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Quantum Field Theory and Statistical Systems

Duality in non-Hermitian random matrix theory

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ABSTRACT

We consider 9 Gaussian matrix ensembles characterized by single symmetry among the 38-fold symmetry classification classes of non-Hermitian random matrices, and establish exact duality formulae of certain observables between them. Particularly, averaged products of K characteristic polynomials in an $N \times N$ matrix ensemble can be expressed in terms of another $K \times K$ matrix ensemble. Our method is to combine matrix-valued heat equations and differential identities for determinants and Pfaffians and has more possible applications.

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Table 1
Single symmetry classes, matrix form and Gaussian ensembles.

Class	Matrix form	Gaussian measure
A	$X : N \times N$ complex	**GinUE (dGinUE)**
AI($D^\dagger$)	$X : N \times N$ real	**GinOE**
AII($C^\dagger$)	$X = \begin{bmatrix} X_1 & X_2 \\ -\overline{X_2} & \overline{X_1} \end{bmatrix}$, $X_1 \& X_2 : N \times N$ complex	**GinSE**
$AI^\dagger$	$X : N \times N$ complex symmetric	**dGinSE**
$AII^\dagger$	$X = \mathbb{J}_N Y, Y : 2N \times 2N$ complex antisymmetric	**dGinOE**
D	$X : 2N \times 2N$ complex antisymmetric	**Antisymmetric GinUE**
C	$X = \mathbb{J}_N Y, Y : 2N \times 2N$ complex symmetric	**symplectic[a] dGinSE**
AIII	$X = I_{k,N-k} H, H : N \times N$ Hermitian[b]	**pseudo GUE**
$AIII^\dagger$	$X = \begin{bmatrix} 0 & X_1 \\ X_2 & 0 \end{bmatrix}$, $X_1 \& X_2 : N \times N$ complex	**Chiral GinUE**

[a] "Symplectic" in the name comes from the left multiplication matrix by $\mathbb{J}_N$.
[b] It is equivalent to pseudo-Hermiticity and the unitary matrix $I_{k,N-k} = \mathrm{diag}(\mathbb{I}_k, -\mathbb{I}_{N-k})$, $1 < k < N$.

1. Introduction

Ginibre in 1965 introduced three classical examples of non-Hermitian random matrices in his seminal work [22], Ginibre's Orthogonal Ensemble (GinOE), Ginibre's Unitary Ensemble (GinUE) and Ginibre's Symplectic Ensemble (GinSE) by imposing Gaussian measures on real, complex or quaternion matrix spaces. These ensembles have complex spectra and can be treated as non-Hermitian counterparts of the Dyson's threefold symmetry classes–Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE), and Gaussian Symplectic Ensemble (GSE), by dropping the Hermiticity constraint. Near the end of the first paragraph in [22], Ginibre wrote, "*Apart from the intrinsic interest of the problem, one may hope that the methods and results will provide further insight in the cases of physical interest or suggest as yet lacking applications.*" Now, non-Hermitian random matrices have proven to be as important as their Hermitian counterparts from the physical point of view, as remarked by Akemann and Kanzieper in [2, Section 2.2]. Their applications include dissipative quantum maps, quantum chromodynamics, quantum chaos, quantum Hall effect, stability of complex ecological and neural networks; see e.g. [2,7,14,15,29] and references therein.

Symmetry and universality are two primary concepts in mathematical and physical systems, especially in Random Matrix Theory (RMT). For Hermitian random matrices (or Hamiltonians), by introducing three key internal symmetries: time-reversal symmetry, particle-hole symmetry and chiral symmetry, Altland and Zirnbauer have extended Dyson's threefold way to tenfold classification classes for topological insulators and superconductors [6]. However, non-Hermiticity crucially changes the nature of symmetry and universality, which causes the two distinctions between complex conjugation and transposition symmetry, and between chiral symmetry (pseudo-Hermiticity) and sublattice symmetry for non-Hermitian random matrices (Hamiltonians). Recently, it is shown in [27,38] that symmetry ramifies the Altland-Zirnbauer 10-fold classification in Hermitian physics to the 38-fold symmetry classification in non-Hermitian physics; see also the most recent arXiv version v2 of [8]. This particularly includes 9 non-Hermitian symmetry classes characterized only by a single symmetry [27], labeled by A, AI, AII, $AI^\dagger$, $AII^\dagger$, D, C, AIII and $AIII^\dagger$, as shown in Table 1; see [25] for more details about relevant symmetry operators. Surprisingly, the level-spacing statistics of all the three Ginibre ensembles in the bulk of the spectrum away from the real axis are characterized by the same universality class. In contrast, the three Wigner-Dyson classes-GOE, GUE and GSE, display distinct universal patterns in the bulk and actually govern all Altland-Zirnbauer 10-fold classes. Quite recently, based on heuristic arguments and numerical simulations, Hamazaki, Kawabata, Kura and Ueda have found that **GinUE class and two further classes $AI^\dagger$ (complex symmetric matrices, dual GinSE) and $AII^\dagger$ (relevant to complex anti-symmetric matrices, dual GinOE), which possess additional transposition symmetries, display only three distinct universal statistics in the bulk of the spectrum among the 38 symmetry classes** [25]. The three universality classes are marked in Red, Green and Blue ensemble, see Table 1. Independently, the breaking of spectral statistics of $AI^\dagger$ towards the Ginibre ensemble was also found in [26]. Besides, the two new classes $AI^\dagger$ and $AII^\dagger$ have been established in 2D Coulomb gases at inverse temperature $\beta = 1.4, 2.6$ [3], in non-Hermitian Dirac operator [28], non-Hermitian many-body quantum chaos [21] and many-body Lindbladians [36].

This HKKU conjecture underscores a fundamental role of transposition symmetry and paves the way toward understanding universality in non-Hermitian systems. However, exact eigenvalue correlation functions exist only for a very few of 38 classes of Gaussian matrix ensembles, for instances, three Ginibre and chiral GinUE ensembles (A, AI, AII, $AIII^\dagger$) in Table 1; see [14,15,34]. Particularly, as far as we know, no exact results are known about the joint densities for complex eigenvalues of classes $AI^\dagger$ and $AII^\dagger$. Our goal in this paper is to investigate duality relationships hidden in the Gaussian matrix ensembles with external source for the 9 single symmetry classes. These further imply auto-correlation functions of characteristic polynomials. Duality and characteristic polynomials are not only very important objects [1,4,5,17–20], but can be used to derive eigenvalue correlation functions [31,32]. Meanwhile, characteristic polynomials are found to be related to number theory [10,11], integrable systems [16] and log-correlated Gaussian field [30,35].

Our long arXiv preprint [31] has been divided into three parts for better readability. Part I and II are devoted to phase transition of eigenvalues at the edge of the spectrum under finite-rank perturbations, respectively in the deformed GinUE and GinSE ensembles. This Part III focuses on duality formulae between Gaussian random matrix ensembles chosen from 9 single symmetry classes.

