Generalized Schwarzians

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In this paper we demonstrate that the different generalizations of the Schwarzians, supersymmetric or purely bosonic, can be easily constructed by using the nonlinear realizations technique.

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I. INTRODUCTION

The Schwarzian derivative (1), or just the Schwarzian, appears in apparently unrelated fields of mathematics: from classical complex analysis to integrable systems [1]. In contrast, in physics the Schwarzian appears either in the transformation properties of the conformal (supersymmetric) stress tensor [2,3] or arises as the low energy limit of the Sachdev-Ye-Kitaev (SYK) model [4]. Schwarzian also appeared as a quantum correction in Hamiltonians of some supersymmetric mechanic models [5]. Therefore, it is not strange that the possible generalizations of the Schwarzian are mainly related to its supersymmetric extensions where the supersymmetric Schwarzian naturally appears in the superconformal transformations of the supercurrent [3,6–8]. However, this generalization quickly stops at $\mathcal{N}=4$ supersymmetry due to appearance of components with negative conformal dimension in the current superfield $J^{(N)}$ for $\mathcal{N}>4$. In addition, the recent construction of the "flat space" version of the Schwarzian [9,10] raised the question about the existence of the systematic way to build the generalized (bosonic ones or possessing higher $\mathcal{N}>4$ supersymmetries) Schwarzians.

The treatment of the supersymmetric Schwarzians as the anomalous terms in the transformations of the currents superfield $J^{(\mathcal{N})}(Z)$ [3] leads to the conclusion that the structure of the (super-)Schwarzians is completely defined by the conformal symmetry and, therefore, it should exist a different, probably purely algebraic, way to define the (super-)Schwarzians. The main property of the (super-)Schwarzians that defines their structure is their invariance with respect to (super)conformal transformations. The suitable way to construct (super)conformal invariants is the method of nonlinear realizations [11,12] equipped with the inverse Higgs phenomenon [13]. However, the natural invariant objects in the nonlinear realization approach are the Cartan forms, which contain the differentials of the coordinates of the (super)space with nontrivial transformation properties. Thus, an additional question concerns the implementation of the inert (super) coordinates in the nonlinear realization approach.

This method was first applied to the $sl(2)$ algebra in [14] to obtain the standard Schwarzian and then extended to different superconformal algebras in [15–18]. Later on, this approach has been applied to the cases of nonrelativistic Schwarzians and Carroll algebra [19]. It should be noted that the constraints proposed in these papers look like the results of an illuminating guess. Moreover, in some practically interesting cases the proposed constraints are too strong to set the recovered supersymmetric Schwarzian as a constant.

In two of our papers [20,21] the method proposed in [14] was modified in two directions. First, we introduced the "inert superspace" as the coordinates of the independent "inert" coset elements. Second, the constraints were imposed on the full Cartan forms by either nullifying them or identifying with the "inert superspace" forms. This last feature gives us the possibility to invoke into the game the powerful method of the Maurer-Cartan equations to analyze the consequences of the constraints, which drastically simplifies calculations.

In this paper, we apply the proposed approach to construct some new generalized Schwarzians. After a short review of the basic steps of our approach in Sec. II, we will construct

(i) A "flat space" variant of the Schwarzian (Sec. III),
(ii) A bosonic variant of the Schwarzian with $su(1,2)$ symmetry (Sec. IV), and
(iii) Schwarzians with $\mathcal{N}$-extended supersymmetry (Sec. V).

We conclude in the Sec. VI with some interesting but unsolved at the moment questions and hypotheses.

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II. SKETCH OF THE IDEA

Before getting to the main results of our paper, let us illustrate how the method of nonlinear realizations works when applied to the Schwarzians in two simpler examples: the standard bosonic Schwarzian ($\mathcal{N}=0$ case) and one of supersymmetric Schwarzians ($\mathcal{N}=2$ case).

A. $\mathcal{N}=0$ case

The Schwarzian derivative $\{t,\tau\}$ is defined by the relation

$$\{t,\tau\}=\frac{\dddot{t}}{\dot{t}}-\frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2, \qquad \dot{f}=\partial_\tau f. \tag{1}$$

Its famous property is the invariance with respect to $SL(2,R)$ Möbius transformations, acting on t:

$$t'=\frac{at+b}{ct+d}\Rightarrow\{t',\tau\}=\{t,\tau\}. \tag{2}$$

Note that the "time" τ is invariant with respect to these $SL(2,R)$ transformations.

The action of the bosonic Schwarzian mechanics (see, e.g., [4])

$$S_{\mathrm{schw}}[t]=-\frac{1}{2}\int d\tau\{t,\tau\} \tag{3}$$

leads to the following equation of motion

$$\frac{d}{d\tau}\{t,\tau\}=0\Rightarrow\{t,\tau\}=2m^2=\mathrm{const}. \tag{4}$$

As $SL(2,R)$ transformations of t are involved, it is natural to look at this system from the nonlinear realization viewpoint. Indeed, one can consider the $sl(2,R)$ algebra, spanned by the Hermitian generators P, D, K

$$\mathrm{i}[D,P]=P, \qquad \mathrm{i}[D,K]=-K, \qquad \mathrm{i}[K,P]=2D \tag{5}$$

and parametrize the group element in the following way:

$$g=e^{\mathrm{i}tP}e^{\mathrm{i}zK}e^{\mathrm{i}uD}. \tag{6}$$

This parametrization is similar to one used in the construction of the conformal mechanics [22], when P, D, and K generate time translations, dilatations, and conformal boosts, respectively. Then the Cartan forms, invariant with respect to left multiplication $g'=g_0g$, read

$$g^{-1}dg=\mathrm{i}\omega_P P+\mathrm{i}\omega_D D+\mathrm{i}\omega_K K\Rightarrow\omega_P=e^{-u}dt,$$
$$\omega_D=du-2zdt, \qquad \omega_K=e^u(dz+z^2dt). \tag{7}$$

The infinitesimal $sl(2,R)$ transformations

$$g_0=e^{\mathrm{i}(\tilde{a}P+\tilde{b}D+\tilde{c}K)}\Rightarrow\delta t=\tilde{a}+\tilde{b}t+\tilde{c}t^2,$$
$$\delta u=\frac{d}{dt}\delta t, \qquad \delta z=\frac{1}{2}\frac{d}{dt}\delta u-\frac{d}{dt}\delta tz \tag{8}$$

are just the ones expected for t (2).

If one continues this way, treating t as time and u and z as functions of t, imposing a covariant condition $\omega_D=0$ would result in the elimination of z as an independent variable, $z=\frac{1}{2}\frac{du}{dt}$. (This is a manifestation of the inverse Higgs phenomenon [13].) Then one can obtain the action of conformal mechanics as [22]

$$\begin{aligned}S_{\mathrm{cf}}&=-\int(\omega_K+m^2\omega_P)\\&=\int dt\left[\frac{1}{4}e^u\left(\frac{du}{dt}\right)^2-m^2e^{-u}\right]\\&=\int dt\left[\left(\frac{dx}{dt}\right)^2-\frac{m^2}{x^2}\right], \qquad x=e^{u/2}.\end{aligned} \tag{9}$$

Note that the equation of motion that follows from the action (9)

$$-e^u\left[\frac{1}{2}\frac{d^2}{dt^2}u+\frac{1}{4}\left(\frac{du}{dt}\right)^2\right]+m^2e^{-u}=0 \tag{10}$$

can be rewritten as the constraint on the Cartan forms [22]

$$\omega_K-m^2\omega_P=0. \tag{11}$$

Thus the Schwarzian mechanics are essentially the conformal mechanics rewritten in the new coordinates.

