Topological classes of thermodynamics of the four-dimensional static accelerating black holes

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(Received 6 July 2023; accepted 27 September 2023; published 20 October 2023)

In this paper, utilizing the generalized off shell Helmholtz free energy, we explore the topological numbers of the four-dimensional static accelerating black hole and its AdS extension, as well as the static charged accelerating black hole and its AdS extension. Our analysis reveals a profound and significant impact of the acceleration parameter on the topological numbers associated with the static black holes; and different values (nonzero) of the acceleration parameter do not affect the topological numbers of the corresponding four-dimensional static accelerating black holes. In addition, we demonstrate that the electric charge parameter has an important effect on the topological number of the static neutral accelerating black holes, and the cosmological constant has a remarkable influence on the topological number of the static accelerating black hole. Furthermore, it is interesting to observe that the difference between the topological number of the asymptotically flat static accelerating black hole and that of its corresponding asymptotically flat static nonaccelerating black hole is always unity, and the difference between the topological number of the asymptotically AdS static accelerating black hole and that of its corresponding asymptotically AdS static nonaccelerating black hole is always −1. This new observation leads us to conjure that it might be valid also for other accelerating black holes. Of course, this captivating conjecture requires empirical verification through comprehensive investigation into the topological numbers of other accelerating black holes and their corresponding usual counterparts.

DOI: 10.1103/PhysRevD.108.084041

I. INTRODUCTION

In the big family of four-dimensional black hole solutions in General Relativity, in addition to the Schwarzschild black hole and Taub–Newman-Unti-Tamburino (Taub-NUT) spacetime [1,2], another simplest exact vacuum solution is the C-metric [3–6], which represents an accelerating black hole. In fact, it had already been shown [3,7] that the C-metric solution describes a pair of causally separated black holes which accelerate away from each other due to the presence of strings or struts that are represented by conical singularities. Later, it was shown [8] that the C-metric can be derived from the metric of two superposed Schwarzschild black holes by assuming that the mass and location of one of them approaches infinity in an appropriate way. In recent years, aspects of the accelerating black holes, including global causal structure [9], quantum thermal properties [10], holographic heat engines [11,12], black hole shadows [13], holographic complexity [14,15], and so on, have been investigated extensively. In particular, thermodynamics of the AdS_4 C-metric were figured out first in Ref. [16] and then well-addressed in Refs. [17–24], where the first law of thermodynamics [25,26] and the Bekentein-Smarr mass formula [27] as well as the Christodoulou-Ruffini-type squared-mass formula [28,29] are properly extended to accelerating, charged, and rotating black holes.

Naturally, the establishment of the above mass formulas is not the only aspect of the investigation of black hole thermodynamics. Recently, topology has attracted a lot of attention as a mathematical tool to explore the thermodynamic properties of black holes [30–46].[1] Remarkably, a novel approach proposed in Ref. [54] has emerged to examine the thermodynamic topological properties of black holes. This approach interprets black hole solutions as topological thermodynamic defects, establishes topological numbers, and subsequently categorizes black holes into three distinct classes based on their respective topological numbers. This groundbreaking methodology has

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[1] One can also apply topology to study the light rings [47–51] and the timelike circular orbits [52,53].

illuminated new facets of our understanding of the fundamental properties of black holes and gravity. The topological approach outlined in Ref. [54] has gained widespread acceptance due to its adaptability and simplicity. Consequently, it has been successfully employed to investigate the topological numbers associated with various well-known black hole solutions [55–68]. However, the topological number of the accelerating black holes remains virgin territory; it deserves to be explored deeply which motivates us to conduct the present work.

In this paper, we shall investigate the topological number associated with the four-dimensional static accelerating black hole and its AdS extension, as well as the static charged accelerating black hole and its AdS extension. This paper aims to fill the gap in the existing literature by examining the influence of the acceleration parameter on the topological number of black holes, a facet that has been overlooked so far. The findings of this research will provide valuable insights into the crucial role played by the acceleration parameter in determining the topological number of static black holes and their AdS counterparts within the framework of the Einstein-Maxwell gravity theory. We shall witness a constant unity in the difference of the topological number between the asymptotically flat static accelerating black hole and its corresponding asymptotically flat static nonaccelerating black hole. Additionally, we shall observe a consistent -1 difference in the topological number between the asymptotically AdS static accelerating black hole and its corresponding asymptotically AdS static nonaccelerating black hole. We conjecture that they may also be valid for other accelerating black holes.

The remaining part of this paper is organized as follows. In Sec. II, we present a brief review of the thermodynamic topological approach outlined in Ref. [54]. In Sec. III, we first investigate the topological number of the four-dimensional static accelerating black hole by considering the simplest static C-metric solution, and then extend it to the case of the static AdS C-metric solution with a nonzero negative cosmological constant. In Sec. IV, we discuss the topological number of the four-dimensional charged accelerating black hole by considering the Reissner-Nordström C-metric (RN C-metric) solution, and then extend it to the RN-AdS C-metric case. Finally, our conclusion and outlook are given in Sec. V.

II. A BRIEF REVIEW OF THERMODYNAMIC TOPOLOGICAL APPROACH

In this section, we give a brief review of the novel thermodynamic topological approach. According to Ref. [54], we begin by introducing the generalized off shell Helmholtz free energy

$$\mathcal{F} = M - \frac{S}{\tau} \tag{1}$$

for a black hole thermodynamical system with the mass M and the entropy S, where τ is an extra variable that can be viewed as the inverse temperature of the cavity surrounding the black hole. Only when $\tau = T^{-1}$, the generalized Helmholtz free energy (1) manifests its on shell characteristics and converges to the standard Helmholtz free energy $F = M - TS$ of the black hole [18,69–72].

In Ref. [54], a key vector ϕ is defined as

$$\phi = \left(\frac{\partial \mathcal{F}}{\partial r_h}, -\cot\Theta\csc\Theta\right). \tag{2}$$

Within the given framework, the parameters are subject to the conditions $0 < r_h < +\infty$ and $0 \le \Theta \le \pi$. It is important to highlight that the component ϕ^Θ exhibits divergence at $\Theta = 0$ and $\Theta = \pi$, implying an outward direction of the vector in these particular scenarios.

