DIRAC SHELL QUARK-CORE MODEL FOR THE STUDY OF NON-STRANGE BARYONIC SPECTROSCOPY*

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A Dirac shell model is developed for the study of baryon spectroscopy, taking into account the most relevant results of the quark–diquark models. The lack of translational invariance of the shell model is avoided in the present work, by introducing a scalar–isoscalar fictitious particle that represents the origin of quark shell interaction; in this way, the states of the system are eigenstates of the total momentum of the baryon. Only one-particle excitations are considered. A two-quark core takes the place of the diquark, while the third quark is excited to reproduce the baryonic resonances. For the $N(939)$ and $\Delta(1232)$, that represent the ground states of the spectra, the three quarks are considered identical particles and the wave functions are completely antisymmetric. The model is used to calculate the spectra of the N and Δ resonances and the nucleon magnetic moments. The results are compared to the present experimental data. Due to the presence of the core and to the one-particle excitations, the structure of the obtained spectra is analogous to that given by the quark–diquark models.

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1. Introduction

Due to the great difficulties in directly solving the field equations of Quantum Chromo-Dynamics (QCD), light baryon spectroscopy has been widely studied by means of a variety of different quark models. Without any attempt to be exhaustive but only with the aim of introducing some relevant concepts for the development of the present work, we tentatively group these models in the following way: single-particle relativistic models (SPRMs), constituent quark models (CQMs) and quark–diquark models (QDMs).

In general, in the SPRMs, the independent motion of the three quarks is considered, so the wave function of the system is given by the product of three single-particle wave functions, where each wave function represents the state of a quark. In particular, we recall the historically relevant MIT relativistic bag models (RBMs) and the relativistic chiral (shell) model (RCM). In the RBMs [1–8], three massless (or light) quarks are moving inside a spherical bag where a field energy density is also present. In consequence, the quark wave functions are given by standard Dirac spinors with spherical Bessel functions. In the original formulation [1–4], energy quantization is obtained by means of a boundary condition that takes into account the energy-momentum conservation at the surface of the bag. A residual quark–quark vector interaction is introduced to remove the degeneracy between the $N(939)$ and the $\Delta(1232)$. A good description of the ground state properties was achieved but the reproduction of the excited state spectra was less accurate. From a fundamental point of view, the RBMs, as all the SPRMs, are not translationally invariant, so the total wave function does not represent an eigenstate of the total momentum of the baryon. *Ad hoc* procedures are used to subtract from the total energy the spurious contributions of the center-of-mass motion [5, 9, 10].

An RCM was proposed in which the pionic field is explicitly introduced [11]. Moreover, the valence quark interaction is based on the one-pion exchange mechanism. That model contains some ideas that, as it will be explained in the following, have been also used to develop the present work. In particular, in Ref. [11], the author makes the hypothesis that two quarks belong to the ground S-wave shell, while the third quark goes to the excited shells in order to reproduce the baryonic spectra. However, the experimental energies of the resonances are not reproduced with high accuracy because no extra quark–quark interaction is introduced. As in the case of RBMs, a particular technique is used to subtract the spurious effects related to the center-of-mass motion. Subsequently, the same author also developed a field theoretical model in which the translational invariance is completely satisfied from the beginning [12].

The CQMs have represented a very successful method for the study of light baryon spectrum [13–30]. In these theoretical models, baryons are described as bound states of three constituent quarks that can be considered as effective degrees of freedom representing the three valence quarks inside baryons, *dressed* by virtual gluons and $q\bar{q}$ sea pairs. In consequence, for their mass, a much higher value than the QCD *current mass* is generally taken. The spatial dynamical variables that are used to study the three-quark systems in the center-of-mass reference frame, are the Jacobi coordinates $\vec{\rho}$ and $\vec{\lambda}$, where the former represents the distance between the first two quarks and the latter the distance between the third quark and the

center of mass of the first two. In this way, the CQMs are translationally invariant and the wave function is an eigenstate of the total momentum of the baryon. We recall that the total wave function is written as the product of two factors: the *first factor* is given by a sum of products of spatial, spin and isospin terms; the *second factor* represents the antisymmetric (white) color term. In consequence, the *first factor* must be symmetric with respect to the exchange of every pair of quarks. Furthermore, the use of the Jacobi variables gives rise to a quite complex (in any case, not "independent particle") structure for the spatial terms of the quark–quark interaction. Due to this complexity, CQMs have been formulated initially by means of nonrelativistic or relativized Hamiltonians. A fully relativistic study by means of Dirac spinors was given in the model of Refs. [29, 30]. However, also in this model, the quark mass is of the order of 300 MeV, which value is, in any case, much higher than the QCD current quark mass.

In general, in the CQMs, the light baryons can be ordered according to the approximate $\mathrm{SU_f}(3)$ symmetry into the multiplets $[\mathbf{1}]_A \oplus [\mathbf{8}]_M \oplus [\mathbf{8}]_M \oplus [\mathbf{10}]_S$. CQMs reproduce with good accuracy several properties of baryons, such as the strong decays, the magnetic moments and the electromagnetic elastic form factors. However, they predict a larger number of states than the experimentally observed resonances, that is known as *the missing resonance problem*. Furthermore, some states with certain quantum numbers appear in the spectrum at excitation energies much lower than predicted [31]. The problem of the missing resonances [31–33] has motivated the realization of several experiments, such as CB-ELSA [34], CBELSA/TAPS [35], TAPS [36–38], GRAAL [39, 40], SAPHIR [41, 42] and CLAS [43–45], which only provided a few weak indications about some states. Even though several experiments have been dedicated to the search of missing resonances, just a small number of them has been included into the resonance list [31].

Three possible solutions have been proposed for the missing resonance problem:

(1) considering the detection mechanism, some resonances may be very weakly coupled to the single pion, but with higher probabilities of decaying into two or more pions or into other mesons [31–33]; further difficulties can be given by the problem of the separation of the experimental data from the background and by the expansion of the differential cross sections into many partial waves;

(2) theoretically, it is possible to construct effective models that are characterized by a smaller number of *active* degrees of freedom with respect to the three quarks of the CQMs; in this way, the majority of the missing resonances, not yet experimentally observed, are simply *not predicted* by these models;

(3) only a selected set of excited states are retained; in particular, the *one-particle* excited states are taken to represent the experimental spectra of light baryons; this choice, proposed in Ref. [11], is made also in the present work.

We highlight that solution (2) represents the basic assumption of the widely developed QDMs [46–60] that are able to reproduce the baryonic spectra with high accuracy.

The notion of diquark dates back to 1964, when its possibility was mentioned by Gell-Mann [61] in his original paper on quarks. Since then, many papers have been written on this topic (for a review, see Ref. [48]) and, more recently, the diquark concept has been applied to various calculations [49–59, 62–71].

For the present study, we only recall that in Ref. [52], there was developed a nonrelativistic interacting quark–diquark model, *i.e.* a potential model based on the effective degrees of freedom of a constituent quark and diquark. In Refs. [56, 57], it was "relativized" and reformulated within the Point Form formalism [72–74]. In Ref. [58], the wave functions of Refs. [56, 57] were used to compute the nucleon electromagnetic form factors. An accurate reproduction of the baryonic spectra was obtained in a relativistic QDM in which a spin–isospin transition interaction was introduced with the aim of mixing the scalar and the axial-vector diquarks [60]. We shall consider mainly that work for a comparison with the results of the present model.

We point out that in the QDMs, the effective degree of freedom of the diquark is introduced to describe baryons as bound states of a constituent diquark and a quark [46, 47]. In more detail, two quarks are supposed to be strongly correlated (say, *frozen*) in the constituent diquark; their relative motion is assumed to have a vanishing relative orbital angular momentum, that is $L_d = 0$. The diquark can be found in two orthogonal states of spin S_d and isospin T_d: the *scalar diquark* with $S_d = T_d = 0$ and the *axial-vector diquark* with $S_d = T_d = 1$. These quantum numbers are determined considering that:

— the *frozen* quarks of the diquark are identical particles that satisfy the Pauli exclusion principle;

— the color factor is given by the standard antisymmetric (white) function, as in CQMs.

On the other hand, the motion of the quark with respect to the diquark is described by the spatial variable $\vec{r}$ that represents their relative distance. The use of only one spatial variable ($\vec{r}$) instead of the two variables ($\vec{\rho}$ and $\vec{\lambda}$) of the CQMs gives rise to a number of excited states that is *substantially*

reduced with respect to the predictions of the three-quark CQMs. Furthermore, the obtained spectrum has a *one-particle* excitation structure, more consistent with the experimental data.

In all the previously mentioned investigations, the diquarks were used as effective degrees of freedom of the baryonic states. However, within the QDMs, the dynamical mechanism that allows for the diquark formation is not specified: the diquark is directly assumed as a new effective particle with the same quantum numbers of two strongly correlated quarks. In consequence, the wave function is not antisymmetrized with respect to the interchange of a quark *belonging* to the diquark and of the *external* quark.

