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Products of current operators in the exact renormalization group formalism

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Given a Wilson action invariant under global chiral transformations, we can construct current composite operators in terms of the Wilson action. The short-distance singularities in the multiple products of the current operators are taken care of by the exact renormalization group. The Ward–Takahashi identity is compatible with the finite momentum cutoff of the Wilson action. The exact renormalization group and the Ward–Takahashi identity together determine the products. As a concrete example, we study the Gaussian fixed-point Wilson action of the chiral fermions to construct the products of current operators.

Subject Index B31, B32

1. Introduction

It is a principle of quantum field theory that the invariance of a theory under a continuous transformation implies the conservation of a current. When a theory is expressed by a Wilson action with a finite momentum cutoff, the principle holds for the Wilson action. In Ref. [1] an energy–momentum tensor was constructed from the invariance of the Wilson action under translations and rotations. In this paper we would like to consider the Wilson action of chiral fermions with global flavor symmetry to construct multiple products of the conserved current operator.

To build the Wilson action, we use the exact renormalization group (ERG) formalism (see, for example, Refs. [2–5] and references therein). The Wilson action satisfies a well-defined differential equation under the continuous change of scale. We adopt a convention that each time we integrate more of the high-momentum fluctuations, we introduce a change of scale to restore the same cutoff function. The continuum limit corresponds to a trajectory parametrized by a logarithmic scale parameter t so that a fixed point is reached in the limit $t \to -\infty$.

The Wilson action of a theory in the continuum limit has all the short-distance physics incorporated into the vertices of the action. The full theory is obtained by further integration of the fields with momenta below the cutoff. The Wilson action is determined by the ERG differential equation whose solution is parametrized by the relevant variables of the theory.

Composite operators can be considered as infinitesimal changes of the Wilson action, and they also obey well-defined differential equations under the change of logarithmic scale. The general properties of products of composite operators have been discussed in Ref. [6]. We follow and extend the discussions there by considering the multiple products of current operators.

The ERG formalism is good at handling the short-distance singularities via ERG differential equations. Well-defined ERG differential equations admit only the solutions consistent with locality,

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i.e. the vertices of the action and composite operators must be analytic at zero momenta. This is the guiding principle we follow throughout the paper.

Though we consider only chiral fermion fields as dynamical fields, our discussion of current operators is easy to modify in the presence of other dynamical fields, for example in the case of quantum chromodynamics with massless quarks.

Our subject obviously overlaps with the construction of chiral gauge theories using the ERG formalism (see, for example, Ref. [7] and references therein). For example, the derivation of the chiral anomaly using the ERG formalism was done in the context of gauge theory (see, for example, Sect. 9 of Ref. [4] and Ref. [8]). The multiple products of current operators require a much lighter formalism.

The paper is organized as follows. In Sect. 2 we introduce a current operator for a generic Wilson action of chiral fermions under the assumption of global continuous symmetry. In Sect. 3 we introduce multiple products of current operators and derive the ERG equations satisfied by them. By coupling the current with an external gauge field, we construct a composite operator in terms of which we can consider all the products of current operators at once. In Sect. 4 we introduce the Ward–Takahashi (WT) identity for the multiple products of currents. This amounts to the commutation relation of currents. The single-current operator, introduced in Sect. 2, satisfies the WT identity by construction. The corresponding identity for the products is very plausible, but we are unable to derive it solely from the assumption of global continuous symmetry. We introduce it here as a working hypothesis. The ERG differential equation and the WT identity thus introduced are mutually consistent, and they together characterize the products of current operators. In Sect. 5 we discuss the changes to the ERG equation and the WT identity caused by the short-distance singularities of the operator products. In Sect. 6 we consider the products of current operators for the free theory. Though this section is all about one-loop diagrams, the example elucidates the general formalism given in the preceding sections.

Please note that we use the following condensed notation for momentum integrals:

$$\int_p \equiv \int \frac{d^D p}{(2\pi)^D}, \qquad \delta(p) \equiv (2\pi)^D \delta^{(D)}(p). \tag{1}$$

2. Current composite operators

We consider a theory of chiral fermion fields $\psi, \bar{\psi}$ satisfying

$$a_R \psi(p) = \psi(p), \qquad \bar{\psi}(-p) a_L = \bar{\psi}(-p), \tag{2}$$

where

$$a_R \equiv \frac{1+\gamma_5}{2}, \qquad a_L \equiv \frac{1-\gamma_5}{2}. \tag{3}$$

The theory is determined by its Wilson action with a fixed ultraviolet (UV) cutoff. The cutoff is given in terms of a smooth momentum cutoff function $K(p)$, such as e^{-p^2}, that is 1 at $p = 0$ and vanishes as $p \to \infty$. We parametrize the Wilson action by a logarithmic scale parameter t and demand that it obey the ERG differential equation

$$\partial_t e^{S_t[\psi,\bar{\psi}]} = \int_p \left[\left(\frac{\Delta(p)}{K(p)} + \frac{D+1}{2} + p \cdot \partial_p - \gamma_t \right) \bar{\psi}(-p) \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(-p)} e^{S_t} \right.$$

$$+ e^{S_t} \frac{\overleftarrow{\delta}}{\delta\psi(p)} \left(\frac{\Delta(p)}{K(p)} + \frac{D+1}{2} + p\cdot\partial_p - \gamma_t \right) \psi(p)$$
$$\left. - \mathrm{Tr}\, \frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)} e^{S_t} \frac{\overleftarrow{\delta}}{\delta\psi(p)} a_R \frac{\Delta(p) - 2\gamma_t K(p)\,(1-K(p))}{\not p} \right], \tag{4}$$

where

$$\Delta(p) \equiv -p\cdot\partial_p K(p), \tag{5}$$

γ_t is an anomalous dimension of the chiral fermion field, the trace is for both spinor and flavor indices, and the minus in front of the trace is due to the Fermi statistics.

We assume that the Wilson action S_t describes a continuum limit; as we take $t \to -\infty$, we obtain a UV fixed point:

$$\lim_{t\to-\infty} S_t = S^*. \tag{6}$$

All the physics beyond the fixed cutoff scale of 1 has been incorporated into the action. By integrating the fluctuations of momenta less than 1, we get full correlation functions of the fields.

We define the correlation functions by

$$\langle\langle \psi(p_1)\cdots\psi(p_n)\bar{\psi}(-q_1)\cdots\bar{\psi}(-q_n)\rangle\rangle_t \equiv \prod_{i=1}^n \frac{1}{K(p_i)K(q_i)}$$
$$\times \left\langle \psi(p_1)\cdots\psi(p_n)\exp\left(-\int_p \frac{\overleftarrow{\delta}}{\delta\psi(p)} K(p) h_F(p) \frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)} \right) \bar{\psi}(-q_1)\cdots\bar{\psi}(-q_n) \right\rangle_{S_t}, \tag{7}$$

where

$$h_F(p) \equiv a_R \frac{1-K(p)}{\not p} \tag{8}$$

is the high-momentum propagator. The correction involving the cutoff function is a technicality typical in the ERG formalism. Thanks to the correction, though, the correlation functions satisfy the simple scaling relation

$$\left\langle\left\langle \psi(p_1 e^{t-t'})\cdots\bar{\psi}(-q_n e^{t-t'}) \right\rangle\right\rangle_t$$
$$= \exp\left(-n(D-1)(t-t') + 2n\int_{t'}^t d\tau\,\gamma_\tau \right) \langle\langle \psi(p_1)\cdots\bar{\psi}(-q_n)\rangle\rangle_{t'}. \tag{9}$$

Another technicality is necessary before we move on to discuss symmetry. A composite operator $\mathcal{O}_t(p)$ is a functional whose correlation functions are defined by

$$\langle\langle \mathcal{O}_t(p)\,\psi(p_1)\cdots\bar{\psi}(-q_n)\rangle\rangle_t \equiv \prod_{i=1}^n \frac{1}{K(p_i)K(q_i)}$$
$$\equiv \left\langle \mathcal{O}_t(p) \left(\psi(p_1)\cdots\exp\left(-\int_q \frac{\overleftarrow{\delta}}{\delta\psi(q)} K(q) h_F(q) \frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-q)} \right)\cdots\bar{\psi}(-q_n) \right) \right\rangle_{S_t}, \tag{10}$$

where the exponentiated differential operator does not act on $\mathcal{O}_t(p)$. We define $\mathcal{O}_t(p)$ so that its correlation functions satisfy the scaling relation

$$\begin{aligned}&\left\langle\!\left\langle \mathcal{O}_t(pe^{t-t'})\psi(p_1e^{t-t'})\cdots\bar{\psi}(-q_ne^{t-t'})\right\rangle\!\right\rangle_t\\&= e^{-y(t-t')}\exp\left(-n(D-1)(t-t')+n\int_{t'}^t d\tau\,\gamma_\tau\right)\langle\!\langle \mathcal{O}_{t'}(p)\psi(p_1)\cdots\bar{\psi}(-q_n)\rangle\!\rangle_{t'}\,. \end{aligned}\tag{11}$$

For simplicity we have taken $-y$, the scale dimension of $\mathcal{O}_t(p)$, independent of t. For Eq. (11) to be valid, $\mathcal{O}_t(p)$ must satisfy the ERG differential equation

$$\left(\partial_t + y + p\cdot\partial_p - \mathcal{D}_t\right)\mathcal{O}_t(p) = 0, \tag{12}$$

where $\mathcal{D}_t$, acting on functionals, is defined by

$$\begin{aligned}\mathcal{D}_t\mathcal{O} \equiv \int_q \Bigg[&\left(\frac{\Delta(q)}{K(q)}+\frac{D+1}{2}-\gamma_t+q\cdot\partial_q\right)\bar{\psi}(-q)\cdot\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-q)}\mathcal{O}\\&+\mathcal{O}\frac{\overleftarrow{\delta}}{\delta\psi(q)}\left(\frac{\Delta(q)}{K(q)}+\frac{D+1}{2}-\gamma_t+q\cdot\partial_q\right)\psi(q)\\&+S_t\frac{\overleftarrow{\delta}}{\delta\psi(q)}\left(a_R\frac{\Delta(q)}{\not{q}}-2\gamma_tK(q)h_F(q)\right)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-q)}\mathcal{O}\\&+\mathcal{O}\frac{\overleftarrow{\delta}}{\delta\psi(q)}\left(a_R\frac{\Delta(q)}{\not{q}}-2\gamma_tK(q)h_F(q)\right)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-q)}S_t\\&-\mathrm{Tr}\left(a_R\frac{\Delta(q)}{\not{q}}-2\gamma_tK(q)h_F(q)\right)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-q)}\mathcal{O}\frac{\overleftarrow{\delta}}{\delta\psi(q)}\Bigg],\end{aligned}\tag{13}$$

The simplest example of a composite operator is

$$\Psi(p) \equiv \frac{1}{K(p)}\left[\psi(p)+h_F(p)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_t\right], \tag{14a}$$

$$\bar{\Psi}(-p) \equiv \frac{1}{K(p)}\left[\bar{\psi}(-p)+S_t\frac{\overleftarrow{\delta}}{\delta\psi(p)}h_F(p)\right]. \tag{14b}$$

Though they are composite operators, they have the same correlation functions as the elementary fields $\psi(p), \bar{\psi}(-p)$:

$$\langle\!\langle \Psi(p_1)\psi(p_2)\cdots\bar{\psi}(-q_n)\rangle\!\rangle_t = \langle\!\langle \psi(p_1)\cdots\bar{\psi}(-q_n)\rangle\!\rangle_t\,, \tag{15}$$

$$\langle\!\langle \psi(p_1)\cdots\bar{\psi}(-q_{n-1})\bar{\Psi}(-q_n)\rangle\!\rangle_t = \langle\!\langle \psi(p_1)\cdots\bar{\psi}(-q_n)\rangle\!\rangle_t\,. \tag{16}$$

We are now ready to discuss symmetry. We assume that the correlation functions have global symmetry:

$$\left\langle\!\left\langle U\psi(p_1)\cdots U\psi(p_n)\,\bar{\psi}(-q_1)U^\dagger\cdots\bar{\psi}(-q_n)U^\dagger\right\rangle\!\right\rangle_t = \langle\!\langle \psi(p_1)\cdots\psi(p_n)\bar{\psi}(-q_1)\cdots\bar{\psi}(-q_n)\rangle\!\rangle_t\,, \tag{17}$$

where U is an arbitrary unitary matrix that acts on the flavor indices of ψ and $\bar{\psi}$. (U may be a U(N) matrix if we have N flavors.) For infinitesimal transformations we obtain

$$\sum_{i=1}^{n}\Big(-\left\langle\!\left\langle\psi(p_1)\cdots T^a\psi(p_i)\cdots\psi(p_n)\bar{\psi}(-q_1)\cdots\bar{\psi}(-q_n)\right\rangle\!\right\rangle_t \\ +\left\langle\!\left\langle\psi(p_1)\cdots\psi(p_n)\bar{\psi}(-q_1)\cdots\bar{\psi}(-q_i)T^a\cdots\bar{\psi}(-q_n)\right\rangle\!\right\rangle_t\Big)=0, \tag{18}$$

where T^a are Hermitian matrices normalized by

$$\mathrm{Tr}\, T^a T^b = \delta^{ab} \tag{19}$$

and satisfying the commutation relation

$$\left[T^a, T^b\right] = i\sum_c f^{abc}T^c. \tag{20}$$

(We will omit the summation symbol for the repeated indices c from now on.)

