\alpha_N,\alpha'_1,\ldots,\alpha'_N\bigr) = \frac{|\Psi_{\mathrm{F}}(\alpha_1,\ldots,\alpha_N)|^2|\Psi_{\mathrm{F}}(\alpha'_1,\ldots,\alpha'_N)|^2}{(\sum_{\alpha''_1,\ldots,\alpha''_N}|\Psi_{\mathrm{F}}(\alpha''_1,\ldots,\alpha''_N)|^2)^2} \tag{55}$$

as a classical probability distribution. The results of the computations are shown as a function of the number of spins along the cut in Fig. 3. The figure provides evidence for $n = 3$ and $n = 4$ that the TEE is $-\gamma = -\ln(n)/2$. This is consistent with the prediction that the states in 2D are chiral spin liquid states, with the SU$(n)_1$ WZW model being their corresponding chiral edge CFT: According to the fusion rule (1) of the SU$(n)_1$ WZW model, the states support n types of Abelian anyons with quantum dimension 1, giving rise to a total quantum dimension $\sqrt{n}$.

4. Parent Hamiltonians for the states from the fundamental representation

In this section, we derive parent Hamiltonians of the states Ψ_{F} in Eq. (33). In 1D, we obtain two-body parent Hamiltonians, including the SU(n) HS model as a special case, and for 2D lattices they are parent Hamiltonians of the SU(n) chiral spin liquid states.

Our starting point is the fact that the operator $\mathcal{C}_i^a$ in (26) annihilates the state (33) as derived above, $\mathcal{C}_i^a|\Psi_{\mathrm{F}}\rangle = 0$. It follows that the positive semi-definite Hermitian operator

$$H = \sum_i \left(\mathcal{C}_i^a\right)^\dagger \mathcal{C}_i^a \tag{56}$$

is therefore a parent Hamiltonian of (33), $H|\Psi_{\mathrm{F}}\rangle = 0$. Inserting (26) in (56) and utilizing the formulas listed in Appendix A, we obtain the explicit expression

$$\begin{aligned} H = {} & \frac{(n-1)(n+2)}{2(n+1)} \sum_{i\neq j} |w_{ij}|^2 (\vec{t}_i \cdot \vec{t}_j) + \frac{1}{2} \sum_{i\neq j\neq k} w_{ij}^* w_{ik} (\vec{t}_j \cdot \vec{t}_k) \\ & + \sum_{i\neq j\neq k} \left(w_{ij}^* w_{ik} - \frac{1}{n+1} w_{ik}^* w_{ij} \right) (\vec{t}_i \cdot \vec{t}_k)(\vec{t}_i \cdot \vec{t}_j) + \frac{(n-1)(n+2)}{4n} \sum_{i\neq j} |w_{ij}|^2, \end{aligned} \tag{57}$$

which is valid for general z_j.

4.1. Exchange form of the parent Hamiltonian

As we shall now show, H can also be expressed in terms of the exchange operator P_{ij}, which swaps the spin states at sites i and j, i.e., $P_{ij} = \sum_{\alpha,\beta=1}^n |\alpha_i, \beta_j\rangle\langle\beta_i, \alpha_j|$. To do so, we define the following fermionic representation of the SU(n) generators

$$t_i^a = \sum_{\alpha\beta} c_{i\alpha}^\dagger \left(t^a\right)_{\alpha\beta} c_{i\beta} \tag{58}$$

with the local constraint $\sum_{\alpha=1}^n c_{i\alpha}^\dagger c_{i\alpha} = 1$ for all i. Using Fierz identity (A.9), we can then express the SU(n) Heisenberg interaction

$$\begin{aligned} \vec{t}_i \cdot \vec{t}_j &= \sum_{\alpha\beta\gamma\delta} c_{i\alpha}^\dagger c_{i\beta} c_{j\gamma}^\dagger c_{j\delta} \left(t^a\right)_{\alpha\beta} \left(t^a\right)_{\gamma\delta} \\ &= \frac{1}{2} \sum_{\alpha\beta} c_{i\alpha}^\dagger c_{i\beta} c_{j\beta}^\dagger c_{j\alpha} - \frac{1}{2n} \sum_{\alpha\gamma} c_{i\alpha}^\dagger c_{i\alpha} c_{j\gamma}^\dagger c_{j\gamma} \\ &= \frac{1}{2} P_{ij} - \frac{1}{2n} \end{aligned} \tag{59}$$

in terms of P_{ij}. Inserting this into (57), we get

$$\begin{aligned} H = {} & \frac{(n-1)(n+2)}{4(n+1)} \sum_{i\neq j} |w_{ij}|^2 P_{ij} + \frac{(n-1)(n+2)}{4(n+1)} \sum_{i\neq j} |w_{ij}|^2 - \frac{1}{4(n+1)} \sum_{i\neq j\neq k} w_{ij}^* w_{ik} \\ & + \frac{1}{4} \sum_{i\neq j\neq k} w_{ij}^* w_{ik} P_{jk} - \frac{1}{4(n+1)} \sum_{i\neq j\neq k} w_{ij}^* w_{ik} P_{ik} - \frac{1}{4(n+1)} \sum_{i\neq j\neq k} w_{ij}^* w_{ik} P_{ij} \\ & + \frac{1}{4} \sum_{i\neq j\neq k} w_{ij}^* w_{ik} P_{ik} P_{ij} - \frac{1}{4(n+1)} \sum_{i\neq j\neq k} w_{ij}^* w_{ik} P_{ij} P_{ik}. \end{aligned} \tag{60}$$

4.2. SU(n) *Hamiltonian in 1D*

In this subsection, we restrict ourselves to 1D systems. This is done by restricting all z_j to lie on the unit circle in the complex plane, i.e. $|z_j| = 1\ \forall j$. When this is the case, we have $w_{ij}^* = -w_{ij}$, and using (A.5), the 1D Hamiltonian therefore takes the form

$$\begin{aligned} H_{\rm 1D} = &-\frac{(n-1)(n+2)}{2(n+1)}\sum_{i\neq j} w_{ij}^2(\vec{t}_i\cdot\vec{t}_j) \\ &-\frac{n+2}{2(n+1)}\sum_{i\neq j\neq k} w_{ij}w_{ik}(\vec{t}_j\cdot\vec{t}_k) - \frac{(n-1)(n+2)}{4n}\sum_{i\neq j} w_{ij}^2 \\ &-\frac{n}{2(n+1)}\sum_{i\neq j\neq k} w_{ij}w_{ik}d_{abc}t_i^a t_j^b t_k^c. \end{aligned} \tag{61}$$

For the particular case $n = 2$, the three-body term vanishes because $d_{abc} = 0$, and we recover the Hamiltonian in Eq. (70) of [56].

In the following, we simplify (61). First, by using the cyclic identity $w_{ij}w_{ik} + w_{ji}w_{jk} + w_{ki}w_{kj} = 1$, we find

$$\sum_{i(\neq j,k)} w_{ij}w_{ik} = 2w_{jk}^2 + w_{jk}(c_j - c_k) + (N-2), \quad c_j = \sum_{i(\neq j)} w_{ij}, \tag{62}$$

and

$$\sum_{i\neq j\neq k} w_{ij}w_{ik}d_{abc}t_i^a t_j^b t_k^c = \frac{1}{3}\sum_{i\neq j\neq k} d_{abc}t_i^a t_j^b t_k^c. \tag{63}$$

Inserting these relations into (61), utilizing that $T^a = \sum_i t_i^a$ and $w_{ij}^2 = 1 + \frac{4z_iz_j}{(z_i-z_j)^2}$, the parent Hamiltonian (61) for the state (33) with $|z_j| = 1$, $\forall j$, can be written as

$$\begin{aligned} H_{\rm 1D} = &-2(n+2)\sum_{i\neq j}\left[\frac{z_iz_j}{(z_i-z_j)^2} + \frac{1}{4(n+1)}w_{ij}(c_i-c_j)\right](\vec{t}_i\cdot\vec{t}_j) \\ &-\frac{n}{6(n+1)}d_{abc}T^aT^bT^c - \frac{n+2}{4(n+1)}(2N+n)T^aT^a - E_{\rm 1D}, \end{aligned} \tag{64}$$

where $E_{\rm 1D}$ is given by

$$E_{\rm 1D} = \frac{(n-1)(n+2)}{4n}\sum_{i\neq j} w_{ij}^2 - \frac{(n+2)(n-1)}{12n}N(3N+2n-1). \tag{65}$$

Here let us remind that (64) directly comes from (56) and $H_{\rm 1D}|\Psi_{\rm F}\rangle = 0$.

Since $|\Psi_{\rm F}\rangle$ is an SU(n) singlet, we have $d_{abc}T^aT^bT^c|\Psi_{\rm F}\rangle = T^aT^a|\Psi_{\rm F}\rangle = 0$. Thus, we could get rid of the three-body and two-body Casimirs in (64) and define a pure two-body parent Hamiltonian

$$H'_{\rm 1D\ nonuniform} = -\sum_{i\neq j}\left[\frac{z_iz_j}{(z_i-z_j)^2} + \frac{1}{4(n+1)}w_{ij}(c_i-c_j)\right](\vec{t}_i\cdot\vec{t}_j), \tag{66}$$

which has (33) as its ground state with ground-state energy

$$E'_{\text{1D nonuniform}} = \frac{(n-1)}{8n}\sum_{i\neq j} w_{ij}^2 - \frac{n-1}{24n}N(3N+2n-1). \tag{67}$$

The Hamiltonian (66) is an inhomogenous generalization of the SU(n) HS model. For $n=2$, it reduces to the SU(2) inhomogenous HS model derived in [55].

4.3. 1D uniform Hamiltonian and the SU(n) HS model

We now further restrict z_j to be uniformly distributed on the unit circle by choosing $z_j = e^{2\pi i j/N}$. This gives a uniform 1D lattice with periodic boundary conditions. In this case,

$$c_j = \sum_{i(\neq j)} w_{ij} = 0 \quad \forall j, \tag{68}$$

$$\sum_{i\neq j} w_{ij}^2 = -\frac{1}{3}N(N-1)(N-2). \tag{69}$$

The 1D uniform parent Hamiltonian is therefore

$$\begin{aligned} H_{\text{1D uniform}} = {} & -2(n+2)\sum_{i\neq j}\frac{z_i z_j}{(z_i-z_j)^2}(\vec{t}_i\cdot\vec{t}_j) - \frac{n}{6(n+1)}d_{abc}T^aT^bT^c \\ & - \frac{n+2}{4(n+1)}(2N+n)T^aT^a - E_{\text{1D uniform}}, \end{aligned} \tag{70}$$

whose ground-state energy is given by

$$E_{\text{1D uniform}} = -\frac{(n-1)(n+2)}{12n}N\left(N^2+2n+1\right). \tag{71}$$

We note that the first term in (70) is given by

$$-\sum_{i\neq j}\frac{z_i z_j}{(z_i-z_j)^2}(\vec{t}_i\cdot\vec{t}_j) = \sum_{i\neq j}\frac{1}{4\sin^2\frac{\pi}{N}(i-j)}(\vec{t}_i\cdot\vec{t}_j) = \frac{1}{2}\left(\frac{N}{\pi}\right)^2 H_{\text{HS}}, \tag{72}$$

where H_{HS} is the 1D SU(n) HS model

$$H_{\text{HS}} = \sum_{i<j}\frac{\vec{t}_i\cdot\vec{t}_j}{(\frac{N}{\pi})^2\sin^2\frac{\pi}{N}(i-j)}. \tag{73}$$

In the thermodynamic limit $N\to\infty$, we have $(\frac{N}{\pi})^2\sin^2\frac{\pi}{N}(i-j)\to 1/(i-j)^2$ and the strength of SU(n) exchange interaction in (73) is inversely proportional to the square of the distance between the spins.

