



# The all-loop non-Abelian Thirring model and its RG flow



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## ABSTRACT

We analyze the renormalization group flow in a recently constructed class of integrable  $\sigma$ -models which interpolate between WZW current algebra models and the non-Abelian T-duals of PCM for a simple group  $G$ . They are characterized by the integer level  $k$  of the current algebra, a deformation parameter  $\lambda$  and they exhibit a remarkable invariance involving the inversion of  $\lambda$ . We compute the  $\beta$ -function for  $\lambda$  to leading order in  $\frac{1}{k}$ . Based on agreement with previous results for the exact  $\beta$ -function of the non-Abelian bosonized Thirring model and matching global symmetries, we state that our integrable models are the resummed version (capturing all counterterms in perturbation theory) of the non-Abelian bosonized Thirring model for a simple group  $G$ . Finally, we present an analogous treatment in a simple example of a closely related class of models interpolating between gauged WZW coset CFTs and the non-Abelian T-duals of PCM for the coset  $G/H$ .

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## 1. Introduction and set-up

A class of integrable  $\sigma$ -models with a group theoretical structure was recently constructed explicitly in [1] (using the algebraic construction set-up in [2]), which we first review. Consider a general compact simple group  $G$ . For a group element  $g \in G$  parametrized by  $X^\mu$ ,  $\mu = 1, 2, \dots, \dim(G)$ , we introduce the right and left invariant Maurer–Cartan forms as follows

$$\begin{aligned} J_+^a &= -i \text{Tr}(t^a \partial_+ g g^{-1}) = R_\mu^a \partial_+ X^\mu, \\ J_-^a &= -i \text{Tr}(t^a g^{-1} \partial_- g) = L_\mu^a \partial_- X^\mu, \end{aligned} \quad (1.1)$$

where the matrices  $t^a$  obey the commutation relations  $[t_a, t_b] = if_{abc} t_c$  and are normalized as  $\text{Tr}(t_a t_b) = \delta_{ab}$ . Hence, there is no difference between upper and lower tangent space indices. The Maurer–Cartan forms are related by an orthogonal matrix  $D$  as

$$R^a = D_{ab} L^b, \quad D_{ab} = \text{Tr}(t_a g t_b g^{-1}). \quad (1.2)$$

Then the form of the integrable  $\sigma$ -model action is<sup>1</sup>

$$S_{k,\lambda}(g) = S_{\text{WZW},k}(g) + \frac{k\lambda}{\pi} \int J_+^a (\mathbb{1} - \lambda D^T)^{-1}_{ab} J_-^b, \quad (1.3)$$

where

$$\begin{aligned} S_{\text{WZW},k}(g) &= -\frac{k}{2\pi} \int \text{Tr}(g^{-1} \partial_+ g g^{-1} \partial_- g) \\ &\quad + \frac{ik}{6\pi} \int_B \text{Tr}(g^{-1} dg)^3 \end{aligned} \quad (1.4)$$

is the Wess–Zumino–Witten (WZW) action at level  $k$  and  $\lambda$  is a real coupling constant.

These models were constructed through a gauging procedure and are invariant under the global symmetry  $g \rightarrow \Lambda_0^{-1} g \Lambda_0$ , where  $\Lambda_0 \in G$ . Moreover, their coupling constant is  $0 \leq \lambda \leq 1$  by construction. For  $\lambda \ll 1$  the action (1.3) corresponds to the WZW theory perturbed by the current bilinear term as

$$S_{k,\lambda}(g) = S_{\text{WZW},k}(g) + \frac{k\lambda}{\pi} \int J_+^a J_-^a + \mathcal{O}(\lambda^2), \quad (1.5)$$

which clearly preserves the above global symmetry. The first two terms in the above expansion define the so-called non-Abelian bosonized Thirring model (in short non-Abelian Thirring model) [3], see also [4]. For this model the  $\beta$ -function for  $\lambda$  has been computed, to leading order in the  $1/k$  expansion, but exactly in  $\lambda$ . The result is [5]

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<sup>1</sup> The action below is the simplest one of a class of multi-parameter models constructed in [1]. However, only in special cases, such as the one below, these  $\sigma$ -models are expected to be integrable.