To express Eq. (17) as an operator equation, we introduce an equation-of-motion composite operator by

$$\mathcal{E}^a(p) \equiv e^{-S_t}\int_q K(q)\mathrm{Tr}\left[\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-q)}\left(\bar{\Psi}(-q+p)T^a e^{S_t}\right)-\left(e^{S_t}T^a\Psi(q+p)\right)\frac{\overleftarrow{\delta}}{\delta\psi(q)}\right], \tag{21}$$

where $\Psi, \bar{\Psi}$ are defined by Eq. (14). $\mathcal{E}^a$ is a total derivative of the exponentiated Wilson action, and it has correlation functions

$$\left\langle\!\left\langle\mathcal{E}^a(p)\,\psi(p_1)\cdots\psi(p_n)\bar{\psi}(-q_1)\cdots\bar{\psi}(-q_n)\right\rangle\!\right\rangle_t \\ =\sum_{i=1}^{n}\Big[-\left\langle\!\left\langle\psi(p_1)\cdots T^a\psi(p+p_i)\cdots\psi(p_n)\bar{\psi}(-q_1)\cdots\bar{\psi}(-q_n)\right\rangle\!\right\rangle_t \\ +\left\langle\!\left\langle\psi(p_1)\cdots\psi(p_n)\bar{\psi}(-q_1)\cdots\bar{\psi}(p-q_i)T^a\cdots\bar{\psi}(-q_n)\right\rangle\!\right\rangle_t\Big]. \tag{22}$$

The symmetry in Eq. (17) is equivalent to

$$\mathcal{E}^a(p=0)=0. \tag{23}$$

In fact, this is equivalent to what we usually consider as the invariance of the action

$$\int_p\left(\bar{\psi}(-p)T^a\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_t - S_t\frac{\overleftarrow{\delta}}{\delta\psi(p)}T^a\psi(p)\right)=0. \tag{24}$$

In Appendix A we show that this is equivalent to Eq. (23).

Since $\mathcal{E}^a(p)$ is a local operator, it must be proportional to the momentum:

$$\mathcal{E}^a(p) = p_\mu J^a_\mu(p), \tag{25}$$

where the current $J^a_\mu(p)$ must be a local composite operator. Unless there is a local operator $j^a_\mu(p)$ orthogonal to p_μ,

$$p_\mu j^a_\mu(p)=0, \tag{26}$$

Eq. (25) defines the current $J^a_\mu(p)$ unambiguously. Since $\mathcal{E}^a(p)$ has scale dimension 0, $J^a_\mu(p)$ must have scale dimension -1. In coordinate space $J^a_\mu(x) = \int_p e^{ipx} J^a_\mu(p)$ has scale dimension $D-1$.

As an example, let us consider the Gaussian fixed-point theory

$$S_{\mathrm{G}} = -\int_p \frac{1}{K(p)} \bar\psi(-p) \not{p}\, a_R \psi(p), \tag{27}$$

for which

$$\Psi(p) = \psi(p), \qquad \bar\Psi(-p) = \bar\psi(-p). \tag{28}$$

We find

$$\begin{aligned}\mathcal{E}^a(p) &= \int_q K(q) \left[-\bar\psi(-q+p) T^a \frac{\overrightarrow{\delta}}{\delta\bar\psi(-q)} S + S \frac{\overleftarrow{\delta}}{\delta\psi(q)} T^a \psi(q+p) \right] \\ &= \int_q \left(\bar\psi(-q+p) T^a a_R \not{q} \psi(q) - \bar\psi(-q) a_R \not{q} T^a \psi(q+p) \right) \\ &= \int_q \bar\psi(-q) T^a a_R \not{p} \psi(q+p). \end{aligned} \tag{29}$$

This implies that

$$J^a_\mu(p) = \int_q \bar\psi(-q) T^a \gamma_\mu a_R \psi(q+p). \tag{30}$$

3. Products of current operators

We wish to define multiple products of currents. The product of two currents is defined as

$$\left[J^a_\mu(p) J^b_\nu(q) \right] \equiv J^a_\mu(p) J^b_\nu(q) + \mathcal{P}^{ab}_{\mu\nu}(p,q), \tag{31}$$

where $\mathcal{P}$ is a local counterterm necessary to make the product a composite operator; the bare product $J^a_\mu(p) J^b_\nu(q)$ is not a composite operator in the sense introduced in the previous section. $\mathcal{P}$ also takes care of the short-distance singularity occurring when the two currents come close together. For the product to be a composite operator of scale dimension -2, it must satisfy

$$\left(\partial_t + p\cdot\partial_p + q\cdot\partial_q + 2 - \mathcal{D}_t \right) \left[J^a_\mu(p) J^b_\nu(q) \right] = 0, \tag{32}$$

where $\mathcal{D}_t$ is given by Eq. (13). This implies

$$\begin{aligned}&\left(\partial_t + p\cdot\partial_p + q\cdot\partial_q + 2 - \mathcal{D}_t \right) \mathcal{P}^{ab}_{\mu\nu}(p,q) \\ &= \int_r \left[J^a_\mu(p) \frac{\overleftarrow{\delta}}{\delta\psi(r)} \left(a_R \frac{\Delta(r)}{\not{r}} - 2\gamma_t K(r) h_F(r) \right) \frac{\overrightarrow{\delta}}{\delta\bar\psi(-r)} J^b_\nu(q) + \left(J^a_\mu(p) \leftrightarrow J^b_\nu(q) \right) \right]. \end{aligned} \tag{33}$$

Similarly, we define the product of three currents as

$$\begin{aligned}&\left[J^{a_1}_{\mu_1}(p_1) J^{a_2}_{\mu_2}(p_2) J^{a_3}_{\mu_3}(p_3) \right] \equiv J^{a_1}_{\mu_1}(p_1) J^{a_2}_{\mu_2}(p_2) J^{a_3}_{\mu_3}(p_3) \\ &\quad + \mathcal{P}^{a_1a_2}_{\mu_1\mu_2}(p_1,p_2) J^{a_3}_{\mu_3}(p_3) + \mathcal{P}^{a_2a_3}_{\mu_2\mu_3}(p_2,p_3) J^{a_1}_{\mu_1}(p_1) + \mathcal{P}^{a_3a_1}_{\mu_3\mu_1}(p_3,p_1) J^{a_2}_{\mu_2}(p_2) \\ &\quad + \mathcal{P}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3), \end{aligned} \tag{34}$$

and so on for the higher-order products. We note that $\mathcal{P}^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p_1,\ldots,p_n)$ gives the short-distance singularity due to all the n currents coming together simultaneously, and it is proportional to the delta function in momentum space,

$$\mathcal{P}^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p_1,\ldots,p_n) \propto \delta\left(\sum_{i=1}^n p_i\right), \tag{35}$$

unless there is a composite operator of scale dimension $-n$ or less available. (That means scale dimension $D-n$ or less in coordinate space.) The ERG equation for $\mathcal{P}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)$ is given by

$$\begin{aligned}
&\left(\partial_t + \sum_{i=1}^3 p_i\cdot\partial_{p_i} + 3 - \mathcal{D}_t\right)\mathcal{P}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)\\
&= \int_q \left[\mathcal{P}^{a_1a_2}_{\mu_1\mu_2}(p_1,p_2)\frac{\overleftarrow{\delta}}{\delta\psi(q)}\left(a_R\frac{\Delta(q)}{\not q} - 2\gamma_t K(q)h_F(q)\right)\frac{\overrightarrow{\delta}}{\delta\bar\psi(-q)}J^{a_3}_{\mu_3}(p_3)\right.\\
&\quad + J^{a_3}_{\mu_3}(p_3)\frac{\overleftarrow{\delta}}{\delta\psi(q)}\left(a_R\frac{\Delta(q)}{\not q} - 2\gamma_t K(q)h_F(q)\right)\frac{\overrightarrow{\delta}}{\delta\bar\psi(-q)}\mathcal{P}^{a_1a_2}_{\mu_1\mu_2}(p_1,p_2)\\
&\quad \left. + \text{(4 other terms)}\right].
\end{aligned} \tag{36}$$

The ERG equations for the higher-order counterterms are given similarly.

To consider all the local products of current operators simultaneously, we introduce a classical gauge field coupled to the current,

$$W_t[A^a_\mu] \equiv \int_p A^a_\mu(-p)J^a_\mu(p) + \sum_{n=2}^\infty \frac{1}{n!}\int_{p_1,\ldots,p_n} A^{a_1}_{\mu_1}(-p_1)\cdots A^{a_n}_{\mu_n}(-p_n)\,\mathcal{P}^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p_1,\ldots,p_n), \tag{37}$$

so that its exponential

$$e^{W_t[A]} \equiv \sum_{n=0}^\infty \frac{1}{n!}\int_{p_1,\ldots,p_n} A^{a_1}_{\mu_1}(-p_1)\cdots A^{a_n}_{\mu_n}(-p_n)\,\left[J^{a_1}_{\mu_1}(p_1)\cdots J^{a_n}_{\mu_n}(p_n)\right] \tag{38}$$

is a composite operator. We assign the scale dimension $-D+1$ to the source field A^a_μ so that $e^{W_t[A]}$ becomes a composite operator of scale dimension 0, satisfying the ERG equation

$$\left(\partial_t + \int_p \left(-p\cdot\partial_p - D + 1\right)A^a_\mu(p)\cdot\frac{\delta}{\delta A^a_\mu(p)} - \mathcal{D}_t\right)e^{W_t[A]} = 0, \tag{39}$$

where $\mathcal{D}_t$ is defined by Eq. (13).

4. Commutation relation: The Ward–Takahashi identity

We now wish to consider the "commutation relation" of two currents. The quotation marks are there because it needs to be explained. Our commutation relation is an operator equation,

$$p_\mu\left[J^a_\mu(p)J^b_\nu(q)\right] = if^{abc}J^c_\nu(p+q) + \mathcal{E}^a(p)\star J^b_\nu(q), \tag{40}$$

which amounts to the WT identity

$$p_\mu\left\langle\!\!\left\langle\left[J^a_\mu(p)J^b_\nu(q)\right]\psi(p_1)\cdots\bar\psi(-q_n)\right\rangle\!\!\right\rangle_t = if^{abc}\left\langle\!\!\left\langle J^c_\nu(p+q)\psi(p_1)\cdots\bar\psi(-q_n)\right\rangle\!\!\right\rangle_t$$

$$+\sum_{i=1}^{n}\Big(-\Big\langle\!\Big\langle J^b_\nu(q)\psi(p_1)\cdots T^a\psi(p+p_i)\cdots\bar\psi(-q_n)\Big\rangle\!\Big\rangle_t + \Big\langle\!\Big\langle J^b_\nu(q)\psi(p_1)\cdots\bar\psi(p-q_i)T^a\cdots\bar\psi(-q_n)\Big\rangle\!\Big\rangle_t\Big). \tag{41}$$

We wish to explain the above and its generalization to higher-order products in this section.

We define an equation-of-motion composite operator by

$$\mathcal{E}^a(p)\star J^b_\alpha(q) \equiv e^{-S}\int_r K(r)\mathrm{Tr}\left[\frac{\overrightarrow{\delta}}{\delta\bar\psi(-r)}\left(\left[\bar\Psi(-r+p)T^a J^b_\alpha(q)\right]e^{S_t}\right) - \left(e^{S_t}\left[T^a\Psi(q+p)J^b_\alpha(q)\right]\right)\frac{\overleftarrow{\delta}}{\delta\psi(r)}\right], \tag{42}$$

where

$$\left[\bar\Psi(-r)J^b_\alpha(q)\right] \equiv \bar\Psi(-r)J^b_\alpha(q) + J^b_\alpha(q)\frac{\overleftarrow{\delta}}{\delta\psi(r)}h_F(r), \tag{43}$$

$$\left[\Psi(r)J^b_\alpha(q)\right] \equiv \Psi(r)J^b_\alpha(q) + h_F(r)\frac{\overrightarrow{\delta}}{\delta\bar\psi(-r)}J^b_\alpha(q) \tag{44}$$

are the composite operators satisfying

$$\Big\langle\!\Big\langle\psi(p_1)\cdots\bar\psi(-q_{n-1})\left[\bar\Psi(-r)J^b_\alpha(q)\right]\Big\rangle\!\Big\rangle_t = \Big\langle\!\Big\langle J^b_\alpha(q)\psi(p_1)\cdots\bar\psi(-q_{n-1})\bar\psi(-r)\Big\rangle\!\Big\rangle_t, \tag{45}$$

$$\Big\langle\!\Big\langle\left[\Psi(r)J^b_\alpha(q)\right]\psi(p_2)\cdots\bar\psi(-q_n)\Big\rangle\!\Big\rangle_t = \Big\langle\!\Big\langle J^b_\alpha(q)\psi(r)\psi(p_2)\cdots\bar\psi(-q_n)\Big\rangle\!\Big\rangle_t. \tag{46}$$

Hence, we obtain

$$\Big\langle\!\Big\langle\mathcal{E}^a(p)\star J^b_\alpha(q)\,\psi(p_1)\cdots\bar\psi(-q_n)\Big\rangle\!\Big\rangle_t = \sum_{i=1}^{n}\Big(-\Big\langle\!\Big\langle J^b_\nu(q)\psi(p_1)\cdots T^a\psi(p+p_i)\cdots\bar\psi(-q_n)\Big\rangle\!\Big\rangle_t + \Big\langle\!\Big\langle J^b_\nu(q)\psi(p_1)\cdots\bar\psi(p-q_i)T^a\cdots\bar\psi(-q_n)\Big\rangle\!\Big\rangle_t\Big). \tag{47}$$

This gives the second term on the right-hand side of Eq. (41).