The Cartan forms in (7) are invariant with respect to $sl(2,R)$ transformations (8), while the time variable t transforms according to (2), (8). At this point one may impose our main condition [14,15,20]

$$\omega_P=e^{-u}dt=d\tau, \tag{12}$$

where τ is a new invariant "time," which is completely inert under $sl(2,R)$ transformations. Treating now t, u, z as the functions of τ, one can express the Goldstone fields u and z in terms of $\dot{t}$, $\ddot{t}$

$$u=\log\dot{t}, \quad \omega_D=du-2ze^u d\tau=0\Rightarrow z=\frac{1}{2}e^{-u}\dot{u}=\frac{\ddot{t}}{2\dot{t}^2}. \tag{13}$$

Putting these relations into the remaining form ω_K, one immediately obtains that it is proportional to the Schwarzian $\{t,\tau\}$:

$$\omega_K=\frac{1}{2}d\tau\left[\ddot{u}-\frac{1}{2}\dot{u}^2\right]=\frac{1}{2}\left[\frac{\dddot{t}}{\dot{t}}-\frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2\right]d\tau=\frac{1}{2}d\tau\{t,\tau\}. \tag{14}$$

The Schwarzian action is, obviously, $S_{\mathrm{schw}}=-\int\omega_K$.

B. $\mathcal{N}=2$ case

This idea can be straightforwardly generalized to the supersymmetric case. To obtain supersymmetric Schwarzians, we should consider the proper superalgebra, which differs from (5) by the presence of supercharges Q^i, superconformal charges S^i, and, possibly, internal symmetry generators J^{ij}. Then one should introduce the superconformally inert superspace coordinates τ and θ using the relations

$$\omega_P = \triangle\tau, \qquad (\omega_Q)^i = d\theta^i, \tag{15}$$

where the forms $\triangle\tau$ and $d\theta^i$ are invariant with respect to standard superspace transformations $\delta\tau \sim \epsilon\theta$, $\delta\theta \sim \epsilon$. After imposing the condition $\omega_D = 0$ also, one realizes that the remaining forms are composed of supersymmetric Schwarzians and their derivatives. As one of the simplest examples, let us consider $\mathcal{N}=2$ Schwarzian mechanics [15,20].

In the case of $\mathcal{N}=2$ supersymmetry we start from $\mathcal{N}=2$ superconformal algebra $su(1,1|1)$ with the following (anti)commutation relations

$$\begin{aligned}
&\mathrm{i}[D,P]=P, \qquad \mathrm{i}[D,K]=-K, \qquad \mathrm{i}[K,P]=2D,\\
&\{Q,\bar{Q}\}=2P, \qquad \{S,\bar{S}\}=2K, \qquad \{Q,\bar{S}\}=-2D+2J, \qquad \{\bar{Q},S\}=-2D-2J,\\
&\mathrm{i}[J,Q]=\frac{1}{2}Q, \qquad \mathrm{i}[J,\bar{Q}]=-\frac{1}{2}\bar{Q}, \qquad \mathrm{i}[J,S]=\frac{1}{2}S, \qquad \mathrm{i}[J,\bar{S}]=-\frac{1}{2}\bar{S},\\
&\mathrm{i}[D,Q]=\frac{1}{2}Q, \qquad \mathrm{i}[D,\bar{Q}]=\frac{1}{2}\bar{Q}, \qquad \mathrm{i}[D,S]=-\frac{1}{2}S, \qquad \mathrm{i}[D,\bar{S}]=-\frac{1}{2}\bar{S},\\
&\mathrm{i}[K,Q]=-S, \qquad \mathrm{i}[K,\bar{Q}]=-\bar{S}, \qquad \mathrm{i}[P,S]=Q, \qquad \mathrm{i}[P,\bar{S}]=\bar{Q}.
\end{aligned} \tag{16}$$

We parametrize the $SU(1,1|1)$ group element in the following way:

$$g = e^{\mathrm{i}tP}e^{\xi Q+\bar{\xi}\bar{Q}}e^{\psi S+\bar{\psi}\bar{S}}e^{\mathrm{i}zK}e^{\mathrm{i}uD}e^{\phi J}. \tag{17}$$

The Cartan forms

$$g^{-1}dg = \mathrm{i}\omega_P P + \omega_Q Q + \bar{\omega}_Q\bar{Q} + \mathrm{i}\omega_D D + \omega_J J + \omega_S S + \bar{\omega}_S\bar{S} + \mathrm{i}\omega_K K \tag{18}$$

explicitly read

$$\begin{aligned}
&\omega_P \equiv e^{-u}\triangle t = e^{-u}(dt + \mathrm{i}(d\bar{\xi}\xi + d\xi\bar{\xi})),\\
&\omega_Q = e^{-\frac{u}{2}+\mathrm{i}\frac{\phi}{2}}(d\xi + \psi\triangle t), \bar{\omega}_Q = e^{-\frac{u}{2}-\mathrm{i}\frac{\phi}{2}}(d\bar{\xi} + \bar{\psi}\triangle t),\\
&\omega_D = du - 2z\triangle t - 2\mathrm{i}(d\xi\bar{\psi} + d\bar{\xi}\psi), \omega_J = d\phi - 2\psi\bar{\psi}\triangle t + 2(d\bar{\xi}\psi - d\xi\bar{\psi}),\\
&\omega_S = e^{\frac{u}{2}+\mathrm{i}\frac{\phi}{2}}(d\psi - \mathrm{i}\psi\bar{\psi}d\xi + z(d\xi + \psi\triangle t)),\\
&\bar{\omega}_S = e^{\frac{u}{2}-\mathrm{i}\frac{\phi}{2}}(d\bar{\psi} + \mathrm{i}\psi\bar{\psi}d\bar{\xi} + z(d\bar{\xi} + \bar{\psi}\triangle t)),\\
&\omega_K = e^{u}(dz + z^2\triangle t - \mathrm{i}(\psi d\bar{\psi} + \bar{\psi}d\psi) + 2\mathrm{i}z(d\xi\bar{\psi} + d\bar{\xi}\psi)).
\end{aligned} \tag{19}$$

Now we impose the following conditions on the forms ω_P, ω_Q, ω_D (19):

$$\omega_P = \triangle\tau, \quad \omega_Q = d\theta, \quad \bar{\omega}_Q = d\bar{\theta}, \quad \omega_D = 0. \tag{20}$$

