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Off-shell CHY amplitudes

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Abstract

The Cachazo–He–Yuan (CHY) formula for on-shell scattering amplitudes is extended off-shell. The off-shell amplitudes (amputated Green's functions) are Möbius invariant, and have the same momentum poles as the on-shell amplitudes. The working principles which drive the modifications to the scattering equations are mainly Möbius covariance and energy momentum conservation in off-shell kinematics. The same technique is also used to obtain off-shell massive scalars. A simple off-shell extension of the CHY gauge formula which is Möbius invariant is proposed, but its true nature awaits further study.

1. Introduction

S-matrix theory was very popular in the late 1950s and early 1960s. It sought to deal more directly with physical observables, and to avoid ultraviolet divergences by staying away from local space-time interactions. Unfortunately, it never got too far because dynamics could not be fully introduced without a Lorentz-invariant interaction Lagrangian density. This problem is now nicely circumvented by the Cachazo–He–Yuan (CHY) scattering theory [1–6], where local Lorentz invariance is supplemented by Möbius invariance of the scattering amplitude in an underlying complex plane. Since its inception, there have been many other papers discussing the

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properties of the scattering equations [7–16], calculations of the amplitude [17–21], its relation to string theory [22–24], the soft and collinear limits [25], and generalization to include massive and/or other particles [20,26–29]. The CHY formula, in its original form, is a tree amplitude for massless particles. In order to implement unitarity, generalization to loop amplitudes [30–37] is required. To facilitate such a generalization and to understand better its connection with local quantum field theory, it is necessary to study the off-shell behavior of these scattering amplitudes. This is what we propose to do in this paper. In Sec. 2, we will extend the CHY on-shell scalar amplitude off-shell to get the amputated Green's functions. We will also use the same technique to extend massless amplitudes to massive scalar amplitudes in Sec. 3, on-shell and off-shell. The on-shell version agrees with the result obtained previously by Dolan and Goddard [20]. The same consideration also yields an off-shell extension of the CHY gauge amplitude, which is Möbius invariant, but the implication of such an extension requires more study as we shall discuss in Sec. 4. Some of the illustrative details are contained in the three appendices.

2. Off-shell massless scalar amplitude

Consider a set of scalar fields ϕ^{ia} in which the first index is in the adjoint representation of some Lie algebra and the second index is in another. If they interact tri-linearly through

$$L_{int} = \frac{1}{3!} f_{ijk} g_{abc} \phi^{ia} \phi^{jb} \phi^{kc}, \tag{1}$$

f's and g's being the structure constants in the Lie algebras, then the Green's function for n particle with momenta k_i, $i = 1 \cdots n$ at the tree level will be a function of products of propagators $\frac{1}{s_{i_1 i_2 \cdots i_m}}$ with $2 \le m \le n-2$ and $s_{i_1 i_2 \cdots i_m} \equiv (k_{i_1} + k_{i_2} + \cdots + k_{i_m})^2$. The coefficients will be a product C_i of $n-2$ f's of the first Lie algebra and another product D_a of $n-2$ g's of the other. For some subsets of indices, they satisfy the Jacobi identities

$$C_i + C_j + C_k = 0, \quad D_a + D_b + D_c = 0. \tag{2}$$

Because of this and because of f and g being totally antisymmetric, only $(n-2)!$ of the C's and $(n-2)!$ of the D's are independent. We can choose an independent set, such that C's are of the form $f_{i_1 j_2 j_3} \cdots f_{j_{n-2} j_{n-1} i_n}$ and D's $g_{a_1 b_2 b_3} \cdots g_{b_{n-2} b_{n-1} a_n}$. The n-particle Green's function will be given as

$$\langle (\phi^{i_1 a_1}(k_1) \phi^{i_2 a_2}(k_2) \cdots \phi^{i_n a_n}(k_n)) \rangle \approx \langle C | M | D \rangle, \tag{3}$$

irrespective of whether k_i are on-shell or not. Here $\langle C|$ is a vector formed from the independent set just mentioned and so is the vector $|D\rangle$. M is the $(n-2)! \times (n-2)!$ symmetric propagating matrix given by CHY formula when all the $k_i^2 = 0$. Explicit expressions for $n = 4, 5$ were given earlier by Vaman and Yao [38]. In the next two sections, we shall explicitly solve for the modifications to the scattering functions f_i in the CHY formulas such that M takes exactly the same form even when $k_i^2 \neq 0$ for $n = 4, 5$. Generalization to any n will then be given in what follows.

2.1. Four particles

When all the particles are on shell, the amplitudes are given by [1–6]

$$M^{1234\ 1ij4} = -\frac{1}{2\pi i} \oint \frac{d\sigma_3}{f_3} \frac{\sigma_{124}^2}{\sigma_{1234} \sigma_{1ij4}}, \tag{4}$$

with

$$\sigma_{i_1 i_2 i_3 \cdots i_m} \equiv \sigma_{i_1 i_2}\sigma_{i_2 i_3}\cdots\sigma_{i_m i_1}, \quad \sigma_{ij} = \sigma_i - \sigma_j, \tag{5}$$

and the scattering function f_i is defined in (30). The integrals are to be evaluated at the pole due to $f_3 = 0$, with $\sigma_{ij} = \sigma_i - \sigma_j$, $\sigma_1 = 0$, $\sigma_2 = 1$ and $\sigma_4 \to \infty$. We need to evaluate the integrals for these configurations only, because from Bose statistics, we are required to obtain

$$M^{1324\ 1324} = M^{1234,\ 1234}|_{2\leftrightarrow 3}, \quad M^{1324\ 1234} = M^{1234,\ 1324}|_{2\leftrightarrow 3}. \tag{6}$$

We shall make changes for f_3 in eq. (4) to give correct $M^{1234,1234}$, and $M^{1234,\ 1324}$ off-shell. The other configurations will be given by the substitutions of eq. (6). To avoid confusion, we will use $\hat{f}_3$ to denote the modified f_3. The modifications we propose are

$$\hat{f}_3 = \frac{s_{31} + x_{31}}{\sigma_{31}} + \frac{s_{32} + x_{32}}{\sigma_{32}}, \tag{7}$$

in which x_{31} and x_{32} are independent of σ's. This assumption of σ independence is predicated by our preference not having to solve a high order algebraic equation for the poles embedded in f_3 otherwise. As we shall see, with this assumption, Möbius covariance and energy momentum conservation for off-shell kinematics will lead to the determination of x_{ij}. However, we shall obtain x_{31} and x_{32} here directly by demanding that the Green's functions from eq. (4) should be the same as those given by the double-color field theory. We postpone to Appendix A to show that such modifications will abide by Mobius invariance, which allows us to set the three σ's to the values we gave. The pole is at

$$\sigma_{31} = \sigma_3 = \frac{s_{31} + x_{31}}{s_{31} + s_{32} + x_{31} + x_{32}}, \tag{8}$$

and therefore

$$\sigma_{32} = \sigma_3 - 1 = -\frac{s_{32} + x_{32}}{s_{31} + s_{32} + x_{31} + x_{32}}. \tag{9}$$

