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A (2+1)-dimensional domain wall at one-loop

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Abstract: We consider the domain wall in the (2+1)-dimensional ϕ^4 double well model, created by extending the ϕ^4 kink in an additional infinite direction. Classically, the tension is $m^3/3\lambda$ where λ is the coupling and m is the meson mass. At order $O(\lambda^0)$ all ultraviolet divergences can be removed by normal ordering, less trivial divergences arrive only at the next order. This allows us to easily quantize the domain wall, working at order $O(\lambda^0)$. We calculate the leading quantum correction to its tension as a two-dimensional integral over a function which is determined analytically. This integral is performed numerically, resulting in $-0.0866m^2$. This correction has previously been computed twice in the literature, and the results of these two computations disagreed. Our result agrees with and so confirms that of Jaimunga, Semenoff and Zarembo. We also find, at this order, the excitation spectrum and a general expression for the one-loop tensions of domain walls in other scalar models.

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Contents

1 Introduction

In his Erice lectures [1], Coleman suggested an open problem. It was already known that in 1+1 dimensional scalar theories, quantum states with solitons correspond to coherent states [2], albeit with perturbative corrections [3, 4]. The coherent state construction works in these theories essentially because their ultraviolet divergences can be removed by normal ordering. Moving beyond this narrow class of theories on the other hand, the coherent state constructon leads to various pathologies, for example, Coleman claims that the expectation value of the Hamiltonian density is infinite. The open question, is how to construct the states corresponding to solitons in this larger class of theories. Coleman writes, “A good place to begin exploring would be a super-renormalizable theory in two spatial dimensions.”

In this paper, we follow Coleman’s suggestion. We try to construct solitons in a scalar theory in 2+1 dimensions. We do not yet complete the problem, rather we push it as far as we can before we run into the troublesome ultraviolet divergences. More precisely, we work to linear order in perturbations about the soliton, corresponding to one loop in the original formulation of the theory. We explicitly construct a perturbative expansion, and use it to calculate the $O(\lambda^0)$ quantum correction to the tension of the domain wall present in this theory.

The next step in this program requires a choice for the construction of the subleading correction to the state. Then several consistency checks will be necessary. One must be sure that the tadpole cancellation present in 1+1 dimensions is not ruined by the renormalization. Also, one must check that a choice of counterterms which cancels the ultraviolet divergences in the vacuum sector, automatically also does so in the soliton sector. The results of the present paper are independent of this choice, and so we believe can serve as a springboard for this next, critical step in Coleman's program.

We begin in section 2 with a review of classical solitons. Our main construction appears in section 3, where we apply canonical quantization. The soliton states are written as a nonperturbative displacement operator, which creates the coherent states, acting on a state. We show that this later state can be constructed and evolved in perturbation theory using an operator called the soliton Hamiltonian, which we construct. This procedure is a straightforward generalization of refs. [5, 6] to more dimensions. Unfortunately Derrick's theorem tells us that higher-dimensional scalar theories, as we have constructed, do not have localized soliton solutions. In section 5 we apply to construction of the previous section to a domain wall solution in the (2+1)-dimensional ϕ^4 double well theory. The solution is just the kink of the (1+1)-dimensional theory lifted up a dimension.

2 Classical solitons

Let us consider a theory in $d+1$ dimensions consisting of a scalar field $\phi(\vec{x})$ with conjugate momentum $\pi(\vec{x})$ and governed by a Hamiltonian which in the Schrodinger picture is

$$H = \int d^d\vec{x} : \mathcal{H} :_a, \qquad \mathcal{H} = \frac{\pi^2(\vec{x}) + \nabla\phi(\vec{x}) \cdot \nabla\phi(\vec{x})}{2} + \frac{V(\sqrt{\lambda}\phi(\vec{x}))}{\lambda}. \tag{2.1}$$

The normal ordering $::_a$ is the usual plane-wave normal ordering, defined at the mass scale corresponding to the mass m of the perturbative meson far from the soliton.

The corresponding classical theory, defined by ignoring the normal ordering and allowing $\phi(\vec{x}, t)$ and $\pi(\vec{x}, t)$ to depend on time, is characterized by the classical equation of motions

$$\ddot{\phi}(\vec{x}, t) = \nabla^2\phi(\vec{x}, t) - \frac{V^{(1)}(\sqrt{\lambda}\phi(\vec{x}, t))}{\sqrt{\lambda}} \tag{2.2}$$

where $V^{(n)}$ is the n-th derivative of V with respect to its argument $\sqrt{\lambda}\phi$.

We will be interested in two solutions. First, consider a time-independent soliton

$$\phi(\vec{x}, t) = f(\vec{x}). \tag{2.3}$$

In this case, the classical equation of motion (2.2) is

$$\nabla^2 f(\vec{x}) = \frac{V^{(1)}(\sqrt{\lambda}f(\vec{x}))}{\sqrt{\lambda}}. \tag{2.4}$$

Second we are interested in small perturbations about this solution

$$\phi(\vec{x}, t) = f(\vec{x}) + \mathfrak{g}(\vec{x}, t). \tag{2.5}$$

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In this case, to linear order in $\mathfrak{g}$, the equation of motion is

$$V^{(2)}(\sqrt{\lambda}f(\vec{x}))\mathfrak{g}(\vec{x},t)+\ddot{\mathfrak{g}}(\vec{x},t)=\nabla^2\mathfrak{g}(\vec{x},t). \tag{2.6}$$

We will decompose $\mathfrak{g}(\vec{x},t)$ into components $\mathfrak{g}(\vec{x})$ with fixed frequencies, which can be taken to be real for a stable soliton

$$\mathfrak{g}_{\vec{k}}(\vec{x},t)=\mathfrak{g}_{\vec{k}}(\vec{x})e^{-i\omega_k t}. \tag{2.7}$$

For each component, labeled by the abstract index $\vec{k}$, the equation of motion becomes

$$V^{(2)}(\sqrt{\lambda}f(\vec{x}))\mathfrak{g}_{\vec{k}}(\vec{x})=\left(\omega_{\vec{k}}^2+\nabla^2\right)\mathfrak{g}_{\vec{k}}(\vec{x}). \tag{2.8}$$

We define the functions $\mathfrak{g}_{\vec{k}}(\vec{x},t)$ to be the solutions of this equation. The index $\vec{k}$ will in general run over discrete and also continuous values. The continuous values include a vector space $\mathbb{R}^d$, which is the reason for the vector symbol on the $\vec{k}$, defined up to signs by $\omega_{\vec{k}}=\sqrt{m^2+\vec{k}^2}$. In the case of the continuous values, we will normalize the $\mathfrak{g}_k(\vec{x})$ via

$$\int d^d\vec{x}\mathfrak{g}_{\vec{k}_1}(\vec{x})\mathfrak{g}_{\vec{k}_2}(\vec{x})=(2\pi)^d\delta^d(\vec{k}_1+\vec{k}_2),\qquad \mathfrak{g}^*_{\vec{k}}(\vec{x})=\mathfrak{g}_{-\vec{k}}(\vec{x}). \tag{2.9}$$

In the case of discrete indices, $\mathfrak{g}_{\vec{k}}(\vec{x})$ will be taken to be real and

$$\int d^d\vec{x}\mathfrak{g}_{\vec{k}_1}(\vec{x})\mathfrak{g}_{\vec{k}_2}(\vec{x})=\delta_{\vec{k}_1,\vec{k}_2}. \tag{2.10}$$

More generally, some values of $\vec{k}$ inhabit lower dimensions submanifolds, and we will use the obvious hybrids in which continuous directions are normalized with Dirac delta functions and discrete labels of manifolds are normalized with Kronecker δ. Often we will use a shorthand in which the normalization condition in (2.9) is written, but it is implied that if some component of $\vec{k}$ is discrete, then the corresponding $2\pi\delta$ should be replaced with a Kronecker δ.