Let

$$\mathcal{O}^{ab}_\nu(p+q) \equiv p_\mu\left[J^a_\mu(p)J^b_\nu(q)\right] - \mathcal{E}^a(p)\star J^b_\nu(q). \tag{48}$$

Equation (41) then amounts to

$$\mathcal{O}^{ab}_\nu(p+q) = if^{abc}J^c_\nu(p+q). \tag{49}$$

This equality is plausible but not obvious, and it needs an explanation. We will check this later explicitly for the Gaussian theory, but we have not been able to derive it on the basis of the global symmetry in Eq. (17). Here we satisfy ourselves by checking the consistency of Eq. (49) with Bose symmetry of the current operator, which requires the product

$$p_\mu q_\nu\left[J^a_\mu(p)J^b_\nu(q)\right]$$

to be symmetric under the interchange. The product may depend on which divergence we calculate first. Calculating $p_\mu J^a_\mu(p)$ first, Eq. (48) gives

$$\begin{aligned} p_\mu q_\nu \left[J^a_\mu(p) J^b_\nu(q) \right] &= q_\nu \mathcal{O}^{ab}_\nu(p+q) + \mathcal{E}^a(p) \star q_\nu J^b_\nu(q) \\ &= q_\nu \mathcal{O}^{ab}_\nu(p+q) + \mathcal{E}^a(p) \star \mathcal{E}^b(q). \end{aligned} \tag{50}$$

Calculating $q_\nu J^b_\nu(q)$ first, we obtain

$$\begin{aligned} p_\mu q_\nu \left[J^a_\mu(p) J^b_\nu(q) \right] &= p_\mu \mathcal{O}^{ba}_\mu(p+q) + \mathcal{E}^b(q) \star p_\mu J^a_\mu(p) \\ &= p_\mu \mathcal{O}^{ba}_\mu(p+q) + \mathcal{E}^b(q) \star \mathcal{E}^a(p). \end{aligned} \tag{51}$$

Hence, for consistency, we must find

$$p_\mu \mathcal{O}^{ba}_\mu(p+q) - q_\mu \mathcal{O}^{ab}_\mu(p+q) = \mathcal{E}^a(p) \star \mathcal{E}^b(q) - \mathcal{E}^b(q) \star \mathcal{E}^a(p). \tag{52}$$

To compute the right-hand side, we consider correlation functions:

$$\begin{aligned} &\left\langle\!\left\langle \mathcal{E}^a(p) \star \mathcal{E}^b(q)\, \psi(p_1) \cdots \psi(p_n) \bar{\psi}(-q_1) \cdots \bar{\psi}(-q_n) \right\rangle\!\right\rangle_t \\ &= \sum_{i=1}^n \left[- \left\langle\!\left\langle \mathcal{E}^b(q)\ \cdots T^a \psi(p+p_i) \cdots \right\rangle\!\right\rangle_t + \left\langle\!\left\langle \mathcal{E}^b(q)\ \cdots \bar{\psi}(p-q_i) T^a \cdots \right\rangle\!\right\rangle_t \right] \\ &= \sum_{i=1}^n \Bigg[\left\langle\!\left\langle \cdots T^a T^b \psi(p+q+p_i) \cdots \right\rangle\!\right\rangle_t + \left\langle\!\left\langle \cdots \bar{\psi}(p+q-q_i) T^b T^a \cdots \right\rangle\!\right\rangle_t \\ &\quad - \sum_{j=1}^n \left(\left\langle\!\left\langle \cdots T^a \psi(p+p_i) \cdots \bar{\psi}(q-q_i) T^b \cdots \right\rangle\!\right\rangle_t + \left\langle\!\left\langle \cdots T^b \psi(q+p_i) \cdots \bar{\psi}(p-q_i) T^b \cdots \right\rangle\!\right\rangle_t \right) \\ &\quad + \sum_{j \neq i} \left(\left\langle\!\left\langle \cdots T^a \psi(p+p_i) \cdots T^b \psi(q+p_j) \cdots \right\rangle\!\right\rangle_t + \left\langle\!\left\langle \cdots \bar{\psi}(p-q_i) T^a \cdots \bar{\psi}(q-q_i) T^b \cdots \right\rangle\!\right\rangle_t \right) \Bigg]. \end{aligned} \tag{53}$$

Hence, we obtain

$$\mathcal{E}^a(p) \star \mathcal{E}^b(q) - \mathcal{E}^b(q) \star \mathcal{E}^a(p) = -i f^{abc} \mathcal{E}^c(p+q). \tag{54}$$

Then, the consistency condition in Eq. (52) gives

$$p_\mu \mathcal{O}^{ba}_\mu(p+q) - q_\mu \mathcal{O}^{ab}_\mu(p+q) = -i f^{abc} \mathcal{E}^c(p+q) = (-i f^{abc})(p+q)_\mu J^c_\mu(p+q), \tag{55}$$

which is indeed satisfied by Eq. (49).

We have thus checked at least that Eq. (40) is consistent with the Bose symmetry of the current. We adopt Eq. (40) and its generalization to higher orders as our working hypothesis:

$$\begin{aligned} p_\mu \left[J^a_\mu(p) J^{a_1}_{\mu_1}(p_1) \cdots J^{a_k}_{\mu_k}(p_k) \right] &= \sum_{i=1}^k i f^{a a_i b} \left[J^{a_1}_{\mu_1}(p_1) \cdots J^b_{\mu_i}(p+p_i) \cdots J^{a_k}_{\mu_k}(p_k) \right] \\ &\quad + \mathcal{E}^a(p) \star \left[J^{a_1}_{\mu_1}(p_1) \cdots J^{a_k}_{\mu_k}(p_k) \right]. \end{aligned} \tag{56}$$

For the correlation functions, this gives

$$p_\mu \left\langle\!\left\langle \left[J^a_\mu(p) J^{a_1}_{\mu_1}(p_1)\cdots J^{a_k}_{\mu_k}(p_k)\right] \psi(q_1)\cdots\psi(q_n)\bar\psi(-r_1)\cdots\bar\psi(-r_n)\right\rangle\!\right\rangle_t$$
$$= \sum_{i=1}^k i f^{a a_i b} \left\langle\!\left\langle \left[J^{a_1}_{\mu_1}(p_1)\cdots J^b_{\mu_i}(p_i+p)\cdots J^{a_k}_{\mu_k}(p_k)\right] \psi(q_1)\cdots\psi(q_n)\bar\psi(-r_1)\cdots\bar\psi(-r_n)\right\rangle\!\right\rangle_t$$
$$+ \sum_{j=1}^n \Big\{ - \left\langle\!\left\langle \left[J^{a_1}_{\mu_1}(p_1)\cdots J^{a_k}_{\mu_k}(p_k)\right]\cdots T^a\psi(q_j+p)\cdots\right\rangle\!\right\rangle_t$$
$$+ \left\langle\!\left\langle \left[J^{a_1}_{\mu_1}(p_1)\cdots J^{a_k}_{\mu_k}(p_k)\right]\cdots\bar\psi(-r_j+p)T^a\cdots\right\rangle\!\right\rangle_t \Big\}. \tag{57}$$

The WT identity in Eq. (56) we just introduced is compactly expressed in terms of the composite operator $e^{W_t[A]}$ as

$$\int_q \left(p_\mu \delta_{ab}\delta(p-q) + i f^{acb} A^c_\mu(-q+p)\right) \frac{\delta}{\delta A^b_\mu(-q)} e^{W_t[A]} = \mathcal{E}^a(p) \star e^{W_t[A]}. \tag{58}$$

Expanding this in powers of the external source A, we can easily check the equivalence to Eq. (56). Multiplying an infinitesimal $\epsilon^a(-p)$ and integrating over p, we can rewrite this as

$$\delta_\epsilon e^{W_t[A]} \equiv e^{W_t[A^\epsilon]} - e^{W_t[A]} = \int_p \epsilon^a(-p)\mathcal{E}^a(p) \star e^{W_t[A]}, \tag{59}$$

where

$$\left(A^\epsilon\right)^a_\mu(-p) \equiv A^a_\mu(-p) + p_\mu \epsilon^a(-p) + i f^{abc} \int_q A^b_\mu(q-p)\epsilon^c(-q) \tag{60}$$

is an infinitesimal gauge transformation.

5. Corrections to the ERG equation and the WT identity

We have identified two important properties of $e^{W_t[A]}$. One is the ERG differential equation, Eq. (39), and the other is the gauge invariance, Eq. (59). Both may receive corrections due to short-distance singularities. Since the nature of singularities depends on the space dimension D, we specify $D = 4$ in the following discussion.

We first consider possible corrections to the ERG equation. The product of n current operators has scale dimension $-n$, and it can mix with operators of the same scale dimension. As for the mixing with the delta function $\delta\left(\sum_i p_i\right)$, we only need to consider

$$\left[J^a_\alpha(p_1)J^b_\beta(p_2)\right], \qquad \left[J^a_\alpha(p_1)J^b_\beta(p_2)J^c_\gamma(p_3)\right], \qquad \left[J^a_\alpha(p_1)J^b_\beta(p_2)J^c_\gamma(p_3)J^d_\delta(p_4)\right],$$

which mix with the delta function $\delta(\sum_i p_i)$ with appropriate powers (quadratic, linear, none) of momenta. This gives a new ERG differential equation:

$$\left(\partial_t + \int_p \left(-p\cdot\partial_p - D + 1\right) A^a_\mu(p)\cdot \frac{\delta}{\delta A^a_\mu(p)} - \mathcal{D}_t\right) e^{W_t[A]} = \int d^D x\, f(t; A(x))\, e^{W_t[A]} \tag{61}$$

where f is a linear combination of the products of two A's with two derivatives, three A's with one derivative, and four A's with no derivative. Consistency with Eq. (59) gives the gauge invariance of

f. Hence, we obtain

$$f(t;A) = b(t)\frac{1}{4}\,\mathrm{Tr}\,\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]\right)^2, \tag{62}$$

where

$$A_\mu \equiv T^a A^a_\mu. \tag{63}$$

In fact, the gauge invariance in Eq. (59) itself may also get corrected as

$$\delta_\epsilon e^{W_t[A]} \equiv e^{W_t[A^\epsilon]} - e^{W_t[A]} = \int_p \epsilon^a(-p)\left(\mathcal{E}^a(p) \star + F^a(p;A)\right) e^{W_t[A]}, \tag{64}$$

where $F^a(p;A)$ is a polynomial of A with scale dimension -4. This is the familiar chiral anomaly [9,10]. Please note that we have used gauge invariance to derive Eqs. (61) and (62). For Eq. (64) to be consistent with Eq. (61), $F^a(p;A)$ must be independent of t, i.e. the anomaly must be scale independent. In other words the t-dependence of $W_t[A]$ is still gauge invariant, as is given by Eq. (61).

The algebraic structure of the anomaly is well known [11]. For completeness, let us derive it using the ERG formalism. By definition of δ_ϵ, we must obtain

$$\left(\delta_\eta\delta_\epsilon - \delta_\epsilon\delta_\eta\right) e^{W_t[A]} = \delta_{[\eta,\epsilon]} e^{W_t[A]}, \tag{65}$$

where

$$[\eta, \epsilon] = \eta^a \epsilon^b \left[T^a, T^b\right] = i f^{abc} \eta^a \epsilon^b T^c. \tag{66}$$

Using Eq. (64) twice, we obtain

$$\begin{aligned}\left(\delta_\epsilon\delta_\eta - \delta_\eta\delta_\epsilon\right) e^{W_t[A]} &= \int_p \epsilon^a(-p) \int_q \eta^b(-q)(-i) f^{abc} \mathcal{E}^c(p+q) \star e^{W_t[A]} \\ &\quad + \int_p \left(\epsilon^a(-p)\delta_\eta F^a(p;A) - \eta^a(-p)\delta_\epsilon F^a(p;A)\right) e^{W_t[A]},\end{aligned} \tag{67}$$

where we have used Eq. (54). Hence, Eq. (65) gives the desired algebraic constraint:

$$\int_p \left(\epsilon^a(-p)\delta_\eta F^a(p;A) - \eta^a(-p)\delta_\epsilon F^a(p;A)\right) = -i f^{abc} \int_p \epsilon^a(-p)\eta^b(-q) F^c(p+q;A). \tag{68}$$

A well-known nontrivial solution to this is given by [12]

$$\int_p \epsilon^a(-p) F^a(p) = \mathrm{const} \times \epsilon_{\alpha\beta\gamma\delta} \int d^4x\, \mathrm{Tr}\, \partial_\alpha \epsilon \cdot \left(A_\beta \partial_\gamma A_\delta + \frac{1}{2i} A_\beta A_\gamma A_\delta\right). \tag{69}$$

(A trivial solution is δ_ϵ of a polynomial of A.)

Concluding this section, we have explained that the ERG equation for $e^{W_t[A]}$ can be modified to Eqs. (61) and (62), and that the WT identity can get an anomaly Eq. (64) where F^a is given by Eq. (69). Differentiating these with respect to the source A, we can get the ERG equation and WT identity satisfied by the products of the current operators. Since their expressions are lengthy, we give them in Appendix C.

