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Scaling limit of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model

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Abstract

The work contains a detailed study of the scaling limit of a certain critical, integrable inhomogeneous six-vertex model subject to twisted boundary conditions. It is based on a numerical analysis of the Bethe ansatz equations as well as the powerful analytic technique of the ODE/IQFT correspondence. The results indicate that the critical behaviour of the lattice system is described by the gauged SL(2) WZW model with certain boundary and reality conditions imposed on the fields. Our proposal revises and extends the conjectured relation between the lattice system and the Euclidean black hole non-linear sigma model that was made in the 2011 paper of Ikhlef, Jacobsen and Saleur.

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1. Introduction

The seminal work of Polyakov on the $O(n)$ models [1] opened an era in the study of quantum Non-Linear Sigma Models (NLSM) in $1+1$ dimensions. Among their most prominent physical applications is the description of the universality class of phase transitions in disordered electronic systems [2–4]. Taking inspiration from the AdS/CFT correspondence [5,6], an interesting proposal was made in ref. [7] for the NLSM that would describe the transition between the plateaus in the quantum Hall effect in a $2D$ disordered electron gas. One of the basic principles for identifying the target space background, in the author's own words, was the following

"*In trying to solve the statistical physics problem at hand, we have to be very discriminating about which functional integral to accept as well-defined and which to not. In concrete terms, we are looking for a field theory defined over* ***Euclidean*** *two-space, and with a target space of* ***Euclidean*** *signature. This constraint eliminates candidate theories with an action functional that is bounded neither from below nor from above. Among these are the above supergroups, the natural supergeometry of which is non-Riemann, or of indefinite signature. (The natural geometry is forced on us by symmetry considerations.)*"

The requirement of Euclidean signature for the target space, which is closely related to the unitarity of the model, is well motivated from the technical point of view. However, the original heuristic treatment of the problem relied on a fermionic version of the replica trick, leading to the Pruisken model – a G/H NLSM where $G=U(2n)$ is gauged by $H=U(n)\times U(n)$ with $n=0$ [3]. In light of this the above requirement may seem as too severe.

As was explained in [7] the Pruisken model shares the same infra-red behaviour as a certain one dimensional spin chain, whose degrees of freedom take values in an alternating sequence of modules V and V^* for the super Lie algebra $\mathfrak{gl}(2,2)$. This super spin chain turns out not to be integrable in the Yang-Baxter sense, and there has been little progress towards its solution. Nevertheless, interest was prompted into studying integrable critical "alternating" spin chains [8–15]. Perhaps the most remarkable output of this study was the conjecture formulated in ref. [11].

The proposal of Ikhlef, Jacobsen and Saleur concerns a critical spin chain, belonging to the integrability class of a $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model, which is a special case of the lattice system introduced by Baxter in 1971 [17]. They present highly non-trivial arguments, including numerical evidence, that the infra-red behaviour of the spin chain is governed by the so-called Euclidean black hole NLSM [18–27]. However their proposal raises an immediate question. For a spin chain of finite length, the energy spectrum is complex so that there does not exist any positive definite inner product w.r.t. which the spin chain Hamiltonian is Hermitian. On the other hand the Euclidean black hole NLSM is a unitary CFT [20,28]. Of course one could argue that unitarity is restored in the scaling limit of the non-unitary lattice model. The same argument can be employed to explain why the infra-red fixed point of the non-unitary Pruisken model is controlled by an NLSM with a Riemannian target space manifold. However, if one were not to simply brush aside this issue, it could be taken as a signal that the conjecture from ref. [11] is not quite correct. An interesting alternative would be that the scaling behaviour of the spin chain is still described by a NLSM, but with a non-Riemannian target space. This would open a way of assigning a meaning to a quantum NLSM whose target space metric has a Lorentzian type signature. Apart from Condensed Matter Physics applications, that would be of interest for understanding the physics of black holes [20,21]. This work was motivated by such an exciting possibility.

Our study essentially employs the Yang-Baxter integrable structures of the lattice system. Due to the heavy amount of technical details involved, we moved the part of the work that considers the formal algebraic aspects of the general inhomogeneous Baxter model to a separate publication [29]. Some formulae from that paper, which are directly relevant to the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model are collected, for the reader's convenience, in the Preliminaries section of this work.

