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$\overline{m}_c$ and $\overline{m}_b$ from M_{B_c} and improved estimates of f_{B_c} and $f_{B_c(2S)}$

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ABSTRACT

We extract (*for the first time*) the correlated values of the running masses $\overline{m}_c$ and $\overline{m}_b$ from M_{B_c} using QCD Laplace sum rules (LSR) within stability criteria where perturbative (PT) expressions at N2LO and non-perturbative (NP) gluon condensates at LO are included. Allowing the values of $\overline{m}_{c,b}(\overline{m}_{c,b})$ to move inside the enlarged range of recent estimates from charmonium and bottomium sum rules (Table 1) obtained using similar stability criteria, we deduce: $\overline{m}_c(\overline{m}_c) = 1286(16)$ MeV and $\overline{m}_b(\overline{m}_b) = 4202(8)$ MeV. Combined with previous estimates (Table 2), we deduce a tentative QCD Spectral Sum Rules (QSSR) average: $\overline{m}_c(\overline{m}_c) = 1266(6)$ MeV and $\overline{m}_b(\overline{m}_b) = 4197(8)$ MeV where the errors come from the precise determinations from J/ψ and Υ sum rules. As a result, we present an improved prediction of $f_{B_c} = 371(17)$ MeV and the tentative upper bound $f_{B_c(2S)} \leq 139(6)$ MeV, which are useful for a further analysis of B_c-decays.

1. Introduction

Extractions of the perturbative (quark masses, α_s) and non-perturbative quark and gluon condensates QCD parameters are very important as they will serve as inputs in different phenomenological applications of the (non)-standard model. Lattice calculations are an useful tool for a such project but alternative analytical approaches based on QCD first principles (Chiral perturbation, effective theory and QCD spectral sum rules (QSSR)) are useful complement and independent check of the previous numerical simulations as they give insights for a better understanding of the (non)-perturbative phenomena inside the hadron "black box".

In this note,[1] we shall use the Laplace version [2–7] of QSSR introduced by Shifman-Vainshtein-Zakharov (SVZ) [2,3,8–21] for a new extraction of the running quark masses $\overline{m}_c$ and $\overline{m}_b$ from the $B_c(0^{-+})$-meson mass which we shall use for improving the prediction on its decay constant f_{B_c} done previously using similar approaches in [22–27] and for deriving a tentative upper bound on $f_{B_c(2S)}$.

2. The QCD Laplace sum rules

• *The QCD interpolating current*

We shall be concerned with the following QCD interpolating current:

$$\langle 0|J_5(x)|P\rangle = f_P M_P^2 \; : \; J_5(x) \equiv (m_c + m_b)\bar{c}(i\gamma_5)b \, , \tag{1}$$

where: $J_5(x)$ is the local heavy-light pseudoscalar current; $m_{c,b}$ are renormalized mass of the QCD Lagrangian; f_P is the decay constant related to the leptonic widths $\Gamma[P \to l^+\nu_l]$ and normalised as $f_\pi = 132$ MeV.

• *Form of the sum rules*

We shall work with the Finite Energy version of the QCD Laplace sum rules (LSR) and their ratios:

$$\mathcal{L}_n^c(\tau,\mu) = \int_{(m_c+m_b)^2}^{t_c} dt\, t^n\, e^{-t\tau}\, \frac{1}{\pi} \mathrm{Im}\, \psi_5(t,\mu)\, , \quad \mathcal{R}_n^c(\tau) = \frac{\mathcal{L}_{n+1}^c}{\mathcal{L}_n^c}\, , \tag{2}$$

where τ is the LSR variable, n is the degree of moments, t_c is the threshold of the "QCD continuum" which parametrizes, from the discontinuity of the Feynman diagrams, the spectral function $\mathrm{Im}\psi_5(t, m_Q^2, \mu)$ where $\psi_5(t, m_Q^2, \mu)$ is the (pseudo)scalar correlator:

$$\psi_5(q^2) = i \int d^4x\, e^{-iqx} \langle 0|\mathcal{T} J_5(x)\, (J_5(0))^\dagger\, |0\rangle . \tag{3}$$

3. QCD expression of the two-point function

Using the SVZ [2] Operator Product Expansion (OPE), the two-point correlator can be written in the form:

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[1] Some preliminary results of this work has been presented in [1].

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$$\psi_5(q^2) = \int_{(m_c+m_b)^2}^{\infty} \frac{dt}{t-q^2}\frac{1}{\pi}\text{Im}\,\psi_5(t,\mu)|_{PT} + \langle\alpha_s G^2\rangle C_{G^2}(q^2,\mu) + \langle g^3 G^3\rangle C_{G^3}(q^2,\mu) + \cdots , \quad (4)$$

where μ is the subtraction scale; Im $\psi_5(t,\mu)|_{PT}$ is the perturbative part of the spectral function; C_{G^2} and C_{G^2} are (perturbatively) calculable Wilson coefficients; $\langle\alpha_s G^2\rangle$ and $\langle g^3G^3\rangle$ are the non-perturbative gluon condensate of dimensions $d=4,6$ contributions where: $G^2 \equiv G^a_{\mu\nu}G_a^{\mu\nu}$ and $g^3G^3 \equiv g^3 f_{abc}G^a_{\mu\nu}G^{\nu\rho,b}G^{\mu,c}_{\rho}$. As explicitly shown in Ref. [23], C_{G^2} and C_{G^2} include the ones of the quark and mixed quark-gluon condensate through the relation [2,28,29]:

$$\langle\bar{Q}\,Q\rangle = -\frac{1}{12\pi m_Q}\langle\alpha_s G^2\rangle - \frac{\langle g^3G^3\rangle}{1440\pi^2 m_Q^3} ,$$
$$\langle\bar{Q}\,G\,Q\rangle = \frac{m_Q}{\pi}\left(\log\frac{m_Q}{\mu}\right)\langle\alpha_s G^2\rangle - \frac{\langle g^3G^3\rangle}{48\pi^2 m_Q} , \quad (5)$$

from the heavy quark mass expansion.