These give

$$M^{1234,\ 1234} = \frac{1}{s_{31} + s_{32} + x_{31} + x_{32}}\left(\frac{s_{31} + x_{31}}{s_{32} + x_{32}}\right). \tag{10}$$

Using the off-shell kinematics

$$s_{31} + s_{32} + s_{12} = \sum_{i=1}^{4} k_i^2, \ s_{32} = s_{14}, \tag{11}$$

we obtain

$$M^{1234,\ 1234} = \frac{1}{s_{14} + x_{32}} + \frac{1}{s_{12} - \sum_{i=1}^{4} k_i^2 - (x_{31} + x_{32})}. \tag{12}$$

To coincide with the field theory result $M^{1234,\ 1234} = \frac{1}{s_{14}} + \frac{1}{s_{12}}$, we need

$$x_{31} = -\sum_{i=1}^{4} k_i^2, \quad x_{32} = 0. \tag{13}$$

With these, it is easy to obtain

$$M^{1234,\ 1324} = -\frac{1}{s_{14}}. \tag{14}$$

If we are to use these x_{ij} for $\hat{f}_3$ and assume that $M^{1324,1324}$ is given by eq. (4), with $\sigma_{1234}\sigma_{1ij4}$ replaced by $\sigma_{1324}\sigma_{1324}$, we would obtain $M^{1324,1324} = \frac{1}{s_{32}} + \frac{1}{s_{31}+x_{31}}$, which is not what field theory gives. Instead, we should use eq. (6)

$$M^{1324,\ 1324} = \frac{1}{s_{13}} + \frac{1}{s_{14}}, \quad M^{1324,\ 1234} = -\frac{1}{s_{14}}. \tag{15}$$

We have not been able to find one single universal $\hat{f}_3$, which can produce all the results we want for all configurations for off-shell Green's functions. This color dependence of the off-shell scattering function is a new feature that will be further discussed in Sec. 2.3.

2.2. Five particles

The amplitudes are given by [1–6]

$$M^{12345,\ 1ijk5} = \left(\frac{-1}{2\pi i}\right)^2 \oint \frac{d\sigma_2 d\sigma_4}{f_2 f_4} \frac{\sigma_{135}^2}{\sigma_{12345}\sigma_{1ijk5}}, \tag{16}$$

in which $i, j, k = 2, 3, 4$ or their permutations, and $\sigma_{1,3,5}$ can take on any fixed values. The scattering functions f_i are defined in (30). We shall obtain the other configurations $M^{1lmn5,\ 1ijk5}$ by appropriately relabeling indices of the results from eq. (16), *e.g.*

$$M^{13425,\ 14235} = M^{12345,\ 13425}|_{3\to4,4\to2,2\to3}. \tag{17}$$

The integrals are to be evaluated at the poles due to $f_2 = 0$ and $f_4 = 0$ simultaneously.

In this case, let us assume that to obtain the Green's function, the modifications are

$$\hat{f}_2 = \frac{\hat{s}_{21}}{\sigma_{21}} + \frac{\hat{s}_{23}}{\sigma_{23}} + \frac{\hat{s}_{24}}{\sigma_{24}} + \frac{\hat{s}_{25}}{\sigma_{25}}, \tag{18}$$

and

$$\hat{f}_4 = \frac{\hat{s}_{41}}{\sigma_{41}} + \frac{\hat{s}_{42}}{\sigma_{42}} + \frac{\hat{s}_{43}}{\sigma_{43}} + \frac{\hat{s}_{45}}{\sigma_{45}}, \tag{19}$$

in which $\hat{s}_{ij} = s_{ij} + x_{ij}$ and $x_{ij} = x_{ji}$ are assumed to be independent of σ.

Then, for the Green's function $M^{12345,\ 12345}$, the expected result from field theory is [38]

$$M^{12345,\ 12345}|_{exp} = \frac{1}{s_{15}s_{34}} + \frac{1}{s_{15}s_{23}} + \frac{1}{s_{12}s_{45}} + \frac{1}{s_{23}s_{45}} + \frac{1}{s_{12}s_{34}}. \tag{20}$$

From Appendix B, we find that

$$M^{12345,\ 12345} = \frac{1}{\hat{s}_{23}+\hat{s}_{34}+\hat{s}_{24}}\left(\frac{1}{\hat{s}_{34}} + \frac{1}{\hat{s}_{23}}\right) + \frac{1}{\hat{s}_{12}\hat{s}_{45}} + \frac{1}{\hat{s}_{23}\hat{s}_{45}} + \frac{1}{\hat{s}_{12}\hat{s}_{34}}. \tag{21}$$

Comparing the two equations above, we conclude that

$$s_{12} = \hat{s}_{12},\ s_{45} = \hat{s}_{45},\ s_{23} = \hat{s}_{23},\ s_{34} = \hat{s}_{34}, \tag{22}$$

which gives

$$x_{12} = x_{23} = x_{34} = x_{45} = 0. \tag{23}$$

Furthermore, we have

$$\hat{s}_{23} + \hat{s}_{34} + \hat{s}_{24} = s_{23} + s_{34} + s_{24} + x_{24} = s_{15}. \tag{24}$$

Upon using kinematics

$$s_{23} + s_{34} + s_{24} = s_{15} + k_2^2 + k_3^2 + k_4^2, \tag{25}$$

we arrive at

$$x_{24} = -(k_2^2 + k_3^2 + k_4^2). \tag{26}$$

Picking other pairs of $\hat{f}_i,\ \hat{f}_j$ to evaluate $M^{12345,\ 12345}$, we also obtain

$$x_{15} = 0,\ x_{14} = -(k_1^2 + k_4^2 + k_5^2),\ x_{13} = -(k_1^2 + k_2^2 + k_3^2), \tag{27}$$

$$x_{25} = -(k_1^2 + k_2^2 + k_5^2),\ x_{35} = -(k_3^2 + k_4^2 + k_5^2). \tag{28}$$

2.3. Any number of particles

The CHY on-shell amplitude for n scalar particles is [1–6]

$$M^{\alpha,\beta} = \left(-\frac{1}{2\pi i}\right)^{n-3} \oint_\Gamma \sigma_{rst}^2 \left(\prod_{i=1,\, i\neq r,s,t}^{n} \frac{d\sigma_i}{f_i}\right) \frac{1}{\sigma_\alpha \sigma_\beta}, \tag{29}$$

where $\sigma_r, \sigma_s, \sigma_t$ are three Möbius constants which will be left arbitrary, $\alpha = [\alpha_1 \alpha_2 \cdots \alpha_n]$ and $\beta = [\beta_1 \beta_2 \cdots \beta_n]$ are the two configurations of colors, with $\sigma_\alpha = \prod_{a=1}^{n} \sigma_{\alpha_a \alpha_{a+1}}$, $\sigma_\beta = \prod_{b=1}^{n} \sigma_{\beta_b \beta_{b+1}}$, and $n+1 \equiv 1$. The contour Γ encloses the $(n-3)!$ zeros of f_i anti-clockwise, and the on-shell scattering functions are

$$f_i = \sum_{j=1,\, j\neq i}^{n} \frac{2k_i \cdot k_j}{\sigma_{ij}}, \quad (1 \le i \le n). \tag{30}$$