A topological current can be established through the utilization of Duan's theory [73–75] on ϕ-mapping topological currents in the following manner:

$$j^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\rho}\varepsilon_{ab}\partial_\nu n^a \partial_\rho n^b. \quad \mu,\nu,\rho = 0,1,2, \tag{3}$$

Here, we have $\partial_\nu = \partial/\partial x^\nu$ and $x^\nu = (\tau, r_h, \Theta)$. The unit vector n is formulated as $n = (n^r, n^\Theta)$, where $n^r = \phi^{r_h}/\|\phi\|$ and $n^\Theta = \phi^\Theta/\|\phi\|$. It is evident that the conservation of the aforementioned current (3) can be easily demonstrated, leading to $\partial_\mu j^\mu = 0$. Furthermore, it can be promptly shown that the topological current represents a δ-function of the field configuration [49,74,75]

$$j^\mu = \delta^2(\phi) J^\mu\left(\frac{\phi}{x}\right), \tag{4}$$

where the three-dimensional Jacobian $J^\mu(\phi/x)$ fulfills $\varepsilon^{ab} J^\mu(\phi/x) = \varepsilon^{\mu\nu\rho}\partial_\nu\phi^a\partial_\rho\phi^b$. It becomes evident that the value of j^μ vanishes only when $\phi^a(x_i) = 0$, allowing us to derive the topological number W in the subsequent manner,

$$W = \int_\Sigma j^0 d^2x = \sum_{i=1}^{N}\beta_i\eta_i = \sum_{i=1}^{N} w_i. \tag{5}$$

In the given context, β_i represents the positive Hopf index, serving as a count for the number of loops formed by the vector ϕ^a within the ϕ-space as x^μ revolves around the zero point (ZP) z_i. Simultaneously, $\eta_i = \mathrm{sign}(J^0(\phi/x)_{z_i}) = \pm 1$ denotes the Brouwer degree, and w_i denotes the winding number associated with the ith zero point of ϕ enclosed within the domain Σ. Furthermore, in the case that two distinct closed curves, denoted as Σ_1 and Σ_2, encompass the identical zero point of ϕ, it follows that the corresponding winding number must be equivalent. Conversely, if there exists no zero point of ϕ within the enclosed region, it is imperative that $W = 0$.

It is important to emphasize that the local winding number w_i can serve as a valuable tool for characterizing the local thermodynamic stability. Thermodynamically stable black holes correspond to positive values of w_i, while unstable black holes correspond to negative values. On the other hand, the global topological number W represents the difference between the numbers of thermodynamically stable and unstable black holes within a classical black hole solution at a fixed temperature [54]. Hence, the local winding number not only allows for differentiation between different phases of black holes (stable or unstable) within the same black hole solution at a specific temperature, but it also facilitates the classification of black hole solutions based on the global topological number. Moreover, based on this classification, black holes with the same global topological number exhibit similar thermodynamic properties, even if they belong to different geometric classes.

III. STATIC NEUTRAL ACCELERATING BLACK HOLES

In this section, we will investigate the topological number of the four-dimensional static neutral accelerating black hole by considering the simplest static C-metric solution, and then extend it to the case of the static AdS C-metric solution with a nonzero negative cosmological constant.

A. C-metric black hole

An accelerating black hole can be described by the metric [6,76,77]

$$ds^2 = \frac{1}{\Omega^2}\left\{-f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left[\frac{d\theta^2}{g(\theta)} + g(\theta)\sin^2\theta\frac{d\varphi^2}{K^2}\right]\right\}, \tag{6}$$

where

$$f(r) = (1 - A^2r^2)\left(1 - \frac{2m}{r}\right),$$
$$g(\theta) = 1 + 2mA\cos\theta, \qquad \Omega = 1 + Ar\cos\theta,$$

in which K is the conical deficit of the spacetime, m and A are the mass and acceleration parameters, respectively.

The thermodynamic quantities are [18,19]

$$M = \frac{m}{K}, \qquad \mu_\pm = \frac{1}{4}\left(1 - \frac{1 \pm 2mA}{K}\right),$$
$$T = \frac{m}{2\pi r_h^2} - \frac{(r_h - m)A^2}{2\pi}, \qquad S = \frac{\pi r_h^2}{K(1 - A^2r_h^2)}, \tag{7}$$

where $\mu_\pm$ are the tensions of the conical deficits on the north and south poles, r_h are the locations of the event and Cauchy horizons that satisfy the equation: $f(r_h) = 0$.

It is a simple matter to check that the above thermodynamic quantities simultaneously fulfil the first law and the Bekenstein-Smarr relation

$$dM = TdS - \lambda_+ d\mu_+ - \lambda_- d\mu_-, \tag{8}$$

$$M = 2TS, \tag{9}$$

where the thermodynamic lengths [17]

$$\lambda_\pm = \frac{r_h}{1 \pm Ar_h} - m \tag{10}$$

are conjugate to the tensions $\mu_\pm$.

In the subsequent step, we will derive the topological number of the four-dimensional C-metric black hole. The evaluation of the Helmholtz free energy for this black hole can be carried out by utilizing the Euclidean action as follows:

$$I_E = \frac{1}{16\pi}\int_M d^4x\sqrt{g}R + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}(\mathcal{K} - \mathcal{K}_0), \tag{11}$$

where h is the determinant of the induced metric h_{ij}, $\mathcal{K}$ is the extrinsic curvature of the boundary, and $\mathcal{K}_0$ is the subtracted one of the massless C-metric solution as the reference background. The calculation of the Euclidean action integral yields the following result for the Helmholtz free energy

$$F = \frac{I_E}{\beta} = \frac{m}{2K} = M - TS, \tag{12}$$

where $\beta = 1/T$ being the interval of the time coordinate. Furthermore, the last equality of Eq. (12) is valid using the results of Eq. (7). Therefore, the conical singularity (the $(\lambda_\pm - \mu_\pm)$-pairs) has no effect on the calculation of the Helmholtz free energy of the C-metric black hole. It is interesting to make a comparison of the above discussions with our recent works [67,68] on the thermodynamic topology of four-dimensional Taub-NUT spacetimes, where the new conjugate $(\psi - \mathcal{N})$-pair is introduced in the expressions of the Taub-NUT spacetimes. Replacing T with $1/\tau$ in Eq. (12) and substituting $m = r_h/2$, thus the generalized off shell Helmholtz free energy is

$$\mathcal{F} = M - \frac{S}{\tau} = \frac{r_h}{2K} - \frac{\pi r_h^2}{(1 - A^2r_h^2)K\tau}. \tag{13}$$