6. Free theory in $D = 4$

As a concrete example, we construct $W[A]$ for the Gaussian fixed-point theory in $D = 4$:

$$W[A] = \int_p A^a_\mu(-p) J^a_\mu(p) + \sum_{n=2}^{\infty} \frac{1}{n!} \int_{p_1,\ldots,p_n} A^{a_1}_{\mu_1}(-p_1) \cdots A^{a_n}_{\mu_n}(-p_n)\, \mathcal{P}^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p_1,\ldots,p_n). \tag{70}$$

The construction of $e^{W[A]}$ is guided by two equations. One is the ERG differential equation,

$$\begin{aligned}&\left(\int_p \left(-p\cdot\partial_p - D + 1\right) A^a_\mu(p) \cdot \frac{\delta}{\delta A^a_\mu(p)} - \mathcal{D}\right) e^{W[A]}\\ &= \frac{b}{4} \int d^4x \, \mathrm{Tr}\, \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha - i\left[A_\alpha, A_\beta\right]\right)^2,\end{aligned} \tag{71}$$

where b is a constant and $\mathcal{D}$ is defined by

$$\begin{aligned}\mathcal{D}\mathcal{O} \equiv \int_q \Bigg[&\left(\frac{D+1}{2} + q\cdot\partial_q\right) \bar\psi(-q) \cdot \frac{\overrightarrow{\delta}}{\delta\bar\psi(-q)} \mathcal{O} + \mathcal{O} \frac{\overleftarrow{\delta}}{\delta\psi(q)} \left(\frac{D+1}{2} + q\cdot\partial_q\right) \psi(q)\\ &- \mathrm{Tr}\, a_R \frac{\Delta(q)}{\not q} \frac{\overrightarrow{\delta}}{\delta\bar\psi(-q)} \mathcal{O} \frac{\overleftarrow{\delta}}{\delta\psi(q)} \Bigg].\end{aligned} \tag{72}$$

The other is the WT identity with anomaly,

$$\delta_\epsilon e^{W[A]} = \left[\int_p \epsilon^a(-p) \mathcal{E}^a(p) \star + \mathcal{A}\, \epsilon_{\alpha\beta\gamma\delta} \int d^4x \, \mathrm{Tr}\, \partial_\alpha \epsilon \cdot \left(A_\beta \partial_\gamma A_\delta + \frac{1}{2i} A_\beta A_\gamma A_\delta\right)\right] e^{W[A]}, \tag{73}$$

where $\mathcal{A}$ is a constant. Both b and $\mathcal{A}$ are determined as we construct $\mathcal{P}^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p_1,\ldots,p_n)$ from $n = 2$ to higher n; b is determined by locality. Locality implies the analyticity of $\mathcal{P}$'s at zero momenta. We must choose b appropriately to guarantee that Eq. (71) admits a solution satisfying locality. Similarly, the coefficient $\mathcal{A}$ of the chiral anomaly is determined by locality. The solution to Eq. (71) admits a couple of free parameters consistent with locality. We tune them to satisfy Eq. (73) as much as possible. What is left is the anomaly.

At the end of Sect. 2 the current was derived as

$$J^a_\mu(p) = \int_q \bar\psi(-q) T^a \gamma_\mu a_R \psi(q+p). \tag{74}$$

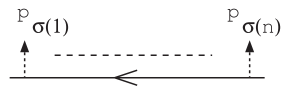
The counterterms $\mathcal{P}$ are quadratic in fields, and we can write them in the form

$$\begin{aligned}\mathcal{P}^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p_1,\ldots,p_n) = &\int_q \bar\psi(-q) c^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p_1,\ldots,p_n; -q, q+p_1+\cdots+p_n) \psi(q+p_1+\cdots+p_n)\\ &+ d^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p_1,\ldots,p_n)\, \delta\left(\sum_{i=1}^n p_i\right),\end{aligned} \tag{75}$$

where

$$\begin{aligned}&c^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p_1,\ldots,p_n; -q, q+p_1+\cdots+p_n)\\ &= \sum_{\sigma\in\mathcal{S}_n} T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}} \gamma_{\mu_{\sigma(1)}} h_F(q+p_{\sigma(1)}) \gamma_{\mu_{\sigma(2)}} h_F(q+p_{\sigma(1)}+p_{\sigma(2)})\end{aligned}$$

$$\cdots \gamma_{\mu_{\sigma(n-1)}} h_F(q + p_{\sigma(1)} + \cdots + p_{\sigma(n-1)}) \gamma_{\mu_{\sigma(n)}} \tag{76}$$

$$= \sum_{\sigma \in \mathcal{S}_n}$$  .

The sum is taken over all the permutations of $1, \ldots, n$.

Similarly, we can write

$$d^{a_1 \cdots a_n}_{\mu_1 \cdots \mu_n}(p_1, \ldots, p_n)$$
$$= \sum_{\sigma \in \mathcal{S}_{n-1}} \mathrm{Tr}\left(T^{a_1} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n-1)}} T^{a_{\sigma(n)}}\right) \cdot d_{\mu_1 \mu_{\sigma(2)} \cdots \mu_{\sigma(n)}}(p_1, p_{\sigma(2)}, \ldots, p_{\sigma(n)}), \tag{77}$$

where the sum is taken over all the permutations of $2, \ldots, n$. The d satisfy the ERG equations

$$\left(\sum_{i=1}^n p_i \cdot \partial_{p_i} + n - 4\right) d_{\alpha_1 \cdots \alpha_n}(p_1, \ldots, p_n)$$
$$= (-) \int_q \mathrm{Tr} f_F(q) \left[\gamma_{\alpha_1} h_F(q + p_1) \gamma_{\alpha_2} \cdots \gamma_{\alpha_{n-1}} h_F(q + p_1 + \cdots + p_{n-1}) \gamma_{\alpha_n}\right.$$
$$\left. + \gamma_{\alpha_2} h_F(q + p_2) \gamma_{\alpha_3} \cdots \gamma_{\alpha_n} h_F(q + p_2 + \cdots + p_n) \gamma_{\alpha_1} + \cdots\right], \tag{78}$$

where

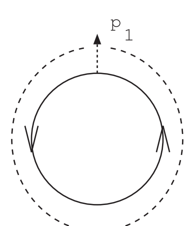
$$h(p) \equiv \frac{1 - K(p)}{p^2}, \tag{79a}$$

$$f(p) \equiv (p \cdot \partial_p + 2) h(p) = \frac{\Delta(p)}{p^2}, \tag{79b}$$

$$f_F(p) \equiv f(p) a_R \not{p} = a_R \frac{\Delta(p)}{\not{p}}. \tag{79c}$$

For $n \geq 5$, the solutions are given by the finite loop integrals:

$$d_{\mu_1 \cdots \mu_n}(p_1, \ldots, p_n)$$
$$= (-) \int_q \mathrm{Tr}\left[\gamma_{\mu_1} h_F(q + p_1) \gamma_{\mu_2} h_F(q + p_1 + p_2) \cdots h_F(q - p_n) \gamma_{\mu_n} h_F(q)\right] \tag{80}$$

$=$  .

For $n = 2, 3, 4$, however, the above loop integrals are UV divergent, and we must define the d's as solutions of Eq. (78). We emphasize that there is no need to introduce an additional UV cutoff to regularize the loop integrals. In fact, we need to modify Eq. (78) first, by adding local terms proportional to the coefficient b so that the solutions become analytic at zero momenta. ERG then determines d_2 up to t-independent terms quadratic in momenta, d_3 up to terms linear in momenta, and d_4 up to a constant. To remove the ambiguities we can resort to the WT identity, which would be given by

$$\begin{aligned} p_{1\alpha} d_{\alpha\alpha_2\cdots\alpha_n}(p_1,\ldots,p_n) &= d_{\alpha_2\cdots\alpha_n}(p_1+p_2,p_3,\cdots,p_n) - d_{\alpha_2\cdots\alpha_n}(p_2,\cdots,p_{n-1},p_n+p_1) \\ &+ \int_q K(q)\mathrm{Tr}\,\big[h_F(q-p_1)\gamma_{\alpha_2}h_F(q+p_2)\cdots h_F(q+p_2+\cdots+p_{n-1})\gamma_{\alpha_n} \\ &\quad - h_F(q+p_1)\gamma_{\alpha_2}h_F(q+p_1+p_2)\cdots h_F(q+p_1+\cdots+p_{n-1})\gamma_{\alpha_n}\big] \end{aligned} \tag{81}$$

if there were no anomaly. This is satisfied by Eq. (80) for $n \geq 5$, but is corrected for $n = 3, 4$ by the anomaly, proportional to $\mathcal{A}$. We can obtain $\mathcal{A}$ by expanding the WT identity in powers of small momenta. This is a straightforward calculation.

In the following we sketch the calculation of $d_{\alpha_1\cdots\alpha_n}$ for $n = 2, 3, 4$. The case $n = 2$ is sufficient to determine the coefficient b, but we need the case $n = 3$ to determine $\mathcal{A}$. We calculate the case $n = 4$ for completeness and to check our formalism. The essential steps are expansions of the d's in small momenta. The calculations are all straightforward, and thanks to the presence of a finite cutoff there is no hidden subtlety. Perhaps we could have condensed this section into a smaller number of pages, but we have decided to give all the details for the reader unfamiliar with calculations with cutoff functions. The more experienced reader may skip what seems trivial or redundant.

6.1. *Product of two, $n = 2$*

$d^{ab}_{\alpha\beta}(p_1,p_2) = \delta_{ab} d_{\alpha\beta}(p,-p)$ satisfies the ERG equation,

$$\begin{aligned} &\left(p\cdot\partial_p - 2\right) d_{\alpha\beta}(p,-p) \\ &= (-)\int_q \mathrm{Tr} f_F(q)\left(\gamma_\alpha h_F(q+p)\gamma_\beta + \gamma_\beta h_F(q-p)\gamma_\alpha\right) + b(p^2\delta_{\alpha\beta} - p_\alpha p_\beta), \end{aligned} \tag{82}$$

and the Ward identity,

$$p_\alpha d_{\alpha\beta}(p,-p) = \int_q K(q)\mathrm{Tr}\,\left[h_F(q-p)\gamma_\beta - h_F(q+p)\gamma_\beta\right]. \tag{83}$$

The analyticity of $d_{\alpha\beta}(p,-p)$ at $p = 0$ demands that the right-hand side of Eq. (82) be free of quadratic terms in p. (If there were any, we would obtain a nonlocal $p^2 \ln p$ dependence.) To expand the integral on the right-hand side of Eq. (82) in powers of p, we use

$$\mathrm{Tr}\, a\!\!\!/ b\!\!\!/ c\!\!\!/ d\!\!\!/ = 2\left[(ab)(cd) + (ad)(bc) - (ac)(bd) + \epsilon_{\alpha\beta\gamma\delta} a_\alpha b_\beta c_\gamma d_\delta\right], \tag{84}$$

where $\epsilon_{1234} = 1$, and

$$h(q+p) = h(q) + (2(qp)+p^2)h'(q) + \frac{1}{2}(2(qp))^2 h''(q) + \mathrm{O}(p^3), \tag{85}$$

where $h'(q) \equiv \frac{d}{dq^2}h(q)$, etc. We obtain

$$\begin{aligned} &(-)\int_q \mathrm{Tr} f_F(q)\left(\gamma_\alpha h_F(q+p)\gamma_\beta + \gamma_\beta h_F(q-p)\gamma_\alpha\right) \\ &\overset{p\to 0}{\longrightarrow} 2\delta_{\alpha\beta}\int_q f(q)h(q)q^2 - b_2(p^2\delta_{\alpha\beta} - p_\alpha p_\beta), \end{aligned} \tag{86}$$

where

$$b_2 \equiv -4 \int_q f(q) \left(q^2 h'(q) + \frac{1}{3} q^4 h''(q) \right) = \frac{1}{(4\pi)^2} \frac{4}{3}. \tag{87}$$

The integrand is a total derivative, and the value of the integral is independent of the choice of the cutoff function $K(p)$. (See Appendix B.3 for the calculation.)

Hence, with the choice

$$b = b_2 = \frac{1}{(4\pi)^2} \frac{4}{3} \tag{88}$$

the general solution of Eq. (82) is given by

$$\begin{aligned} d_{\alpha\beta}(p, -p) = & -\delta_{\alpha\beta} \int_q f(q) h(q) q^2 \\ & + \int_{-\infty}^0 dt\, e^{-2t} \Big[(-) \int_q \mathrm{Tr}\, f_F(q) \left(\gamma_\alpha h_F(q + pe^t) \gamma_\beta + \gamma_\beta h_F(q - pe^t) \gamma_\alpha \right) \\ & \quad + \int_q \mathrm{Tr}\, f_F(q) \left(\gamma_\alpha h_F(q) \gamma_\beta + (\alpha \leftrightarrow \beta) \right) + 4b \left(p^2 \delta_{\alpha\beta} - p_\alpha p_\beta \right) e^{2t} \Big] \\ & + A p^2 \delta_{\alpha\beta} + B \left(p_\alpha p_\beta - p^2 \delta_{\alpha\beta} \right), \end{aligned} \tag{89}$$

where A and B are free parameters. The subtractions make the integrand of order e^{2t} as $t \to -\infty$, and the integral is convergent.