To get an off-shell amplitude, we assume the only change needed is to replace f_i by an off-shell version given by

$$\hat{f}_i = \sum_{j=1,\, j\neq i}^{n} \frac{\hat{s}_{ij}}{\sigma_{ij}}, \quad (1 \le i \le n),$$
$$\hat{s}_{ij} = s_{ij} + x_{ij} = 2k_i \cdot k_j + k_i^2 + k_j^2 + x_{ij}, \tag{31}$$

where $x_{ij} = x_{ji}$ is assumed to be σ-independent. We also assume that the contour Γ is replaced by $\hat{\Gamma}$ to enclose $\hat{f}_i = 0$ $(i \neq r, s, t)$ anti-clockwise. Since x_{ii} do not appear, we may and will set them equal to zero. The rest of the parameters $x_{ij} = x_{ji}$ are determined by requiring the off-shell amplitudes to be Möbius invariant, and to have propagators identical to those given by field theory.

Under a Möbius transformation $\sigma_j \to (\alpha\sigma_j + \beta)/(\gamma\sigma_j + \delta)$, with $\alpha\delta - \beta\gamma = 1$, the off-shell amplitude is invariant if $\hat{f}_i \to \hat{f}_i(\gamma\sigma_i + \delta)^2$. A simple calculation shows that this is fulfilled if

$$\sum_{j\neq i} \hat{s}_{ij} = 0, \tag{32}$$

which, by using momentum conservation, implies

$$\sum_{j=1}^{n} x_{ij} = -(n-4)k_i^2 - \sum_{j=1}^{n} k_j^2. \tag{33}$$

There are many solutions to this equation, so Möbius invariance alone is too general to fix the off-shell amplitude, and that is where the propagator requirement mentioned two paragraphs above comes in. First consider the case $\alpha = \beta = [123\cdots n]$. Then any $(\ell+1)$ $(1 \le \ell \le n-3)$ consecutive lines may form a propagator, with an inverse factor

$$s_{i,i+1,i+2,\cdots,i+\ell} = \sum_{r=i}^{i+\ell-1} \sum_{t=r+1}^{i+\ell} s_{rt} - (\ell-1) \sum_{r=i}^{i+\ell} k_r^2, \tag{34}$$

for some i and some ℓ. Here and after the line indices are understood to be mod n. For on-shell amplitudes, the inverse propagators $\sum_{r<t} 2k_r \cdot k_t$ can be obtained by carrying out the integration of eq. (29). For off-shell amplitudes, the only change is to replace f_i in eq. (29) by $\hat{f}_i$, namely, by replacing $2k_r \cdot k_t$ by $\hat{s}_{rt}$, hence the inverse propagator is $\sum_{r<t} \hat{s}_{rt}$. Equating this with eq. (34), we get

$$\sum_{r=i}^{i+\ell-1} \sum_{t=r+1}^{i+\ell} s_{rt} - (\ell-1) \sum_{r=i}^{i+\ell} k_r^2 = \sum_{r=i}^{i+\ell-1} \sum_{t=r+1}^{i+\ell} (s_{rt} + x_{rt}), \tag{35}$$

or equivalently,

$$\sum_{r=i}^{i+\ell-1} \sum_{t=r+1}^{i+\ell} x_{rt} = -(\ell-1) X_i^{i+\ell}, \quad X_i^{i+\ell} = \sum_{r=i}^{i+\ell} k_r^2, \quad (1 \le \ell \le n-3). \tag{36}$$

In particular, if $\ell = 1$, then

$$x_{i,i+1} = 0, \tag{37}$$

and

$$x_{i,i+n-1} = x_{i+n-1,i+n} = x_{i-1,i} = 0. \tag{38}$$

To obtain solutions for other x_{ij}, subtract eq. (36) from the same equation with ℓ replaced by $\ell - 1$ to get

$$\sum_{r=i}^{i+\ell-1} x_{r,i+\ell} = -(\ell-1) X_i^{i+\ell} + (\ell-2) X_i^{i+\ell-1}, \quad (2 \le \ell \le n-3). \tag{39}$$

For $\ell = 2$, it gives

$$x_{i,i+2} = -X_i^{i+2} = -(k_i^2 + k_{i+1}^2 + k_{i+2}^2), \quad (n \ge 5), \tag{40}$$

and

$$x_{i,i+n-2} = x_{i+n-2,i} = x_{i+n-2,i+n} = x_{i-2,i} = X_{i+n-2}^{i+n} = -(k_{i+n-2}^2 + k_{i+n-1}^2 + k_{i+n}^2), \quad (n \ge 5). \tag{41}$$

The restriction $n \geq 5$ comes about because the requirement that $2 = \ell \leq n-3$. In case $n = 4$, eqs. (40) and (41) are no longer valid. They would be replaced by a relation obtained from eqs. (33) and (37) to be

$$x_{i,i+2} = -\sum_{r=1}^{4} k_r^2, \quad (n=4). \tag{42}$$

This agrees with the result (13) obtained previously by direct calculation.

For $\ell \geq 3$, the solution can be obtained by subtracting (39) from the same equation with (i, ℓ) replaced by $(i+1, \ell-1)$ to get

$$\begin{aligned} x_{i,i+\ell} &= -(\ell-1)X_i^{i+\ell} + (\ell-2)X_i^{i+\ell-1} + (\ell-2)X_{i+1}^{i+\ell} - (\ell-3)X_{i+1}^{i+\ell-1} \\ &= -(k_i^2 + k_{i+\ell}^2), \quad (3 \leq \ell \leq n-3). \end{aligned} \tag{43}$$

To summarize, the solutions are

$$x_{ij} = \left\{ \begin{array}{ll} 0 & \text{if } |j-i| = 0 \text{ or } 1 \\ -(k_i^2 + k_m^2 + k_j^2) & \text{if } |j-i| = 2 \\ -(k_i^2 + k_j^2) & \text{if } |j-i| \geq 3 \end{array} \right\}, \quad \alpha = \beta = (123\cdots n), \tag{44}$$

where m in the middle equation is the line between i and j. These solutions are symmetric in i and j, as they should be, and automatically satisfy the gauge-covariant condition eq. (33) because

$$\sum_{j\neq i} x_{ij} = -X_{i-2}^{i} - X_i^{i-2} - \sum_{j=i+3}^{i+n-3} \left(k_i^2 + k_j^2\right) = (n-4)k_i^2 - \sum_{j=1}^{n} k_j^2. \tag{45}$$

More generally, if $\alpha = \beta = [\alpha_1\alpha_2\alpha_3\cdots\alpha_n]$, then the inverse propagators allowed would be $s_{\alpha_i\alpha_{i+1}\alpha_{i+2}\cdots\alpha_{i+\ell}}$, and the solution of x_{ij} can be obtained from eq. (44) by a substitution to get

$$x^{\alpha,\alpha}_{\alpha_i\alpha_j} = \left\{ \begin{array}{ll} 0 & \text{if } |j-i| = 0 \text{ or } 1 \\ -(k_{\alpha_i}^2 + k_{\alpha_m}^2 + k_{\alpha_j}^2) & \text{if } |j-i| = 2 \\ -(k_{\alpha_i}^2 + k_{\alpha_j}^2) & \text{if } |j-i| \geq 3 \end{array} \right\}, \quad \alpha = \beta = [\alpha_1\alpha_2\alpha_3\cdots\alpha_n], \tag{46}$$

where the colors are now indicated in the superscripts.