These theoretical difficulties motivated the development of the present model with the objective of reproducing the same structure of the QDM spectra, that is a one-particle excitation structure and no missing resonances. However, in our model, the three quarks are present as *real* degrees of freedom. In more detail, we construct a quasi-independent particle shell model in which the resonances of the spectra are given by the excitation of one quark, while the other two quarks always remain in the first shell with vanishing orbital angular momentum. These two quarks form a *core*, that replaces the diquark of the QDMs. In particular, due to the antisymmetry of the wave function, the two quarks of the core (analogously to the diquark case) can be found in two orthogonal states: the state with $S_c = T_c = 1$ and the state with $S_c = T_c = 0$, where S_c and T_c, respectively, represent spin and isospin of the core. Above, we have used the definition of "quasi-independent" particle model because, as it will be explained in the following, we introduce a fictitious particle in order to obtain a translational invariant model. In consequence, the kinetic energy associated to that particle does give rise to a non-independent particle operator that, however, will be treated perturbatively.

In Sect. 2, we shall give an overall description of the model. In Sect. 3, starting with the definition of the spatial variables, we shall formally construct the Hamiltonian of the model. In Sect. 4, we shall explain the calculation of the magnetic moments of the proton and neutron. In Sect. 5, the results for the spectra and magnetic moments will be shown and commented. Finally, in Sect. 6, some conclusions will be drawn and some possible perspectives will be illustrated.

The technical details of the model will be analyzed in the appendices. In Appendix A, the main properties of the one-body Dirac equation with spin symmetry will be discussed. In Appendix B, the same formalism will be applied to the case of a harmonic oscillator interaction. The magnetic dipole operator, for the spin symmetry one-body Dirac equation, will be studied in Appendix C. Finally, in Appendix D, the three-quark complete wave

functions will be constructed and the numerical procedure for the solution will be synthetically described. For the calculations, we use the so-called natural units, that is $\hbar = c = 1$.

2. General description of the model

In this section we discuss, at a general level, the different parts of the model with respect to its objectives, making also a critical comparison with the choices of other studies; on the other hand, more details about the formulation of the model will be explained in the next section.

The construcion of the model is based on the five points that are illustrated in the following:

(i) The effective particles of the model are represented by three very *light quarks* and a fictitious scalar–isoscalar particle, denoted, in the following, as "x-particle". Pictorially, we can say that in our model, the baryon "looks like" a lithium atom, in which the electrons are replaced by the quarks and the nucleus is replaced by the x-particle. (To avoid misunderstanding, note that, in the ground states, the three quarks belong to the same shell, while this configuration is forbidden for the electrons of the lithium atom.)

In a pure shell model, the quark interaction would be referred to the origin of the coordinates, violating the translational invariance. This difficulty is avoided here assuming that the quark interaction depends on the distance between the quark and the x-particle. Obviously, the x-particle possesses a momentum and a kinetic energy. In the present model, its momentum, in the Center-of-Mass (CM) reference frame of the baryon, is *opposite* to the *sum* of the quark momenta; in consequence, the baryon state is an eigenstate of the total momentum; in particular, the total momentum is zero in the CM. For the kinetic energy of the x-particle, in the case of a sufficiently high mass, a non-relativistic expansion can be performed and the contributions of this term can be calculated *perturbatively*, without spoiling the independent particle character of the model. For this reason, we introduced above the definition of "quasi-independent" particle model. Furthermore, the mass of the x-particle also represents, in this work, the zero point energy of the spectrum. The contribution of the kinetic energy of the x-particle will be studied in Subsect. 3.5.

Without attempting to attribute a *real* character to the x-particle, we recall that the hypothesis of an effective bosonic particle (the so-called pomeron) is not unusual in the study of other problems of hadronic physics. It was introduced to study baryonic scattering, also in the

framework of QCD; see, for example, Ref. [75]. Some works have identified the pomeron as a tensorial particle [76, 77] and a model has been proposed in which it is represented as bound state of two effective gluons [78].

To avoid confusion, we give here a brief terminological explanation: we shall use the term *core* (introduced above for the two quarks of the first shell) without including the x-particle, for the two following reasons: *(i)* in the present work, the x-particle is essentially considered as a *fictitious* particle; *(ii)* in any case, it interacts also with the quark not belonging to the *core*. We also note that the x-particle does not bring angular momentum that, as it will be explained in the following, is brought by the two quarks of the core and by the third quark.

(ii) Another relevant objective of our model is the use of very *light quarks* without appealing for a mechanism that generates the constituent mass. More precisely, we consider the standard value of the "current-quark mass" that is estimated by means of a mass-independent subtraction scheme in the QCD theory. In particular, for this model, we take the mean value of the up and down quark, that is $m_q = (m_u + m_d)/2 = 3.5$ MeV [79].

Due to this hypothesis, the quark motion is extremely relativistic. Correspondingly, the formulation of a model with a three-body ultra-relativistic equation, suitable for this choice, would involve a high level of complexity. This is another argument (beyond the structure of the spectrum) to prefer, at this stage, a quasi-single-particle model where the quark motion is described, in a first approximation, by independent Dirac equations.

(iii) As for the interaction in the Dirac equation, we take two central terms of *equal magnitude*: a scalar term and the zero component of a vectorial interaction. With this choice, that corresponds to the so-called "spin symmetry case" [80–86], the quark orbital angular momentum and its spin are decoupled and no spin–orbit interaction is produced. Due to this property, the Dirac equation with spin symmetry was used to study mesonic spectra [82, 84, 86]. Also in the present case of baryonic spectra, this option is strongly favored by the experimental data that show only a very small spin–orbit splitting of the baryonic resonances. Moreover, we point out that, theoretically, in QDMs, the spin–orbit interaction is usually neglected in a first approximation; see, for example, Ref. [60].

Another interesting property of the Dirac equation with spin symmetry is that it can be transformed into a Schrödinger-like, energy-dependent equation, reducing the numerical complexity of the solution procedure. In the present work, we take a harmonic oscillator interaction to represent the main contribution to quark confinement; we obtain, in this way, an analytically solvable equation. Other contributions, all with spin symmetry, are added to reproduce in more detail the structure of the spectra. The Dirac quark Hamiltonian is introduced in Subsect. 3.3. The Dirac equation with spin symmetry is studied in Appendix A and specialized to the harmonic oscillator interaction in Appendix B.

The present choice of a Dirac equation with spin symmetry strongly differs from that of Ref. [11] where a pseudoscalar interaction related to one-pion exchange was considered.

A phenomenological spin–spin interaction is also introduced to remove the degeneracy between the $N(939)$ and the $\Delta(1232)$ and to reproduce in more detail the resonance levels. For simplicity, also in this case, the spatial dependence is taken as a central function of the quark distance with respect to the position of the x-particle; for this interaction, see Subsect. 3.4.

(iv) We now discuss the implementation in the model of the Pauli exclusion principle that implies the antisymmetric character of the quark wave function. As seen before, this principle is considered as a basic assumption in the SPRMs and CQMs. On the other hand, in QDMs, the quarks inside the diquark do not appear as dynamical degrees of freedom, so, for the quark outside the diquark, no antisymmetrization is required.

For the ground states, *i.e.* the $N(939)$ and the $\Delta(1232)$, within the present model (in which all the tree quarks belong to the first shell) no reason can be found (within the model) to refuse this basic principle. In this sense, we recall that, historically, it compelled the introduction of the color quantum number, when applied to the wave function of the $\Delta(1232)$.

Let us analyze the case of $N(939)$. Assuming, in a standard way, that the spatial term of the wave function is symmetric, the total antisymmetry of the wave function requires a symmetric spin–isospin factor. This symmetric factor must contain the core states $|S_c = T_c = 0\rangle$ and $|S_c = T_c = 1\rangle$ with *equal amplitudes*, that is $a_0 = a_1 = 1/\sqrt{2}$. (We have used S_c and T_c to denote, as before, the spin and the isospin of the pair of quarks 1 and 2 belonging to the core.)

Generally, in QDMs (where the total antisymetrization is not required), the two amplitudes can be not equal; in some QDMs, see, for example, Ref. [56], the amplitude of the state $|S_c = T_c = 1\rangle$ is vanishing. However, if a spin–isospin transition interaction is introduced into the dynamics of the model, we highlight that the solution of the eigenvalue equation gives two amplitudes having very similar values, that is $a_0 = (a_S) = 0.727$, $a_1 = (a_V) = 0.687$ [60], suggesting that, also in QDMs, a symmetric spin–isospin factor can be a good approximation for the $N(939)$ wave function.

The case of the $\Delta(1232)$ is even more obvious: the spatial, spin and isospin factors must be, *all*, symmetric with respect to quark interchange.

Concluding, in the present work, we consider the three quarks, in the ground states, as identical particles, with the standard consequences, discussed above, for the wave functions. These wave functions will be given explicitly in Eqs. (D.5a), (D.5b) and (D.6), for $N(939)$ and $\Delta(1232)$, respectively.

