Asymptotic representations and q-oscillator solutions of the graded Yang–Baxter equation related to Baxter Q-operators

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Abstract

We consider a class of asymptotic representations of the Borel subalgebra of the quantum affine superalgebra $U_q(\hat{gl}(M|N))$. This is characterized by Drinfeld rational fractions. In particular, we consider contractions of $U_q(gl(M|N))$ in the FRT formulation and obtain explicit solutions of the graded Yang–Baxter equation in terms of q-oscillator superalgebras. These solutions correspond to L-operators for Baxter Q-operators. We also discuss an extension of these representations to the ones for contracted algebras of $U_q(\hat{gl}(M|N))$ by considering the action of renormalized generators of the other side of the Borel subalgebra. We define model independent universal Q-operators as the supertrace of the universal R-matrix and write universal T-operators in terms of these Q-operators based on shift operators on the supercharacters. These include our previous work on $U_q(\hat{sl}(2|1))$ case [1] in part, and also give a cue for the operator realization of our Wronskian-like formulas on T- and Q-functions in [2,3].

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1. Introduction

The Baxter Q-operators were introduced [4] by Baxter when he solved the 8-vertex model. Nowadays his method of Q-operators is recognized as one of the most powerful tools in quantum integrable systems. In particular, Bazhanov, Lukyanov and Zamolodchikov [5] defined Q-operators as the trace of the universal R-matrix over q-oscillator representations of the Borel subalgebra of the quantum affine algebra $U_q(\hat{sl}(2))$. Their work based on the q-oscillator algebra was generalized and developed for various directions [6–9,1,10].

In our previous paper [1], we gave Q-operators for the quantum affine superalgebra $U_q(\hat{sl}(2|1))$. Our Q-operators in [1] are universal in the sense that they do not depend on the quantum space and can be applied for both lattice models and quantum field theoretical models as well. We also proposed [2] an idea that there are 2^{M+N} kind of Baxter Q-functions for $U_q(\hat{gl}(M|N))$ case and gave Wronskian-like formulas on T- and Q-functions for finite [2] and infinite [3] dimensional representations for any (M, N).[2] The Q-function in [2] is labeled by the index set I, which is a subset of the set $\{1, 2, \ldots, M + N\}$. In this paper, we continue these our previous works and define model independent universal Q-operators for $U_q(\hat{gl}(M|N))$ (or $U_q(\hat{sl}(M|N))$) as the supertrace of the universal R-matrix for any (M, N). This gives a cue for the operator realization of the Wronskian-like formulas in [2,3].

In Section 2, we define the quantum affine superalgebra (or rather quantum loop superalgebra) $U_q(\hat{sl}(M|N))$ in terms of the Chevalley generators and the universal R-matrix associated with it. We also mention their extension to $U_q(\hat{gl}(M|N))$. Our task is basically evaluate this universal R-matrix for q-oscillator representations of the Borel subalgebra. As is well known, the Yang–Baxter equation follows from the defining relations of the universal R-matrix. The images of the universal R-matrix for particular representations give the so-called L-operators and R-matrices. The Yang–Baxter equations for the L-operators and the R-matrix ($RLL = LLR$ relations), which are also image of the Yang–Baxter equation for the universal R-matrix, give another realization of the quantum affine superalgebra (FRT realization, [12]). In accordance with the quantum affine superalgebra, the quantum (finite) superalgebra $U_q(gl(M|N))$ also have these two realizations. In Section 3, we consider 2^{M+N} kind of contractions of the L-operator for $U_q(gl(M|N))$, which define contracted algebras $U_q(gl(M|N; I))$. A preliminary form of these contractions for $(M, N) = (3, 0)$ case was previously considered in [13]. We also reported such contractions for $(M, N) = (2, 1)$ case in conferences [14].

Next, we consider q-oscillator realizations of these contracted algebras. These induce representations of the Borel subalgebra of the quantum affine superalgebra (or q-super-Yangian) via the evaluation map. We remark that these representations cannot be straightforwardly extended to the full quantum affine superalgebra. These are examples of asymptotic representations characterized by the Drinfeld rational fractions[3] [15]. They are certain limits of the Kirillov–Reshetikhin modules (or their extension). The hart of an idea is to synchronize the highest weight of the representations and automorphisms of the algebra in the limit so that one can obtain finite quantities. In this way, we obtain spectral parameter dependent L-operators whose matrix elements are written in terms of the q-oscillator superalgebras. Similar L-operators for

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[2] We also proposed [11] Wronskian-like formulas for infinite dimensional representations for $(M, N) = (4, 4)$ case in the context of the AdS_5/CFT_4 spectral problem.

[3] They considered [15] an asymptotic algebra associated with the Drinfeld's second realization of the non-twisted quantum loop algebra. In this paper, we did not consider Ding–Frenkel type isomorphism from their algebra to our's, but rather developed our preliminary discussions [14,1,16] on L-operators for the Q-operators.

$(M, N) = (3, 0)$ were previously considered in [16] and [10]. We also reported such L-operators for $(M, N) = (2, 1)$ in [14,1]. All these L-operators satisfy the defining relations of the universal R-matrix (mentioned in Section 2) evaluated by the tensor product of the q-oscillator representations and the fundamental representation of the Borel subalgebras. It should be remarked here that the above q-oscillator representations of the Borel subalgebra can be extended to those of contracted algebras $U_q(\hat{gl}(M|N; I))$ of $U_q(\hat{gl}(M|N))$. For example for $U_q(\hat{gl}(2))$ case, the contracted algebra $U_q(\hat{gl}(2; \{1\}))$ in terms of the Chevalley generators is defined by the following commutation relations[4]:

$$[e_0, f_0] = \frac{q^{h_0}}{q - q^{-1}}, \qquad [e_1, f_1] = -\frac{q^{-h_1}}{q - q^{-1}},$$

$$[e_0, f_1] = [e_1, f_0] = [k_i, k_j] = 0,$$

$$[k_i, e_0] = (\delta_{i,2} - \delta_{i,1})e_0, \qquad [k_i, e_1] = (\delta_{i,1} - \delta_{i,2})e_1,$$

$$[k_i, f_0] = -(\delta_{i,2} - \delta_{i,1})f_0, \qquad [k_i, f_1] = -(\delta_{i,1} - \delta_{i,2})f_1,$$

$$\big[e_0, [e_0, e_1]_{q^{-2}}\big] = \big[e_1, [e_1, e_0]_{q^2}\big] = \big[f_0, [f_0, f_1]_{q^2}\big] = \big[f_1, [f_1, f_0]_{q^{-2}}\big] = 0, \qquad (1.1)$$

where $i, j \in \{1, 2\}$, $h_1 = -h_0 := k_1 - k_2$, $[X, Y]_q := XY - qYX$, $[X, Y] := [X, Y]_1$. The generators of the Borel subalgebra of $U_q(\hat{gl}(2; \{1\}))$ automatically satisfy the defining relations of the Borel subalgebras of $U_q(\hat{gl}(2))$. The restriction of the above relations to the generators $\{e_1, f_1, k_1, k_2\}$ gives $U_q(gl(2; \{1\}))$. Then we can consider evaluation representations of $U_q(\hat{gl}(2; \{1\}))$ in terms of the representations of $U_q(gl(2; \{1\}))$. The q-oscillator representations of the Borel subalgebra of $U_q(\hat{sl}(2))$ introduced by Bazhanov, Lukyanov and Zamolodchikov [5] are special cases of this type of representations.

In Section 4, we define the universal Q-operators as the supertrace of the universal R-matrix over the q-oscillator representations defined in the previous section. The T-operators are written in terms of these Q-operators. In the same way as previous paper [1], our Q-operators here are universal in the sense that they do not depend on the quantum space. As an example, we write Q-operators whose quantum space is the fundamental representation on each lattice site based on the L-operators derived in Section 3. Section 5 is devoted to concluding remarks. Technical details are tucked into the appendices and a number of footnotes.

There are many literatures on Q-operators related to $sl(2)$, which we could not refer. However there are not so many references for the higher rank case or superalgebras case, which are our main subjects of this paper; and here we only mention some of them for rational models. In the rational limit ($q \to 1$; after multiplying diagonal matrices for the renormalization), our L-operators naturally reduce to L-operators which are similar to the ones proposed recently in [17] for rational lattice models. However, our L-operators are not simple generalization of the rational ones since many of the non-zero matrix elements of our L-operators become zero in the rational limit. Thus the q-deformation of the rational L-operators is not trivial. There are also Q-operators for infinite dimensional representations on the quantum space [18,19]. It will be interesting to evaluate our universal Q-operators for infinite dimensional representations on the quantum space and to see how (or if) their formulas are lifted to the trigonometric case. We also proposed [20] Q-operators based on the co-derivative [21] on the supercharacters of $gl(M|N)$.

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[4] In this paper, we need the level 0 case of the algebra. One may extend this by adding the central element c and the degree operator d: $h_0 = k_2 - k_1 + c$, $[d, e_i] = \delta_{i,0}e_i$, $[d, f_i] = -\delta_{i,0}f_i$. The same remark can be applied for the higher rank case.

This construction of the Q-operators is useful to discuss [20,22] functional relations among T- and Q-operators and embed them into the soliton theory. It is desirable to generalize this for the trigonometric case.

2. The quantum affine superalgebra $U_q(\hat{sl}(M|N))$ and the universal R-matrix, and their extension to $U_q(\hat{gl}(M|N))$

2.1. The quantum affine superalgebra $U_q(\hat{sl}(M|N))$

Let us introduce a grading parameter $p(i) = 0$ for $i \in \{1, 2, \ldots, M\}$ and $p(i) = 1$ for $i \in \{M+1, M+2, \ldots, M+N\}$. The quantum affine superalgebra $U_q(\hat{sl}(M|N))$ [23] (see also [24]) is a $\mathbb{Z}_2$-graded Hopf algebra generated by the generators[5] e_i, f_i, h_i, where $i \in \{0, 1, \ldots, M+N-1\}$. We assign the parity for these generators so that $p(e_0) = p(e_M) = p(f_0) = p(f_M) = 1$ for $MN \neq 0$ and $p(X) = 0$ for all the other generators X. For any $X, Y \in U_q(\hat{sl}(M|N))$, we define $p(XY) = p(X) + p(Y) \pmod 2$. We introduce the generalized commutator $[X, Y]_q = XY - (-1)^{p(X)p(Y)} qYX$. In particular, we set $[X, Y]_1 = [X, Y]$. For $i, j \in \{0, 1, 2, \ldots, M+N-1\}$, the defining relations of the algebra $U_q(\hat{sl}(M|N))$ are given by

$$[h_i, h_j] = 0, \qquad [h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j, \tag{2.1}$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \tag{2.2}$$

$$[e_i, e_j] = [f_i, f_j] = 0 \quad \text{for } a_{ij} = 0, \tag{2.3}$$

where $(a_{ij})_{0 \leq i,j \leq M+N-1}$ is the Cartan matrix

$$a_{ij} = \big((-1)^{p(i)} + (-1)^{p(i+1)}\big)\delta_{ij} - (-1)^{p(i+1)}\delta_{i,j-1} - (-1)^{p(i)}\delta_{i,j+1}, \tag{2.4}$$

here i, j should be interpreted modulo $M + N$: $p(M + N) = p(0)$, $\delta_{i,-1} = \delta_{i,M+N-1}, \delta_{i,M+N} = \delta_{i,0}$. In addition to the above relations, there are Serre relations

$$\big[e_i, [e_i, e_j]_q\big]_{q^{-1}} = 0, \qquad \big[f_i, [f_i, f_j]_{q^{-1}}\big]_q = 0 \quad \text{for } |a_{ij}| = 1, \; a_{ii} \neq 0, \tag{2.5}$$

$$\big[e_i, [e_i, [e_i, e_j]_{q^2}]\big]_{q^{-2}} = 0, \qquad \big[f_i, [f_i, [f_i, f_j]_{q^{-2}}]\big]_{q^2} = 0$$
$$\text{for } (M, N) = (2, 0), (0, 2), \; i \neq j, \tag{2.6}$$

and also for the superalgebra case ($MN \neq 0$), the extra Serre relations[6]:

$$\big[[[e_i, e_j]_q, e_k]_{q^{-1}}, e_j\big] = 0, \qquad \big[[[f_i, f_j]_{q^{-1}}, f_k]_q, f_j\big] = 0$$
$$\text{for } M + N \geq 4, \; (i, j, k) = (M + N - 1, 0, 1), \; (M - 1, M, M + 1), \tag{2.7}$$

$$\big[e_0, [e_2, [e_0, [e_2, e_1]_{q^{-1}}]]\big]_q = \big[e_2, [e_0, [e_2, [e_0, e_1]_{q^{-1}}]]\big]_q,$$
$$\big[f_0, [f_2, [f_0, [f_2, f_1]_{q^{-1}}]]\big]_q = \big[f_2, [f_0, [f_2, [f_0, f_1]_{q^{-1}}]]\big]_q \quad \text{for } (M, N) = (2, 1), \tag{2.8}$$

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[5] In this paper, we do not use the degree operator d.

[6] (2.7) is equivalent to $[[e_i, e_j]_q, [e_k, e_j]_{q^{-1}}] = [[f_i, f_j]_{q^{-1}}, [f_k, f_j]_q] = 0$ under $(e_j)^2 = (f_j)^2 = 0$. We heard from Hiroyuki Yamane that we will need infinitely many Serre relations for $M = N = 2$ case, due to the Lusztig isomorphism (see [23], for more details).

$$[e_0, [e_1, [e_0, [e_1, e_2]_{q^{-1}}]]]_q = [e_1, [e_0, [e_1, [e_0, e_2]_{q^{-1}}]]]_q,$$
$$[f_0, [f_1, [f_0, [f_1, f_2]_{q^{-1}}]]]_q = [f_1, [f_0, [f_1, [f_0, f_2]_{q^{-1}}]]]_q \quad \text{for } (M, N) = (1, 2). \quad (2.9)$$

In this paper, we consider the case where the following central element is zero (level zero condition):

$$h_0 + h_1 + \cdots + h_{M+N-1} = 0. \quad (2.10)$$

The algebra has the co-product $\Delta : U_q(\hat{sl}(M|N)) \to U_q(\hat{sl}(M|N)) \otimes U_q(\hat{sl}(M|N))$ defined by

$$\Delta(e_i) = e_i \otimes 1 + q^{-h_i} \otimes e_i, \quad (2.11)$$
$$\Delta(f_i) = f_i \otimes q^{h_i} + 1 \otimes f_i, \quad (2.12)$$
$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad (2.13)$$

where the tensor product is the graded one: $(A \otimes B)(C \otimes D) = (-1)^{p(B)p(C)}(AC \otimes BD)$. We assume that every tensor product $\otimes$ in this paper is the graded one. We will also use an opposite co-product defined by

$$\Delta' = \sigma \circ \Delta, \qquad \sigma \circ (X \otimes Y) = (-1)^{p(X)p(Y)} Y \otimes X, \quad X, Y \in U_q(\hat{sl}(M|N)). \quad (2.14)$$

In addition to these, there are anti-poide and co-unit, which will not be used in this paper.

The Borel subalgebras $\mathcal{B}_+$ (resp. $\mathcal{B}_-$) is generated by e_i, h_i (resp. f_i, h_i), where $i \in \{0, 1, \ldots, M + N - 1\}$. Let us take complex numbers $c_i \in \mathbb{C}$ which obey a relation $\sum_{i=0}^{M+N-1} c_i = 0$. Then the following transformation

$$h_i \mapsto h_i + c_i \quad \text{for } 0 \leq i \leq M + N - 1 \quad (2.15)$$

gives a shift automorphism of $\mathcal{B}_+$ or $\mathcal{B}_-$. Here we omit the unit element multiplied by the above complex numbers. This automorphism played a role[7] in the construction of the Q-operators in [5,6,1].