So far we have considered diagonal colors, with $\beta = \alpha$. In general, we should take $x_{ij}^{\alpha,\beta} = (x_{ij}^{\alpha,\alpha} + x_{ij}^{\beta,\beta})/2$ to ensure $M^{\alpha,\beta} = M^{\beta,\alpha}$. This gives the right answer because when $\alpha = \beta$, it returns to eq. (46). When $\alpha \neq \beta$, an $(\ell+1)$-line pole is present in $M^{\alpha,\beta}$ only when $s_{\alpha_i\alpha_{i+1}\alpha_{i+2}\cdots\alpha_{i+\ell}} = s_{\beta_i\beta_{i+1}\beta_{i+2}\cdots\beta_{i+\ell}}$, which demands the unordered set of momenta $\{k_{\alpha_i}, k_{\alpha_{i+1}}, \cdots, k_{\alpha_{i+\ell}}\}$ to be identical to the unordered set $\{k_{\beta_i}, k_{\beta_{i+1}}, \cdots, k_{\beta_{i+\ell}}\}$. In that case the propagator requirement eq. (34) and eq. (36) are automatically satisfied.

It is interesting to note that the on-shell scattering functions f_i do not depend on the colors, but the off-shell functions $\hat{f}_i$ do.

2.4. *Off-shell amplitude and off-shell extension of on-shell amplitudes*

Take the on-shell amplitude in eq. (29). There is a way to extend $M^{\alpha,\beta}$ such that three of the particles are off-shell while the rest are on-shell. Let us call r, s, t constant lines and the others $i \neq r, s, t$ variable lines. As long as all the variable lines are on-shell, *i.e.,* $k_i^2 = 0$ for all $i \neq r, s, t$, then the amplitude is Möbius invariant no matter whether k_r, k_s, k_t are on-shell or not. In this way

we can define a Möbius-invariant amplitude using (29) for up to three off-shell lines. We shall refer to this as the off-shell extension of the on-shell amplitude.

What we want to point out is that this off-shell extension is generally different from the off-shell amplitude considered above, by keeping all but at most three lines on-shell, because the off-shell extension amplitude may not satisfy the propagator requirement.

For example, suppose line r is off-shell and a variable line i is next to it in an amplitude with diagonal colors. The off-shell amplitude satisfies the propagator requirement and gives rise to a propagator $1/s_{ir}$, whereas the corresponding contribution from the off-shell extension amplitude is $1/2k_i \cdot k_r = 1/(s_{ir} - k_r^2)$.

In the case of diagonal colors, the only time an off-shell extension amplitude coincides with an off-shell amplitude is when there is only one off-shell line, say r, shielded on either side of it by the other two on-shell constant lines s and t. In this way no variable line can get next to the off-shell line to produce a different propagator.

3. Off-shell massive scalar amplitude

We want to explore whether the amplitude of a double-color scalar theory with mass m can also be given by eq. (29), with f_i replaced by $\hat{f}_i$ of eq. (31), but with a different x_{ij} than the massless case. The general solution is given below in this section, but to make it more concrete and easier to understand, explicit evaluations for $n = 5$ and $n = 6$ are given in Appendix C.

To be Möbius invariant the condition eq. (33) must be satisfied. For $\alpha = \beta = [123\cdots n]$, the inverse propagators are $s_{i,i+1,i+2,\cdots,i+\ell} - m^2$, so eq. (35) should be replaced by

$$\sum_{r=i}^{i+\ell-1}\sum_{t=r+1}^{i+\ell} s_{rt} - (\ell-1)\sum_{r=i}^{i+\ell} k_r^2 - m^2 = \sum_{r=i}^{i+\ell-1}\sum_{t=r+1}^{i+\ell}(s_{rt} + x_{rt}), \tag{47}$$

and eq. (36) should be replaced by

$$\sum_{r=i}^{i+\ell-1}\sum_{t=r+1}^{i+\ell} x_{rt} = -(\ell-1)\sum_{r=i}^{i+\ell} k_r^2 - m^2 := -(\ell-1)X_i^{i+\ell} - m^2, \quad (1 \le \ell \le n-3). \tag{48}$$

Setting $\ell = 1$, we get

$$x_{i,i+1} = x_{i,i+n-1} = -m^2. \tag{49}$$

Eq. (39) is still valid in the massive case, because subtraction cancels the m^2 terms. Setting $\ell = 2$, we get

$$x_{i,i+2} = -X_i^{i+2} - x_{i,i+1} = -(k_i^2 + k_{i+1}^2 + k_{i+2}^2) + m^2, \quad (n \ge 5). \tag{50}$$

Similarly,

$$x_{i,i+n-2} = X_{i+n-2}^{i+n} - x_{i,i+n-1} = -(k_{i+n-2}^2 + k_{i+n-1}^2 + k_{i+n}^2) + m^2, \quad (n \ge 5). \tag{51}$$

As in the massless case, $n = 4$ must be treated separately. There, we need to use eq. (33) to get

$$x_{i,i+2} = -x_{i,i+1} - x_{i,i-1} - \sum_{r=1}^{4} k_r^2 = -\sum_{r=1}^{4} k_r^2 + 2m^2, \quad (n = 4). \tag{52}$$

For $\ell \ge 3$, the solution can be obtained by subtracting (39) from the same equation with (i, ℓ) replaced by $(i+1, \ell-1)$ to get

$$x_{i,i+\ell} = -(\ell-1)X_i^{i+\ell} + (\ell-2)X_i^{i+\ell-1} + (\ell-2)X_{i+1}^{i+\ell} - (\ell-3)X_{i+1}^{i+\ell-1}$$
$$= -(k_i^2 + k_{i+\ell}^2), \quad (3 \le \ell \le n-3). \tag{53}$$

The final solution is therefore

$$x_{ij} = \left\{ \begin{array}{ll} 0 & \text{if } |j-i| = 0 \\ -m^2 & \text{if } |j-i| = 1 \\ -(k_i^2 + k_m^2 + k_j^2) + m^2 & \text{if } |j-i| = 2 \\ -(k_i^2 + k_j^2) & \text{if } |j-i| \ge 3 \end{array} \right\}, \quad \alpha = \beta = (123\cdots n). \tag{54}$$

It can easily be checked that the Möbius-invariant condition eq. (33) is also automatically satisfied. For general colors α and β, the solution can again be obtained from eq. (54) by a substitution as in the massless case.