We can fix A using the WT identity in Eq. (83). First, note that

$$p_\alpha d_{\alpha\beta}(p, -p) \xrightarrow{p \to 0} -p_\alpha d_{\alpha\beta}(0, 0) + A p^2 p_\beta. \tag{90}$$

To determine A we compute the right-hand side of Eq. (83):

$$\begin{aligned} & \int_q K(q) \mathrm{Tr} \left(h_F(q - p) \gamma_\beta - h_F(q + p) \gamma_\beta \right) = -2 \int_q K(q) h(q + p) \mathrm{Tr} \left(\not{q} + \not{p} \right) \gamma_\beta a_R \\ & \xrightarrow{p \to 0} p_\beta (-4) \int_q K(q) \left(h(q) + \frac{1}{2} q^2 h'(q) \right) \\ & \quad + p^2 p_\beta (-4) \int_q K(q) \left\{ h'(q) + q^2 h''(q) + \frac{1}{6} q^4 h'''(q) \right\}. \end{aligned} \tag{91}$$

Consistency with Eq. (90) demands

$$\int_q f(q) h(q) q^2 = \int_q K(q) \left(4h(q) + 2q^2 h'(q) \right) \tag{92}$$

and

$$A = -4 \int_q K(q) \left\{ h'(q) + q^2 h''(q) + \frac{1}{6} q^4 h'''(q) \right\}. \tag{93}$$

The first equation must hold since the WT identity is an operator equation consistent with ERG; we verify it explicitly in Appendix B.5. In Appendix B.4 we also compute

$$A = \frac{1}{(4\pi)^2}\frac{1}{3}. \tag{94}$$

B is left arbitrary.

Let us stop here to examine the asymptotic behavior of $d_{\alpha\beta}(p,-p)$ for large p. In principle we could obtain the asymptotic behavior using the solution in Eq. (89). Instead, it is easier to go back to Eqs. (82) and (83), which give

$$\left(p\cdot\partial_p - 2\right) d_{\alpha\beta}(p,-p) \overset{p\to\infty}{\longrightarrow} b\left(p^2\delta_{\alpha\beta} - p_\alpha p_\beta\right), \tag{95}$$

$$p_\alpha d_{\alpha\beta}(p,-p) \overset{p\to\infty}{\longrightarrow} 0. \tag{96}$$

Hence, we obtain the asymptotic behavior

$$d_{\alpha\beta}(pe^t,-pe^t) \overset{t\to+\infty}{\longrightarrow} b\,t\,e^{2t}\left(p^2\delta_{\alpha\beta} - p_\alpha p_\beta\right), \tag{97}$$

determined by the constant b. Using this, we can construct the continuum limit as [13]

$$D_{\alpha\beta}(p,-p) \equiv \lim_{t\to\infty} e^{-2t}\left(d_{\alpha\beta}(pe^t,-pe^t) - bte^{2t}\left(p^2\delta_{\alpha\beta} - p_\alpha p_\beta\right)\right). \tag{98}$$

This satisfies

$$\left(p\cdot\partial_p - 2\right) D_{\alpha\beta}(p,-p) = b\left(p^2\delta_{\alpha\beta} - p_\alpha p_\beta\right). \tag{99}$$

Since $D_{\alpha\beta}$ depends on the constant B, we can rewrite this as

$$\left(p\cdot\partial_p - 2 + b\partial_B\right) D_{\alpha\beta}(p,-p) = 0. \tag{100}$$

$D_{\alpha\beta}(p,-p)$ is also transverse:

$$p_\alpha D_{\alpha\beta}(p,-p) = 0. \tag{101}$$

The two-point function of the current is now obtained as

$$\left\langle\!\left\langle J^a_\alpha(p) J^b_\beta(q)\right\rangle\!\right\rangle_B = \delta^{ab}\delta(p+q)\, D_{\alpha\beta}(p,-p), \tag{102}$$

which is transverse, and satisfies the scaling relation

$$\left(p\cdot\partial_p + q\cdot\partial_q + 2 + b\partial_B\right)\left\langle\!\left\langle J^a_\alpha(p) J^b_\beta(q)\right\rangle\!\right\rangle_B = 0. \tag{103}$$

6.2. *Product of three, $n = 3$*

$d_{\alpha\beta\gamma}(p_1,p_2,p_3)$ satisfies the ERG equation

$$\left(\sum_{i=1}^{3} p_i\cdot\partial_{p_i} - 1\right) d_{\alpha\beta\gamma}(p_1,p_2,p_3)$$
$$= (-)\int_q \mathrm{Tr}\, f_F(q)\left[\gamma_\alpha h_F(q+p_1)\gamma_\beta h_F(q+p_1+p_2)\gamma_\gamma\right.$$

$$+\gamma_\beta h_F(q+p_2)\gamma_\gamma h_F(q+p_2+p_3)\gamma_\alpha + \gamma_\gamma h_F(q+p_3)\gamma_\alpha h_F(q+p_3+p_1)\gamma_\beta\Big]$$
$$+ b\left[\delta_{\alpha\beta}(p_1-p_2)_\gamma + \delta_{\beta\gamma}(p_2-p_3)_\alpha + \delta_{\gamma\alpha}(p_3-p_1)_\beta\right], \qquad (104)$$

where b is given by Eq. (88), and the WT identity

$$p_{1\alpha} d_{\alpha\beta\gamma}(p_1,p_2,p_3) = d_{\beta\gamma}(p_1+p_2,p_3) - d_{\beta\gamma}(p_2,p_3+p_4)$$
$$+ \int_q K(q)\mathrm{Tr}\left[h_F(q-p_1)\gamma_\beta h_F(q+p_2)\gamma_\gamma - h_F(q+p_1)\gamma_\beta h_F(q+p_1+p_2)\gamma_\gamma\right]$$
$$- \frac{1}{2}\mathcal{A}\epsilon_{\alpha\beta\gamma\delta} p_{1\alpha}(p_2-p_3)_\delta, \qquad (105)$$

where $\mathcal{A}$ is to be determined.

Analyticity of $d_{\alpha\beta\gamma}$ at $p_i = 0$ requires the absence of terms linear in p_i from the right-hand side of Eq. (104) (p_i would imply nonlocal $p_i \ln p_j$). Let us check it. Expanding the integral in momenta, we obtain the linear terms as

$$(-)2\int_q f(q)h(q)^2 q^2 \left[\delta_{\alpha\beta}(p_1-p_2)_\gamma + \delta_{\beta\gamma}(p_2-p_3)_\alpha + \delta_{\gamma\alpha}(p_3-p_1)_\beta\right]. \qquad (106)$$

The integrand is a total derivative, and we obtain

$$-2\int_q f(q)h(q)^2 q^2 = -\frac{1}{(4\pi)^2}\frac{4}{3} = -b \qquad (107)$$

(see Appendix B.1). Hence, Eq. (104) is consistent with locality.

The general solution is given by

$$d_{\alpha\beta\gamma}(p_1,p_2,p_3)$$
$$= \int_{-\infty}^0 dt\, e^{-t}\Big[(-)\int_q \mathrm{Tr} f_F(q)\left\{\gamma_\alpha h_F(q+p_1e^t)\gamma_\beta h_F(q+(p_1+p_2)e^t)\gamma_\gamma\right.$$
$$+\gamma_\beta h_F(q+p_2e^t)\gamma_\gamma h_F(q+(p_2+p_3)e^t)\gamma_\alpha + \gamma_\gamma h_F(q+p_3e^t)\gamma_\alpha h_F(q+(p_3+p_1)e^t)\gamma_\beta\big\}$$
$$- b\left\{\delta_{\alpha\beta}(p_1-p_2)_\gamma + \delta_{\beta\gamma}(p_2-p_3)_\alpha + \delta_{\gamma\alpha}(p_3-p_1)_\beta\right\}e^t\Big]$$
$$+ c_{\alpha\beta\gamma\delta}p_{1\delta} + c_{\beta\gamma\alpha\delta}p_{2\delta} + c_{\gamma\alpha\beta\delta}p_{3\delta}, \qquad (108)$$

where $c_{\alpha\beta\gamma\delta}$ are arbitrary constants, not determined by Eq. (104). We note that the integrand behaves as e^t as $t \to -\infty$, and the integral is convergent. The particular form of the linear terms is required by the cyclic symmetry

$$d_{\alpha\beta\gamma}(p_1,p_2,p_3) = d_{\beta\gamma\alpha}(p_2,p_3,p_1) = d_{\gamma\alpha\beta}(p_3,p_1,p_2). \qquad (109)$$

The most general form of $c_{\alpha\beta\gamma\delta}$ is given by

$$c_{\alpha\beta\gamma\delta} = s\,\delta_{\alpha\beta}\delta_{\gamma\delta} + t\,\delta_{\alpha\gamma}\delta_{\beta\delta} + u\,\delta_{\alpha\delta}\delta_{\beta\gamma}, \qquad (110)$$

where s, t, u are constants.[1]

[1] Cyclic symmetry allows a term proportional to $\epsilon_{\alpha\beta\gamma\delta}$, but it does not contribute to $d_{\alpha\beta\gamma}$. Similarly, $c_{\alpha\beta\gamma\delta}$ does not change if we change s, t, u by the same amount. So, we could set u to zero.

We now wish to show that we can choose s, t, u, and $\mathcal{A}$ so that Eq. (105) is valid. Since Eq. (105) is consistent with Eq. (104), we only need to check the terms quadratic in momenta. Using

$$d_{\alpha\beta}(p,-p) \stackrel{p\to 0}{\longrightarrow} -\delta_{\alpha\beta}\int_q f(q)h(q)q^2 + Ap^2\delta_{\alpha\beta} + B\left(p_\alpha p_\beta - p^2\delta_{\alpha\beta}\right), \tag{111}$$

we obtain

$$\begin{aligned} d_{\beta\gamma}(p_1+p_2,p_3) - d_{\beta\gamma}(p_2,p_3+p_1) &\stackrel{p_i\to 0}{\longrightarrow} \\ & A(p_3^2-p_2^2)\delta_{\beta\gamma} + B\{p_{3\beta}p_{3\gamma} - p_3^2\delta_{\beta\gamma} - p_{2\beta}p_{2\gamma} + p_2^2\delta_{\beta\gamma}\}, \end{aligned} \tag{112}$$

where A is given by Eq. (94).

We next consider the small-momentum behavior of the integral on the right-and side of Eq. (105):

$$\begin{aligned} &\int_q K(q)\mathrm{Tr}\left[h_F(q-p_1)\gamma_\beta h_F(q+p_2) - h_F(q+p_1)\gamma_\beta h_F(q-p_3)\right]\gamma_\gamma \\ &= \int_q K(q)\mathrm{Tr}\left[h_F(q+p_1)\gamma_\beta\left(h_F(q-p_2)-h_F(q-p_3)\right)\gamma_\gamma\right] \\ &= \int_q K(q)\left[h(q+p_1)h(q-p_2)\mathrm{Tr}\, a_R(\not{q}+\not{p}_1)\gamma_\beta(\not{q}-\not{p}_2)\gamma_\gamma\right. \\ &\qquad \left. -h(q+p_1)h(q-p_3)\mathrm{Tr}\, a_R(\not{q}+\not{p}_1)\gamma_\beta(\not{q}-\not{p}_3)\gamma_\gamma\right] \\ &\stackrel{p_i\to 0}{\longrightarrow} \left(-2\epsilon_{\alpha\beta\delta\gamma}p_{1\alpha}(p_2-p_3)_\delta - 2(p_3^2-p_2^2)\delta_{\beta\gamma}\right)\int_q K(q)\left(h(q)^2 + h(q)q^2h'(q)\right) \\ &\quad + \left(p_{3\beta}p_{3\gamma} - p_3^2\delta_{\beta\gamma} - p_{2\beta}p_{2\gamma} + p_2^2\delta_{\beta\gamma}\right) \\ &\qquad \times \int_q K(q)\left(-4h(q)^2 - 6h(q)q^2h'(q) - \frac{4}{3}(q^2h'(q))^2 - \frac{2}{3}h(q)q^4h''(q)\right), \end{aligned} \tag{113}$$

where the first integral, whose integrand is a total derivative, can be calculated as

$$\int_q K(q)\left(h(q)^2 + h(q)q^2h'(q)\right) = \frac{1}{(4\pi)^2}\frac{1}{6} \tag{114}$$

(see Appendix B.2). Hence, we obtain the right-hand side of Eq. (105) as

$$\begin{aligned} \mathrm{RHS} &\stackrel{p_i\to 0}{\longrightarrow} \left(p_{3\beta}p_{3\gamma} - p_3^2\delta_{\beta\gamma} - p_{2\beta}p_{2\gamma} + p_2^2\delta_{\beta\gamma}\right) \\ &\qquad \times\left[B - \int_q K\left(4h^2 + 6hq^2h' + \frac{4}{3}(q^2h')^2 + \frac{2}{3}hq^4h''\right)\right] \\ &\quad + \left(\frac{1}{(4\pi)^2}\frac{1}{3} - \frac{1}{2}\mathcal{A}\right)\epsilon_{\alpha\beta\gamma\delta}p_{1\alpha}(p_2-p_3)_\delta. \end{aligned} \tag{115}$$

We next compute the small-momentum behavior of the left-hand side of Eq. (105). From

$$\begin{aligned} &d_{\alpha\beta\gamma}(p_1,p_2,p_3) \\ &\stackrel{p_i\to 0}{\longrightarrow} \delta_{\alpha\beta}(sp_1+tp_2+up_3)_\gamma + \delta_{\beta\gamma}(up_1+sp_2+tp_3)_\alpha + \delta_{\gamma\alpha}(tp_1+up_2+sp_3)_\beta, \end{aligned} \tag{116}$$

we obtain

$$p_{1\alpha}d_{\alpha\beta\gamma}(p_1,p_2,p_3) \stackrel{p_i\to 0}{\longrightarrow} p_{1\alpha}\left(c_{\alpha\beta\gamma\delta}p_{1\delta} + c_{\beta\gamma\alpha\delta}p_{2\delta} + c_{\gamma\alpha\beta\delta}p_{3\delta}\right)$$

$$= \left(p_{2\beta}p_{2\gamma} - \delta_{\beta\gamma}p_2^2\right)(s-u) + \left(p_{3\beta}p_{3\gamma} - p_3^2\delta_{\beta\gamma}\right)(t-u)$$
$$+ (s+t-2u)\left(p_{2\beta}p_{3\gamma} - \delta_{\beta\gamma}(p_2p_3)\right). \tag{117}$$