There exists a unique element [25,26] $\mathcal{R} \in \mathcal{B}_+ \otimes \mathcal{B}_-$ called the universal R-matrix which satisfies the following relations

$$\Delta'(a)\mathcal{R} = \mathcal{R}\Delta(a) \quad \text{for } \forall a \in U_q(\hat{sl}(M|N)),$$
$$(\Delta \otimes 1)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{23},$$
$$(1 \otimes \Delta)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{12} \quad (2.16)$$

where[8] $\mathcal{R}^{12} = \mathcal{R} \otimes 1, \mathcal{R}^{23} = 1 \otimes \mathcal{R}, \mathcal{R}^{13} = (\sigma \otimes 1)\mathcal{R}^{23}$. The (graded) Yang–Baxter equation

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}, \quad (2.17)$$

is a corollary of these relations (2.16). The universal R-matrix can be written in the form

$$\mathcal{R} = \overline{\mathcal{R}} q^{\mathcal{K}}, \quad \mathcal{K} = \sum_{i,j=1}^{M+N-1} d_{ij} h_i \otimes h_j, \quad (2.18)$$

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[7] When one takes a limit of the highest weight, one has to take a limit of these shift parameters at the same time to obtain a q-oscillator representation for the Q-operators.

[8] We will use similar notations for the L-operators to indicate the space where they non-trivially act on.

where $(d_{ij})_{1 \leq i,j \leq M+N-1}$ is the inverse of the Cartan matrix $(a_{ij})_{1 \leq i,j \leq M+N-1}$ of $sl(M|N)$. In case this Cartan matrix is degenerated ($M = N$), we have to consider an extended matrix[9] and take the inverse of it [27]. Here $\overline{\mathcal{R}}$ is the reduced universal R-matrix, which is a series in $e_j \otimes 1$ and $1 \otimes f_j$ and does not contain Cartan elements. Thus the reduced universal R-matrix is unchanged under the shift automorphism (2.15), while the pre-factor of the universal R-matrix (2.18) is shifted as

$$\mathcal{K} \mapsto \mathcal{K} + \sum_{i,j=1}^{M+N-1} d_{ij} c_i (1 \otimes h_j), \tag{2.19}$$

where we considered a shift on $\mathcal{B}_+$.

2.2. The quantum superalgebra $U_q(gl(M|N))$

There is a (finite) quantum superalgebra $U_q(gl(M|N))$, which is generated by the elements $\{e_{ij}\}_{i,j=1}^{M+N}$. We assign the parity of these generators as $p(e_{ij}) = p(i) + p(j) \bmod 2$. Let us introduce the notations: $e_{\alpha_i} = e_{i,i+1}$, $e_{-\alpha_i} = e_{i+1,i}$ for $i \in \{1, 2, \ldots, M+N-1\}$. Then the defining relations of $U_q(gl(M|N))$ (for the distinguished simple root system) are (cf. [27])[10]

$$[e_{ii}, e_{jj}] = 0, \qquad [e_{ii}, e_{\pm \alpha_j}] = \pm(\delta_{i,j} - \delta_{i,j+1}) e_{\pm \alpha_j},$$

$$[e_{\alpha_i}, e_{-\alpha_j}] = (-1)^{p(i)} \delta_{ij} \frac{q^{(-1)^{p(i)} e_{ii} - (-1)^{p(i+1)} e_{i+1,i+1}} - q^{-(-1)^{p(i)} e_{ii} + (-1)^{p(i+1)} e_{i+1,i+1}}}{q - q^{-1}},$$

$$[e_{\alpha_i}, e_{\alpha_j}] = [e_{-\alpha_i}, e_{-\alpha_j}] = 0 \quad \text{for } |i - j| \geq 2,$$

$$\left[e_{\alpha_i}, [e_{\alpha_i}, e_{\alpha_j}]_q\right]_{q^{-1}} = \left[e_{-\alpha_i}, [e_{-\alpha_i}, e_{-\alpha_j}]_{q^{-1}}\right]_q = 0 \quad \text{for } |i - j| = 1 \quad \text{and} \quad p(e_{\pm \alpha_i}) = 0,$$

$$(e_{\pm \alpha_M})^2 = 0 \quad \text{for } p(e_{\pm \alpha_M}) = 1,$$

$$\left[e_{\alpha_M}, \left[e_{\alpha_{M+1}}, [e_{\alpha_M}, e_{\alpha_{M-1}}]_{q^{-1}}\right]_q\right] = \left[e_{-\alpha_M}, \left[e_{-\alpha_{M+1}}, [e_{-\alpha_M}, e_{-\alpha_{M-1}}]_q\right]_{q^{-1}}\right] = 0$$

$$\text{for } p(e_{\pm \alpha_M}) = 1. \tag{2.20}$$

The other elements are defined by

$$e_{ij} = [e_{ik}, e_{kj}]_{q^{(-1)^{p(k)}}} \quad \text{for } i > k > j,$$

$$e_{ij} = [e_{ik}, e_{kj}]_{q^{-(-1)^{p(k)}}} \quad \text{for } i < k < j. \tag{2.21}$$

The other relations can also be obtain by (3.23)–(3.25) and (A.1)–(A.15). Let E_{ij} be a $(M+N) \times (M+N)$ matrix unit whose (k,l)-element is $\delta_{i,k}\delta_{j,l}$. $\pi(e_{ij}) = E_{ij}$ gives the fundamental representation of $U_q(gl(M|N))$. There is an evaluation map[11] $\text{ev}_x : U_q(\hat{sl}(M|N)) \mapsto U_q(gl(M|N))$:

$$e_0 \mapsto x q^{-(-1)^{p(1)} e_{11}} e_{M+N,1} q^{-(-1)^{p(M+N)} e_{M+N,M+N}},$$

$$f_0 \mapsto (-1)^{p(M+N)} x^{-1} q^{(-1)^{p(M+N)} e_{M+N,M+N}} e_{1,M+N} q^{(-1)^{p(1)} e_{1,1}},$$

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[9] This may be achieved by adding an extra Cartan element $\sum_{j=1}^{M+N} (-1)^{p(j)} k_j$ to $U_q(\hat{sl}(M|N))$. Here k_j are Cartan elements of $U_q(\hat{gl}(M|N))$, which we will introduce later.

[10] The last two relations $[e_{\pm \alpha_M}, [e_{\pm \alpha_{M+1}}, [e_{\pm \alpha_M}, e_{\pm \alpha_{M-1}}]_{q^{\mp 1}}]_{q^{\pm 1}}] = 0$ are equivalent to $[[e_{\pm \alpha_M}, e_{\pm \alpha_{M-1}}]_{q^{\mp 1}}, [e_{\alpha_M}, e_{\alpha_{M+1}}]_{q^{\pm 1}}] = 0$ under the condition $(e_{\pm \alpha_M})^2 = 0$.

[11] $M = N = 1$ case is special since (2.22) does not satisfy (2.3) for $(i, j) = (0, 1), (1, 0)$, in general.

$$h_0 \mapsto (-1)^{p(M+N)} e_{M+N,M+N} - (-1)^{p(1)} e_{1,1},$$

$$e_i \mapsto e_{i,i+1}, \qquad f_i \mapsto (-1)^{p(i)} e_{i+1,i}, \qquad h_i \mapsto (-1)^{p(i)} e_{ii} - (-1)^{p(i+1)} e_{i+1,i+1}$$

$$\text{for } 1 \le i \le M + N - 1, \tag{2.22}$$

where $x \in \mathbb{C}$ is a spectral parameter.

2.3. Representations

Let π_λ be an irreducible representation of $U_q(gl(M|N))$ with the highest weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{M+N})$ and the highest weight vector $|\lambda\rangle$ defined by

$$e_{ii}|\lambda\rangle = \lambda_i |\lambda\rangle, \qquad e_{jk}|\lambda\rangle = 0 \quad \text{for } j < k, \ i, j, k \in \{1, 2, \ldots, M + N\}. \tag{2.23}$$

Then the composition $\pi_\lambda(x) = \pi_\lambda \circ \mathsf{ev}_x$ gives an evaluation representation of $U_q(\hat{sl}(M|N))$. For the fundamental representation, we will use a notation $\pi(x) = \pi_{(1,0,\ldots,0)}(x)$. We also use a notation $\pi_\lambda^+(x)$ for the evaluation representation based on the Verma module defined by the free action of the generators on the highest weight vector (2.23). In this case, the representation is not necessary irreducible. Our main task is basically to evaluate the universal R-matrix for various representations of $U_q(\hat{sl}(M|N))$ (or $U_q(\hat{gl}(M|N))$). Namely, to find matrices of the form (2.18) which satisfy (2.16) for various representations of $\mathcal{B}_+$ and $\mathcal{B}_-$. The simplest example is the R-matrix for the Perk–Schultz model [28] (see [29] for $N = 0$ case), which is a multi-component generalization of the six-vertex model. Namely, the image of the universal R-matrix for $\pi(x_1) \otimes \pi(x_2)$ gives (up to an overall factor $N(x_1, x_2)$; $x_1, x_2 \in \mathbb{C}$):

$$\mathbf{R}(x_1, x_2) = N(x_1, x_2)\big(\pi(x_1) \otimes \pi(x_2)\big)\mathcal{R} = \mathbf{R} - \frac{x_1}{x_2}\bar{\mathbf{R}}, \tag{2.24}$$

$$\mathbf{R} = \sum_{i=1}^{M+N} q^{1-2p(i)} E_{ii} \otimes E_{ii} + \sum_{i \ne j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i < j} (-1)^{p(j)} E_{ij} \otimes E_{ji}, \tag{2.25}$$

$$\bar{\mathbf{R}} = \sum_{i=1}^{M+N} q^{-1+2p(i)} E_{ii} \otimes E_{ii} + \sum_{i \ne j} E_{ii} \otimes E_{jj} - (q - q^{-1}) \sum_{i > j} (-1)^{p(j)} E_{ij} \otimes E_{ji}. \tag{2.26}$$

This obeys the graded Yang–Baxter equation

$$\mathbf{R}^{12}(x_1, x_2)\mathbf{R}^{13}(x_1, x_3)\mathbf{R}^{23}(x_2, x_3) = \mathbf{R}^{23}(x_2, x_3)\mathbf{R}^{13}(x_1, x_3)\mathbf{R}^{12}(x_1, x_2), \tag{2.27}$$

which is an image of (2.17) for $\pi(x_1) \otimes \pi(x_2) \otimes \pi(x_3)$, where $x_1, x_2, x_3 \in \mathbb{C}$.

2.4. Extension to $U_q(\hat{gl}(M|N))$

Let us introduce Cartan elements $\{k_i\}_{i=1}^{M+N}$ of $U_q(\hat{gl}(M|N))$, which is related to the generators of $U_q(\hat{sl}(M|N))$ under (2.10) as

$$h_i = (-1)^{p(i)} k_i - (-1)^{p(i+1)} k_{i+1},$$

$$[k_i, k_j] = 0, \qquad [k_i, e_j] = (\delta_{ij} - \delta_{i,j+1}) e_j, \qquad [k_i, f_j] = -(\delta_{ij} - \delta_{i,j+1}) f_j, \tag{2.28}$$

where the indices i, j should be interpreted modulo $M + N$. These are even elements $p(k_i) = 0$. It is sometimes convenient to define

$$\bar{k}_i = -k_i, \tag{2.29}$$

and rewrite (2.2) as

$$[e_i, f_j] = \delta_{ij} \frac{q^{(-1)^{p(i)}k_i + (-1)^{p(i+1)}\bar{k}_{i+1}} - q^{(-1)^{p(i)}\bar{k}_i + (-1)^{p(i+1)}k_{i+1}}}{q - q^{-1}}. \tag{2.30}$$

Later on, we will renormalize the generators and consider the case where $\bar{k}_i$ differs from $-k_i$ (cf. Appendix B). Moreover, this difference can be infinite in some limit. Now the Borel subalgebras $\mathcal{B}_+$ and $\mathcal{B}_-$ are generated by $\{e_i, k_i\}$ and $\{f_i, \bar{k}_i\}$, respectively. The co-product is defined as $\Delta(k_i) = k_i \otimes 1 + 1 \otimes k_i$. For $M \neq N$, the pre-factor of the universal R-matrix (2.18) can be rewritten as

$$\mathcal{K} = \tilde{\mathcal{K}} - \frac{1}{M - N} \mathcal{C} \otimes \mathcal{C}, \tag{2.31}$$

where $\mathcal{C} = \sum_{i=1}^{M+N} k_i$ is a central element and

$$\tilde{\mathcal{K}} = \sum_{i=1}^{M+N} (-1)^{p(i)} k_i \otimes k_i. \tag{2.32}$$

Note that

$$\tilde{\mathcal{R}} = \overline{\mathcal{R}} q^{\tilde{\mathcal{K}}} \tag{2.33}$$

satisfies (2.16) for $U_q(\hat{gl}(M|N))$ generators. Then we regard[12] this renormalized universal R-matrix $\tilde{\mathcal{R}}$ as a universal R-matrix for $U_q(\hat{gl}(M|N))$ (under the condition (2.10)). For $M \neq N$, $\tilde{\mathcal{R}}$ is related to $\mathcal{R}$ via an overall central element: $\tilde{\mathcal{R}} = \mathcal{R} q^{\frac{1}{M-N}\mathcal{C} \otimes \mathcal{C}}$. However, $\tilde{\mathcal{R}}$ itself is well-defined for $M = N$ case as well. For any $c_i \in \mathbb{C}$ (multiplied by a unit element), the following transformation

$$k_i \mapsto k_i + (-1)^{p(i)} c_i \quad \text{for } 1 \leq i \leq M + N \tag{2.34}$$

gives the shift automorphism of the Borel subalgebra. This keeps the level zero condition (2.10) for any c_i. The pre-factor of the universal R-matrix (2.33) is shift by the shift automorphism (2.34) for $\mathcal{B}_+$ as

$$\tilde{\mathcal{K}} \mapsto \tilde{\mathcal{K}} + \sum_{i=1}^{M+N} c_i (1 \otimes k_i). \tag{2.35}$$

The evaluation map for the Cartan elements is defined by

$$\mathsf{ev}_x(k_i) = e_{ii} \quad \text{for } 1 \leq i \leq M + N. \tag{2.36}$$

The evaluation representations are defined via this map in the same way as $U_q(\hat{sl}(M|N))$ case (the same symbols will be used). In particular $\pi(x)(\mathcal{C})$ is a $(M+N) \times (M+N)$ unit matrix. In the subsequent sections, the contribution of the difference between $\mathcal{R}$ and $\tilde{\mathcal{R}}$ to each formula will be absorbed into a (representation dependent) overall factor of it. For example, the factor $(\pi(x_1) \otimes \pi(x_2))(q^{\frac{1}{M-N}\mathcal{C} \otimes \mathcal{C}}) = q^{\frac{1}{M-N}}$ for (2.24) can be absorbed into $N(x_1, x_2)$.

---

[12] To be precise, $\tilde{\mathcal{R}} q^{a\mathcal{C} \otimes \mathcal{C}}$ for any $a \in \mathbb{C}$ will be the universal R-matrix of $U_q(\hat{gl}(M|N))$ for (2.10). Here we normalized this for $a = 0$.