$$\frac{d\lambda}{dt} = -\frac{c_G \lambda^2}{2k(1+\lambda)^2}, \tag{1.6}$$

where  $t = 2\pi \ln \mu$ , with  $\mu$  being the energy scale and where  $c_G$  is the quadratic Casimir in the adjoint representation, defined from the relation  $f_{acd}f_{bcd} = c_G \delta_{ab}$ . Note that this equation is invariant under the transformation  $\lambda \rightarrow 1/\lambda$  and  $k \rightarrow -k$ , which is a symmetry of (1.3) in a way that is made precise in (2.1) below. Moreover this map exists also in the non-Abelian Thirring model for large values of  $k$  [6]. In that sense (1.3) captures all counterterms in perturbation theory corresponding to the coupling  $\lambda$  and to leading order in  $1/k$ .<sup>2</sup> We also note that the necessary conditions for one-loop conformal invariance of a general class of models, which includes (1.3), were derived in [7].

For  $\lambda \rightarrow 1$  the  $\sigma$ -model action is effectively described by the non-Abelian T-dual of the Principal Chiral Model (PCM) for the group  $G$ .<sup>3</sup> The correspondence involves a limiting procedure for the coordinates  $X^\mu$  parametrizing the group element  $g \in G$  and the details can be found in [1]. The value  $\lambda = 1$  is special since once crossed from below the  $\sigma$ -model metric changes its signature from Euclidean by picking up an overall sign. It is also a self-dual point of the above  $\lambda \rightarrow 1/\lambda$  transformation.

## 2. Renormalization group flow restricted by symmetries

The overall coupling constant  $k$  is not expected, being an integer, to get renormalized, a fact that will be confirmed by our computation. In contrast, the coupling constant  $\lambda$  is expected to have a non-trivial running since the perturbation  $J_+^a J_-^a$  is not exactly marginal. The purpose of this section is to restrict the form of the corresponding  $\beta$ -function  $\beta_\lambda = \mu \frac{d\lambda}{d\mu}$  by symmetry considerations. In the next section we will explicitly compute the  $\beta$ -function and prove that it is compatible with symmetry arguments.

It is useful to extend the range of the coupling constant  $\lambda$  so that  $0 \leq \lambda < \infty$ . Then the following remarkable property

$$S_{-k, 1/\lambda}(g^{-1}) = S_{k, \lambda}(g), \tag{2.1}$$

holds true. This implies a large/small coupling duality under a simultaneous flipping of the sign of the overall coupling  $k$ . This duality severely restricts the form of the RG flow equation for  $\lambda$ . This equation is of the form

$$\beta_\lambda = \mu \frac{d\lambda}{d\mu} = -\frac{1}{2\pi} \frac{f(\lambda)}{k}, \tag{2.2}$$

where  $f(\lambda)$  is a function to be determined. Due to the above duality symmetry the relation

$$f(1/\lambda) = \lambda^{-2} f(\lambda), \tag{2.3}$$

should hold, which severely constrains the function  $f(\lambda)$ . From the structure of the action (1.3) and in particular the fact that it is built up by finite dimensional matrices, it is clear that the function  $f(\lambda)$  should be the ratio of two polynomials. The coefficients of these polynomials can be almost completely determined as follows: Let us first recall that when perturbing a CFT by terms of the form  $\lambda_i \Phi_i$ , where the operators  $\Phi_i$  have anomalous dimensions  $\Delta_i$ , the  $\beta$ -functions for the couplings  $\lambda_i$ , up to two-loops in perturbation theory, are of the form (see, for instance, [13])

$$\frac{d\lambda_i}{dt} = -(2 - \Delta_i)\lambda_i - C_i^{jk} \lambda_j \lambda_k + \mathcal{O}(\lambda^3), \tag{2.4}$$

where  $C_{ijk}$  are the coefficients of the operator product expansions of the operators  $\Phi_i$  among themselves. In our case we have a single operator, namely that  $\Phi_1 = J_+^a J_-^a$  with  $\Delta_1 = 2$ . Using that the  $J_\pm^a$ 's obey two mutually commuting current algebras, we easily compute that  $C_i^{11} = c_G \delta_{i,1}$ , where, as noted,  $c_G$  is the quadratic Casimir in the adjoint representation. That means in our case  $\frac{d\lambda}{dt} = -c_G \lambda^2 + \mathcal{O}(\lambda^3)$ . Then the function  $f(\lambda)$  will be the ratio of two polynomials whose degrees as well as their coefficients, for each one of them separately, will be related via to the above large/small coupling duality symmetry. Clearly, if we know the structure of the zeros and the poles of  $f(\lambda)$  we can determine (almost) completely the RG flow equation for  $\lambda$ . We know that there is only one conformal point in which  $\beta_\lambda = 0$ , i.e. when  $\lambda \rightarrow 0$ , reached in the UV. Therefore  $f(\lambda)$  cannot have any zeros for real  $\lambda$ . There is also no reason to reach a conformal point for  $\lambda$  complex. Hence, we end up with the expression