Using the definition of Eq. (2), the components of the vector ϕ can be easily obtained as follows:

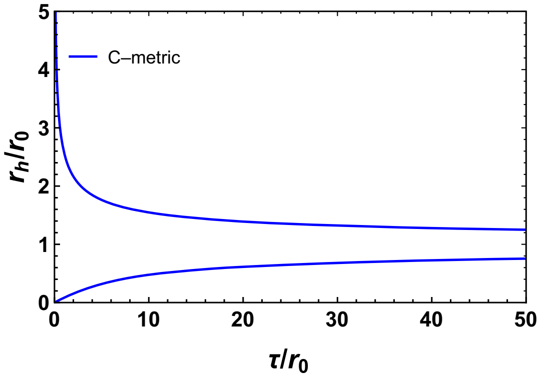


FIG. 1. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $Ar_0 = 1$. There are one thermodynamically stable and one thermodynamically unstable four-dimensional C-metric black hole for any value of τ. Obviously, the topological number is: $W = 1 - 1 = 0$.

$$\phi^{r_h} = \frac{1}{2K} - \frac{2\pi r_h}{(A^2 r_h^2 - 1)^2 K \tau}, \tag{14}$$

$$\phi^{\Theta} = -\cot\Theta \csc\Theta. \tag{15}$$

By solving the equation: $\phi^{r_h} = 0$, one can get a curve on the $r_h - \tau$ plane. For the four-dimensional static accelerating black hole, one can arrive at

$$\tau = \frac{4\pi r_h}{(A^2 r_h^2 - 1)^2}. \tag{16}$$

Note that Eq. (16) consistently reduces to the result obtained in the case of the four-dimensional Schwarzschild black hole [54] when the acceleration parameter A vanishes.

Under the assumption of $Ar_0 = 1$ for the four-dimensional C-metric black hole (other values of A do not affect the topological number of this black hole), we plot Figs. 1 and 2 to visualize key aspects. These figures depict the zero points of the component ϕ^{r_h} and the behavior of the unit vector field n on a portion of the $\Theta - r_h$ plane, with $\tau = 20r_0$. Here, r_0 corresponds to an arbitrary length scale determined by the size of a cavity that encloses the static accelerating black hole. From Fig. 1, one can easily observe that there are one thermodynamically stable and one thermodynamically unstable four-dimensional C-metric black hole for any value of τ. Therefore, it is evident that the C-metric black hole exhibits distinct behavior compared to the Schwarzschild black hole (which always exist one Schwarzschild black hole for any value of τ) [54], emphasizing the significant impact of the acceleration parameter on the thermodynamical stability of the static neutral black hole. Consequently, it would be intriguing to explore deeper into the topological properties of black holes with unusual horizon topologies, such as planar [78], toroidal [79], hyperbolic [80], ultraspinning black holes [81–90], and NUT-charged spacetimes [91–95].

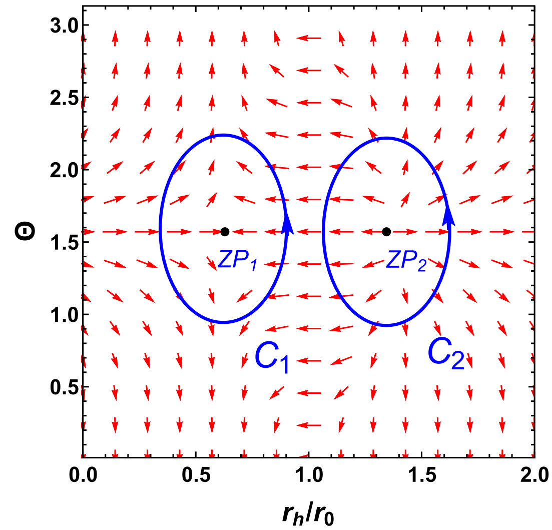


FIG. 2. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the four-dimensional C-metric black hole with $Ar_0 = 1$ and $\tau/r_0 = 20$. The ZPs marked with black dots are at $(r_h/r_0, \Theta) = (0.62, \pi/2)$, and $(1.39, \pi/2)$, respectively. The blue contours C_i are closed loops enclosing the zero points.

In Fig. 2, the zero points are located at $(r_h/r_0, \Theta) = (0.62, \pi/2)$, and $(1.39, \pi/2)$, respectively. Consequently, the winding numbers w_i for the blue contours C_i can be interpreted as follows: $w_1 = -1$, $w_2 = 1$, which deviate from those associated with the Schwarzschild black hole [54]. Regarding the topological global properties, the topological number $W = 0$ for the four-dimensional C-metric black hole can be readily observed from Fig. 2, which also distinguishes it from the topological number of the Schwarzschild black hole ($W = -1$). Thus, it can be indicated that not only do the C-metric black hole and the Schwarzschild black hole exhibit clear differences in terms of geometric topology, but they also belong to different categories from the thermodynamic topological perspective.

B. AdS C-metric black hole

In this subsection, we will extend the above discussions to the cases of the static neutral AdS accelerating black hole by considering the four-dimensional AdS C-metric black hole, whose metric is still given by Eq. (6), but now

$$f(r) = (1 - A^2 r^2)\left(1 - \frac{2m}{r}\right) + \frac{r^2}{l^2},$$

in which the AdS radius l is associated with the thermodynamic pressure $P = 3/(8\pi l^2)$ of the four-dimensional AdS black hole [96–98].

The thermodynamic quantities are [18]

$$M=\frac{m\alpha}{K},\qquad \mu_{\pm}=\frac{1}{4}\left(1-\frac{1\pm 2mA}{K}\right),$$
$$T=\frac{r_h^3+ml^2}{2\pi\alpha r_h^2 l^2}-\frac{(r_h-m)A^2}{2\pi\alpha},\qquad S=\frac{\pi r_h^2}{K(1-A^2r_h^2)},$$
$$V=\frac{4\pi}{3\alpha K}\left[\frac{r_h^3}{(1-A^2r_h^2)^2}+mA^2l^4\right],\qquad P=\frac{3}{8\pi l^2},$$
$$\lambda_{\pm}=\frac{1}{\alpha}\left[\frac{r_h}{1-A^2r_h^2}-m\left(1\pm\frac{2Al^2}{r_h}\right)\right],\tag{17}$$

where the rescaled factor $\alpha=\sqrt{1-A^2l^2}$.