3 Quantum solitons

3.1 Soliton Hamiltonian

Define the displacement operator

$$\mathcal{D}_f=\exp\left[-i\int d^d\vec{x}f(\vec{x})\pi(\vec{x})\right] \tag{3.1}$$

and the soliton Hamiltonian

$$H'=\mathcal{D}_f^\dagger H\mathcal{D}_f. \tag{3.2}$$

Explicitly, the soliton Hamiltonian is

$$H'[\phi(\vec{x}),\pi(\vec{x})]=H[\phi(\vec{x})+f(\vec{x}),\pi(\vec{x})]. \tag{3.3}$$

One can check that this identity holds despite the normal ordering. We will expand H' in powers of the coupling $\sqrt{\lambda}$

$$H'=\sum_{j=0}^{\infty}H'_j \tag{3.4}$$

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where H'_j is a functional of the fields times of a coefficient of order $\lambda^{j/2-1}$. It is defined to consist of terms which, when normal ordered using $::_a$, are n-linear in $\phi(x)$ and $\pi(x)$. One easily finds

$$H'_0 = Q_0, \qquad H'_1 = 0 \tag{3.5}$$

where Q_0 is the energy of the classical solution $\phi(\vec{x},t) = f(\vec{x})$.

The most important step in perturbation theory is order $O(\lambda^0)$, as any failure to diagonalize the Hamiltonian exactly at this order will not be suppressed at small λ. The contribution to the Hamiltonian at this order is

$$\begin{aligned} H'_2 &= A + B + C, & A &= \frac{1}{2}\int d^d\vec{x} :\pi^2(\vec{x}):_a \\ B &= \frac{1}{2}\int d^d\vec{x} :(\nabla\phi)^2(\vec{x}):_a, & C &= \frac{1}{2}\int d^d\vec{x} V^{(2)}(\sqrt{\lambda}f(\vec{x})) :\phi^2(\vec{x}):_a . \end{aligned} \tag{3.6}$$

3.2 Decompositions

We will consider two decompositions of the fields

$$\begin{aligned} \phi(\vec{x}) &= \int \frac{d^d\vec{p}}{(2\pi)^d} e^{-i\vec{x}\cdot\vec{p}}\phi_{\vec{p}} = \sum\!\!\!\!\!\!\int \frac{d^d\vec{k}}{(2\pi)^d} \mathfrak{g}_{\vec{k}}(\vec{x})\phi_{\vec{k}} \\ \pi(\vec{x}) &= \int \frac{d^d\vec{p}}{(2\pi)^d} e^{-i\vec{x}\cdot\vec{p}}\pi_{\vec{p}} = \sum\!\!\!\!\!\!\int \frac{d^d\vec{k}}{(2\pi)^d} \mathfrak{g}_{\vec{k}}(\vec{x})\pi_{\vec{k}}. \end{aligned} \tag{3.7}$$

To avoid a proliferation of hats and tildes, we use the same notation for $\phi_{\vec{p}}$ and $\phi_{\vec{k}}$ although they represent distinct bases of the space of operators, they will be distinguished only by the letter used for the index. The $\sum\!\!\!\!\!\!\int$ symbol is an integration over continuous indices $\vec{k}$, dividing by 2π for each dimension, plus a sum over discrete indices. In general, the space of $\vec{k}$ has components of various dimensions and these are each integrated over and the integrals are summed.

The completeness relations (2.9) allow these decompositions to be inverted. The canonical commutation relations

$$[\phi(\vec{x}_1), \pi(\vec{x}_2)] = i\delta^d(\vec{x}_1 - \vec{x}_2) \tag{3.8}$$

then lead to the usual commutation relations in the plane wave and normal mode bases

$$[\phi_{\vec{p}_1}, \pi_{\vec{p}_2}] = i(2\pi)^d\delta^d(\vec{p}_1 + \vec{p}_2), \qquad [\phi_{\vec{k}_1}, \pi_{\vec{k}_2}] = i(2\pi)^d\delta^d(\vec{k}_1 + \vec{k}_2) \tag{3.9}$$

where again it is implicit that in the case of lower dimensional submanifolds in the $\vec{k}$ space, the transverse $2\pi\delta$ should be replaced with Kronecker deltas.

3.3 Harmonic oscillators

Using the $\vec{k}$ decompositions in eq. (3.7) one finds

$$A = \frac{1}{2}\sum\!\!\!\!\!\!\int dk_1 \sum\!\!\!\!\!\!\int dk_2 \int d^d\vec{x}\mathfrak{g}_{\vec{k}_1}(x)\mathfrak{g}_{\vec{k}_2}(x) :\pi_{\vec{k}_1}\pi_{\vec{k}_2}:_a = \frac{1}{2}\sum\!\!\!\!\!\!\int dk :\pi_{\vec{k}}\pi_{-\vec{k}}:_a \tag{3.10}$$

where we have used the completeness relation (2.9). Similarly, integrating by parts and dropping a rapidly oscillating boundary term

$$B = -\frac{1}{2} \sumint dk_1 \sumint dk_2 \int d^d\vec{x}\mathfrak{g}_{\vec{k}_1}(x)\nabla^2\mathfrak{g}_{\vec{k}_2}(x) :\phi_{\vec{k}_1}\phi_{\vec{k}_2}:_a \tag{3.11}$$

while the defining equation (2.8) leads to

$$C = \frac{1}{2} \sumint dk_1 \sumint dk_2 \int d^d\vec{x}\mathfrak{g}_{\vec{k}_1}(x)\left(\nabla^2 + \omega^2_{\vec{k}_2}\right)\mathfrak{g}_{\vec{k}_2}(x) :\phi_{\vec{k}_1}\phi_{\vec{k}_2}:_a . \tag{3.12}$$