• $q^2=0$ *behaviour of the correlator*

To NLO, the perturbative part of $\psi_5(0)$ reads [6,10,11,30]:

$$\psi_5(0)|_{PT} = \frac{3}{4\pi^2}(m_b+m_c)\left(m_b^3 Z_b + m_c^3 Z_c\right) , \quad (6)$$

with:

$$Z_i = \left(1-\log\frac{m_i^2}{\mu^2}\right)\left(1+\frac{10}{3}a_s\right) + 2a_s\log^2\frac{m_i^2}{\mu^2} , \quad (7)$$

where $i\equiv c,b$; μ is the QCD subtraction constant and $a_s \equiv \alpha_s/\pi$ is the QCD coupling. This PT contribution which is present here has to be added to the well-known non-perturbative contribution:

$$\psi_5(0)|_{NP} = -(m_b+m_c)\langle\bar{c}c+\bar{b}b\rangle , \quad (8)$$

for absorbing $log^n(-m_i^2/q^2)$ mass singularities appearing during the evaluation of the PT two-point function, a technical point not often carefully discussed in some papers. Working with $\psi_5(q^2)$ is safe as $\psi_5(0)$, which disappears after successive derivatives, does not affect the pseudoscalar sum rule. This is not the case of the longitudinal part of the axial-vector two-point function $\Pi_A^{(0)}(q^2)$ built from the axial-vector current:

$$J_A^\mu \equiv \bar{c}(\gamma^\mu\gamma_5)b , \quad (9)$$

which is related to $\psi_5(q^2)$ through the Ward identity [6,10,11]:

$$\Pi_A^{(0)}(q^2) = \frac{1}{q^2}\left[\psi_5(q^2)-\psi_5(0)\right] , \quad (10)$$

and which is also often (uncorrectly) used in literature.

• *LO and NLO Perturbative contribution at large* q^2

The complete expressions of the PT spectral function has been obtained to LO in [31], to NLO in [30] and explicitly written in [23]. It reads ($i\equiv c,b$):

$$\text{Im}\psi_5(t)|_{PT} = \frac{3(m_b+m_c)^2}{8\pi t}\bar{q}^4 v\left\{1+\frac{4}{3}\left(\frac{\bar{\alpha}_s}{\pi}\right)\left\{\frac{3}{8}(7-v^2) + \sum_{i=b,c}\left[(v+v^{-1})\,(\text{L}_2(\alpha_1\alpha_2)-\text{L}_2(-\alpha_\text{i})-\log\alpha_1\,\log\beta_\text{i}) + A_i\log\alpha_i + B_i\log\beta_i\right]\right\} + O(\alpha_s^2)\right\} \quad (11)$$

where

$$\text{L}_2(\text{x}) = -\int_0^{\text{x}}\frac{\text{dy}}{\text{y}}\log(1-\text{y}) \quad (12)$$

and

$$A_i = \frac{3}{4}\frac{3m_i+m_j}{m_i+m_j} - \frac{19+2v^2+3v^4}{32v} - \frac{m_i(m_i-m_j)}{\bar{q}^2 v(1+v)}\left(1+v+\frac{2v}{1+\alpha_i}\right);$$
$$B_i = 2+2\frac{m_i^2-m_j^2}{\bar{q}^2 v};$$
$$\alpha_i = \frac{m_i}{m_j}\frac{1-v}{1+v};\qquad \beta_i = \sqrt{1+\alpha_i}\,\frac{(1+v)^2}{4v}$$
$$\bar{q}^2 = t-(m_b-m_c)^2;\qquad v = \sqrt{1-4\frac{m_b m_c}{\bar{q}^2}}, \quad (13)$$

where m_i is the on-shell/pole mass.

• *Higher Orders Perturbative contributions at large* q^2

In the absence of a complete result to order α_s^2, we shall approximatively use the expression of the spectral function for $m_c=0$:

$$\text{Im}\psi_5(t)|_{PT}^{N2LO} \simeq \frac{m_b^2 t}{8\pi}\left(\frac{\bar{\alpha}_s}{\pi}\right)^2 R_{2s} , \quad (14)$$

where R_{2s} has been obtained semi-analytically in [32,33] and is available as a Mathematica package program Rvs.m.

We expect that it is a good approximation because we shall see that the NLO contributions induce (as expected) small corrections in the ratio of moments used to determine $\overline{m}_{c,b}$ due to the partial cancellation of this contribution.

We estimate the accuracy of this approximation by comparing this N2LO approximation with the one obtained assuming a geometric growth of the PT coefficients [34].

We estimate the error due to the truncation of the PT series from the N3LO contribution estimated, as above, from a geometric growth of the PT series which is expected to mimic the phenomenological $1/q^2$ dimension-two contribution parametrizing the uncalculated large order terms of PT series [35–38].

• $\langle\alpha_s G^2\rangle$ *contribution at large* q^2

We shall use the complete expression of the gluon condensate $\langle\alpha_s G^2\rangle$ contribution to the two-point correlator given in [23], which agrees with known results for $m_b=m_c$ [10,11]. The Wilson coefficient reads:

$$C_{G^2}(q^2,\mu) = \frac{1}{\pi}\int_{(m_b+m_c)^2}^{\infty}\left\{-\frac{m_b m_c\,t\,\left(t-m_b^2-m_b m_c-m_c^2\right)}{2\,(Q^2+t)\,[t-(m_b-m_c)^2]^{3/2}} - \sqrt{m_b m_c}\left\{\frac{(m_b+m_c)^2[t-(m_b+m_c)^2]}{16[Q^2+(m_b+m_c)^2]^2} - \frac{1}{16[Q^2+(m_b+m_c)^2]} \times\left[(m_b+m_c)^2 + \frac{(5m_b^2+18m_b m_c+5m_c^2)[t-(m_b+m_c)^2]}{8m_b m_c}\right]\right\}\right\} \times\frac{dt}{[t-(m_b+m_c)^2]^{5/2}}, \quad (15)$$

where $Q^2 \equiv -q^2$, from which we can easily deduce the Laplace transform.