It is easy to verify that the above thermodynamic quantities obey the differential first law and integral Bekenstein-Smarr mass formula simultaneously,

$$dM=TdS+VdP-\lambda_+d\mu_+-\lambda_-d\mu_-,\tag{18}$$

$$M=2TS-2VP.\tag{19}$$

Now, we explore the topological number of the four-dimensional static-neutral accelerating AdS black hole. In order to get the result of the Helmholtz free energy, one can calculate the Euclidean action integral [18]

$$I_E=\frac{1}{16\pi}\int_M d^4x\sqrt{g}\left(R+\frac{6}{l^2}\right)+\frac{1}{8\pi}\int_{\partial M}d^3x\sqrt{h}\left[\mathcal{K}-\frac{2}{l}-\frac{l}{2}\mathcal{R}(h)\right],\tag{20}$$

where $\mathcal{K}$ and $\mathcal{R}(h)$ are the extrinsic curvature and Ricci scalar of the boundary metric $h_{\mu\nu}$, respectively. To eliminate the divergence, the action encompasses not only the standard Einstein-Hilbert term but also includes the Gibbons-Hawking boundary term and the corresponding AdS boundary counterterms [99–103].

With the help of the above expressions in Eq. (17) and utilizing $m=r_h/2+r_h^3/[2(1-\alpha^2r_h^2)l^2]$, the Helmholtz free energy of the four-dimensional AdS C-metic black hole reads

$$F=\frac{I_E}{\beta}=\frac{m\alpha}{2K}-\frac{1}{2\alpha Kl^2}\left[\frac{r_h^3}{(1-A^2r_h^2)^2}+m\alpha^2l^4\right]=M-TS=\frac{M}{2}-VP,\tag{21}$$

where, similar to the case of the four-dimensional C-metric black hole in the last subsection, the conical singularity also has no effect on the calculation of the Helmholtz free energy.

Replacing T with $1/\tau$ and substituting $l^2=3/(8\pi P)$ into Eq. (21), then the generalized off shell Helmholtz free energy simply reads

$$\mathcal{F}=M-\frac{S}{\tau}=\frac{r_h}{24K}\sqrt{16-\frac{6A^2}{\pi P}}\left(3+\frac{8\pi Pr_h^2}{1-A^2r_h^2}\right)-\frac{\pi r_h^2}{(1-A^2r_h^2)K\tau}.\tag{22}$$

Therefore, the components of the vector ϕ are computed as follows:

$$\phi^{r_h}=\frac{1}{24K(A^2r_h^2-1)^2}\left[-8\sqrt{2\pi}Pr_h^2\sqrt{8\pi-\frac{3A^2}{P}}(A^2r_h^2-3)+3\sqrt{16-\frac{6A^2}{\pi P}}(A^2r_h^2-1)^2\right]-\frac{2\pi r_h}{(A^2r_h^2-1)^2K\tau},\tag{23}$$

$$\phi^{\Theta}=-\cot\Theta\csc\Theta,\tag{24}$$

thus one can calculate the zero point of the vector field ϕ^{r_h} as

$$\tau=-\frac{24\pi^{\frac{3}{2}}r_h}{\sqrt{4\pi-\frac{3A^2}{2P}}[8\pi Pr_h^2(A^2r_h^2-3)-3(A^2r_h^2-1)^2]},\tag{25}$$

which consistently reduces to the one obtained in the four-dimensional Schwarzschild-AdS$_4$ black hole case [58] when the acceleration parameter A is turned off. We point out that the generation point satisfies the constraint conditions given by

$$\frac{\partial\tau}{\partial r_h}=0,\qquad \frac{\partial^2\tau}{\partial r_h^2}>0,\tag{26}$$

and the annihilation point obeys the constraint conditions as follows:

$$\frac{\partial\tau}{\partial r_h}=0,\qquad \frac{\partial^2\tau}{\partial r_h^2}<0.\tag{27}$$

Considering the pressure as $Pr_0^2=0.01$ and the acceleration parameter $Ar_0=0.2$ for the four-dimensional AdS C-metric black hole (other values of A do not influence its topological number), we illustrate the zero points of ϕ^{r_h} in the $r_h-\tau$ plane in Fig. 3, and the unit vector field n in Fig. 4 with $\tau=20r_0$, $21r_0$, and $22r_0$, respectively. From Figs. 3 and 4, one can observe that for these values of Pr_0^2 and Ar_0, one generation point and one annihilation point can be found at $\tau/r_0=\tau_a/r_0=20.75$ and $\tau/r_0=\tau_b/r_0=21.77$, respectively. It is evident that there exists a small black hole branch for $\tau<\tau_a$, three distinct black hole branches for $\tau_a<\tau<\tau_b$, and one large black hole branch for $\tau>\tau_b$. Computing the winding number w for these three black hole branches, we find that both the small and large black hole branches have $w=-1$ (thermodynamically unstable), while the intermediate black hole

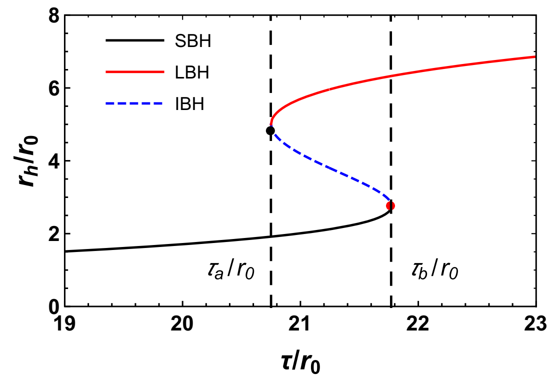


FIG. 3. Zero points of the vector ϕ^{r_h} shown on the $r_h - \tau$ plane with $Pr_0^2 = 0.01$, and $Ar_0 = 0.2$ for the four-dimensional AdS C-metric black hole. The red solid, blue dashed, and black solid lines are for the large black hole (LBH), intermediate black hole (IBH), and small black hole (SBH), respectively. The annihilation and generation points are represented by red and black dots, respectively. Clearly, the topological number is: $W = -1 + 1 - 1 = -1$.

branch has $w = 1$ (thermodynamically stable). Thus, the AdS$_4$ C-metric black hole consistently maintains a topological number of $W = -1$, in contrast to the four-dimensional C-metric black hole discussed in the previous subsection, which possesses a topological number of zero. Therefore, from a thermodynamic topological standpoint, these aforementioned two black holes represent distinct categories of black hole solutions, indicating the importance of the cosmological constant in determining the topological number for the static neutral accelerating black hole. Furthermore, since the topological number of the Schwarzschild-AdS$_4$ black hole is zero, while that of the AdS$_4$ C-metric black hole is -1, it can be inferred that the acceleration parameter has a remarkable effect on the topological classification of the four-dimensional static uncharged AdS black hole.