The key tool in our analysis of the scaling limit is the ODE/IQFT correspondence. The first part of the paper serves to illustrate the technique for the critical homogeneous six-vertex model. No original results are contained therein. It gives us an opportunity to explain the ODE/IQFT approach [30–36] and to set-up the notation. Moreover, following the recent paper [37], we discuss the Hermitian structures consistent with the integrable one for the homogeneous six-vertex model. Then the link is explained between these Hermitian structures and those that they induce in the scaling limit. Our elaboration of this example would be important for the conceptual understanding of the non-unitarity issue for the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model.

Part II contains the main results of this paper. Using the ODE/IQFT correspondence we identify the algebra of extended conformal symmetry and describe the linear and Hermitian structures of the space of states occurring in the scaling limit of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model. The final Part III is devoted to a discussion of the CFT underlying the critical behaviour of the lattice model. In particular, we put forward a modified version of the conjecture of Ikhlef, Jacobsen and Saleur. A list of the central results of this work is given in the Summary section.

2. Preliminaries

In this work we follow the conventions and use the results of [29], which discusses some general aspects of the inhomogeneous six vertex model. Here, for the convenience of the reader, we collect some basic formulae from that paper.

Let σ_m^A $(A = \pm, z)$ be the standard Pauli matrices acting on the m-th factor of the tensor product

$$\mathscr{V}_N = \mathbb{C}_N^2 \otimes \mathbb{C}_{N-1}^2 \otimes \cdots \otimes \mathbb{C}_1^2 \, . \tag{2.1}$$

Introduce the monodromy matrix

$$\boldsymbol{M}(\zeta) = q^{-\frac{N}{2}} \, \boldsymbol{R}_N\big(q\zeta/\eta_N\big) \, \boldsymbol{R}_{N-1}\big(q\zeta/\eta_{N-1}\big) \cdots \boldsymbol{R}_1\big(q\zeta/\eta_1\big) \, , \tag{2.2}$$

where the N complex numbers $\{\eta_J\}_{J=1}^N$ parameterize the inhomogeneities, while $\boldsymbol{R}_m$ stands for the 2×2 matrix

$$\boldsymbol{R}_m(q\zeta) = \begin{pmatrix} q^{\frac{1}{2}(1+\sigma_m^z)} + q^{\frac{1}{2}(1-\sigma_m^z)}\,\zeta & -(q-q^{-1})\,q\,\zeta\,\sigma_m^- \\ (q-q^{-1})\,\sigma_m^+ & q^{\frac{1}{2}(1-\sigma_m^z)} + q^{\frac{1}{2}(1+\sigma_m^z)}\,\zeta \end{pmatrix} , \tag{2.3}$$

whose entries act in the $\mathbb{C}_m^2$ factor in the tensor product (2.1). We'll be considering twisted boundary conditions parameterized by ω. Then the transfer matrix for the inhomogeneous six vertex model on the square lattice with N columns is given by the trace

$$\mathbb{T}(\zeta) = \mathrm{Tr}\big[\,\omega^{\sigma^z}\,\boldsymbol{M}(\zeta)\,\big]\,. \tag{2.4}$$

The transfer matrix satisfies a number of operator valued relations. The latter involve the matrices $\mathbb{A}_\pm(\zeta)$ which together with $\mathbb{T}(\zeta)$ form a commuting family

$$[\mathbb{A}_\pm(\zeta), \mathbb{A}_\mp(\zeta')] = [\mathbb{A}_\pm(\zeta), \mathbb{A}_\pm(\zeta')] = [\mathbb{A}_\pm(\zeta), \mathbb{T}(\zeta')] = 0\,. \tag{2.5}$$

The construction of $\mathbb{A}_\pm(\zeta)$ along with their properties may be found in sec. 3 of [29]. Here we just mention that

$$\mathbb{T}(\zeta)\,\mathbb{A}_\pm(\zeta) = \omega^{\pm1}\,q^{\pm\mathbb{S}^z}\,f(q^{-1}\zeta)\,\mathbb{A}_\pm(q^2\zeta) + \omega^{\mp1}\,q^{\mp\mathbb{S}^z}\,f(q^{+1}\zeta)\,\mathbb{A}_\pm(q^{-2}\zeta)\,, \tag{2.6}$$

where $f(\zeta)$ is given by

$$f(\zeta) = \prod_{J=1}^{N}(1+\zeta/\eta_J) \tag{2.7}$$

and $\mathbb{S}^z$ stands for the z projection of the total spin operator,

$$\mathbb{S}^z = \tfrac{1}{2}\sum_m \sigma_m^z\ :\quad [\mathbb{S}^z,\,\mathbb{A}_\pm(\zeta)] = [\mathbb{S}^z,\,\mathbb{T}(\zeta)] = 0\,. \tag{2.8}$$