Then, we can write the uniform 1D parent Hamiltonian as

$$\begin{aligned} H_{\text{1D uniform}} = {} & (n+2)\left(\frac{N}{\pi}\right)^2 H_{\text{HS}} - \frac{n}{6(n+1)}d_{abc}T^aT^bT^c \\ & - \frac{n+2}{4(n+1)}(2N+n)T^aT^a - E_{\text{1D uniform}}. \end{aligned} \tag{74}$$

Since $T^a|\Psi_{\text{F}}\rangle = 0$, the ground-state energy of H_{HS} is given by

$$E_{\text{HS}} = \frac{1}{n+2}\left(\frac{\pi}{N}\right)^2 E_{\text{1D uniform}} = -\frac{n-1}{12n}\pi^2\left(N+\frac{2n+1}{N}\right). \tag{75}$$

The 1D uniform parent Hamiltonian thus practically reduces to the 1D SU(n) HS model.

4.3.1. Energy spectra of the SU(n) HS model

For the SU(n) HS model, it has been shown [72] that it has a hidden Yangian symmetry, generated by the total spin operator T^a and the operator

$$\Lambda^a = \frac{i}{2}\sum_{i\neq j} w_{ij} f_{abc} t_i^b t_j^c. \tag{76}$$

We note that $\Lambda^a = \frac{n+1}{n+2}\sum_i \mathcal{C}_i^a$, which thus annihilates $|\Psi_{\mathrm{F}}\rangle$ as well. It is known [72] that T^a and Λ^b both commute with H_{HS}, but they do not mutually commute, which is responsible for the huge degeneracies in the spectra of H_{HS}.

The eigenvalues of the SU(n) HS model have been obtained in [72]. Combining (59) and (72), we rewrite the SU(n) HS Hamiltonian as

$$\begin{aligned} H_{\mathrm{HS}} &= -\left(\frac{\pi}{N}\right)^2 \sum_{i\neq j} \frac{z_i z_j}{(z_i - z_j)^2}\left(P_{ij} - \frac{1}{n}\right) \\ &= \left(\frac{\pi}{N}\right)^2 H_{\mathrm{Haldane}} + \left(\frac{\pi}{N}\right)^2 \frac{n-1}{2n}\frac{N(N^2-1)}{6}, \end{aligned} \tag{77}$$

where

$$H_{\mathrm{Haldane}} = -\sum_{i\neq j} \frac{z_i z_j}{(z_i - z_j)^2}(P_{ij} - 1). \tag{78}$$

It has been shown [72] that the complete set of eigenvalues of H_{Haldane} can be obtained by the simple formula

$$H_{\mathrm{Haldane}}\big|\{m_i\}\big\rangle = \sum_i \epsilon(m_i)\big|\{m_i\}\big\rangle, \tag{79}$$

where

$$\epsilon(m_i) = m_i(m_i - N). \tag{80}$$

Here m_i are distinct integer rapidities satisfying $m_i \in [0, N]\ \forall i$. Physically, the sum of these rapidities is proportional to the lattice momenta of the energy eigenstate $|\{m_i\}\rangle$ [72]

$$P = \frac{2\pi}{N}\sum_{\{m_i\}} m_i \ (\mathrm{mod}\ 2\pi). \tag{81}$$

According to [72], there is a simple rule for finding physically allowed sets of rapidities

$$\{m_i\} = \{m_1, m_2, \ldots, m_M\}$$

with $m_1 < m_2 < \cdots < m_M$ and M is an integer satisfying $M \in [0, \frac{n-1}{n}N]$. The rule is that, all possible sets $\{m_1, m_2, \ldots, m_M\}$ without n or more consecutive integers are allowed and correspond to an eigenstate of H_{Haldane}. For example, the ground state is represented by the sequence

$$\{m_i\} = \{1, 2, \ldots, n-1, n+1, n+2, \ldots 2n-1, 2n+1, \ldots, N-1\}. \tag{82}$$

Using (79) and (81), the energy and lattice momenta of the ground state are therefore given by

$$E_{\mathrm{Haldane}} = -\frac{n-1}{6n}\big(N^3 + nN\big), \tag{83}$$

and

$$P_{\mathrm{GS}} = \frac{n-1}{n} N\pi \ (\mathrm{mod}\ 2\pi) = \begin{cases} 0 & N/n \text{ even} \\ \pi & N/n \text{ odd and } n \text{ even} \\ 0 & N/n \text{ odd and } n \text{ odd.} \end{cases} \tag{84}$$

Note that the ground-state energy E_{Haldane} determined in this way is consistent with (75) by taking into account the constant term in (77).

4.3.2. Identifying CFT from finite-size spectra

CFT gives a powerful prediction for the spectra of 1D critical spin chains. In particular, it is known that the eigenenergies of a critical quantum chain with N sites and with periodic boundary conditions are given by [73,74]

$$E = \varepsilon_\infty N - \frac{\pi v c}{6N} + \frac{2\pi v}{N}(h + \bar{h} + n_l + n_r), \tag{85}$$

where ε_∞ is the ground-state energy per site in the thermodynamic limit, v is the spin-wave velocity, c is the central charge, h and $\bar{h}$ are conformal weights of the primary fields, and n_l and n_r are non-negative integers.

For the SU(n) HS model, the spin-wave velocity and the conformal weights of the primary fields can be determined directly by the finite-size spectra obtained from (79). To show this, we consider the SU(n) HS Hamiltonian in (77). Let us start with an excitation defined through the rapidities

$$\{m_i\} = \{2, \ldots, n-1, n+1, n+2, \ldots, 2n-1, 2n+1, \ldots, N-1\}, \tag{86}$$

which is obtained by removing the particle "1" in the ground-state configuration (82). Using (79) and (81), we obtain the excitation energy E' and the lattice momentum P' of this excitation

$$\begin{aligned} E' &= E_{\mathrm{HS}} + \left(\frac{\pi}{N}\right)^2 (N-1) \\ &= E_{\mathrm{HS}} + \frac{\pi^2}{N} - \mathcal{O}\big(1/N^2\big), \end{aligned} \tag{87}$$

and

$$P' = P_{\mathrm{GS}} - \frac{2\pi}{N}. \tag{88}$$

Comparing to the CFT prediction of the finite-size spectra (85), this excited state corresponds to $h = \bar{h} = n_r = 0$ and $n_l = 1$. Thus, we obtain the spin-wave velocity

$$v = \frac{\pi}{2}. \tag{89}$$

However, let us note that the central charge c cannot be obtained using (85). The reason is that the SU(n) HS Hamiltonian has long-range interactions, which allow an N-dependent constant term and the ground-state energy as a function of N could violate the CFT prediction (85).

Now we consider other excited states of H_{HS}, by shifting the sequence of ground-state rapidities in (82) by a, with $a \in \{1, \ldots, n-1\}$. The corresponding rapidity sets are given by

$$\{m_i\} = \big\{1, \ldots, [a], \ldots, [n+a], \ldots, [N-n+a], \ldots, N\big\}, \tag{90}$$

where $[qn+a]$, with $q=0,\ldots,\frac{N}{n}-1$, denotes the missing rapidities in the rapidity set. By using (79) and (81), we obtain the excitation energies and lattice momenta of the corresponding excited states

$$E_a = E_{\rm HS} + \frac{a(n-a)}{n}\frac{\pi^2}{N}, \tag{91}$$

and

$$P_a = P_{\rm GS} - a\frac{2\pi}{n}. \tag{92}$$

Note that these excitations are gapless in the thermodynamic limit $N\to\infty$. Compared to the CFT prediction (85), these excited states correspond to $h_a=\bar{h}_a$ and $n_l=n_r=0$. Comparing with (85), we obtain the conformal weights $h_a=\frac{a(n-a)}{2n}$, which correspond to the primary fields Λ_a of the $\mathrm{SU}(n)_1$ WZW model. This also agrees with the known results [72,75,76] that the $\mathrm{SU}(n)_1$ WZW model describes the low-energy physics of the $\mathrm{SU}(n)$ HS model.

Regarding the excited states of the $\mathrm{SU}(n)$ HS model, one remaining interesting question is to obtain their explicit form and to relate them with the rapidity description in (79). Some of these excited states have already been obtained in Refs. [77,78]. As a further remark, we note that the gapless excitations at lattice momenta $P=a\frac{2\pi}{n}$ with $a\in\{1,\ldots,n\}$ are also known to exist in the $\mathrm{SU}(n)$ ULS model [39–41], which belongs to the same $\mathrm{SU}(n)_1$ WZW universality class [79–82].

4.4. SU(n) Hamiltonian in 2D

In this subsection, we discuss the parent Hamiltonian in 2D. After multiplying by an overall constant $\frac{2(n+1)}{(n-1)(n+2)}$, the 2D Hamiltonian in (57) can be written as

$$\begin{aligned} H_{\rm 2D} = &\sum_{i\neq j}\left[|w_{ij}|^2+\left(\sum_{k(\neq i,j)} w_{ki}^* w_{kj}\right)\right](\vec{t}_i\cdot\vec{t}_j) \\ &-\frac{1}{n-1}\sum_{i\neq j\neq k} w_{ij}^* w_{ik}\left(if_{abc}-\frac{n}{n+2}d_{abc}\right)t_i^a t_j^b t_k^c + \frac{n+1}{2n}\sum_{i\neq j}|w_{ij}|^2. \end{aligned} \tag{93}$$

Note that this Hamiltonian can be defined on any 2D lattice (both regular or irregular) and does not rely on a particular lattice geometry.

For SU(2), we have $d_{abc}=0$ and $f_{abc}t_i^a t_j^b t_k^c=\vec{t}_i\cdot(\vec{t}_j\times\vec{t}_k)$. Then, (93) reduces to the parent Hamiltonian in [57] for the $\nu=1/2$ lattice Laughlin state. This state is also known as the Kalmeyer–Laughlin state [64,65], whose parent Hamiltonian has been extensively studied [57, 83–89]. From the parent Hamiltonian, it becomes transparent that the chiral three-spin interaction term $\vec{t}_i\cdot(\vec{t}_j\times\vec{t}_k)$, which explicitly breaks time-reversal and parity symmetries, stabilizes the spin-1/2 Kalmeyer–Laughlin state. Recently, it has been found [61,90,91] that Hamiltonians with short-range chiral three-spin interactions can already stabilize the Kalmeyer–Laughlin state. This is very encouraging, as such short-range Hamiltonian might be realized in cold atomic systems in optical lattices [61,92].