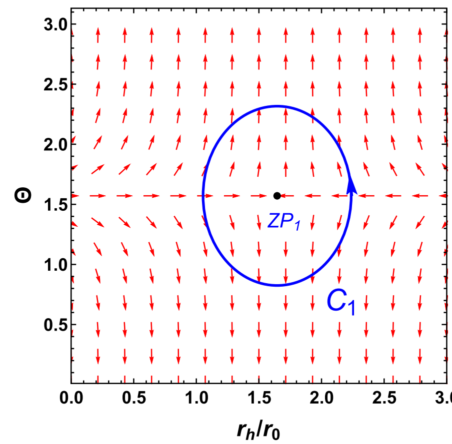
IV. STATIC CHARGED ACCELERATING BLACK HOLES

In this section, we turn to discuss the topological number of the four-dimensional charged accelerating black hole by considering the RN C-metric solution, and then extend it to the RN-AdS C-metric case with a nonzero negative cosmological constant.

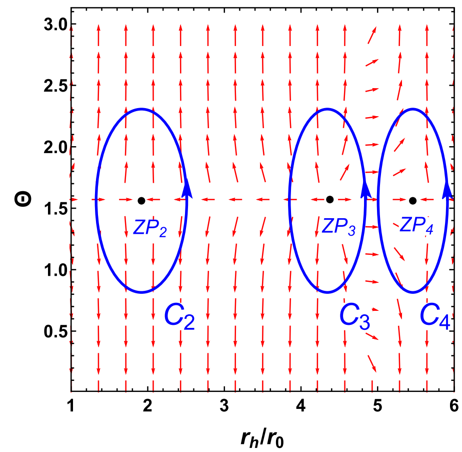
A. RN C-metric black hole

A static charged accelerating black hole is represented by the metric and the Abelian gauge potential [6]

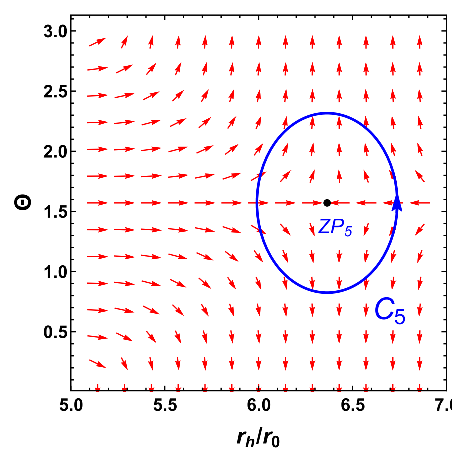
$$ds^2 = \frac{1}{\Omega^2}\left\{-f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left[\frac{d\theta^2}{g(\theta)} + g(\theta)\sin^2\theta\frac{d\varphi^2}{K^2}\right]\right\}, \tag{28}$$



(a) The unit vector field for the four-dimensional AdS-C-metric black hole with $\tau/r_0 = 20$, $Ar_0 = 0.2$, and $Pr_0^2 = 0.01$.



(b) The unit vector field for the four-dimensional AdS-C-metric black hole with $\tau/r_0 = 21$, $Ar_0 = 0.2$, and $Pr_0^2 = 0.01$.



(c) The unit vector field for the four-dimensional AdS-C-metric black hole with $\tau/r_0 = 22$, $Ar_0 = 0.2$, and $Pr_0^2 = 0.01$.

FIG. 4. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane. The ZPs marked with black dots are at $(r_h/r_0, \Theta) = (1.70, \pi/2)$, $(1.98, \pi/2)$, $(4.37, \pi/2)$, $(5.57, \pi/2)$, $(6.41, \pi/2)$, for ZP$_1$, ZP$_2$, ZP$_3$, ZP$_4$, and ZP$_5$, respectively. The blue contours C_i are closed loops surrounding the zero points.

$$F = dB, \qquad B = \frac{q}{r}dt, \tag{29}$$

where

$$f(r) = (1 - A^2r^2)\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right),$$
$$g(\theta) = 1 + 2mA\cos\theta + q^2A^2\cos^2\theta, \quad \Omega = 1 + Ar\cos\theta,$$

in which K is the conical deficit of the charged accelerating black hole, m, q, A are the mass, the electric charge and the acceleration parameters, respectively.

The thermodynamic quantities are [19]

$$M = \frac{m}{\alpha K}, \qquad \mu_\pm = \frac{1}{4}\left(1 - \frac{1 \pm 2mA + q^2A^2}{K}\right),$$
$$T = \frac{mr_h - q^2}{2\pi\alpha r_h^3} - \frac{(r_h - m)A^2}{2\pi\alpha}, \qquad Q = \frac{q}{K},$$
$$S = \frac{\pi r_h^2}{K(1 - A^2r_h^2)}, \qquad \Phi = \frac{q}{\alpha r_h}, \tag{30}$$

where the factor $\alpha = \sqrt{1 + q^2A^2}$, r_h are the locations of the event and Cauchy horizons that obey the horizon equation, $f(r_h) = 0$.

It is easy to verify that the above thermodynamic quantities satisfy the differential first law and the integral Bekenstein-Smarr relation simultaneously

$$dM = TdS + \Phi dQ - \lambda_+ d\mu_+ - \lambda_- d\mu_-, \tag{31}$$

$$M = 2TS + \Phi Q, \tag{32}$$

with the thermodynamic lengths

$$\lambda_\pm = \frac{1 \mp Ar_h}{\alpha(1 + q^2A^2)}\left(r_h - 2m + \frac{m}{1 \pm Ar_h}\right) \tag{33}$$

being conjugate to the tensions $\mu_\pm$.