Adding (3.11) and (3.12) and again using the completeness (2.9) one obtains

$$B+C = \frac{1}{2} \sumint dk_1 \sumint dk_2\omega^2_{\vec{k}_2} :\phi_{\vec{k}_1}\phi_{\vec{k}_2}:_a \int d^d\vec{x}\mathfrak{g}_{\vec{k}_2}(x)\mathfrak{g}_{\vec{k}_1}(x) = \frac{1}{2} \sumint dk\omega^2_{\vec{k}} :\phi_{\vec{k}}\phi_{-\vec{k}}:_a . \tag{3.13}$$

Adding all of these contributions we find the soliton Hamiltonian at order $O(\lambda^0)$

$$H_2' = \frac{1}{2} \sumint dk \left(:\pi_{\vec{k}}\pi_{-\vec{k}}:_a + \omega^2_{\vec{k}} :\phi_{\vec{k}}\phi_{-\vec{k}}:_a \right). \tag{3.14}$$

If it were not for the normal ordering, this would be a sum of harmonic oscillators, one at each $\vec{k}$. The ground state at leading order in perturbation theory would be the ground state of each harmonic oscillator, while the excited states would be created by the corresponding creation operators $B^\ddagger_{\vec{k}}$. There in general will be zero modes, for example if the Hamiltonian is translation invariant or has some similar internal symmetry. For these, $\omega_{\vec{k}} = 0$ and so only the corresponding $\pi^2_{\vec{k}}$ term is present. This describes the quantum mechanics of a free particle describing the position with respect to that symmetry, and one must impose that the ground state is annihilated by each such $\pi_{\vec{k}}$, while excited states correspond to exponentials in $i\phi_{\vec{k}}$.

What is the effect of the normal ordering? Since these operators are linear, it can only add a constant. We refer to this constant as Q_1 when H_2' is ordered in the form $B^\ddagger B$. It is the one-loop correction to the soliton mass [5, 6]. We will now compute it.

4 One-loop mass correction

4.1 Plane-wave decomposition

The normal ordering is defined in terms of the usual plane wave decomposition of the fields, corresponding to the middle expressions in (3.7). Using this decomposition, one easily finds

$$A = \frac{1}{2} \int \frac{d^d\vec{p}_1}{(2\pi)^d} \int \frac{d^d\vec{p}_2}{(2\pi)^d} \int d^d\vec{x} e^{-ix(\vec{p}_1+\vec{p}_2)} :\pi_{\vec{p}_1}\pi_{\vec{p}_2}:_a = \frac{1}{2} \int \frac{d^d\vec{p}}{(2\pi)^d} :\pi_{\vec{p}}\pi_{-\vec{p}}:_a . \tag{4.1}$$

As the decompositions are both in complete bases, one may map from one to the other via

$$\phi_{\vec{k}} = \int \frac{d^d\vec{p}}{(2\pi)^d}\tilde{\mathfrak{g}}_{-\vec{k}}(\vec{p})\phi_{\vec{p}}, \qquad \pi_{\vec{k}} = \int \frac{d^d\vec{p}}{(2\pi)^d}\tilde{\mathfrak{g}}_{-\vec{k}}(\vec{p})\pi_{\vec{p}} \tag{4.2}$$

where we have defined the Fourier transform

$$\tilde{\mathfrak{g}}_{\vec{k}}(\vec{p}) = \int d^d\vec{x}\mathfrak{g}_{\vec{k}}(\vec{x})e^{-i\vec{p}\cdot\vec{x}}. \tag{4.3}$$

This allows us to rewrite $B+C$, given in eq. (3.13), in the plane-wave basis

$$B+C=\frac{1}{2}\sum\!\!\!\!\!\!\int dk\omega_{\vec{k}}^2\int\frac{d^d\vec{p}_1}{(2\pi)^d}\int\frac{d^d\vec{p}_2}{(2\pi)^d}\tilde{\mathfrak{g}}_{-\vec{k}}(\vec{p}_1)\tilde{\mathfrak{g}}_{\vec{k}}(\vec{p}_2):\phi_{\vec{p}_1}\phi_{\vec{p}_2}:_a. \tag{4.4}$$

Now we use the standard Schrodinger picture decomposition into creation and annihilation operators

$$\phi_{\vec{p}}=A^{\ddagger}_{\vec{p}}+\frac{A_{-\vec{p}}}{2\omega_{\vec{p}}},\qquad \pi_{\vec{p}}=i\omega_{\vec{p}}A^{\ddagger}_{\vec{p}}-\frac{iA_{-\vec{p}}}{2},\qquad A^{\ddagger}_{\vec{p}}=\frac{A^{\dagger}_{\vec{p}}}{2\omega_{\vec{p}}}. \tag{4.5}$$

The normal ordering $::_a$ is defined to be the operation that places all $A^{\ddagger}$ to the left of all A. And so we may finally evaluate the normal ordering

$$A=\frac{1}{2}\int\frac{d^d\vec{p}}{(2\pi)^d}\left[-\omega_{\vec{p}}^2A^{\ddagger}_{\vec{p}}A^{\ddagger}_{-\vec{p}}+\omega_{\vec{p}}A^{\ddagger}_{\vec{p}}A_{\vec{p}}-\frac{A_{\vec{p}}A_{-\vec{p}}}{4}\right] \tag{4.6}$$

$$B+C=\frac{1}{2}\sum\!\!\!\!\!\!\int dk\omega_{\vec{k}}^2\int\frac{d^d\vec{p}_1}{(2\pi)^d}\int\frac{d^d\vec{p}_2}{(2\pi)^d}\tilde{\mathfrak{g}}_{-\vec{k}}(\vec{p}_1)\tilde{\mathfrak{g}}_{\vec{k}}(\vec{p}_2)$$
$$\times\left[A^{\ddagger}_{\vec{p}_1}A^{\ddagger}_{\vec{p}_2}+\frac{A^{\ddagger}_{\vec{p}_1}A_{-\vec{p}_2}}{2\omega_{\vec{p}_2}}+\frac{A^{\ddagger}_{\vec{p}_2}A_{-\vec{p}_1}}{2\omega_{\vec{p}_1}}+\frac{A_{-\vec{p}_1}A_{-\vec{p}_2}}{4\omega_{\vec{p}_1}\omega_{\vec{p}_2}}\right].$$

4.2 Back to the normal mode basis

Now that the normal ordering symbol has disappeared, we can freely move between bases with Bogoliubov transforms. We will now need to move back to the normal mode basis. We will decompose the index $\vec{k}$ into zero modes, for which $\omega_{\vec{k}}=0$, and nonzero modes, for which it is taken to be positive. We will now consider nonzero modes. With a page of calculations, following the example worked out in ref. [6], the argument below can be easily modified to the case of zero modes, and one can derive that the final results below will hold for zero modes just by setting $\omega_{\vec{k}}=0$, although this substitution cannot be used at intermediate steps.