Although the most results in the present paper are taken from [31], there are some differences: (1) the dual ensemble of GinOE is redefined as AII† such that the dual Ginibre ensembles consist of the universal classes, see Definition 2.2; (2) Duality formulae related to classes D, C, AIII and AIII† are new.

This paper is organized as follows. In Section 2, the Gaussian matrix ensembles are defined and duality formulae between them are stated. In Section 3, we provide the proofs of main results via matrix-valued heat equations and elementary algebraic and differential operations. In the last section 4, we will conclude with some remarks on possible applications and with future questions for further investigation.

2. Duality formulae

In Hermitian RMT there exist certain dual relationships between β and $4/\beta$ matrix ensembles, e.g., GOE, GUE and GSE ensembles; see [10,11,17] and references therein. Also, some analogous dual relationships are known in non-Hermitian RMT, see [12,13,18,23]. This section is devoted to duality formulae of certain observables (or functionals) between deformed GinOE, GinUE, GinSE, complex symmetric, antisymmetric matrix ensembles and the other four Gaussian matrix ensembles.

2.1. Non-Hermitian Gaussian ensembles

By convention, the conjugate, transpose, and conjugate transpose of a complex matrix A are denoted by $\overline{A}$, A^t and A^*, respectively. The trace of a square matrix with real, complex or real quaternion elements (corresponding to Dyson index $\beta = 1, 2, 4$) is redefined as

$$\mathrm{Tr}_\beta := \begin{cases} \mathrm{Tr}, & \beta = 1,2; \\ \frac{1}{2}\mathrm{Tr}, & \beta = 4. \end{cases}$$

Throughout the present paper, a real quaternion matrix of size N will always be identified as its complex representation, that is, a complex $2N \times 2N$ matrix X that satisfies the relation $X\mathbb{J}_N = \mathbb{J}_N\overline{X}$, where

$$\mathbb{J}_N = J_{N,N}, \quad J_{k,l} = \begin{bmatrix} 0_{l\times k} & \mathbb{I}_l \\ -\mathbb{I}_k & 0_{k\times l} \end{bmatrix}. \tag{2.1}$$

In this case it has the form

$$X = \begin{bmatrix} X^{(1)} & X^{(2)} \\ -\overline{X^{(2)}} & \overline{X^{(1)}} \end{bmatrix}, \tag{2.2}$$

where both $X^{(1)}$ and $X^{(2)}$ are $N \times N$ complex matrices; see e.g. [33,37].

Our objects in considerations are additive and multiplicative deformations of the Ginibre ensembles.

Definition 2.1. A random real, complex or real quaternion $N \times N$ matrix X, denoted respectively by Dyson index $\beta = 1, 2, 4$, is said to belong to the deformed Ginibre ensemble, if the joint density function for matrix entries is given by

$$P_{N,\beta}(\tau; X, X_0) = \frac{1}{Z_{N,\beta}} \exp\Big\{-\frac{1}{\tau}\mathrm{Tr}_\beta\Big(\Sigma^{-1}(X - X_0)\Gamma^{-1}(X - X_0)^*\Big)\Big\}, \tag{2.3}$$

with the normalization constant

$$Z_{N,\beta} = \begin{cases} (\det(\Sigma\Gamma))^{\beta N/2}\,(\pi\tau)^{\beta N^2/2}, & \beta = 1,2; \\ (\det(\Sigma\Gamma))^{N}\,(\pi\tau)^{2N^2}, & \beta = 4, \end{cases} \tag{2.4}$$

where $\tau > 0$, correspondingly Σ, Γ and X_0 are real, complex or real quaternion $N \times N$ matrices, and both Σ and Γ are positive definite. Particularly when both Σ and Γ are identity matrices, the density is rewritten as $P^{(\mathrm{null})}_{N,\beta}(\tau; X, X_0)$.

Definition 2.2. (i) A complex symmetric, complex, or complex $K \times K$ matrix Y such that $J_{[K/2],K-[K/2]}Y$ is antisymmetric, is said to be the dual Ginibre orthogonal, unitary or symplectic ensemble (dGinOE, dGinUE or dGinSE, Dyson index $\beta = 1, 2, 4$) with mean Y_0, if the joint density function is given by

$$\widehat{P}_{K,\beta}(\tau; Y, Y_0) = \frac{1}{\widehat{Z}_{K,\beta}} \exp\left\{-\frac{1}{\tau}\mathrm{Tr}_{4/\beta}(Y - Y_0)(Y - Y_0)^*\right\}, \tag{2.5}$$

where the normalization constant

$$\widehat{Z}_{K,\beta} = \begin{cases} 2^K\,(\pi\tau)^{K(K+1)/2}, & \beta = 1; \\ (\pi\tau)^{K^2}, & \beta = 2; \\ (\pi\tau/2)^{K(K-1)/2}, & \beta = 4. \end{cases} \tag{2.6}$$

(ii) A complex antisymmetric $K \times K$ matrix Y is said to be the antisymmetric Ginibre unitary ensemble, if the joint density function is given by

$$P_K^{(\mathrm{CA})}(\tau; Y, Y_0) = \frac{1}{Z_K^{(\mathrm{CA})}} \exp\left\{-\frac{1}{2\tau}\mathrm{Tr}(Y - Y_0)(Y - Y_0)^*\right\}, \tag{2.7}$$

where the normalization constant

$$Z_K^{(\mathrm{CA})} = (2\pi\tau)^{K(K-1)/2}. \tag{2.8}$$

Also, we need to introduce certain observables. For this, let's recall the notions of the Pfaffian and the tensor product. For a $2m \times 2m$ complex anti-symmetric matrix $A = [a_{i,j}]$, the Pfaffian

$$\mathrm{Pf}(A) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \mathrm{sgn}(\sigma) \prod_{i=1}^{m} a_{\sigma(2i-1),\sigma(2i)}.$$

A well-known transformation property by a $2m \times 2m$ complex matrix B is

$$\mathrm{Pf}(BAB^t) = \det(B)\,\mathrm{Pf}(A).$$

The tensor product, or Kronecker product, of an $m \times m$ matrix $A = [a_{ij}]$ and a $q \times q$ matrix B is a block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{bmatrix}.$$

Moreover, $A \otimes B$ is similar to $B \otimes A$ by a permutation matrix.