Here, $\triangle\tau = d\tau + \mathrm{i}(d\theta\bar{\theta} + d\bar{\theta}\theta)$. The forms $\triangle\tau$, $d\theta$, $d\bar{\theta}$ are invariant with respect to $\mathcal{N}=2$ supersymmetry transformations

$$\delta\tau = \mathrm{i}(\epsilon\bar{\theta} + \bar{\epsilon}\theta), \qquad \delta\theta = \epsilon, \qquad \delta\bar{\theta} = \bar{\epsilon}. \tag{21}$$

Covariant derivatives with respect to τ, θ, $\bar{\theta}$ are

$$D = \frac{\partial}{\partial\theta} - i\bar{\theta}\frac{\partial}{\partial\tau}, \quad \bar{D} = \frac{\partial}{\partial\bar{\theta}} - i\theta\frac{\partial}{\partial\tau}, \quad \{D,\bar{D}\} = -2i\partial_\tau. \tag{22}$$

The constraints on the Cartan forms (20), expanded in projections $\triangle\tau$, $d\theta$, $d\bar{\theta}$ with the help of (22), are

$$\begin{aligned}
&\dot{t} + \mathrm{i}(\dot{\bar{\xi}}\xi + \dot{\xi}\bar{\xi}) = e^{u}, \quad z = \frac{1}{2}e^{-u}\dot{u}\\
&\dot{\xi} + e^{u}\psi = 0, \qquad D\xi = e^{\frac{1}{2}(u-\mathrm{i}\phi)}, \qquad \bar{D}\xi = 0,\\
&\dot{\bar{\xi}} + e^{u}\bar{\psi} = 0, \qquad \bar{D}\bar{\xi} = e^{\frac{1}{2}(u+\mathrm{i}\phi)}, \qquad D\bar{\xi} = 0.
\end{aligned} \tag{23}$$

Using (23), we express all Cartan forms in the terms of $\mathcal{N}=2$ Schwarzian

$$\mathcal{S}_{\mathcal{N}=2}=\frac{D\dot{\xi}}{D\xi}-\frac{\bar{D}\dot{\bar{\xi}}}{\bar{D}\bar{\xi}}-2\mathrm{i}\frac{\dot{\xi}\dot{\bar{\xi}}}{D\xi\bar{D}\bar{\xi}} \tag{24}$$

as

$$\begin{aligned}
\omega_J &= \mathrm{i}\mathcal{S}_{\mathcal{N}=2}\triangle\tau,\\
\omega_S &= -\frac{1}{2}\mathcal{S}_{\mathcal{N}=2}d\theta-\frac{\mathrm{i}}{2}\bar{D}\mathcal{S}_{\mathcal{N}=2}\triangle\tau,\\
\bar{\omega}_S &= \frac{1}{2}\mathcal{S}_{\mathcal{N}=2}d\bar{\theta}+\frac{\mathrm{i}}{2}D\mathcal{S}_{\mathcal{N}=2}\triangle\tau,\\
\omega_K &= -\frac{1}{2}d\theta D\mathcal{S}_{\mathcal{N}=2}+\frac{1}{2}d\bar{\theta}\,\bar{D}\,\mathcal{S}_{\mathcal{N}=2}\\
&\quad+\frac{1}{4}\triangle\tau(\mathrm{i}[D,\bar{D}]\mathcal{S}_{\mathcal{N}=2}-\mathcal{S}^2_{\mathcal{N}=2}).
\end{aligned} \tag{25}$$

Note that the same conclusion about the structure of the forms can be achieved by the analysis of Maurer-Cartan equations the forms of (18) satisfy. We will use such equations in Sec. V to study a system with $\mathcal{N}$ supersymmetries.

The constructed $\mathcal{N}=2$ Schwarzian (24) is invariant with respect to superconformal transformations that explicitly read

$$g'=e^{\epsilon Q+\bar{\epsilon}\bar{Q}}e^{\varepsilon S+\bar{\varepsilon}\bar{S}}g\Rightarrow\begin{cases}\delta t=\mathrm{i}(\bar{\epsilon}\xi+\epsilon\bar{\xi})-\mathrm{i}t(\bar{\varepsilon}\xi+\varepsilon\bar{\xi})\\ \delta\xi=\epsilon-\varepsilon t+\mathrm{i}\varepsilon\xi\bar{\xi}\\ \delta\bar{\xi}=\bar{\epsilon}-\bar{\varepsilon}t-\mathrm{i}\bar{\varepsilon}\xi\bar{\xi}\end{cases}. \tag{26}$$

Thus one can construct the supersymmetric Schwarzian action as

$$\begin{aligned}
S_{N2\mathrm{schw}} &= -\frac{\mathrm{i}}{2}\int d\tau\,d\theta\,d\bar{\theta}\,\mathcal{S}_{\mathcal{N}=2}\\
&= -\frac{1}{2}\int\omega_J\wedge\omega_Q\wedge\bar{\omega}_Q\\
&= \mathrm{i}\int\omega_P\wedge\omega_S\wedge\bar{\omega}_Q\\
&= -\mathrm{i}\int\omega_P\wedge\omega_Q\wedge\bar{\omega}_S.
\end{aligned} \tag{27}$$

It is matter of a quite lengthy calculation to check that the equations of motion that follow from the action (27) can be written as

$$\frac{d}{d\tau}\mathcal{S}_{\mathcal{N}=2}=0\Rightarrow\mathcal{S}_{\mathcal{N}=2}=\mathrm{const}=-2m. \tag{28}$$

Looking at the Cartan forms (25), one may note that Eq. (28) reduces them to the forms on the subalgebra that were formed by the following generators:

$$\begin{aligned}
&R=P+m^2K-2\mathrm{i}mJ, \qquad \Gamma=Q+\mathrm{i}mS,\\
&\bar{\Gamma}=\bar{Q}-\mathrm{i}m\bar{S}, \qquad \{\Gamma,\bar{\Gamma}\}=2R.
\end{aligned} \tag{29}$$

The reduction of the Cartan forms on the algebra $su(1,1|1)$ to the forms on the subalgebra (29) is the key ingredient of the covariant reduction used in [23] to construct $\mathcal{N}=2$ superconformal mechanics. Thus, in the $\mathcal{N}=2$ supersymmetric case the Schwarzian mechanics is nothing but the superconformal mechanics written in the superfields $\{t,\xi,\bar{\xi}\}$ depending on the coordinates of the inert superspace $\{\tau,\theta,\bar{\theta}\}$. Unfortunately, this relation does not work beyond the $\mathcal{N}=2$ case with $m\neq0$.

III. FLAT SPACE ANALOG OF THE SCHWARZIAN

As the first example of the generalized Schwarzian in this section we will consider the use nonlinear realizations to construct the so-called flat space analog of the Schwarzian. The latter was discovered in [9] by study of coadjoint orbits of product of Virasoro group with functions on the circle. Later it was found [10] to play a role in the holographic description of two-dimensional gravity in flat space, just like the original Schwarzian, which is related to the Sachdev-Ye-Kitaev model that provides a holographic dual to the Jackiw-Teitelboim gravity [24,25] in anti–de Sitter space.