In the case of nonzero modes, we will make the decomposition into creation and annihilation operators

$$\phi_{\vec{k}}=B^{\ddagger}_{\vec{k}}+\frac{B_{-\vec{k}}}{2\omega_{\vec{k}}},\qquad \pi_{\vec{k}}=i\omega_{\vec{k}}B^{\ddagger}_{\vec{k}}-\frac{iB_{-\vec{k}}}{2},\qquad B^{\ddagger}_{\vec{k}}=\frac{B^{\dagger}_{\vec{k}}}{2\omega_{\vec{k}}}. \tag{4.7}$$

In the case of discrete modes, it is understood that $B_{-\vec{k}}$ is defined to be $B_{\vec{k}}$ as the corresponding $\mathfrak{g}_{\vec{k}}$ were taken to be real. Define the state $|0\rangle_0$ by

$$B_{\vec{k}}|0\rangle_0=0. \tag{4.8}$$

We will also impose that it is annihilated by $\pi_{\vec{k}}$ for each zero mode $\vec{k}$, but that will not be relevant now.

The one-loop correction to the soliton mass, Q_1, is the eigenvalue of H_2'

$$H_2'|0\rangle_0=Q_1|0\rangle_0. \tag{4.9}$$

Therefore we are only interested in those terms in H_2' which do not annihilate $|0\rangle_0$. We have already seen in eq. (3.14) that any such term must be a scalar, and so there cannot be any $B^\ddagger B^\ddagger$ terms. This leaves terms of the form $BB^\ddagger$. We can simplify these using (4.8) which implies the identity

$$B_{\vec{k}_1} B^\ddagger_{\vec{k}_2}|0\rangle_0 = [B_{\vec{k}_1}, B^\ddagger_{\vec{k}_2}]|0\rangle_0 = (2\pi)^d \delta^d(\vec{k}_1 - \vec{k}_2)|0\rangle_0. \tag{4.10}$$

Our strategy will therefore be to calculate Q_1 by isolating all $B^\ddagger B|0\rangle_0$ terms $H_2'|0\rangle_0$ and applying the identity (4.10) to simplify them. We will obtain these terms by plugging the Bogoliubov transform[1]

$$\begin{aligned} A^\ddagger_{\vec{p}} &= \frac{1}{2} \sum\!\!\!\!\!\!\int dk \frac{\tilde{\mathfrak{g}}_{\vec{k}}(-\vec{p})}{\omega_{\vec{p}}} \left[(\omega_{\vec{p}} + \omega_{\vec{k}}) B^\ddagger_{\vec{k}} + (\omega_{\vec{p}} - \omega_{\vec{k}}) \frac{B_{-\vec{k}}}{2\omega_{\vec{k}}}\right] \\ \frac{A_{-\vec{p}}}{2\omega_{\vec{p}}} &= \frac{1}{2} \sum\!\!\!\!\!\!\int dk \frac{\tilde{\mathfrak{g}}_{\vec{k}}(-\vec{p})}{\omega_{\vec{p}}} \left[(\omega_{\vec{p}} - \omega_{\vec{k}}) B^\ddagger_{\vec{k}} + (\omega_{\vec{p}} + \omega_{\vec{k}}) \frac{B_{-\vec{k}}}{2\omega_{\vec{k}}}\right] \end{aligned} \tag{4.11}$$

into eq. (4.6).

Only two combinations of ladder operators do not annihilate the kink ground state, namely $B^\ddagger B^\ddagger$ and $BB^\ddagger$. The $B^\ddagger B^\ddagger$ terms in A and $B + C$ are

$$A \supset -\frac{1}{2} \sum\!\!\!\!\!\!\int dk \omega^2_{\vec{k}} B^\ddagger_{\vec{k}} B^\ddagger_{-\vec{k}}, \qquad B + C \supset \frac{1}{2} \sum\!\!\!\!\!\!\int dk \omega^2_{\vec{k}} B^\ddagger_{\vec{k}} B^\ddagger_{-\vec{k}}. \tag{4.12}$$

Therefore $H_2' = A + B + C$ contains no $B^\ddagger B^\ddagger$ terms, and only $BB^\ddagger$ terms remain.

Restricting our attention to terms proportional to $BB^\ddagger$, in the case of the π^2 term, one finds

$$\begin{aligned} A|0\rangle_0 = \frac{1}{8} \int \frac{d^d\vec{p}}{(2\pi)^d} \sum\!\!\!\!\!\!\int dk_1 \sum\!\!\!\!\!\!\int dk_2 \frac{\tilde{\mathfrak{g}}_{\vec{k}_1}(-\vec{p})}{\omega_{\vec{p}}} \frac{\tilde{\mathfrak{g}}_{\vec{k}_2}(\vec{p})}{\omega_{\vec{p}}} \frac{\omega^2_{\vec{p}}}{2\omega_{\vec{k}_1}} \Big[&-(\omega_{\vec{p}} + \omega_{\vec{k}_2})(\omega_{\vec{p}} - \omega_{\vec{k}_1}) \\ +2(\omega_{\vec{p}} - \omega_{\vec{k}_2})(\omega_{\vec{p}} - \omega_{\vec{k}_1}) - (\omega_{\vec{p}} - \omega_{\vec{k}_2})(\omega_{\vec{p}} + \omega_{\vec{k}_1}) \Big] & B_{-\vec{k}_1} B^\ddagger_{\vec{k}_2}|0\rangle_0. \end{aligned} \tag{4.13}$$

The identity (4.10) then yields

$$A|0\rangle_0 = \frac{1}{4} \int \frac{d^d\vec{p}}{(2\pi)^d} \sum\!\!\!\!\!\!\int dk \tilde{\mathfrak{g}}_{\vec{k}}(-\vec{p}) \tilde{\mathfrak{g}}_{-\vec{k}}(\vec{p}) (\omega_{\vec{k}} - \omega_{\vec{p}})|0\rangle_0. \tag{4.14}$$

Similarly

$$\begin{aligned} (B + C)|0\rangle_0 = \frac{1}{8} \sum\!\!\!\!\!\!\int dk \int \frac{d^d\vec{p}_1}{(2\pi)^d} \int \frac{d^d\vec{p}_2}{(2\pi)^d} \sum\!\!\!\!\!\!\int dk' \omega^2_{\vec{k}} \tilde{\mathfrak{g}}_{-\vec{k}}(\vec{p}_1) \tilde{\mathfrak{g}}_{\vec{k}}(\vec{p}_2) \tilde{\mathfrak{g}}_{\vec{k}'}(-\vec{p}_1) \tilde{\mathfrak{g}}_{-\vec{k}'}(-\vec{p}_2) \\ \times \left[\frac{2}{\omega_{\vec{k}'}} - \frac{\omega_{\vec{p}_1} + \omega_{\vec{p}_2}}{\omega_{\vec{p}_1}\omega_{\vec{p}_2}}\right] |0\rangle_0 = (D + E)|0\rangle_0 \end{aligned} \tag{4.15}$$

where D and E correspond to the first and second terms in the square bracket. In the case of the term D, we perform the $\vec{p}_2$ integral using the completeness relation, which yields a $(2\pi)^d \delta^d(\vec{k} - \vec{k}')$, which is used to perform the k' integration. We thus find

$$D = \frac{1}{4} \sum\!\!\!\!\!\!\int dk \int \frac{d^d\vec{p}}{(2\pi)^d} \omega_{\vec{k}} \tilde{\mathfrak{g}}_{-\vec{k}}(\vec{p}) \tilde{\mathfrak{g}}_{\vec{k}}(-\vec{p}). \tag{4.16}$$

[1] This is derived from eqs. (4.2), (4.5) and (4.7).