For the four-dimensional RN C-metric black hole, the expression of the Gibbs free energy can be obtained from the Euclidean action

$$I_E = \frac{1}{16\pi}\int_M d^4x\sqrt{g}(R - F^2) + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}(\mathcal{K} - \mathcal{K}_0), \tag{34}$$

where h represents the determinant of the induced metric h_{ij}, $\mathcal{K}$ denotes the extrinsic curvature of the boundary, and $\mathcal{K}_0$ signifies the subtracted value of the massless C-metric solution used as the reference background. Thus, the Gibbs free energy reads

$$G = \frac{I_E}{\beta} = \frac{r_h^2 - q^2}{4\alpha K r_h} = M - TS - \Phi Q = \frac{M - \Phi Q}{2}, \tag{35}$$

where $\beta = T^{-1}$ denotes the interval of the time coordinate, the last two equalities are valid with the thermodynamic variables in Eq. (30) as required. In addition, it is easy to see that the conical singularity also has no effect on the calculation of the Gibbs free energy in this case.

Next, we will investigate the topological number of the four-dimensional static charged accelerating black hole. We note that the Helmholtz free energy is given by

$$F = G + \Phi Q = M - TS. \tag{36}$$

It is very easy to obtain the generalized off shell Helmholtz free energy as

$$\mathcal{F} = M - \frac{S}{\tau} = \frac{r_h^2 + q^2}{2\alpha K r_h} - \frac{\pi r_h^2}{(1 - A^2r_h^2)K\tau}. \tag{37}$$

Then, the components of the vector ϕ are

$$\phi^{r_h} = \frac{r_h^2 - q^2}{2\alpha K r_h^2} - \frac{2\pi r_h}{(A^2r_h^2 - 1)^2 K\tau}, \tag{38}$$

$$\phi^\Theta = -\cot\Theta\csc\Theta. \tag{39}$$

Thus, by solving the equation, $\phi^{r_h} = 0$, one can compute the zero point of the vector field ϕ as

$$\tau = \frac{4\pi\alpha r_h^3}{(r_h^2 - q^2)(A^2r_h^2 - 1)^2}. \tag{40}$$

We point out that Eq. (40) consistently reduces to the one obtained in the case of the four-dimensional RN black hole [54] when the accelerating parameter A vanishes.

For the four-dimensional RN C-metric black hole, we take $Ar_0 = 1$, $q/r_0 = 1$, and plot the zero points of the component ϕ^{r_h} in Fig. 5, and the unit vector field n with $\tau/r_0 = 20$ in Fig. 6, respectively. By the way, we also point out that different values (nonzero) of the acceleration parameter A do not affect the topological number of the four-dimensional RN C-metric black hole. Obviously, there is only one thermodynamically stable four-dimensional RN C-metric black hole for any value of τ. Based upon the local property of the zero point, one can obtain the topological number $W = 1$ for the four-dimensional RN C-metric black hole, which is different from that of the four-dimensional RN black hole ($W = 0$) [54]. This fact indicates that the acceleration parameter parameter plays an crucial role in determining the topological number of the four-dimensional static-charged black hole. In addition, compared with the four-dimensional RN black hole which has a topological number of zero, it can be inferred that the electric charge parameter also has an important effect on the

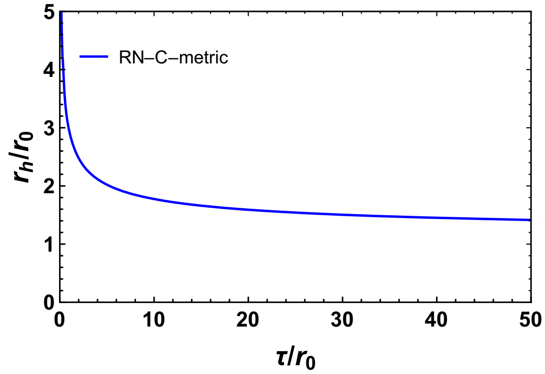


FIG. 5. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $Ar_0 = 1$ and $q/r_0 = 1$ for the four-dimensional RN C-metric black hole. There is one thermodynamically stable four-dimensional RN-C-metric black hole for any value of τ. Obviously, the topological number is $W = 1$.

topological number for the four-dimensional C-metric black hole.

B. RN-AdS C-metric black hole

In this subsection, we will extend the discussions in the last subsection to the cases of the static charged accelerating AdS black hole by considering the four-dimensional RN-AdS C-metric black hole, whose metric and the Abelian gauge potential are still given by Eqs. (28) and (29), but now

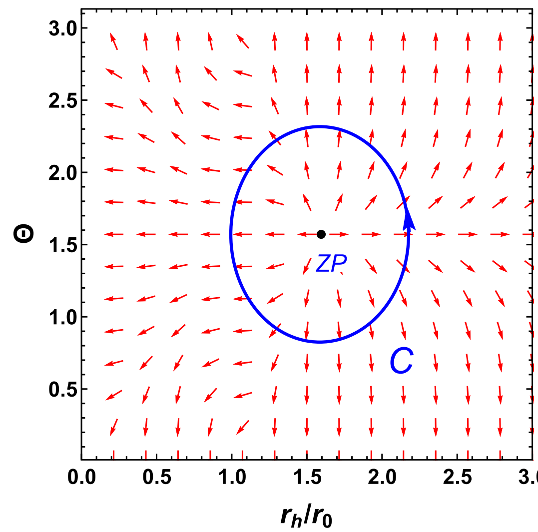


FIG. 6. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the four-dimensional RN C-metric solution with $Ar_0 = 1$, $q/r_0 = 1$, and $\tau/r_0 = 20$. The ZP marked with black dot is at $(r_h/r_0, \Theta) = (1.59, \pi/2)$. The blue contour C is a closed loop enclosing the zero point.

$$f(r) = (1 - A^2r^2)\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) + \frac{r^2}{l^2}.$$