The $\mathrm{SU}(n)$ parent Hamiltonian (93) also has three-body interactions. Compared to the SU(2) case, one remarkable feature is that, the three-body coupling is suppressed by a factor of $1/(n-1)$. This gives us a hint that, for large n, one may have a chance to drop the three-body terms and the (long-range) Hamiltonian with two-body Heisenberg interactions may stabilize the lattice Halperin state (50) as its ground state. However, as the number of terms in the three-body interactions $d_{abc}t_i^a t_j^b t_k^c$ and $f_{abc}t_i^a t_j^b t_k^c$ also increases with n, it is unclear whether the parent

Hamiltonian can be adiabatically connected to the long-range Heisenberg model without closing the gap. Clarifying whether the gap closes in this interpolation is an interesting problem and certainly deserves further investigation.

Finally, for the 2D SU(n) Heisenberg model on a square lattice with only nearest-neighbor interactions, it has been argued [93,94] that chiral spin liquid supporting Abelian anyons becomes stable in the large n limit. Thus, it would be interesting to further explore its possible connection with our wave function (33).

5. Quantum states from the fundamental and conjugate representations of SU(n)

In this section, we turn to the more general situation, where we use both the fundamental and the conjugate representation to construct IDMPSs. In this case, the chiral correlator (7) evaluates to

$$\Psi(\alpha_1,\alpha_2,\ldots,\alpha_{N+\bar{N}}) = \chi(\alpha_1,\alpha_2,\ldots,\alpha_{N+\bar{N}})\delta_{\sum_{i=1}^{N+\bar{N}} r_i \vec{m}_{\alpha_i}=0} \prod_{i<j}^{N+\bar{N}} (z_i - z_j)^{2r_i r_j \vec{m}_{\alpha_i}\cdot\vec{m}_{\alpha_j}}, \tag{94}$$

where $\chi(\alpha_1,\alpha_2,\ldots,\alpha_{N+\bar{N}}) = \kappa_{\alpha_1}\kappa_{\alpha_2}\cdots\kappa_{\alpha_{N+\bar{N}}}$ is a z_j-independent phase factor that we shall determine below and

$$r_j = \begin{cases} +1 & \text{for } j \in A \\ -1 & \text{for } j \in B. \end{cases} \tag{95}$$

By using (32), we can also express the chiral correlator in the simpler form

$$\Psi(\alpha_1,\alpha_2,\ldots,\alpha_{N+\bar{N}}) \propto \chi(\alpha_1,\alpha_2,\ldots,\alpha_{N+\bar{N}})\delta_{\sum_{i=1}^{N+\bar{N}} r_i \vec{m}_{\alpha_i}=0} \prod_{i<j} (z_i - z_j)^{r_i r_j \delta_{\alpha_i\alpha_j}}. \tag{96}$$

Considering (27), we observe that the charge neutrality condition $\delta_{\sum_{i=1}^{N+\bar{N}} r_i \vec{m}_{\alpha_i}=0}$ yields

$$N_1 - N_2 = N_{\bar{1}} - N_{\bar{2}} \tag{97}$$

$$N_1 + N_2 - 2N_3 = N_{\bar{1}} + N_{\bar{2}} - 2N_{\bar{3}} \tag{98}$$

$$\vdots$$

$$N_1 + N_2 + \cdots - (n-1)N_n = N_{\bar{1}} + N_{\bar{2}} + \cdots - (n-1)N_{\bar{n}}, \tag{99}$$

where N_α ($N_{\bar{\alpha}}$) is the number of spins in the fundamental (conjugate) representation in the state $|\alpha\rangle$. Together with the conditions

$$N_1 + N_2 + \cdots + N_n = N, \tag{100}$$

$$N_{\bar{1}} + N_{\bar{2}} + \cdots + N_{\bar{n}} = \bar{N}, \tag{101}$$

we thus conclude that

$$N_1 = N_{\bar{1}} + \frac{N-\bar{N}}{n}, \qquad N_2 = N_{\bar{2}} + \frac{N-\bar{N}}{n}, \quad \cdots, \quad N_n = N_{\bar{n}} + \frac{N-\bar{N}}{n} \tag{102}$$

must hold for all nonzero terms in the wave function. This is consistent with our previous observation that $(N-\bar{N})/n$ must be an integer.

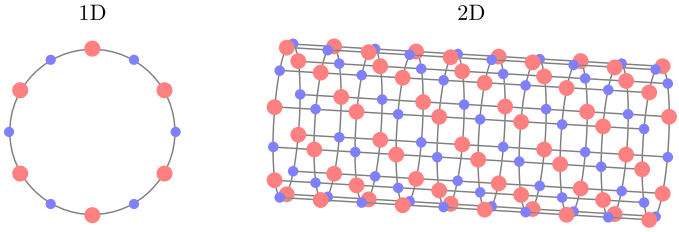

Fig. 4. Plots of the 1D and 2D lattices with alternating fundamental and conjugate representations.

χ is determined from the requirement that $|\Psi\rangle$ must be a singlet state, i.e. $T^a|\Psi\rangle = 0$, where $T^a = \sum_{i=1}^{N+\bar{N}} t_i^a$. We show explicitly in Appendix B that this condition is fulfilled for

$$\chi(\alpha_1, \alpha_2, \ldots, \alpha_{N+\bar{N}}) = \operatorname{sgn}\big(x_1^{(1,\bar{1})}, \ldots, x_{N_1+N_{\bar{1}}}^{(1,\bar{1})}, x_1^{(2,\bar{2})}, \ldots, x_{N_2+N_{\bar{2}}}^{(2,\bar{2})}, \ldots, x_1^{(n,\bar{n})}, \ldots, x_{N_n+N_{\bar{n}}}^{(n,\bar{n})}\big), \tag{103}$$

where $x_i^{(\alpha,\bar{\alpha})}$ is the position within the ket of the ith spin that is in the state $|\alpha\rangle$ without distinguishing between the fundamental and the conjugate representation. As for the case, where only the fundamental representation is used, we can obtain (103) by demanding κ_α in (28) to be Klein factors.

5.1. Numerical results

We next investigate the states numerically for lattices with alternating fundamental and conjugate representations. We start with the uniform 1D case, where we use the fundamental representation on all the odd sites and the conjugate representation on all the even sites (see Fig. 4). Let us consider the entanglement entropy of a block of L consecutive spins, where L is even. We compute this quantity by Monte Carlo simulations as explained in Section 3.2, and the result is shown in Fig. 5. We observe that the entanglement entropy grows logarithmically. The CFT prediction for the entanglement entropy of a critical 1D system is [95–97]

$$S_L^{(2)} = \frac{c}{4} \ln\left[\sin\left(\frac{\pi L}{N+\bar{N}}\right)\frac{N+\bar{N}}{\pi}\right] + \text{constant}, \tag{104}$$

and by using this formula as a fit, we obtain the central charges $c = 1.5$ for $n = 3$ and $c = 1.7$ for $n = 4$, respectively.

Next we compute the correlation function

$$c(k) = \langle t_i^3 t_{i+k}^3\rangle - \langle t_i^3\rangle\langle t_{i+k}^3\rangle \tag{105}$$

by using the Metropolis Monte Carlo algorithm. Here, t^3 is the third SU(n) generator, which we choose such that $(t^3)_{11} = -(t^3)_{22} = 1/2$ in the fundamental representation and $(t^3)_{11} = -(t^3)_{22} = -1/2$ in the conjugate representation, whereas all other matrix elements of t^3 are zero in both the fundamental and the conjugate representation (see Appendix A). The correlator is plotted in Fig. 6 for $n = 3$ and $n = 4$. It is seen to decay algebraically as a function of the chord distance $\sin[\pi k/(N+\bar{N})]$ with an exponent that is -1.20 and -1.34, respectively.

The logarithmic growth of entanglement entropy and powerlaw decaying correlation functions both suggest that the 1D state (96) with alternating fundamental and conjugate representations

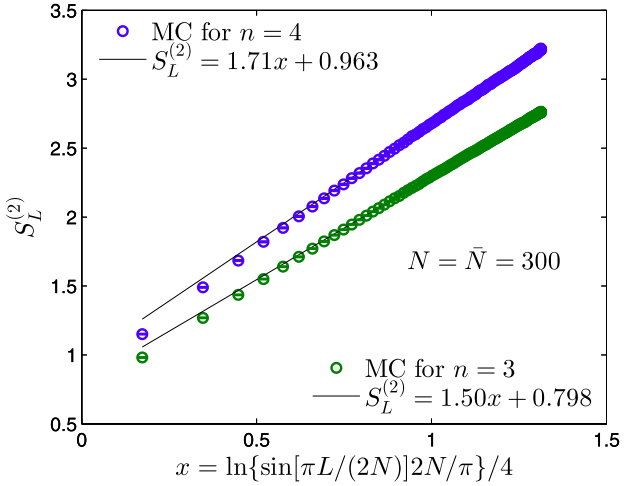

Fig. 5. Plot of the entanglement entropy of blocks of L consecutive spins for $N = \bar{N} = 300$ and $n = 3$ and $n = 4$ in 1D. Note that we plot results only for L even, since the state is only translationally invariant under translation by an even number of lattice sites. The points are obtained from Monte Carlo simulations, and the solid lines are linear fits. In computing the fits, we ignore the 10 leftmost points, since the fits are only expected to be valid for long distances.

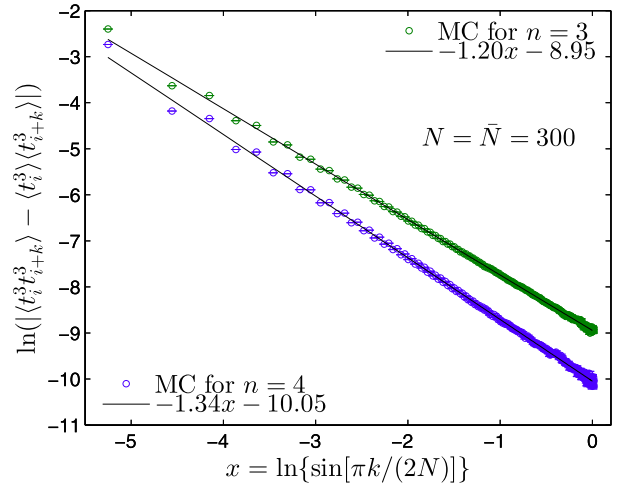

Fig. 6. Plot of the logarithm of the absolute value of the correlator (105) for $N = \bar{N} = 300$ and $n = 3$ and $n = 4$ in 1D. The sign of the correlator is $(-1)^k$. The points are obtained from Monte Carlo simulations, and the solid lines are linear fits. In computing the fits, we ignore the 10 leftmost points, since the fits are only expected to be valid for long distances.

describes a critical spin chain. However, the numerical estimations of the central charges for $n = 3$ and 4 show a clear deviation from the $\mathrm{SU}(n)_1$ WZW model with $c = n - 1$. The numerically estimated critical exponents of the two-point correlation function also differ from $2(n-1)/n$, the expected value for critical spin chains described by the $\mathrm{SU}(n)_1$ WZW model. One possibility for these deviations is that the system is still described by the $\mathrm{SU}(n)_1$ WZW model, but in the presence of marginally irrelevant perturbations. Another possibility is that the system belongs to another universality class which is sharply different from the $\mathrm{SU}(n)_1$ WZW model.