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At this point the $\vec{p}$ integral could be performed, yielding an infinite answer. This is to be expected, only the sum of all terms in H_2' is finite and the integral should not be performed until the sum is taken.

The term E may be evaluated by performing the $\vec{k}'$ integral, which yields a $(2\pi)^d\delta^d(\vec{p}_1-\vec{p}_2)$, which in turn is used to perform the $\vec{p}_2$ integral. One finds

$$E = -\frac{1}{4}\sumint dk \int \frac{d^d\vec{p}}{(2\pi)^d}\tilde{\mathfrak{g}}_{-\vec{k}}(\vec{p})\tilde{\mathfrak{g}}_{\vec{k}}(-\vec{p})\frac{\omega_{\vec{k}}^2}{\omega_{\vec{p}}}. \tag{4.17}$$

Adding the scalars D and E to the eigenvalue of A, one finds our main result for the one-loop mass correction

$$Q_1 = -\frac{1}{4}\sumint dk \int \frac{d^d\vec{p}}{(2\pi)^d}\tilde{\mathfrak{g}}_{-\vec{k}}(\vec{p})\tilde{\mathfrak{g}}_{\vec{k}}(-\vec{p})\frac{(\omega_{\vec{k}}-\omega_{\vec{p}})^2}{\omega_{\vec{p}}}. \tag{4.18}$$

This generalizes the famous formula from ref. [5] to solitons in arbitrary dimensions.

One can now write the leading order soliton Hamiltonian as

$$H_2' = Q_1 + \frac{\pi_0^2}{2} + \sumint dk\omega_{\vec{k}}B_{\vec{k}}^\ddagger B_{\vec{k}} \tag{4.19}$$

where π_0 is $\pi_{\vec{k}}$ in the case in which $\omega_{\vec{k}} = 0$. If there are multiple such values of $\vec{k}$, corresponding to zero modes of various classically broken symmetries, then the corresponding π_0 terms should be summed.

4.3 Limitations

Our master formula (4.18) appears very general. We have not even assumed that the theory is renormalizable. However some caution is in order. First of all, one needs to check that the expression for Q_1 is convergent. As we describe below, this is not the case for infinitely extended solitons, but that is not a problem as one instead is interested in densities or tensions in such cases. There may also be ultraviolet divergences if, for example, $\tilde{\mathfrak{g}}_{-\vec{k}}(\vec{p})$ does not fall faster than $\vec{p}^{(3-d)/2}$ when $\vec{k}-\vec{p}$ is held fixed.

Another problem is that Derrick's theorem tells us that there are no localized solitons in the scalar theories that we have considered beyond the 1+1 dimensional case, which was handled already in ref. [5]. In practice this exercise has been useful for settings which are somewhat different. First, one may stabilize the solutions with additional fields, such as gauge fields. In this case the one-loop correction Q_1 will arise, but one must add the corrections arising from the gauge fields. We intend to study such theories in future work. Second, one may consider time-dependent solutions such as oscillons and Q-balls. We have recently shown that the generalization to such cases is feasible. Finally, one may consider extended solutions. For these Q_1 will be infinite, but the mass per volume will be finite, and can be calculated via a straightforward generalization of the argument above. This case will be considered in the next section.

The more serious problem is mass renormalization. We have used the bare mass in the normal ordered Hamiltonian to construct $\mathcal{D}_f$ and so H'. In a theory that requires mass renormalization, this bare mass is infinite. As a result, the factors of ω in (4.18) are all infinite and the argument makes no sense. Of course, we are working at order $O(\lambda^0)$ and such divergences appear at order $O(\lambda)$, so formally in the sense of an asymptotic expansion this is not a problem. Whether it nonetheless leads to physical results in such theories is unclear. We will investigate this problem in future work, in which we will explore higher order corrections with the necessary counterterms introduced, following ref. [7]. We will need to determine, in particular, how $\mathcal{D}_f$ is to be renormalized. This is in fact the open problem posed by Coleman in ref. [1]. Our hope is that the correct handling of the renormalization of $\mathcal{D}_f$ will imply that eliminating divergences in the vacuum sector eliminates them in the soliton secctor, and that (4.18) proves to be the correct one-loop mass correction in all renormalizable theories, as the naive asymptotic expansion suggests.

For now, we note that there are a few theories that are not finite yet do not require mass normalization, to which the above treatment may be applied immediately. We will provide an example in the following section.

We note that at one loop, spectral methods are also available to calculate mass corrections [8] and even form factors [9]. However, these do not generalize in any obvious way to higher loops, whereas in future work we intend to demonstrate that our approach can be extended to higher-loop corrections.

5 Example: ϕ^4 domain wall

Needless to say, Q_1 is divergent for a soliton that extends along an infinite direction. In this case one should calculate not the infinite mass correction, but rather the correction to the tension. Let us first see this divergence in the case of the ϕ^4 double-well model in $2+1$ dimensions. Renormalizability tells us that more generally one may consider a potential which is at most sextic in $\phi(\vec{x})$, and the generalization to this case will be obvious.

5.1 The classical domain wall

Let the spatial directions be x and y. Consider the potential

$$V(\sqrt{\lambda}\phi) = \frac{\lambda\phi^2}{4}\left(\sqrt{\lambda}\phi - \sqrt{2}m\right)^2 \tag{5.1}$$

and the classical solution

$$f(x,y) = \frac{m}{\sqrt{2\lambda}}\left(1 + \tanh\left(\frac{mx}{2}\right)\right). \tag{5.2}$$

The solution is identical to that of the kink in 1+1 dimensions, but now it is infinitely extended in the y direction and we will call it a domain wall [10].

The classical domain wall tension is [11, 12]

$$\rho_0 = \frac{m^3}{3\lambda}. \tag{5.3}$$

This is the same formula as the mass Q_0 of the ϕ^4 kink in 1+1 dimensions. However, one should recall that in $2+1$ dimensions λ has dimensions of mass whereas in $1+1$ dimensions it has dimensions of mass^2.