$$f(\lambda) = -\frac{c_G \lambda^2}{1 + a\lambda + \lambda^2}, \tag{2.5}$$

for some constant  $a$ , which clearly exhibits the correct perturbative behavior. The form of the background's details as discussed in Appendix A suggests that  $a = \pm 2$ . Of course continuing with such type of arguments can leave unsatisfied a skeptical reader. In the next section we will take up the task of actually explicitly proving (1.6), a result corresponding to the value of the constant  $a = 2$ .

## 3. Tour de force

In this section we explicitly compute the  $\beta$ -function for  $\lambda$  using the form of the background.

Being an integer we expect that  $k$  is not running with the energy scale. This is not an assumption as we are going to prove it at the end of this section. It is convenient to write the metric using a frame  $e^a = e_\mu^a dX^\mu$ , as  $g_{\mu\nu} = e_\mu^a e_\nu^a$ . Any other frame  $\tilde{e}^a$  will be related to this one by an orthogonal transformation, i.e.  $\tilde{e}^a = \Lambda^{ab} e^b$ . In our case these are given by [1]<sup>4</sup>

$$e^a = \sqrt{k(1-\lambda^2)} (D - \lambda \mathbb{I})_{ab}^{-1} R^b, \quad \Lambda = \frac{D - \lambda \mathbb{I}}{\mathbb{1} - \lambda D}. \tag{3.1}$$

The one-loop  $\beta$ -function equations are given by [16–18]

$$\frac{dg_{\mu\nu}}{dt} - \frac{dB_{\mu\nu}}{dt} = R_{\mu\nu}^+ + \nabla_\nu^+ \xi_\mu, \tag{3.2}$$

where the second term corresponds to diffeomorphisms along  $\xi^\mu$ . Passing to the tangent space indices and using the definitions

$$\frac{dg_{\mu\nu}}{dt} = \beta_{ab}^g e_\mu^a e_\nu^b, \quad \frac{dB_{\mu\nu}}{dt} = \beta_{ab}^B e_\mu^a e_\nu^b, \tag{3.3}$$

we have that

$$\beta_{ab}^g - \beta_{ab}^B = R_{ab}^+ + \nabla_b^+ \xi_a. \tag{3.4}$$

We next compute the left hand side of this equation. Since the WZW model first term in (1.3) does not depend on the parameter  $\lambda$  we immediately have that

<sup>2</sup> It is tempting to suggest that the exact to all orders in  $1/k$  action is given by (1.3), but with  $k$  replaced by  $k + c_G$ . This replacement is in accordance with the exact map  $k \rightarrow -k - 2c_G$  of [6] we mentioned above.

<sup>3</sup> For recent developments and the usage of non-Abelian T-duality in supergravity, string theory and the gauge/gravity correspondence, as well as additional references in the literature, the reader is advised to consult [8–10].

<sup>4</sup> We reinstate the overall factor  $\sqrt{k(1-\lambda^2)}$  as compared to the correspondent expression in [1].

$$\begin{aligned} \frac{dg_{\mu\nu}}{dt} + \frac{dB_{\mu\nu}}{dt} &= 2k \frac{d}{dt} (\lambda R_{\mu}^a (1 - \lambda D^T)^{-1} L_{ab}^b) \\ &= 2k \frac{d\lambda}{dt} R_{\mu}^a [(1 - \lambda D^T)^{-1} (1 - \lambda D^T)^{-1}]_{ab} L_{\nu}^b \\ &= \frac{2}{1 - \lambda^2} \frac{d\lambda}{dt} e_{\mu}^a \Lambda_{ba} e_{\nu}^b, \end{aligned} \quad (3.5)$$