• *$\langle g^3G^3\rangle$ contribution at large q^2*

A similar (but lengthy) expression of the $\langle g^3G^3\rangle$ condensate contribution can also be obtained from [23], where it has been checked that it agrees with known result for $m_b = m_c$ [29]. It reads:

$$C_{G^3}(q^2) = \frac{1}{\pi}\int_{\Sigma^2}^{\infty} \frac{dt}{[t-\Sigma^2]^{9/2}} \Bigg\{ \frac{m_b m_c\, t}{6\,(t+Q^2)\,[t-(m_b-m_c)^2]^{7/2}}$$
$$\times \Big\{ 3t^4 - 2(3m_b^2 + 2m_b m_c + 3m_c^2)t^3$$
$$+(5m_b^3 m_c + 18 m_b^2 m_c^2 + 5 m_b m_c^3)\, t^2$$
$$+2(3m_b^6 + m_b^5 m_c - 6m_b^4 m_c^2 - 6m_b^3 m_c^3 - 6m_b^2 m_c^4 + m_b m_c^5 + 3m_c^6)\, t$$
$$-3(m_b^8 + m_b^7 m_c - m_b^5 m_c^3 - 2m_b^4 m_c^4 - m_b^3 m_c^5 + m_b m_c^7 + m_c^8)\Big\}$$
$$-\sqrt{m_b m_c}\Bigg\{ \frac{-7\Sigma^4\,(t-\Sigma^2)^3}{192\,(Q^2+\Sigma^2)^4}$$
$$+\left[\frac{7\Sigma^2\,(t-\Sigma^2)^2}{192} + \frac{A\,(t-\Sigma^2)^3}{1536 m_b m_c}\right]\frac{\Sigma^2}{(Q^2+\Sigma^2)^3}$$
$$+\left[\frac{-7\Sigma^4\,(t-\Sigma^2)}{192} - \frac{\Sigma^2 A\,(t-\Sigma^2)^2}{1536 m_b m_c} + \frac{B\,(t-\Sigma^2)^3}{24576 m_b^2 m_c^2}\right]$$
$$\times \frac{1}{(Q^2+\Sigma^2)^2}$$
$$+\left[\frac{7\Sigma^4}{192} + \frac{\Sigma^2 A\,(t-\Sigma^2)}{1536 m_b m_c} - \frac{B\,(t-\Sigma^2)^2}{24576 m_b^2 m_c^2} - \frac{C\,(t-\Sigma^2)^3}{196608 m_b^3 m_c^3}\right]$$
$$\times \frac{1}{Q^2+\Sigma^2}\Bigg\}\Bigg\},$$

with:

$$\Sigma = m_b + m_c$$
$$A = 51m_b^2 + 166 m_b m_c + 51 m_c^2$$
$$B = 31m_b^4 - 836 m_b^3 m_c - 1862 m_b^2 m_c^2 - 836 m_b m_c^3 + 31 m_c^4$$
$$C = 277m_b^4 + 596 m_b^3 m_c - 514 m_b^2 m_c^2 + 596 m_b m_c^3 + 277 m_c^4. \quad (16)$$

• *From On-shell to the $\overline{MS}$-scheme*

We transform the pole masses m_Q to the running masses $\overline{m}_Q(\mu)$ using the known relation in the $\overline{MS}$-scheme to order α_s^2 [39–47]:

$$m_Q = \overline{m}_Q(\mu)\Big[1 + \frac{4}{3}a_s + (16.2163 - 1.0414 n_l)a_s^2$$
$$+ \ln\frac{\mu^2}{m_Q^2}\left(a_s + (8.8472 - 0.3611 n_l)a_s^2\right)$$
$$+ \ln^2\frac{\mu^2}{m_Q^2}\,(1.7917 - 0.0833 n_l)\,a_s^2 \ldots\Big], \quad (17)$$

for $n_l = 3$ light flavours. In the following, we shall use $n_f = 5$ total number of flavours for the numerical value of α_s.

4. QCD input parameters

The QCD parameters which shall appear in the following analysis will be the charm and bottom quark masses $m_{c,b}$, the gluon condensates $\langle\alpha_s G^2\rangle$ and $\langle g^3G^3\rangle$.

Table 1
QCD input parameters from recent QSSR analysis based on stability criteria. The values of $\overline{m}_c(m_c)$ and $\overline{m}_b(m_b)$ come from recent moments SR and their ratios [48] where the errors have been multiplied by a factor 2 to be conservative.

Parameters	Values	Sources	Ref.
$\alpha_s(M_Z)$	0.1181(16)(3)	$M_{\chi_{0c,b}} - M_{\eta_{c,b}}$	Ratios of LSR [49]
$\overline{m}_c(m_c)$	1264(12) MeV	J/ψ family	Mom. [48]
$\overline{m}_b(m_b)$	4188(16) MeV	Υ family	Mom. [48]
$\langle\alpha_s G^2\rangle$	$(6.35 \pm 0.35)\times 10^{-2}$ GeV^4	Hadrons	QSSR average [49]
$\langle g^3G^3\rangle$	(8.2 ± 2.0) GeV^2 $\times \langle\alpha_s G^2\rangle$	J/ψ family	Mom. [50,51]
			Ratios of LSR [52]

• *QCD coupling α_s*

We shall use from the $M_{\chi_{0c}} - M_{\eta_c}$ mass-splitting sum rule [49]:

$$\alpha_s(2.85) = 0.262(9) \rightsquigarrow \alpha_s(M_\tau) = 0.318(15)$$
$$\rightsquigarrow \alpha_s(M_Z) = 0.1183(19)(3) \quad (18)$$

which is more precise than the one from $M_{\chi_{0b}} - M_{\eta_b}$ [49]:

$$\alpha_s(9.50) = 0.180(8) \rightsquigarrow \alpha_s(M_\tau) = 0.312(27)$$
$$\rightsquigarrow \alpha_s(M_Z) = 0.1175(32)(3). \quad (19)$$

These lead to the mean value quoted in Table 1, which is in complete agreement with the world average [53]:

$$\alpha_s(M_Z) = 0.1181(11), \quad (20)$$

but with a larger error.