Matching this with Eq,. (115), we obtain

$$u = \frac{1}{2}(s+t), \tag{118a}$$

$$\frac{1}{2}(s-t) = -B + \int_q K(q)\left(4h^2 + 6hq^2h' + \frac{4}{3}(q^2h')^2 + \frac{2}{3}hq^4h''\right), \tag{118b}$$

which determine the low-momentum behavior,

$$d_{\alpha\beta\gamma}(p_1,p_2,p_3) \stackrel{p_i\to 0}{\longrightarrow} \frac{1}{2}(s-t)\left(\delta_{\alpha\beta}(p_1-p_2)_\gamma + \delta_{\beta\gamma}(p_2-p_3)_\alpha + \delta_{\gamma\alpha}(p_3-p_1)_\beta\right). \tag{119}$$

We also obtain the coefficient of the anomaly as[2]

$$\mathcal{A} = \frac{1}{(4\pi)^2}\frac{2}{3}. \tag{120}$$

Let us stop here to examine the asymptotic behavior of $d_{\alpha\beta\gamma}(p_1,p_2,p_3)$ for large momenta. Instead of taking the asymptotic limit of Eq. (108), we go back to Eqs. (104) and (105). For large momenta, Eq. (104) gives

$$\left(\sum_{i=1}^3 p_i\cdot\partial_{p_i} - 1\right) d_{\alpha\beta\gamma}(p_1,p_2,p_3) \stackrel{p_i\to\infty}{\longrightarrow} b\left[\delta_{\alpha\beta}(p_1-p_2)_\gamma + \delta_{\beta\gamma}(p_2-p_3)_\alpha + \delta_{\gamma\alpha}(p_3-p_1)_\beta\right], \tag{121}$$

and Eq. (105) gives

$$p_{1\alpha}d_{\alpha\beta\gamma}(p_1,p_2,p_3) \stackrel{p_i\to\infty}{\longrightarrow} d_{\beta\gamma}(-p_3,p_3) - d_{\beta\gamma}(p_2,-p_2) - \frac{1}{2}\mathcal{A}\epsilon_{\alpha\beta\gamma\delta}p_{1\alpha}(p_2-p_3)_\delta. \tag{122}$$

Equation (121) gives the dominant asymptotic behavior,

$$d_{\alpha\beta\gamma}(p_1e^t,p_2e^t,p_3e^t) \stackrel{t\to\infty}{\longrightarrow} b\,t\,e^t\left[\delta_{\alpha\beta}(p_1-p_2)_\gamma + \delta_{\beta\gamma}(p_2-p_3)_\alpha + \delta_{\gamma\alpha}(p_3-p_1)_\beta\right], \tag{123}$$

which is proportional to the coefficient b. Hence, we can construct the continuum limit as

$$D_{\alpha\beta\gamma}(p_1,p_2,p_3) \equiv \lim_{t\to+\infty} e^{-t}\Big[d_{\alpha\beta\gamma}(p_1e^t,p_2e^t,p_3e^t)$$
$$- bte^t\left\{\delta_{\alpha\beta}(p_1-p_2)_\gamma + \delta_{\beta\gamma}(p_2-p_3)_\alpha + \delta_{\gamma\alpha}(p_3-p_1)_\beta\right\}\Big]. \tag{124}$$

This satisfies the scaling relation

$$\left(\sum_{i=1}^3 p_i\cdot\partial_{p_i} - 1\right) D_{\alpha\beta\gamma}(p_1,p_2,p_3)$$
$$= b\left\{\delta_{\alpha\beta}(p_1-p_2)_\gamma + \delta_{\beta\gamma}(p_2-p_3)_\alpha + \delta_{\gamma\alpha}(p_3-p_1)_\beta\right\}$$

[2] It was first pointed out in Ref. [14] that the chiral anomaly comes from the short-distance singularity of three currents. The calculation following this suggestion was completed in coordinate space in Ref. [15].

$$= -b\partial_B D_{\alpha\beta\gamma}(p_1, p_2, p_3) \tag{125}$$

and the WT identity

$$p_{1\alpha} D_{\alpha\beta\gamma}(p_1, p_2, p_3) = D_{\beta\gamma}(-p_3, p_3) - D_{\beta\gamma}(p_2, -p_2) - \frac{1}{2}\mathcal{A}\epsilon_{\alpha\beta\gamma\delta} p_{1\alpha}\,(p_2 - p_3)_\delta\,. \tag{126}$$

The continuum limit of the connected three-point function defined by

$$\begin{aligned}&\left\langle\!\left\langle J^a_\alpha(p_1) J^b_\beta(p_2) J^c_\gamma(p_3)\right\rangle\!\right\rangle^{\mathrm{conn}}_B\\ &\equiv \delta(p_1+p_2+p_3)\left[\mathrm{Tr}\, T^a T^b T^c\, D_{\alpha\beta\gamma}(p_1,p_2,p_3) + \mathrm{Tr}\, T^a T^c T^b\, D_{\alpha\gamma\beta}(p_1,p_3,p_2)\right]\end{aligned} \tag{127}$$

satisfies the scaling relation

$$\left(\sum_{i=1}^3 p_i\cdot\partial_{p_i} + 3 + b\partial_B\right)\left\langle\!\left\langle J^a_\alpha(p_1) J^b_\beta(p_2) J^c_\gamma(p_3)\right\rangle\!\right\rangle^{\mathrm{conn}}_B = 0 \tag{128}$$

and the WT identity

$$\begin{aligned}&p_{1\alpha}\left\langle\!\left\langle J^a_\alpha(p_1) J^b_\beta(p_2) J^c_\gamma(p_3)\right\rangle\!\right\rangle^{\mathrm{conn}}_B\\ &= if^{abd}\left\langle\!\left\langle J^d_\beta(p_1+p_2) J^c_\gamma(p_3)\right\rangle\!\right\rangle^{\mathrm{conn}}_B + if^{acd}\left\langle\!\left\langle J^b_\beta(p_2) J^d_\gamma(p_1+p_3)\right\rangle\!\right\rangle^{\mathrm{conn}}_B\\ &\quad - \frac{1}{2}\mathcal{A}\,\mathrm{Tr}\, T^a\left\{T^b, T^c\right\}\,\epsilon_{\alpha\beta\gamma\delta} p_{1\alpha}(p_2-p_3)_\delta\,\delta(p_1+p_2+p_3).\end{aligned} \tag{129}$$

6.3. *Product of four, $n = 4$*

$d_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$ must satisfy the ERG equation

$$\begin{aligned}&\sum_{i=1}^4 p_i\cdot\partial_{p_i}\, d_{\alpha\beta\gamma\delta}(p_1,p_2,p_3,p_4)\\ &= (-)\int_q \mathrm{Tr}\, f_F(q)\left[\gamma_\alpha h_F(q+p_1)\gamma_\beta h_F(q+p_1+p_2)\gamma_\gamma h_F(q+p_1+p_2+p_3)\gamma_\delta\right.\\ &\quad + \gamma_\beta h_F(q+p_2)\gamma_\gamma h_F(q+p_2+p_3)\gamma_\delta h_F(q+p_2+p_3+p_4)\gamma_\alpha\\ &\quad + \gamma_\gamma h_F(q+p_3)\gamma_\delta h_F(q+p_3+p_4)\gamma_\alpha h_F(q+p_3+p_4+p_1)\gamma_\beta\\ &\quad \left.+\gamma_\delta h_F(q+p_4)\gamma_\alpha h_F(q+p_4+p_1)\gamma_\beta h_F(q+p_4+p_1+p_2)\gamma_\gamma\right]\\ &\quad + b\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\beta\gamma}\delta_{\alpha\delta} - 2\delta_{\alpha\gamma}\delta_{\beta\delta}\right),\end{aligned} \tag{130}$$

where b is given by Eq. (88), and the WT identity

$$\begin{aligned}p_{1\alpha} d_{\alpha\beta\gamma\delta}(p_1,p_2,p_3,p_4) &= d_{\beta\gamma\delta}(p_1+p_2,p_3,p_4) - d_{\beta\gamma\delta}(p_2,p_3,p_4+p_1)\\ &\quad + \int_q K(q)\mathrm{Tr}\,\left[h_F(q-p_1)\gamma_\beta h_F(q+p_2)\gamma_\gamma h_F(q+p_2+p_3)\right.\\ &\qquad \left.-h_F(q+p_1)\gamma_\beta h_F(q+p_1+p_2)\gamma_\gamma h_F(q+p_1+p_2+p_3)\right]\gamma_\delta\\ &\quad - \frac{1}{2}\mathcal{A} p_{1\alpha}\epsilon_{\alpha\beta\gamma\delta},\end{aligned} \tag{131}$$

where $\mathcal{A}$ is given by Eq. (120).

We would like to check two things. As for Eq. (130), we would like to check the vanishing of the right-hand side at zero momenta (a constant would imply nonlocal $\ln p$). As for Eq. (131), we would like to check its validity at the first order in momenta.

The right-hand side of Eq. (130) gives

$$(\mathrm{RHS}) \xrightarrow{p_i\to 0} \left[(-)\frac{1}{6}\int_q f(q)h(q)^3 q^4 \times 16 + b\right]\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\beta\gamma}\delta_{\alpha\delta} - 2\delta_{\alpha\gamma}\delta_{\beta\delta}\right). \tag{132}$$

The integrand is a total derivative, and we obtain

$$\frac{8}{3}\int_q f(q)h(q)^3 q^4 = \frac{1}{(4\pi)^2}\frac{4}{3} = b \tag{133}$$

(see Appendix B.1). Hence, the right-hand side vanishes at zero momenta as desired.

We now wish to check Eq. (131) to first order in momenta. Equation (130) determines only the momentum dependence of $d_{\alpha\beta\gamma\delta}$, but its value at $p_i = 0$ is left undetermined. The most general form, consistent with cyclic symmetry, is

$$d_{\alpha\beta\gamma\delta}(0,0,0,0) = s_4\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\beta\gamma}\delta_{\delta\alpha}\right) + t_4\,\delta_{\alpha\gamma}\delta_{\beta\delta}, \tag{134}$$

where s_4, t_4 are constants such that

$$p_{1\alpha}d_{\alpha\beta\gamma\delta}(0,0,0,0) = s_4\left(p_{1\beta}\delta_{\gamma\delta} + p_{1\delta}\delta_{\beta\gamma}\right) + t_4 p_{1\gamma}\delta_{\beta\delta}. \tag{135}$$

To compare this with the right-hand side, we first compute

$$d_{\beta\gamma\delta}(p_1+p_2,p_3,p_4) - d_{\beta\gamma\delta}(p_2,p_3,p_4+p_1) \xrightarrow{p_i\to 0} \frac{1}{2}(s-t)\left(p_{1\delta}\delta_{\beta\gamma} + p_{1\beta}\delta_{\gamma\delta} - 2p_{1\gamma}\delta_{\beta\delta}\right), \tag{136}$$

where we have used Eq. (119), and $\frac{1}{2}(s-t)$ is given by Eq. (118b). We next compute

$$\begin{aligned}
&\int_q K(q)\mathrm{Tr}\,\big[h_F(q-p_1)\gamma_\beta h_F(q+p_2)\gamma_\gamma h_F(q+p_2+p_3)\\
&\qquad -h_F(q+p_1)\gamma_\beta h_F(q+p_1+p_2)\gamma_\gamma h_F(q+p_1+p_2+p_3)\big]\gamma_\delta\\
&= \int_q K(q)\big[h(q-p_1)h(q+p_2)h(q+p_2+p_3)\mathrm{Tr}\,(\not{q}-\not{p}_1)\gamma_\beta(\not{q}+\not{p}_2)\gamma_\gamma(\not{q}+\not{p}_2+\not{p}_3)\\
&\qquad - h(q+p_1)h(q+p_1+p_2)h(q+p_1+p_2+p_3)\\
&\qquad\quad \times\mathrm{Tr}\,(\not{q}+\not{p}_1)\gamma_\beta(\not{q}+\not{p}_1+\not{p}_2)\gamma_\gamma(\not{q}+\not{p}_1+\not{p}_2+\not{p}_3)\big]\gamma_\delta a_R\\
&\xrightarrow{p_i\to 0} p_{1\alpha}\epsilon_{\alpha\beta\gamma\delta}4\int_q K(q)h(q)^2q^2\left(h(q)+q^2h'(q)\right)\\
&\qquad + \left(p_{1\beta}\delta_{\gamma\delta}+p_{1\delta}\delta_{\beta\gamma}\right)\int_q K(q)q^2h(q)^2\left(2h(q)+\frac{4}{3}q^2h'(q)\right)\\
&\qquad + p_{1\gamma}\delta_{\beta\delta}\frac{4}{3}\int_q K(q)h(q)^2q^4h'(q).
\end{aligned} \tag{137}$$