Definition 2.3. For an $N \times N$ matrix X and a $K \times K$ matrix Y respectively corresponding to Dyson index β and $4/\beta$, define an observable $Q_\beta(A; X, Y)$ as

$$Q_1(A; X, Y) := \mathrm{Pf}\left(A + \begin{bmatrix} \left(J_{[K/2],K-[K/2]}Y\right) \otimes \Sigma & -\mathbb{I}_K \otimes X \\ \mathbb{I}_K \otimes X^t & \left(J_{[K/2],K-[K/2]}Y\right)^* \otimes \Gamma \end{bmatrix} \right), \tag{2.9}$$

$$Q_2(A; X, Y) := \det\left(A + \begin{bmatrix} -\mathbb{I}_K \otimes X & -Y^* \otimes \Sigma \\ Y \otimes \Gamma & -\mathbb{I}_K \otimes X^* \end{bmatrix} \right), \tag{2.10}$$

and

$$Q_4(A; X, Y) := \mathrm{Pf}\left(A + \begin{bmatrix} iY \otimes \left(\Sigma \mathbb{J}_N\right) & -\mathbb{I}_K \otimes X \\ \mathbb{I}_K \otimes X^t & iY^* \otimes \left(\mathbb{J}_N \Gamma\right) \end{bmatrix} \right), \tag{2.11}$$

where a complex square matrix A is anti-symmetric when $\beta = 1, 4$.

It is worth emphasizing that the observable Q_2 in (2.10) has essentially been introduced in [12,23], where $\Sigma = \Gamma = \mathbb{I}_N$, and $A = \mathrm{diag}(A_1, A_2)$ with A_1, A_2 being tensor products of two matrices; see [23, eqn (36)].

2.2. Duality between Gaussian ensembles

Duality formulas between the Ginibre ensembles and the dual Ginibre ensembles given in Definition 2.2 can be stated as follows.

Theorem 2.4. *With the same notations as in Definitions 2.1, 2.2 and 2.3, for $\beta = 1, 2$ and 4 we have*

$$\int Q_\beta(A; X, Y_0) P_{N,\beta}(\tau; X, X_0) \mathrm{d}X = \int Q_\beta(A; X_0, Y) \widehat{P}_{K,4/\beta}(\tau; Y, Y_0) \mathrm{d}Y, \tag{2.12}$$

where $\mathrm{d}X$ and $\mathrm{d}Y$ denote Lebesgue measures on associated linear matrix spaces.

Theorem 2.5. *With the deformed GinSE in Definition 2.1 and with the antisymmetric GinUE and the dual GinOE in Definition 2.2, we have*

$$\int \widetilde{Q}_4(A; X, Y_0) P_{N,4}^{(\mathrm{null})}(\tau; X, X_0) \mathrm{d}X = \int \widetilde{Q}_4(A; X_0, Y) P_K^{(\mathrm{CA})}(\tau; Y, Y_0) \mathrm{d}Y, \tag{2.13}$$

and with even K

$$\int \widehat{Q}_4(A; X, Y_0) P_{N,4}^{(\mathrm{null})}(\tau; X, X_0) \mathrm{d}X = \int \widehat{Q}_4(A; X_0, Y) \hat{P}_{K,1}(\tau; Y, Y_0) \mathrm{d}Y, \tag{2.14}$$

where

$$\widetilde{Q}_4(A; X, Y) := \mathrm{Pf}\left(A + \begin{bmatrix} 0 & -Y \otimes \mathbb{I}_N & \mathbb{I}_K \otimes X^{(1)} & \mathbb{I}_K \otimes X^{(2)} \\ Y^t \otimes \mathbb{I}_N & 0 & -\mathbb{I}_K \otimes \overline{X^{(2)}} & \mathbb{I}_K \otimes \overline{X^{(1)}} \\ -\mathbb{I}_K \otimes X^{(1)t} & -\mathbb{I}_K \otimes X^{(2)*} & 0 & -Y^* \otimes \mathbb{I}_N \\ -\mathbb{I}_K \otimes X^{(2)t} & -\mathbb{I}_K \otimes X^{(1)*} & \overline{Y} \otimes \mathbb{I}_N & 0 \end{bmatrix} \right), \tag{2.15}$$

and

$$\widehat{Q}_4(A; X, Y) := \mathrm{Pf}\left(A + \begin{bmatrix} 0 & -\left(\mathbb{J}_{\frac{K}{2}} Y\right) \otimes \mathbb{I}_N & \mathbb{I}_K \otimes X^{(1)} & \mathbb{J}_{\frac{K}{2}} \otimes X^{(2)} \\ \left(\mathbb{J}_{\frac{K}{2}} Y\right)^t \otimes \mathbb{I}_N & 0 & -\mathbb{J}_{\frac{K}{2}} \otimes \overline{X^{(2)}} & \mathbb{I}_K \otimes \overline{X^{(1)}} \\ -\mathbb{I}_K \otimes X^{(1)t} & -\mathbb{J}_{\frac{K}{2}} \otimes X^{(2)*} & 0 & -\left(\mathbb{J}_{\frac{K}{2}} Y\right)^* \otimes \mathbb{I}_N \\ \mathbb{J}_{\frac{K}{2}} \otimes X^{(2)t} & -\mathbb{I}_K \otimes X^{(1)*} & \overline{\left(\mathbb{J}_{\frac{K}{2}} Y\right)} \otimes \mathbb{I}_N & 0 \end{bmatrix} \right), \tag{2.16}$$

with a complex antisymmetric matrix A and X given in (2.2).

Recall that the GUE matrix ensemble with source refers to the joint density for matrix entries

$$P_N^{(\text{gue})}(\tau; H, H_0) = \frac{1}{(\sqrt{2\pi\tau})^{N^2}} \exp\left\{ -\frac{1}{2\tau} \mathrm{Tr}(H - H_0)^2 \right\}, \tag{2.17}$$

where H_0 is a fixed $N \times N$ matrix.

Theorem 2.6. *Let A and B be complex matrices of size KN and N respectively, then*

$$\int \det\left(A - \mathbb{I}_K \otimes (BH) + \sqrt{-1}\widehat{H}_0 \otimes B \right) P_N^{(\text{gue})}(\tau; H, H_0) \mathrm{d}H = \int \det\left(A - \mathbb{I}_K \otimes (BH_0) + \sqrt{-1}\widehat{H} \otimes B \right) P_K^{(\text{gue})}(\tau; \widehat{H}, \widehat{H}_0) \mathrm{d}\widehat{H}. \tag{2.18}$$