On the other hand, considering that in our model only *one-quark* excitations are taken into account, we make the same hypothesis of the QDMs: the excited quark is considered *not necessarily* identical to the two quarks of the core. This assumption gives the correct spectroscopy, analogously to the QDMs, with no missing resonances. Phenomenologically, this assumption can be justified observing that the excited quark is in a different energy state with respect to the two quarks of the core. In consequence, its *effective* properties, in particular the interaction, are modified by the strong field and its effective interaction is different with respect to the interaction of the two quarks of the core. For this reason, we shall take different parametrizations for the interaction of the quarks in the core and for the interaction of the excited quark.

(v) With the assumptions discussed above, we can introduce here the basic structure of the spectroscopy of the model. Preliminarly, we assign the indices 1 and 2 to the two quarks of the core and the index 3 to the quark that can be excited to higher levels.

In the first place, we analyze the coupling scheme for the angular momenta. For the orbital angular momentum of the core, we have

$$\vec{L}_c = \vec{l}_1 + \vec{l}_2 \,. \tag{1}$$

For the total orbital angular momentum, one has

$$\vec{L} = \vec{L}_c + \vec{l}_3 \,. \tag{2a}$$

However, given that we always have $L_c = 0$, the total orbital angular momentum simply is

$$\vec{L} = \vec{l}_3 \,. \tag{2b}$$

The core spin is

$$\vec{S}_c = \vec{s}_1 + \vec{s}_2 \tag{3}$$

being $S_c = 0, 1$; the total spin is

$$\vec{S} = \vec{S}_c + \vec{s}_3 \,; \tag{4}$$

the possible values for S are $S = 1/2,\ 3/2$.

Finally, the total angular momentum is

$$\vec{J} = \vec{L} + \vec{S} \,. \tag{5}$$

For the isospin of the core, we have

$$\vec{T}_c = \vec{t}_1 + \vec{t}_2 \,, \tag{6}$$

being $T_c = 0, 1$. As discussed above, the Pauli exclusion priciple requires $S_c = T_c$; the total isospin is

$$\vec{T} = \vec{T}_c + \vec{t}_3 \tag{7}$$

and the possible values for T are $T = 1/2$ (N states), and $T = 3/2$ (Δ states); these latter states only have $S_c = T_c = 1$.

We also introduce the parity of the state

$$P = (-1)^L \tag{8}$$

and, finally, $n_r = 0, 1, 2, \ldots$ that represents the radial excitation number of the quark 3.

The states are identified by the following notation:

$$|\Psi\rangle = \left|T; n_r, L, S_c, S, J^P\right\rangle \,, \tag{9}$$

where, for simplicity, the "third components" M_T and M_J have been omitted.

We now consider the list of the "first" states of the model; their quantum numbers are displayed in Table I and Table II, for the N and Δ spectrum, respectively. We have taken the states with $L \leq 2$. For $L = 0$, we have taken $n_r = 0,\ 1$; for $L = 1, 2$, $n_r = 0$ only. With the previous choices, we have taken all the possible values for S_c, S and J. For the case of the N resonances, we have also considered one state with $n_r = 2$ (the last of Table I in order to reproduce the $N(1880)\frac{1}{2}^+$).

The excitation energies of the states of Table I and Table II roughly correspond to the energies of the states with $N \leq 2$ in the standard CQMs [79] but our model predicts *less* states than the CQMs.

TABLE I

Quantum numbers of the first N states of the model.

n_r	L	S_c	S	J^P			
0	0	(0, 1)	1/2	$1/2^+$			
1	0	0	1/2	$1/2^+$			
1	0	1	1/2	$1/2^+$			
1	0	1	3/2	$3/2^+$			
0	1	0	1/2	$1/2^-$	$3/2^-$		
0	1	1	1/2	$1/2^-$	$3/2^-$		
0	1	1	3/2	$1/2^-$	$3/2^-$	$5/2^-$	
0	2	0	1/2	$3/2^+$	$5/2^+$		
0	2	1	1/2	$3/2^+$	$5/2^+$		
0	2	1	3/2	$1/2^+$	$3/2^+$	$5/2^+$	$7/2^+$
2	0	0	1/2	$1/2^+$			

TABLE II

Quantum numbers of the first Δ states of the model.

n_r	L	S_c	S	J^P			
0	0	1	3/2	$3/2^+$			
1	0	1	1/2	$1/2^+$			
1	0	1	3/2	$3/2^+$			
0	1	1	1/2	$1/2^-$	$3/2^-$		
0	1	1	3/2	$1/2^-$	$3/2^-$	$5/2^-$	
0	2	1	1/2	$3/2^+$	$5/2^+$		
0	2	1	3/2	$1/2^+$	$3/2^+$	$5/2^+$	$7/2^+$

The previous states will be used to reproduce the experimental baryonic spectra, without missing resonances, up to 2000 MeV.

The mass values of each state will be determined by the model calculations. Being absent a spin–orbit interaction, the states with different J but with the same values for the other quantum numbers are degenerate. The first state of Table I and that of Table II, respectively, represent the $N(939)$ and the $\Delta(1232)$. As discussed above, the $N(939)$ wave function is completely antisymmetric, requiring $S_c = 0$ and $S_c = 1$ (with equal amplitudes) as it is indicated in the first line of Table I.

3. Construction of the Hamiltonian of the model

3.1. The coordinates and conjugate momenta

The first task is to define the coordinates and the conjugate momenta of the constituents of the model. In a generic frame, we introduce the coordinates $\vec{x}_i$, $\vec{x}_x$ that, respectively, represent the position of the three quarks ($i = 1, 2, 3$) and of the x-particle. The corresponding canonical conjugate momenta are $\vec{k}_i$, $\vec{k}_x$. The three quarks have equal mass m_q; the x-particle mass is m_x.

We now define the intrinsic coordinates $\vec{r}_i$ that will be used in the calculation, and the position of the center of mass $\vec{R}$, in the following way:

$$\vec{r}_i = \vec{x}_i - \vec{x}_x \,, \tag{10a}$$

$$\vec{R} = \frac{m_q \left(\vec{x}_1 + \vec{x}_2 + \vec{x}_3\right) + m_x \vec{x}_x}{m_{\mathrm{t}}} \,, \tag{10b}$$

where, for convenience, we have also introduced the total mass of the constituents

$$m_{\mathrm{t}} = 3m_q + m_x \,. \tag{11}$$

The previous Eqs. (10a) and (10b) can be inverted, giving

$$\vec{x}_i = \vec{R} + \frac{\left(m_x + 2m_q\right)\vec{r}_i - m_q\left(\vec{r}_j + \vec{r}_k\right)}{m_{\mathrm{t}}} \,, \qquad i \neq j \neq k \,, \tag{12a}$$

$$\vec{x}_x = \vec{R} - \frac{m_q\left(\vec{r}_1 + \vec{r}_2 + \vec{r}_3\right)}{m_{\mathrm{t}}} \,. \tag{12b}$$

From the previous Eqs. (12a), (12b), we obtain the intrinsic momenta $\vec{p}_i$, conjugate to $\vec{r}_i$, and the total momentum $\vec{P}$, conjugate to $\vec{R}$, in the following form:

$$\vec{p}_i = \frac{\left(m_x + 2m_q\right)\vec{k}_i - m_q\left(\vec{k}_j + \vec{k}_k + \vec{k}_x\right)}{m_{\mathrm{t}}} \,, \qquad i \neq j \neq k \,, \tag{13a}$$

$$\vec{P} = \vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_x \,. \tag{13b}$$

Finally, inverting the previous equations, one has

$$\vec{k}_i = \vec{p}_i + \frac{m_q}{m_{\mathrm{t}}}\vec{P} \,, \tag{14a}$$

$$\vec{k}_x = -\left(\vec{p}_1 + \vec{p}_2 + \vec{p}_3\right) + \frac{m_x}{m_{\mathrm{t}}}\vec{P} \,. \tag{14b}$$

3.2. The total Hamiltonian

In the following, we shall always work in the CM frame of the baryon, where $\vec{P} = 0$. The Hamiltonian of the model (whose eigenvalues give the baryonic mass spectra) can be schematically written in the following form:

$$H = H_{\mathrm{D}} + H_{ST} + H_x \,, \tag{15}$$

where H_{D}, H_{ST} and H_x, respectively, represent the Dirac quark Hamiltonian, the spin- and isospin-dependent Hamiltonian and the kinetic contribution of the x-particle.

3.3. The Dirac term

The Dirac quark term is

$$H_{\mathrm{D}} = \sum_{i=1}^{3} h\left(\vec{p}_i, \vec{r}_i\right) \tag{16}$$

that represents a sum of three single-particle operators, related to each quark. Note that, as given by Eq. (14a), the $\vec{p}_i$ represent the quark momenta in the CM; the $\vec{r}_i$ are the corresponding conjugate coordinates.

The single-quark Hamiltonian operator has the form of

$$h\left(\vec{p}_i, \vec{r}_i\right) = \vec{\alpha}_i \cdot \vec{p}_i + \beta_i m_q + \omega_i U(r_i) \,. \tag{17}$$

The properties of this single-particle Hamiltonian and its solutions are studied in detail in Appendix A.