• *c and b quark masses*

For the c and b quarks, we shall use the recent determinations [48] of the running masses and the corresponding value of α_s evaluated at the scale μ obtained using the same sum rule approach from charmonium and bottomium systems. The range of values given in Table 1 enlarged by a factor 2 are within the PDG average [53].

• *Gluon and quark-gluon mixed condensates*

For the $\langle\alpha_s G^2\rangle$ condensate, we use the recent estimate obtained from a correlation with the values of the heavy quark masses and α_s which can be compared with the QSSR average from different channels [49].

The one of $\langle g^3G^3\rangle$ comes from a QSSR analysis of charmonium systems. Their values are given in Table 1.

5. Parametrisation of the spectral function

– In the present case, where no complete data on the B_c spectral function are available, we use the duality ansatz:

$$\mathrm{Im}\psi_5(t) \simeq f_P^2 M_P^4 \delta(t - M_P^2) + \Theta(t - t_c)\text{“}QCD \text{ continuum”}, \quad (21)$$

for parametrizing the spectral function. M_P and f_P are the lowest ground state mass and coupling analogue to f_π. The “QCD continuum” is the imaginary part of the QCD two-point function while t_c is its threshold. Within a such parametrization, one obtains:

$$\mathcal{R}_n^c \equiv \mathcal{R} \simeq M_P^2\,, \quad (22)$$

indicating that the ratio of moments appears to be a useful tool for extracting the mass of the hadron ground state [10–14].

– This simple model has been tested in different channels where complete data are available (charmonium, bottomium and $e^+e^- \to I = 1$ hadrons) [9–11]. It was shown that, within the model, the sum rule reproduces well the one using the complete data, while the masses of the lowest ground state mesons (J/ψ, Υ and ρ) have been predicted with a good accuracy. In the extreme case of the Goldstone pion, the sum rule using the spectral function parametrized by this simple model [10,11] and the more complete one by ChPT [54] lead to similar values of the sum of light quark masses $(m_u + m_d)$ indicating the efficiency of this simple parametrization.

– An eventual violation of the quark-hadron duality (DV) [55, 56] has been frequently tested in the accurate determination of $\alpha_s(\tau)$ from hadronic τ-decay data [56–58], where its quantitative effect in the spectral function was found to be less than 1%. Typically, the DV has the form:

$$\Delta \mathrm{Im}\psi_5(t) \sim (m_c + m_b)^2 t\, \mathrm{e}^{-\kappa t} \sin(\alpha + \beta t)\theta(t - t_c)\ , \tag{23}$$

where κ, α, β are model-dependent fitted parameters but not based from first principles. Within this model, where the contribution is doubly exponential suppressed in the Laplace sum rule analysis, we expect that in the stability regions where the QCD continuum contribution to the sum rule is minimal and where the optimal results in this paper will be extracted, such duality violations can be safely neglected.

– Therefore, we (a priori) expect that one can extract with a good accuracy the c and b running quark masses and the B_c decay constant within the approach. An eventual improvement of the results can be done after a more complete measurement of the B_c pseudoscalar spectral function which is not an easy task though the recent discovery by CMS [59] of the $B_c(2S)$ state at 6872(1.5) MeV is a good starting point in this direction.

– In the following, in order to minimize the effects of unknown higher radial excitations smeared by the QCD continuum and some eventual quark-duality violations, we shall work with the lowest ratio of moments $\mathcal{R}_0^c$ for extracting the quark masses and with the lowest moment $\mathcal{L}_0^c$ for estimating the decay constant f_{B_c}. Moment with negative n will not be considered due to their sensitivity on the non-perturbative contributions such as $\psi_5(0)$.

6. Optimization criteria

For extracting the optimal results from the analysis, we shall use optimization criteria (minimum sensitivity) of the observables versus the variation of the external variables namely the τ sum rule parameter, the QCD continuum threshold t_c and the subtraction point μ.

Results based on these criteria have led to successful predictions in the current literature [10,11]. τ-stability has been introduced and tested by Bell-Bertlmann using the toy model of harmonic oscillator [9] and applied successfully in the heavy [4,5,9, 60–68] and light quarks systems [2,3,10–14,69]. It has been extended later on to the t_c-stability [10–13] and to the μ-stability criteria [27,49,63,69,70].

Stability on the number n of heavy quark moments have also been used [48,50–52].

One can notice in the previous works that these criteria have led to more solid theoretical basis and noticeable improved results from the sum rule analysis.

7. $\overline{m}_c$ and $\overline{m}_b$ from M_{B_c}

In the following, we look for the stability regions of the external parameters τ, t_c and μ where we shall extract our optimal result.

• *τ stability*

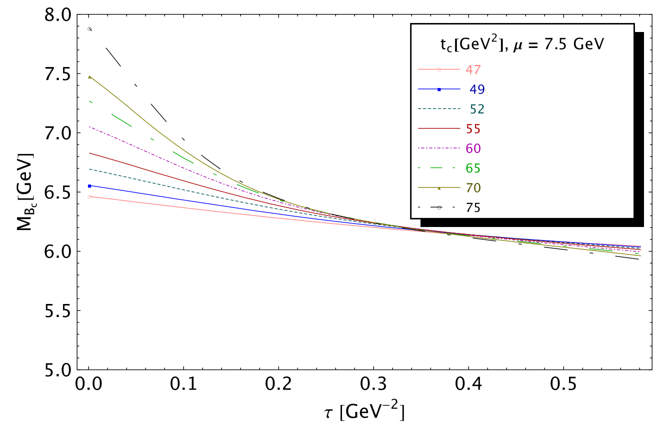

Fig. 1. M_{B_c} as function of τ for different values of t_c, for μ=7.5 GeV and for given values of $\overline{m}_{c,b}(\overline{m}_{c,b})$ in Table 1.