3. L-operators from FRT realization of the quantum affine superalgebra $U_q(\hat{gl}(M|N))$

3.1. FRT realization of $U_q(\hat{gl}(M|N))$

The quantum affine superalgebra $U_q(\hat{gl}(M|N))$ (and its subalgebra $U_q(gl(M|N))$) has another realization, called FRT realization [12] (see also, [30]), based on the Yang–Baxter equation ($RLL = LLR$ relation). In this section we use this realization. The (centerless) quantum affine superalgebra $U_q(\hat{gl}(M|N))$ is defined by

$$L_{ij}^{(0)} = \overline{L}_{ji}^{(0)} = 0, \quad \text{for } 1 \le i < j \le M + N \tag{3.1}$$

$$L_{ii}^{(0)} \overline{L}_{ii}^{(0)} = \overline{L}_{ii}^{(0)} L_{ii}^{(0)} = 1 \quad \text{for } 1 \le i \le M + N, \tag{3.2}$$

$$\mathbf{R}^{23}(x, y)\mathbf{L}^{13}(y)\mathbf{L}^{12}(x) = \mathbf{L}^{12}(x)\mathbf{L}^{13}(y)\mathbf{R}^{23}(x, y), \tag{3.3}$$

$$\mathbf{R}^{23}(x, y)\overline{\mathbf{L}}^{13}(y)\overline{\mathbf{L}}^{12}(x) = \overline{\mathbf{L}}^{12}(x)\overline{\mathbf{L}}^{13}(y)\mathbf{R}^{23}(x, y), \tag{3.4}$$

$$\mathbf{R}^{23}(x, y)\mathbf{L}^{13}(y)\overline{\mathbf{L}}^{12}(x) = \overline{\mathbf{L}}^{12}(x)\mathbf{L}^{13}(y)\mathbf{R}^{23}(x, y), \tag{3.5}$$

where $x, y \in \mathbb{C}$ and

$$\mathbf{L}(x) = \sum_{i,j=1}^{M+N} L_{ij}(x) \otimes E_{ij}, \qquad \overline{\mathbf{L}}(x) = \sum_{i,j=1}^{M+N} \overline{L}_{ij}(x) \otimes E_{ij}, \tag{3.6}$$

and

$$L_{ij}(x) = \sum_{n=0}^{\infty} L_{ij}^{(n)} x^{-n}, \qquad \overline{L}_{ij}(x) = \sum_{n=0}^{\infty} \overline{L}_{ij}^{(n)} x^n. \tag{3.7}$$

The above relations came from the graded Yang–Baxter equation (2.17) for the universal R-matrix under the specialization (2.24) and $\mathbf{L}(x) = N(x)(1 \otimes \pi(x))\tilde{\mathcal{R}}$, $\overline{\mathbf{L}}(x) = \overline{N}(x)(1 \otimes \pi(x))(\tilde{\mathcal{R}}^{21})^{-1}$, where $N(x)$ and $\overline{N}(x)$ are overall factors. In order to obtain the defining relations for $U_q(\hat{sl}(M|N))$, we will have to impose a condition that the quantum super-determinants of the above L-operators are 1. But we do not impose this explicitly here. Let us introduce a function: $\theta(\text{True}) = 1$, $\theta(\text{False}) = 0$. One can rewrite (3.3) as

$$
\begin{aligned}
&(-1)^{(p(a)+p(b))p(c)} \big(q^{(2p(a)-1)\delta_{ac}} x - q^{(1-2p(a))\delta_{ac}} y\big) L_{cd}(y) L_{ab}(x) \\
&\quad - (-1)^{(p(a)+p(b))p(d)} \big(q^{(2p(b)-1)\delta_{bd}} x - q^{(1-2p(b))\delta_{bd}} y\big) L_{ab}(x) L_{cd}(y) \\
&= (-1)^{p(a)p(b)} \big(q - q^{-1}\big) \big[\big(\theta(a > c)x + \theta(a < c)y\big) L_{ad}(y) L_{cb}(x) \\
&\quad - \big(\theta(d > b)x + \theta(d < b)y\big) L_{ad}(x) L_{cb}(y)\big],
\end{aligned}
\tag{3.8}
$$

and (3.4) as

$$
\begin{aligned}
&(-1)^{(p(a)+p(b))p(c)} \big(q^{(2p(a)-1)\delta_{ac}} x - q^{(1-2p(a))\delta_{ac}} y\big) \overline{L}_{cd}(y) \overline{L}_{ab}(x) \\
&\quad - (-1)^{(p(a)+p(b))p(d)} \big(q^{(2p(b)-1)\delta_{bd}} x - q^{(1-2p(b))\delta_{bd}} y\big) \overline{L}_{ab}(x) \overline{L}_{cd}(y) \\
&= (-1)^{p(a)p(b)} \big(q - q^{-1}\big) \big[\big(\theta(a > c)x + \theta(a < c)y\big) \overline{L}_{ad}(y) \overline{L}_{cb}(x) \\
&\quad - \big(\theta(d > b)x + \theta(d < b)y\big) \overline{L}_{ad}(x) \overline{L}_{cb}(y)\big],
\end{aligned}
\tag{3.9}
$$

and (3.5) as

$$(-1)^{(p(a)+p(b))p(c)}\big(q^{(2p(a)-1)\delta_{ac}}x - q^{(1-2p(a))\delta_{ac}}y\big)L_{cd}(y)\overline{L}_{ab}(x)$$
$$- (-1)^{(p(a)+p(b))p(d)}\big(q^{(2p(b)-1)\delta_{bd}}x - q^{(1-2p(b))\delta_{bd}}y\big)\overline{L}_{ab}(x)L_{cd}(y)$$
$$= (-1)^{p(a)p(b)}\big(q - q^{-1}\big)\big[\big(\theta(a > c)x + \theta(a < c)y\big)L_{ad}(y)\overline{L}_{cb}(x)$$
$$- \big(\theta(d > b)x + \theta(d < b)y\big)\overline{L}_{ad}(x)L_{cb}(y)\big]. \tag{3.10}$$

For any $c \in \mathbb{C} \setminus \{0\}$,

$$\mathbf{L}(x) \mapsto \mathbf{L}(cx), \qquad \overline{\mathbf{L}}(x) \mapsto \overline{\mathbf{L}}(cx) \tag{3.11}$$

gives an automorphism of $U_q(\hat{gl}(M|N))$ since $\mathbf{R}(cx_1, cx_2) = \mathbf{R}(x_1, x_2)$. The restriction of the relations (3.1)–(3.5) to the relation for $\mathbf{L}(x)$ defines a sort of Borel subalgebra of $U_q(\hat{gl}(M|N))$ called q-super-Yangian. Note that the following transformation (multiplication of diagonal matrices in the second space)

$$\mathbf{L}(x) \mapsto (1 \otimes \mathcal{H}_L)\mathbf{L}(x)(1 \otimes \mathcal{H}_R), \qquad \overline{\mathbf{L}}(x) \mapsto (1 \otimes \mathcal{H}_L)\overline{\mathbf{L}}(x)(1 \otimes \mathcal{H}_R),$$
$$\mathcal{H}_L = \sum_i \mathcal{H}_L^{(i)} E_{ii}, \qquad \mathcal{H}_R = \sum_i \mathcal{H}_R^{(i)} E_{ii}, \quad \mathcal{H}_L^{(i)}, \mathcal{H}_R^{(i)} \in \mathbb{C} \setminus \{0\} \tag{3.12}$$

keeps[13] the relations (3.1) and (3.3)–(3.5). However it changes (3.2) as

$$L_{ii}^{(0)}\overline{L}_{ii}^{(0)} = \overline{L}_{ii}^{(0)}L_{ii}^{(0)} = \big(\mathcal{H}_L^{(i)}\mathcal{H}_R^{(i)}\big)^2 \quad \text{for } 1 \le i \le M + N. \tag{3.13}$$

Then the inverse of $L_{ii}^{(0)}$ are not $\overline{L}_{ii}^{(0)}$ but renormalized generators $\overline{L}_{ii}^{(0)}(\mathcal{H}_L^{(i)}\mathcal{H}_R^{(i)})^{-2}$. We will meet a situation where some of $\overline{L}_{ii}^{(0)}$ diverge but $\overline{L}_{ii}^{(0)}(\mathcal{H}_L^{(i)}\mathcal{H}_R^{(i)})^{-2}$ remain finite in some limit. The restriction of this transformation to the q-super-Yangian gives an automorphism of it. In addition, if we consider a 'bigger' algebra (a kind of an asymptotic algebra [15]) which does not assume (3.2), it can be an automorphism of such algebra.

3.2. FRT realization of $U_q(gl(M|N))$

The quantum affine superalgebra $U_q(\hat{gl}(M|N))$ has a finite subalgebra $U_q(gl(M|N))$ defined by

$$L_{ij} = \overline{L}_{ji} = 0, \quad \text{for } 1 \le i < j \le M + N \tag{3.14}$$
$$L_{ii}\overline{L}_{ii} = \overline{L}_{ii}L_{ii} = 1 \quad \text{for } 1 \le i \le M + N, \tag{3.15}$$
$$\mathbf{R}^{23}\mathbf{L}^{13}\mathbf{L}^{12} = \mathbf{L}^{12}\mathbf{L}^{13}\mathbf{R}^{23}, \tag{3.16}$$
$$\mathbf{R}^{23}\overline{\mathbf{L}}^{13}\overline{\mathbf{L}}^{12} = \overline{\mathbf{L}}^{12}\overline{\mathbf{L}}^{13}\mathbf{R}^{23}, \tag{3.17}$$
$$\mathbf{R}^{23}\mathbf{L}^{13}\overline{\mathbf{L}}^{12} = \overline{\mathbf{L}}^{12}\mathbf{L}^{13}\mathbf{R}^{23}, \tag{3.18}$$

where

---

[13] This is related to the parameters c_i in the shift automorphism (2.34)–(2.35) via $\mathcal{H}_R^{(i)} = q^{c_i}$. This also came from the first relation for the Cartan elements of $U_q(\hat{gl}(M|N))$ in (2.16). If we restrict these Cartan elements to the ones for $U_q(\hat{sl}(M|N))$, we will obtain a restriction $\prod_{i=1}^{M+N}(\mathcal{H}_L^{(i)})^{(-1)^{p(i)}} = \prod_{i=1}^{M+N}(\mathcal{H}_R^{(i)})^{(-1)^{p(i)}} = 1$. In this case, (3.12) (for $\mathcal{H}_L^{(i)} = 1$) should correspond to the shift automorphism (2.15) and (2.19). Here we assumed that these parameters are not 0 at first. However, we will have to consider limits that some of these go to ∞ or 0.

$$\mathbf{L} = \sum_{i,j=1}^{M+N} L_{ij} \otimes E_{ij}, \qquad \bar{\mathbf{L}} = \sum_{i,j=1}^{M+N} \bar{L}_{ij} \otimes E_{ij}. \tag{3.19}$$

Then the relation (3.16) leads

$$(-1)^{(p(a)+p(b))p(c)} q^{(1-2p(a))\delta_{ac}} L_{cd} L_{ab} - (-1)^{(p(a)+p(b))p(d)} q^{(1-2p(b))\delta_{bd}} L_{ab} L_{cd}$$
$$= (-1)^{p(a)p(b)} (q - q^{-1}) (\theta(d < b) - \theta(a < c)) L_{ad} L_{cb}, \tag{3.20}$$

the relation (3.17) leads

$$(-1)^{(p(a)+p(b))p(c)} q^{(1-2p(a))\delta_{ac}} \bar{L}_{cd} \bar{L}_{ab} - (-1)^{(p(a)+p(b))p(d)} q^{(1-2p(b))\delta_{bd}} \bar{L}_{ab} \bar{L}_{cd}$$
$$= (-1)^{p(a)p(b)} (q - q^{-1}) (\theta(d < b) - \theta(a < c)) \bar{L}_{ad} \bar{L}_{cb}, \tag{3.21}$$

and the relation (3.18) leads

$$(-1)^{(p(a)+p(b))p(c)} q^{(1-2p(a))\delta_{ac}} L_{cd} \bar{L}_{ab} - (-1)^{(p(a)+p(b))p(d)} q^{(1-2p(b))\delta_{bd}} \bar{L}_{ab} L_{cd}$$
$$= (-1)^{p(a)p(b)} (q - q^{-1}) (\theta(d < b) \bar{L}_{ad} L_{cb} - \theta(a < c) L_{ad} \bar{L}_{cb}). \tag{3.22}$$

For convenience, we list a more explicit form of these relations in Appendix A. These generators are related to the generators $\{e_{ij}\}$ in Section 2 as

$$L_{ii} = q^{(-1)^{p(i)} e_{ii}}, \qquad \bar{L}_{ii} = q^{-(-1)^{p(i)} e_{ii}}, \tag{3.23}$$

$$L_{ij} = (-1)^{p(i)} (q - q^{-1}) e_{ji} q^{(-1)^{p(j)} e_{jj}} \quad \text{for } i > j, \tag{3.24}$$

$$\bar{L}_{ij} = -(-1)^{p(i)} (q - q^{-1}) q^{-(-1)^{p(i)} e_{ii}} e_{ji} \quad \text{for } i < j. \tag{3.25}$$

3.3. Representations

The action of generators of $U_q(gl(M|N))$ on the highest weight vector corresponding to (2.23) is

$$L_{ii} |\lambda\rangle = q^{(-1)^{p(i)} \lambda_i} |\lambda\rangle, \quad \bar{L}_{ii} |\lambda\rangle = q^{-(-1)^{p(i)} \lambda_i} |\lambda\rangle \quad \text{for } 1 \le i \le M+N,$$
$$L_{kj} |\lambda\rangle = 0 \quad \text{for } 1 \le j < k \le M+N. \tag{3.26}$$

There is an evaluation map from $U_q(\hat{gl}(M|N))$ to $U_q(gl(M|N))$ such that

$$\mathbf{L}(x) \mapsto \mathbf{L} - \bar{\mathbf{L}} x^{-1}, \tag{3.27}$$

$$\bar{\mathbf{L}}(x) \mapsto \bar{\mathbf{L}} - \mathbf{L} x. \tag{3.28}$$

Apparently, the difference between $\mathbf{L}(x)$ and $\bar{\mathbf{L}}(x)$ are not very important under the evaluation map. Let us consider an irreducible representation of $U_q(\hat{gl}(M|N))$ with the highest weight $(\nu(x), \bar{\nu}(x))$ and the highest weight vector $|\nu, \bar{\nu}\rangle$ defined by

$$L_{ii}(x)|\nu, \bar{\nu}\rangle = \nu_i(x)|\nu, \bar{\nu}\rangle, \quad \bar{L}_{ii}(x)|\nu, \bar{\nu}\rangle = \bar{\nu}_i(x)|\nu, \bar{\nu}\rangle \quad \text{for } 1 \le i \le M+N, \tag{3.29}$$

$$L_{ij}(x)|\nu, \bar{\nu}\rangle = 0, \qquad \bar{L}_{ij}(x)|\nu, \bar{\nu}\rangle = 0 \quad \text{for } i > j, \tag{3.30}$$

where $\nu(x) = (\nu_1(x), \nu_2(x), \dots, \nu_{M+N}(x))$, $\bar{\nu}(x) = (\bar{\nu}_1(x), \bar{\nu}_2(x), \dots, \bar{\nu}_{M+N}(x))$ are tuples of formal power series in x^{-1} and x respectively. For the evaluation representation based on (3.26)–(3.28), (3.29) becomes

$$L_{ii}(x)|\lambda\rangle = \left(q^{(-1)^{p(i)}\lambda_i} - x^{-1}q^{-(-1)^{p(i)}\lambda_i}\right)|\lambda\rangle, \tag{3.31}$$

$$\overline{L}_{ii}(x)|\lambda\rangle = \left(q^{-(-1)^{p(i)}\lambda_i} - xq^{(-1)^{p(i)}\lambda_i}\right)|\lambda\rangle \quad \text{for } 1 \le i \le M + N. \tag{3.32}$$

For the finite dimensional representations, there exist monic polynomials in x, called Drinfeld polynomials[14] $P_i(x)$, such that

$$\frac{v_i(x^{-1})}{v_{i+1}(x^{-1})} = q^{(-1)^{p(i)}\deg P_i(x)} \frac{P_i(xq^{-2(-1)^{p(i)}})}{P_i(x)} = \frac{\overline{v}_i(x^{-1})}{\overline{v}_{i+1}(x^{-1})} \quad \text{for } 1 \le i \le M + N - 1. \tag{3.33}$$

For the evaluation modules whose highest weights are given by (3.31) and (3.32), the Drinfeld polynomials have the form (if $\lambda_i - (-1)^{p(i)+p(i+1)}\lambda_{i+1} \in \mathbb{Z}_{\ge 0}$)

$$P_i(x) = \prod_{k=1}^{\lambda_i-(-1)^{p(i)+p(i+1)}\lambda_{i+1}} \left(1 - xq^{-2(-1)^{p(i+1)}\lambda_{i+1}-2(-1)^{p(i)}(k-1)}\right)$$
$$\text{for } 1 \le i \le M + N - 1. \tag{3.34}$$

For $N = 0$ case, finite dimensional modules which are characterized by the Drinfeld polynomials with the condition $\lambda_i - \lambda_{i+1} = m\delta_{ik}$ (for all $1 \le i \le M - 1$, and some $m \in \mathbb{Z}_{\ge 0}$ and $1 \le k \le M - 1$) are called Kirillov–Reshetikhin modules.