The Schwarzian appearing in [9,10] is connected to the Maxwell algebra. The latter contains the Hermitian generators of translation P, the analog of the dilatation-central-charge generator Z, the analog of the conformal boost K, and the generator of $U(1)$ rotations obeying the following relations:

$$\mathrm{i}[J,P]=P, \qquad \mathrm{i}[J,K]=-K, \qquad \mathrm{i}[K,P]=2Z. \tag{30}$$

If we parametrize the Maxwell group element g as

$$g=e^{\mathrm{i}t(P+qJ+m^2K)}e^{\mathrm{i}zK}e^{\mathrm{i}uZ}e^{\mathrm{i}\phi J}, \tag{31}$$

then the Cartan forms

$$g^{-1}dg=\mathrm{i}\omega_PP+\mathrm{i}\omega_ZZ+\mathrm{i}\omega_KK+\mathrm{i}\omega_JJ \tag{32}$$

will read

$$\begin{aligned}
&\omega_P=e^{-\phi}dt, \qquad \omega_Z=du-2zdt,\\
&\omega_K=e^{\phi}(dz-qzdt+m^2dt), \quad \omega_J=d\phi.
\end{aligned} \tag{33}$$

The constraints

$$\omega_P = d\tau, \qquad \omega_Z = 0 \tag{34}$$

result in the following relations:

$$\dot{t} = e^{\phi}, \qquad z = \frac{\dot{u}}{2\dot{t}}. \tag{35}$$

Finally,

$$\omega_K = d\tau\, \dot{t}\left[\frac{1}{2}\left(\frac{\ddot{u}}{\dot{t}} - \frac{\dot{u}\,\ddot{t}}{\dot{t}^2}\right) + m^2\dot{t} - \frac{1}{2}q\dot{u}\right] \equiv d\tau\mathcal{S}_{\text{flat}}. \tag{36}$$

This is exactly the flat space analog of the Schwarzian constructed in [9,10]

$$\mathcal{S} \sim \int \omega_K.$$

It is possible that other related Schwarzian actions, such as one combining standard and flat Schwarzians [26] or the supersymmetric version of the flat action [27] can also be obtained in algebraic way.

IV. SCHWARZIAN WITH $su(1,2)$ SYMMETRY

In this section we will consider the bosonic version of the $\mathcal{N}=2$ superconformal mechanics system with $su(1,2)$ symmetry.

The $su(1,2)$ algebra includes the following generators:

(i) The generators P, D, K, forming $sl(2,\mathbb{R})$ subalgebra.

(ii) The generators $Q, \bar{Q}$, and $S, \bar{S}$: the bosonic analogs of the supersymmetric and conformal supersymmetry generators.

(iii) $U(1)$ generator U.

The generators P, D, K, and U are Hermitian, while the Q and S generators obey the conjugation rules $(Q)^\dagger = \bar{Q}, (S)^\dagger = \bar{S}$. The nonzero commutators read

$$\begin{gathered}
\mathrm{i}[P,K] = -2D, \qquad \mathrm{i}[P,D] = -P, \qquad \mathrm{i}[K,D] = K, \\
\mathrm{i}[P,S] = -Q, \qquad \mathrm{i}[P,\bar{S}] = -\bar{Q}, \qquad \mathrm{i}[K,Q] = S, \qquad \mathrm{i}[K,\bar{Q}] = \bar{S}, \\
\mathrm{i}[D,Q] = \frac{1}{2}Q, \qquad \mathrm{i}[D,\bar{Q}] = \frac{1}{2}\bar{Q}, \qquad \mathrm{i}[D,S] = -\frac{1}{2}S, \qquad \mathrm{i}[D,\bar{S}] = -\frac{1}{2}\bar{S}, \\
[U,Q] = Q, \qquad [U,\bar{Q}] = -\bar{Q}, \qquad [U,S] = S, \qquad [U,\bar{S}] = -\bar{S}, \\
[Q,\bar{Q}] = -\gamma P, \qquad \mathrm{i}[Q,\bar{S}] = -\frac{3}{2}\gamma U - \mathrm{i}\gamma D, \qquad [S,\bar{S}] = -\gamma K, \qquad \mathrm{i}[S,\bar{Q}] = \frac{3}{2}\gamma U - \mathrm{i}\gamma D.
\end{gathered} \tag{37}$$

We parametrize the group element in a standard way as

$$g = e^{\mathrm{i}tP}e^{\mathrm{i}(\phi Q + \bar{\phi}\bar{Q})}e^{\mathrm{i}(vS + \bar{v}\bar{S})}e^{\mathrm{i}zK}e^{\mathrm{i}uD}e^{\mathrm{i}\varphi U}. \tag{38}$$

The Cartan forms read

$$\begin{aligned}
\omega_P &= e^{-u}\left(dt + \frac{\mathrm{i}}{2}\gamma(\phi d\bar{\phi} - \bar{\phi}d\phi)\right) \equiv e^{-u}\triangle t, \\
\omega_D &= du - \mathrm{i}\gamma(\bar{v}d\phi - v d\bar{\phi}) - 2z\triangle t, \\
\omega_K &= e^{u}\left[dz + \left(z^2 + \frac{\gamma^2}{4}v^2\bar{v}^2\right)\triangle t - \mathrm{i}\gamma z(v d\bar{\phi} - \bar{v}d\phi) + \frac{\mathrm{i}}{2}\gamma(v d\bar{v} - \bar{v}dv) - \frac{\gamma^2}{2}v\bar{v}(v d\bar{\phi} + \bar{v}d\phi)\right] \\
\omega_Q &= e^{-\frac{u}{2}-\mathrm{i}\varphi}[d\phi - v\triangle t], \qquad \bar{\omega}_Q = e^{-\frac{u}{2}-\mathrm{i}\varphi}[d\bar{\phi} - \bar{v}\triangle t], \\
\omega_S &= e^{\frac{u}{2}-\mathrm{i}\varphi}\left[dv - \left(z + \frac{\mathrm{i}}{2}\gamma v\bar{v}\right)(d\phi - v\triangle t) - \mathrm{i}\gamma v^2 d\bar{\phi}\right], \\
\bar{\omega}_S &= e^{\frac{u}{2}+\mathrm{i}\varphi}\left[d\bar{v} - \left(z - \frac{\mathrm{i}}{2}\gamma v\bar{v}\right)(d\bar{\phi} - \bar{v}\triangle t) + \mathrm{i}\gamma\bar{v}^2 d\phi\right], \\
\omega_U &= d\varphi - \frac{3}{2}\gamma(v d\bar{\phi} + \bar{v}d\phi - v\bar{v}\triangle t).
\end{aligned} \tag{39}$$

The constraints we are going to impose are of three different types:

(i) The constraints that introduce the inert "time" τ: $\omega_P = d\tau$.

(ii) The constraints realizing the inverse Higgs phenomenon [13]: $\omega_D = \omega_Q = \bar{\omega}_Q = 0$.

(iii) The dynamical constraints that produce the equations of motion: $\omega_K = \omega_S = \bar{\omega}_S = 0$.