4. An off-shell extension of the gauge amplitude

Similar to (29), the on-shell color-stripped n-gluon scattering amplitude is given by the CHY formula [1–6] to be

$$M^\alpha = \left(-\frac{1}{2\pi i}\right)^{n-3} \oint_\Gamma \sigma_{rst}^2 \left(\prod_{a=1, a\neq r,s,t}^{n} \frac{d\sigma_a}{f_a} \right) \frac{\mathrm{Pf}'\Psi}{\sigma_\alpha}, \tag{55}$$

with the reduced Pfaffian $\mathrm{Pf}'\Psi$ replacing the factor $1/\sigma_\beta$ in the scalar theory. The reduced Pfaffian is related to the Pfaffian of Ψ_{ij}^{ij} by

$$\mathrm{Pf}\Psi' = 2\frac{(-1)^{i+j}}{\sigma_{ij}} \mathrm{Pf}\left(\Psi_{ij}^{ij}\right), \tag{56}$$

the matrix Ψ_{ij}^{ij} is obtained from the matrix Ψ by deleting its ith column and row and its jth column and row, and the antisymmetric matrix Ψ is made up of three $n \times n$ matrices A, B, C,

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}. \tag{57}$$

The non-diagonal elements of these three sub-matrices are

$$A_{ab} = \frac{2k_a \cdot k_b}{\sigma_{ab}}, \quad B_{ab} = \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}}, \quad C_{ab} = \frac{\epsilon_a \cdot k_b}{\sigma_{ab}}, \quad 1 \le a \ne b \le n, \tag{58}$$

where ϵ_a is the polarization of the ath gluon, satisfying $\epsilon_a \cdot k_a = 0$. The diagonal elements of A and B are zero, and that of C is given by

$$C_{aa} = -\sum_{b=1}^{n} C_{ab}, \tag{59}$$

so that the column and row sums of C is zero. A similar property is true for A if the scattering equations $f_a = 0$ are obeyed, which is the case because the integration contour Γ encloses these zeros anti-clockwise.

Under a Möbius transformation, $\sigma_b \to (\alpha\sigma_b + \beta)/(\gamma\sigma_b + \delta)$, with $\alpha\delta - \beta\gamma = 1$, the CHY amplitude eq. (55) is Möbius invariant if the momenta are massless and conserved. The gluon

amplitude is gauge invariant and independent of the choice of λ and ν if the row sum and column sum of the sub-matrices of A and C are zero, as mentioned in (59) and the paragraph below it.

An off-shell extension of (55) can be obtained if we replace f_i by $\hat{f}_i$ obtained in the previous sections, the contour Γ by $\hat{\Gamma}$ enclosing $\hat{f}_i = 0$, and the elements of A in (58) by

$$A_{ij} = \sum_{j \neq i} \frac{\hat{s}_{ij}}{\sigma_{ij}}. \tag{60}$$

The row and column sums of A are still zero because $\hat{f}_i = 0$ and because (32) is satisfied. This off-shell extension is Möbius invariant and is independent of the choice of λ, ν as before, because all the conditions necessary to prove these properties for the on-shell amplitude have been preserved with the change.

These changes are the simplest extensions of the on-shell scattering formula to off-shell, but whether it is the amputated Green's function of an Yang–Mills field theory is not immediately clear. The reason is, off-shell Yang–Mills theory is gauge dependent, and in our extension gauges do not enter. It is possible that this extension determines a particular 'CHY gauge', or that the true off-shell extension is much more complicated than what is discussed in order to reflect the freedom of gauge choice. It is even possible that field-theoretical off-shell expression is not Möbius invariant. Further study is required to know what is the truth.

5. Conclusion

The CHY scattering formulas for massless scalar particles are extended off-shell by changing $2k_i \cdot k_j$ in the scattering function f_i to $(k_i + k_j)^2 + x_{ij}$, where $x_{ij} = x_{ji}$ is independent of σ. It can be determined uniquely from the requirement that off-shell amplitudes are Möbius invariant and have exactly the same invariant-momentum poles as the on-shell amplitudes. The same requirements also allow us to extend the formula to massive scalar and vector amplitudes, on-shell and off-shell. A simple off-shell extension of the CHY gauge amplitude is also proposed, with many nice properties including Möbius invariance and the independence of λ and ν, but the true nature of this extension formula requires further study.

Appendix A. Möbius invariance of the $n = 4$ and $n = 5$ amplitudes

One motivating and intriguing feature of the CHY formulas is that the on-shell amplitudes for scalar, gauge, and gravitational interactions are all invariant under Möbius transformations

$$\sigma_i = \frac{\alpha\sigma_i' + \beta}{\gamma\sigma_i' + \delta}, \quad \alpha\delta - \beta\gamma = 1. \tag{A.1}$$

In our extending the CHY formula to off-shell, it is very natural to ask if such invariance still holds. In fact, this is required of us, because if it were not so, then we would not have the freedom to fix three of the σ's to the values we used in Sections 2.1 and 2.2. Gladly, the answer is in the affirmative, although with some restrictions (eq. (6), eq. (17)). Let us begin with the case of four double-color scalars. The invariance of the off-shell Green's functions is intimately tied up with the transformation property of the scattering equations. Let us generalize these slightly before we fix three of the σ's

$$\hat{f}_1(\sigma) = \frac{s_{12}}{\sigma_{12}} + \frac{s_{13} + x_{13}}{\sigma_{13}} + \frac{s_{14}}{\sigma_{14}},$$

$$\hat{f}_2(\sigma) = \frac{s_{21}}{\sigma_{21}} + \frac{s_{23}}{\sigma_{23}} + \frac{s_{24}+x_{31}}{\sigma_{24}},$$
$$\hat{f}_3(\sigma) = \frac{s_{31}+x_{31}}{\sigma_{31}} + \frac{s_{32}}{\sigma_{32}} + \frac{s_{34}}{\sigma_{34}},$$
$$\hat{f}_4(\sigma) = \frac{s_{41}}{\sigma_{41}} + \frac{s_{42}+x_{31}}{\sigma_{42}} + \frac{s_{43}}{\sigma_{43}}. \tag{A.2}$$

One can verify that they satisfy

$$\sum_{i=1}^{4} \hat{f}_i = \sum_{i=1}^{4} \sigma_i \hat{f}_i = \sum_{i=1}^{4} \sigma_i^2 \hat{f}_i = 0, \tag{A.3}$$

as a result of momentum conservation, and

$$\sum_{j\neq i} s_{ij} = \sum_{j=1}^{4} k_j^2 = -x_{31} = -x_{13}, \tag{A.4}$$

which in turn means that only one of the $\hat{f}_i$ is independent. Let us take this to be $\hat{f}_3$. Then, we find that under Möbius transformation

$$\hat{f}_3(\sigma) = \gamma(\gamma\sigma_3' + \delta)[-(s_{31}+x_{13}) - s_{32} - s_{34}] + (\gamma\sigma_3' + \delta)^2 \hat{f}_3(\sigma'). \tag{A.5}$$