The thermodynamic quantities are [19]

$$\begin{aligned}
M &= \frac{m[1 - A^2l^2(1 + A^2q^2)]}{\alpha K}, \qquad S = \frac{\pi r_h^2}{K(1 - A^2r_h^2)},\\
T &= \frac{1}{4\pi\alpha}\left[-2A^2r_h\left(1 - \frac{2m}{r_h} + \frac{q^2}{r_h^2}\right) + 2(1 - A^2r_h^2)\left(\frac{m}{r_h^2} - \frac{q^2}{r_h^3}\right) + \frac{2r_h}{l^2}\right],\\
Q &= \frac{q}{K}, \qquad \Phi = \frac{q}{\alpha r_h}, \qquad P = \frac{3}{8\pi l^2},\\
V &= \frac{4\pi l^4}{3\alpha K r_h^5}\left\{4m^2r_h^2 + mr_h\left[A^2(1 + A^2q^2)r_h^4 - 4r_h^2 - 4q^2\right] + (r_h^2 + q^2)^2\right\},\\
\mu_\pm &= \frac{1}{4}\left(1 - \frac{1 \pm 2mA + q^2A^2}{K}\right),
\end{aligned} \tag{41}$$

where the factor $\alpha = \sqrt{1 - A^2l^2(1 + A^2q^2)}\sqrt{1 + A^2q^2}$, and r_h is the largest root of the horizon equation, $f(r_h) = 0$. Then one can verify that the above thermodynamical quantities completely satisfy both the first law and the Bekenstein-Smarr mass formula

$$dM = TdS + \Phi dQ + VdP - \lambda_+ d\mu_+ - \lambda_- d\mu_-, \tag{42}$$

$$M = 2TS + \Phi Q - 2VP, \tag{43}$$

with the thermodynamic lengths

$$\begin{aligned}
\lambda_\pm = &-\frac{l^2}{(1 + A^2q^2)\alpha r_h^4}\Big[2m^2r_h(1 - A^2r_h^2)\\
&+ m(Ar_h \mp 1)(Ar_h^3 \pm 2A^2q^2r_h^2 \pm 3r_h^2 + Aq^2r_h \pm q^2)\\
&\pm r_h(1 + A^2q^2)(A^3q^2r_h^3 \pm r_h^2 - Aq^2r_h \pm q^2)\Big],
\end{aligned} \tag{44}$$

being conjugate to the tensions $\mu_\pm$.

Now, we investigate the Gibbs free energy of the four-dimensional RN-AdS C-metric black hole via the Euclidean action integral. The expression of the Euclidean action is given as

$$I_E = \frac{1}{16\pi}\int_M d^4x\sqrt{g}\left(R+\frac{6}{l^2}-F^2\right) + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}\left[\mathcal{K}-\frac{2}{l}-\frac{l}{2}\mathcal{R}(h)\right], \tag{45}$$

where $\mathcal{K}$ and $\mathcal{R}(h)$ are the extrinsic curvature and Ricci scalar of the boundary metric $h_{\mu\nu}$, respectively. Along with the standard Einstein-Hilbert term, the action also contains the Gibbons-Hawking boundary term and the corresponding AdS boundary counterterms in order to eliminate the divergence. Thus, the Gibbs free energy can simply obtain as

$$G = \frac{I}{\beta} = \frac{mr_h - q^2}{2\alpha K r_h} - \left\{\frac{2m^2}{\alpha K r_h^3} + \frac{(r_h^2+q^2)^2}{2\alpha K r_h^5} + \frac{m[A^2(1+A^2q^2)r_h^4 - 2r_h^2 - 2q^2]}{\alpha K r_h^4}\right\} l^2$$
$$= M - TS - \Phi Q = \frac{M-\Phi Q}{2} - VP, \tag{46}$$

where $\beta = T^{-1}$ denotes the interval of the time coordinate, the last two equalities are valid with the thermodynamic variables in Eq. (41) as required. Furthermore, it is easy to see that the conical singularity does not affect the calculation of the Gibbs free energy in this case either.

In order to establish the thermodynamic topological number of the four-dimensional static charged accelerating AdS black hole, we need to obtain the expression of the generalized off shell Helmholtz free energy in advance. The Helmholtz free energy is given by

$$F = G + \Phi Q = M - TS. \tag{47}$$

Using the definition of the generalized off shell Helmholtz free energy (1) and $l^2 = 3/(8\pi P)$, one can easily get

$$\mathcal{F} = \frac{\sqrt{8\pi - 3(A^4q^2+A^2)P^{-1}}}{4\sqrt{2\pi(1+A^2q^2)}Kr_h}\left[r_h^2 + q^2 + \frac{8\pi P r_h^4}{3(1-A^2r_h^2)}\right] - \frac{\pi r_h^2}{(1-A^2r_h^2)K\tau}. \tag{48}$$

Thus, the components of the vector ϕ are computed as follows:

$$\phi^{r_h} = \frac{\sqrt{16\pi - 6(A^4q^2+A^2)P^{-1}}}{24\sqrt{\pi(1+A^2q^2)}(A^2r_h^2-1)^2Kr_h^2}\left[3(r_h^2-q^2)(A^2r_h^2-1)^2 - 8\pi P r_h^4(A^2r_h^2-3)\right] - \frac{2\pi^{\frac{3}{2}}r_h\sqrt{1+A^2q^2}}{\sqrt{\pi(1+A^2q^2)}(A^2r_h^2-1)^2K\tau},$$
$$\phi^{\Theta} = -\cot\Theta\csc\Theta.$$

So the zero point of the vector field ϕ is

$$\tau = -\frac{24\pi^{\frac{3}{2}}r_h^3\sqrt{2(1+A^2q^2)}}{\sqrt{8\pi-3(A^4q^2+A^2)P^{-1}}}\left[8\pi P r_h^4(A^2r_h^2-3) + 3(q^2-r_h^2)(A^2r_h^2-1)^2\right]. \tag{49}$$

Similar to the procedure adopted before, for the four-dimensional RN-AdS C-metric black hole, we show the zero points of the component ϕ^{r_h} with $q = r_0$, $Ar_0 = 0.2$, and $Pr_0^2 = 0.01$ in Fig. 7, and the unit vector field n with $\tau = 22r_0$, $q = r_0$, $Ar_0 = 0.2$, and $Pr_0^2 = 0.01$ in Fig. 8. In addition, we also point out that different values of A have no effect on the topological number of the four-dimensional RN-AdS C-metric black hole. Note that for these values of $q = r_0$, $Ar_0 = 0.2$ and $Pr_0^2 = 0.01$, in Fig. 7, one generation point can be found at $\tau/r_0 = \tau_c/r_0 = 21.56$, and there are a thermodynamically unstable four-dimensional RN-AdS-C-metric black hole and a thermodynamically stable four-dimensional RN-AdS-C-metric black hole when $\tau = \tau_1$. Based on the local property of the zero points, we obtain the topological number of the four-dimensional RN-AdS C-metric black hole is $W = 0$, while that of the RN-AdS black hole is $W = 1$ [54]. Consequently, the introduction of the acceleration parameter brings about a substantial transformation in the topological number of the four-dimensional RN-AdS black hole. Moreover, the contrasting topological numbers