Hence, the right-hand side of Eq. (131) is

$$\mathrm{RHS} \xrightarrow{p_i\to 0} \frac{1}{2}(s-t)\left(p_{1\beta}\delta_{\gamma\delta} + p_{1\delta}\delta_{\beta\gamma} - 2p_{1\gamma}\delta_{\beta\delta}\right)$$

$$+ (p_{1\beta}\delta_{\gamma\delta} + p_{1\delta}\delta_{\beta\gamma}) \int_q K(q)h(q)^2 q^2 \left(2h(q) + \frac{4}{3}q^2 h'(q)\right)$$
$$+ p_{1\gamma}\delta_{\beta\delta}\frac{4}{3}\int_q K(q)h(q)^2 q^4 h'(q)$$
$$+ p_{1\alpha}\epsilon_{\alpha\beta\gamma\delta}\left(4\int_q K(q)h(q)^2 q^2 \left(h(q) + q^2 h'(q)\right) - \frac{1}{2}\mathcal{A}\right). \tag{138}$$

The last term vanishes because

$$\int_q K(q)h(q)^2 q^2 \left(h(q) + q^2 h'(q)\right) = \frac{1}{(4\pi)^2}\frac{1}{12} \tag{139}$$

(see Appendix B.2.) We can make Eq. (138) match Eq. (135) by choosing

$$s_4 = \frac{1}{2}(s-t) + \int_q K(q)h(q)^2 q^2 \left(2h(q) + \frac{4}{3}q^2 h'(q)\right) \tag{140a}$$
$$= -B + \int_q K\left(4h^2 + 6hq^2h' + \frac{4}{3}(q^2h')^2 + \frac{2}{3}hq^4h'' + 2h^3q^2 + \frac{4}{3}h^2q^4h'\right),$$
$$t_4 = -(s-t) + \frac{4}{3}\int_q K(q)h(q)^2 q^4 h'(q)$$
$$= -2s_4 + 4\int_q K(q)h(q)^2 q^2 \left(h(q) + q^2 h'(q)\right) = -2s_4 + \frac{1}{(4\pi)^2}\frac{1}{3}. \tag{140b}$$

We have thus checked the validity of Eq. (131).

Finally, we examine the asymptotic behavior of $d_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$ for large momenta. Equations (130) and (131) give

$$\sum_{i=1}^4 p_i \cdot \partial_{p_i} d_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) \overset{p_i\to\infty}{\longrightarrow} b\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\beta\gamma}\delta_{\alpha\delta} - 2\delta_{\alpha\gamma}\delta_{\beta\delta}\right), \tag{141}$$

$$p_{1\alpha} d_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) \overset{p_i\to\infty}{\longrightarrow} d_{\beta\gamma\delta}(-p_3 - p_4, p_3, p_4) - d_{\beta\gamma\delta}(p_2, p_3, -p_2 - p_3)$$
$$- \frac{1}{2}\mathcal{A} p_{1\alpha}\epsilon_{\alpha\beta\gamma\delta}. \tag{142}$$

The first equation gives the asymptotic behavior

$$d_{\alpha\beta\gamma\delta}(p_1e^t, p_2e^t, p_3e^t, p_4e^t) \overset{t\to\infty}{\longrightarrow} b\,t\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\beta\gamma}\delta_{\alpha\delta} - 2\delta_{\alpha\gamma}\delta_{\beta\delta}\right). \tag{143}$$

Hence, a continuum limit is obtained as

$$D_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$$
$$\equiv \lim_{t\to+\infty}\left[d_{\alpha\beta\gamma\delta}(p_1e^t, p_2e^t, p_3e^t, p_4e^t) - bt\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\beta\gamma}\delta_{\alpha\delta} - 2\delta_{\alpha\gamma}\delta_{\beta\delta}\right)\right], \tag{144}$$

which satisfies the scaling relation

$$\sum_{i=1}^4 p_i \cdot \partial_{p_i} D_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = b\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\beta\gamma}\delta_{\alpha\delta} - 2\delta_{\alpha\gamma}\delta_{\beta\delta}\right)$$
$$= -b\partial_B D_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) \tag{145}$$

and the WT identity

$$p_{1\alpha} D_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = D_{\beta\gamma\delta}(-p_3 - p_4, p_3, p_4) - D_{\beta\gamma\delta}(p_2, p_3, -p_2 - p_3) - \frac{1}{2}\mathcal{A} p_{1\alpha} \epsilon_{\alpha\beta\gamma\delta}. \tag{146}$$

Hence, the connected four-point function defined by

$$\left\langle\!\left\langle J^a_\alpha(p_1) J^b_\beta(p_2) J^c_\gamma(p_3) J^d_\delta(p_4)\right\rangle\!\right\rangle^{\mathrm{conn}}_B \equiv \delta(p_1+p_2+p_3+p_4)\Big[\mathrm{Tr}\, T^a T^b T^c T^d D_{\alpha\beta\gamma\delta}(p_1,p_2,p_3,p_4) + \mathrm{Tr}\, T^a T^b T^d T^c\, D_{\alpha\beta\delta\gamma}(p_1,p_2,p_4,p_3) + \cdots\Big] \tag{147}$$

satisfies the scaling relation

$$\left(\sum_{i=1}^4 p_i \cdot \partial_{p_i} + 4 + b\partial_B\right)\left\langle\!\left\langle J^a_\alpha(p_1) J^b_\beta(p_2) J^c_\gamma(p_3) J^d_\delta(p_4)\right\rangle\!\right\rangle^{\mathrm{conn}}_B = 0 \tag{148}$$

and the WT identity

$$p_{1\alpha}\left\langle\!\left\langle J^a_\alpha(p_1) J^b_\beta(p_2) J^c_\gamma(p_3) J^d_\delta(p_4)\right\rangle\!\right\rangle^{\mathrm{conn}}_B = if^{abe}\left\langle\!\left\langle J^e_\beta(p_1+p_2) J^c_\gamma(p_3) J^d_\delta(p_4)\right\rangle\!\right\rangle^{\mathrm{conn}}_B + \cdots - \frac{1}{2}\mathcal{A} p_{1\alpha}\epsilon_{\alpha\beta\gamma\delta}\,\delta(p_1+p_2+p_3+p_4) \times \mathrm{Tr}\, T^a\left(T^b[T^c,T^d] + T^c[T^d,T^b] + T^d[T^b,T^c]\right). \tag{149}$$

6.4. *Recapitulation*

Let us recapitulate the results of this section by writing down equations for $e^{W[A]}$, a composite operator of scale dimension 0. The ERG differential equation is given by

$$\left(\int_p \left(-p\cdot\partial_p - D + 1\right) A^a_\mu(p)\cdot\frac{\delta}{\delta A^a_\mu(p)} - \mathcal{D}\right) e^{W[A]} = \frac{1}{(4\pi)^2}\frac{4}{3}\cdot\frac{1}{4}\int d^4x\, \mathrm{Tr}\,\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]\right) e^{W[A]}. \tag{150}$$

The WT identity is given by

$$\delta_\epsilon e^{W[A]} \equiv \int_p \left(-p_\mu \epsilon^a(p) + if^{abc}\int_q A^b_\mu(p+q)\epsilon^c(-q)\right)\frac{\delta}{\delta A^a_\mu(p)} e^{W[A]} = \left[\int_p \epsilon^a(-p)\mathcal{E}^a(p)\star + \frac{1}{(4\pi)^2}\frac{2}{3}\int d^4x\, \epsilon_{\alpha\beta\gamma\delta}\mathrm{Tr}\,\partial_\alpha\epsilon\left(A_\beta\partial_\gamma A_\delta - i\frac{1}{2}A_\beta A_\gamma A_\delta\right)\right] e^{W[A]}. \tag{151}$$

$W[A]$ is determined uniquely by the above two equations up to a constant multiple of the gauge invariant

$$\frac{1}{4}\int d^4x\, \mathrm{Tr}\,\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]\right).$$

If we define

$$W_g[A] = -\frac{1}{4g}\int d^4x\, \mathrm{Tr}\,\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]\right) + W[A], \tag{152}$$

we can rewrite the ERG equation as

$$\left(\beta(g)\partial_g + \int_p \left(-p\cdot\partial_p - D + 1\right) A^a_\mu(p)\cdot\frac{\delta}{\delta A^a_\mu(p)} - \mathcal{D}\right) e^{W_g[A]} = 0, \tag{153}$$

where

$$\beta(g) = -\frac{1}{(4\pi)^2}\frac{4}{3}g^2 \tag{154}$$

is the one-loop beta function.

7. Conclusions

We have discussed the multiple products of current operators using the exact renormalization group formalism. The multiple products are characterized by two mutually consistent equations: one is the ERG differential equation and the other is the Ward–Takahashi identity. We have argued that these two equations suffer changes due to the short-distance singularities of the products, and the revised equations are given by Eq. (61) for ERG and Eq. (64) for the WT identity. In Sect 6 we have calculated the multiple products explicitly by solving these equations for the Gaussian fixed point. The guiding principle in these calculations is the locality of the operators. Since the momenta below the cutoff have not been integrated, the coefficient functions for the products of the current are analytic at zero momenta.

There are some future directions we can consider. We may consider a theory such as quantum chromodynamics with fields other than the chiral fermions. Or we may consider a more nontrivial fixed-point Wilson action. We also think it interesting to study the multiple products of other composite operators such as the energy–momentum tensor.

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Appendix A. Invariance of the Wilson action

Given a Wilson action $S[\psi,\bar{\psi}]$, its invariance under global flavor transformations is most straightforwardly given by

$$\int_p \left[\bar{\psi}(-p)T^a \frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)} S_t - S_t \frac{\overleftarrow{\delta}}{\delta\psi(p)} T^a \psi(p)\right] = 0. \tag{A.1}$$

We wish to show that this is equal to Eq. (23), which is

$$\mathcal{E}^a(0) \equiv e^{-S_t}\int_p K(p)\mathrm{Tr}\left[\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}\left(\bar{\Psi}(-p)T^a e^{S_t}\right) - \left(e^{S_t}T^a\Psi(p)\right)\frac{\overleftarrow{\delta}}{\delta\psi(p)}\right] = 0, \tag{A.2}$$

where

$$\Psi(p) = \frac{1}{K(p)}\left(\psi(p) + h_F(p)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_t\right), \tag{A.3a}$$

$$\bar{\Psi}(-p) = \frac{1}{K(p)}\left(\bar{\psi}(-p) + S_t\frac{\overleftarrow{\delta}}{\delta\psi(p)}h_F(p)\right). \tag{A.3b}$$

Substituting Eq. (A.3) into Eq. (A.2), we obtain

$$\begin{aligned}
\mathcal{E}^a(0) &= \int_p \left[-\left(\bar{\psi}(-p) + S_t\frac{\overleftarrow{\delta}}{\delta\psi(p)}h_F(p)\right)T^a\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_t \right.\\
&\quad \left. + S_t\frac{\overleftarrow{\delta}}{\delta\psi(p)}T^a\left(\psi(p) + h_F(p)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_t\right)\right]\\
&\quad + \int_p \mathrm{Tr}\left[\delta(0)T^a - \delta(0)T^a\right]\\
&\quad + \int_p \mathrm{Tr}\left[\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_t\frac{\overleftarrow{\delta}}{\delta\psi(p)}h_F(p)T^a - T^a h_F(p)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_t\frac{\overleftarrow{\delta}}{\delta\psi(p)}\right]\\
&= \int_p\left[-\bar{\psi}(-p)T^a\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_t + S_t\frac{\overleftarrow{\delta}}{\delta\psi(p)}T^a\psi(p)\right] = 0,
\end{aligned} \tag{A.4}$$

which is Eq. (A.1).