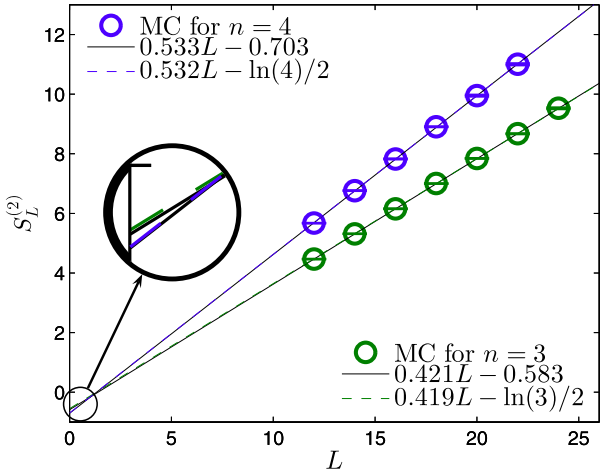

Fig. 7. Renyi entanglement entropy $S_L^{(2)}$ of the 2D IDMPSs (6) obtained from the fundamental and conjugate representations of SU(n) for $n = 3$ and $n = 4$. The states are defined on an $R \times L$ square lattice on the cylinder, the cylinder is cut into two halves in the direction perpendicular to the axis of the cylinder, and L is the number of spins along the cut. The length of the cylinder is $R = 12$ lattice sites, and we use the fundamental and conjugate representation on every second site in a checkerboard pattern. The intersection with the y-axis gives the TEE. The points are obtained from Monte Carlo simulations, and the lines are linear fits with the constant term being a free parameter (solid lines) or being fixed at $-\ln(n)/2$ (dashed lines). The inset is an enlarged view.

In the present framework, it is rather difficult to distinguish these possibilities. In Section 6.3, we propose a short-range Hamiltonian where critical ground states belonging to the same universality class are likely to appear and which is easier to analyze in practice and may shed light on the correct critical theory. Another integrable $U_q[sl(2|1)]$ superspin chain with alternating representations **3** and $\bar{\mathbf{3}}$ has been studied in [98], which exhibits several critical theories depending on the parameters of the Hamiltonian. There could be a connection between these results and our results.

We now turn to the 2D state on a square lattice on the cylinder, with fundamental and conjugate representations in a checkerboard pattern (see Fig. 4). In Fig. 7, we compute the TEE following the same approach as in Section 3.2. The results are in agreement with $-\gamma = -\ln(n)/2$ within the precision of the computation. Similar to the SU(n) state with only fundamental representations, this indicates that the states (96) in 2D are chiral spin liquids and have the SU(n)$_1$ WZW model as their chiral edge CFT.

6. Parent Hamiltonians for the states from the fundamental and conjugate representations

As for the case where only the fundamental representation is used, we can construct a positive semi-definite parent Hamiltonian $H = \sum_i (\mathcal{C}_i^a)^\dagger \mathcal{C}_i^a$ of the state $|\Psi\rangle$ in (6) from the operator (26) with the property $H|\Psi\rangle = 0$. Utilizing the formulas listed in Appendix A, we obtain

$$H = \frac{n(n+2)}{4(n+1)} \sum_{i\neq j} |w_{ij}|^2 \left(1 + r_i r_j \frac{n-2}{n}\right)(\vec{t}_i \cdot \vec{t}_j) + \frac{1}{2} \sum_{i\neq j\neq k} w_{ij}^* w_{ik} (\vec{t}_j \cdot \vec{t}_k)$$
$$+ \sum_{i\neq j\neq k} \left(w_{ij}^* w_{ik} - \frac{1}{n+1} w_{ik}^* w_{ij}\right)(\vec{t}_i \cdot \vec{t}_k)(\vec{t}_i \cdot \vec{t}_j) + \frac{(n-1)(n+2)}{4n} \sum_{i\neq j} |w_{ij}|^2, \quad (106)$$

which is valid for general z_j. Note that this reduces to our previous result (57) for $r_j = +1\ \forall j$. We also observe that (106) does not depend on r_j for $n = 2$. This happens because the fundamental representation and the conjugate representation are the same representation for $n = 2$.

6.1. 1D parent Hamiltonian

We now specialize to 1D by forcing all z_j to fulfill $|z_j| = 1$. This gives $w_{ij}^* = -w_{ij}$. We therefore obtain the 1D parent Hamiltonian

$$H_{\mathrm{1D}} = -\frac{n(n+2)}{4(n+1)} \sum_{i\neq j} w_{ij}^2 \left(1 + r_i r_j \frac{n-2}{n}\right)(\vec{t}_i \cdot \vec{t}_j) - \frac{1}{2} \sum_{i\neq j\neq k} w_{ij} w_{ik} (\vec{t}_j \cdot \vec{t}_k) - \frac{n}{n+1} \sum_{i\neq j\neq k} w_{ij} w_{ik} (\vec{t}_i \cdot \vec{t}_j)(\vec{t}_i \cdot \vec{t}_k) - \frac{(n-1)(n+2)}{4n} \sum_{i\neq j} w_{ij}^2. \tag{107}$$

By using (A.5) and (A.7), we find

$$\sum_{i\neq j\neq k} w_{ij} w_{ik} (\vec{t}_i \cdot \vec{t}_j)(\vec{t}_i \cdot \vec{t}_k) = \frac{1}{2n} \sum_{i\neq j\neq k} w_{ij} w_{ik} (\vec{t}_j \cdot \vec{t}_k) + \frac{1}{2} \sum_{i\neq j\neq k} w_{ij} w_{ik} r_i d_{abc} t_i^a t_j^b t_k^c, \tag{108}$$

and by using (62) and the definition of T^a, we get

$$\sum_{i\neq j\neq k} w_{ij} w_{ik} (\vec{t}_j \cdot \vec{t}_k) = \sum_{i\neq j} \left[2w_{ij}^2 + w_{ij}(c_i - c_j)\right](\vec{t}_i \cdot \vec{t}_j) + (N_{\mathrm{T}} - 2) T^a T^a - \frac{n^2 - 1}{2n} N_{\mathrm{T}}(N_{\mathrm{T}} - 2). \tag{109}$$

Inserting these expressions in the expression for the Hamiltonian leads to

$$H_{\mathrm{1D}} = -\frac{(n+2)}{2(n+1)} \sum_{i\neq j} \left[w_{ij}^2 \left(\frac{n+4}{2} + r_i r_j \frac{n-2}{2} \right) + w_{ij}(c_i - c_j) \right] (\vec{t}_i \cdot \vec{t}_j) - \frac{n}{2(n+1)} \sum_{i\neq j\neq k} w_{ij} w_{ik} r_i d_{abc} t_i^a t_j^b t_k^c - \frac{n+2}{2(n+1)} (N_{\mathrm{T}} - 2) T^a T^a - E_{\mathrm{1D}}, \tag{110}$$

where

$$E_{\mathrm{1D}} = \frac{(n-1)(n+2)}{4n} \left[\sum_{i\neq j} w_{ij}^2 - N_{\mathrm{T}}(N_{\mathrm{T}} - 2) \right]. \tag{111}$$

6.2. 1D uniform parent Hamiltonian

For the 1D uniform case, $z_j = \exp(i \frac{2\pi}{N_{\mathrm{T}}} j)$. By using (68) and (69), the parent Hamiltonian therefore simplifies to

$$H_{\mathrm{1D\ uniform}} = -\frac{(n+2)}{4(n+1)} \sum_{i\neq j} w_{ij}^2 \left[n + 4 + r_i r_j (n-2) \right] (\vec{t}_i \cdot \vec{t}_j)$$

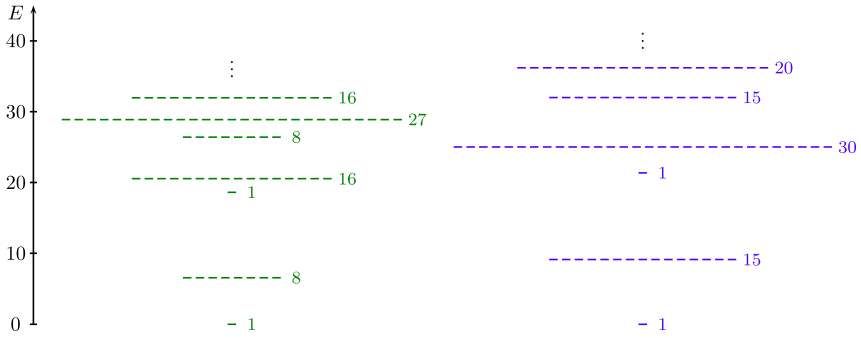

Fig. 8. Low lying part of the spectrum of $H_{\text{1D uniform}}$ for $N = \bar{N} = 4$ and fundamental (conjugate) representations on the odd (even) sites. The results are obtained by exact diagonalization, and the plot on the left (right) is for $n = 3$ ($n = 4$). The numbers written next to the multiplets are the degeneracies.

$$-\frac{n}{2(n+1)}\sum_{i\neq j\neq k} w_{ij}w_{ik}r_i d_{abc}t_i^a t_j^b t_k^c$$
$$-\frac{n+2}{2(n+1)}(N_{\mathrm{T}}-2)T^aT^a - E_{\text{1D uniform}}, \tag{112}$$

where

$$E_{\text{1D uniform}} = -\frac{(n-1)(n+2)}{12n}N_{\mathrm{T}}\left(N_{\mathrm{T}}^2-4\right). \tag{113}$$

We plot examples of spectra of $H_{\text{1D uniform}}$ in Fig. 8. The spectra show that the ground state is unique.

6.3. $SU(n)$ J_2–J_3 chain

One important motivation of studying long-range parent Hamiltonians is that they may shed light on the physics of some short-range realistic Hamiltonians. As we already mentioned, the SU(n) HS Hamiltonian with inverse-square interactions and the SU(n) ULS model with only nearest-neighbor interactions belong to the same $\mathrm{SU}(n)_1$ WZW universality class. For other long-range parent Hamiltonians constructed for the $\mathrm{SU}(2)_k$ and $\mathrm{SO}(n)_1$ WZW models [56,59,99,100], the corresponding short-range Hamiltonians are the SU(2) spin-$\frac{k}{2}$ Takhtajan–Babujian models [101,102] and the SO(n) Reshetikhin models [103,104], respectively. Regarding the SU(n) parent Hamiltonian (112) with both fundamental and conjugate representations, the natural question one may ask is whether there exist short-range Hamiltonians belonging to the same universality class. In fact, finding such short-range Hamiltonians can also be very useful for clarifying the unsolved issue in Section 5.1 on identifying the critical theory of these models.