3.4. Contraction of $U_q(gl(M|N))$

Let us take a subset I of the set $\{1, 2, \dots, M + N\}$ and its complement set $\overline{I} := \{1, 2, \dots, M + N\} \setminus I$. There are 2^{M+N} choices of the subsets in this case. Corresponding to the set I, we consider 2^{M+N} kind of representations of the q-super-Yangian. For this purpose, we consider 2^{M+N} kind of contractions of $U_q(gl(M|N))$. At first, we change the condition (3.15) and define a contracted algebra as follows.

Definition 3.1. The contracted algebra $\tilde{U}_q(gl(M|N; I))$ is an associative algebra over $\mathbb{C}$ with a unit element 1 and generators $L_{ij}, \overline{L}_{ij}$ obeying the relations (3.14), (3.16)–(3.19) and

$$L_{ii}\overline{L}_{ii} = \overline{L}_{ii}L_{ii} = 1 \quad \text{for } i \in I, \tag{3.35}$$
$$\overline{L}_{ii} = 0 \quad \text{for } i \in \overline{I}. \tag{3.36}$$

In addition, we assume the existence of an inverse element L_{ii}^{-1} of L_{ii} for any $i \in \{1, 2, \dots, M + N\}$.

$$L_{ii}L_{ii}^{-1} = L_{ii}^{-1}L_{ii} = 1. \tag{3.37}$$

Note that L_{ii}^{-1} coincides with $\overline{L}_{ii}$ only for $i \in I$. Then one can obtain 2^{M+N} kind of algebraic solutions of the graded Yang–Baxter equation via the map (3.27). In addition to the contraction (3.36), we consider the following subsidiary contraction and define a contracted algebra which is smaller than $\tilde{U}_q(gl(M|N; I))$.

---

[14] Here we define these so that these become monic polynomials of the spectral parameter from $\mathcal{B}_+$. We can also define them so that they are monic polynomials of the spectral parameter from $\mathcal{B}_-$. In this case, q in (3.34) will be replaced by q^{-1}. In addition, the definition for (3.33) for $p(i) = 1$ will have to be modified for the case where the Kac–Dynkin label take continuous number (typical representation).

Definition 3.2. Suppose the set I has the form $I = \{k+1, k+2, \ldots, k+n\}$ for some $k \geq 0$, $n > 0$, then the contracted algebra $U_q(gl(M|N; I))$ is defined by adding the following relations to $\tilde{U}_q(gl(M|N; I))$.

$$L_{ij} = 0 \quad \text{for } k+n < i \leq M+N \text{ and } 1 \leq j \leq k, \tag{3.38}$$

$$\overline{L}_{ij} = 0 \quad \text{for } 1 < i < j \leq k \text{ or } k+n < i < j \leq M+N. \tag{3.39}$$

One may consider different contractions than (3.38), (3.39). Here we consider a contraction so that the location of the zeros becomes cyclic with respect to the shift of the suffixes by an operation: $a \mapsto a+1$ for $a < M+N$ and $M+N \mapsto 1$. Namely, the contraction for $k > 0$ can be given by applying this operation k-times for the case $k = 0$. What is important here is to respect the relations among the generators (3.20)–(3.22). Let us apply the contraction (3.36) to the relation (A.14) for the case $a, b \in \bar{I}$. Then we obtain $[L_{ba}, \overline{L}_{ab}] = 0$. This relation holds true automatically if $L_{ba} = 0$ or $\overline{L}_{ab} = 0$. This is an origin of our subsidiary contractions (3.38)–(3.39). Thus the contractions (corresponding to (3.38)–(3.39)) for a generic set would be realized by putting one of L_{ab} and $\overline{L}_{ab}$ to 0 for $a, b \in \bar{I}$. Whether the contracted algebras for the generic sets have non-trivial useful representations is an open problem. For the contracted algebra $\tilde{U}_q(gl(M|N; I))$, the conditions (3.38)–(3.39) may hold true only on the level of representation. We remark that these contractions on the L-operator for $U_q(\hat{gl}(3))$ (written in terms of the generators e_{ij} and substituted into (3.27)) was previously considered in [13]. We also reported these contractions for $U_q(\hat{gl}(2|1))$ in conferences [14].

3.5. Representations of the contracted algebras

The next task is to consider representations of these contracted algebras. We are interested in q-oscillator representations. The q-oscillator (super)algebra (see for example, [31]) is generated by the generators $\mathbf{c}_{ai}, \mathbf{c}_{ia}^{\dagger}, \mathbf{n}_{ia}$ for $i \in I$, $a \in \bar{I}$, whose parities are defined by $p(\mathbf{c}_{ai}) = p(\mathbf{c}_{ia}^{\dagger}) = p(a) + p(i) \bmod 2$, $p(\mathbf{n}_{ia}) = 0$. They obey the following defining relations:

$$\left[\mathbf{c}_{ai}, \mathbf{c}_{jb}^{\dagger}\right]_{q^{(-1)^{p(a)}\delta_{ab}\delta_{ij}}} = \delta_{ab}\delta_{ij}q^{-(-1)^{p(i)}\mathbf{n}_{ia}},$$

$$\left[\mathbf{c}_{ai}, \mathbf{c}_{jb}^{\dagger}\right]_{q^{-(-1)^{p(a)}\delta_{ab}\delta_{ij}}} = \delta_{ab}\delta_{ij}q^{(-1)^{p(i)}\mathbf{n}_{ia}}, \tag{3.40}$$

$$[\mathbf{n}_{ia}, \mathbf{c}_{bj}] = -\delta_{ij}\delta_{ab}\mathbf{c}_{bj}, \qquad \left[\mathbf{n}_{ia}, \mathbf{c}_{jb}^{\dagger}\right] = \delta_{ij}\delta_{ab}\mathbf{c}_{jb}^{\dagger},$$

$$[\mathbf{n}_{ia}, \mathbf{n}_{jb}] = [\mathbf{c}_{ai}, \mathbf{c}_{bj}] = \left[\mathbf{c}_{ia}^{\dagger}, \mathbf{c}_{jb}^{\dagger}\right] = 0, \tag{3.41}$$

where $i, j \in I$, $a, b \in \bar{I}$. From (3.40), we can derive the relations: $\mathbf{c}_{ai}\mathbf{c}_{ia}^{\dagger} = [\mathbf{n}_{ia}+1]_q$, $\mathbf{c}_{ia}^{\dagger}\mathbf{c}_{ai} = [\mathbf{n}_{ia}]_q$ for $p(i) + p(a) = 0 \bmod 2$, and $\mathbf{c}_{ai}\mathbf{c}_{ia}^{\dagger} = [1 - \mathbf{n}_{ia}]_q$, $\mathbf{c}_{ia}^{\dagger}\mathbf{c}_{ai} = [\mathbf{n}_{ia}]_q$ for[15] $p(i) + p(a) = 1 \bmod 2$, where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. Note that the following transformation

$$\mathbf{n}_{ia} \mapsto \mathbf{n}_{ia}, \qquad \mathbf{c}_{ai} \mapsto \xi_{ia}\mathbf{c}_{ai}q^{\sum_{(j,b)\in I \times \bar{I}} \eta_{ia}^{jb}\mathbf{n}_{jb}}, \qquad \mathbf{c}_{ia}^{\dagger} \mapsto \xi_{ia}^{-1}q^{-\sum_{(j,b)\in I \times \bar{I}} \eta_{ia}^{j,b}\mathbf{n}_{jb}}\mathbf{c}_{ia}^{\dagger},$$

$$\eta_{ia}^{jb} = \eta_{jb}^{ia} \in \mathbb{C}, \qquad \xi_{ia} \in \mathbb{C} \setminus \{0\}, \quad i, j \in I, \quad a, b \in \bar{I} \tag{3.42}$$

---

[15] We consider these generators on the Fock space fixed by the vacuum (3.76). Then for the fermionic case $p(i) + p(a) = 1 \bmod 2$, these relation effectively becomes $\mathbf{c}_{ai}\mathbf{c}_{ia}^{\dagger} = 1 - \mathbf{n}_{ia}$, $\mathbf{c}_{ia}^{\dagger}\mathbf{c}_{ai} = \mathbf{n}_{ia}$.

gives a $|I||\bar{I}|(|I||\bar{I}| + 3)/2$ parameter continuous automorphism of the q-oscillator algebra (3.40). We also remark that the following transformation

$$\mathbf{n}_{ia} \mapsto -\mathbf{n}_{ia} - (-1)^{p(i)+p(a)}, \qquad \mathbf{c}_{ai} \mapsto \mathbf{c}_{ia}^{\dagger}, \qquad \mathbf{c}_{ia}^{\dagger} \mapsto -(-1)^{p(i)+p(a)}\mathbf{c}_{ai} \qquad (3.43)$$

gives a discrete automorphism of the q-oscillator algebra (3.40) for any $i \in I$ and $a \in \bar{I}$. For the diagonal part, we consider the following[16]

$$L_{ii} = q^{-(-1)^{p(i)} \sum_{b\in\bar{I}} \mathbf{n}_{ib}} \quad \text{for } i \in I, \qquad (3.44)$$

$$L_{aa} = q^{(-1)^{p(a)} \sum_{j\in I} \mathbf{n}_{ja}} \quad \text{for } a \in \bar{I}, \qquad (3.45)$$

$$\bar{L}_{ii} = q^{(-1)^{p(i)} \sum_{b\in\bar{I}} \mathbf{n}_{ib}} \quad \text{for } i \in I. \qquad (3.46)$$

Let us look for q-oscillator realization of the non-diagonal part, which are compatible with the defining relations with the diagonal part (3.44)–(3.46). Let us introduce notations $\mathbf{n}_{[i,j],a} = \sum_{k=i}^{j} \mathbf{n}_{k,a}$, $\mathbf{n}_{i,[a,b]} = \sum_{c=a}^{b} \mathbf{n}_{i,c}$, $\mathbf{n}_{I,a} = \sum_{k\in I} \mathbf{n}_{k,a}$, $\mathbf{n}_{i,\bar{I}} = \sum_{c\in\bar{I}} \mathbf{n}_{i,c}$. We find the following solutions.[17]

(i) The case $I = \emptyset$, $\bar{I} = \{1, 2, \ldots, M + N\}$: for $a, b \in \bar{I}$,

$$L_{ab} = 0 \quad \text{for } a \neq b \quad \text{and} \quad L_{aa} = 1, \qquad (3.47)$$

$$\bar{L}_{ab} = 0. \qquad (3.48)$$

(ii) The case $I = \{i\}$, $\bar{I} = \{1, 2, \ldots, M + N\} \setminus \{i\}$:

$$L_{\alpha\beta} = 0 \quad \text{for } \alpha < \beta \text{ or } 1 \leq \beta < i < \alpha \leq M + N, \qquad (3.49)$$

$$L_{ii} = q^{-(-1)^{p(i)}\mathbf{n}_{i,\bar{I}}}, \qquad (3.50)$$

$$L_{aa} = q^{(-1)^{p(a)}\mathbf{n}_{i,a}} \quad \text{for } a \in \bar{I}, \qquad (3.51)$$

$$L_{ai} = (-1)^{p(a)}\mathbf{c}_{ai}q^{(-1)^{p(i)}\mathbf{n}_{i,[i+1,a-1]}} \quad \text{for } i + 1 \leq a \leq M + N, \qquad (3.52)$$

$$L_{ib} = (q - q^{-1})\mathbf{c}_{ib}^{\dagger}q^{(-1)^{p(i)}\mathbf{n}_{i,[b,i-1]}} \quad \text{for } 1 \leq b \leq i - 1, \qquad (3.53)$$

$$L_{ab} = (-1)^{(p(a)+p(b))(p(a)+p(i))+p(i)}(q - q^{-1})\mathbf{c}_{ai}\mathbf{c}_{ib}^{\dagger}q^{(-1)^{p(i)}\mathbf{n}_{i,[b,a-1]}}$$
$$\text{for } 1 \leq b < a \leq i - 1 \text{ or } i + 1 \leq b < a \leq M + N, \qquad (3.54)$$

$$\bar{L}_{\alpha\beta} = 0 \quad \text{for } \alpha > \beta \text{ or } 1 \leq \alpha \leq \beta \leq i - 1 \text{ or } i + 1 \leq \alpha \leq \beta \leq M + N, \qquad (3.55)$$

$$\bar{L}_{ii} = q^{(-1)^{p(i)}\mathbf{n}_{i,\bar{I}}}, \qquad (3.56)$$

$$\bar{L}_{ai} = (-1)^{p(a)}\mathbf{c}_{ai}q^{(-1)^{p(i)}(\mathbf{n}_{i,[1,a-1]}+\mathbf{n}_{i,[i+1,M+N]})} \quad \text{for } 1 \leq a \leq i - 1, \qquad (3.57)$$

$$\bar{L}_{ib} = (q - q^{-1})\mathbf{c}_{ib}^{\dagger}q^{(-1)^{p(i)}(\mathbf{n}_{i,[1,i-1]}+\mathbf{n}_{i,[b,M+N]})} \quad \text{for } i + 1 \leq b \leq M + N, \qquad (3.58)$$

$$\bar{L}_{ab} = (-1)^{(p(a)+p(b))(p(a)+p(i))+p(i)}(q - q^{-1})\mathbf{c}_{ai}\mathbf{c}_{ib}^{\dagger}q^{(-1)^{p(i)}(\mathbf{n}_{i,[1,a-1]}+\mathbf{n}_{i,[b,M+N]})}$$
$$\text{for } 1 \leq a < i < b \leq M + N. \qquad (3.59)$$

(iii) The case $I = \{1, 2, \ldots, M + N\} \setminus \{a\}$, $\bar{I} = \{a\}$:

---

[16] For $\mathbf{L}$, this satisfies a $U_q(sl(M|N))$-type relation $\prod_{i\in I} L_{ii}^{(-1)^{p(i)}} \prod_{a\in\bar{I}} L_{aa}^{(-1)^{p(a)}} = 1$, but for $\bar{\mathbf{L}}$, it does not.