Theorem 2.7. *Let*

$$\widetilde{Q}_2(A; X_1, X_2, Y_1, Y_2) := \det\left(A - \begin{bmatrix} 0 & \mathbb{I}_K \otimes X_1 & Y_1 \otimes \mathbb{I}_N & 0 \\ \mathbb{I}_K \otimes X_2 & 0 & 0 & Y_2 \otimes \mathbb{I}_N \\ Y_2^* \otimes \mathbb{I}_N & 0 & 0 & \mathbb{I}_K \otimes X_2^* \\ 0 & Y_1^* \otimes \mathbb{I}_N & \mathbb{I}_K \otimes X_1^* & 0 \end{bmatrix} \right) \tag{2.19}$$

where A is a complex matrix of size $4KN$, then

$$\iint \widetilde{Q}_2(A; X_1, X_2, Y_{10}, Y_{20}) P_{N,2}^{(\text{null})}(\tau; X_1, X_{10}) P_{N,2}^{(\text{null})}(\tau; X_2, X_{20}) \mathrm{d}X_1 \mathrm{d}X_2 = \iint \widetilde{Q}_2(A; X_{10}, X_{20}, Y_1, Y_2) P_{K,2}^{(\text{null})}(\tau; Y_1, Y_{10}) P_{K,2}^{(\text{null})}(\tau; Y_2, Y_{20}) \mathrm{d}Y_1 \mathrm{d}Y_2. \tag{2.20}$$

By choosing an appropriate form of A, we can immediately derive autocorrelation functions of characteristic polynomials in all nine classes of Gaussian matrix ensembles from the above theorems. Here we just state corresponding results in the three Ginibre ensembles.

Corollary 2.8. *Let $Z = \mathrm{diag}\left(z_1, \cdots, z_{K_1}\right)$ and $W = \mathrm{diag}\left(w_1, \cdots, w_{K_2}\right)$ with complex diagonal entries, then*

$$\begin{aligned} &\int \prod_{j=1}^{K_1} \det\left(z_j - X\right) \prod_{k=1}^{K_2} \det\left(\overline{w_k} - X^*\right)\ P_{N,\beta}(\tau; X, X_0) \mathrm{d}X \\ &= c_\beta \int Q_\beta(A; X_0, Y) \widehat{P}_{K,4/\beta}(\tau; Y, 0) \mathrm{d}Y, \end{aligned} \tag{2.21}$$

with

$$A = \begin{cases} \begin{bmatrix} 0 & \mathrm{diag}\left(Z \otimes \mathbb{I}_N, W^* \otimes \mathbb{I}_N\right) \\ -\mathrm{diag}\left(Z \otimes \mathbb{I}_N, W^* \otimes \mathbb{I}_N\right) & 0 \end{bmatrix}, & \beta = 1; \\ \mathrm{diag}\left(Z \otimes \mathbb{I}_N, W^* \otimes \mathbb{I}_N\right), & \beta = 2; \\ \begin{bmatrix} 0 & \mathrm{diag}\left(Z \otimes \mathbb{I}_{2N}, W^* \otimes \mathbb{I}_{2N}\right) \\ -\mathrm{diag}\left(Z \otimes \mathbb{I}_{2N}, W^* \otimes \mathbb{I}_{2N}\right) & 0 \end{bmatrix}, & \beta = 4. \end{cases} \tag{2.22}$$

Here $K = K_1 + K_2$ for $\beta = 1, 4$ and $K = K_1 = K_2$ for $\beta = 2$, while $c_\beta = (-1)^{KN(KN-1)/2}, 1, (-1)^{KN}$ for $\beta = 1, 2, 4$.

Proof. Choose Y_0 as a zero matrix in (2.12) and use the identity for the Pfaffian

$$\mathrm{Pf}\begin{bmatrix} 0_n & B \\ -B^t & 0_n \end{bmatrix} = (-1)^{n(n-1)/2} \det B, \tag{2.23}$$

where B is a complex $n \times n$ matrix, we can rewrite the product of the determinants as Q_β up to some sign. This thus completes the proof. □

Remark 2.9. Some special cases of Corollary 2.8 have been studied by Afanasiev [1], Akemann-Phillips-Sommers [5], Grela [23], Forrester-Rains [18], Fyodorov [19] and so on. In the case of the real Ginibre ensemble, that is, $\beta = 1$, $\Sigma = \Gamma = \mathbb{I}_N, X_0 = 0$, see [5] for the product of two characteristic polynomials and [1] for the product of arbitrarily finite characteristic polynomials. In the case of $\beta = 2$, $\Sigma = \Gamma = \mathbb{I}_N$ and any finite K, or general Σ, Γ and $K = 1$, see [23]. For $\beta = 1, 2, 4$ but with $X_0 = 0, \Gamma = \mathbb{I}_N$ and general Σ, see [18] for the moments of characteristic polynomials. Besides, for the deformed complex Ginibre ensemble given in Definition 2.1, the average ratio of two "generalized" characteristic polynomials and the spectral density has been exactly evaluated as a double integral [24]; for certain rank-one deviation from the real Ginibre ensemble the average modulus of the characteristic polynomial is also evaluated [9, Appendix B]. These very interesting results are expected to hold in the more general ensemble as in Definition 2.1.

3. Diffusion method without SUSY

3.1. New ideas and differential identities

Diffusion method that is based on matrix-valued heat equations has important applications in RMT, see Mehta's classical book [33] or Grela's recent survey [23]. Supersymmetry (SUSY) method which concern Grassmann variables and integrals also seems indispensable for many problems in RMT, particularly for the first finding of dual formulas, see e.g. [10,11] for Hermitian RMT and [5,12,13,23] for non-Hermitian RMT. For the deformed GinUE ensemble in Definition 2.1 but with $\Sigma = \Gamma = \mathbb{I}_N$, Grela in [23] combines the diffusion method and SUSY technique to obtain Corollary 2.8. In the present paper we introduce a new method that is the combination of diffusion method and elementary matrix operations for determinants and Pfaffians, instead of SUSY technique. A key step is to verify the identity

$$\Delta_X Q(X,Y) = c\Delta_Y Q(X,Y), \tag{3.1}$$

for some $c > 0$, proper observables $Q(X,Y)$ and certain matrix Laplacians Δ_X, Δ_Y.

This new idea has many potential applications both in Hermitian and non-Hermitian Gaussian matrix ensembles, except for all the stated results in Section 2.2. For instances, duality formulae in the GOE, GUE and GSE ensembles [10,11,17] can also been derived too. **Very detailed proofs will be presented only for Theorem 2.4 with $\beta = 1, 2, 4$ in this section, since the most interesting Ginibre and dual Ginibre ensembles are involved and Theorems 2.5, 2.6 and 2.7 can be tackled in similar ways.** The following two differential identities are of primary importance.

For a matrix-valued function $A := A(x) = \left[a_{ij}(x)\right]_{i,j=1}^n$ of variable x without symmetry, and for some $1 \le p \le n$ deleting p rows and p columns respectively indexed by $I = \{i_1, \cdots, i_p\}$ and $J = \{j_1, \cdots, j_p\}$ as two subsets of $\{1, 2, \ldots, n\}$, the resulting sub-matrix is denoted by $A[I; J]$. By definition of determinant it's easy to find an identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\det(A) = \sum_{i,j=1}^{n} (-1)^{i+j} \frac{\mathrm{d}a_{ij}(x)}{\mathrm{d}x} \det(A[i;j]). \tag{3.2}$$

For the Pfaffian, we have a similar identity.