The results of two first constraints are

$$\dot t + \frac{\mathrm{i}}{2}\gamma(\phi\dot{\bar\phi} - \bar\phi\dot\phi) = e^{u}, \qquad v = e^{-u}\dot\phi,$$
$$\bar v = e^{-u}\dot{\bar\phi}, \qquad z = \frac{1}{2}e^{-u}\dot u. \tag{40}$$

The dynamical constraints give the following equations of motion:

$$\ddot\phi = \dot u\,\dot\phi + \mathrm{i}e^{-u}\gamma\dot\phi^2\dot{\bar\phi}, \qquad \ddot{\bar\phi} = \dot u\,\dot{\bar\phi} - \mathrm{i}e^{-u}\gamma\dot\phi\dot{\bar\phi}^2,$$
$$\ddot u = \frac{1}{2}(\dot u^2 - e^{-2u}\gamma^2\dot\phi^2\dot{\bar\phi}^2). \tag{41}$$

To get the Schwarzian-like system one has to pass from the variable u to the "old" time t (40). The result

$$\frac{\dddot t + \frac{\mathrm{i}\gamma}{2}(\dot\phi\ddot{\bar\phi} - \dot{\bar\phi}\ddot\phi + \phi\dddot{\bar\phi} - \bar\phi\dddot\phi)}{\dot t + \frac{\mathrm{i}\gamma}{2}(\phi\dot{\bar\phi} - \bar\phi\dot\phi)} - \frac{3}{2}\frac{(\ddot t + \frac{\mathrm{i}\gamma}{2}(\phi\ddot{\bar\phi} - \bar\phi\ddot\phi))^2}{(\dot t + \frac{\mathrm{i}\gamma}{2}(\phi\dot{\bar\phi} - \bar\phi\dot\phi))^2}$$
$$= -\frac{1}{2}\frac{\gamma^2\dot\phi^2\dot{\bar\phi}^2}{(\dot t + \frac{\mathrm{i}\gamma}{2}(\phi\dot{\bar\phi} - \bar\phi\dot\phi))^2} \tag{42}$$

is somewhat complicated, but it evidently generalizes the equation of motion of the standard Schwarzian mechanics

$$\frac{\dddot t}{\dot t} - \frac{3}{2}\left(\frac{\ddot t}{\dot t}\right)^2 = 2m^2. \tag{43}$$

V. SCHWARZIANS WITH HIGHER ($\mathcal{N} > 4$) SUPERSYMMETRY

It has been well known for a long time that the supersymmetric Schwarzians appear in the transformations of the current superfield $J^{(N)}(Z)$ under $\mathcal{N}$-extended superconformal algebra [3]. In fact, such an appearance of the supersymmetric Schwarzians can be considered as their definition. However, $\mathcal{N}$ extended superconformal theories have the natural upper bound $\mathcal{N} = 4$, since for $\mathcal{N} > 4$ the current superfield $J^{(N)}(Z)$ has components with negative conformal dimension. In the series of the previous papers [20,21] and related but using a slightly different approach [15,16], all such $\mathcal{N} = 4$ super-Schwarzians were reproduced. As expected, they coincide with the Schwarzians from the seminal paper by K. Schoutens [3].

However, the approach developed in [15,16,20,21], and which we advocated here, does not possess the upper bound on the number of supersymmetries. So, it is natural to try to construct some analogs of the Schwarzian with higher $\mathcal{N} > 4$ supersymmetry. Alas, our first attempts in this direction were failures. The analysis of the superalgebras $osp(4^\star|4)$, and $su(1,1|\mathcal{N}/2 > 2)$ leads to the conclusion that, as a result of standard constraints imposed on the differential forms

$$\omega_P = \triangle\tau, \quad (\omega_Q)^\alpha = d\theta^\alpha, \quad (\bar\omega_Q)_\alpha = d\bar\theta_\alpha, \quad \omega_D = 0, \tag{44}$$

all others, in contrast to the already studied cases with $\mathcal{N} \le 4$, are put to zero leaving no room for the Schwarzians in the standard sense. Instead, in such an approach we obtain a set of higher-order differential equations on the fields involved, which can be treated as describing some dynamical (and quite possibly integrable) system. Though discussion of these dynamical systems is beyond the scope of this paper, let us note that there exists at least one possibility when the standard constraints are not strong enough to put the Schwarzian to zero for $\mathcal{N} \ge 4$. It is given by the series of $osp(\mathcal{N}|2)$ superalgebras, which we discuss in detail.

A. Superalgebra $osp(\mathcal{N}|2)$

The bosonic part of the superalgebra $osp(\mathcal{N}|2)$ contains among the subgroups $sl(2)\times so(\mathcal{N})$ with the generators (P,D,K) and $J_{ij} = -J_{ji}, i,j = 1,2,\ldots,\mathcal{N}$, respectively [28]. The fermionic part of this algebra includes $2\cdot\mathcal{N}$ fermionic generators Q_i, S_i forming the vectors with respect to $so(\mathcal{N})$ algebra and doublet with respect to $sl(2)$ subalgebra. The commutation relations have a rather compact form:

$$[D,P] = -\mathrm{i}P, \qquad [D,K] = \mathrm{i}K, \qquad [P,K] = 2\mathrm{i}D,$$
$$[D,Q_i] = -\frac{\mathrm{i}}{2}Q_i, \qquad [D,S_i] = \frac{\mathrm{i}}{2}S_i, \qquad [K,Q_i] = \mathrm{i}S_i, \qquad [P,S_i] = -\mathrm{i}Q_i,$$
$$\{Q_i,Q_j\} = 2\delta_{ij}P, \qquad \{S_i,S_j\} = 2\delta_{ij}K, \qquad \{Q_i,S_j\} = -2\delta_{ij}D + J_{ij},$$
$$[J_{ij},J_{kl}] = \mathrm{i}(\delta_{ik}J_{jl} - \delta_{jk}J_{il} - \delta_{il}J_{jk} + \delta_{jl}J_{ik}),$$
$$[J_{ij},Q_k] = \mathrm{i}(\delta_{ik}Q_j - \delta_{jk}Q_i), \qquad [J_{ij},S_k] = \mathrm{i}(\delta_{ik}S_j - \delta_{jk}S_i). \tag{45}$$

The group element can be defined as

$$g = e^{\mathrm{i}tP} e^{\xi_i Q_i} e^{\psi_i S_i} e^{\mathrm{i}zK} e^{\mathrm{i}uD} e^{\lambda_{ij} J_{ij}}. \tag{46}$$

Here, the superfields t, ξ_i, u, z, ψ_i, λ_{ij} depend on the coordinates of $\mathcal{N}$-extended superspace τ, θ_i. Defining the Cartan forms as

$$\Omega = g^{-1}dg = \mathrm{i}\omega_P P + \mathrm{i}\omega_K K + \mathrm{i}\omega_D D + (\omega_Q)_i Q_i + (\omega_S)_i S_i + \mathrm{i}(\omega_J)_{ij} J_{ij}, \tag{47}$$

one may impose the standard constraints of our approach [20,21]:

$$\omega_P = \triangle\tau \equiv d\tau + \mathrm{i}d\theta_i\theta_i, \quad (\omega_Q)_i = d\theta_i, \quad \omega_D = 0. \tag{48}$$