To address this problem, we restrict ourselves to the 1D uniform case with alternating fundamental and conjugate representations (see Fig. 4). Following the strategy in [61], we truncate the long-range interactions in (112) by keeping only two-body interactions between nearest-neighbor and next-nearest-neighbor sites, as well as three-body interaction terms among three consecutive sites. In the thermodynamic limit, $N_{\mathrm{T}} \to \infty$, this procedure yields the following Hamiltonian:

$$H_{\text{truncated}} = \frac{3(n+2)}{n+1}\sum_i \vec{t}_i\cdot\vec{t}_{i+1} + \frac{n+2}{4}\sum_i \vec{t}_i\cdot\vec{t}_{i+2} + \frac{2n}{n+1}\sum_i r_i d_{abc}t_i^a t_{i+1}^b t_{i+2}^c. \tag{114}$$

By using (A.4) and (A.6), the three-body interaction term can be rewritten as

$$\sum_i r_i d_{abc} t_i^a t_{i+1}^b t_{i+2}^c = \frac{1}{n}\sum_i \vec{t}_i \cdot \vec{t}_{i+2} - \sum_i \big[(\vec{t}_i \cdot \vec{t}_{i+1})(\vec{t}_{i+1} \cdot \vec{t}_{i+2}) + \text{h.c.}\big], \tag{115}$$

and then the truncated Hamiltonian is expressed as

$$H_{\text{truncated}} = \frac{3(n+2)}{n+1}\sum_i \vec{t}_i \cdot \vec{t}_{i+1} + \frac{n^2+3n+10}{4(n+1)}\sum_i \vec{t}_i \cdot \vec{t}_{i+2} - \frac{2n}{n+1}\sum_i \big[(\vec{t}_i \cdot \vec{t}_{i+1})(\vec{t}_{i+1} \cdot \vec{t}_{i+2}) + \text{h.c.}\big]. \tag{116}$$

There is no guarantee that the truncated Hamiltonian with precisely the coupling constants in (116) has the same physics as the long-range parent Hamiltonian (112). However, the form of (116) suggests that a candidate short-range Hamiltonian which shares the same physics might be found in the $J_2 - J_3$ SU(n) spin chain

$$H_{J_2-J_3} = \sum_i \vec{t}_i \cdot \vec{t}_{i+1} + J_2 \sum_i \vec{t}_i \cdot \vec{t}_{i+2} + J_3 \sum_i \big[(\vec{t}_i \cdot \vec{t}_{i+1})(\vec{t}_{i+1} \cdot \vec{t}_{i+2}) + \text{h.c.}\big] \tag{117}$$

with J_2, J_3 being close to the couplings in (116).

We have performed an exact diagonalization of the Hamiltonian in (117) for $n = 3$ and $N_{\text{T}} = 10$ sites. Fig. 9 shows the overlap $|\langle \Psi_{J_2-J_3} | \Psi \rangle|$ between the ground state $|\Psi_{J_2-J_3}\rangle$ of (117) and the state $|\Psi\rangle$ defined in (96). The maximum overlap (marked with a circle in Fig. 9) is 0.9998 and occurs for $J_2 = 0.557$ and $J_3 = -0.536$. These values are quite close to $J_2 = 0.467$ and $J_3 = -0.400$ predicted by the truncated Hamiltonian (116).

Let us also mention several solvable cases in (117), which are useful for understanding the phase diagram and are also interesting on their own right. One known solvable point in (117) is the pure SU(n) Heisenberg chain with $J_2 = J_3 = 0$, which has gapped dimerized ground states for $n \geq 3$ [105,106]. In Fig. 9, this Heisenberg point is marked with a plus sign. Motivated by a recent work [107], we have also identified another class of solvable cases in (117), which have *perfectly* dimerized ground states and can be viewed as SU(n) generalizations of the spin-1/2 Majumdar–Ghosh model [108]. These SU(n) Majumdar–Ghosh Hamiltonians are written as

$$H_{\text{MG}} = \frac{2}{n}\left(\frac{n-2}{n-1}K_1 + \frac{n+2}{n+1}K_2\right)\sum_i \vec{t}_i \cdot \vec{t}_{i+1} + (K_2 - K_1)\sum_i \vec{t}_i \cdot \vec{t}_{i+2} + 2\left(\frac{1}{n-1}K_1 - \frac{1}{n+1}K_2\right)\sum_i \big[(\vec{t}_i \cdot \vec{t}_{i+1})(\vec{t}_{i+1} \cdot \vec{t}_{i+2}) + \text{h.c.}\big], \tag{118}$$

where $K_1, K_2 > 0$ and which, on a periodic chain with even N_{T} sites, have ground-state energy $E_{\text{MG}} = [(n+1)(n-2)K_1 + (n-1)(n+2)K_2]N_{\text{T}}/(2n^2)$. In Fig. 9, the Majumdar–Ghosh Hamiltonian (118) is shown as a straight line terminated at $J_2 = -3$, $J_3 = 3$ and $J_2 = 6/5$, $J_3 = -3/5$. This line seems to be at a phase boundary between two different phases. Fully clarifying the phase diagram of (117) requires extensive numerics. This is beyond the scope of the present work and we leave it for a future study.

7. Conclusion

In summary, we have constructed a family of spin wave functions with SU(n) symmetry from CFT, and we have used the CFT properties of the states to derive parent Hamiltonians in both

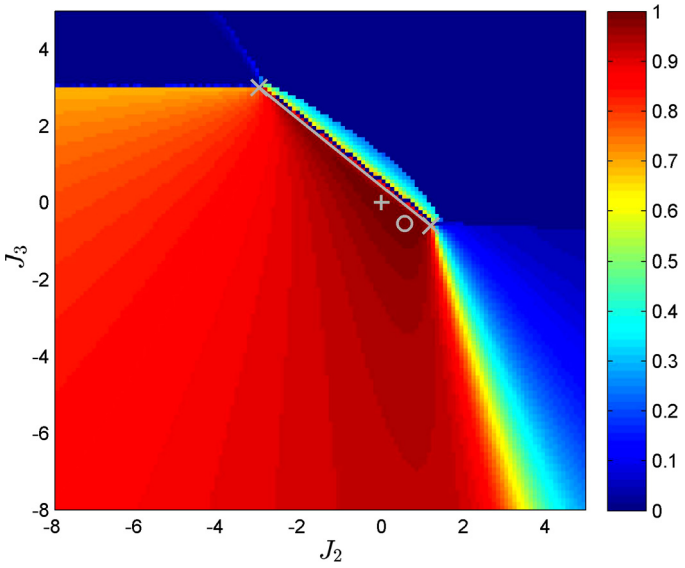

Fig. 9. Overlap $|\langle\Psi_{J_2-J_3}|\Psi\rangle|$ between the ground state of (117) and the state (96) as a function of J_2 and J_3 for $n = 3$ and $N_T = 10$ sites. The circle denotes the point with $J_2 = 0.557$ and $J_3 = -0.536$, where the maximal overlap 0.9998 is achieved. The plus sign corresponds to the pure Heisenberg model with $J_2 = J_3 = 0$. The Majumdar–Ghosh model (118) is shown by a straight line terminated at $J_2 = -3$, $J_3 = 3$ and $J_2 = 6/5$, $J_3 = -3/5$.

1D and 2D. The states are defined on arbitrary lattices, and each of the spins transforms under either the fundamental or the conjugate representation of SU(n). For the case, where all spins in the model transform under the fundamental representation, our results provide a natural generalization of the SU(n) HS model from a uniform lattice in 1D to nonuniform lattices in 1D and to 2D. For the nonuniform 1D case, the Hamiltonian can be chosen to consist of only two-body terms. In 2D, the states reduce to Halperin type wave functions in the thermodynamic limit. This suggests that these states are chiral spin liquids with Abelian anyons, and we find numerically that the total quantum dimension is close to $\sqrt{n}$. It also shows that a class of Halperin states have an SU(n) symmetry and provides parent Hamiltonians that can stabilize these topological states.

We have also investigated the case with alternating fundamental and conjugate representations numerically. In 1D, our results suggest that the state is critical, but the central charges and the exponents of the correlation functions deviate from the results expected for the SU$(n)_1$ WZW model. In 2D, we find a nonzero TEE, and the extracted total quantum dimension is $\sqrt{n}$, which is consistent with the SU$(n)_1$ WZW model predictions.

For the case with alternating fundamental and conjugate representations, we have proposed a short-range Hamiltonian for the 1D uniform case and solved it exactly for particular choices of the parameters. Given that it is possible in many related models with long-range Hamiltonians to find short-range Hamiltonians that describe practically the same low-energy physics, it is likely that the proposed short-range Hamiltonian has a ground state in the same universality class as the constructed SU(n) wave functions for certain choices of the parameters.

8. Note added

During the preparation of this manuscript, we learned that related results have been obtained by R. Bondesan and T. Quella [109].

Acknowledgements

The authors acknowledge discussions with Holger Frahm and Kareljan Schoutens. Our special thanks go to J. Ignacio Cirac for his collaborations on related topics and for his helpful comments on the present work. This work has been supported by the EU project SIQS, the DFG cluster of excellence NIM, FIS2012-33642, QUITEMAD (CAM), and the Severo Ochoa Program.

Appendix A. Some useful identities for SU(n)

The SU(n) Lie algebra is formed by n^2-1 Hermitian and traceless generators t^a ($a = 1,\ldots,n^2-1$). They satisfy the commutation relations

$$\left[t^a, t^b\right] = i f_{abc} t^c, \tag{A.1}$$

where f_{abc} is the antisymmetric structure constant of SU(n). For SU(2), we have $f_{abc} = \varepsilon_{abc}$.

In the fundamental representation, the generators t^a are $n \times n$ matrices that we shall denote by τ^a, and in the conjugate representation the generators are $-(\tau^a)^*$. For SU(2), a familiar choice is $\tau^a = \frac{1}{2}\sigma^a$, where σ^a are Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.2}$$

For SU(3), it is convenient to define $\tau^a = \frac{1}{2}\lambda^a$, where λ^a are the following eight Gell-Mann matrices:

$$\begin{aligned}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{A.3}$$

The SU(3) Gell-Mann matrices can be straightforwardly generalized to SU(n) [110]. In our present work, we normalize the SU(n) generators t^a as $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta_{ab}$.