Proposition 3.1. *Let $A = \left[a_{ij}(x)\right]_{i,j=1}^{2m}$ be an antisymmetric complex matrix, whose entries are differentiable functions of variable x, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Pf}(A) = \sum_{1\le i<j\le 2m} (-1)^{i+j+1} \frac{\mathrm{d}a_{ij}(x)}{\mathrm{d}x} \mathrm{Pf}(A[i,j;i,j]). \tag{3.3}$$

Proof. Introduce independent variables $\{x_{ij} : 1 \le i < j \le 2m\}$ and treat $a_{ij} = -a_{ji} = a_{ij}(x_{ij})$ as a function of x_{ij} whenever $i < j$, while all $a_{ii} = 0$. By definition of the Pfaffian,

$$\frac{\partial}{\partial x_{ij}}\mathrm{Pf}\left(\left[a_{kl}(x_{kl})\right]_{k,l=1}^{2m}\right) = (-1)^{i+j+1}\frac{\mathrm{d}a_{ij}(x_{ij})}{\mathrm{d}x_{ij}}\mathrm{Pf}\left(\left[a_{kl}(x_{kl})\right]_{k,l\neq i,j}\right),$$

so it suffices to prove

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Pf}\,(A) = \left(\sum_{i<j}\frac{\partial}{\partial x_{ij}}\mathrm{Pf}\left(\left[a_{kl}(x_{kl})\right]_{k,l=1}^{2m}\right)\right)\Bigg|_{\text{all } x_{ij}=x}. \tag{3.4}$$

Taking the derivative with respect to x_{ij} on both sides

$$\mathrm{Pf}\left([a_{kl}(x_{kl})]_{k,l=1}^{2m}\right) = \sum_{\sigma\in A_{2m}} \mathrm{sgn}(\sigma)\prod_{k=1}^{m} a_{\sigma(2k-1)\sigma(2k)}\left(x_{\sigma(2k-1)\sigma(2k)}\right),$$

where

$$A_{2m} = \left\{\sigma \in S_{2m} : \sigma(1) < \sigma(3) < \cdots < \sigma(2m-1), \sigma(2k-1) < \sigma(2k), \forall k\right\},$$

and summing them together, we have

$$\sum_{i<j}\frac{\partial}{\partial x_{ij}}\mathrm{Pf}\left([a_{kl}(x_{kl})]_{k,l=1}^{2m}\right) = \sum_{\sigma\in A_{2m}}\mathrm{sgn}(\sigma)\sum_{i<j}\frac{\partial}{\partial x_{ij}}\left(\prod_{k=1}^{m} a_{\sigma(2k-1)\sigma(2k)}(x_{\sigma(2k-1)\sigma(2k)})\right)$$
$$= \sum_{\sigma\in A_{2m}}\mathrm{sgn}(\sigma)\sum_{k=1}^{m}\frac{\mathrm{d}a_{\sigma(2k-1)\sigma(2k)}(x_{\sigma(2k-1)\sigma(2k)})}{\mathrm{d}x_{\sigma(2k-1)\sigma(2k)}}\prod_{l=1,l\neq k}^{m} a_{\sigma(2l-1)\sigma(2l)}(x_{\sigma(2l-1)\sigma(2l)}).$$

On the right hand side, set $x_{ij} = x$ for all $i < j$, (3.4) immediately follows from the definition of the Pfaffian. This thus completes the proof. □

The Laplace operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial \overline{z}} \tag{3.5}$$

will be used frequently, where $z = x + \mathrm{i}y$ and $\overline{z} = x - \mathrm{i}y$.

3.2. *Proof of Theorem 2.4: $\beta = 1$*

On the left-hand side of (2.12) use change of variables X to $\Sigma^{1/2}X\Gamma^{1/2}$, while on both sides of (2.12) replace nonrandom matrices X_0 by $\Sigma^{1/2}X_0\Gamma^{1/2}$ and A by $\mathrm{diag}\left(\mathbb{I}_K\otimes\Sigma^{1/2}, \mathbb{I}_K\otimes\Gamma^{1/2}\right) A\,\mathrm{diag}\left(\mathbb{I}_K\otimes\Sigma^{1/2}, \mathbb{I}_K\otimes\Gamma^{1/2}\right)$, respectively, divide both sides by $\sqrt{\det(\Sigma\Gamma)}$, we then see that the resulting duality identity is independent of Σ and Γ. Without loss of generality, we may assume $\Sigma = \Gamma = \mathbb{I}_N$.

On the other hand, on both sides of (2.12) change Y, Y_0 to $-J_{K-[K/2],[K/2]}Y, -J_{K-[K/2],[K/2]}Y_0$ so that the resulting Y and Y_0 are antisymmetric. Moreover, the density (2.5) has the same form and $Q_1(A;X,Y)$ reduces to

$$Q_1(A;X,Y) := \mathrm{Pf}\left(A + \begin{bmatrix} Y\otimes\Sigma & -\mathbb{I}_K\otimes X \\ \mathbb{I}_K\otimes X^t & Y^*\otimes\Gamma\end{bmatrix}\right). \tag{3.6}$$

We will derive the desired result in the simplified situation.

Write

$$\overline{Q_1}(\tau;A,Y_0) := \int Q_1(A;X,Y_0)P_{N,1}(\tau;X,X_0)\mathrm{d}X, \tag{3.7}$$

noting that when $\Sigma = \Gamma = \mathbb{I}_N$ the density given in (2.3) satisfies the heat equation

$$\partial\tau P_{N,1}(\tau;X,X_0) = \frac{1}{4}\Delta_{1,X}P_{N,2}(\tau;X,X_0), \quad \Delta_{1,X} := \sum_{a,b=1}^{N}\partial^2_{x_{a,b}}, \tag{3.8}$$

using integration by parts one finds

$$\begin{aligned}\partial\tau\overline{Q_1}(\tau;A,Y_0) &= \int Q_1(A;X,Y_0)\partial\tau P_{N,1}(\tau;X,X_0)\mathrm{d}X \\ &= \frac{1}{4}\int Q_1(A;X,Y_0)\left(\Delta_{1,X}P_{N,1}(\tau;X,X_0)\right)\mathrm{d}X \\ &= \frac{1}{4}\int\left(\Delta_{1,X}Q_1(A;X,Y_0)\right)P_{N,1}(\tau;X,X_0)\mathrm{d}X.\end{aligned} \tag{3.9}$$

Rewrite the antisymmetric complex matrix Y_0 as $Y_0 = [y_{j,k}] = [a_{j,k} + \mathrm{i}b_{j,k}]$, and let

$$\Delta_{Y_0} := \sum_{1\le j<k\le K}(\partial^2_{a_{j,k}} + \partial^2_{b_{j,k}}).$$

We claim that

$$\Delta_{1,X}Q_1(A;X,Y_0) = \frac{1}{2}\Delta_{Y_0}Q_1(A;X,Y_0). \tag{3.10}$$

If so, one obtains a heat equation from (3.9) that

$$\partial\tau\overline{Q_1}(\tau;A,Y_0) = \frac{1}{8}\Delta_{Y_0}\overline{Q_1}(\tau;A,Y_0).$$

Together with the initial boundary condition

$$\overline{Q_1}(0; A, Y_0) = Q_1(A; X_0, Y_0),$$

the solution is thus given by

$$\overline{Q_1}(\tau; A, Y_0) = \int Q_1(A; X_0, Y)\widehat{P}_{K,4}(\tau; Y, Y_0)\mathrm{d}Y,$$

from which the desired result immediately follows.