where in the last step we used the definition of the frame (3.1). Then by letting the group element  $g \rightarrow g^{-1}$  which reverses the sign of  $B_{\mu\nu}$ , we obtain that

$$\begin{aligned} \frac{dg_{\mu\nu}}{dt} - \frac{dB_{\mu\nu}}{dt} &= \frac{2}{1 - \lambda^2} \frac{d\lambda}{dt} \tilde{e}_{\mu}^a \Lambda_{ab} \tilde{e}_{\nu}^b \\ &= \frac{2}{1 - \lambda^2} \frac{d\lambda}{dt} e_{\mu}^a \Lambda_{ab} e_{\nu}^b, \end{aligned} \quad (3.6)$$

from which

$$\beta_{ab}^g - \beta_{ab}^B = \frac{2}{1 - \lambda^2} \frac{d\lambda}{dt} \Lambda_{ab}. \quad (3.7)$$

The right hand side of the one-loop RG flow equation (3.4) can also be worked out. Indeed, by choosing for  $\xi_a = -c_2 f_{abc} \Lambda_{bc}$  and using (A.8), (A.6) and (A.12), we can prove that

$$R_{ab}^+ + \nabla_b^+ \xi_a = -c_G c_2^2 \Lambda_{ab}. \quad (3.8)$$

Plugging (3.8) and (3.7) in (3.4) and using the expression (A.9) for the constant  $c_2$ , we readily find that the RG flow equation reads

$$\frac{d\lambda}{dt} = -\frac{c_G \lambda^2}{2k(1 + \lambda)^2}, \quad (3.9)$$

which is nothing but (1.6).<sup>5</sup> This is a quite simple formula, valid for all simple compact groups and constitutes one of the main results of present paper. It is essentially universal in the sense that its dependence on the group is only through the overall coefficient  $c_G$ . In fact (3.9) can be solved explicitly, leading to

$$\lambda - \frac{1}{\lambda} + \ln \lambda^2 = -\frac{c_G}{2k} (t - t_0), \quad (3.10)$$

where  $t_0$  is an integration constant. In the UV at  $t \rightarrow \infty$ , we have that  $\lambda \rightarrow 0$  and one reaches the conformal point described by the WZW action. Towards the IR at  $t = t_0$  one reaches the self-dual point  $\lambda = 1$  corresponding to the non-Abelian T-dual of the PCM as mentioned above.

As was discussed, the form of the RG flow equations is such that  $k$  does not run. Its quantization of topological nature [15] remains unaltered at one-loop, a fact which is expected to hold true to all orders in perturbation theory. For completeness we note that if we had not assumed that  $k$  would remain fixed, we would have obtained instead of (3.7) that

$$\beta_{ab}^g - \beta_{ab}^B = \frac{2}{1 - \lambda^2} \frac{d\lambda}{dt} \Lambda_{ab} + \frac{1}{k} \frac{dk}{dt} (\delta_{ab} - b_{ab}), \quad (3.11)$$

where  $b$  is a matrix defined from the antisymmetric two-form as  $B_{\mu\nu} = b_{ab} e_{\mu}^a e_{\nu}^b$ . Clearly, only by requiring that  $\frac{dk}{dt} = 0$  we can match with (3.8).

#### 4. Renormalization group flow on cosets

Closely related to (1.3) there is a class of models interpolating between exact coset  $G/H$  CFT realized by gauged WZW models and the non-Abelian T-duals of the PCM for the geometric coset  $G/H$  spaces [1]. These models have not been shown to be integrable, though we expect integrability for the cases that  $G/H$  is a symmetric space. For the case of  $G = SU(2)$  and  $H = U(1)$  the details have been worked out [1]. The result for the  $\sigma$ -model action can be presented as

$$\begin{aligned} S &= \frac{k}{\pi} \int \left[ \frac{1 - \lambda}{1 + \lambda} (\partial_+ \omega \partial_- \omega + \cot^2 \omega \partial_+ \phi \partial_- \phi) \right. \\ &\quad + 4 \frac{\lambda}{1 - \lambda^2} (\cos \phi \partial_+ \omega + \sin \phi \cot \omega \partial_+ \phi) (\cos \phi \partial_- \omega \\ &\quad \left. + \sin \phi \cot \omega \partial_- \phi) \right]. \end{aligned} \quad (4.1)$$

This action is invariant under the large/small duality symmetry for which  $\lambda \rightarrow 1/\lambda$  and  $k \rightarrow -k$ . It has been shown in [1] that for  $\lambda \ll 1$  this represents the corresponding  $\sigma$ -model action for the coset  $SU(2)_k/U(1)$  CFT perturbed by the parafermion bilinears  $\frac{k\lambda}{\pi} \int (\psi \bar{\psi} + \psi^\dagger \bar{\psi}^\dagger)$ .