For the specific model, we take the interaction $U(r_i)$ in the form of

$$U(r_i) = \tfrac{1}{2} k r_i^2 + U^{(1)}(r_i) \,, \tag{18}$$

where the first term represents the confining harmonic oscillator interaction that will be analyzed in Appendix B. The second term $U^{(1)}(r_i)$ is taken phenomenologically in the form of

$$U^{(1)}(r_i) = -\frac{\tau_{\mathrm{C}}}{r_i}\left[1 - \exp\left(-\frac{r_i}{r_{\mathrm{C}}}\right)\right] + \Delta_i \left[\lambda r_i - \frac{\tau_{\mathrm{G}}}{r_i}\exp\left(-\left(\frac{r_i}{r_{\mathrm{G}}}\right)^2\right)\right], \tag{19a}$$

with

$$\begin{aligned} \Delta_i &= 1 \text{ for } i = 3 \textit{ and} \text{ excited states}, \\ \Delta_i &= 0 \text{ otherwise}. \end{aligned} \tag{19b}$$

The contribution of the first line represents a regularized Coulombic interaction, where $\tau_{\rm C}$ and $r_{\rm C}$ are the effective coupling constant and the regularization radius, respectively. The interaction of the second line, due to the factor Δ_i, is nonvanishing only for quark 3 when it is in an excited state. Moreover, the first term represents a linear confining term, besides the harmonic oscillator interaction of Eq. (18); the second term is a short-range Gaussian interaction that has been introduced to reproduce in detail the energy levels of the spectra. Due to its short range, it is more effective for the states with $L = l_3 = 0$. We recall that also in QDMs, see, for example, [60], a special term, denoted $M_c(q, r)$, was introduced for the states with $L = 0$. The coupling costant of the short-range interaction has been taken as $\tau_{\rm G} = \tau_{\rm C}$ without introducing a new parameter; finally, the constant $r_{\rm G}$ represents the radius of the short-range Gaussian interaction.

3.4. The spin–isospin-dependent term of the Hamiltonian

The spin–isospin-dependent interaction, that is mainly required to reproduce the spin splittings of the spectra, is introduced in a phenomenological way, with a spatial factor that only depends on $\vec{r}_i$, that is the single quark coordinate. In this way, we try to simulate the quark–quark (residual) interaction whose effects cannot be reproduced by the potentials of Eqs. (18) and (19a).

The present interaction term, beyond the standard spin–spin and isospin–isospin operators, also depends on $S_{\rm c}$, S, T and l_i; the last quantity is the angular momentum quantum number of the $i^{\rm th}$ quark. Its expression is inspired by analogous terms of the QDMs.

For clarity, we introduce the following spin–spin operators:

$$\mathcal{S}_1 = \mathcal{S}_2 = \tfrac{1}{2}\left[(\vec{s}_1 + \vec{s}_3)\cdot\vec{s}_2 + (\vec{s}_2 + \vec{s}_3)\cdot\vec{s}_1\right] \tag{20a}$$

for the interaction of the quarks of the core, and

$$\mathcal{S}_3 = (\vec{s}_1 + \vec{s}_2)\cdot\vec{s}_3 \tag{20b}$$

for the interaction of quark 3.

These operators, by definition, are symmetric with respect to interchange of quarks 1 and 2 according to the general properties of the model. We also introduce, by replacing $\vec{s}_i$ with $\vec{t}_i$ in Eqs. (20a) and (20b), the isospin operators $\mathcal{T}_i$.

With those definitions, the spin–isospin-dependent Hamiltonian takes the form of

$$\begin{aligned} H_i^{ST} \;=\; & e^{-\sigma r_i}(-1)^{l_i+1}\{(1-\Delta_i)\left[\mathcal{S}_i A_S + \mathcal{T}_i A_T + \mathcal{S}_i\mathcal{T}_i A_{ST}\right] \\ & +[1+(-1)^{l_i+1}]\Delta_i[\mathcal{S}_i \bar{A}_S + \mathcal{T}_i \bar{A}_T + \mathcal{S}_i\mathcal{T}_i \bar{A}_{ST} \\ & +B_S S(S+1) + B_{Sc} S_{\rm c}(S_{\rm c}+1) + B_T T(T+1)]\}\,, \end{aligned} \tag{21a}$$

and finally,

$$H_{ST} = \sum_{i=1}^{3} H_i^{ST} \,. \tag{21b}$$

We note that the term proportional to $(1-\Delta_i)$, due to *this* factor, is active for the quarks of the core 1, 2; for quark 3, it gives a nonvanishing contribution only in the ground states. On the other hand, the term proportional to Δ_i gives a contribution for quark 3, only in the excited states. The matrix elements of the spin and isospin operators are easily calculated for each state of the model allowing to determine the total contribution of Eq. (21b).

We have taken $\bar{A}_T = \bar{A}_S$, $\bar{A}_{ST} = A_{ST}$ to reduce the number of free parameters of the model without worsening the reproduction of the experimental spectra.

3.5. The kinetic Hamiltonian of the x-particle

The x-particle is assumed to be a scalar particle. In consequence, its kinetic energy is written in the form of

$$H_x = \sqrt{m_x^2 + (\vec{p}_1 + \vec{p}_2 + \vec{p}_3)^2} \,, \tag{22}$$

where we have used Eq. (14b) for the momentum $\vec{k}_x$ of the x-particle, in the CM; furthermore, the product of three Dirac identity operators is understood.

If the mean value of the quark momenta is smaller than m_x, the standard nonrelativistic expansion can be performed as

$$H_x \simeq m_x + \frac{1}{2m_x} \cdot (\vec{p}_1 + \vec{p}_2 + \vec{p}_3)^2 \,. \tag{23}$$

Note that the products of the momenta of different quarks, that is $\vec{p}_i \cdot \vec{p}_j$ with $i \neq j$, give vanishing matrix elements with the wave functions of the model. For this reason, the nonvanishing matrix elements of H_x of Eq. (23) are proportional to the squared quark momenta $\vec{p}_i^{\,2}$ that are single particle operators. Their contributions are calculated perturbatively and added to the total energy of the resonances. We finally note that the x-particle mass, m_x, as shown by Eq. (23), represents, at the same time, the zero point energy of the spectrum.

4. The magnetic moment of the nucleon

In this section, we study the static magnetic properties of the $N(939)$. The interaction of the system with an external electromagnetic three-vector

field $\vec{A}$ is introduced by means of the minimal substitution on the quark momenta; the x-particle, being electrically neutral, does not contribute. The minimal substitution has the standard form: $\vec{k}_i \to \vec{k}_i - e_i\vec{A}(\vec{x}_i)$, where $\vec{x}_i$ and $\vec{k}_i$ are the quark coordinate and momenta in a generic frame and e_i represents the electric charge of the i^{th} quark. However, considering that in our model $m_q \ll m_x$ (see the numerical values of the parameters in Table III) by means of Eq. (13a), one finds that the minimal substitution can be performed directly on the intrinsic quark momenta $\vec{p}_i$ that appear in the Hamiltonian of the model. For the same reason, by means of Eq. (12a), with $\vec{R} = 0$, one can also approximate the generic frame coordinates with the relative ones, that is $\vec{x}_i \simeq \vec{r}_i$.

In more detail, we make the minimal substitution in the Dirac term of the Hamiltonian, given by Eq. (17); the spin–isospin term H_{ST} of Eq. (21a) does not contain the quark momenta and, in consequence, gives no contribution; the kinetic operator of the x-particle does contain the quark momenta and, in principle, could only give a contribution to the *orbital terms* of the magnetic moment of the nucleon. However, we recall that for the $N(939)$, all the quark orbital angular momenta are vanishing; in consequence, only the *spin terms* derived from H_{D}, give a contribution to the nucleon magnetic moment.

For the reasons discussed above, we can take the total magnetic dipole of the system as the sum of the single-quark contributions and make use of the results obtained in Appendix C, in particular the development of Eq. (C.8) and the final result of Eq. (C.11). In this way, we can write the total magnetic dipole operator in the spin–isospin space, in the following form:

$$\vec{\mu} = \sum_{i=1}^{3} e_i G_{0,i}^{(d)} \vec{\sigma}_i \to 3e_3 G_0^{(d)} \vec{\sigma}_3 \,, \tag{24}$$

where, in the last expression on the right-hand side, we have taken into account the (anti)symmetry of the nucleon wave function; we have also dropped the quark index i in $G_0^{(d)}$ recalling that the three quarks have the same spatial wave function, that is $\frac{1}{\sqrt{4\pi}} R_{0,0}(r_i)$, as explained in Appendix D.

The magnetic moments of the nucleon are obtained calculating the mean values of μ_z of Eq. (24) with the spin and isospin factor of the wave function of Eq. (D.5b), taking $M_J = M_s = 1/2$ and $M_T = \pm 1/2$ for the proton and neutron, respectively.