The remaining task is to verify the identity (3.10). Put

$$\widetilde{T}_1 = A + \begin{bmatrix} Y_0 \otimes \mathbb{I}_N & -\mathbb{I}_K \otimes X \\ \mathbb{I}_K \otimes X^t & Y_0^* \otimes \mathbb{I}_N \end{bmatrix},$$

one sees from (3.5) that it suffices to prove

$$\Delta_{1,X}\mathrm{Pf}\left(\widetilde{T}_1\right) = 2 \sum_{1\le\alpha<\beta\le K} \frac{\partial^2}{\partial y_{\alpha,\beta}\partial \overline{y_{\alpha,\beta}}}\mathrm{Pf}\left(\widetilde{T}_1\right). \tag{3.11}$$

By use of (3.3), for $1 \le a, b \le N$ one has

$$\partial_{x_{a,b}}\mathrm{Pf}\left(\widetilde{T}_1\right) = (-1)^{KN+a+b} \sum_{\alpha=1}^{K} \mathrm{Pf}\left(\widetilde{T}_1\left[I_{0,1}; I_{0,1}\right]\right),$$

where

$$I_{0,1} = \{(\alpha-1)N + a, (K+\alpha-1)N + b\}.$$

Further use leads to

$$\partial_{x_{a,b}}\mathrm{Pf}\left(\widetilde{T}_1\left[I_{0,1}; I_{0,1}\right]\right) = (-1)^{KN+a+b+1} \sum_{\beta\neq\alpha}^{K} \mathrm{Pf}\left(\widetilde{T}_1\left[I_1; I_1\right]\right),$$

where

$$I_1 = \{(\alpha-1)N + a, (\beta-1)N + a, (\alpha+K-1)N + b, (\beta+K-1)N + b\}.$$

Hence,

$$\sum_{a,b=1}^{N} \partial^2_{x_{ab}}\mathrm{Pf}\left(\widetilde{T}_1\right) = -2 \sum_{1\le\alpha<\beta\le K} \sum_{a,b=1}^{N} \mathrm{Pf}\left(\widetilde{T}_1\left[I_1; I_1\right]\right).$$

On the other hand, for $\alpha < \beta$ it's easy to get

$$\frac{\partial^2}{\partial y_{\alpha,\beta}\partial \overline{y_{\alpha,\beta}}}\mathrm{Pf}\left(\widetilde{T}_1\right) = -\sum_{a,b=1}^{N} \mathrm{Pf}\left(\widetilde{T}_1\left[I_1; I_1\right]\right),$$

from which the summation over α and β implies (3.11). This thus completes the proof.

3.3. Proof of Theorem 2.4: $\beta = 4$

We proceed in a similar way as in the real case. On the left-hand side of (2.12) use change of variables X to $\Sigma^{1/2}X\Gamma^{1/2}$, while on both sides of (2.12) replace nonrandom matrices X_0 and A by $\Sigma^{1/2}X_0\Gamma^{1/2}$ and $\mathrm{diag}\left(\mathbb{I}_K \otimes \Sigma^{1/2}, \mathbb{I}_K \otimes \Gamma^{1/2}\right) A\, \mathrm{diag}\left(\mathbb{I}_K \otimes \Sigma^{1/2}, \mathbb{I}_K \otimes \Gamma^{1/2}\right)$, respectively, divide both sides by $\sqrt{\det(\Sigma\Gamma)}$, we then see from the relations

$$\Sigma \mathbb{J}_N = \Sigma^{1/2}\mathbb{J}_N\left(\Sigma^{1/2}\right)^t, \qquad \mathbb{J}_N\Gamma = \left(\Gamma^{1/2}\right)^t\mathbb{J}_N\Gamma^{1/2}$$

that the resulting duality identity is independent of Σ and Γ. Without loss of generality, we may assume $\Sigma = \Gamma = \mathbb{I}_{2N}$.

Write

$$\overline{Q_4}(\tau; A, Y_0) := \int Q_4(A; X, Y_0)P_{N,4}(\tau; X, X_0)\mathrm{d}X, \tag{3.12}$$

where X is given in (2.2) with $X^{(j)} = [x^{(j)}_{a,b}]$ $(j = 1, 2)$, noting that when $\Sigma = \Gamma = \mathbb{I}_{2N}$ the density given in (2.3) satisfies the heat equation

$$\partial\tau P_{N,4}(\tau; X, X_0) = \frac{1}{4}\Delta_{4,X}P_{N,4}(\tau; X, X_0), \quad \Delta_{4,X} := \sum_{j=1}^{2}\sum_{a,b=1}^{N}\left(\partial^2_{\Re x^{(j)}_{a,b}} + \partial^2_{\Im x^{(j)}_{a,b}}\right). \tag{3.13}$$

By integration by parts one finds

$$\begin{aligned}\partial\tau\overline{Q_4}(\tau;A,Y_0) &= \int Q_4(A;X,Y_0)\partial\tau P_{N,4}(\tau;X,X_0)\mathrm{d}X \\ &= \frac{1}{4}\int Q_4(A;X,Y_0)\Big(\Delta_{4,X}P_{N,1}(\tau;X,X_0)\Big)\mathrm{d}X \\ &= \frac{1}{4}\int \Big(\Delta_{4,X}Q_4(A;X,Y_0)\Big)P_{N,4}(\tau;X,X_0)\mathrm{d}X.\end{aligned} \tag{3.14}$$

Rewrite the complex symmetric matrix Y_0 as $Y_0 = [y_{jk}] = [a_{j,k} + \mathrm{i}b_{j,k}]$, and let

$$\Delta_{Y_0} := \sum_{1\le j\le k\le K}\left(\partial^2_{a_{j,k}} + \partial^2_{b_{j,k}}\right) + \sum_{j=1}^{K}\left(\partial^2_{a_{j,j}} + \partial^2_{b_{j,j}}\right).$$

We claim that

$$\Delta_{4,X}Q_4(A;X,Y_0) = \Delta_{Y_0}Q_4(A;X,Y_0). \tag{3.15}$$

If so, one obtains a heat equation from (3.14) that

$$\partial\tau\overline{Q_4}(\tau;A,Y_0) = \frac{1}{4}\Delta_{Y_0}\overline{Q_4}(\tau;A,Y_0).$$

Together with the initial boundary condition

$$\overline{Q_4}(0;A,Y_0) = Q_4(A;X_0,Y_0),$$

the solution is thus given by

$$\overline{Q_4}(\tau;A,Y_0) = \int Q_4(A;X_0,Y)\widehat{P}_{K,1}(\tau;Y,Y_0)\mathrm{d}Y,$$

from which the desired result immediately follows.