The SU(n) generators in the fundamental representation fulfill

$$\left\{\tau^a, \tau^b\right\} = \frac{1}{n}\delta_{ab} + d_{abc}\tau^c, \tag{A.4}$$

where d_{abc} is symmetric in all indices, and from (A.1) and (A.4), it follows that

$$\tau^a \tau^b = \frac{1}{2n}\delta_{ab} + \frac{d_{abc} + i f_{abc}}{2}\tau^c. \tag{A.5}$$

For the conjugate representation, we have

$$\left\{\left(-\left(\tau^a\right)^*\right), \left(-\left(\tau^b\right)^*\right)\right\} = \frac{1}{n}\delta_{ab} - d_{abc}\left(-\left(\tau^c\right)^*\right), \tag{A.6}$$

and hence

$$\left(-\left(\tau^a\right)^*\right)\left(-\left(\tau^b\right)^*\right) = \frac{1}{2n}\delta_{ab} + \frac{-d_{abc} + i f_{abc}}{2}\left(-\left(\tau^c\right)^*\right). \tag{A.7}$$

The Casimir charge for both the fundamental and the conjugate representations is given by

$$t^a t^a = \frac{n^2-1}{2n}, \tag{A.8}$$

and the SU(n) Fierz identity states that

$$\left(t^a\right)_{\alpha\beta}\left(t^a\right)_{\gamma\delta} = \frac{1}{2}\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{2n}\delta_{\alpha\beta}\delta_{\gamma\delta}. \tag{A.9}$$

The tensors d_{abc} and f_{abc} satisfy

$$d_{aab} = 0, \tag{A.10}$$

$$d_{abc}d_{abd} = \frac{n^2-4}{n}\delta_{cd}, \tag{A.11}$$

$$d_{abc}d_{abc} = \frac{(n^2-1)(n^2-4)}{n}, \tag{A.12}$$

$$f_{abc}f_{abd} = n\delta_{cd}. \tag{A.13}$$

Additionally, their threefold products are given by [111]

$$f_{gae}f_{ebh}f_{hcg} = -\frac{n}{2}f_{abc}, \tag{A.14}$$

$$d_{gae}f_{ebh}f_{hcg} = -\frac{n}{2}d_{abc}, \tag{A.15}$$

$$d_{gae}d_{ebh}f_{hcg} = \frac{n^2-4}{2n}f_{abc}, \tag{A.16}$$

$$d_{gae}d_{ebh}d_{hcg} = \frac{n^2-12}{2n}d_{abc}. \tag{A.17}$$

Using the above identities, we find that

$$t^a t^b t^a = -\frac{1}{2n}t^b, \tag{A.18}$$

$$t^a t^b t^c t^a = -\frac{1}{2n}t^b t^c + \frac{1}{4}\delta_{bc}, \tag{A.19}$$

$$(\vec{t}_i\cdot\vec{t}_j)^2 = \frac{n^2-1}{4n^2} - \frac{n}{4}\left(1 - r_i r_j\frac{n^2-4}{n^2}\right)(\vec{t}_i\cdot\vec{t}_j) \quad (i\neq j), \tag{A.20}$$

$$t_j^a(\vec{t}_i\cdot\vec{t}_j)t_i^a = \frac{n^2-1}{4n^2} + \frac{n}{4}\left(1 + r_i r_j\frac{n^2-4}{n^2}\right)(\vec{t}_i\cdot\vec{t}_j) \quad (i\neq j). \tag{A.21}$$

In the last two equations we consider two copies $\vec{t}_i$ and $\vec{t}_j$ of SU(n) generators acting on different sites i and j, and r_k is $+1$ (-1) if $\vec{t}_k$ belongs to the fundamental (conjugate) representation.

Appendix B. Global singlet condition

In this appendix, we prove that the state (96) with χ given by (103) fulfills $T^a|\Psi\rangle = 0$, where $T^a = \sum_{j=1}^{N+\bar{N}} t_j^a$. First we note that the charge neutrality condition ensures that the wave function is invariant under the $\mathrm{U}(1)^{\otimes(n-1)}$ subgroup of $\mathrm{SU}(n)$. It is then sufficient to prove that the operator $\sum_{j=1}^{N+\bar{N}} S_j^{12}$ annihilates the state, where $S_j^{12} = t_j^1 + it_j^2$. The first two generators in the fundamental representation have the form

$$\tau^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \ddots \end{pmatrix}, \qquad \tau^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & \ddots \end{pmatrix},$$

where all elements that are not shown are zero. For the A sites (fundamental representation), we therefore have

$$S_j^{12} = \tau_j^1 + i\tau_j^2 = \begin{pmatrix} 0 & 1 & \\ 0 & 0 & \\ & & \ddots \end{pmatrix}_j = |1\rangle\langle 2|_j, \quad j \in A, \tag{B.1}$$

and for the B sites (conjugate representation), we have

$$S_j^{12} = \left(-\left(\tau_j^1\right)^*\right) + i\left(-\left(\tau_j^2\right)^*\right) = \begin{pmatrix} 0 & 0 & \\ -1 & 0 & \\ & & \ddots \end{pmatrix}_j = -|2\rangle\langle 1|_j, \quad j \in B. \tag{B.2}$$

Altogether,

$$\sum_{j=1}^{N+\bar{N}} S_j^{12} = \sum_{j\in A} |1\rangle\langle 2|_j - \sum_{j\in B} |2\rangle\langle 1|_j. \tag{B.3}$$

Let us define

$$\left|\Psi'\right\rangle = \sum_{j=1}^{N+\bar{N}} S_j^{12}|\Psi\rangle. \tag{B.4}$$

The term in $|\Psi'\rangle$ having N_1+1 spins in the state $|1\rangle$ in the fundamental representation, $N_{\bar{1}}$ spins in the state $|1\rangle$ in the conjugate representation, $N_2 - 1$ spins in the state $|2\rangle$ in the fundamental representation, and $N_{\bar{2}}$ spins in the state $|2\rangle$ in the conjugate representation at given positions has coefficient

$$\begin{aligned}
&\Psi'\left(\left\{x_{1\to N_1+1}^{(1)}\right\}, \left\{x_{1\to N_2-1}^{(2)}\right\}, \ldots, \left\{x_{1\to N_{\bar{1}}}^{(\bar{1})}\right\}, \left\{x_{1\to N_{\bar{2}}}^{(\bar{2})}\right\}, \ldots\right) \\
&\quad = \sum_{j=1}^{N_1+1} \Psi\left(\left\{x_{1\to j-1}^{(1)}, x_{j+1\to N_1+1}^{(1)}\right\}, \left\{x_{1\to N_2-1}^{(2)}, x_j^{(1)}\right\}, \ldots, \left\{x_{1\to N_{\bar{1}}}^{(\bar{1})}\right\}, \left\{x_{1\to N_{\bar{2}}}^{(\bar{2})}\right\}, \ldots\right) \\
&\qquad - \sum_{j=1}^{N_{\bar{2}}} \Psi\left(\left\{x_{1\to N_1+1}^{(1)}\right\}, \left\{x_{1\to N_2-1}^{(2)}\right\}, \ldots, \left\{x_{1\to N_{\bar{1}}}^{(\bar{1})}, x_j^{(\bar{2})}\right\}, \left\{x_{1\to j-1}^{(\bar{2})}, x_{j+1\to N_{\bar{2}}}^{(\bar{2})}\right\}, \ldots\right),
\end{aligned}$$

where $x_j^{(\alpha)}$ ($x_j^{(\bar{\alpha})}$) is the index of the jth spin in the state $|\alpha\rangle$ in the fundamental (conjugate) representation. We define the order operator O as

$$O(z_j - z_k) = \begin{cases} z_j - z_k & \text{for } j < k \\ 0 & \text{for } j = k \\ z_k - z_j & \text{for } j > k. \end{cases} \tag{B.5}$$

Note that

$$\begin{aligned}
&\Psi\big(\{x^{(1)}_{1\to j-1}, x^{(1)}_{j+1\to N_1+1}\}, \{x^{(2)}_{1\to N_2-1}, x^{(1)}_j\}, \ldots, \{x^{(\bar 1)}_{1\to N_{\bar 1}}\}, \{x^{(\bar 2)}_{1\to N_{\bar 2}}\}, \ldots\big) \\
&= \frac{\operatorname{sgn}(x_1^{(1,\bar 1)}, \ldots \{x^{(1)}_j \text{ missing}\} \ldots, x^{(1,\bar 1)}_{N_1+N_{\bar 1}+1}, x_1^{(2,\bar 2)}, \ldots, x^{(1)}_j, \ldots, x^{(2,\bar 2)}_{N_2+N_{\bar 2}-1}, \ldots)}{\operatorname{sgn}(x_1^{(1,\bar 1)}, \ldots, x^{(1,\bar 1)}_{N_1+N_{\bar 1}+1}, x_1^{(2,\bar 2)}, \ldots, x^{(2,\bar 2)}_{N_2+N_{\bar 2}-1}, \ldots)} \\
&\quad \times \frac{\prod_{i=1}^{N_2-1} O(z_{x^{(1)}_j} - z_{x^{(2)}_i})}{\prod_{i=1(i\neq j)}^{N_1+1} O(z_{x^{(1)}_j} - z_{x^{(1)}_i})} \frac{\prod_{i=1}^{N_{\bar 1}} O(z_{x^{(1)}_j} - z_{x^{(\bar 1)}_i})}{\prod_{i=1}^{N_{\bar 2}} O(z_{x^{(1)}_j} - z_{x^{(\bar 2)}_i})} \\
&\quad \times \Psi\big(\{x^{(1)}_{1\to N_1+1}\}, \{x^{(2)}_{1\to N_2-1}\}, \ldots, \{x^{(\bar 1)}_{1\to N_{\bar 1}}\}, \{x^{(\bar 2)}_{1\to N_{\bar 2}}\}, \ldots\big) \\
&= \frac{\operatorname{sgn}(x_1^{(1,\bar 1)}, \ldots \{x^{(1)}_j \text{ missing}\} \ldots, x^{(1,\bar 1)}_{N_1+N_{\bar 1}+1}, x^{(1)}_j, x_1^{(2,\bar 2)}, \ldots, x^{(2,\bar 2)}_{N_2+N_{\bar 2}-1}, \ldots)}{\operatorname{sgn}(x_1^{(1,\bar 1)}, \ldots, x^{(1,\bar 1)}_{N_1+N_{\bar 1}+1}, x_1^{(2,\bar 2)}, \ldots, x^{(2,\bar 2)}_{N_2+N_{\bar 2}-1}, \ldots)} \\
&\quad \times \frac{\prod_{i=1}^{N_2-1} (z_{x^{(1)}_j} - z_{x^{(2)}_i})}{\prod_{i=1(i\neq j)}^{N_1+1} O(z_{x^{(1)}_j} - z_{x^{(1)}_i})} \frac{\prod_{i=1}^{N_{\bar 1}} O(z_{x^{(1)}_j} - z_{x^{(\bar 1)}_i})}{\prod_{i=1}^{N_{\bar 2}} (z_{x^{(1)}_j} - z_{x^{(\bar 2)}_i})} \\
&\quad \times \Psi\big(\{x^{(1)}_{1\to N_1+1}\}, \{x^{(2)}_{1\to N_2-1}\}, \ldots, \{x^{(\bar 1)}_{1\to N_{\bar 1}}\}, \{x^{(\bar 2)}_{1\to N_{\bar 2}}\}, \ldots\big) \\
&= \frac{\prod_{i=1}^{N_2-1} (z_{x^{(1)}_j} - z_{x^{(2)}_i})}{\prod_{i=1(\neq j)}^{N_1+1} (z_{x^{(1)}_i} - z_{x^{(1)}_j})} \frac{\prod_{i=1}^{N_{\bar 1}} (z_{x^{(\bar 1)}_i} - z_{x^{(1)}_j})}{\prod_{i=1}^{N_{\bar 2}} (z_{x^{(1)}_j} - z_{x^{(\bar 2)}_i})} \\
&\quad \times \Psi\big(\{x^{(1)}_{1\to N_1+1}\}, \{x^{(2)}_{1\to N_2-1}\}, \ldots, \{x^{(\bar 1)}_{1\to N_{\bar 1}}\}, \{x^{(\bar 2)}_{1\to N_{\bar 2}}\}, \ldots\big).
\end{aligned} \tag{B.6}$$