Here, τ and θ_i are the coordinates of the "inert" superspace. The covariant (with respect to the flat $\mathcal{N}$-extended supersymmetry, generated by Q_i and P) differentials $\triangle\tau$ and $d\theta_i$ can be used to define the covariant derivatives as

$$d\mathcal{A} = \triangle\tau D_\tau \mathcal{A} + d\theta_i D_i \mathcal{A}, \tag{49}$$

with

$$D_\tau = \partial_\tau, \quad D_i = \frac{\partial}{\partial\theta_i} - \mathrm{i}\theta_i\partial_\tau \Rightarrow \{D_i, D_j\} = -2\mathrm{i}\delta_{ij}\partial_\tau. \tag{50}$$

B. Maurer-Cartan equations

One may explicitly calculate the forms (47) and analyze the consequences of the constraints (48). However, in practice this way is a rather cumbersome and involved. The simplification comes from the evident statement that our constraints include the Cartan forms themselves, and, therefore, it makes sense to use the Maurer-Cartan equations to analyze their consequences.

If the Cartan form Ω is defined as in (47), then by construction it obeys the Maurer-Cartan equation[1]

$$d_2\Omega_1 - d_1\Omega_2 = [\Omega_1, \Omega_2], \quad \Omega_1 = \Omega(d_1), \quad \Omega_2 = \Omega(d_2). \tag{51}$$

This equation can be expanded into following set of equations:

$$\begin{aligned}
\mathrm{i}(d_2\omega_{1P} - d_1\omega_{2P}) &= -\mathrm{i}(\omega_{1P}\omega_{2D} - \omega_{1D}\omega_{2P}) - 2(\omega_{1Q})_i(\omega_{2Q})_i,\\
\mathrm{i}(d_2\omega_{1K} - d_1\omega_{2K}) &= \mathrm{i}(\omega_{1K}\omega_{2D} - \omega_{1D}\omega_{2K}) - 2(\omega_{1S})_i(\omega_{2S})_i,\\
\mathrm{i}(d_2\omega_{1D} - d_1\omega_{2D}) &= -2\mathrm{i}(\omega_{1P}\omega_{2K} - \omega_{1K}\omega_{2P}) + 2(\omega_{1Q})_i(\omega_{2S})_i - 2(\omega_{2Q})_i(\omega_{1S})_i,\\
\mathrm{i}(d_2(\omega_{1J})_{ij} - d_1(\omega_{2J})_{ij}) &= 2\mathrm{i}(\omega_{1J})_{ik}(\omega_{2J})_{kj} - 2\mathrm{i}(\omega_{2J})_{ik}(\omega_{1J})_{kj} - (\omega_{1Q})_{[i}(\omega_{2S})_{j]} + (\omega_{2Q})_{[i}(\omega_{1S})_{j]},\\
d_2(\omega_{1Q})_i - d_1(\omega_{2Q})_i &= \omega_{1P}(\omega_{2S})_i - \omega_{2P}(\omega_{1S})_i + \frac{1}{2}(\omega_{1D}(\omega_{2Q})_i - \omega_{2D}(\omega_{1Q})_i) + 2(\omega_{1J})_{ik}(\omega_{2Q})_k - 2(\omega_{2J})_{ik}(\omega_{1Q})_k,\\
d_2(\omega_{1S})_i - d_1(\omega_{2S})_i &= -\omega_{1K}(\omega_{2Q})_i + \omega_{2K}(\omega_{1Q})_i - \frac{1}{2}(\omega_{1D}(\omega_{2S})_i - \omega_{2D}(\omega_{1S})_i) + 2(\omega_{1J})_{ik}(\omega_{2S})_k - 2(\omega_{2J})_{ik}(\omega_{1S})_k.
\end{aligned} \tag{52}$$

To analyze the consequences of these constraints let us represent other forms in most general way as

$$(\omega_S)_i = \triangle\tau\Psi_i + d\theta_j A_{ij}, \qquad (\omega_J)_{ij} = \triangle\tau X_{ij} + \mathrm{i}d\theta_k\Sigma_{kij}, \qquad \omega_K = \triangle\tau C + \mathrm{i}d\theta_i\Xi_i. \tag{53}$$

Substituting constraints (48) and the anzatz for other forms (53) into Eq. (52), one finds that the $d\omega_P$ equation is satisfied identically, and the $d\omega_Q$ equation implies that

$$A_{ij} + 2X_{ij} = 0, \qquad \Sigma_{kil} + \Sigma_{lik} = 0. \tag{54}$$

As by definition $\Sigma_{kij} = -\Sigma_{kji}$, the second equation implies that Σ_{kij} is completely antisymmetric. The second $d\omega_J$ equation reads

$$\begin{aligned}
\mathrm{i}D_k X_{ij} + \dot{\Sigma}_{kij} &= -2X_{\mathrm{in}}\Sigma_{knj} + 2X_{jn}\Sigma_{kni} + \frac{1}{2}(\delta_{ik}\Psi_j - \delta_{jk}\Psi_i),\\
-2\delta_{kl}X_{ij} - \delta_{ik}X_{jl} + \delta_{jk}X_{il} - \delta_{il}X_{jk} + \delta_{jl}X_{ik} &= D_k\Sigma_{lij} + D_l\Sigma_{kij} - 2\mathrm{i}\Sigma_{kin}\Sigma_{ljn} - 2\mathrm{i}\Sigma_{lin}\Sigma_{kjn}.
\end{aligned} \tag{55}$$

[1]Here, d_1 and d_2 are mutually commuting differentials, $d\tau$ is the commuting bosonic object, while $d\theta$ is the anticommuting fermionic one.

The $d\omega_D$ equation implies that

$$2\Xi_k - 2\Psi_k = 0, \qquad A_{ij} + A_{ji} = 0. \tag{56}$$

The second equation is satisfied due to $A_{ij} = -2X_{ij} = -2X_{[ij]}$. The $d\omega_S$ equation reads

$$\begin{aligned} &D_k\Psi_i - \dot{A}_{ik} = -\delta_{ik}C + 2X_{ij}A_{jk} - 2\mathrm{i}\Sigma_{kij}\Psi_j, \\ &D_lA_{ik} + D_kA_{il} + 2\mathrm{i}\delta_{kl}\Psi_i \\ &\quad = \mathrm{i}(\delta_{il}\Xi_k + \delta_{ik}\Xi_l) - 2\mathrm{i}\Sigma_{kij}A_{jl} - 2\mathrm{i}\Sigma_{lij}A_{jk}. \end{aligned} \tag{57}$$

Finally, the $d\omega_K$ equation reads

$$\begin{aligned} \mathrm{i}D_kC + \dot{\Xi}_k = 2\Psi_iA_{ik}, \\ -2\delta_{kl}C - D_k\Xi_l - D_l\Xi_k = -2A_{km}A_{lm}. \end{aligned} \tag{58}$$