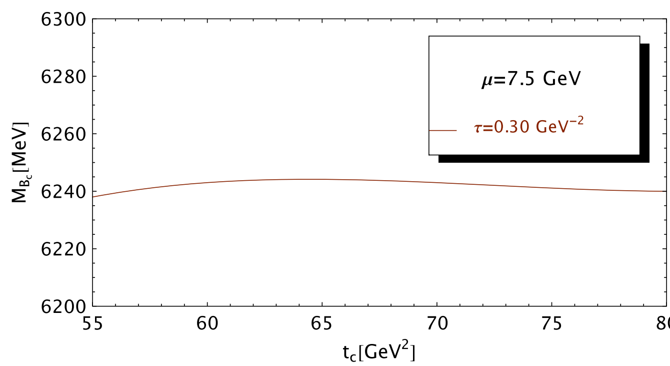

Fig. 2. M_{B_c} as function of t_c for $\mu = 7.5$ GeV and for the range of τ-stability values. We use the central values of $\overline{m}_{c,b}(\overline{m}_{c,b})$ given in Table 1.

In a first step, fixing the value of $\mu = 7.5$ GeV which we shall justify later and which is the central value of $\mu = (7.5 \pm 0.5)$ GeV obtained in [27], we show in Fig. 1 the behaviour of M_{B_c} for different values of t_c where the central values of $\overline{m}_c(\overline{m}_c)$=1264 MeV and $\overline{m}_b(\overline{m}_b)$=4188 MeV given in Table 1 have been used. We see that the inflexion points at $\tau \simeq (0.30 \sim 0.32)$ GeV^{-2} appear for $t_c \geq$ 52 GeV^2. The smallest value of $\sqrt{t_c}$ is around the $B_c(2S)$ mass of 6872(1.5) MeV recently discovered by CMS [59] but does not necessarily coïncide with it as the QCD continuum is expected to smear all higher states contributions to the spectral function. Instead, its value is related by duality to the ground state parameters as discussed in [71] from a FESR analysis of the ρ-meson channel.

• *t_c stability*

We show in Fig. 2 the behaviour of M_{B_c} versus t_c which is very stable. For definiteness, we take t_c in the range 52 to 79 GeV^2 where we have a slight maximum at $t_c \simeq 60$ GeV^2. At this range of t_c values, one can easily check that the QCD continuum contribution to the sum rule is (almost) negligible. To have more insights on this contribution, we show in Fig. 3 the ratio of the continuum over the lowest ground state contribution as predicted by QCD:

$$r_c \equiv \frac{\int_{t_c}^{\infty} dt e^{-t\tau}\, \Delta \mathrm{Im}\psi_5^{cont}}{\int_{(m_c+m_b)^2}^{t_c} dt e^{-t\tau}\, \Delta \mathrm{Im}\psi_5^{B_c}} \tag{24}$$

where one can indeed see that the QCD continuum to the moment sum rule $\mathcal{L}_0^c$ is negligible in this range of t_c values. This contribution is even less in the ratio of moments $\mathcal{R}_0^c$ used to get M_{B_c}.

• *μ stability*

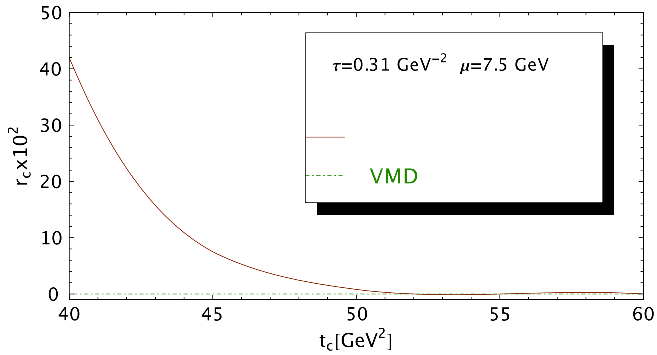

Fig. 3. Ratio r_c of the continuum over the lowest ground state contribution as function of t_c at the corresponding τ-stability points for $\mu = 7.5$ GeV and for given values of $\overline{m}_{c,b}(\overline{m}_{c,b})$ in Table 1. The dashed-dotted line is the contribution for a Vector Meson Dominance (VDM) assumption of the spectral function.

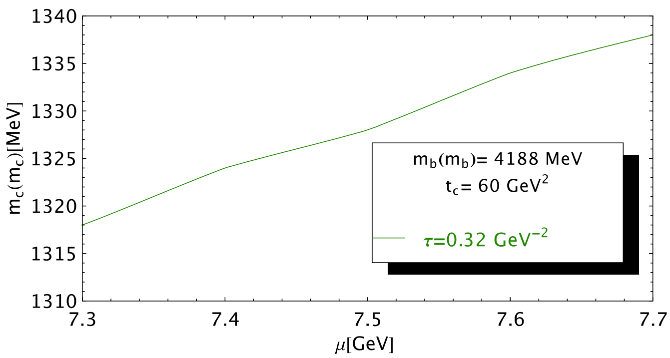

Fig. 4. $\overline{m}_c(\overline{m}_c)$ as function of μ for $\tau \simeq 0.32$ GeV^2 and for the central value of $\overline{m}_b(\overline{m}_b)$ given in Table 1.