[17] We used relations in Appendix A for the direct calculations.

$$L_{\alpha\beta} = 0 \quad \text{for } \alpha < \beta, \tag{3.60}$$

$$L_{aa} = q^{(-1)^{p(a)}\mathbf{n}_{I,a}}, \tag{3.61}$$

$$L_{ii} = q^{-(-1)^{p(i)}\mathbf{n}_{i,a}} \quad \text{for } i \in I, \tag{3.62}$$

$$L_{ia} = (-1)^{p(a)}\big(q - q^{-1}\big)\mathbf{c}_{ia}^{\dagger}q^{(-1)^{p(a)}(\mathbf{n}_{[1,a-1],a}+\mathbf{n}_{[i+1,M+N],a})}$$
$$\text{for } a + 1 \le i \le M + N, \tag{3.63}$$

$$L_{aj} = q^{-(-1)^{p(a)}}\mathbf{c}_{aj}q^{(-1)^{p(a)}(\mathbf{n}_{[1,j],a}+\mathbf{n}_{[a+1,M+N],a})} \quad \text{for } 1 \le j \le a - 1, \tag{3.64}$$

$$L_{ij} = (-1)^{(p(i)+p(j))p(a)+p(i)p(j)+1}\big(q - q^{-1}\big)\mathbf{c}_{ia}^{\dagger}\mathbf{c}_{aj}q^{-(-1)^{p(a)}\mathbf{n}_{[j+1,i],a}}$$
$$\text{for } 1 \le j < i \le a - 1 \text{ or } a + 1 \le j < i \le M + N, \tag{3.65}$$

$$L_{ij} = (-1)^{(p(i)+p(j))p(a)+p(i)p(j)}q^{-(-1)^{p(a)}}\big(q - q^{-1}\big)\mathbf{c}_{ia}^{\dagger}\mathbf{c}_{aj}q^{(-1)^{p(a)}(\mathbf{n}_{[1,j],a}+\mathbf{n}_{[i+1,M+N],a})}$$
$$\text{for } 1 \le j < a < i \le M + N, \tag{3.66}$$

$$\overline{L}_{\alpha\beta} = 0 \quad \text{for } \alpha > \beta \text{ or } \alpha = \beta = a, \tag{3.67}$$

$$\overline{L}_{ii} = q^{(-1)^{p(i)}\mathbf{n}_{i,a}} \quad \text{for } i \in I, \tag{3.68}$$

$$\overline{L}_{ia} = (-1)^{p(a)}\big(q - q^{-1}\big)\mathbf{c}_{ia}^{\dagger}q^{(-1)^{p(a)}\mathbf{n}_{[i+1,a-1],a}} \quad \text{for } 1 \le i \le a - 1, \tag{3.69}$$

$$\overline{L}_{aj} = q^{-(-1)^{p(a)}}\mathbf{c}_{aj}q^{(-1)^{p(a)}\mathbf{n}_{[a+1,j],a}} \quad \text{for } a + 1 \le j \le M + N, \tag{3.70}$$

$$\overline{L}_{ij} = (-1)^{(p(i)+p(j))p(a)+p(i)p(j)+1}\big(q - q^{-1}\big)\mathbf{c}_{ia}^{\dagger}\mathbf{c}_{aj}q^{-(-1)^{p(a)}(\mathbf{n}_{[1,i],a}+\mathbf{n}_{[j+1,M+N],a})}$$
$$\text{for } 1 \le i < a < j \le M + N, \tag{3.71}$$

$$\overline{L}_{ij} = (-1)^{(p(i)+p(j))p(a)+p(i)p(j)}q^{-(-1)^{p(a)}}\big(q - q^{-1}\big)\mathbf{c}_{ia}^{\dagger}\mathbf{c}_{aj}q^{(-1)^{p(a)}\mathbf{n}_{[i+1,j],a}}$$
$$\text{for } 1 \le i < j \le a - 1 \text{ or } a + 1 \le i < j \le M + N. \tag{3.72}$$

(iv) The case $I = \{1, 2, \dots, M + N\}$, $\overline{I} = \emptyset$: for $i, j \in I$,

$$L_{ij} = \overline{L}_{ij} = 0 \quad \text{for } i \ne j \quad \text{and} \quad L_{ii} = \overline{L}_{ii} = 1. \tag{3.73}$$

Expressions of $L_{ij}, \overline{L}_{ij}$ for the generic set I in terms of the oscillator algebras for the case $M + N \ge 4$ are involved especially for $|i - j| \ge 2$, and their explicit forms are unknown.

One may also apply the transformations (3.42) or (3.43) to these solutions to get many parameter solutions. The q-oscillator solutions of the graded Yang–Baxter equation are given by substituting the above q-oscillator realizations of the L-operators into the map (3.27). We denote the corresponding solutions as

$$\mathbf{L}_I(x) = \mathbf{L} - \overline{\mathbf{L}}x^{-1}. \tag{3.74}$$

We remark that the following renormalized L-operators

$$\mathcal{L}_I(v) := 1 \otimes \left(\sum_{i \in I}\big(q - q^{-1}\big)^{-1}E_{ii} + \sum_{b \in \overline{I}}E_{bb} \right)q^v \mathbf{L}_I\big(q^{2v}\big), \quad v \in \mathbb{C} \tag{3.75}$$

reduce to L-operators similar to the ones in [17] in the rational limit $q \to 1$.

Now (3.74) defines an evaluation map from the q-super-Yangian to the contracted algebra. Let us calculate the actions of generators on the vacuum defined by

$$\mathbf{n}_{ia}|0\rangle = \mathbf{c}_{ai}|0\rangle = 0 \quad \text{for all } i \in I, a \in \overline{I}. \tag{3.76}$$

They lead

$$L_{ii}(x)|0\rangle = (1 - x^{-1})|0\rangle \quad \text{for } i \in I,$$
$$L_{aa}(x)|0\rangle = |0\rangle \quad \text{for } a \in \bar{I}. \tag{3.77}$$

In particular for $I = \{1, 2, \ldots, n\} \subset \{1, 2, \ldots, M + N\}$, we find

$$L_{ij}(x)|0\rangle = 0 \quad \text{for } i > j. \tag{3.78}$$

Thus the corresponding representation is a highest weight representation of the q-super-Yangian with the highest weight vector $|0\rangle$ and the highest weight given by (3.77). In addition, the ratio of the eigenvalues $\nu_i(x)$ of $L_{ii}(x)$ on $|0\rangle$ is $\nu_i(x)/\nu_{i+1}(x) = 1 - x^{-1}\delta_{n,i}$ for $1 \le i \le M + N - 1$. This is a kind of Drinfeld rational fraction[18] introduced in [15]. The finite dimensional representations of the quantum affine algebras are characterized by the Drinfeld polynomials. In contrast, q-oscillator representations given as limits of the Kirillov–Reshetikhin modules[19] of the Borel subalgebra of the quantum affine algebras are characterized by the Drinfeld rational fractions. One may regard (3.77)–(3.78) as a new definition of this type of representations in the FRT formulation, which seems to be unknown in the literatures. For the other sets I, the highest weight condition (3.78) will have to be changed since they should be interpreted as representations permuted by automorphisms of $U_q(\hat{gl}(M|N))$. Let us consider a renormalized L-operator

$$\tilde{\mathbf{L}}(x) = \mathbf{L}(xq^{-2m})(1 \otimes q^{-m\sum_{i \in I} E_{ii}}) \tag{3.79}$$

for the q-super-Yangian shifted by the automorphisms (3.11) and (3.12). The latter corresponds to

$$c_i = -m \quad \text{for } i \in I, \qquad c_i = 0 \quad \text{for } i \in \bar{I} \tag{3.80}$$

in (2.34)–(2.35). For an evaluation representation based on the map (3.27) and the highest weight representation of $U_q(gl(M|N))$ with the highest weight

$$\lambda_i = (-1)^{p(i)}m \quad \text{for } i \in I, \quad \text{and} \quad \lambda_a = 0 \quad \text{for } a \in \bar{I} \tag{3.81}$$

(cf. (3.31)), the eigenvalues of the diagonal part of $\tilde{\mathbf{L}}(x)$ on the highest weight vector coincides with the ones in (3.77) in the limit[20] $m \to \infty$ for $|q| < 1$ (or $m \to -\infty$ for $|q| > 1$).

3.6. Toward contraction of $U_q(\hat{gl}(M|N))$

It will be natural to consider an affine analogue $U_q(\hat{gl}(M|N; I))$ (or $\tilde{U}_q(\hat{gl}(M|N; I))$) of $U_q(gl(M|N; I))$ (or $\tilde{U}_q(gl(M|N; I))$). We will discuss how they will look like.

The evaluation map (2.22) has another presentation of the form:

$$e_0 \mapsto -(-1)^{p(1)}x(q - q^{-1})^{-1}\overline{L}_{1,M+N}L_{M+N,M+N}^{-1},$$
$$f_0 \mapsto x^{-1}(q - q^{-1})^{-1}L_{M+N,M+N}L_{M+N,1},$$

---

[18] Here the spectral parameter x came from $\mathcal{B}_-$. To interpret it as the one from $\mathcal{B}_+$, we have to replace x with x^{-1}.

[19] The q-characters or the T-functions for the Kirillov–Reshetikhin modules solve the T-system for $MN = 0$ [32] and for $MN \ne 0$ [33].

[20] The opposite limit $m \to -\infty$ for $|q| < 1$ (or $m \to \infty$ for $|q| > 1$) [without the shift of the spectral parameter in (3.79)] will effectively interchange the role of I and $\bar{I}$.

$$h_0 \mapsto \frac{\log(L_{M+N,M+N}L_{1,1}^{-1})}{\log q},$$

$$e_i \mapsto (-1)^{p(i+1)}(q-q^{-1})^{-1}L_{i+1,i}L_{ii}^{-1},$$

$$f_i \mapsto -(q-q^{-1})^{-1}L_{ii}\overline{L}_{i,i+1},$$

$$h_i \mapsto \frac{\log(L_{ii}L_{i+1,i+1}^{-1})}{\log q} \quad \text{for } 1 \le i \le M+N-1. \tag{3.82}$$

In addition, the map (2.36) becomes:

$$k_i \mapsto (-1)^{p(i)}\frac{\log L_{ii}}{\log q} \quad \text{for } 1 \le i \le M+N. \tag{3.83}$$

We also define

$$\bar{k}_i \mapsto (-1)^{p(i)}\frac{\log \overline{L}_{ii}}{\log q} \quad \text{for } 1 \le i \le M+N. \tag{3.84}$$

Due to the relation (3.15), (3.83) and (3.84) are consistent with (2.29). Let us substitute L_{ij} given by (3.44)–(3.73) (for a fixed I) into the right hand side of (3.82)–(3.83). This gives an evaluation map from $\mathcal{B}_+$ or $\mathcal{B}_-$ to the q-oscillator superalgebra. We denote this map as $\rho_I(x)$. Similar maps from (restricted to) $\mathcal{B}_+$ to the q-oscillator (super)algebra were considered for $U_q(\hat{sl}(2))$ [5], $U_q(\hat{sl}(3))$ [6], $U_q(\hat{sl}(M))$ [9] and $U_q(\hat{sl}(2|1))$ [1]. Here we used L_{ii}^{-1} in (3.82) instead of $\overline{L}_{ii}$ since L_{ii}^{-1} (for $\in \bar{I}$) do not coincide with $\overline{L}_{ii}$ for the contracted algebras $U_q(gl(M|N;I))$. We remark that $\rho_I(x)$ is not an evaluation map from $U_q(\hat{gl}(M|N))$ to the q-oscillator superalgebra but rather should be interpreted as a map from a certain contracted algebra $U_q(\hat{gl}(M|N;I))$ on $U_q(\hat{gl}(M|N))$. We do not have a rigorous definition of $U_q(\hat{gl}(M|N;I))$ in full generality. Here we mention relations for $U_q(\hat{gl}(M|N;I))$, which we observe through examples.

First, we find that the following contracted commutation relations hold true under the map.

$$[e_i, f_j] = \begin{cases} \delta_{ij}\frac{q^{h_i}-q^{-h_i}}{q-q^{-1}} & \text{for } i, i+1 \in I, \\ \frac{\delta_{ij}q^{h_i}}{q-q^{-1}} & \text{for } i \in \bar{I}, \ i+1 \in I, \\ \frac{-\delta_{ij}q^{-h_i}}{q-q^{-1}} & \text{for } i \in I, \ i+1 \in \bar{I}, \\ 0 & \text{for } i, i+1 \in \bar{I}, \end{cases} \tag{3.85}$$

where i, j should be interpreted under mod $M+N$. The other commutation relations hold true basically in the same way as the ones in Section 2. However, some of the relations become trivial $(0=0)$ when the generator f_i vanishes[21]:

$$f_i = 0 \quad \text{for } i, i+1 \in \bar{I}. \tag{3.86}$$

To be precise, we observed the following non-trivial Serre-type relations in addition to (2.3) and (2.5)–(2.9).

---

[21] These f_i are not original generators of $U_q(\hat{gl}(M|N))$ but the limit of renormalized generators of it (see Appendix B). Original generators f_i of $U_q(\hat{gl}(M|N))$ can diverge. Then these q-oscillator representations of $\mathcal{B}_+$ cannot be straight-forwardly extended to the ones for the whole algebra $U_q(\hat{gl}(M|N))$. We can still extend them for the contracted algebra $U_q(\hat{gl}(M|N;I))$ instead.

The case $M + N \geq 3$:

$$[e_i, e_{i+1}]_{q^{-a_{i,i+1}}} = [f_i, f_{i+1}]_{q^{a_{i,i+1}}} = 0 \quad \text{for } i, i+2 \in I, \; i+1 \in \bar{I}, \tag{3.87}$$

$$[e_i, e_{i+1}]_{q^{a_{i,i+1}}} = [f_i, f_{i+1}]_{q^{-a_{i,i+1}}} = 0 \quad \text{for } i, i+2 \in \bar{I}, \; i+1 \in I, \tag{3.88}$$

where a_{ij} is the Cartan matrix (2.4) and the indices should be interpreted under mod $M + N$.