Taking into account simple equations (54), (56), one may note that the second equation (57) is a direct consequence of the first in (55), and the second equation (58) follows from the first ones in (57) and (54). Therefore, the really independent variables are $\Sigma_{ijk} = \Sigma_{[ijk]}$, $X_{ij} = X_{[ij]}$, Ψ_i, and C. They satisfy the following set of equations:

$$\begin{aligned} &-2\delta_{kl}X_{ij} - \delta_{ik}X_{jl} + \delta_{jk}X_{il} - \delta_{il}X_{jk} + \delta_{jl}X_{ik} \\ &\quad = D_k\Sigma_{lij} + D_l\Sigma_{kij} - 2\mathrm{i}\,\Sigma_{kin}\Sigma_{ljn} - 2\mathrm{i}\,\Sigma_{lin}\Sigma_{kjn}, \end{aligned} \tag{59}$$

$$\mathrm{i}D_kX_{ij} + \dot{\Sigma}_{kij} = -2X_{\mathrm{in}}\Sigma_{knj} + 2X_{jn}\Sigma_{kni} + \frac{1}{2}(\delta_{ik}\Psi_j - \delta_{jk}\Psi_i), \tag{60}$$

$$D_k\Psi_i + 2\dot{X}_{ik} = -\delta_{ik}C - 4X_{ij}X_{jk} - 2\mathrm{i}\,\Sigma_{kij}\Psi_j, \tag{61}$$

$$\mathrm{i}\,D_kC + \dot{\Psi}_k = -4\Psi_iX_{ik}. \tag{62}$$

The first of these equations (59) defines X_{ij} in terms of Σ_{ijk} and its derivative. Using this solution and Eq. (59) again, one can find D_kX_{ij} and substitute it to the next Eq. (60). This reduces (60) to terms with δ_{ij} symbols, which allow us to find Ψ_k. Continuing down this road, one can simplify (61) to find C and check that the last one (62) becomes just an identity. Therefore, all the superfields X_{ij}, Ψ_i, C can be expressed in terms of Σ_{ijk} satisfying (59). It is remarkable that this can be done for an arbitrary number of supersymmetries $\mathcal{N}$. The solution explicitly reads

$$\begin{aligned} X_{ij} &= \frac{1}{2-\mathcal{N}}(D_m\Sigma_{mij} - 2\mathrm{i}\Sigma_{imn}\Sigma_{jmn}), \\ \Psi_i &= -\frac{2\mathrm{i}}{\mathcal{N}-1}(D_lX_{il} + 2\mathrm{i}X_{mn}\Sigma_{imn}), \\ C &= -\frac{1}{\mathcal{N}}(D_j\Psi_j - 4X_{mn}X_{mn}). \end{aligned} \tag{63}$$

Thus, all the Cartan forms can be expressed in terms of a unique object: superfield Σ_{ijk}. This superfield Σ_{ijk}, being fully antisymmetric over permutations of the indices, appeared as the $d\theta$ projection of the form $(\omega_J)_{ij}$. Due to these properties, one can call this superfield Σ_{ijk} as the supersymmetric $\mathcal{N}$-extended Schwarzian. It satisfies the nonlinear constraint given by Eq. (59), where X_{ij} is expressed in terms of Σ_{ijk} by (63).

C. Explicit form of the supersymmetric $\mathcal{N}$-extended Schwarzian

From the previous subsection, we see that the supersymmetric $\mathcal{N}$-extended Schwarzian Σ_{ijk} we are looking for appears as a $d\theta$ projection of the form $(\omega_J)_{ij}$. Thus, the final task is to express Σ_{ijk} in terms of the parameters of the group element (46), depending, in virtue of our constraints (48), on the coordinates of the flat inert superspace τ, θ_i.

The Cartan forms $\Omega = g^{-1}dg$, explicitly calculated for the group element (46), read

$$\begin{aligned} \omega_P &= e^{-u}\triangle t = e^{-u}(dt + \mathrm{i}d\xi_j\xi_j), \\ (\omega_Q)_i &= e^{-u/2}(d\xi_j + \triangle t\psi_j)M_{ji}, \\ \omega_D &= du - 2\mathrm{i}d\xi_k\psi_k - 2z\triangle t, \\ (\omega_S)_i &= e^{u/2}(d\psi_j + \mathrm{i}d\xi_k\psi_k\psi_j + z(d\xi_j + \triangle t\psi_j))M_{ji}, \\ \omega_K &= e^u(dz + z^2\triangle t + \mathrm{i}d\psi_j\psi_j + 2\mathrm{i}zd\xi_j\psi_j), \\ (\omega_J)_{kl} &= \frac{1}{2}(M^{-1})_{km}dM_{ml} + \frac{\mathrm{i}}{2}(M^{-1})_{km}(M^{-1})_{nl}e^{-u}(d\xi_m\psi_n \\ &\quad - d\xi_n\psi_m + \triangle t\psi_m\psi_n). \end{aligned} \tag{64}$$

Here, the $so(\mathcal{N})$ matrix M_{ij} is defined as

$$\begin{aligned} M_{ij} &= (e^{2\lambda})_{ij} \\ &= \delta_{ij} + 2\lambda_{ij} + \frac{4\lambda_{ik}\lambda_{kj}}{2!} + \frac{8\lambda_{ik}\lambda_{kl}\lambda_{lj}}{3!} + \ldots, (M^{-1})_{ij} = M_{ji}. \end{aligned} \tag{65}$$

The constraints (48) imply

$$\begin{aligned} \dot{t} + \mathrm{i}\dot{\xi}_k\xi_k = e^u, &\qquad D_it + \mathrm{i}D_i\xi_k\xi_k = 0, \\ D_m\xi_k = e^{u/2}(M^{-1})_{mk}, &\qquad \psi_i = -e^{-u}\dot{\xi}_i, \\ z = \frac{1}{2}e^{-u}\dot{u}, &\qquad D_iu = 2\mathrm{i}D_i\xi_j\psi_j. \end{aligned} \tag{66}$$

Some of these relations define some of the Goldstone fields in terms of derivatives of others, and some are not independent. For example, acting by D_j on the second relation and symmetrizing with respect to i, j one can obtain

$$\begin{aligned} &D_i(D_jt + \mathrm{i}D_j\xi_k\xi_k) + D_j(D_it + \mathrm{i}D_i\xi_k\xi_k) = 0 \\ &\quad \Rightarrow (\dot{t} + \mathrm{i}\dot{\xi}_k\xi_k)\delta_{ij} = D_i\xi_kD_j\xi_k. \end{aligned} \tag{67}$$

Therefore, $D_i\xi_k$ has structure implied by the third equation of (66). From this, it can also be derived that

$$\begin{aligned} D_i e^u &= \frac{1}{\mathcal{N}} D_i(D_m\xi_n D_m\xi_n) = \frac{2}{\mathcal{N}} D_i D_m\xi_n D_m\xi_n = \frac{2}{\mathcal{N}}(-2\mathrm{i}D_i\xi_n\dot{\xi}_n - D_m D_i\xi_n D_m\xi_n) \\ &= \frac{2}{\mathcal{N}}(-2\mathrm{i}D_i\xi_n\dot{\xi}_n - \mathrm{i}\mathcal{N}D_i\xi_n\dot{\xi}_n - D_i e^u) \Rightarrow D_i e^u = -2\mathrm{i}D_i\xi_m\dot{\xi}_m. \end{aligned} \tag{68}$$