Given e.g. the central value of $\overline{m}_b(\overline{m}_b) = 4188$ MeV from Table 1 and using $\tau = .32$ GeV^{-2} and $t_c = 60$ GeV^2, we show in Fig. 4 the correlated values of $\overline{m}_c(\overline{m}_c)$ at different values of μ needed for reproducing M_{B_c}. We obtain an inflexion point at:

$$\mu = (7.5 \pm 0.1)\ \text{GeV}\,, \tag{25}$$

which we shall use in the following. This value agrees with the one $\mu = (7.5 \pm 0.5)$ GeV quoted in [27] using different ways.

• *Extracting the set* $(\overline{m}_c, \overline{m}_b)$

In the following, we study the correlation between $\overline{m}_c$ and $\overline{m}_b$ needed for reproducing the experimental mass [53]:

$$M_{B_c}^{exp} = 6275.6(1.1)\ \text{MeV}\,, \tag{26}$$

from the ratio $\mathcal{R}_0^c$ of Laplace sum rules defined in Eqs. (2) and (22).

– Allowing $\overline{m}_c(\overline{m}_c)$ to move in the range:

$$\overline{m}_c(\overline{m}_c) \simeq (1252 - 1282)\ \text{MeV} \tag{27}$$

from the J/ψ and $M_{\chi_{0c}-M_{\eta_c}}$ mass-splitting sum rules, we show in Fig. 5, the predictions for M_{B_c} as a function of $\overline{m}_b(\overline{m}_b)$. The band is the error induced by the choice of the stability points $\tau = (0.30 - 0.32)$ GeV^{-2} which is about (12-13) MeV, while the error due to some other parameters are negligible. Then, we deduce:

$$\overline{m}_b(\overline{m}_b) = (4195 - 4245)\ \text{MeV}\,, \tag{28}$$

which leads to the correlated set of values in units of MeV:

$$[\overline{m}_b(\overline{m}_b), \overline{m}_c(\overline{m}_c)] = [4195, 1282] \ldots [4245, 1252]\,. \tag{29}$$

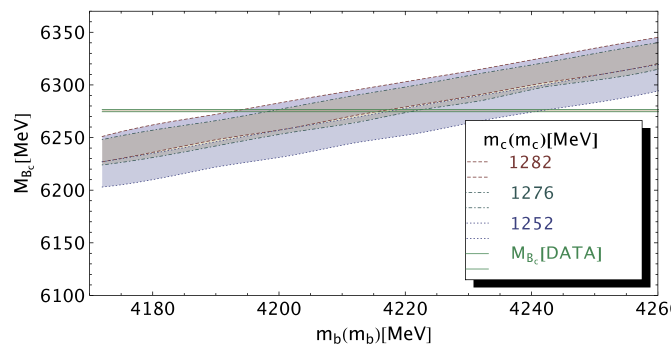

Fig. 5. M_{B_c} as function of $\overline{m}_b(\overline{m}_b)$ for different values of $\overline{m}_c(\overline{m}_c)$, for $\mu = 7.5$ GeV and for the range of τ-stability values $\tau = (0.30 - 0.32)$ GeV^{-2}.

This result shows that a small value of m_c is correlated to a large value of m_b and vice-versa.

– Scrutinizing Fig. 5, one can see that, at fixed m_b, e.g. 4220 MeV, M_{B_c} increases with the m_c values given in the legend (vertical line), as intuitively expected. On the other, fixing the value of m_c at the one in the legend say 1282 MeV, one can see (straight-line with a positive slope) that M_{B_c} increases when m_b increases on the m_b-axis as also intuitively expected.

– Considering that the values of $\overline{m}_b(\overline{m}_b)$ are inside the range:

$$\overline{m}_b(\overline{m}_b) = (4176 - 4209)\ \text{MeV}\,, \tag{30}$$

allowed from the Υ sum rules as given in Table 2, we can deduce from Fig. 5 stronger constraints on $\overline{m}_b(\overline{m}_b)$:

$$\begin{aligned}\overline{m}_b(\overline{m}_b) &= (4195 - 4209)\ \text{MeV}\\ &= 4202(7)\ \text{MeV}\,,\end{aligned} \tag{31}$$

where the error is similar to the accurate value from the Υ sum rule in Table 2. This is due to the small intersection region of the results from the J/ψ, Υ and the B_c sum rules. With this range of values, we deduce:

$$\begin{aligned}\overline{m}_c(\overline{m}_c) &= 1286(8)_{fig.4}(14)_\tau(1)_{t_c}\ \text{MeV}\,,\\ &= 1286(16)\ \text{MeV}\,,\end{aligned} \tag{32}$$

where the errors due to some other parameters and to the truncation of the PT series are negligible. The susbscript $fig.4$ indicates that the error comes from the intersection region between the Υ and B_c sum rules in Fig. 5.

We consider the values in Eqs. (31) and (32) as our final determinations of $\overline{m}_b(\overline{m}_b)$ and $\overline{m}_c(\overline{m}_c)$ from M_{B_c} and combined constraints from the J/ψ and Υ sum rules.

• *Comments*

– One can notice that the effect of the PT radiative corrections are quite small in the ratio of moments because the ones of the absolute moments $\mathcal{L}_{0,1}$ tend to compensate each others. This fact can be checked from a numerical parametrization of the LSR ratio. At the optimization scale $\tau \simeq 0.32$ GeV^{-2} and $\mu = 7.5$ GeV, it reads ($a_s \equiv \alpha_s/\pi$):

$$\sqrt{\mathcal{R}_0}|_{PT}^{N2LO} \simeq \sqrt{\mathcal{R}_0}|_{PT}^{LO}(1 - 0.16a_s - 0.42a_s^2)\,, \tag{33}$$

while the LSR lowest moment is:

$$\sqrt{\mathcal{L}_0}|_{PT}^{N2LO} \simeq \sqrt{\mathcal{L}_0}|_{PT}^{LO}(1 + 6a_s + 26.4a_s^2)\,. \tag{34}$$

– One can also notice that the approximate N2LO contribution obtained for $m_c = 0$ in the lowest moment is about the same as

Table 2
Values of $\overline{m}_c(m_c)$ and $\overline{m}_b(m_b)$ coming from our recent QSSR analysis based on stability criteria. Some other determinations can be found in [53].