Appendix B. Universal cutoff integrals

We give four integrals involving a cutoff function $K(p)$. The values of these integrals are universal in the sense that they do not depend on the choice of $K(p)$ as long as $K(0) = 1$ and $K(p)$ vanishes asymptotically as $p^2 \to \infty$. The functions h and f are defined by

$$h(p) \equiv \frac{1 - K(p)}{p^2}, \tag{B.1a}$$

$$f(p) \equiv \left(p \cdot \partial_p + 2\right) h(p) = \frac{\Delta(p)}{p^2}, \tag{B.1b}$$

$$\Delta(p) \equiv -p \cdot \partial_p K(p). \tag{B.1c}$$

Appendix B.1. $\int_q f(q)h(q)\left(q^2h(q)\right)^n$

For $n = 0, 1, 2, \ldots$, we obtain

$$\begin{aligned}
\int_q f(q)h(q)\left(q^2h(q)\right)^n &= \int_q \left(q\cdot\partial_q + 2\right)h(q)\cdot\frac{1}{q^2}\left(q^2h(q)\right)^{n+1}\\
&= \int_q \frac{1}{q^4}q\cdot\partial_q\left\{\frac{\left(q^2h(q)\right)^{n+2}}{n+2}\right\}\\
&= \frac{2\pi^2}{(2\pi)^4}\int_0^\infty dq^2\frac{d}{dq^2}\left\{\frac{\left(q^2h(q)\right)^{n+2}}{n+2}\right\}
\end{aligned}$$

$$= \frac{1}{(4\pi)^2} \frac{2}{n+2}. \tag{B.2}$$

Appendix B.2. $\int_q K(q)h(q)\left(q^2h(q)\right)^n \left(h(q)+q^2h'(q)\right)$

For $n = 0, 1, 2, \ldots$, we obtain

$$\begin{aligned}
\int_q K(q)h(q)\left(q^2h(q)\right)^n \left(h(q)+q^2h'(q)\right) &= \int_q \left(1-q^2h(q)\right)h(q)\left(q^2h(q)\right)^n \frac{d}{dq^2}\left(q^2h(q)\right)\\
&= \int_q \frac{1}{q^2}\left(1-q^2h(q)\right)\left(q^2h(q)\right)^{n+1} \frac{d}{dq^2}\left(q^2h(q)\right)\\
&= \frac{1}{(4\pi)^2}\int_0^\infty dq^2 \frac{d}{dq^2}\left(\frac{\left(q^2h(q)\right)^{n+2}}{n+2} - \frac{\left(q^2h(q)\right)^{n+3}}{n+3}\right)\\
&= \frac{1}{(4\pi)^2}\frac{1}{(n+2)(n+3)}.
\end{aligned} \tag{B.3}$$

Appendix B.3. $\int_q f(q)\left(q^2h'(q)+\frac{1}{3}q^4h''(q)\right)$

$$\begin{aligned}
\int_q f(q)\left(q^2h'(q)+\frac{1}{3}q^4h''(q)\right) &= \frac{1}{(4\pi)^2}\int_0^\infty xdx \underbrace{f(x)}_{=2\left(x\frac{d}{dx}+1\right)h(x)} \left(x\frac{d}{dx}+\frac{1}{3}x^2\frac{d^2}{dx^2}\right)h(x)\\
&= \frac{2}{(4\pi)^2}\int_0^\infty dx\, x\left(1+x\frac{d}{dx}\right)h(x)\cdot x\left(\frac{d}{dx}+\frac{1}{3}x\frac{d^2}{dx^2}\right)h(x)\\
&= \frac{2}{(4\pi)^2}\int_0^\infty dx\,\frac{d}{dx}\left(\frac{1}{3}x^3h(x)h'(x)+\frac{1}{6}x^4h'(x)^2\right)\\
&= \frac{2}{(4\pi)^2}\left(-\frac{1}{3}+\frac{1}{6}\right) = -\frac{1}{(4\pi)^2}\frac{1}{3}.
\end{aligned} \tag{B.4}$$

Appendix B.4. $\int_q K(q)\left(h'(q)+q^2h''(q)+\frac{1}{6}q^4h'''(q)\right)$

$$\begin{aligned}
&\int_q K(q)\left(h'(q)+q^2h''(q)+\frac{1}{6}q^4h'''(q)\right)\\
&\quad= \frac{1}{(4\pi)^2}\int_0^\infty dq^2\, q^2K(q)\left(h'(q)+q^2h''(q)+\frac{1}{6}h''(q)\right)\\
&\quad= \frac{1}{(4\pi)^2}\int_0^\infty dq^2\,\frac{d}{dq^2}\left[-\frac{1}{6}q^4K(q)K''(q)+\frac{1}{6}\left(q^4\frac{1}{2}K'(q)^2-q^2K(q)K'(q)\right)+\frac{1}{12}K(q)^2\right]\\
&\quad= -\frac{1}{(4\pi)^2}\frac{1}{12}.
\end{aligned} \tag{B.5}$$

Appendix B.5. Check of Eq. (92)

We wish to check Eq. (92) in Sect. 6, which can be written as

$$\int_q \left(f(q)h(q)q^2 - K(q)(2h(q)+f(q))\right) = 0. \tag{B.6}$$

The integrand is a total derivative:

$$\left(q \cdot \partial_q + 4\right) (K(q)h(q)) = -\Delta(q)h(q) + K(q)f(q) + 2K(q)h(q) \\ = -q^2 f(q)h(q) + K(q)\,(2h(q) + f(q))\,. \tag{B.7}$$

Since $K(q)h(q)$ vanishes at $q^2 = 0\,, \infty$, the integral vanishes.

Appendix C. Corrections to the ERG equation and the WT identity

Differentiating Eqs. (61) and (64) with respect to the source A, we obtain the ERG equation and the WT identity for the products of current operators.

Appendix C.1. Product of two

The ERG differential equation is

$$\left(\partial_t + p_1 \cdot \partial_{p_1} + p_2 \cdot \partial_{p_2} + 2 - \mathcal{D}_t\right) \left[J^a_\alpha(p_1) J^b_\beta(p_2)\right] = b(t)\delta(p_1 + p_2)\,\delta^{ab}\left(p_{1\alpha}p_{1\beta} - p_1^2\delta_{\alpha\beta}\right). \tag{C.1}$$

The WT identity has no anomaly:

$$p_\alpha \left[J^a_\alpha(p) J^b_\beta(p_1)\right] = i f^{abc} J^c_\beta(p + p_1) + \mathcal{E}^a(p) \star J^b_\beta(p_1). \tag{C.2}$$

Appendix C.2. Product of three

The ERG differential equation is

$$\left(\sum_{i=1}^3 p_i \cdot \partial_{p_i} + 3 - \mathcal{D}_t\right) \left[J^a_\alpha(p_1) J^b_\beta(p_2) J^c_\gamma(p_3)\right] \\ = b(t)\Bigg[\delta\left(\sum_{i=1}^3 p_i\right) \mathrm{Tr}\, T^a \left[T^b, T^c\right] \left\{\delta_{\alpha\beta}(p_1 - p_2)_\gamma + \delta_{\beta\gamma}(p_2 - p_3)_\alpha + \delta_{\gamma\alpha}(p_3 - p_1)_\beta\right\} \\ - \Big\{\delta(p_1 + p_2)\left(p_{1\alpha}p_{1\beta} - p_1^2\delta_{\alpha\beta}\right)\delta^{ab} J^c_\gamma(p_3) + \delta(p_2 + p_3)\left(p_{2\beta}p_{2\gamma} - p_2^2\delta_{\beta\gamma}\right)\delta^{bc} J^a_\alpha(p_1) \\ + \delta(p_3 + p_1)\left(p_{3\gamma}p_{3\alpha} - p_3^2\delta_{\gamma\alpha}\right)\delta^{ca} J^b_\beta(p_2)\Big\}\Bigg]. \tag{C.3}$$

The WT identity can be anomalous:

$$p_\alpha \left[J^a_\alpha(p) J^b_\beta(q) J^c_\gamma(r)\right] = i f^{abd}\left[J^d_\beta(p+q) J^c_\gamma(r)\right] + i f^{acd}\left[J^b_\beta(q) J^d_\gamma(p+r)\right] \\ + \mathcal{E}^a(p) \star \left[J^b_\beta(q) J^c_\gamma(r)\right] \\ - \frac{\mathcal{A}}{2}\delta\,(p + q + r)\,\mathrm{Tr}\, T^a \left\{T^b, T^c\right\}\,\epsilon_{\alpha\beta\gamma\delta}\, p_\alpha (q - r)_\delta. \tag{C.4}$$

Appendix C.3. Product of four

The ERG differential equation is

$$\left(\sum_{i=1}^4 p_i \cdot \partial_{p_i} + 4 - \mathcal{D}_t\right) \left[J^a_\alpha(p_1) J^b_\beta(p_2) J^c_\gamma(p_3) J^d_\delta(p_4)\right]$$

$$
\begin{aligned}
= b(t)\Bigg[& \delta(p_1+p_2+p_3+p_4)\Big[\mathrm{Tr}\, T^a T^b T^c T^d \left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\beta\gamma}\delta_{\delta\alpha} - 2\delta_{\alpha\gamma}\delta_{\beta\delta}\right) \\
& \quad + \mathrm{Tr}\, T^a T^b T^d T^c \left(\delta_{\alpha\beta}\delta_{\delta\gamma} + \delta_{\beta\delta}\delta_{\gamma\alpha} - 2\delta_{\alpha\delta}\delta_{\beta\gamma}\right) + \mathrm{Tr}\, T^a T^c T^b T^d \left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\gamma\beta}\delta_{\delta\alpha} - 2\delta_{\alpha\beta}\delta_{\gamma\delta}\right) \\
& \quad + \mathrm{Tr}\, T^a T^c T^d T^b \left(\delta_{\alpha\gamma}\delta_{\delta\beta} + \delta_{\gamma\delta}\delta_{\beta\alpha} - 2\delta_{\alpha\delta}\delta_{\gamma\beta}\right) + \mathrm{Tr}\, T^a T^d T^b T^c \left(\delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\delta\beta}\delta_{\gamma\alpha} - 2\delta_{\alpha\beta}\delta_{\delta\gamma}\right) \\
& \quad + \mathrm{Tr}\, T^a T^d T^c T^b \left(\delta_{\alpha\delta}\delta_{\gamma\beta} + \delta_{\delta\gamma}\delta_{\beta\alpha} - 2\delta_{\alpha\gamma}\delta_{\delta\beta}\right)\Big] \\
& + \delta(p_1+p_2+p_3)\mathrm{Tr}\, T^a\left[T^b, T^c\right]\left\{\delta_{\alpha\beta}(p_1-p_2)_\gamma + \delta_{\beta\gamma}(p_2-p_3)_\alpha + \delta_{\gamma\alpha}(p_3-p_1)_\beta\right\} J^d_\delta(p_4) \\
& + \delta(p_1+p_2+p_4)\mathrm{Tr}\, T^a\left[T^b, T^d\right]\left\{\delta_{\alpha\beta}(p_1-p_2)_\delta + \delta_{\beta\delta}(p_2-p_4)_\alpha + \delta_{\delta\alpha}(p_4-p_1)_\beta\right\} J^c_\gamma(p_3) \\
& + \delta(p_1+p_3+p_4)\mathrm{Tr}\, T^a\left[T^c, T^d\right]\left\{\delta_{\alpha\gamma}(p_1-p_3)_\delta + \delta_{\gamma\delta}(p_3-p_4)_\alpha + \delta_{\delta\alpha}(p_4-p_1)_\gamma\right\} J^b_\beta(p_2) \\
& + \delta(p_2+p_3+p_4)\mathrm{Tr}\, T^b\left[T^c, T^d\right]\left\{\delta_{\beta\gamma}(p_2-p_3)_\delta + \delta_{\gamma\delta}(p_4-p_4)_\beta + \delta_{\delta\beta}(p_4-p_2)_\gamma\right\} J^a_\alpha(p_1) \\
& + \delta(p_1+p_2)\left(p_1^2\delta_{\alpha\beta} - p_{1\alpha}p_{1\beta}\right)\delta^{ab}\left[J^c_\gamma(p_3)J^d_\delta(p_4)\right] \\
& + \delta(p_1+p_3)\left(p_1^2\delta_{\alpha\gamma} - p_{1\alpha}p_{1\gamma}\right)\delta^{ac}\left[J^b_\beta(p_2)J^d_\gamma(p_4)\right] \\
& + \delta(p_1+p_4)\left(p_1^2\delta_{\alpha\delta} - p_{1\alpha}p_{1\delta}\right)\delta^{ad}\left[J^b_\beta(p_2)J^c_\gamma(p_3)\right] \\
& + \delta(p_2+p_3)\left(p_2^2\delta_{\beta\delta} - p_{2\beta}p_{2\delta}\right)\delta^{bc}\left[J^a_\alpha(p_1)J^d_\delta(p_4)\right] \\
& + \delta(p_2+p_4)\left(p_2^2\delta_{\beta\delta} - p_{2\beta}p_{2\delta}\right)\delta^{bd}\left[J^a_\alpha(p_1)J^c_\gamma(p_4)\right] \\
& + \delta(p_3+p_4)\left(p_3^2\delta_{\gamma\delta} - p_{3\gamma}p_{3\delta}\right)\delta^{cd}\left[J^a_\alpha(p_1)J^b_\beta(p_2)\right]\Bigg].
\end{aligned} \tag{C.5}
$$

The WT identity can be anomalous:

$$
\begin{aligned}
& p_\alpha\left[J^a_\alpha(p)J^b_\beta(q)J^c_\gamma(r)J^d_\delta(s)\right] \\
& = if^{abe}\left[J^e_\beta(q+p)J^c_\gamma(r)J^d_\delta(s)\right] + if^{ace}\left[J^b_\beta(q)J^e_\gamma(r+p)J^d_\delta(s)\right] + if^{ade}\left[J^b_\beta(q)J^c_\gamma(r)J^e_\delta(s+p)\right] \\
& \quad + \mathcal{E}^a(p)\star\left[J^b_\beta(q)J^c_\gamma(r)J^d_\delta(s)\right] \\
& \quad - \frac{\mathcal{A}}{2}\Big[\delta(p+q+r+s)\, p_{1\alpha}\epsilon_{\alpha\beta\gamma\delta}\mathrm{Tr}\, T^a\left(T^b\left[T^c, T^d\right] + T^c\left[T^d, T^b\right] + T^d\left[T^b, T^c\right]\right) \\
& \qquad + \delta(p+q+r)\mathrm{Tr}\, T^a\left\{T^b, T^c\right\}\epsilon_{\alpha\beta\gamma\epsilon}p_\alpha(q-r)_\epsilon J^d_\delta(s) \\
& \qquad + \delta(p+q+s)\mathrm{Tr}\, T^a\left\{T^b, T^d\right\}\epsilon_{\alpha\beta\delta\epsilon}p_\alpha(q-s)_\epsilon J^c_\gamma(r) \\
& \qquad + \delta(p+r+s)\mathrm{Tr}\, T^a\left\{T^c, T^d\right\}\epsilon_{\alpha\gamma\delta\epsilon}p_\alpha(r-s)_\epsilon J^b_\beta(q)\Big].
\end{aligned} \tag{C.6}
$$

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