The calculation is performed in the same way as in the CQMs, replacing $\frac{1}{2m}$ with $G_0^{(d)}$. The results, in nuclear magneton units, are

$$\mu_p = 2M_p G_0^{(d)} \tag{25a}$$

for the proton, and

$$\mu_n = -\tfrac{4}{3} M_p G_0^{(d)} \tag{25b}$$

for the neutron, where M_p represents the proton mass.

Note that the ratio of the proton and neutron magnetic moments does not depend on the value of $G_0^{(d)}$ and is, in any case, $\mu_p/\mu_n = -3/2$. The numerical results, obtained with the solutions of the Hamiltonian wave equation, for $G_0^{(d)}$, μ_p and μ_n, are given in Table III.

TABLE III

Comparison between the experimental values [79] of the N resonance masses up to 2 GeV and the results of the model (all mass values are expressed in MeV). Two resonances predicted by the model, with experimental masses above 2 GeV, are shown at the bottom of the table. The quantum numbers J^P, n_r, L, S_c and S have been introduced in Sect. 2; they represent the total angular momentum and parity, the radial excitation, the total orbital angular momentum, the core spin and the total spin, respectively. The states with $S = \frac{3}{2}$ have necessarily $S_\mathrm{c} = 1$.

Resonance	Status	M^{exp} [MeV]	J^P	n_r	L	S_c	S	$M_\mathrm{a}^{\mathrm{calc}}$ [MeV]	$M_\mathrm{b}^{\mathrm{calc}}$ [MeV]
$N(939)$	****	939	$\frac{1}{2}^+$	0	0	$(0,1)$	$\frac{1}{2}$	939	938
$N(1440)$	****	1410–1470	$\frac{1}{2}^+$	1	0	0	$\frac{1}{2}$	1429	1446
$N(1520)$	****	1510–1520	$\frac{3}{2}^-$	0	1	0	$\frac{1}{2}$	1510	1517
$N(1535)$	****	1515–1545	$\frac{1}{2}^-$	0	1	0	$\frac{1}{2}$	1510	1517
$N(1650)$	****	1635–1665	$\frac{1}{2}^-$	0	1	1	$\frac{3}{2}$	1672	1677
$N(1675)$	****	1665–1680	$\frac{5}{2}^-$	0	1	1	$\frac{3}{2}$	1672	1677
$N(1680)$	****	1680–1690	$\frac{5}{2}^+$	0	2	0	$\frac{1}{2}$	1698	1710
$N(1700)$	***	1650–1800	$\frac{3}{2}^-$	0	1	1	$\frac{3}{2}$	1672	1677
$N(1710)$	****	1680–1740	$\frac{1}{2}^+$	1	0	1	$\frac{1}{2}$	1700	1719
$N(1720)$	****	1680–1750	$\frac{3}{2}^+$	0	2	0	$\frac{1}{2}$	1699	1710
$N(1875)$	***	1850–1920	$\frac{3}{2}^-$	0	1	1	$\frac{1}{2}$	1871	1882
$N(1880)$	***	1830–1930	$\frac{1}{2}^+$	2	0	0	$\frac{1}{2}$	1847	1865
$N(1895)$	****	1870–1920	$\frac{1}{2}^-$	0	1	1	$\frac{1}{2}$	1871	1882
$N(1900)$	****	1890–1950	$\frac{3}{2}^+$	1	0	1	$\frac{3}{2}$	1820	1840
$N(2000)$	**	1950–2150	$\frac{5}{2}^+$	0	2	1	$\frac{1}{2}$	1970	1983
$N(2040)$	*	2010–2070	$\frac{3}{2}^+$	0	2	1	$\frac{1}{2}$	1970	1983
$N(2100)$	***	2050–2150	$\frac{1}{2}^+$	0	2	1	$\frac{3}{2}$	2090	2104

5. The results for the spectra and the nucleon magnetic moments

The results of our Dirac shell-core model calculation, compared with the experimental data [79], are shown in Table III and Table IV, for the N and Δ states, respectively. The theoretical results are obtained with two sets of slightly different parameters, namely (a) and (b), given in Table V.

TABLE IV

Comparison between the experimental values [79] of the Δ resonance masses up to 2 GeV and the results of the model. At the bottom of the table, we also show a resonance, predicted by the model, whose real experimental mass is greater than 2 GeV. The units for the masses and the quantum numbers are as in Table III. For all the resonances, one has necessarily $S_c = 1$, that has been omitted in the table.

Resonance	Status	M^{exp} [MeV]	J^P	n_r	L	S	M_a^{calc} [MeV]	M_b^{calc} [MeV]
$\Delta(1232)$	****	1230–1234	$\frac{3}{2}^+$	0	0	$\frac{3}{2}$	1230	1230
$\Delta(1600)$	****	1500–1640	$\frac{3}{2}^+$	1	0	$\frac{3}{2}$	1678	1698
$\Delta(1620)$	****	1590–1630	$\frac{1}{2}^-$	0	1	$\frac{1}{2}$	1687	1672
$\Delta(1700)$	****	1690–1730	$\frac{3}{2}^-$	0	1	$\frac{1}{2}$	1687	1672
$\Delta(1750)$	*	1680–1782	$\frac{1}{2}^+$	1	1	$\frac{1}{2}$	1759	1779
$\Delta(1900)$	***	1840–1920	$\frac{1}{2}^-$	0	1	$\frac{3}{2}$	1902	1903
$\Delta(1905)$	****	1855–1910	$\frac{5}{2}^+$	0	2	$\frac{3}{2}$	1949	1962
$\Delta(1910)$	****	1850–1950	$\frac{1}{2}^+$	0	2	$\frac{3}{2}$	1949	1962
$\Delta(1920)$	****	1870–1970	$\frac{3}{2}^+$	0	2	$\frac{3}{2}$	1949	1962
$\Delta(1930)$	***	1900–2000	$\frac{5}{2}^-$	0	1	$\frac{3}{2}$	1902	1903
$\Delta(1940)$	**	1940–2060	$\frac{3}{2}^-$	0	1	$\frac{3}{2}$	1902	1903
$\Delta(1950)$	****	1915–1950	$\frac{7}{2}^+$	0	2	$\frac{3}{2}$	1949	1962
$\Delta(2000)$	**	2075–2325	$\frac{5}{2}^+$	0	2	$\frac{1}{2}$	2030	2043

The relatively high number of parameters is related to the phenomenological character of the model in which different effects of the interaction are parametrized by means of the potential terms introduced above. In the present work, we have, totally, 14 free parameters, considering that the quark mass $m_q = 3.5$ MeV is obtained from QCD extimations, as explained in Sect. 2; for a comparison, in the QDM of Ref. [60], 15 parameters were used to reproduce the N and Δ spectra. The value of m_x in the present work (see Table V) is greater but of the same order of magnitude as the zero point energy used in Ref. [60], that was $E_0 = 826$ MeV.

TABLE V

Values of the model parameters.

	Set (a)	Set (b)	Units
m_q	3.5	3.5	MeV
m_x	1.574	1.570	GeV
k	0.1611	0.1627	GeV fm^{-2}
$\tau_{\rm C}$	4.292	4.283	
$r_{\rm C}$	0.6695	0.6692	fm
$r_{\rm G}$	0.3322	0.3478	fm
λ	0.2572	0.2442	GeV fm^{-1}
σ	2.326	2.325	fm $^{-1}$
A_S	−0.3293	−0.3326	GeV
A_T	52.01	50.62	MeV
A_{ST}	1.568	1.569	GeV
$\bar{A}_S$	0.2971	0.2952	GeV
B_S	75.15	74.38	MeV
B_{Sc}	−0.1845	−0.1845	GeV
B_T	0.1593	0.1562	GeV

All the experimental data of Tables III and IV have been taken into account to determine (by means of a complex fit procedure) the free parameters of the model. We point out that the quantum number *assignations* of Table I and Table II represent a crucial element to perform the whole process. Moreover, for the degenerate multiplets (with respect to J), the central values of the corresponding experimental mass data have been used.

Our model reproduces all the 3* and 4* resonances up to 2 GeV using the states listed in Table I. For the $N(1880)\frac{1}{2}^+$ only, we have used $n_{\rm r} = 2$.

The experimental masses are reproduced with acceptable accuracy. A slight improvement is obtained with respect to the QDM of Ref. [60]. In general, some discrepancies with the experimental data are found in the degenerate multiplets, given that the spin–orbit interaction has not been included in the model.

Analyzing the N resonances of Table III, we note that the theoretical mass for the $N(1900)\frac{3}{2}^+$ is lower than the experimental data; a better extimation is given by the parameter set (b). In any case, an improvement is obtained with respect to Ref. [60], where the result of the calculation was 1780 MeV. We also note that, for this resonance, the experimental mass interval passed from 1870–1930 MeV of the previous Particle Data Group [87], to the actual value of 1890–1950 MeV.