In two target space dimensions the one-loop RG flow equation is simply given by

$$\frac{dg_{\mu\nu}}{dt} = \frac{R}{2} g_{\mu\nu} + \nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu}. \quad (4.2)$$

It turns that the above  $\sigma$ -model is one-loop renormalizable and the corresponding RG flow equation for  $\lambda$  is simply given by

$$\frac{d\lambda}{dt} = -\frac{2\lambda}{k}, \quad (4.3)$$

where we also found necessary to employ a diffeomorphism with  $\xi_{\omega} = -\cot \omega$  and  $\xi_{\phi} = 0$ . It is remarkable that this result coincides to the one-loop perturbative result in  $\lambda$ . This follows directly from the general expression (2.4) with scaling dimension  $2 - 2/k$  and, as it turns out, vanishing operator product expansion structure constants. It is also invariant under the large/small symmetry  $\lambda \rightarrow 1/\lambda$  and  $k \rightarrow -k$ , as expected. In the UV one reaches the exact  $SU(2)_k/U(1)$  CFT and towards the IR at  $\lambda = 1$  the theory corresponds to the non-Abelian T-dual of  $S^2$  with respect to  $SU(2)$  via a limiting procedure involving also the coordinates  $\omega$  and  $\phi$ . The details can be found in [1].

#### 5. Concluding remarks and outlook

We have computed the one-loop renormalization group flow for the integrable  $\sigma$ -model action (1.3) interpolating between WZW current algebra models and the non-Abelian T-duals of PCM for a group  $G$ . The  $\beta$ -function for the deformation parameter  $\lambda$  coincided with that computed in the past, and argued to be exact, for the non-Abelian Thirring model. Based on the fact that the two models have the same global symmetries it is natural to suggest that the  $\sigma$ -model action (1.3) is a resummed version of the non-Abelian Thirring model action (given by the first two terms in (1.5)) in which all perturbative, in the deformation parameter  $\lambda$ , effects have been taken into account. To further support our suggestion one could compute using the general results of [1] the analog of the  $\sigma$ -model action (1.3) but with more than one deformation parameters such that when they are small it yields the form of an “anisotropic” non-Abelian Thirring model

$$S_{k,\lambda}(g) = S_{WZW,k}(g) + \frac{k}{\pi} \sum_{a=1}^{\dim G} \lambda_a \int J_+^a J_-^a + \mathcal{O}(\lambda_a^2). \quad (5.1)$$

<sup>5</sup> It turns out that the system of beta-function equations computed for the  $SU(2)$  case in [2] and [14] is consistent with (3.9).

The result for the running of the  $\lambda_a$ 's under the renormalization group flow can then be compared to that obtained in [19] for the case of the  $SU(2)$  group for the “anisotropic” non-Abelian Thirring model with couplings  $\lambda_1 = \lambda_2 \neq \lambda_3$ .

Finally, we note that the integrable models described by the  $\sigma$ -model action (1.3) are distinct from the ones constructed in [11], for which the running of the deformation parameter was computed in [12].

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## Appendix A. The generalized curvature and Ricci tensors

In this appendix we derive the expressions for the generalized Riemann and Ricci tensors constructed using the torsion. We follow the conventions of [1].

The torsionless and metric compatible spin connection is defined by

$$de^a + \omega_{ab} \wedge e^b = 0, \quad \omega_{ab} = -\omega_{ba}. \quad (\text{A.1})$$

The torsionless Riemann 2-forms are constructed as

$$\Omega_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega_{cb} = \frac{1}{2} R_{ab|cd} e^c \wedge e^d. \quad (\text{A.2})$$

In addition, the definition of the torsionfull Riemann 2-forms is

$$\Omega_{ab}^\pm = d\omega_{ab}^\pm + \omega_{ac}^\pm \wedge \omega_{cb}^\pm = \frac{1}{2} R_{ab|cd}^\pm e^c \wedge e^d. \quad (\text{A.3})$$