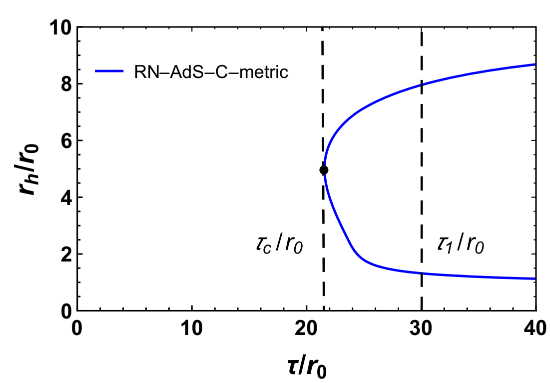


FIG. 7. Zero points of the vector ϕ^{r_h} shown on the $r_h - \tau$ plane with $q/r_0 = 1$, $Ar_0 = 0.2$, and $Pr_0^2 = 0.01$ for the four-dimensional RN-AdS C-metric solution. The generation point for this black hole is represented by the black dot with τ_c. There are two four-dimensional RN-AdS C-metric black holes when $\tau = \tau_1$. Clearly, the topological number is $W = -1 + 1 = 0$.

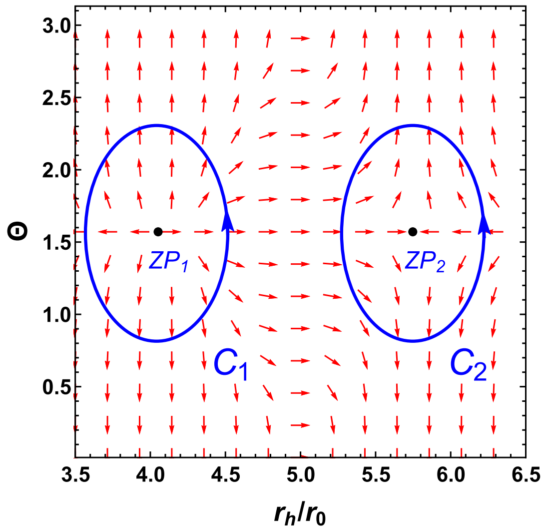


FIG. 8. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane with $q/r_0 = 1, Ar_0 = 0.2, Pr_0^2 = 0.01$, and $\tau/r_0 = 22$ for the four-dimensional RN-AdS C-metric black hole. The ZPs marked with black dots are at $(r_h/r_0, \Theta) = (4.11, \pi/2)$, $(5.78, \pi/2)$ for ZP_1 and ZP_2, respectively. The blue contours C_i are closed loops surrounding the zero points.

between the four-dimensional AdS C-metric black hole ($W = -1$) and the four-dimensional RN-AdS C-metric black hole ($W = 0$) underscores the noteworthy impact of the electric charge parameter on the topological number for the former. Furthermore, the topological number of $W = 1$ exhibited by the RN-C-metric black hole distinguishes it from the RN-AdS C-metric black hole ($W = 0$), emphasizing the important role played by the cosmological constant in determining the topological number for the four-dimensional RN-C-metric black hole.

V. CONCLUSIONS

The results we found in the current paper are presented in Table I. Note that we have also included some known results in the table for comparison purposes.

In this paper, employing the generalized off shell Helmholtz free energy, we investigate the topological numbers of the four-dimensional static accelerating black hole and its AdS extension, along with the static charged accelerating black hole and its AdS extension. We observe that the four-dimensional C-metric black hole and the AdS_4 RN-C-metric black hole fall under the same category of topological classifications, as evidenced by their same topological number of $W = 0$. On the other hand, the four-dimensional AdS C-metric black hole and the four-dimensional RN C-metric black hole belong to other two distinct topological categories, distinguished by their topological numbers of $W = -1$ and $W = 1$, respectively. By the way, it is worth to noting that different values (nonzero) of the acceleration parameter do not influence the topological number of the corresponding four-dimensional static accelerating black hole.

TABLE I. The topological number W, numbers of generation and annihilation points for static accelerating black holes and their usual nonaccelerating counterparts.

BH solution	W	Generation point	Annihilation point
C-metric	0	0	0
Schwarzschild [54]	−1	0	0
AdS C-metric	−1	1 or 0	1 or 0
Schwarzschild-AdS [41]	0	0	1
RN C-metric	1	0	0
RN [54]	0	1	0
RN-AdS C-metric	0	1	0
RN-AdS [54]	1	1 or 0	1 or 0

Furthermore, it will become apparent that the difference in the topological number between the asymptotically flat static accelerating black hole and its corresponding asymptotically flat static nonaccelerating black hole is consistently unity. Moreover, we will take note of the difference in the topological number between the asymptotically AdS static accelerating black hole and its corresponding asymptotically AdS static nonaccelerating black hole, which is always −1. We conjecture that they might also hold for other accelerating black holes. However, the conjecture need to be tested by further investigating the topological numbers of many other accelerating black holes and their common counterparts. Through our analysis, we uncover a profound and significant impact of the acceleration parameter on the topological characteristics of the static black holes.

Additionally, we provide evidence of the crucial role played by the electric charge parameter in determining the topological number for the static neutral accelerating black holes. What is more, we emphasize the remarkable influence exerted by the cosmological constant on the topological number of the static accelerating black hole. A most related issue is to extend the present work to the more general rotating charged accelerating black holes.

ACKNOWLEDGMENTS

We are greatly indebted to the two anonymous referees for their constructive comments to improve the presentation of this work. This work is supported by the National Natural Science Foundation of China (NSFC) under Grants No. 12205243, No. 12375053, and No. 11675130, by the Sichuan Science and Technology Program under Grant No. 2023NSFSC1347, and by the Doctoral Research Initiation Project of China West Normal University under Grant No. 21E028.

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