5.2 Normal modes

The normal modes $\mathfrak{g}_{\vec{k}}(x,y)$ can be factorized

$$\mathfrak{g}_{k_xk_y}(x,y)=\mathfrak{g}_{k_x}(x)e^{-ik_yy} \tag{5.4}$$

where the normal modes in the x direction are those of the ϕ^4 kink, described by the exact solutions of the Poschl-Teller potential

$$\begin{aligned}\mathfrak{g}_k(x)&=\frac{e^{-ikx}}{\omega_k\sqrt{m^2+4k^2}}\left[2k^2-m^2+(3/2)m^2\text{sech}^2(mx/2)-3imk\tanh(mx/2)\right]\\ \mathfrak{g}_S(x)&=\frac{\sqrt{3m}}{2}\tanh(mx/2)\text{sech}(mx/2),\qquad \mathfrak{g}_B(x)=-\sqrt{\frac{3m}{8}}\text{sech}^2(mx/2).\end{aligned} \tag{5.5}$$

Here the indices B and S represent the zero mode and shape mode of the kink in 1+1 dimensions. The corresponding frequencies of the $\mathfrak{g}_{k_xk_y}(x,y)$ are

$$\omega_{Bk_y}=|k_y|,\qquad \omega_{Sk_y}=\sqrt{\frac{3m^2}{4}+k_y^2},\qquad \omega_{k_xk_y}=\sqrt{m^2+k_x^2+k_y^2}. \tag{5.6}$$

There is a single zero mode, corresponding to the case $k_y=0$ of $\mathfrak{g}_{Bk_y}$.

Unlike the 1+1 dimensional case, there is no mass gap, as k_y can be arbitrarily small but positive leading to an arbitrarily small ω_{Bky}. These correspond to long wavelength vibrations of the domain wall x coordinate. At every step it is essential to check that they do not lead to infrared divergences.

As in previous sections, we use the abstract vector notation $\vec{k}$ to represent pairs (k_x,k_y) of continuum modes as well as pairs (B,k_y) and (S,k_y).

The Fourier transform is

$$\tilde{\mathfrak{g}}_{k_xk_y}(p_x,p_y)=\int dx\int dy\mathfrak{g}_{k_x}(x)e^{-i(p_xx+(p_y+k_y)y)}=2\pi\delta(p_y+k_y)\tilde{\mathfrak{g}}_{k_x}(p_x). \tag{5.7}$$

This can be easily substituted into our master formula for the one-loop mass correction (4.18). The mass correction is quadratic in $\tilde{\mathfrak{g}}$, yielding two factors of $\delta(p_y+k_y)$. The first may be used to perform the k_y or p_y integration. The second, leaves an infinity. This, of course, is the expected answer for the mass correction to a domain wall of infinite length.

5.3 The one-loop tension

Can one modify the derivation to obtain the one-loop correction not to the mass, but rather the tension? Recall that in a local quantum field theory, the $\vec{x}$ integration in the Hamiltonian should be performed last. Therefore, ideally, one could write the Hamiltonian as an integral $\int dy$, perform the entire derivation to arrive at the density as a function of y and declare that the answer is the tension. Unfortunately, this method of handling the infrared divergence from the infinite y integration is not compatible with our normal ordering, which is defined in terms of momenta and not positions.

The normal ordering was implemented in eq. (4.6). And so any modification must be made after that point in the derivation. The divergent Dirac δ in $\tilde{\mathfrak{g}}$ entered our derivation through the Bogoliubov transformation in eq. (4.11), which in turn obtained $\tilde{\mathfrak{g}}$ from the field transformation (4.2). This was derived by inverting decompositions (3.7). The inversion required an integral of the field over all y coordinates. This integral is not defined in the present case, leading to the above infrared divergence. Moreover, as a result of the locality of the theory, this integral should be performed after the momentum integrals. We are justified in reordering the integrals only when they satisfy the usual Fubini's Theorem conditions, which is not the case here.

Therefore, each $\tilde{\mathfrak{g}}_{\vec{k}}(\vec{p})$ should in general be expanded following the derivation of eq. (5.7) as

$$\tilde{\mathfrak{g}}_{k_x k_y}(p_x, p_y) = \int dx \int dy \mathfrak{g}_{k_x}(x) e^{-i(p_x x + (p_y + k_y) y)} = \tilde{\mathfrak{g}}_{k_x}(p_x) \int dy e^{-i(p_y + k_y) y}. \tag{5.8}$$

Clearly, this oscillates rapidly when $k_y \neq -p_y$ and so the k_y integration will have support at $-p_y$ and we may safely use one of the Dirac δ functions to perform the k_y integral.

Therefore we conclude that (4.18) becomes

$$\begin{aligned} Q_1 &= -\frac{1}{4} \sumint 2k \int \frac{d^2\vec{p}}{(2\pi)^2} \tilde{\mathfrak{g}}_{-k_x}(p_x) 2\pi\delta(k_y - p_y) \tilde{\mathfrak{g}}_{k_x}(-p_x) \int dy e^{-i(-p_y + k_y) y} \frac{(\omega_{\vec{k}} - \omega_{\vec{p}})^2}{\omega_{\vec{p}}} \\ &= \int dy \left[-\frac{1}{4} \sumint \frac{dk_x}{2\pi} \int \frac{d^2\vec{p}}{(2\pi)^2} \tilde{\mathfrak{g}}_{-k_x}(p_x) \tilde{\mathfrak{g}}_{k_x}(-p_x) \frac{(\omega_{k_x p_y} - \omega_{p_x p_y})^2}{\omega_{p_x p_y}} \right]. \end{aligned} \tag{5.9}$$

Identifying the term in square brackets with the one-loop correction to the domain wall tension $\rho_1(y)$ one obtains

$$Q_1 = \int dy \rho_1(y), \qquad \rho_1(y) = -\frac{1}{4} \sumint \frac{dk_x}{2\pi} \int \frac{d^2\vec{p}}{(2\pi)^2} \tilde{\mathfrak{g}}_{-k_x}(p_x) \tilde{\mathfrak{g}}_{k_x}(-p_x) \frac{(\omega_{k_x p_y} - \omega_{p_x p_y})^2}{\omega_{p_x p_y}}. \tag{5.10}$$

This formula is valid for 2+1 dimensional scalar models with quartic or sextic potentials. We remind the reader that

$$\omega_{p_x p_y} = \sqrt{m^2 + p_x^2 + p_y^2} \tag{5.11}$$

while $\omega_{k_x p_y}$ is given in eq. (5.6). Clearly $\rho_1(y)$ is independent of y, due to the flatness of the wall.