The first term vanishes as indicated by eq. (A.4), which makes $\hat{f}_3$ Möbius covariant. Now, we also set $\sigma_1 = 0,\ \sigma_2 = 1$ and $\sigma_4 \to \infty$. Using

$$d\sigma_3 = \frac{d\sigma_3'}{(\gamma\sigma_3' + \delta)^2}, \tag{A.6}$$

and

$$\frac{\sigma_{124}}{\sigma_{1234}\sigma_{1ij4}} = (\gamma\sigma_3' + \delta)^4 \frac{\sigma_{124}'}{\sigma_{1234}'\sigma_{1ij4}'}, \quad i,j = 2,3, \tag{A.7}$$

we have

$$\oint \frac{d\sigma_3}{\hat{f}_3(\sigma)} \frac{\sigma_{124}}{\sigma_{1234}\sigma_{1ij4}} = \oint \frac{d\sigma_3'}{\hat{f}_3(\sigma')} \frac{\sigma_{124}'}{\sigma_{1234}'\sigma_{1ij4}'}. \tag{A.8}$$

The caveat here is that in order to have poles at the correct place, we have committed to the form of $\hat{f}_3$ as determined.

For the double-color five particle amplitudes, let us generalize the functions slightly to

$$\hat{f}_1 = \frac{s_{12}}{\sigma_{12}} + \frac{s_{13}+x_{13}}{\sigma_{13}} + \frac{s_{14}+x_{14}}{\sigma_{14}} + \frac{s_{15}}{\sigma_{15}},$$
$$\hat{f}_2 = \frac{s_{21}}{\sigma_{21}} + \frac{s_{23}}{\sigma_{23}} + \frac{s_{24}+x_{24}}{\sigma_{24}} + \frac{s_{25}+x_{25}}{\sigma_{25}},$$
$$\hat{f}_3 = \frac{s_{31}+x_{31}}{\sigma_{31}} + \frac{s_{32}}{\sigma_{32}} + \frac{s_{34}}{\sigma_{34}} + \frac{s_{35}+x_{35}}{\sigma_{35}},$$
$$\hat{f}_4 = \frac{s_{41}+x_{41}}{\sigma_{41}} + \frac{s_{42}+x_{42}}{\sigma_{42}} + \frac{s_{43}}{\sigma_{43}} + \frac{s_{45}}{\sigma_{45}},$$
$$\hat{f}_5 = \frac{s_{51}}{\sigma_{51}} + \frac{s_{52}+x_{52}}{\sigma_{52}} + \frac{s_{53}+x_{53}}{\sigma_{53}} + \frac{s_{54}}{\sigma_{54}}. \tag{A.9}$$

It is easy to check that

$$\sum_{i=1}^{5} \hat{f}_i = \sum_{i=1}^{5} \sigma_i \hat{f}_i = \sum_{i=1}^{5} \sigma_i^2 \hat{f}_i = 0, \tag{A.10}$$

because of momentum conservation, similar to eq. (A.4), when we take

$$\begin{aligned} &x_{31} = -(k_1^2 + k_2^2 + k_3^2), \quad x_{41} = -(k_1^2 + k_4^2 + k_5^2), \quad x_{42} = -(k_2^2 + k_3^2 + k_4^2), \\ &x_{52} = -(k_1^2 + k_2^2 + k_5^2), \quad x_{53} = -(k_3^2 + k_4^2 + k_5^2), \end{aligned} \tag{A.11}$$

which implies that only two of the $\hat{f}_i's$ are independent. We choose them to be $\hat{f}_3$ and $\hat{f}_4$, which are Möbius covariant, in the sense that

$$\hat{f}_3(\sigma) = (\gamma\sigma_3' + \delta)^2 \hat{f}_3(\sigma'), \quad \hat{f}_4(\sigma) = (\gamma\sigma_4' + \delta)^2 \hat{f}_4(\sigma'), \tag{A.12}$$

and we are led to

$$\oint \frac{d\sigma_3 d\sigma_4}{\hat{f}_3(\sigma)\hat{f}_4(\sigma)} \frac{\sigma_{125}^2}{\sigma_{12345}\sigma_{1ijk5}} = \oint \frac{d\sigma_3' d\sigma_4'}{\hat{f}_3(\sigma')\hat{f}_4(\sigma')} \frac{(\sigma_{125}')^2}{\sigma_{12345}'\sigma_{1ijk5}'}. \tag{A.13}$$

Appendix B. A diagonal element of the $n = 5$ amplitude

In this note one of $n = 5$ scalar amplitudes is calculated. The others can be done in a similar fashion. For our purpose here, let us consider the most complicated case with diagonal colors, say with 1, 3, 5 as the constant lines.

$$I_5 = \left(\frac{-1}{2\pi i}\right)^2 \oint_\Gamma \frac{\sigma_{135}^2 d\sigma_2 d\sigma_4}{f_2 f_4 \sigma_{12345}\sigma_{12345}}, \tag{B.1}$$

$$\begin{aligned} f_2 &= \frac{s_{21}}{\sigma_{21}} + \frac{s_{23}}{\sigma_{23}} + \frac{s_{24}}{\sigma_{24}} + \frac{s_{25}}{\sigma_{25}}, \\ f_4 &= \frac{s_{41}}{\sigma_{41}} + \frac{s_{42}}{\sigma_{42}} + \frac{s_{43}}{\sigma_{43}} + \frac{s_{45}}{\sigma_{45}}. \end{aligned} \tag{B.2}$$

Poles are from {21}, {23}, {234}, {43}, {45}. We denote contributions from (({21}, {43}), ({21}, {45})), ({23}, {45}), and {234} by I_{5a}, I_{5b}, I_{5c} respectively.