The case $(M, N) = (2, 0)$ or $(0, 2)$:

$$\big[e_0, [e_0, e_1]_{q^{a_{01}}}\big] = \big[e_1, [e_1, e_0]_{q^{-a_{10}}}\big] = \big[f_0, [f_0, f_1]_{q^{-a_{01}}}\big] = \big[f_1, [f_1, f_0]_{q^{a_{10}}}\big] = 0$$
$$\text{for } 1 \in I, \; 2 \in \bar{I}, \tag{3.89}$$

$$\big[e_0, [e_0, e_1]_{q^{-a_{01}}}\big] = \big[e_1, [e_1, e_0]_{q^{a_{10}}}\big] = \big[f_0, [f_0, f_1]_{q^{a_{01}}}\big] = \big[f_1, [f_1, f_0]_{q^{-a_{10}}}\big] = 0$$
$$\text{for } 1 \in \bar{I}, \; 2 \in I. \tag{3.90}$$

The case[22] $(M, N) = (2, 1)$:

$$\big[e_2, \big[e_0, [e_2, e_1]_q\big]\big] = \big[f_2, \big[f_0, [f_2, f_1]_{q^{-1}}\big]\big] = 0 \quad \text{for } 1 \in \bar{I}, \; 2, 3 \in I, \tag{3.91}$$

$$\big[e_2, \big[e_0, [e_2, e_1]_{q^{-1}}\big]\big] = 0 \quad \text{for } 1 \in I, \; 2, 3 \in \bar{I}, \tag{3.92}$$

$$\big[e_0, \big[e_2, [e_0, e_1]_{q^{-1}}\big]\big] = \big[f_0, \big[f_2, [f_0, f_1]_q\big]\big] = 0 \quad \text{for } 2 \in \bar{I}, \; 1, 3 \in I, \tag{3.93}$$

$$\big[e_0, \big[e_2, [e_0, e_1]_q\big]\big] = 0 \quad \text{for } 2 \in I, \; 1, 3 \in \bar{I}. \tag{3.94}$$

The case $(M, N) = (1, 2)$:

$$\big[e_1, \big[e_0, [e_1, e_2]_q\big]\big] = \big[f_1, \big[f_0, [f_1, f_2]_{q^{-1}}\big]\big] = 0 \quad \text{for } 3 \in \bar{I}, \; 1, 2 \in I, \tag{3.95}$$

$$\big[e_1, \big[e_0, [e_1, e_2]_{q^{-1}}\big]\big] = 0 \quad \text{for } 3 \in I, \; 1, 2 \in \bar{I}, \tag{3.96}$$

$$\big[e_0, \big[e_1, [e_0, e_2]_{q^{-1}}\big]\big] = \big[f_0, \big[f_1, [f_0, f_2]_q\big]\big] = 0 \quad \text{for } 2 \in \bar{I}, \; 1, 3 \in I, \tag{3.97}$$

$$\big[e_0, \big[e_1, [e_0, e_2]_q\big]\big] = 0 \quad \text{for } 2 \in I, \; 1, 3 \in \bar{I}. \tag{3.98}$$

The first equation in (3.87) (or (3.88)) for $i = 0$ and $(M, N) = (3, 0)$ case corresponds to the second equation[23] in Eq. (4.45) in [6] (see also [34]). Some of the Serre-type relations in Section 2 automatically hold true under these relations. For example, we find the following relations[24]:

$$\big[[e_i, e_{i+1}]_{q^{-a_{i,i+1}}}, e_{i+2}\big]_{q^{-a_{i+1,i+2}}} = \big[[f_i, f_{i+1}]_{q^{a_{i,i+1}}}, f_{i+2}\big]_{q^{a_{i+1,i+2}}} = 0$$
$$\text{for } i, i+1, i+3 \in I, \; i+2 \in \bar{I}, \; M + N \geq 4, \tag{3.99}$$

$$\big[[e_i, e_{i+1}]_{q^{a_{i,i+1}}}, e_{i+2}\big]_{q^{a_{i+1,i+2}}} = 0$$
$$\text{for } i, i+1, i+3 \in \bar{I}, \; i+2 \in I, \; M + N \geq 4. \tag{3.100}$$

---

[22] For $1 \in \bar{I}$, $2, 3 \in I$ case, (3.94) follows from (3.87); for $1 \in I$, $2, 3 \in \bar{I}$ case, (3.93) follows from (3.88); for $2 \in \bar{I}$, $1, 3 \in I$ case, (3.92) follows from (3.87); for $2 \in I$, $1, 3 \in \bar{I}$ case, (3.91) follows from (3.88); for $3 \in \bar{I}$, $1, 2 \in I$ case, (3.91) (resp. (3.93)) follows from (3.87) and $(e_2)^2 = 0$ (resp. $(e_0)^2 = 0$); for $3 \in I$, $1, 2 \in \bar{I}$ case, (3.92) (resp. (3.94)) follows from (3.88) and $(e_2)^2 = 0$ (resp. $(e_0)^2 = 0$). A similar remark can be applied for $(M, N) = (1, 2)$ case as well.

[23] We did not consider the first equation in Eq. (4.45) in [6]. It looks like a statement that the relation is a center rather than it is a Serre-type relation.

[24] If we can relax the conditions on the indices for (3.99)–(3.100) (in particular, if we can drop the condition $i + 3 \in I$ in (3.99), and the condition $i + 3 \in \bar{I}$ in (3.100)), then these become independent of the relations (3.87) and (3.88).

(3.99) (resp. (3.100)) follows from (2.3) and (3.87) (resp. (3.88)). Then, (2.7) follows from these if $i = M - 1$ or $M + N - 1$. Note that these relations (3.87)–(3.94) are not symmetric under $q \leftrightarrow q^{-1}$, although the original Serre-type relations in Section 2 are symmetric under this.

Our L-operators (3.74) satisfy the defining relations of the universal R-matrix. In particular, the following relations are valid

$$\begin{aligned}
&\big(1 \otimes \pi(y)(k_i) + \rho_I(x)(k_i) \otimes 1\big) \mathbf{L}_I(y/x) \\
&\quad = \mathbf{L}_I(y/x)\big(\rho_I(x)(k_i) \otimes 1 + 1 \otimes \pi(y)(k_i)\big),
\end{aligned} \tag{3.101}$$

$$\begin{aligned}
&\big(1 \otimes \pi(y)(e_i) + \rho_I(x)(e_i) \otimes \pi(y)(q^{-h_i})\big) \mathbf{L}_I(y/x) \\
&\quad = \mathbf{L}_I(y/x)\big(\rho_I(x)(e_i) \otimes 1 + \rho_I(x)(q^{-h_i}) \otimes \pi(y)(e_i)\big),
\end{aligned} \tag{3.102}$$

where $0 \le i \le M + N - 1$ ($k_0 = k_{M+N}$). This is because our L-operators are image of the universal R-matrix (up to an overall factor $N_I(x, y)$): $\mathbf{L}_I(y/x) = N_I(x, y)(\rho_I(x) \otimes \pi(y))(\tilde{\mathcal{R}})$ (see also discussions on the universal R-matrix in [34]). Note that the relation for f_i, namely

$$\begin{aligned}
&\big(\rho_I(x)(q^{h_i}) \otimes \pi(y)(f_i)\theta(i + 1 \in I) + \rho_I(x)(f_i) \otimes 1\big) \mathbf{L}_I(y/x) \\
&\quad = \mathbf{L}_I(y/x)\big(\rho_I(x)(f_i) \otimes \pi(y)(q^{h_i}) + 1 \otimes \pi(y)(f_i)\theta(i \in I)\big)
\end{aligned} \tag{3.103}$$

has the standard form only for the case $i, i + 1 \in I$ ($0 \equiv M + N$) since we are considering a contracted algebra $U_q(\hat{gl}(M|N; I))$. In particular, this can be $0 = 0$ for $i, i + 1 \notin I$ case.

We have observed the relations (3.85)–(3.103) under the map $\rho_I(x)$. To be precise, (3.85) follows from the maps (3.82)–(3.84) for (2.30) with the contraction (3.36). (3.86) follows from the map (3.82) with the contraction (3.38)–(3.39). (3.89)–(3.90) follow from the map (3.82) with the contraction (3.36). Thus, the map is an algebra homomorphism. However, (3.87)–(3.88) and (3.91)–(3.98) seem to be true only under the map $\rho_I(x)$, and thus can be representation theoretical relations rather than algebraic relations.

Now we want to consider these from an opposite direction. Namely, we may interpret some of the relations (3.85)–(3.100), (2.3)–(2.9), and (2.28) as the defining relations of the contracted algebras $\tilde{U}_q(\hat{gl}(M|N; I))$ and $U_q(\hat{gl}(M|N; I))$. There is a certain arbitrariness on which relations should be included in the defining relations. Apparently, (3.85) (resp. (3.86)) is a consequence of an affine analogue of the contraction (3.36) (resp. subsidiary contraction (3.39)). Then we propose to include (3.85), (2.3)–(2.9) and (2.28) in the defining relations of $\tilde{U}_q(\hat{gl}(M|N; I))$; and (3.85)–(3.86), (3.89)–(3.90), (2.3)–(2.5), (2.7)–(2.9) and (2.28) in the defining relations of $U_q(\hat{gl}(M|N; I))$. We expect these fix the whole contracted algebras for the case $|\bar{I}| = 1$. However, we may have to add more generators and relations[25] for the case $|\bar{I}| \ge 2$. The restriction of the generators to $\{e_i, f_i\}_{i=1}^{M+N-1}$ and $\{k_i\}_{i=1}^{M+N}$ gives relations of $U_q(gl(M|N; I))$. Then we can consider evaluation representations of $U_q(\hat{gl}(M|N; I))$ based on the representations of $U_q(gl(M|N; I))$. The co-product[26] $\Delta : U_q(\hat{gl}(M|N; I)) \mapsto U_q(\hat{gl}(M|N; I)) \otimes U_q(\hat{gl}(M|N))$ for e_i and k_i is the same as the one in Section 2, while the one for f_i is (as observed from (3.103)) contracted as

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[25] This can be guessed from an example on the finite algebra $U_q(gl(M|N; I))$. For $U_q(gl(M|N))$, the generator $\bar{L}_{ij}$ ($|i - j| \ge 2$) can be fixed by the relation (A.7) and the generators $\bar{L}_{k,k+1}, \bar{L}_{k+1,k+1}$ ($i \le k \le j - 1$), which are directly related to the Chevalley type generators. However this is not always the case for $U_q(gl(M|N; I))$ since the relation (A.7) can be trivial (from $\bar{L}_{j-1,j} = 0$, $\bar{L}_{j,j} = 0$) while $\bar{L}_{ij}$ is not for $i \in I$, $j - 1, j \in \bar{I}$. Then the Chevalley type generators may not be enough to fix the whole contracted algebra (explicit relations among $\bar{L}_{ij}, L_{ij}$ ($|i - j| \ge 2$) may be necessary).

[26] This 'co-product' is different from the usual one in that $\mathcal{A}$ and $\mathcal{B}$ for $\Delta : \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{B}$ are different algebras.

$$\Delta(f_i) = f_i \otimes q^{h_i} + \theta(i \in I)(1 \otimes f_i), \tag{3.104}$$

$$\Delta'(f_i) = \theta(i + 1 \in I)\big(q^{h_i} \otimes f_i\big) + f_i \otimes 1. \tag{3.105}$$

This may be rewritten as

$$\Delta(f_i) = f_i \otimes q^{(-1)^{p(i)}k_i + (-1)^{p(i+1)}\bar{k}_{i+1}} + q^{(-1)^{p(i)}k_i + (-1)^{p(i)}\bar{k}_i} \otimes f_i \tag{3.106}$$

since[27]

$$q^{(-1)^{p(i)}\bar{k}_i} = \begin{cases} \theta(i \in I)q^{-(-1)^{p(i)}k_i} & \text{for } U_q(\hat{gl}(M|N; I)) \\ q^{-(-1)^{p(i)}k_i} & \text{for } U_q(\hat{gl}(M|N)). \end{cases} \tag{3.107}$$

The co-product $\Delta(\bar{k}_i) = \bar{k}_i \otimes 1 + 1 \otimes \bar{k}_i$ is well defined only for $i \in I$ since $\bar{k}_i \in U_q(\hat{gl}(M|N; I))$ diverges for $i \in \bar{I}$. However $\Delta(q^{\bar{k}_i}) = q^{\bar{k}_i} \otimes q^{\bar{k}_i}$ is still well defined even for $i \in \bar{I}$ (it just becomes 0).

We may be able to define contracted universal R-matrices in $U_q(\hat{gl}(M|N; I)) \otimes \mathcal{B}_-$ by the contracted co-products for the contracted algebras and (2.16). They will be the universal R-matrices for the Q-operators. Of course, the existence of such an object is not a trivial issue. The universal R-matrix for $U_q(\hat{gl}(M|N))$ is a sort of a power series of the generators of $U_q(\hat{gl}(M|N))$. The generators of $U_q(\hat{gl}(M|N; I))$ are considered to be reductions of the generators of $U_q(\hat{gl}(M|N))$. Thus, the universal R-matrix for $U_q(\hat{gl}(M|N; I))$ may be a reduction of the universal R-matrix for $U_q(\hat{gl}(M|N))$ as a power series on the generators (up to the normalization). More formally, this may be shown by realizing $U_q(\hat{gl}(M|N; I))$ as a kind of Drinfeld double.[28] Furthermore, it will be important to construct and evaluate a contracted universal R-matrix in $U_q(\hat{gl}(M|N; I)) \otimes U_q(\hat{gl}(M|N; J))$. For this, we may have to repeat similar calculations discussed in Appendix B for $\mathcal{B}_+$ as well as $\mathcal{B}_-$. The original universal R-matrix (under a certain condition) may be factorized with respect to contracted universal R-matrices. This could be a step toward the construction of the Q-operators for the generic representations on the quantum space.

We may also interpret $U_q(\hat{gl}(M|N; I))$ as a subalgebra of an asymptotic algebra (cf. [15]) associated with $U_q(\hat{gl}(M|N))$. In terms of the asymptotic algebra, the vanishing of the action of the Cartan generator $q^{\bar{k}_i}$ for $i \in \bar{I}$ in (2.30) occurs on the level of the representation. Here we regarded this as a phenomenon on the level of the algebra and discussed the contracted algebra $U_q(\hat{gl}(M|N; I))$.

As for the FRT formulation of $U_q(\hat{gl}(M|N; I))$, we will have to replace the condition (3.2) with

$$L_{ii}^{(0)} \bar{L}_{ii}^{(0)} = \bar{L}_{ii}^{(0)} L_{ii}^{(0)} = 1 \quad \text{for } i \in I, \tag{3.108}$$

$$\bar{L}_{ii}^{(0)} = 0 \quad \text{for } i \in \bar{I}. \tag{3.109}$$

On the other hand, in the context of the asymptotic algebra, we just forget about (3.2) and interpret that (3.109) occurs on the level of the representation.

In this paper, we consider contractions defined by (3.35)–(3.36). Instead of (3.36), one can consider the following:

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[27] Here (2.29) is not always true since the generators are renormalized.

[28] We thank a referee for this comment.

$$L_{ii} = 0 \quad \text{for } i \in \bar{I}. \tag{3.110}$$

The L-operators based on this contraction have one to one correspondence to the ones proposed in this paper. They seem to be the image of the Cartan anti-involution for our L-operators. One may also consider more general contractions than (3.36) and (3.110):

$$L_{ii} = 0 \quad \text{for } i \in \bar{I}_1, \qquad \bar{L}_{ii} = 0 \quad \text{for } i \in \bar{I}_2, \qquad \bar{I}_1, \bar{I}_2 \subset \bar{I}. \tag{3.111}$$

This defines more degenerated algebras and gives degenerated solutions of the graded Yang–Baxter equation.

4. T- and Q-operators

In this section, we define Q-operators based on the q-oscillator representations introduced in the previous section and sketch an idea how to write the T-operators in terms of them. This gives a cue for operator realization of the formulas in our previous papers [2,3].