5.4 Numerical one-loop tension correction

In this subsection we will numerically evaluate our expression (5.10) for the tension ρ_1 in the case of the ϕ^4 double-well model. In this case k_x needs to be integrated over all real values, corresponding to continuum modes, plus it should be summed over the discrete zero mode and shape mode. The relevant Fourier transformations have been computed in ref. [6]

$$\begin{aligned} \tilde{\mathfrak{g}}_B(p) &= -\frac{\sqrt{6}\pi p}{m^{3/2}} \operatorname{csch}\left(\frac{\pi p}{m}\right), \qquad \tilde{\mathfrak{g}}_S(p) = -\frac{2i\sqrt{3}\pi p}{m^{3/2}} \operatorname{sech}\left(\frac{\pi p}{m}\right) \\ \tilde{\mathfrak{g}}_k(p) &= \frac{2k^2 - m^2}{\omega_k \sqrt{m^2 + 4k^2}} 2\pi\delta(p + k) + \frac{6\pi p}{\omega_k \sqrt{m^2 + 4k^2}} \operatorname{csch}\left(\frac{\pi(p + k)}{m}\right). \end{aligned} \tag{5.12}$$

We will decompose the tension into contributions from distinct normal modes k_x

$$\begin{aligned}
\rho_1 &= \rho_{1B} + \rho_{1S} + \int \frac{dk_x}{2\pi}\rho_{1k_x} \qquad (5.13)\\
\rho_{1B} &= -\frac{1}{4}\int \frac{d^2\vec{p}}{(2\pi)^2}\tilde{\mathfrak{g}}_B(p_x)\tilde{\mathfrak{g}}_B(-p_x)\frac{\left(|p_y| - \sqrt{m^2+p_x^2+p_y^2}\right)^2}{\sqrt{m^2+p_x^2+p_y^2}}\\
\rho_{1S} &= -\frac{1}{4}\int \frac{d^2\vec{p}}{(2\pi)^2}\tilde{\mathfrak{g}}_S(p_x)\tilde{\mathfrak{g}}_S(-p_x)\frac{\left(\sqrt{\frac{3m^2}{4}+p_y^2} - \sqrt{m^2+p_x^2+p_y^2}\right)^2}{\sqrt{m^2+p_x^2+p_y^2}}\\
\rho_{1k_x} &= -\frac{1}{4}\int \frac{d^2\vec{p}}{(2\pi)^2}\tilde{\mathfrak{g}}_{-k_x}(p_x)\tilde{\mathfrak{g}}_{k_x}(-p_x)\frac{\left(\sqrt{m^2+k_x^2+p_y^2} - \sqrt{m^2+p_x^2+p_y^2}\right)^2}{\sqrt{m^2+p_x^2+p_y^2}}.
\end{aligned}$$

The p_y integrations may be performed analytically using the identity

$$\int \frac{dp_y}{2\pi}\frac{\left(\sqrt{a+p_y^2} - \sqrt{m^2+p_x^2+p_y^2}\right)^2}{\sqrt{m^2+p_x^2+p_y^2}} = \frac{m^2+p_x^2+a\left(\ln\left(\frac{a}{m^2+p_x^2}\right)-1\right)}{2\pi}. \qquad (5.14)$$

In the case of the zero mode, which will turn out to be the dominant contribution, $a=0$ and one easily evaluates all integrals analytically

$$\begin{aligned}
\rho_{1B} &= -\frac{1}{8\pi}\int \frac{dp_x}{2\pi}\left(m^2+p_x^2\right)\tilde{\mathfrak{g}}_B(p_x)\tilde{\mathfrak{g}}_B(-p_x) \qquad (5.15)\\
&= -\frac{3\pi}{4m^3}\int \frac{dp_x}{2\pi}\left(m^2+p_x^2\right)p_x^2\mathrm{csch}^2\left(\frac{\pi p_x}{m}\right)\\
&= -\frac{3}{20\pi}m^2.
\end{aligned}$$

In the case of the shape mode, $a = 3m^2/4$ and so

$$\begin{aligned}
\rho_{1S} &= -\frac{1}{8\pi}\int \frac{dp_x}{2\pi}\left(m^2+p_x^2+\frac{3m^2}{4}\left[\ln\left(\frac{m^2}{m^2+p_x^2}\right)+\ln\left(\frac{3}{4}\right)-1\right]\right)\tilde{\mathfrak{g}}_S(p_x)\tilde{\mathfrak{g}}_S(-p_x)\\
&= -\frac{3\pi}{2m^3}\int \frac{dp_x}{2\pi}\left(m^2+p_x^2+\frac{3m^2}{4}\left[\ln\left(\frac{m^2}{m^2+p_x^2}\right)+\ln\left(\frac{3}{4}\right)-1\right]\right)p_x^2\mathrm{sech}^2\left(\frac{\pi p_x}{m}\right)\\
&= -\frac{27}{160\pi}m^2 + \frac{9\pi}{8m}\int \frac{dp_x}{2\pi}p_x^2\ln\left(\frac{4e(m^2+p_x^2)}{3m^2}\right)\mathrm{sech}^2\left(\frac{\pi p_x}{m}\right). \qquad (5.16)
\end{aligned}$$

In the case of the continuum modes, our identity (5.14) becomes

$$\int \frac{dp_y}{2\pi}\frac{\left(\sqrt{m^2+k_x^2+p_y^2} - \sqrt{m^2+p_x^2+p_y^2}\right)^2}{\sqrt{m^2+p_x^2+p_y^2}} = \frac{p_x^2-k_x^2-(m^2+k_x^2)\ln\left(\frac{m^2+p_x^2}{m^2+k_x^2}\right)}{2\pi}. \qquad (5.17)$$

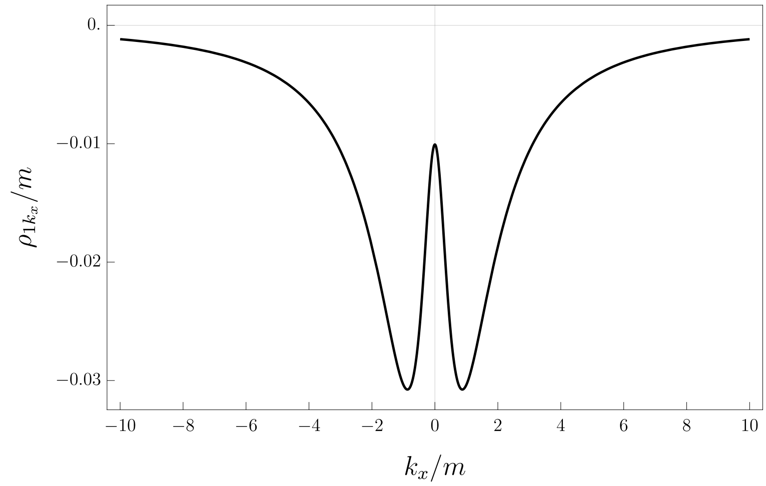

Figure 1. The contribution ρ_{1k_x} to the one loop tension arising from each continuum normal mode k_x. The global minima are obtained at $k_x/m \approx \pm 0.87$ with $\rho_{1k_x} \approx -0.03$.

This leaves

$$\begin{aligned}
\rho_{1k_x} &= -\frac{1}{8\pi}\int \frac{dp_x}{2\pi}\tilde{\mathfrak{g}}_{-k_x}(p_x)\tilde{\mathfrak{g}}_{k_x}(-p_x)\left[p_x^2 - k_x^2 - (m^2+k_x^2)\ln\left(\frac{m^2+p_x^2}{m^2+k_x^2}\right)\right]. \\
&= -\frac{9\pi}{2(m^2+k_x^2)(m^2+4k_x^2)}\int \frac{dp_x}{2\pi}p_x^2\mathrm{csch}^2\left(\frac{\pi(k_x+p_x)}{m}\right) \\
&\quad\times\left[p_x^2 - k_x^2 - (m^2+k_x^2)\ln\left(\frac{m^2+p_x^2}{m^2+k_x^2}\right)\right]
\end{aligned} \tag{5.18}$$

which is plotted in figure 1.