Our model predicts, with a theoretical mass of 1970 MeV (set (a)) and 1983 MeV (set (b)), a state $N\frac{5}{2}^+$. This state is associated to the resonance $N(2000)\frac{5}{2}^+$, that is a 2* resonance. We also have a $N\frac{3}{2}^+$, degenerate with the

former. This state is tentatively assigned to the $N(2040)\frac{3}{2}^+$ resonance that is a 1^* resonance. Below 2 GeV, no other missing resonances are predicted. Finally, the model predicts a positive parity multiplet, with $\frac{1}{2} \geq J \geq \frac{7}{2}$ at 2090 MeV (set (a)) and 2104 MeV (set (b)). Experimentally, only the $N(2100)\frac{1}{2}^+$ is observed. The $N(2040)\frac{3}{2}^+$ and the $N(2000)\frac{5}{2}^+$ are the only 1^* and 2^* N resonances reported in Table III.

We now analyze the Δ resonances of Table IV.

We note that the $\Delta(1600)\frac{3}{2}^+$ is not reproduced accurately by our model, in particular by set (b). We note that, for this resonance, the experimental mass interval passed from 1500–1700 MeV of the previous Particle Data Group [87] to the actual value of 1500–1640 MeV. The model predicts, besides the 3^* and 4^* Δ resonances up to 2 GeV, a state $\Delta\frac{1}{2}^+$ with a theoretical mass of 1759 MeV (set (a)) and 1779 MeV (set (b)). This state is associated to the $\Delta(1750)\frac{1}{2}^+$, that is a 1^* resonance.

Considering the triplet with $L = 1$ and $S = 3/2$ and $1/2 \geq J \geq 5/2$, the member with $J = 3/2$ is associated to the $\Delta(1940)\frac{3}{2}^-$, 2^* resonance; for this state, our model predicts a mass value of 1902 MeV and 1903 MeV, with set (a) and set (b), respectively.

Finally, our model predicts a doublet with $L = 2$ and $S = 1/2$ at 2030 MeV (set (a)) and 2043 MeV (set (b)). The member of the doublet with $J = 5/2$ is associated to the $\Delta(2000)\frac{5}{2}^+$, 2^* resonance. The other member of the doublet, with $J = 3/2$ is not observed experimentally.

In Table IV, we have reported only the three 1^* and 2^* resonances mentioned above.

In Table VI, we give the results for the factor $G_0^{(d)}$ and the magnetic moments of the nucleon. As in CQMs, the results favourably compare with the experimental data.

TABLE VI

Results for $G_0^{(d)}$ and proton and neutron magnetic moments.

	Set (a)	Set (b)	Exp.	Units
$G_0^{(d)}$	1.506	1.502		GeV^{-2}
μ_p	2.826	2.819	2.793	n.m.u.
μ_n	−1.884	−1.879	−1.913	n.m.u.

6. Conclusions and outlook

In this work, we have developed a Dirac quark shell model to study the baryonic spectra. The experimental data are well-reproduced taking into account only the one-quark excitations.

With respect to CQMs, our model does not introduce any missing resonance up to 2000 MeV. With respect to the QDMs, we obtain baryonic spectra of the same quality. However, our model presents some relevant improvements considering its theoretical consistency. Namely, the diquark is replaced by the two unexcited quarks of the core, without the necessity of introducing a specific *freezing* hypothesis. Moreover, the quark not belonging to the core, having different physical *effective* properties, is not identical to the two quarks of the core and does not require wave fuction antisymmetrization. Finally, the quark wave function is completely relativistic. The use of the Dirac equation with equal scalar and vector potentials (spin symmetry case), avoids the spin–orbit splittings and, moreover, allows to take the QCD value for the quark mass without the necessity of introducing a *dressing* mechanism for this parameter. The same structure of the adopted Dirac equation also gives, with a *small* quark mass, the correct value for the nucleon magnetic moment.

Further investigation is needed to understand, at a more fundamental level, the reason why only one-quark excitations reproduce the baryonic spectra. Finally, the spin–orbit and tensorial interactions should be introduced and the quark–quark residual interaction should be also studied to construct a complete model for the baryonic spectroscopy.

Appendix A

Dirac equation with spin symmetry

The Hamiltonian of Eq. (17) represents the sum of three *independent* Hamiltonian operators (with spin symmetry) for the quarks of the baryon. We now discuss some general properties of a *single particle* Hamiltonian in the case of spin symmetry. For simplicity, in this discussion, we shall drop the quark index i and also put $m_q = m$.

The Hamiltonian operator has the form of

$$h\left(\vec{p},\vec{r}\right) = \vec{\alpha}\cdot\vec{p} + \beta m + \omega U(r)\,, \tag{A.1}$$

where we have introduced the standard Dirac matrices $\vec{\alpha} = \gamma^0\vec{\gamma}$, $\beta = \gamma^0$, in the standard representation, and the following projection operator:

$$\omega = \tfrac{1}{2}(1+\beta)\,. \tag{A.2}$$

Preliminarily, we split the Dirac spinor into two upper and two lower components, that is

$$\psi(\vec{r}) = \begin{pmatrix} \hat{\varphi}(\vec{r}) \\ \hat{\eta}(\vec{r}) \end{pmatrix}\,. \tag{A.3}$$

By applying the ω projection operator to the Dirac spinor, one immediately finds

$$\omega\ \psi(\vec{r}) = \begin{pmatrix} \hat{\varphi}(\vec{r}) \\ 0 \end{pmatrix} . \tag{A.4}$$

We note that the projection operator ω annihilates the lower components of a Dirac spinor. Going back to Eq. (A.1), we also note that the interaction operator $\omega U(r)$ contains a time component of a vector interaction and a scalar interaction, respectively, given by the first and second term of ω, as shown by Eq. (A.2). These two terms have the same spatial dependence

$$V_s(r) = V_v^0(r) = \tfrac{1}{2}U(r) . \tag{A.5}$$

The eigenvalue equation corresponding to the Hamiltonian of Eq. (A.1) is

$$h(\vec{p},\vec{r})\psi(\vec{r}) = E\psi(\vec{r}) . \tag{A.6}$$

Taking into account Eq. (A.4), we can rewrite Eq. (A.6) as two coupled equations, in the following form:

$$\vec{\sigma}\cdot\vec{p}\,\hat{\eta}(\vec{r}) + (m + U(r))\hat{\varphi}(\vec{r}) \ = \ E\hat{\varphi}(\vec{r}) , \tag{A.7a}$$

$$\vec{\sigma}\cdot\vec{p}\,\hat{\varphi}(\vec{r}) - m\hat{\eta}(\vec{r}) \ = \ E\hat{\eta}(\vec{r}) , \tag{A.7b}$$

where $\vec{\sigma}$ represents the vector of the three Pauli matrices. The previous equations can be solved expressing the lower components $\hat{\eta}(\vec{r})$ of Eq. (A.7b) by means of the upper ones $\hat{\varphi}(\vec{r})$; replacing the result in Eq. (A.7a), without approximations, one obtains a Schrödinger-like, *energy-dependent*, equation in the form of [88]

$$\left(\frac{\vec{p}^{\,2}}{E+m} + U(r) + m\right)\hat{\varphi}(\vec{r}) = E\hat{\varphi}(\vec{r}) , \tag{A.8}$$

where we require $E \neq -m$. See also, in the following, Eq. (B.2). Note that, in Eq. (A.8), the spin does not appear explicitly; in consequence, *no spin–orbit effect* is introduced and the spin dependence can be completely factorized. We have

$$\hat{\varphi}(\vec{r}) = \varphi(\vec{r})\chi_{m_s}, \tag{A.9}$$

where $\varphi(\vec{r})$ is a one-component function and χ_{m_s} is a standard Pauli spinor corresponding to the state $|1/2, m_s\rangle$.

The lower components of the Dirac spinor $\hat{\eta}(\vec{r})$ can be obtained straightforwardly from Eq. (A.7b); in this way, the complete four-component Dirac spinor takes the form of

$$\psi(\vec{r}) = N\begin{pmatrix} 1 \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} \end{pmatrix}\varphi(\vec{r})\chi_{m_s} \tag{A.10a}$$

$$= D(\vec{p}\cdot\vec{\sigma}; E, m)\varphi(\vec{r})\chi_{m_s} . \tag{A.10b}$$

In Eq. (A.10a), we have introduced the normalization constant N that will be determined in the following. Equation (A.10b) synthetically defines the operator $D(\vec{p}\cdot\vec{\sigma};E,m)$ that constructs the four components Dirac spinor when it is applied to the corresponding two-component spinor. This operator will be used when studying the complete wave function of the baryonic system in Appendix D.