We introduce the universal boundary operator

$$\mathcal{D} = q^{\sum_{i=1}^{M+N} \varphi_i k_i}, \tag{4.1}$$

where $\varphi_i \in \mathbb{C}$. This boundary operator is a Cartan element of $U_q(\hat{gl}(M|N))$. Due to the first relation in (2.16), its co-product commutates with the universal R-matrix

$$\tilde{\mathcal{R}}(\mathcal{D} \otimes \mathcal{D}) = (\mathcal{D} \otimes \mathcal{D})\tilde{\mathcal{R}}. \tag{4.2}$$

The images of the evaluation map (2.22) and $\rho_I(x)$ are given as

$$\mathbf{D} := \mathrm{ev}_x(\mathcal{D}) = q^{\sum_{i=1}^{M+N} \varphi_i e_{ii}}, \tag{4.3}$$

$$\mathbf{D}_I := \rho_I(x)(\mathcal{D}) = q^{\sum_{i \in I, a \in \bar{I}} (\varphi_i - \varphi_a) \mathbf{n}_{ia}}. \tag{4.4}$$

We define the universal T-operator by

$$\mathbb{T}_\lambda(x) = (\mathrm{Str}_{\pi_\lambda(x)} \otimes 1)\left[\tilde{\mathcal{R}}(\mathcal{D} \otimes 1)\right]. \tag{4.5}$$

Note that $\mathbb{T}_\lambda(x)$ is an element of $\mathcal{B}_-$ and this definition of the T-operator does not depend on the particular representation of the quantum space. It is convenient to introduce operators

$$z_i = q^{(-1)^{p(i)} k_i + \varphi_i}, \tag{4.6}$$

where $1 \le k \le M + N$. Then the T-operator (4.5) can be rewritten as

$$\mathbb{T}_\lambda(x) = (\mathrm{Str}_{\pi_\lambda(x)} \otimes 1)[\overline{\mathcal{R}}\overline{\mathcal{D}}], \tag{4.7}$$

where

$$\overline{\mathcal{D}} := q^{\tilde{\mathcal{K}}}(\mathcal{D} \otimes 1) = \prod_{j=1}^{M+N} (1 \otimes z_j)^{k_j \otimes 1}, \tag{4.8}$$

where $\tilde{\mathcal{K}}$ is introduced in (2.32). Here we have renormalized the boundary operator (4.1) by the pre-factor of the universal R-matrix (2.18) as in [1].

In the $U_q(\hat{sl}(M|N))$-picture, we may define (4.1), (4.6) and (4.8) respectively as

$$\mathcal{D} = q^{\sum_{k=1}^{M+N-1}\sum_{i=1}^{k}(-1)^{p(i)}\varphi_i h_k}, \qquad z_k = q^{\varphi_k + (-1)^{p(k)}\sum_{j=1}^{M+N-1}(d_{kj}-d_{k-1,j})h_j}, \tag{4.9}$$

$$\overline{\mathcal{D}} := q^{\mathcal{K}}(\mathcal{D} \otimes 1) = \prod_{k=1}^{M+N-1}\left(\prod_{i=1}^{k}\left(1 \otimes z_i^{(-1)^{p(i)}}\right)\right)^{h_k \otimes 1}, \tag{4.10}$$

where $d_{k0} = d_{M+N,j} = 0$, and the parameter φ_{M+N} is defined by the relation $\sum_{i=1}^{M+N}(-1)^{p(i)} \times$ $\varphi_i = 0$. In this case, the following relation holds: $\prod_{k=1}^{M+N} z_k^{(-1)^{p(k)}} = 1$.

If there is no reduced universal R-matrix in (4.7), the following quantity

$$\mathbb{Z}(\lambda) = (\mathrm{Str}_{\pi_\lambda(x)} \otimes 1)[\overline{\mathcal{D}}], \tag{4.11}$$

gives the supercharacter. For finite dimensional modules, it is a supersymmetric Schur function on the variables (4.6). In particular for the Verma module, it leads

$$\mathbb{Z}^+(\lambda) := (\mathrm{Str}_{\pi_\lambda^+(x)} \otimes 1)[\overline{\mathcal{D}}] = \frac{\prod_{j=1}^{M} z_j^{\lambda_j + M - N - j} \prod_{k=M+1}^{M+N}(-z_k)^{\lambda_k + N - M - k}}{\mathsf{D}}, \tag{4.12}$$

$$\mathsf{D} := \frac{\prod_{1 \le b < b' \le M}(z_b - z_{b'}) \prod_{M+1 \le f < f' \le M+N}(z_{f'} - z_f)}{\prod_{b=1}^{M}\prod_{f=M+1}^{M+N}(z_b - z_f)}. \tag{4.13}$$

In the above formulas, the reduced universal R-matrix plays a role to put the spectral parameter into the supercharacters, or to change the supercharacters to the q-supercharacters. This induces sort of shits on the parameters (4.6) in the supercharacters. Let $\mathcal{F}_I$ be the Fock space defined by the action of the generators $\{\mathbf{c}_{ai}, \mathbf{c}_{ia}^\dagger, \mathbf{n}_{ia}\}$ $(i \in I, a \in \bar{I})$ of the q-oscillator superalgebras on the vacuum (3.76). We define the universal Q-operator by

$$\mathbb{Q}_I(x) = \mathbb{Z}_I^{-1}(\mathrm{Str}_{\mathcal{F}_I} \otimes 1)(\rho_I(x) \otimes 1)[\overline{\mathcal{R}}\,\overline{\mathcal{D}}], \tag{4.14}$$

where the normalization function reads

$$\mathbb{Z}_I = (\mathrm{Str}_{\mathcal{F}_I} \otimes 1)(\rho_I(x) \otimes 1)[\overline{\mathcal{D}}]. \tag{4.15}$$

Note that these are elements of $\mathcal{B}_-$. We remark that (4.14) is basically fixed by the map $\rho_I(x)$ and the defining relations of the q-oscillator superalgebra (3.40) and does not depend on the definition of the vacuum (see Section 5.2.3 in [1] for more details). Due to the commutation relation (4.2) and (2.17), the universal T- and Q-operators are commutative.[29]

$$\mathbb{T}_\lambda(x)\mathbb{T}_\mu(y) = \mathbb{T}_\mu(y)\mathbb{T}_\lambda(x), \qquad \mathbb{T}_\lambda(x)\mathbb{Q}_I(y) = \mathbb{Q}_I(y)\mathbb{T}_\lambda(x),$$
$$\mathbb{Q}_I(x)\mathbb{Q}_J(y) = \mathbb{Q}_J(y)\mathbb{Q}_I(x), \tag{4.16}$$

where $x, y \in \mathbb{C}$, $I, J \subset \{1, 2, \ldots, M + N\}$ and λ, μ are any highest weights.

Let us calculate the supertrace (4.15) over the Fock space $\mathcal{F}_I$. Explicitly, it leads

$$\mathbb{Z}_I = \prod_{i \in I}\prod_{a \in \bar{I}}\left(1 - \frac{z_a}{z_i}\right)^{-(-1)^{p(i)+p(a)}}. \tag{4.17}$$

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[29] To prove the commutativity of the Q-operators algebraically, we need (2.17) for the contracted universal R-matrix in $U_q(gl(M|N; I)) \otimes U_q(gl(M|N; J))$, which we do not discuss in this paper. Or, one may prove this on the level of the representation (an isomorphism between the tensor product of two auxiliary spaces).

As expected, this coincides with a limit of a normalized character of the Kirillov–Reshetikhin module at least for the case[30] $N = 0$ (cf. [15]):

$$\mathbb{Z}_I = \lim_{m \to \infty} \frac{\mathrm{S}_\lambda(z_1, z_2, \ldots, z_M)}{\prod_{k=1}^{M} z_k^{\lambda_k}}, \quad |z_i| > |z_a| \quad \text{for all } i \in I, a \in \bar{I},$$

$$m := \lambda_k \quad \text{for } k \in I, \qquad \lambda_k = 0 \quad \text{for } k \in \bar{I}, \tag{4.18}$$

where $\mathrm{S}_\lambda(z_1, z_2, \ldots, z_M) = \det_{1 \le i, j \le M}(z_i^{M+\lambda_j - j})/\det_{1 \le i, j \le M}(z_i^{M-j})$ is the Schur function. Here we meant the equality in (4.18) by the substitution of elements of $\mathcal{B}_-$ (4.6) for the complex numbers $\{z_k\}$ on the right hand side after the limit. The normalization factor in (4.18) came from the shift automorphism (2.34) on $\mathcal{B}_+$ for the parameters in (3.80). We expect [2,3] that the T-operator is given by the Baxterization of the supercharacter[31]

$$\mathbb{T}_\lambda(x) = \frac{1}{\mathsf{D}} \prod_{k=1}^{M+N} \mathbb{Q}_{\{k\}}(xq^{-d_k}) \cdot [\mathsf{D}\,\mathbb{Z}(\lambda)] \tag{4.19}$$

where d_k are differential operators which evaluate the degrees of the monomials on $\{z_j\}$ in the right of the dot $\cdot$. They effectively act as $d_k = 2(-1)^{p(k)} z_k \frac{\partial}{\partial z_k}$ in $[\cdots]$. We assume d_k act on the functions in the left of the dot $\cdot$ as just an identity, although $\{\mathbb{Q}_{\{k\}}\}$ are also functions of $\{z_k\}$. In particular for the Verma module,[32] we have [3]

$$\mathbb{T}_\lambda^+(x) = \mathbb{Z}^+(\lambda) \prod_{j=1}^{M+N} \mathbb{Q}_{\{j\}}\big(xq^{-2((-1)^{p(j)}\lambda_j - \sum_{k=1}^{j-1}(-1)^{p(k)})}\big). \tag{4.20}$$

We remark that the most of the T-operators can be written as summations of the above formula (4.20) based on the Bernstein–Gelfand–Gelfand resolution and rewritten as Wronskian-like determinants (see [5] for $U_q(\hat{sl}(2))$, [6] for $U_q(\hat{sl}(3))$, [9] for finite dimensional representations of $U_q(\hat{sl}(M))$ (see also a Wronskian like determinant in [35]), [1] for $U_q(\hat{sl}(2|1))$; [2,3] for the Wronskian-like determinants for any $U_q(\hat{gl}(M|N))$). We expect our universal Q-operators obey functional relations of the form: for $p(i) = p(j)$:

$$(z_i - z_j)\mathbb{Q}_I\big(xq^{1-2p(i)}\big)\mathbb{Q}_{I \cup \{i,j\}}\big(xq^{-1+2p(i)}\big)$$
$$= z_i \mathbb{Q}_{I \cup \{i\}}\big(xq^{-1+2p(i)}\big)\mathbb{Q}_{I \cup \{j\}}\big(xq^{1-2p(i)}\big)$$
$$- z_j \mathbb{Q}_{I \cup \{i\}}\big(xq^{1-2p(i)}\big)\mathbb{Q}_{I \cup \{j\}}\big(xq^{-1+2p(i)}\big), \tag{4.21}$$

and for $p(i) \ne p(j)$:

$$(z_i - z_j)\mathbb{Q}_{I \cup \{i\}}\big(xq^{-1+2p(i)}\big)\mathbb{Q}_{I \cup \{j\}}\big(xq^{1-2p(i)}\big)$$
$$= z_i \mathbb{Q}_I\big(xq^{1-2p(i)}\big)\mathbb{Q}_{I \cup \{i,j\}}\big(xq^{-1+2p(i)}\big)$$
$$- z_j \mathbb{Q}_I\big(xq^{-1+2p(i)}\big)\mathbb{Q}_{I \cup \{i,j\}}\big(xq^{1-2p(i)}\big). \tag{4.22}$$

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[30] We have also checked that a normalized Sergeev–Pragacz formula produces (4.17) in the large Young diagram limit under a similar condition for the case $MN \ne 0$.

[31] The shift of the spectral parameter of the Q-operators in [2,3] can be recovered by putting $q \to q^{-1}$ after the replacement $\mathbb{Q}_I(x) \mapsto \mathbb{Q}_I(xq^{\sum_{k \in I}(-1)^{p(k)}})$.

[32] This formula (4.20) was presented first as a poster at a conference 'Integrability in Gauge and String Theory 2010', Nordita, Sweden, 28 June 2010–2 July. To fit the formula in [3], one has to make an overall shift of the spectral parameter $x \to zq^{2(M-N)}$ after the manipulation in the footnote 31.

At the moment, these functional relations are fully proven for $U_q(\hat{sl}(2))$ [5], for $U_q(\hat{sl}(3))$ [6] and for $U_q(\hat{sl}(2|1))$ [1]. Their proof is based on decompositions of q-oscillator representations of $\mathcal{B}_+$ and does not rely on the representation of $\mathcal{B}_-$ on the quantum space. See also [20,17] for discussions on rational models ($q = 1$). On the level of the eigenvalues of Q-operators for rational models, (4.22) were discussed in details in relation to the Bäcklund transformations [36]. Here we used expressions based on the 2^{M+N} index sets on the Hasse diagram presented in [2].

Now that we have the universal T- and Q-operators (4.5), (4.14), our next task is to evaluate these for particular representations of $\mathcal{B}_-$ on the quantum space of the model. For example, the T-operator for the lattice model whose quantum space is the fundamental representation on each site is given as

$$\mathbf{T}_\lambda(x) = N_\lambda^{(L)}(x)\big(\pi(\xi_1) \otimes \pi(\xi_2)\cdots \otimes \pi(\xi_L)\big)\big[\Delta^{(L-1)}\mathbb{T}_\lambda(x)\big] \tag{4.23}$$

$$= \mathrm{Str}_{\pi_\lambda}\big[\mathbf{L}^{0L}(\xi_L/x)\cdots \mathbf{L}^{02}(\xi_2/x)\mathbf{L}^{01}(\xi_1/x)\big(\mathbf{D} \otimes 1^{\otimes L}\big)\big], \tag{4.24}$$

where L is the number of the lattice site; the complex parameters $\{\xi_j\}_{j=1}^L$ are inhomogeneities on the spectral parameter; and $N_\lambda^{(L)}(x)$ is a function for the normalization. In (4.24), the evaluation map (3.27) is used and the supertrace is taken over the auxiliary space denoted as '0'. The Q-operators for the same system are given by

$$\mathbf{Q}_I(x) = N_I^{(L)}(x)\big(\pi(\xi_1) \otimes \pi(\xi_2)\cdots \otimes \pi(\xi_L)\big)\big[\Delta^{(L-1)}\mathbb{Q}_I(x)\big] \tag{4.25}$$

$$= \mathbf{Z}_I^{-1}\mathrm{Str}_{\mathcal{F}_I}\big[\mathbf{L}_I^{0L}(\xi_L/x)\cdots \mathbf{L}_I^{02}(\xi_2/x)\mathbf{L}_I^{01}(\xi_1/x)\big(\mathbf{D}_I \otimes 1^{\otimes L}\big)\big], \tag{4.26}$$

where $\mathbf{Z}_I := (\pi(\xi_1) \otimes \pi(\xi_2)\cdots \otimes \pi(\xi_L))[\Delta^{(L-1)}\mathbb{Z}_I]$ and the normalization function is $N_I^{(L)}(x) := \prod_{k=1}^L N_I(x, \xi_k)$. It is instructive to calculate the lattice T-operator (4.24) for the Verma module[33] and the lattice Q-operator (4.26) even for one site $L = 1$ case. Let us introduce a notation $\mathbf{Z}^+(\lambda) := \pi(\xi_1)(\mathbb{Z}^+(\lambda))$. Then we obtain

$$\big[\mathbf{T}_\lambda^+(x)\big]_{ii} = \mathbf{Z}^+(\lambda) - \frac{x}{\xi_1}q^{-d_i} \cdot \mathbf{Z}^+(\lambda)$$

$$= \mathbf{Z}^+(\lambda)\left(1 - \frac{xq^{-2((-1)^{p(i)}\lambda_i - \sum_{k=1}^{i-1}(-1)^{p(k)})}}{\xi_1}\prod_{\substack{b=1 \\ b \neq i}}^{M+N}\left(\frac{1 - \frac{z_b}{z_i}}{1 - \frac{z_b q^{2(-1)^{p(i)}}}{z_i}}\right)^{(-1)^{p(i)+p(b)}}\right)$$

for $1 \leq i \leq M + N$,

$$\big[\mathbf{T}_\lambda^+(x)\big]_{\alpha\beta} = 0 \quad \text{for } \alpha \neq \beta, \tag{4.27}$$

$$\big[\mathbf{Q}_I(x)\big]_{ii} = 1 - \frac{x}{\xi_1}\frac{q^{-d_i} \cdot \mathbf{Z}_I}{\mathbf{Z}_I} = 1 - \frac{x}{\xi_1}\prod_{b \in \bar{I}}\left(\frac{1 - \frac{z_b}{z_i}}{1 - \frac{z_b q^{2(-1)^{p(i)}}}{z_i}}\right)^{(-1)^{p(i)+p(b)}} \quad \text{for } i \in I,$$

$$\big[\mathbf{Q}_I(x)\big]_{aa} = 1 \quad \text{for } a \in \bar{I},$$

$$\big[\mathbf{Q}_I(x)\big]_{\alpha\beta} = 0 \quad \text{for } \alpha \neq \beta, \tag{4.28}$$

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[33] We remark that a formula similar to the first equality in (4.27) (for the characters of finite dimensional representations of $U_q(gl(M))$) was previously derived by Anton Zabrodin in 2007 based on the trigonometric version of the co-derivative for $L = 1$ case.

where $[\mathbf{M}]_{\alpha\beta}$ denotes the (α, β) matrix element of a $((M + N) \times (M + N))$ matrix $\mathbf{M}$. In (4.27) and (4.28), the twist parameters should be interpreted as (i, i)-matrix element of them[34]:

$$z_k = [z_k]_{ii} = q^{\varphi_k + (-1)^{p(k)} \delta_{ik}}. \tag{4.29}$$

The above example gives a non-trivial support to the QQ-relations (4.21)–(4.22) and the factorization formulas (4.20) for the Verma module as the shape of these equations will be essentially independent of the quantum space of the model. This also agrees with examples in Eqs. (3.38)–(3.43) in [1] up to a transformation $q \to q^{-1}$ and a rescaling of the spectral parameter.