Parameters	Values [MeV]	Sources	Ref.
$\overline{m}_c(m_c)$	1256(30)	J/ψ family	Ratios of LSR17 [49]
	1266(16)	$M_{\chi_{0c}-M_{\eta_c}}$	Ratios of LSR [49]
	1264(6)	J/ψ family	MOM & Ratios of MOM [48]
	1286(66)	M_D	Ratios of LSR [70]
	1286(16)	M_{B_c}	Ratios of LSR *(This work)*
	1266(6)	*Average*	*This work*
$\overline{m}_b(m_b)$	4192(17)	Υ family	Ratios of LSR [49]
	4188(8)	Υ family	MOM & Ratios of MOM [48]
	4236(69)	M_B	Ratios of MOM & of LSR [70]
	4213(59)	M_B	Ratio of HQET-LSR [63]
	4202(7)	M_{B_c}	Ratios of LSR *(This work)*
	4196(8)	*Average*	*This work*

the one $36a_s^2$ which one would obtain using a geometric growth of the a_s PT coefficients [36]. Therefore, the error induced by the difference of the two N2LO approximations is negligible.

– The contribution of the gluon condensates is also small at the optimization scale as $\langle\alpha_s G^2\rangle$ increases M_{B_c} by about 5 MeV while $\langle g^3G^3\rangle$ decreases it by 1 MeV. These contributions are small and also show the good convergence of the OPE. Then, its induces an increase of about 6 MeV in the quark mass values and introduces a negligible error of 1 MeV. However, the non-perturbative contributions are important for having the τ-stability region.

– As the QCD continuum contribution which is expected to smear all radial excitation contributions is negligible at the optimization region due to the exponential dumping factor of the sum rule, we expect that some eventual DV discussed previously can be safely neglected due to its doubly exponential suppression in the LSR analysis. We also expect that the effects of higher radial excitations can be similarly neglected like the one of the QCD continuum.

8. Comparison with some other QSSR determinations

We compare the previous results with the ones in Table 2 obtained from some other QSSR analysis using the same stability criteria. A tentative average of the central values and using the error from the most precise predictions from J/ψ and Υ families leads to the averages quoted in Table 2.

9. Revisiting f_{B_c}

Using the previous correlated values of $(\overline{m}_c, \overline{m}_b)$, we reconsider the estimate of f_{B_c} recently done in Ref. [27].

• *τ and t_c stabilities*

We show the τ-behaviour of f_{B_c} in Fig. 6 for different values of t_c for $\mu = 7.5$ GeV and for $[\overline{m}_c(\overline{m}_c), \overline{m}_b(\overline{m}_b)] = [1264, 4188]$ MeV from Table 1. We start to have τ-stability (minimum) for the set $[t_c, \tau]$=[47 GeV^2, 0.22 GeV^{-2}] and t_c-stability for the set [60 GeV^2, (0.30 − 0.32) GeV^{-2}]. One can notice that the τ-stability starts earlier for smaller t_c-value than to the case of the ratio of moments used in the preceding sections. To be conservative, we shall consider the value of f_{B_c} obtained inside this larger range of t_c-values and take as a final value its mean. In this range, the value of f_{B_c} increases by about 15 MeV.

• *μ stability*

We study the influence of μ on f_{B_c} in Fig. 7 given the values of τ and $m_{b,c}$. We see a clear stability for $\mu = (7.2 - 7.5)$ GeV which is consistent with the one for M_{B_c} and with the one obtained in [27] indicating the self-consistency of the analysis.

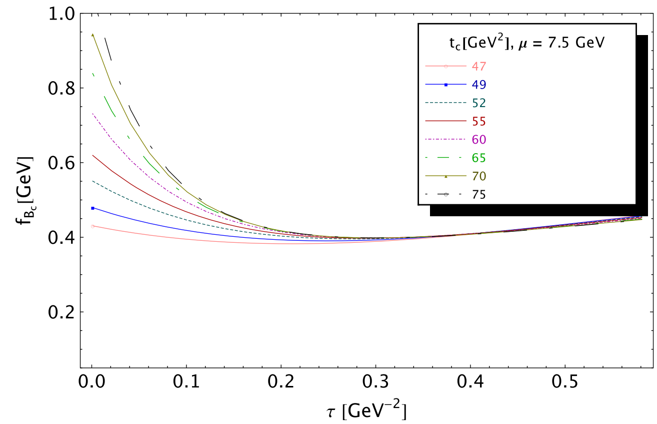

Fig. 6. f_{B_c} versus τ given $\mu = 7.5$ GeV and $[\overline{m}_c(\overline{m}_c), \overline{m}_b(\overline{m}_b)] = [1264, 4188]$ MeV from Table 1.

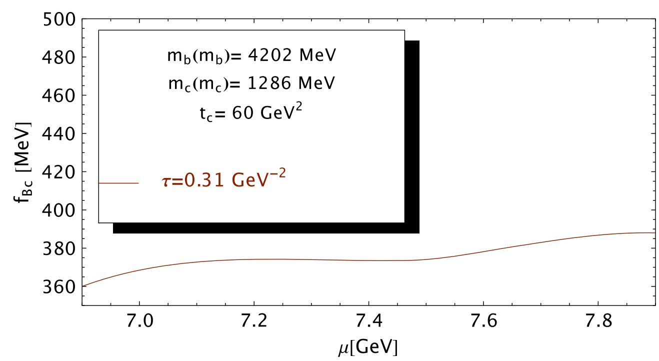

Fig. 7. f_{B_c} as function of μ for $\tau \simeq 0.31$ GeV^{-2} and for the central value of $\overline{m}_{c,b}(\overline{m}_{c,b})$ given in Eqs. (31) and (32).