In a similar way, we find

$$\begin{aligned}
&\Psi\big(\{x^{(1)}_{1\to N_1+1}\}, \{x^{(2)}_{1\to N_2-1}\}, \ldots, \{x^{(\bar 1)}_{1\to N_{\bar 1}}, x^{(\bar 2)}_j\}, \{x^{(\bar 2)}_{1\to j-1}, x^{(\bar 2)}_{j+1\to N_{\bar 2}}\}, \ldots\big) \\
&= \frac{\prod_{i=1}^{N_{\bar 1}} (z_{x^{(\bar 1)}_i} - z_{x^{(\bar 2)}_j})}{\prod_{i=1(\neq j)}^{N_{\bar 2}} (z_{x^{(\bar 2)}_j} - z_{x^{(\bar 2)}_i})} \frac{\prod_{i=1}^{N_2-1} (z_{x^{(\bar 2)}_j} - z_{x^{(2)}_i})}{\prod_{i=1}^{N_1+1} (z_{x^{(1)}_i} - z_{x^{(\bar 2)}_j})} \\
&\quad \times \Psi\big(\{x^{(1)}_{1\to N_1+1}\}, \{x^{(2)}_{1\to N_2-1}\}, \ldots, \{x^{(\bar 1)}_{1\to N_{\bar 1}}\}, \{x^{(\bar 2)}_{1\to N_{\bar 2}}\}, \ldots\big).
\end{aligned}$$

To prove that $\Psi'(\{x^{(1)}_{1\to N_1+1}\}, \{x^{(2)}_{1\to N_2-1}\}, \ldots, \{x^{(\bar 1)}_{1\to N_{\bar 1}}\}, \{x^{(\bar 2)}_{1\to N_{\bar 2}}\}, \ldots)$ vanishes, we thus need to proof that

$$\sum_{j=1}^{N_1+1} \frac{\prod_{i=1}^{N_2-1}(z_{x_j^{(1)}} - z_{x_i^{(2)}})}{\prod_{i=1(\neq j)}^{N_1+1}(z_{x_i^{(1)}} - z_{x_j^{(1)}})} \frac{\prod_{i=1}^{N_{\bar{1}}}(z_{x_i^{(\bar{1})}} - z_{x_j^{(1)}})}{\prod_{i=1}^{N_{\bar{2}}}(z_{x_j^{(1)}} - z_{x_i^{(\bar{2})}})}$$
$$- \sum_{j=1}^{N_{\bar{2}}} \frac{\prod_{i=1}^{N_{\bar{1}}}(z_{x_i^{(\bar{1})}} - z_{x_j^{(\bar{2})}})}{\prod_{i=1(\neq j)}^{N_{\bar{2}}}(z_{x_j^{(\bar{2})}} - z_{x_i^{(\bar{2})}})} \frac{\prod_{i=1}^{N_2-1}(z_{x_j^{(\bar{2})}} - z_{x_i^{(2)}})}{\prod_{i=1}^{N_1+1}(z_{x_i^{(1)}} - z_{x_j^{(\bar{2})}})} = 0. \tag{B.7}$$

We rewrite the left-hand side (LHS) of Eq. (B.7) into

$$\text{LHS} = (-1)^{N_{\bar{1}}-N_1} \sum_{j=1}^{N_1+1} \frac{\prod_{i=1}^{N_{\bar{1}}}(z_{x_j^{(1)}} - z_{x_i^{(\bar{1})}}) \prod_{i=1}^{N_2-1}(z_{x_j^{(1)}} - z_{x_i^{(2)}})}{\prod_{i=1(\neq j)}^{N_1+1}(z_{x_j^{(1)}} - z_{x_i^{(1)}}) \prod_{i=1}^{N_{\bar{2}}}(z_{x_j^{(1)}} - z_{x_i^{(\bar{2})}})}$$
$$+ (-1)^{N_{\bar{1}}-N_1} \sum_{j=1}^{N_{\bar{2}}} \frac{\prod_{i=1}^{N_{\bar{1}}}(z_{x_j^{(\bar{2})}} - z_{x_i^{(\bar{1})}}) \prod_{i=1}^{N_2-1}(z_{x_j^{(\bar{2})}} - z_{x_i^{(2)}})}{\prod_{i=1(\neq j)}^{N_{\bar{2}}}(z_{x_j^{(\bar{2})}} - z_{x_i^{(\bar{2})}}) \prod_{i=1}^{N_1+1}(z_{x_j^{(\bar{2})}} - z_{x_i^{(1)}})}. \tag{B.8}$$

Let us denote $z_{x_1^{(1)}}, \ldots, z_{x_{N_1+1}^{(1)}}, z_{x_1^{(\bar{2})}}, \ldots, z_{x_{N_{\bar{2}}}^{(\bar{2})}}$ as z_p $(p = 1, 2, \ldots, N_1 + N_{\bar{2}} + 1)$ and $z_{x_1^{(\bar{1})}}, \ldots, z_{x_{N_{\bar{1}}}^{(\bar{1})}}, z_{x_1^{(2)}}, \ldots, z_{x_{N_2-1}^{(2)}}$ as w_l $(l = 1, 2, \ldots, N_{\bar{1}} + N_2 - 1)$. Then

$$\text{LHS} = (-1)^{N_{\bar{1}}-N_1} \sum_{p=1}^{N_1+N_{\bar{2}}+1} \frac{\prod_{l=1}^{N_{\bar{1}}+N_2-1}(z_p - w_l)}{\prod_{q=1(\neq p)}^{N_1+N_{\bar{2}}+1}(z_p - z_q)}. \tag{B.9}$$

From (97) it follows that $N_1 + N_{\bar{2}} = N_{\bar{1}} + N_2$. Multiplying out the polynomial in the numerator, we observe that (B.9) is zero for all choices of w_l if we have

$$\sum_{p=1}^{K} \frac{z_p^m}{\prod_{q=1(\neq p)}^{K}(z_p - z_q)} = 0 \quad \text{for } m = 0, 1, \ldots, K-2, \tag{B.10}$$

where $K = N_1 + N_{\bar{2}} + 1$. To prove (B.10), we first multiply by the nonzero factor $(-1)^{K-1} \prod_{1 \leq q < l \leq K}(z_l - z_q)$, which transforms the left-hand side of (B.10) into

$$\sum_{p=1}^{K} (-1)^{p-1} z_p^m \prod_{q<l(\neq p)}^{K} (z_l - z_q). \tag{B.11}$$

This expression contains the Vandermonde determinant. Specifically,

$$\sum_{p=1}^{K} (-1)^{p-1} z_p^m \prod_{q<l(\neq p)}^{K} (z_l - z_q)$$
$$= \sum_{p=1}^{K} (-1)^{p-1} z_p^m \det \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ z_1 & \cdots & z_{p-1} & z_{p+1} & \cdots & z_K \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_1^{K-2} & \cdots & z_{p-1}^{K-2} & z_{p+1}^{K-2} & \cdots & z_K^{K-2} \end{pmatrix}$$

$$= \det \begin{pmatrix} z_1^m & z_2^m & \cdots & z_K^m \\ 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_K \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{K-2} & z_2^{K-2} & \cdots & z_K^{K-2} \end{pmatrix}$$
$$= 0, \tag{B.12}$$

where the last equality follows because the determinant has two identical rows for all $m \in \{0, 1, \ldots, K-2\}$. This completes the proof of the singlet property.