The other interesting examples of the Q-operators are the ones for the conformal filed theory (CFT). The monodromy matrix of the CFT can be expressed as an ordered exponential of the form $\overline{\mathcal{L}} = \mathcal{P} \exp(\sum_{i=0}^{M+N-1} \int_0^{2\pi} du\, e_i \otimes V_i(u))$, where $V_i(u)$ are q-vertex operators obeying $V_i(u)V_j(v) = (-1)^{p(i)p(j)} q^{a_{ij}} V_j(v) V_i(u)$ for $u > v$ and e_i are the generators of $\mathcal{B}_+$. Thus, if we substitute our q-oscillator realizations of $\mathcal{B}_+$ through (3.82) into the formula and taking the supertrace over the Fock space for $\mathcal{B}_+$ we will obtain Q-operators for the CFT. Examples of such Q-operators can be seen for $(M, N) = (2, 0)$ in [5], $(M, N) = (3, 0)$ in [6], $N = 0$ in [9], $(M, N) = (2, 1)$ in [1] and for $U_q(C(2)^{(2)})$ in [7]. See also a related recent paper [34].

Finally, we can define the universal master T-operator [22] by

$$\tau(x, t) = \sum_\lambda S_\lambda(t) \mathbb{T}_\lambda(x), \tag{4.30}$$

where $t = (t_1, t_2, \ldots)$ are time variables in the KP hierarchy and $S_\lambda(t)$ is the Schur function labeled by the Young diagram λ. This is a τ-function of the modified KP hierarchy and allows embedding of the quantum integrable system into the soliton theory. Basically, all the functional relations among T- and Q-operators in the Hirota form can be derived from this (see [22,20] for more details).

5. Concluding remarks

In this paper, we have developed our preliminary discussions on L-operators for the Baxter Q-operators for $U_q(\hat{sl}(2|1))$ [14,1] and $U_q(\hat{gl}(3))$ [16], and generalized them to the higher rank case $U_q(\hat{gl}(M|N))$. The contraction of the algebra related to these L-operators was discussed. The model independent universal Q-operators are defined as supertrace of the universal R-matrix. This is a step toward our trial [1–3] (also [20,22]) to construct systematically Q-operators and Wronskian-like expressions of T-operators in terms of them. The L-operators given in this paper can be building blocks of them. Our next task [37] directly related to this paper will be mainly two fold: to generalize our q-oscillator realization of the L-operators for the Q-operators to all the intermediate ones labeled by any 2^{M+N} index set I introduced in [2], and to generalize these for more general representations on the quantum space. All these will be basically accomplished by evaluating the universal R-matrix in the light of asymptotic representations of the quantum affine algebra [15]. We find that a fusion method [17,19] on L-operators for Q-operators developed for rational models is also helpful for this.

A generalization to the elliptic case is perhaps interesting. Although whether the contraction of the Sklyanin algebra works is not clear at the moment, elliptic L-operators may be given by

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[34] In the $U_q(\hat{sl}(M|N))$ picture, this is $z_k = [z_k]_{ii} = q^{\varphi_k + (-1)^{p(k) + p(i)} (d_{ki} - d_{k-1,i} - d_{k,i-1} + d_{k-1,i-1})}$.

twists[35] of our trigonometric L-operators since the elliptic algebras (for both vertex type models and face type models) can be obtained by twists on the quantum affine algebras [38].

The other obvious direction of further development will be a generalization to the other quantum affine superalgebras. For this, it will be helpful to characterize our *L*-operators as sort of Lax operators for the generalized Toda system [39] in terms of the asymptotic algebra [15] and investigate the system in the light of the soliton theory [20,22].

Note added for arXiv:1205.1471v2

In this version, we made some revisions to the version 1 (arXiv:1205.1471v1) of our paper. The revisions are mainly devoted to corrections of misprints and additions of details. Although the q-oscillator representations of the Borel subalgebra $\mathcal{B}_+$ for the Q-operators cannot be straightforwardly extended to the whole algebra $U_q(\hat{gl}(M|N))$, they still can be extended to those of the contracted algebra of $U_q(\hat{gl}(M|N))$. In version 1, we exemplified this by considering contracted commutation relations (3.85) and a part of the intertwining relations (that accompany the co-product and the opposite co-product) for the generators f_i, which come from the other side of the Borel subalgebra $\mathcal{B}_-$ (after the renormalization), in addition to the generators of $\mathcal{B}_+$. In this version, we made these more precise by adding some details. The fact that Serre-type relations for oscillator representations for the Q-operators can be simpler than the original ones was pointed out first by [6] for $\mathcal{B}_+$ of $U_q(\hat{sl}(3))$. However, a systematic study on this (for $U_q(\hat{gl}(M|N; I))$) was missing in the literatures. After version 1 of our paper appeared in May 2012, we received a note from Alessandro Torrielli in November 2012. He discussed an algebra (the co-product, Serre-type relations, etc.) related to L-operators for the Q-operators associated with Yangian $Y(sl(2))$. However, it is not very clear at the moment how (or if) his result is related to our's.

Acknowledgements

Appendix A. Relations for $U_q(gl(M|N))$

(3.20) can be rewritten as:

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[35] This is based on an averaging procedure on the periods with respect to the spectral parameter. A similar procedure may also work to lift rational L-operators to trigonometric ones.

[36] He is supported by the Australian Research Council at ANU and at The University of Melbourne.

$$[L_{cd}, L_{ab}] = 0 \quad \text{for } b < d \leq c < a \text{ or } d < b \leq a < c \text{ or } d \leq c < b \leq a$$
$$\text{or } b \leq a < d \leq c, \tag{A.1}$$

$$[L_{cd}, L_{ab}] = (-1)^{(p(a)+p(b))p(c)+p(a)p(b)} (q - q^{-1}) L_{ad} L_{cb} \quad \text{for } d < b \leq c < a, \tag{A.2}$$

$$[L_{ab}, L_{ad}]_{q^{2p(a)-1}} = 0 \quad \text{for } d < b \leq a, \tag{A.3}$$

$$[L_{cb}, L_{ab}]_{q^{1-2p(b)}} = 0 \quad \text{for } b \leq c < a, \tag{A.4}$$

$$(L_{ab})^2 = 0 \quad \text{for } p(a) + p(b) = 1. \tag{A.5}$$

(3.21) can be rewritten as:

$$[\bar{L}_{cd}, \bar{L}_{ab}] = 0 \quad \text{for } a < c \leq d < b \text{ or } c < a \leq b < d \text{ or } a \leq b < c \leq d$$
$$\text{or } c \leq d < a \leq b, \tag{A.6}$$

$$[\bar{L}_{ab}, \bar{L}_{cd}] = (-1)^{(p(a)+p(b))p(d)+p(a)p(b)} (q - q^{-1}) \bar{L}_{ad} \bar{L}_{cb} \quad \text{for } a < c \leq b < d, \tag{A.7}$$

$$[\bar{L}_{ad}, \bar{L}_{ab}]_{q^{2p(a)-1}} = 0 \quad \text{for } a \leq b < d, \tag{A.8}$$

$$[\bar{L}_{cb}, \bar{L}_{ab}]_{q^{1-2p(b)}} = 0 \quad \text{for } c < a \leq b, \tag{A.9}$$

$$(\bar{L}_{ab})^2 = 0 \quad \text{for } p(a) + p(b) = 1. \tag{A.10}$$

(3.22) can be rewritten as:

$$[L_{cd}, \bar{L}_{ab}] = 0 \quad \text{for } d < a \leq b < c \text{ or } a < d \leq c < b \text{ or } d \leq c < a \leq b$$
$$\text{or } a \leq b < d \leq c \text{ or } a = b = c = d, \tag{A.11}$$

$$[L_{cd}, \bar{L}_{ab}] = (-1)^{(p(a)+p(b))p(c)+p(a)p(b)} (q - q^{-1}) \bar{L}_{ad} L_{cb} \quad \text{for } a \leq d < b < c$$
$$\text{or } a < d < b \leq c, \tag{A.12}$$

$$[L_{cd}, \bar{L}_{ab}] = (-1)^{(p(a)+p(b))p(c)+p(a)p(b)+1} (q - q^{-1}) L_{ad} \bar{L}_{cb} \quad \text{for } d \leq a < c < b$$
$$\text{or } d < a < c \leq b, \tag{A.13}$$

$$[L_{ba}, \bar{L}_{ab}] = (-1)^{p(b)} (q - q^{-1}) (\bar{L}_{aa} L_{bb} - L_{aa} \bar{L}_{bb}) \quad \text{for } a < b, \tag{A.14}$$

$$[L_{ad}, \bar{L}_{ab}]_{q^{2p(a)-1}} = 0 \quad \text{for } d \leq a \leq b \text{ and } d \neq b, \tag{A.15}$$

$$[L_{cb}, \bar{L}_{ab}]_{q^{1-2p(b)}} = 0 \quad \text{for } a \leq b \leq c \text{ and } a \neq c. \tag{A.16}$$

The relations for the contracted algebra $U_q(gl(M|N; I))$ can be obtained by applying (3.35)–(3.39) for the above relations.

Appendix B. Renormalization of generators

The effect of the renormalization for the L-operator (3.79) to the generators of $U_q(\hat{gl}(M|N))$ can be seen from (3.27), (3.82), (3.83) and (3.84):

$$\tilde{e}_i = e_i, \tag{B.1}$$

$$\tilde{f}_i = q^{(2-\theta(i \in I)-\theta(i+1 \in I))m} f_i, \tag{B.2}$$

$$\tilde{h}_i = h_i - (\theta(i \in I) - \theta(i + 1 \in I))m, \tag{B.3}$$

$$\tilde{k}_i = k_i - (-1)^{p(i)} \theta(i \in I)m, \tag{B.4}$$

$$\tilde{\bar{k}}_i = \bar{k}_i + (-1)^{p(i)} (2 - \theta(i \in I))m, \tag{B.5}$$

where the right hand side of these should be understood under the evaluation map $\mathsf{ev}_{xq^{2m}}$ (3.82)–(3.83); the effect of the renormalization is denoted by tilde; and the suffix i should be interpreted under modulo $M + N$. (B.3) and (B.4) came from the transformations for the shift automorphisms (2.15) and (2.34), respectively. Then the commutation relations become

$$[\tilde{e}_i, \tilde{f}_j] = \delta_{ij} \frac{q^{2(1-\theta(i+1\in I))m+\tilde{h}_i} - q^{2(1-\theta(i\in I))m-\tilde{h}_i}}{q - q^{-1}}. \tag{B.6}$$

$$= \delta_{ij} \frac{q^{(-1)^{p(i)}\tilde{k}_i + (-1)^{p(i+1)}\tilde{\tilde{k}}_{i+1}} - q^{(-1)^{p(i)}\tilde{\tilde{k}}_i + (-1)^{p(i+1)}\tilde{k}_{i+1}}}{q - q^{-1}}. \tag{B.7}$$

Let us consider the limit $m \to \infty$ for $|q| < 1$ (or $m \to -\infty$ for $|q| > 1$). We assume the renormalized generators except for (B.5) do not diverge in this limit at least for the evaluation representation $\pi_I(xq^{2m})$ in an appropriate basis, where π_I is the highest weight representation of $U_q(gl(M|N))$ with the highest weight (3.81). Then, in the limit, we obtain:

$$q^{(-1)^{p(i)}\tilde{k}_i} q^{(-1)^{p(i)}\tilde{\tilde{k}}_i} = q^{2m(1-\theta(i\in I))} \to \theta(i \in I), \tag{B.8}$$

and in particular

$$q^{(-1)^{p(i)}\tilde{\tilde{k}}_i} \to 0 \quad \text{for } i \in \bar{I}. \tag{B.9}$$

The inverse of $q^{(-1)^{p(i)}\tilde{k}_i}$, namely $q^{-(-1)^{p(i)}\tilde{k}_i}$ coincides with $q^{(-1)^{p(i)}\tilde{\tilde{k}}_i}$ only for $i \in I$ in the limit. Then the commutation relations (B.6) reduce to the contracted commutation relations (3.85) in the limit. Note that the limit of (B.6) automatically hold true if $\tilde{f}_i = 0$ for $i, i+1 \in \bar{I}$ in the limit.

Let us multiply $(U_q(\hat{gl}(M|N))$ case of) the first relation in (2.16) for f_i by $q^{(2-\theta(i\in I)-\theta(i+1\in I))m}(1 \otimes q^{-m\sum_{j\in I} k_j})$ from the right:

$$\left(q^{2(1-\theta(i+1\in I))m} q^{\tilde{h}_i} \otimes f_i + \tilde{f}_i \otimes 1\right)\tilde{\mathcal{R}}\left(1 \otimes q^{-m\sum_{j\in I} k_j}\right)$$
$$= \tilde{\mathcal{R}}\left(1 \otimes q^{-m\sum_{j\in I} k_j}\right)\left(\tilde{f}_i \otimes q^{h_i} + q^{2(1-\theta(i\in I))m} 1 \otimes f_i\right), \tag{B.10}$$

where $\tilde{R}$ is defined in (2.33). One can see an effect of the shift automorphism for $\mathcal{B}_+$ by the transformation (2.34) with the parameters (3.80). Then this relation for $\pi_I(xq^{2m}) \otimes \pi(y)$ suggests (3.103) in the limit.

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