• *Higher orders (HO) PT corrections*

– The N2LO contribution increases the prediction from LO ⊕ NLO by 24 MeV. We estimate the error induced by using the result at $m_c = 0$ by comparing it with the one obtained from the estimate of the N2LO coefficient using a geometric growth of the PT series (see Eq. (34)). The difference induces an error of $\pm 9.6a_s^2$ which corresponds to ±9 MeV.

– We estimate the error due to the uncalculated higher order (HO) part of the PT series from the N3LO contribution estimated by using the geometric growth of the coefficients given in the numerical expression in Eq. (34) which is $\pm 158a_s^3$. It introduces an error of about 10 MeV.

• *Gluon condensate contributions*

The inclusion of the $\langle\alpha_s G^2\rangle$ condensate increases the sum of the PT contributions by 3 MeV, while the inclusion of the $\langle g^3G^3\rangle$ decreases the prediction by 1 MeV. These contributions and the induced error are negligible.

• *Result*

– The result of the analysis in units of MeV is:

$$f_{B_c} = 368(1)_\tau(8)_{t_c}(7)_{m_c}(5)_{m_b}(1)_{\alpha_s}(1)_\mu 9)_{N2LO}(10)_{HO}$$
$$= 368(18)\text{ MeV}, \tag{35}$$

if one uses the mass values obtained in Eqs. (31) and (32), while it is:

$$f_{B_c} = 381(1)_\tau(8)_{t_c}(3)_{m_c}(5)_{m_b}(1)_{\alpha_s}(1)_\mu(9)_{N2LO}(10)_{HO}$$
$$= 381(17)\text{ MeV}, \tag{36}$$

if one uses the tentative mass averages given in Table 2. We take as a final result the mean of the two determinations:

$$f_{B_c} = 371(17)\text{ MeV}\,. \tag{37}$$

This result improves the previous one $f_{B_c} = 436(40)$ MeV obtained recently in Ref. [27] and the earlier results in [22,24,25]. It confirms and improves the one $f_{B_c} = 388(29)$ MeV averaged from moments and LSR in [23] where the values of the pole masses have been used. However, it disagrees with some results including the lattice one reviewed in Table 3 of [27]. New estimates from the lattice approach is needed for clarifying the issue. Comments related to some of the previous works have been already addressed in [27] and can be consulted there.

10. Attempted upper bound for $f_{B_c(2S)}$

We attempt to give an upper bound to $f_{B_c(2S)}$ by using a "two resonances + QCD continuum" parametrization of the spectral function. However, we are aware on the fact that due to the exponential suppression of the $B_c(2S)$ contribution compared to $B_c(1S)$ and of its eventual smaller coupling as expected for the observed radial excitations in some other channels, we may not extract with a good precision the $B_c(2S)$ decay constant from this approach. Instead by using the positivity of the QCD continuum contribution for $t_c \geq 47$ GeV^2 just above $M^2_{B_c(2S)}$, one obtains the semi-analytic expression from $\mathcal{L}^c_0$:

$$\rho_{B_c} \equiv \left(\frac{f_{B_c(2S)}}{f_{B_c}}\right)^2 \left(\frac{M_{B_c(2S)}}{M_{B_c}}\right)^4 e^{-\left(M^2_{B_c(2S)} - M^2_{B_c}\right)\tau} \leq 3.6\%, \tag{38}$$

at the τ-stability of about 0.22 GeV^{-2} as can be deduced from Fig. 3. Using the previous value of f_{B_c} in Eq. (37), we deduce:

$$f_{B_c(2S)} \leq (139 \pm 6)\text{ MeV}\,, \tag{39}$$

indicating that the radial excitation couples weaker to the corresponding quark current than the ground state meson. This feature has been already observed experimentally in the case of light (π, ρ) and heavy (ψ, Υ) mesons.

11. Summary and conclusions

• *$\overline{m}_c$ and $\overline{m}_b$*

We have used QCD Laplace sum rules to estimate (*for the first time*) the correlated values of $\overline{m}_c(\overline{m}_c)$ and $\overline{m}_b(\overline{m}_b)$ from the B_c-meson mass allowing them to move inside the extended (multiplied by a factor 2: see Fig. 5 and Table 1) range of values allowed by charmonium and bottomium sum rules. These values:

$$\overline{m}_b(\overline{m}_b) = 4202(7)\text{ MeV (Eq. (31))},$$

$$\overline{m}_c(\overline{m}_c) = 1286(16)\text{ MeV (Eq. (32))},$$

agree with previous recent ones from charmonium and bottomium systems quoted in Table 2. The errors are similar to the ones from J/ψ and Υ sum rules. They have been relatively reduced compared to the ones from the D and B meson masses thanks to the extra constraints on the range of variations of $\overline{m}_{c,b}(\overline{m}_{c,b})$ used in Fig. 5 from J/ψ and Υ sum rules. Using these values and the ones from recent different QSSR determinations collected in Table 2, we deduce the QSSR average:

$$\overline{m}_c(\overline{m}_c) = 1266(6)\text{ MeV and } \overline{m}_b(\overline{m}_b) = 4196(8)\text{ MeV}\,,$$

where the error comes from the most accurate determinations.

• *Decay constants f_{B_c} and $f_{B_c(2S)}$*

Using the new results in Eqs. (31) and (32), we improve our previous predictions of f_{B_c} [22,23,27] which becomes more accurate due to the inclusion of HO PT corrections and to the use of modern values of the QCD input parameters:

$$f_{B_c} = 371(17)\text{ MeV (Eq. (37))}$$

An upper bound for the $B_c(2S)$ decay constant is also derived:

$$f_{B_c}(2S) \leq (139 \pm 6)\text{ MeV (Eq. (39))}\,.$$

These new results will be useful for further phenomenological analysis.

Improvement of our results requires a complete evaluation of the spectral function at N2LO and a future measurement of the $B_c(2S)$ decay constant. We plan to extend the analysis in this paper to some other B_c-like mesons in a future work.

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