Naively this looks logarithmically divergent. At large k_x, the csch in $\tilde{\mathfrak{g}}_{k_x}(p_x)$ has support at $p_x = -k_x + C$ where C is fixed in this limit. Then the argument of the logarithm in ρ_{1k_x} becomes

$$\frac{p_x^2}{k_x^2} = 1 - \frac{2C}{k_x} \tag{5.19}$$

whose logarithm contributes a $-2C$ to the term in square brackets, canceling the term from $p_x^2 - k_x^2$. Therefore the term in square brackets remains finite in the ultraviolet, while the $\tilde{\mathfrak{g}}^2$ prefactor scales as $1/k^2$. As the p_x integration has fixed support in the support of the csch term, the scaling is that of $\int dk 1/k^2$ which is convergent. Thus we have not yet encountered the ultraviolet divergence in the Hamiltonian density predicted in ref. [1], it will need to wait for the next order.

We note that, as in the case of the kink, the Dirac δ term in $\tilde{\mathfrak{g}}_k$ does not contribute, as it is multiplied by a double zero in the squared frequency difference. Each factor of the Dirac δ vanishes when folded in to the corresponding zero.

Numerically we find

$$\rho_{1B} = -0.0477465m^2, \qquad \rho_{1S} = -0.0072502m^2, \qquad \int \frac{dk_x}{2\pi}\rho_{1k_x} = -0.03156m^2. \tag{5.20}$$

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In all $\rho_1 = -0.08656m^2$. Similarly to the case of the kink mass in 1+1 dimensions [6], the largest contribution arises from the zero mode, followed by the continuum modes, and last the shape mode. However, in the case of the domain wall, the contribution from the continuum modes and zero mode differ by less than a factor of two, in contrast with the factor of eight in the case of the kink.

This tension has been computed previously in refs. [13, 14] using spectral methods. Our result agrees with that of ref. [13], obtained using spectral methods, considering that our definitions of m and λ are twice the definitions there. It does not agree with that of ref. [14], obtained using the zeta function regularization methods of ref. [15]. As noted in ref. [13], this discrepancy arises because the bound modes have not been included in the zeta function approach. This is potentially dangerous, as ref. [15] has enjoyed a resurgence in popularity in the last decade as it has been applied repeatedly to false vacuum decay in an interesting series of papers by the Munich [16] and Ljubljana [17] groups.

5.5 Excited states

The spectrum of excited states of the domain wall is now obvious. Begin with the ground state $|0\rangle_0$ which is annihilated by all B operators, and the zero mode $\pi_{\vec{k}}$ with $k_x = k_y = 0$. At order $O(\lambda^0)$ excited states are created by acting with $B^\ddagger$. Each $B^\ddagger_{\vec{k}}$ increases the energy by $\omega_{\vec{k}}$.

Unlike the case of the kink, some of these have degenerate energy while not being related by any symmetry. For example, if $k_y \geq m$ then $B^\ddagger_{Bk_y}|0\rangle_0$, which describes physical vibrations of the domain wall in the x direction, has the same energy as a continuum state with the same frequency. However the former has more y-momentum, which is separately conserved, and so this state does not unbind from the wall and escape into the bulk. The same argument applies to states $B^\ddagger_{Sk_y}|0\rangle_0$. It does not apply to multiple excitations of bound states, as in some cases these have both degenerate energy and also degenerate momentum with bulk states. However, they will mix only at order $O(\lambda)$, and so their decay to bulk modes will be slow. This is similar to the case of the doubly-excited shape mode of the kink [18].

6 Remarks

A word of caution is in order. As there is no mass gap, in general one expects various infrared divergences. In particular, states of interest will generally have infinite numbers of excitations of $B^\ddagger_{Bk_y}$ with small values of k_y. While these do not appear to lead to any divergences in the $O(\lambda^0)$ study presented here, infrared divergences may be expected at higher orders. Indeed, the domain wall worldsheet theory is a 1+1 dimensional field theory with a massless scalar, which leads to various infinite matrix elements [19]. These will need to be treated as they arise in the computations of observables.

This is not to say that the state eliminated by all B operators is not a Hamiltonian eigenstate, it is the ground state of the leading order soliton Hamiltonian H'_2. However, consider the following argument. Eq. (3.14) shows that each mode k is described by a Harmonic oscillator with frequency squared equal to ω^2, which, except in the ultrarelativistic case, is of order $O(m^2)$ in the case of the kink. Now, consider an incoming meson. The leading interaction H'_3, in the presence of an incoming kink that is not ultrarelativistic, leads to another quadratic term in the field with coefficient of order $O(\sqrt{\lambda}m)$. In the semiclassical approximation this is much smaller than $O(m^2)$ and so the amplitude for an interaction is small and perturbation theory is valid.

What about the case of the domain wall? Now, eq. (3.14) tells us that the frequency squared coefficient of $\phi^2_{Bk_y}$ is equal to k_y^2. When $|k_y|$ is small enough, this is of the same order as the $O(\sqrt{\lambda}m)$ interaction contribution. Therefore, the probability of the oscillator at each such $\vec{k}$ being excited is of order unity. In general, one then expects infinitely many excitations, taking the state out of the Fock space of finite meson-number excitations. Of course this is the usual situation in the presence of massless particles [20, 21], in recent times, at least in four or more dimensions, being attributed to a memory effect [22–24]. It is not a pathology, but rather an interesting part of the physics to which we hope to turn in the near future when we extend the present study to include interactions H'_3.

Our next step will be to proceed to the next order, where a loop diagram leads to a divergence in the meson propagator and a two-loop diagram leads to a divergence in the tension of the domain wall, just the divergence noted by Coleman. In the vacuum sector, these divergences are well-known [25, 26] and they can treated with standard counterterms. We will need to formulate the appropriate renormalization conditions in the Schrodinger picture, choose a displacement operator, and check that H'_1 continues to vanish and also that the ultraviolet divergences are removed in the soliton sector. The key step of course will be finding a displacement operator with these two properties, if it exists.

If this step is successful, then one can proceed to calculate quantum corrections to systems of phenomenological interest. The urgent need for such corrections in the case of models of nuclei has recently been highlighted in refs. [27–29]. Recently, there has even been progress in understanding quantum corrections to Q-balls [30]. A successful treatment of ultraviolet divergences will also allow us to treat fermions, allowing us to study fermion-soliton scattering [31–33]. Besides allowing for more dimensions and richer field content, once we are able to tame ultraviolet divergences in the soliton sector, we may also approach models with nontrivial kinetic terms, such as the modified dilaton, allowing an application to the kinks in refs. [34–36].

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