Taking into account all known kinematic equations, the $d\theta$ projection of the form ω_J reads

$$(\omega_J)_{kl} = \ldots + d\theta_p\left[\frac{1}{2}(M^{-1})_{kn}D_pM_{nl} - \frac{\mathrm{i}}{2}(D_p\xi_m\dot{\xi}_n - D_p\xi_n\dot{\xi}_m)e^{-u}(M^{-1})_{km}(M^{-1})_{ln}\right] \equiv \ldots + \mathrm{i}d\theta_p\Sigma_{pkl}. \tag{69}$$

This expression can be further simplified leading to the following supersymmetric Schwarzian:

$$\Sigma_{ijk} = \frac{\mathrm{i}}{2}e^{-u}D_{[i}D_j\xi_m D_{k]}\xi_m = \frac{\mathrm{i}\mathcal{N}}{2}\frac{D_{[i}D_j\xi_m D_{k]}\xi_m}{D_p\xi_q D_p\xi_q}. \tag{70}$$

The standard bosonic Schwarzian is hidden inside the components of the third derivative of Σ_{ijk}. Roughly speaking, the bosonic part of the Schwarzian reads

$$\begin{aligned} &\frac{2\mathrm{i}D_m D_n D_p\Sigma_{mnp}}{\mathcal{N}(\mathcal{N}-1)(\mathcal{N}-2)} \\ &\quad\approx -\frac{1}{2}\left(\frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\frac{\ddot{t}^2}{\dot{t}^2}\right) + 4\frac{D_{[k}\Sigma_{lij]}D_{[k}\Sigma_{lij]}}{\mathcal{N}(\mathcal{N}-1)(\mathcal{N}-2)} \\ &\qquad + 3\frac{\mathcal{N}-2}{\mathcal{N}(\mathcal{N}-1)}\dot{M}_{mn}\dot{M}_{mn}. \end{aligned} \tag{71}$$

Note that $D_{[k}\Sigma_{lij]}$ is absent in Eq. (59) and starts from an independent component.

As we are discussing arbitrarily high supersymmetries, it is natural to ask whether the main constraint $D_i t + \mathrm{i}D_i\xi_j\xi_j = 0$ puts the system on shell. Explicit component analysis of this constraint for some values of $\mathcal{N}$ indicates, however, that it is essentially an algebraic one, defining superfields t and ξ_i in terms of some unconstrained scalar superfield.

D. Properties with respect to coordinate changes

The supersymmetric Schwarzian should possess a special property with respect to coordinate changes, known as the composition law. The coordinate changes are diffeomorphism transformations

$$\theta_i \to \theta'_i = \tilde{\theta}_i(\tau,\theta), \qquad \tau' = \tilde{\tau}(\tau,\theta), \tag{72}$$

constrained by $D_i\tilde{\tau} + \mathrm{i}D_i\tilde{\theta}_j\tilde{\theta}_j = 0$, so that the derivative D_i transforms homogeneously, $D_i = D_i\tilde{\theta}_j D'_j$. If the composition law holds for the Schwarzian, it should have the form

$$\begin{aligned} \Sigma_{ijk}[\zeta(\tau',\theta');\tau,\theta] &= \Sigma_{ijk}[\tilde{\theta}(\tau',\theta');\tau,\theta] \\ &\quad + M_{[ijk]}{}^{mnp}\Sigma_{mnp}[\zeta(\tau',\theta');\tau',\theta'] \end{aligned} \tag{73}$$

with some matrix $M_{[ijk]}{}^{mnp}$. The Schwarzian reads

$$\Sigma_{ijk} = \frac{\mathrm{i}\mathcal{N}}{2}\frac{D_{[i}D_j\zeta_m D_{k]}\zeta_m}{D_p\zeta_q D_p\zeta_q}. \tag{74}$$

As for $D_i\zeta_j$ and $D_i\tilde{\theta}_j$ the relations $D_i\zeta_k D_j\zeta_k \sim \delta_{ij}$ and $D_i\tilde{\theta}_k D_j\tilde{\theta}_k \sim \delta_{ij}$ hold, one can shortly obtain

$$D_p\zeta_q D_p\zeta_q = \frac{D_k\tilde{\theta}_l D_k\tilde{\theta}_l}{\mathcal{N}} D'_p\zeta_q D'_p\zeta_q. \tag{75}$$

Then, directly substituting $D_i\zeta_k = D_i\tilde{\theta}_j D'_j\zeta_k$ into (74), we obtain

$$\begin{aligned} &\Sigma_{ijk}[\zeta(\tau',\theta');\tau,\theta] \\ &\quad = \Sigma_{ijk}[\tilde{\theta}(\tau',\theta');\tau,\theta] \\ &\qquad + \frac{\mathcal{N}}{D_r\tilde{\theta}_s D_r\tilde{\theta}_s} D_i\tilde{\theta}_m D_j\tilde{\theta}_n D_k\tilde{\theta}_p\Sigma_{mnp}[\zeta(\tau',\theta');\tau',\theta']. \end{aligned} \tag{76}$$

Thus the Schwarzian transforms as in (73), as it should be.

VI. CONCLUSION

In this paper we applied the method of nonlinear realization to some bosonic [Maxwell algebra and $su(1,2)$ one] and supersymmetric $osp(\mathcal{N}|2)$ algebras. After introducing the coordinates of the inert (super)spacetime and imposing the proper constraints,

Cartan forms = Cartan forms on the flat superspace,

we expressed all the Cartan forms of the initial (super)algebra through a single object: a generalized Schwarzian. While doing so, we were able to construct the Schwarzians with $\mathcal{N}$-extended supersymmetry.

The obtained results have to be treated as the first steps in the complete analysis of the Schwarzian systems. Two immediate, but still unanswered questions, concern the following:

(i) The existence of other $\mathcal{N}$-extended systems for $\mathcal{N} > 4$, such as related to $F(4)$ superalgebra, and

(ii) The structure of the equations of motion in Schwarzian supersymmetric mechanics.

It is clear that there is no hope to have the superfield actions for the theories with $\mathcal{N}$-extended supersymmetry. However, the question of the equations of motion for such a system is not trivial. As we know, in the bosonic case the equations of motion of the Schwarzian mechanics reduces to the condition

$$\text{Schwarzian} = \text{const}.$$

It is interesting to understand whether this property can be extended to the supersymmetric case. Another interesting continuation concerns the supersymmetric Maxwell group, its analysis, and possible relation of the corresponding Schwarzians with the flat-space analogs of the Sachdev-Ye-Kitaev model.

Finally, there is a strong expectation that all models constructed in a such manner have to be integrable. It would be interesting to analyze the situation with integrability, at least for the simplest models.

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