We also use the symbols  $\omega_{ab|c}$  and  $\omega_{ab|c}^\pm$  defined by

$$\begin{aligned} \omega_{ab} &= \omega_{ab|c} e^c, & \omega_{ab}^\pm &= \omega_{ab|c}^\pm e^c \\ \omega_{ab|c}^\pm &= \omega_{ab|c} \pm \frac{1}{2} H_{abc}. \end{aligned} \quad (\text{A.4})$$

Using the above conventions we can rewrite  $\Omega_{ab}^+$  in the following form

$$\begin{aligned} \Omega_{ab}^+ &= d\omega_{ab|d}^+ \wedge e^d + \omega_{ab|f}^+ \left( \omega_{fe|d}^+ - \frac{1}{2} H_{fed} \right) e^e \wedge e^d \\ &+ \omega_{ac|e}^+ \omega_{cb|d}^+ e^e \wedge e^d. \end{aligned} \quad (\text{A.5})$$

In our case the components  $\omega_{ab|c}^+$  are given by

$$\omega_{ab|c}^+ = -c_2 f_{abd} \Lambda_{dc}, \quad (\text{A.6})$$

while the components of  $H = \frac{1}{6} H_{abc} e^a \wedge e^b \wedge e^c$  are

$$H_{abc} = -c_1 f_{abc} - c_2 (\Lambda_{da} f_{dbc} + \Lambda_{db} f_{dca} + \Lambda_{dc} f_{dab}). \quad (\text{A.7})$$

In order to compute  $\Omega_{ab}^+$  we also need the expression

$$\begin{aligned} d\Lambda_{ab} &= c_1 \Lambda_{ae} f_{ebc} e^c \\ &+ c_2 (f_{abc} - f_{adc} \Lambda_{db} + \Lambda_{ae} f_{edc} \Lambda_{db}) e^c, \end{aligned} \quad (\text{A.8})$$

where

$$c_1 = \frac{1}{\sqrt{k(1-\lambda^2)}} \frac{1+\lambda+\lambda^2}{1+\lambda}, \quad c_2 = \frac{1}{\sqrt{k(1-\lambda^2)}} \frac{\lambda}{1+\lambda}. \quad (\text{A.9})$$

After some algebra the torsionfull Riemann 2-form is found to be

$$\begin{aligned} \Omega_{ab}^+ &= \frac{1}{2} (c_2^2 f_{abe} f_{ecd} + c_1 c_2 f_{abe} f_{cdf} \Lambda_{ef} + 2c_2^2 f_{abe} f_{efc} \Lambda_{fd} \\ &- 2c_2^2 f_{abe} f_{fgc} \Lambda_{ef} \Lambda_{gd} \\ &- c_2^2 f_{abe} f_{efg} \Lambda_{fc} \Lambda_{gd}) e^c \wedge e^d. \end{aligned} \quad (\text{A.10})$$

From the latter we can read off the generalized Riemann tensor

$$\begin{aligned} R_{ab|cd}^+ &= c_2^2 f_{abe} f_{ecd} + c_1 c_2 f_{abe} f_{cdf} \Lambda_{ef} \\ &+ c_2^2 f_{abe} (f_{efc} \Lambda_{fd} - f_{efd} \Lambda_{fc}) \\ &- c_2^2 f_{abe} ((f_{fgc} \Lambda_{gd} - f_{fgd} \Lambda_{gc}) \Lambda_{ef} \\ &+ f_{efg} \Lambda_{fc} \Lambda_{gd}). \end{aligned} \quad (\text{A.11})$$

Then one computes the generalized Ricci tensor. The result is given by

$$\begin{aligned} R_{ab}^+ &= c_G c_2^2 (\delta_{ab} - \Lambda_{ab}) + f_{age} f_{bce} (c_1 c_2 \Lambda_{gc} - c_2^2 \Lambda_{cg}) \\ &+ c_2^2 (f_{ahe} f_{bcg} \Lambda_{hc} \Lambda_{ge} + f_{age} f_{hce} \Lambda_{gb} \Lambda_{hc}). \end{aligned} \quad (\text{A.12})$$

Equivalently the latter could be computed directly through the torsionfull Riemann 2-forms (A.10), i.e.  $e^c \lrcorner \Omega_{ca}^+ = R_{ab}^+ e^b$ .

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