For the first case,

$$\begin{aligned} I_{5a} &= -\frac{1}{2\pi i} \oint_{\Gamma_4} \frac{\sigma_{135}^2 d\sigma_4}{f_{2a} f_{4a} \sigma_{1345}^2}, \\ f_{2a} &= s_{21}, \\ f_{4a} &= \frac{s_{41} + s_{42}}{\sigma_{41}} + \frac{s_{43}}{\sigma_{43}} + \frac{s_{45}}{\sigma_{45}}. \end{aligned} \tag{B.3}$$

In the subsequent σ_4 integration, both the $\sigma_{43} = 0$ and the $\sigma_{45} = 0$ poles contribute to give two terms. The final result is

$$I_{5a} = \frac{1}{s_{21}} \left(\frac{1}{s_{43}} + \frac{1}{s_{45}}\right). \tag{B.4}$$

Similarly, for the second case

$$I_{5b} = \frac{1}{s_{23}}\frac{1}{s_{45}}. \tag{B.5}$$

As for I_{5c}, there are two regions of contribution. Carry out the change of variables $\sigma_{23} = s\sigma'_{23}$ and $\sigma_{34} = s\sigma'_{34}$, from σ_2 and σ_4 to s and some linear combination of σ'_{23} and σ'_{34}. In the vicinity of $s = 0$, the factor σ^2_{12345} becomes $s^4\sigma^2_{135}(\sigma'_{23}\sigma'_{34})^2$, which shows two zeros, the first at $\sigma'_{43} = 0$ and the second at $\sigma'_{23} = 0$. In the region around the first zero, we fix $\sigma'_{23} = 1$, so the integration measure becomes $d\sigma_2 d\sigma_4 = sdsd\sigma'_4$. After the s integration, we end up with

$$\begin{aligned} I_{5c} &= -\frac{1}{2\pi i}\oint_{\Gamma_4}\frac{\sigma^2_{135}d\sigma'_4}{f_{2c}f_{4c}\sigma^2_{135}(\sigma'_{23}\sigma'_{34})^2},\\ f_{2c} &= \frac{s_{23}}{\sigma'_{23}} + \frac{s_{24}}{\sigma'_{24}},\\ f_{4c} &= \frac{s_{42}}{\sigma'_{42}} + \frac{s_{43}}{\sigma'_{43}}. \end{aligned} \tag{B.6}$$

The contour Γ_4 forces $f_{4c} = 0$. With $\sigma'_{23} = 1$, this yields $\sigma'_{43} = s_{43}/(s_{42} + s_{43})$, and hence $f_{2c} = s_{23} + s_{34} + s_{24} = s_{15}$. Now reverse and distort the σ'_4 contour to surround the pole at $\sigma'_{43} = 0$. In this way we get

$$I^{1st}_{5c} = \frac{1}{s_{15}s_{43}}. \tag{B.7}$$

Contribution from the second region is obtained by $2 \leftrightarrow 4$

$$I^{2nd}_{5c} = \frac{1}{s_{15}s_{23}}. \tag{B.8}$$

Thus $I_5 = I_{5a} + I_{5b} + I_{5c}$ consists of 5 terms, corresponding to five Feynman diagrams.

Appendix C. Off-shell scattering equations for $n = 5, 6$ massive scalars

In the following, we are going to give two examples to illustrate how to construct the scattering equations which will give the same amplitudes as from field theory. There, the massive scalar propagators are $\frac{1}{s_{i_1 i_2\cdots i_p} - m^2}$, $2 \le p \le n-2$, instead of $\frac{1}{s_{i_1 i_2\cdots i_p}}$ as in the massless case. Hence, the obvious rule is first to replace $s_{i_1 i_2\cdots i_p}$ with $s_{i_1 i_2\cdots i_p} - m^2$ in the scattering equations of Section 2. Consider the case $n = 5$, where now eq. (A.2) becomes

$$\begin{aligned} \hat{f}_1 &= \frac{s_{12}-m^2}{\sigma_{12}} + \frac{s_{13}-m^2+x'_{13}}{\sigma_{13}} + \frac{s_{14}-m^2+x'_{14}}{\sigma_{14}} + \frac{s_{15}-m^2}{\sigma_{15}},\\ \hat{f}_2 &= \frac{s_{21}-m^2}{\sigma_{21}} + \frac{s_{23}-m^2}{\sigma_{23}} + \frac{s_{24}-m^2+x'_{24}}{\sigma_{24}} + \frac{s_{25}-m^2+x'_{25}}{\sigma_{25}},\\ \hat{f}_3 &= \frac{s_{31}-m^2+x'_{31}}{\sigma_{31}} + \frac{s_{32}-m^2}{\sigma_{32}} + \frac{s_{34}-m^2}{\sigma_{34}} + \frac{s_{35}-m^2+x'_{35}}{\sigma_{35}},\\ \hat{f}_4 &= \frac{s_{41}-m^2+x'_{41}}{\sigma_{41}} + \frac{s_{42}-m^2+x'_{42}}{\sigma_{42}} + \frac{s_{43}-m^2}{\sigma_{43}} + \frac{s_{45}-m^2}{\sigma_{45}},\\ \hat{f}_5 &= \frac{s_{51}-m^2}{\sigma_{51}} + \frac{s_{52}-m^2+x'_{52}}{\sigma_{52}} + \frac{s_{53}-m^2+x'_{53}}{\sigma_{53}} + \frac{s_{54}-m^2}{\sigma_{54}}, \end{aligned} \tag{C.1}$$

where we have written in the notation of Section 3 $x_{ij} = x'_{ij} - m^2$. Modular covariance of $\hat{f}_1 = 0$ gives the condition

$$(s_{12} - m^2) + (s_{13} - m^2 + x'_{13}) + (s_{14} - m^2 + x'_{14}) + (s_{15} - m^2) = 0. \tag{C.2}$$

Using momentum conservation, we have

$$s_{12} + s_{13} + s_{14} + s_{15} = \sum_{i=1}^{5} k_i^2 + k_1^2, \tag{C.3}$$

which results in

$$x'_{13} + x'_{14} = -(\sum_{i=1}^{5} k_i^2 + k_1^2) + 4m^2. \tag{C.4}$$

Similar consideration gives

$$\begin{aligned}
x'_{24} + x'_{25} &= -(\sum_{i=1}^{5} k_i^2 + k_2^2) + 4m^2, \\
x'_{13} + x'_{35} &= -(\sum_{i=1}^{5} k_i^2 + k_3^2) + 4m^2, \\
x'_{14} + x'_{24} &= -(\sum_{i=1}^{5} k_i^2 + k_4^2) + 4m^2, \\
x'_{25} + x'_{35} &= -(\sum_{i=1}^{5} k_i^2 + k_5^2) + 4m^2,
\end{aligned} \tag{C.5}$$

which lead to the solution

$$\begin{aligned}
x'_{13} &= -(k_1^2 + k_2^2 + k_3^2) + 2m^2, \\
x'_{14} &= -(k_1^2 + k_4^2 + k_5^2) + 2m^2, \\
x'_{24} &= -(k_2^2 + k_3^2 + k_4^2) + 2m^2, \\
x'_{25} &= -(k_1^2 + k_2^2 + k_5^2) + 2m^2, \\
x'_{35} &= -(k_3^2 + k_4^2 + k_5^2) + 2m^2.
\end{aligned} \tag{C.6}$$

These shifts will give us the amplitudes $M^{12345,\ 1ijk5}$, where i, j, k are any permutations of $2, 3, 4$. Particularly, we have

$$\begin{aligned}
M^{12345\ 12345} = &\frac{1}{(s_{15} - m^2)(s_{34} - m^2)} + \frac{1}{(s_{15} - m^2)(s_{23} - m^2)} + \frac{1}{(s_{12} - m^2)(s_{45} - m^2)} \\
&+ \frac{1}{(s_{23} - m^2)(s_{45} - m^2)} + \frac{1}{(s_{12} - m^2)(s_{34} - m^2)}.
\end{aligned} \tag{C.7}$$