To calculate the normalization constant N, we preliminarily introduce the normalization integral

$$I = \langle\varphi|1 + \frac{\vec{p}^{\,2}}{(E+m)^2}|\varphi\rangle\,, \tag{A.11a}$$

and, consequently

$$N = I^{-1/2}\,. \tag{A.11b}$$

We recall that, when solving Eq. (A.8), we shall also diagonalize the orbital angular momentum. The indices $n,\ l, m_l$ respectively denote the number of nodes in the radial wave function and the quantum numbers of the orbital angular momentum. In particular, for the upper component wave function, we have

$$\varphi(\vec{r}\,) = \varphi_{n,l,m_l}(\vec{r}\,) = R_{n,l}(r)Y_{l,m_l}(\hat{r})\,. \tag{A.12}$$

Note also that Eq. (A.8), due to its energy-dependence, does *not* represent an eigenvalue equation for a Hermitian operator, for this reason, its solutions are *not* orthogonal with respect to index n: $\langle\varphi_{n',l,m_l}|\varphi_{n,l,m_l}\rangle \neq \delta_{n',n}$; on the contrary, the Dirac spinors of Eqs. (A.10a), (A.10b), with the normalization of Eq. (A.11b), being the eigenstates of the Hermitian Dirac Hamiltonian of Eq. (17), do satisfy standard orthonormality

$$\langle\psi_{n',l',m_l',m_s'}|\psi_{n,l,m_l,m_s}\rangle = \delta_{n',n}\cdot\delta_{l',l}\cdot\delta_{m_l',m_l}\cdot\delta_{m_s',m_s}\,. \tag{A.13}$$

Appendix B

The Harmonic Oscillator interaction

For some forms of $U(r)$, Eq. (A.8) can be solved analytically by using the results of the "corresponding" nonrelativistic equation. In this section, we study in detail the case of a harmonic interaction.

In general, we introduce for convenience the "subtracted" energy $\bar{E}$

$$\bar{E} = E - m\,. \tag{B.1}$$

In this way, Eq. (A.8) can be written as

$$\left(\frac{\vec{p}^{\,2}}{2\left(m+\frac{\bar{E}}{2}\right)} + U(r)\right)\varphi(\vec{r}\,) = \bar{E}\varphi(\vec{r}\,)\,, \tag{B.2}$$

where we have also discarded the Pauli spinor χ_{m_s}, taking into account Eq. (A.9). We note that Eq. (B.2) can be obtained from the nonrelativistic Schrödinger equation by replacing

$$E^{(nr)} \rightarrow \bar{E}\,, \tag{B.3a}$$

$$m \rightarrow m + \frac{\bar{E}}{2}\,. \tag{B.3b}$$

In particular, we consider a harmonic oscillator (HO) interaction

$$U(r) = \frac{1}{2}kr^2\,. \tag{B.4}$$

We recall that in the nonrelativitic case, the energy eigenvalues are

$$E^{(nr)}_{n_e} = \left(n_e + \frac{3}{2}\right)\sqrt{\frac{k}{m}}\,, \tag{B.5}$$

where we have introduced for convenience the energy quantum number n_e that is related to the number of nodes n by the standard equation

$$n_e = 2n + l\,. \tag{B.6}$$

To solve the relativistic problem of Eq. (B.2), with the interaction of Eq. (B.4), we make the replacement of Eq. (B.3b) in Eq. (B.5), obtaining

$$\bar{E} = \left(n_e + \frac{3}{2}\right)\sqrt{\frac{k}{m + \frac{\bar{E}}{2}}}\,, \tag{B.7a}$$

or, equivalently,

$$\bar{E} = E^{(nr)}_{n_e}\sqrt{\frac{m}{m + \frac{\bar{E}}{2}}}\,. \tag{B.7b}$$

This equation can be transformed into a cubic equation for $\bar{E}$ and solved analytically, finding the energy values $\bar{E}_{n_e}$. By means of Eq. (B.1), one has $E_{n_e} = \bar{E}_{n_e} + m$.

In order to determine the form of the radial wave functions, we recall that in the nonrelativistic HO case, these functions depend on the dimensional constant $\bar{r}$, that is given by the equation

$$\bar{r} = (mk)^{-1/4}\,. \tag{B.8}$$

We also write the nonrelativistic harmonic HO radial wave functions as

$$R^{(nr)}_{n,l}(r;\bar{r}) = (\bar{r})^{-3/2}S_{n,l}(x) \tag{B.9a}$$

with

$$x = \frac{r}{\bar{r}}\,. \tag{B.9b}$$

For completeness, we also recall that

$$S_{n,l}(x) = \left[\frac{2(n!)}{\Gamma\left(n+l+\frac{3}{2}\right)}\right]^{\frac{1}{2}} x^l \mathcal{L}_n^{l+\frac{1}{2}}\left(x^2\right) \exp\left(-\frac{1}{2}x^2\right)\,, \tag{B.10}$$

where $\mathcal{L}_n^{l+\frac{1}{2}}(x^2)$ are the generalized Laguerre polynomials. We can now calculate the $R_{n,l}(r)$ for our relativistic problem by performing the substitution of Eq. (B.3b) for the mass m in Eq. (B.8); then, by using Eq. (B.7b), one obtains

$$\bar{r}_{n_e} = \sqrt{\frac{\bar{E}_{n_e}}{E_{n_e}^{(nr)}}} \cdot \bar{r} \tag{B.11}$$

and, in consequence,

$$R_{n,l}(r;\bar{r}_{n_e}) = (\bar{r}_{n_e})^{-3/2} S_{n,l}(x_{n_e}) \tag{B.12a}$$

with

$$x_{n_e} = \frac{r}{\bar{r}_{n_e}}\,. \tag{B.12b}$$

Note that in the relativistic case, the dimensional constant $\bar{r}_{n_e}$ is energy-dependent.

Finally, also the normalization integral of Eq. (A.11a) can be calculated analytically. From this quantity, one obtains the normalization constant of Eq. (A.11b), in the form of

$$N_{n_e} = \left[1 + \left(\bar{E}_{n_e} + 2m\right)^{-3/2} \left(\frac{m}{2}\right)^{1/2} E_{n_e}^{(nr)}\right]^{-1/2}\,. \tag{B.13}$$

Collecting all the results obtained above, we can write the Dirac orthonormal wave functions for HO interaction in the form of

$$\psi_{n;l,m_l;m_s}(\vec{r}) = N_{n_e} \begin{pmatrix} 1 \\ \frac{\vec{p}\cdot\vec{\sigma}}{E_{n_e}+m} \end{pmatrix} R_{n,l}(r;\bar{r}_{n_e}) Y_{l,m_l}(\hat{r}) \chi_{m_s}. \tag{B.14}$$

As in Eq. (A.10b), the operator $D(\vec{p}\cdot\vec{\sigma}; E_{n_e}, m)$ can be introduced.

A wave function with the same spin-angular quantum numbers (l, m_l, m_s), but with a different radial dependence can be expanded by means of the eigenfunctions of Eq. (B.14) in the form of

$$\psi^{(g)}_{n';l,m_l;m_s}(\vec{r}) = \sum_{n=0}^{n_{\max}} a^{n'}_{n;l,m_l;m_s} \psi_{n;l,m_l;m_s}(\vec{r})\,, \tag{B.15}$$

where the upper index g in the wave function of the l.h.s. denotes its general character.

In the present work, we use that expansion for the single-quark wave functions. The amplitudes $a^{n'}_{n;l,m_l;m_s}$ are determined by diagonalyzing the interaction operator in the relativistic HO basis given by the wave functions of Eq. (B.14). We point out that the index n' corresponds to the radial excitation number $n_{\rm r}$ introduced in Eq. (9) of Sect. 2.

Appendix C

The magnetic dipole operator

We now study the magnetic dipole operator for the one-particle Dirac equation with spin symmetry. We recall that in the case of a free Dirac equation, when the interaction with a magnetic field is introduced, one obtains the well-known result for the magnetic dipole operator of a point-like, *free* particle

$$\vec{\mu} = e\, G^{(f)} \vec{\sigma}\,, \tag{C.1a}$$

with

$$G^{(f)} = \frac{1}{2m}\,. \tag{C.1b}$$

In this section, we shall derive an analogous expression for the Dirac equation with interaction in the case of spin symmetry. We start by performing the minimal coupling substitution in Eq. (A.1). We obtain the standard result for the interaction Hamiltonian with an external three-vector field $\vec{A}(\vec{r})$, that is

$$H_{\rm int} = -e\vec{\alpha}\cdot\vec{A}(\vec{r})\,. \tag{C.2}$$

For studying our shell quark model, we consider two different Dirac wave functions $|\psi_a\rangle$, $|\psi_b\rangle$ and calculate the matrix element of the operator of Eq. (C.2) between these wave functions. We do not include in the matrix element the two component spinors $\chi_{m_{s_a}}$, $\chi_{m_{s_b}}$, in order to highlight, as in Eq. (C.1a), the dependence of the magnetic dipole operator on the Pauli matrices $\vec{\sigma}$. Furthermore, we shall use the spatial wave functions $|\varphi_a\rangle$, $|\varphi_b\rangle$ introduced in Eqs. (A.9), (A.10a) and (A.10b). With standard handlings, one obtains

$$\langle\psi_b|H_{\rm int}|\psi_a\rangle = \langle\psi_b|H_{\rm o}|\psi_a\rangle + \langle\psi_b|H_{\rm s}|\psi_a\rangle\,, \tag{C.3}$$

where the first term represents the *orbital contribution*, of the form of

$$\langle\psi_b|H_{\rm o}|\psi_a\rangle = -eN_bN_a\langle\varphi_b|\frac{\vec{A}\cdot\vec{p}}{E_a+m} + \frac{\vec{p}\cdot\vec{A}}{E_b+m}|\varphi_a\rangle\,. \tag{C.4}$$