The remaining task is to verify the identity (3.15). Put

$$\widetilde{T}_4 = A + \begin{bmatrix}\mathrm{i}Y_0\otimes\mathbb{J}_N & -\mathbb{I}_K\otimes X \\ \mathbb{I}_K\otimes X^t & \mathrm{i}Y_0^*\otimes\mathbb{J}_N\end{bmatrix},$$

one sees from (3.5) that it suffices to prove

$$\begin{aligned}&\sum_{a,b=1}^{N}\left(\frac{\partial^2}{\partial x^{(1)}_{a,b}\partial\overline{x^{(1)}_{a,b}}} + \frac{\partial^2}{\partial x^{(2)}_{a,b}\partial\overline{x^{(2)}_{a,b}}}\right)\mathrm{Pf}\left(\widetilde{T}_4\right) = \\ &\left(\sum_{\alpha<\beta}\frac{\partial^2}{\partial y_{\alpha,\beta}\partial\overline{y_{\alpha,\beta}}} + 2\sum_{\alpha=1}^{K}\frac{\partial^2}{\partial y_{\alpha,\alpha}\partial\overline{y_{\alpha,\alpha}}}\right)\mathrm{Pf}\left(\widetilde{T}_4\right).\end{aligned} \tag{3.16}$$

By use of (3.3), for $\alpha < \beta$ one has

$$\frac{\partial}{\partial y_{\alpha,\beta}}\mathrm{Pf}\left(\widetilde{T}_4\right) = \sum_{a=1}^{N}(-1)^N\mathrm{i}\left\{-\mathrm{Pf}\left(\widetilde{T}_4\left[I_{4,0};I_{4,0}\right]\right) + \mathrm{Pf}\left(\widetilde{T}_4\left[J_{4,0};J_{4,0}\right]\right)\right\},$$

where

$$I_{4,0} = \{2(\alpha-1)N + a, (2\beta-1)N + a\}, \quad J_{4,0} = \{(2\alpha-1)N + a, 2(\beta-1)N + a\},$$

and further

$$\begin{aligned}\frac{\partial^2}{\partial y_{\alpha,\beta}\partial\overline{y_{\alpha,\beta}}}\mathrm{Pf}\left(\widetilde{T}_4\right) &= \sum_{a,b=1}^{N}\left\{\mathrm{Pf}\left(\widetilde{T}_4\left[J^4_{\alpha\beta};J^4_{\alpha\beta}\right]\right) - \mathrm{Pf}\left(\widetilde{T}_4\left[I^4_{\alpha\beta};I^4_{\alpha\beta}\right]\right)\right\} \\ &+ \sum_{a,b=1}^{N}\left\{\mathrm{Pf}\left(\widetilde{T}_4\left[J^4_{\beta\alpha};J^4_{\beta\alpha}\right]\right) - \mathrm{Pf}\left(\widetilde{T}_4\left[I^4_{\beta\alpha};I^4_{\beta\alpha}\right]\right)\right\},\end{aligned}$$

where

$$\begin{aligned}I^4_{\alpha\beta} &= \{2(\alpha-1)N + a, (2\beta-1)N + a, 2(\alpha+K-1)N + b, (2\beta+2K-1)N + b\}, \\ J^4_{\alpha\beta} &= \{2(\alpha-1)N + a, (2\beta-1)N + a, (2\alpha+2K-1)N + b, 2(\beta+K-1)N + b\}.\end{aligned}$$

For $1 \le \alpha \le K$, one has

$$\frac{\partial^2}{\partial y_{\alpha,\alpha}\partial \overline{y_{\alpha,\alpha}}}\mathrm{Pf}\left(\widetilde{T}_4\right) = -\sum_{a,b=1}^{N}\mathrm{Pf}\left(\widetilde{T}_4\left[I^4_{\alpha\alpha}; I^4_{\alpha\alpha}\right]\right).$$

Hence,

$$\left(\sum_{\alpha<\beta}\frac{\partial^2}{\partial y_{\alpha,\beta}\partial \overline{y_{\alpha,\beta}}} + 2\sum_{\alpha=1}^{K}\frac{\partial^2}{\partial y_{\alpha,\alpha}\partial \overline{y_{\alpha,\alpha}}}\right)\mathrm{Pf}\left(\widetilde{T}_4\right) = -2\sum_{a,b=1}^{N}\sum_{\alpha=1}^{K}\mathrm{Pf}\left(\widetilde{T}_4\left[I^4_{\alpha\alpha}; I^4_{\alpha\alpha}\right]\right)$$
$$+\sum_{a,b=1}^{N}\sum_{\alpha\neq\beta}\left\{\mathrm{Pf}\left(\widetilde{T}_4\left[J^4_{\alpha\beta}; J^4_{\alpha\beta}\right]\right) - \mathrm{Pf}\left(\widetilde{T}_4\left[I^4_{\alpha\beta}; I^4_{\alpha\beta}\right]\right)\right\}.$$

On the other hand, for $1 \le a, b \le N$ one gets

$$\frac{\partial^2}{\partial x^{(1)}_{a,b}\partial \overline{x^{(1)}_{a,b}}}\mathrm{Pf}\left(\widetilde{T}_4\right) = -\sum_{\alpha,\beta=1}^{K}\mathrm{Pf}\left(\widetilde{T}_4\left[I^4_{\alpha\beta}; I^4_{\alpha\beta}\right]\right)$$

and

$$\frac{\partial^2}{\partial x^{(2)}_{a,b}\partial \overline{x^{(2)}_{a,b}}}\mathrm{Pf}\left(\widetilde{T}_4\right) = -\sum_{\alpha=1}^{K}\mathrm{Pf}\left(\widetilde{T}_4\left[I^4_{\alpha\alpha}; I^4_{\alpha\alpha}\right]\right) + \sum_{\alpha\neq\beta}\mathrm{Pf}\left(\widetilde{T}_4\left[J^4_{\alpha\beta}; J^4_{\alpha\beta}\right]\right),$$

from which the summation over a and b implies (3.16). This thus completes the proof.