Let us take note that in the scattering equations eq. (C.1), the set of invariants which do not require x'_{ij} shifts other than $-m^2$ are $s_{12},\ s_{23},\ s_{34},\ s_{45},\ s_{51}$ and they all appear as propagators in $M^{12345\ 12345}$, whereas the complement set consisting of $s_{13},\ s_{14},\ s_{24},\ s_{25},\ s_{35}$, require non-trivial shifts. This will prove to be true for all n. With this in mind, we now look at $n = 6$. The

invariants which appear as propagators in $M^{123456,1ijklm6}$, where i, j, k, l are permutations of $2, 3, 4, 5$, are $s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, s_{123} = s_{456}, s_{234} = s_{561}, s_{345} = s_{612}$. They come with consecutive indices. The complement to this set is $s_{13}, s_{14}, s_{15}, s_{24}, s_{25}, s_{26}, s_{35}, s_{36}, s_{46}$. Thus the modified scattering equations are

$$\begin{aligned}
\hat{f}_1 &= \frac{s_{12}-m^2}{\sigma_{12}} + \frac{s_{13}-m^2+x'_{13}}{\sigma_{13}} + \frac{s_{14}-m^2+x'_{14}}{\sigma_{14}} + \frac{s_{15}-m^2+x'_{15}}{\sigma_{15}} + \frac{s_{16}-m^2}{\sigma_{16}},\\
\hat{f}_2 &= \frac{s_{21}-m^2}{\sigma_{21}} + \frac{s_{23}-m^2}{\sigma_{23}} + \frac{s_{24}-m^2+x'_{24}}{\sigma_{24}} + \frac{s_{25}-m^2+x'_{25}}{\sigma_{25}} + \frac{s_{26}-m^2+x'_{26}}{\sigma_{26}},\\
\hat{f}_3 &= \frac{s_{31}-m^2+x'_{31}}{\sigma_{31}} + \frac{s_{32}-m^2}{\sigma_{32}} + \frac{s_{34}-m^2}{\sigma_{34}} + \frac{s_{35}-m^2+x'_{35}}{\sigma_{35}} + \frac{s_{36}-m^2+x'_{36}}{\sigma_{36}},\\
\hat{f}_4 &= \frac{s_{41}-m^2+x'_{41}}{\sigma_{41}} + \frac{s_{42}-m^2+x'_{42}}{\sigma_{42}} + \frac{s_{43}-m^2}{\sigma_{43}} + \frac{s_{45}-m^2}{\sigma_{45}} + \frac{s_{46}-m^2+x'_{46}}{\sigma_{46}},\\
\hat{f}_5 &= \frac{s_{51}-m^2+x'_{51}}{\sigma_{51}} + \frac{s_{52}-m^2+x'_{52}}{\sigma_{52}} + \frac{s_{53}-m^2+x'_{53}}{\sigma_{53}} + \frac{s_{54}-m^2}{\sigma_{54}} + \frac{s_{56}-m^2}{\sigma_{56}},\\
\hat{f}_6 &= \frac{s_{61}-m^2}{\sigma_{61}} + \frac{s_{62}-m^2+x'_{62}}{\sigma_{62}} + \frac{s_{63}-m^2+x'_{63}}{\sigma_{63}} + \frac{s_{64}-m^2+x'_{64}}{\sigma_{64}} + \frac{s_{65}-m^2}{\sigma_{65}}.
\end{aligned} \tag{C.8}$$

The requirement of Möbius covariance leads to

$$\begin{aligned}
x'_{13}+x'_{14}+x'_{15} &= -(\sum_{j=1}^{6} k_i^2 + 2k_1^2) + 5m^2,\\
x'_{24}+x'_{25}+x'_{26} &= -(\sum_{j=1}^{6} k_i^2 + 2k_2^2) + 5m^2,\\
x'_{13}+x'_{35}+x'_{36} &= -(\sum_{j=1}^{6} k_i^2 + 2k_3^2) + 5m^2,\\
x'_{14}+x'_{24}+x'_{46} &= -(\sum_{j=1}^{6} k_i^2 + 2k_4^2) + 5m^2,\\
x'_{15}+x'_{25}+x'_{35} &= -(\sum_{j=1}^{6} k_i^2 + 2k_5^2) + 5m^2,\\
x'_{26}+x'_{36}+x'_{46} &= -(\sum_{j=1}^{6} k_i^2 + 2k_6^2) + 5m^2.
\end{aligned} \tag{C.9}$$

Now consider a propagator which involves three momenta, such as $\frac{1}{s_{123}-m^2}$. We write

$$s_{123} = s_{12} + s_{13} + s_{23} - (k_1^2 + k_2^2 + k_3^2), \tag{C.10}$$

and acknowledge that the induced dependence on the kinematical invariants in the scattering amplitudes due to CHY integrations must be in the form of those combinations $s_{ij} - m^2 + x'_{ij}$ which appear in $\hat{f}_i$ of eq. (C.8). Therefore

$$s_{123} - m^2 = (s_{12} - m^2) + (s_{13} - m^2 + x'_{13}) + (s_{23} - m^2). \tag{C.11}$$

These two equations immediately above give us a consistency condition

$$x'_{13} = -(k_1^2 + k_2^2 + k_3^2) + 2m^2. \tag{C.12}$$

Similar considerations lead to

$$\begin{aligned} &x'_{24} = -(k_2^2 + k_3^2 + k_4^2) + 2m^2, \quad x'_{35} = -(k_3^2 + k_4^2 + k_5^2) + 2m^2, \\ &x'_{46} = -(k_4^2 + k_5^2 + k_6^2) + 2m^2, \quad x'_{15} = -(k_1^2 + k_5^2 + k_6^2) + 2m^2, \\ &x'_{26} = -(k_1^2 + k_2^2 + k_6^2) + 2m^2. \end{aligned} \tag{C.13}$$

Note that we have twelve equations in eqs. (C.9), (C.12), (C.13) for nine x'_{ij}. In other words, there are three consistency checks. The rest of the solution are

$$x'_{14} = -(k_1^2 + k_4^2) + m^2, \quad x'_{25} = -(k_2^2 + k_5^2) + m^2, \quad x'_{36} = -(k_3^2 + k_6^2) + m^2. \tag{C.14}$$

When all the particles are on-shell ($k_i^2 = m^2,\ i = 1, 2, \cdots n$), we find that all the non-trivial $x'_{ij} = -m^2$ in eqs. (C.6), (C.12), (C.13), (C.14). This result was obtained earlier by Dolan and Goddard [20] for $n = 4, 5$. We must reiterate that the scattering equations with the general values given above for x'_{ij} should be used only for $M^{12\cdots,n\ 1i_2i_3\cdots i_{n-1}n}$, where $i_2i_3\cdots i_{n-1}$ are permutations of $2, 3, \cdots, n-1$. The other color amplitudes should be obtained from these by appropriate relabeling of indices, or by following the rule we gave after eq. (49).

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