References

[1] I. Affleck, J.B. Marston, Phys. Rev. B 37 (1988) 3774.
[2] J.B. Marston, I. Affleck, Phys. Rev. B 39 (1989) 11538.
[3] N. Read, S. Sachdev, Nucl. Phys. B 316 (1989) 609.
[4] N. Read, S. Sachdev, Phys. Rev. B 42 (1990) 4568.
[5] D.P. Arovas, A. Auerbach, Phys. Rev. B 38 (1988) 316.
[6] A. Auerbach, Interacting Electrons and Quantum Magnetism, Springer-Verlag, New York, 1994.
[7] Y. Li, M. Ma, D. Shi, F. Zhang, Phys. Rev. Lett. 81 (1998) 3527.
[8] Y. Yamashita, N. Shibata, K. Ueda, Phys. Rev. B 58 (1998) 9114.
[9] K. Kugel, D. Khomskii, Zh. Èksp. Teor. Fiz. 64 (1973) 1429.
[10] P. Azaria, A.O. Gogolin, P. Lecheminant, A.A. Nersesyan, Phys. Rev. Lett. 83 (1999) 624.
[11] R. Assaraf, P. Azaria, M. Caffarel, P. Lecheminant, Phys. Rev. B 60 (1999) 2299.
[12] M. van den Bossche, P. Azaria, P. Lecheminant, F. Mila, Phys. Rev. Lett. 86 (2001) 4124.
[13] G.-M. Zhang, S.-Q. Shen, Phys. Rev. Lett. 87 (2001) 157201.
[14] T.A. Tóth, A.M. Läuchli, F. Mila, K. Penc, Phys. Rev. Lett. 105 (2010) 265301.
[15] P. Corboz, A.M. Läuchli, K. Penc, M. Troyer, F. Mila, Phys. Rev. Lett. 107 (2011) 215301.
[16] P. Corboz, K. Penc, F. Mila, A.M. Läuchli, Phys. Rev. B 86 (2012) 041106.
[17] P. Corboz, M. Lajkó, A.M. Läuchli, K. Penc, F. Mila, Phys. Rev. X 2 (2012) 041013.
[18] B. Bauer, P. Corboz, A.M. Läuchli, L. Messio, K. Penc, M. Troyer, F. Mila, Phys. Rev. B 85 (2012) 125116.
[19] P. Corboz, M. Lajkó, K. Penc, F. Mila, A.M. Läuchli, Phys. Rev. B 87 (2013) 195113.
[20] S. Taie, Y. Takasu, S. Sugawa, R. Yamazaki, T. Tsujimoto, R. Murakami, Y. Takahashi, Phys. Rev. Lett. 105 (2010) 190401.
[21] S. Taie, R. Yamazaki, S. Sugawa, Y. Takahashi, Nat. Phys. 8 (2012) 825.
[22] X. Zhang, M. Bishof, S.L. Bromley, C.V. Kraus, M.S. Safronova, P. Zoller, A.M. Rey, J. Ye, arXiv:1403.2964, 2014.
[23] F. Scazza, C. Hofrichter, M. Höfer, P.C. De Groot, I. Bloch, S. Fölling, arXiv:1403.4761, 2014.
[24] I. Bloch, J. Dalibard, W. Zwerger, Rev. Mod. Phys. 80 (2008) 885.
[25] M.A. Cazalilla, A.M. Rey, arXiv:1403.2792, 2014.
[26] C. Wu, J. Hu, S.-C. Zhang, Phys. Rev. Lett. 91 (2003) 186402.
[27] C. Honerkamp, W. Hofstetter, Phys. Rev. Lett. 92 (2004) 170403.
[28] M. Cazalilla, A. Ho, M. Ueda, New J. Phys. 11 (2009) 103033.
[29] A. Gorshkov, M. Hermele, V. Gurarie, C. Xu, P. Julienne, J. Ye, P. Zoller, E. Demler, M. Lukin, A. Rey, Nat. Phys. 6 (2010) 289.
[30] C. Xu, Phys. Rev. B 81 (2010) 144431.
[31] S.R. Manmana, K.R. Hazzard, G. Chen, A.E. Feiguin, A.M. Rey, Phys. Rev. A 84 (2011) 043601.
[32] L. Messio, F. Mila, Phys. Rev. Lett. 109 (2012) 205306.
[33] H. Nonne, M. Moliner, S. Capponi, P. Lecheminant, K. Totsuka, Europhys. Lett. 102 (2013) 37008.
[34] Z. Cai, H.-H. Hung, L. Wang, D. Zheng, C. Wu, Phys. Rev. Lett. 110 (2013) 220401.
[35] B. Frischmuth, F. Mila, M. Troyer, Phys. Rev. Lett. 82 (1999) 835.
[36] K. Harada, N. Kawashima, M. Troyer, Phys. Rev. Lett. 90 (2003) 117203.
[37] N. Kawashima, Y. Tanabe, Phys. Rev. Lett. 98 (2007) 057202.
[38] R.K. Kaul, arXiv:1403.5678, 2014.
[39] G. Uimin, JETP Lett. 12 (1970) 225.

[40] C. Lai, J. Math. Phys. 15 (1974) 1675.
[41] B. Sutherland, Phys. Rev. B 12 (1975) 3795.
[42] N. Kawakami, Phys. Rev. B 46 (1992) 3191(R).
[43] Z.N.C. Ha, F.D.M. Haldane, Phys. Rev. B 46 (1992) 9359.
[44] H. Kiwata, Y. Akutsu, J. Phys. Soc. Jpn. 61 (1992) 1441.
[45] F.D.M. Haldane, Phys. Rev. Lett. 60 (1988) 635.
[46] B.S. Shastry, Phys. Rev. Lett. 60 (1988) 639.
[47] I. Affleck, D. Arovas, J. Marston, D. Rabson, Nucl. Phys. B 366 (1991) 467.
[48] S. Chen, C. Wu, S.-C. Zhang, Y. Wang, Phys. Rev. B 72 (2005) 214428.
[49] M. Greiter, S. Rachel, D. Schuricht, Phys. Rev. B 75 (2007) 060401.
[50] M. Greiter, S. Rachel, Phys. Rev. B 75 (2007) 184441.
[51] D.P. Arovas, Phys. Rev. B 77 (2008) 104404.
[52] R. Orús, H.-H. Tu, Phys. Rev. B 83 (2011) 201101.
[53] I. Affleck, T. Kennedy, E.H. Lieb, H. Tasaki, Phys. Rev. Lett. 59 (1987) 799.
[54] I. Affleck, T. Kennedy, E.H. Lieb, H. Tasaki, Commun. Math. Phys. 115 (1988) 477.
[55] J.I. Cirac, G. Sierra, Phys. Rev. B 81 (2010) 104431.
[56] A.E.B. Nielsen, J.I. Cirac, G. Sierra, J. Stat. Mech. 2011 (2011) P11014.
[57] A.E.B. Nielsen, J.I. Cirac, G. Sierra, Phys. Rev. Lett. 108 (2012) 257206.
[58] G. Moore, N. Read, Nucl. Phys. B 360 (1991) 362.
[59] H.-H. Tu, Phys. Rev. B 87 (2013) 041103.
[60] H.-H. Tu, A.E.B. Nielsen, J.I. Cirac, G. Sierra, New J. Phys. 16 (2014) 033025.
[61] A.E.B. Nielsen, G. Sierra, J.I. Cirac, Nat. Commun. 4 (2013) 2864.
[62] A. Kitaev, J. Preskill, Phys. Rev. Lett. 96 (2006) 110404.
[63] M. Levin, X.-G. Wen, Phys. Rev. Lett. 96 (2006) 110405.
[64] V. Kalmeyer, R. Laughlin, Phys. Rev. Lett. 59 (1987) 2095.
[65] V. Kalmeyer, R. Laughlin, Phys. Rev. B 39 (1989) 11879.
[66] X.-G. Wen, F. Wilczek, A. Zee, Phys. Rev. B 39 (1989) 11413.
[67] P.D. Francesco, P. Mathieu, D. Sénéchal, Conformal Field Theory, Springer-Verlag, New York, 1997.
[68] P. Bouwknegt, K. Schoutens, Nucl. Phys. B 547 (1999) 501.
[69] B.I. Halperin, Helv. Phys. Acta 56 (1983) 75.
[70] H.-C. Jiang, Z. Wang, L. Balents, Nat. Phys. 8 (2012) 902.
[71] M.B. Hastings, I. González, A.B. Kallin, R.G. Melko, Phys. Rev. Lett. 104 (2010) 157201.
[72] F.D.M. Haldane, Z.N.C. Ha, J.C. Talstra, D. Bernard, V. Pasquier, Phys. Rev. Lett. 69 (1992) 2021.
[73] H. Blöte, J.L. Cardy, M. Nightingale, Phys. Rev. Lett. 56 (1986) 742.
[74] I. Affleck, Phys. Rev. Lett. 56 (1986) 746.
[75] K. Schoutens, Phys. Lett. B 331 (1994) 335.
[76] P. Bouwknegt, K. Schoutens, Nucl. Phys. B 482 (1996) 345.
[77] D. Schuricht, M. Greiter, Europhys. Lett. 71 (2005) 987.
[78] D. Schuricht, M. Greiter, Phys. Rev. B 73 (2006) 235105.
[79] I. Affleck, Nucl. Phys. B 265 (1986) 409.
[80] A. Läuchli, G. Schmid, S. Trebst, Phys. Rev. B 74 (2006) 144426.
[81] M. Führinger, S. Rachel, R. Thomale, M. Greiter, P. Schmitteckert, Ann. Phys. 17 (2008) 922.
[82] M. Aguado, M. Asorey, E. Ercolessi, F. Ortolani, S. Pasini, Phys. Rev. B 79 (2009) 12408.
[83] R.B. Laughlin, Ann. Phys. 191 (1989) 163.
[84] D.F. Schroeter, Ann. Phys. 310 (2004) 155.
[85] D.F. Schroeter, E. Kapit, R. Thomale, M. Greiter, Phys. Rev. Lett. 99 (2007) 097202.
[86] R. Thomale, E. Kapit, D.F. Schroeter, M. Greiter, Phys. Rev. B 80 (2009) 104406.
[87] E. Kapit, E. Mueller, Phys. Rev. Lett. 105 (2010) 215303.
[88] M. Greiter, Mapping of parent Hamiltonians: From Abelian and Non-Abelian Quantum Hall States to Exact Models of Critical Spin Chains, Springer Tracts in Modern Physics, vol. 244, Springer, Berlin, 2011.
[89] M. Greiter, D.F. Schroeter, R. Thomale, Phys. Rev. B 89 (2014) 165125.
[90] B. Bauer, B.P. Keller, M. Dolfi, S. Trebst, A.W.W. Ludwig, arXiv:1303.6963, 2013.
[91] B. Bauer, L. Cincio, B.P. Keller, M. Dolfi, G. Vidal, S. Trebst, A.W.W. Ludwig, arXiv:1401.3017, 2014.
[92] A.E.B. Nielsen, G. Sierra, J.I. Cirac, Phys. Rev. A 90 (2014) 013606.
[93] M. Hermele, V. Gurarie, A.M. Rey, Phys. Rev. Lett. 103 (2009) 135301.
[94] M. Hermele, V. Gurarie, Phys. Rev. B 84 (2011) 174441.

[95] C. Holzhey, F. Larsen, F. Wilczek, Nucl. Phys. B 424 (1994) 443.
[96] G. Vidal, J.I. Latorre, E. Rico, A. Kitaev, Phys. Rev. Lett. 90 (2003) 227902.
[97] P. Calabrese, J. Cardy, J. Stat. Mech. 2004 (2004) P06002.
[98] H. Frahm, M.J. Martins, Nucl. Phys. B 862 (2012) 504.
[99] R. Thomale, S. Rachel, P. Schmitteckert, M. Greiter, Phys. Rev. B 85 (2012) 195149.
[100] B. Paredes, Phys. Rev. B 85 (2012) 195150.
[101] L.A. Takhtajan, Phys. Lett. A 87 (1982) 479.
[102] H. Babujian, Phys. Lett. A 90 (1982) 479.
[103] N.Y. Reshetikhin, Theor. Math. Phys. 63 (1985) 555.
[104] H.-H. Tu, R. Orús, Phys. Rev. Lett. 107 (2011) 077204.
[105] I. Affleck, Phys. Rev. Lett. 54 (1985) 966.
[106] I. Affleck, J. Phys. Condens. Matter 2 (1990) 405.
[107] F. Michaud, F. Vernay, S.R. Manmana, F. Mila, Phys. Rev. Lett. 108 (2012) 127202.
[108] C.K. Majumdar, D.K. Ghosh, J. Math. Phys. 10 (1969) 1388.
[109] R. Bondesan, T. Quella, Nucl. Phys. B (2014), in press, arXiv:1405.2971, http://www.sciencedirect.com/science/article/pii/S0550321314002211.
[110] H. Georgi, Lie Algebras in Particle Physics, Perseus Books, Reading, MA, 1999.
[111] A. Macfarlane, A. Sudbery, P. Weisz, Commun. Math. Phys. 11 (1968) 77.