3.4. *Proof of Theorem 2.4: $\beta = 2$*

On the left-hand side of (2.12) use change of variables X to $\Sigma^{1/2}X\Gamma^{1/2}$, while on both sides of (2.12) replace nonrandom matrices X_0, A_1 and A_2 by $\Sigma^{1/2}X_0\Gamma^{1/2}$, $\left(\mathbb{I}_{K_1}\otimes\Sigma^{1/2}\right)A_1\left(\mathbb{I}_{K_1}\otimes\Gamma^{1/2}\right)$ and $\left(\mathbb{I}_{K_2}\otimes\Gamma^{1/2}\right)A_2\left(\mathbb{I}_{K_2}\otimes\Sigma^{1/2}\right)$ respectively, divide both sides by $\det(\Sigma\Gamma)$, we then see the resulting duality identity is independent of Σ and Γ. Without loss of generality, we may assume $\Sigma = \Gamma = \mathbb{I}_N$ and general A.

Write

$$\overline{Q_2}(\tau; A, Y_0) := \int Q_2(A; X, Y_0)P_{N,2}(\tau; X, X_0)\mathrm{d}X, \tag{3.17}$$

noting that when $\Sigma = \Gamma = \mathbb{I}_N$ the density given in (2.3) satisfies the heat equation

$$\partial\tau P_{N,2}(\tau; X, X_0) = \frac{1}{4}\Delta_{2,X}P_{N,2}(\tau; X, X_0), \quad \Delta_{2,X} := \sum_{a,b=1}^{N}(\partial^2_{\Re x_{a,b}} + \partial^2_{\Im x_{a,b}}), \tag{3.18}$$

using integration by parts one finds

$$\begin{aligned}\partial\tau\overline{Q_2}(\tau; A, Y_0) &= \int Q_2(A; X, Y_0)\partial\tau P_{N,2}(\tau; X, X_0)\mathrm{d}X\\ &= \frac{1}{4}\int Q_2(A; X, Y_0)\Big(\Delta_{2,X}P_{N,2}(\tau; X, X_0)\Big)\mathrm{d}X\\ &= \frac{1}{4}\int \Big(\Delta_{2,X}Q_2(A; X, Y_0)\Big)P_{N,2}(\tau; X, X_0)\mathrm{d}X.\end{aligned} \tag{3.19}$$

Rewrite the $K_2 \times K_1$ complex matrix Y_0 as $Y_0 = [y_{j,k}] = [a_{j,k} + \mathrm{i}b_{j,k}]$, and let

$$\Delta_{Y_0} := \sum_{j=1}^{K_2}\sum_{k=1}^{K_1}\left(\partial^2_{a_{j,k}} + \partial^2_{b_{j,k}}\right).$$

We claim that

$$\Delta_{2,X}Q_2(A; X, Y_0) = \Delta_{Y_0}Q_2(A; X, Y_0). \tag{3.20}$$

If so, one obtains a heat equation from (3.19) that

$$\partial\tau\overline{Q_2}(\tau; A, Y_0) = \frac{1}{4N}\Delta_{Y_0}\overline{Q_2}(\tau; A, Y_0).$$

Together with the initial boundary condition

$$\overline{Q_2}(0; A, Y_0) = Q_2(A; X_0, Y_0),$$

the unique solution is thus given by

$$\overline{Q_2}(\tau; A, Y_0) = \int Q_2(A; X_0, Y)\hat{P}_{K,2}(\tau; Y, Y_0)\mathrm{d}Y,$$

from which the desired result follows.

The remaining task is to verify the identity (3.20). Denote

$$\widetilde{T}_2 = A + \begin{bmatrix} -\mathbb{I}_{K_1} \otimes X & -Y_0^* \otimes \mathbb{I}_N \\ Y_0 \otimes \mathbb{I}_N & -\mathbb{I}_{K_2} \otimes X^* \end{bmatrix},$$

by use of (3.5), it suffices to prove

$$\sum_{a,b=1}^{N} \frac{\partial^2}{\partial x_{ab}\partial \overline{x_{ab}}} \det\left(\widetilde{T}_2\right) = \sum_{\alpha,\beta=1}^{K} \frac{\partial^2}{\partial y_{\alpha,\beta}\partial \overline{y_{\alpha,\beta}}} \det\left(\widetilde{T}_2\right). \tag{3.21}$$

One uses (3.2) to obtain

$$\sum_{a,b=1}^{N} \frac{\partial^2}{\partial x_{ab}\partial \overline{x_{ab}}} \det\left(\widetilde{T}_2\right) = \sum_{a,b=1}^{N} \sum_{\alpha,\beta=1}^{K} \det\left(\widetilde{T}_2\left[I_2; J_2\right]\right),$$

where

$$I_2 = \{(\beta - 1)N + a, (K + \alpha - 1)N + b\}, \quad J_2 = \{(\beta - 1)N + b, (K + \alpha - 1)N + a\}.$$

On the other hand, for $1 \le \alpha, \beta \le K$ simple calculation shows

$$\frac{\partial}{\partial y_{\alpha,\beta}} \det\left(\widetilde{T}_2\right) = \sum_{b=1}^{N} (-1)^{K_1 N + (\alpha+\beta)N} \det\left(\widetilde{T}_2\left[I_{2,0}; J_{2,0}\right]\right),$$

where

$$I_{2,0} = \left\{(K_1 + \alpha - 1)N + b\right\}, \quad J_{2,0} = \{(\beta - 1)N + b\},$$

and

$$\frac{\partial}{\partial \overline{y_{\alpha,\beta}}} \det\left(\widetilde{T}_2\left[I_{2,0}; J_{2,0}\right]\right) = \sum_{a=1}^{N} (-1)^{KN + (\alpha+\beta)N} \det\left(\widetilde{T}_2\left[I_2; J_2\right]\right).$$

So it's easy to obtain (3.21). The proof is thus complete.

4. Conclusion

Non-Hermitian random matrices have been classified into the 38-fold symmetry classes and 9 of them are characterized by single symmetry classes. Only three universality classes due to transposition symmetry are conjectured to exist for the dual Ginibre ensembles. We have studied 9 corresponding Gaussian matrix ensembles and established duality formulae of certain observables. These allow us to evaluate averaged products of K characteristic polynomials in an $N \times N$ matrix ensemble in terms of another $K \times K$ matrix ensemble, from which asymptotic analysis in the large matrix limit looks feasible. Our method is to make full use of two differential identities for determinants and Pfaffians and has more possible applications.

We just establish duality formulae for 9 Gaussian matrix ensembles and will return to the other 29 symmetry classes in the future. The most challenging open problems are to evaluate scaling limits for averaged products of K characteristic polynomials. In particular, whether or not new universality classes appear in the dual GinOE, GinUE and GinSE ensembles for local limits of characteristic polynomials?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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