We shall not develop further this term and focus our attention on the second term that gives the *spin contribution*

$$\langle\psi_b|H_{\rm s}|\psi_a\rangle = -ieN_bN_a\vec{\sigma}\cdot\langle\varphi_b|\frac{\vec{A}\times\vec{p}}{E_a+m} + \frac{\vec{p}\times\vec{A}}{E_b+m}|\varphi_a\rangle\,. \tag{C.5}$$

We consider the case of $|\psi_a\rangle = |\psi_b >= |\psi\rangle$ and, in consequence, $E_a = E_b = E$, *etc.* Furthermore, we take a uniform magnetic field $\vec{B}$, given by $\vec{B} = \vec{\nabla} \times \vec{A}(\vec{r})$. In this way, one easily finds

$$\langle \psi | H_{\rm s} | \psi \rangle = -\vec{\mu} \cdot \vec{B} \tag{C.6a}$$

with

$$\vec{\mu} = e\, G^{(d)} \vec{\sigma} \tag{C.6b}$$

and

$$G^{(d)} = \frac{N^2}{E+m}\,, \tag{C.7}$$

where we have assumed that the wave function $|\varphi\rangle$ is standardly normalized: $\langle \varphi | \varphi \rangle = 1$.

In the case of our model, the expression of Eq. (C.7) cannot be used in a straightforward way for the following reason. The one-particle wave function is expressed as an expansion in the relativistic HO basis, as shown in Eq. (B.15). From that expression, one cannot determine analytically the total normalization constant N; on the other hand, a numerical calculation of that quantity would be affected by numerical uncertainties. In consequence, to calculate the magnetic dipole operator, we prefer to follow a different procedure.

For the calculation, we have in mind the case of the $N(939)$. In consequence, we consider a state with $n' = 0$, $l = m_l = 0$, denoted by $|\psi_0\rangle$. Starting from Eq. (C.5), using the expansion of the Dirac wave function given by Eq. (B.15), and also Eq. (B.14), with standard handlings, one finds

$$\begin{aligned} \langle \psi_0 | H_{\rm s} | \psi_0 \rangle \;=\; & \vec{\sigma} \cdot \frac{e}{4\pi} \sum_{n_a, n_b = 0}^{n_{\max}} a^*_{n_b} a_{n_a} N_{n_{eb}} N_{n_{ea}} \\ & \times \int {\rm d}^3 r \left(\hat{\vec{r}} \times \vec{A}(\vec{r}) \right) \left(\frac{R'_{n_b,0} R_{n_a,0}}{E_{n_{eb}} + m} + \frac{R_{n_b,0} R'_{n_a,0}}{E_{n_{ea}} + m} \right) , \end{aligned} \tag{C.8}$$

where all the indices not relevant for the calculation have been dropped; for brevity, we have also dropped the argument of the radial wave functions: $R_{n,0} = R_{n,0}(r)$; finally, the apex denotes the derivative with respect to the radial coordinate r. For a uniform magnetic field $\vec{B}$, we set

$$\vec{A}(\vec{r}) = \tfrac{1}{2} \vec{B} \times \vec{r}\,. \tag{C.9}$$

By using standard vectorial identities and replacing, under spherical integration, $(\vec{\sigma} \cdot \hat{\vec{r}})(\vec{B} \cdot \hat{\vec{r}}) \to \frac{1}{3} \vec{\sigma} \cdot \vec{B}$, one finally obtains

$$\langle \psi_0 | H_{\rm s} | \psi_0 \rangle = -\vec{\mu} \cdot \vec{B} \tag{C.10a}$$

with

$$\vec{\mu} = e\, G_0^{(d)} \vec{\sigma} \tag{C.10b}$$

and

$$G_0^{(d)} = -\frac{1}{3} \sum_{n_a, n_b=0}^{n_{\max}} a^*_{n_b} a_{n_a} N_{n_{eb}} N_{n_{ea}} \times \int_0^\infty \mathrm{d}r\; r^3 \left(\frac{R'_{n_b,0} R_{n_a,0}}{E_{n_{eb}} + m} + \frac{R_{n_b,0} R'_{n_a,0}}{E_{n_{ea}} + m} \right) . \tag{C.11}$$

We recall again that this expression has been derived for the case of $l = 0$. Taking only one term in the expansion of the wave function, one recovers, with standard handling, the expression of Eq. (C.7). The single-particle spatial matrix element of Eq. (C.11) is used to calculate the magnetic moment of the $N(939)$.

Appendix D

Wave functions and solutions of the Hamiltonian equation

We now specify the form of the total wave functions of the model. To this aim, we take into account the coupling scheme discussed in Sect. 2 and synthetized in Eq. (9). We start with the two-component (Pauli) wave function that (omitting the color factor) can be written by means of the *four factors* given in the following:

(i) We start with the *radial factor*

$$\mathcal{R}_{n_\mathrm{r},L} = R_{0,0}(r_1) R_{0,0}(r_2) R_{n_\mathrm{r},L}(r_3)\,, \tag{D.1}$$

where the first two terms correspond to the quarks 1 and 2; the third term corresponds to the quark 3; for the ground states, one has $n_\mathrm{r} = 0$, $L = 0$;

(ii) The *angular factor* is

$$\mathcal{Y}_{L,M_L} = Y_{0,0}(\hat{r}_1) Y_{0,0}(\hat{r}_2) Y_{L,M_L}(\hat{r}_3) = \frac{1}{4\pi} Y_{L,M_L}(\hat{r}_3)\ ; \tag{D.2}$$

one has $Y_{0,0}(\hat{r}_3) = \frac{1}{\sqrt{4\pi}}$ for the ground states.

(iii) The *spin factor* has the form of

$$\mathcal{X}^{S_\mathrm{c}}_{S,M_S} = [[\chi_{1/2}(1) \otimes \chi_{1/2}(2)]_{S_\mathrm{c}} \otimes \chi_{1/2}(3)]_{S,M_S}\,. \tag{D.3}$$

(iv) Analogously, the *isospin factor* is

$$\mathcal{P}^{T_c}_{T,M_T} = [[\phi_{1/2}(1) \otimes \phi_{1/2}(2)]_{T_c} \otimes \phi_{1/2}(3)]_{T,M_T} \,. \tag{D.4}$$

In Eqs. (D.3) and (D.4), S_c and T_c respectively represent the spin and the isospin quantum numbers of the core.

For the $N(939)$ (ground state, with $S = T = 1/2$), the total wave function can be written in the form of

$$\Phi_N = \mathcal{R}_{0,0} \cdot \mathcal{Y}_{0,0} \cdot \mathcal{Q}^{(N)}_{M_S M_T} \tag{D.5a}$$

with

$$\mathcal{Q}^{(N)}_{M_S,M_T} = \frac{1}{\sqrt{2}} \left[\mathcal{X}^0_{1/2,M_S} \mathcal{P}^0_{1/2,M_T} + \mathcal{X}^1_{1/2,M_S} \mathcal{P}^1_{1/2,M_T} \right] . \tag{D.5b}$$

Note that the spin–isospin factor of Eq. (D.5b) has the same form as the corresponding factor of the CQMs and is *completely symmetric* with respect to quark interchange.

For the $\Delta(1232)$, one has

$$\Phi_\Delta = \mathcal{R}_{0,0} \cdot \mathcal{Y}_{0,0} \cdot \mathcal{X}^1_{3/2,M_S} \cdot \mathcal{P}^1_{3/2,M_T} \tag{D.6}$$

that is also *completely symmetric*.

For the excited states, we have

$$\Phi_{\mathrm{E}} = \mathcal{R}_{n_r,L} \cdot \mathcal{Y}_{L,M_L} \cdot \mathcal{X}^{S_c}_{S,M_S} \cdot \mathcal{P}^{T_c}_{T,M_T} \,. \tag{D.7}$$

The Dirac wave function is constructed applying to these functions the Dirac operators introduced in Eq. (A.10b)

$$\Psi_\Lambda = D_1 D_2 D_3 \Phi_\Lambda \,, \tag{D.8}$$

where Λ stands for N, Δ and E; also $D_i = D(\vec{p}_i \cdot \vec{\sigma}_i; E_i, m_q)$.

The one-body Dirac equation is solved analytically for the harmonic interaction of Eq. (18). With these harmonic oscillator eigenfunctions, we calculate the matrix elements of the interaction $U^{(1)}(r_i)$. We also add the spin–isospin-dependent interaction of Eqs. (21a) and (21b). Then, we diagonalize the total Hamiltonian matrix obtaining the approximate eigenvalues and eigenfunctions of the relativistic equation. For each resonance, we take 10 oscillator eigenfunctions.

Finally, we calculate perturbatively the contributions of H_x with the nonrelativistic expansion of Eq. (23). Due to the single-particle character of the model, the total mass of each resonance is obtained summing the contributions of the three quarks.

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