It follows from the definition (2.2) - (2.4) that the matrix elements of $\mathbb{T}(\zeta)$ are polynomials of order N in the variable ζ. Due to the mutual commutativity, $[\mathbb{T}(\zeta),\,\mathbb{T}(\zeta')] = 0$, the eigenvectors of the transfer matrix do not depend on this variable and hence its eigenvalues are also N-th order polynomials in ζ. It turns out that the eigenvalues of $\mathbb{A}_\pm(\zeta)$, which will be denoted as $A_\pm(\zeta)$ below, are polynomials of order $N/2 \mp S^z$, respectively, where S^z denotes the eigenvalue of $\mathbb{S}^z$. Let $\{\zeta_m\}_{m=1}^M$ with $M = N/2 - S^z$ be the set of roots of $A_+(\zeta)$. For generic values of the parameters, none of the ζ_m are equal to zero, and it will be convenient to choose the normalization convention for $\mathbb{A}_+$ such that

$$A_+(\zeta) = \prod_{m=1}^{M}(1-\zeta/\zeta_m)\,, \qquad M = \tfrac{1}{2}\,N - S^z\,. \tag{2.9}$$

Applying both sides of the relation (2.6) to a common eigenvector and setting $\zeta = \zeta_m$, yields the system of algebraic equations [17,38]

$$\prod_{J=1}^{N}\frac{\eta_J + q\,\zeta_m}{q\,\eta_J + \zeta_m} = -\,\omega^2\prod_{j=1}^{M}\frac{q^{-1}\,\zeta_j - q^{+1}\,\zeta_m}{q^{+1}\,\zeta_j - q^{-1}\,\zeta_m} \qquad (m = 1, 2, \ldots, M) \tag{2.10}$$

for the set of zeroes of $A_+(\zeta)$. Having a solution of the above equations, the eigenvalue of the transfer matrix is given by

$$\begin{aligned} T^{(N)}(\zeta) = {}& \omega^{+1}\,q^{+S^z}\,\Big(\prod_{J=1}^{N}\big(1+q^{-1}\,\zeta/\eta_J\big)\Big)\prod_{j=1}^{M}\frac{\zeta_j - q^{+2}\,\zeta}{\zeta_j-\zeta} \\ & + \omega^{-1}\,q^{-S^z}\,\Big(\prod_{J=1}^{N}\big(1+q^{+1}\,\zeta/\eta_J\big)\Big)\prod_{j=1}^{M}\frac{\zeta_j - q^{-2}\,\zeta}{\zeta_j-\zeta}\,. \end{aligned} \tag{2.11}$$

Of course, there are similar formulae involving the roots of $A_-(\zeta)$. However, we will mainly focus on $\mathbb{A}_+(\zeta)$ for the following reason. It will be assumed that the inhomogeneities satisfy the constraints

$$\eta_{N+1-J} = \eta_J^{-1} \qquad (J = 1, 2, \ldots, N) \,. \tag{2.12}$$

In this case the model possesses the so-called global $\mathcal{CP}$ invariance (the explicit formula for the generators $\hat{\mathcal{C}}$ and $\hat{\mathcal{P}}$ are quoted in eqs. (17.46) and (17.67), respectively). The $\mathcal{CP}$ transformation intertwines the sectors with S^z and $-S^z$. Moreover, it relates the operators $\mathbb{A}_+(\zeta)$ and $\mathbb{A}_-(\zeta)$ as

$$\mathbb{A}_-(\zeta) = \zeta^{\frac{N}{2}-\mathbb{S}^z}\, \hat{\mathcal{C}}\hat{\mathcal{P}}\, \mathbb{A}_+\big(\zeta^{-1}\big)\, \hat{\mathcal{C}}\hat{\mathcal{P}}\, \mathbb{A}_+^{(\infty)} \,, \tag{2.13}$$

where

$$\mathbb{A}_+^{(\infty)} = \lim_{\zeta\to\infty} \zeta^{-\frac{1}{2}N+\mathbb{S}^z}\, \mathbb{A}_+(\zeta) \,. \tag{2.14}$$

Therefore, for the diagonalization problem of the commuting family (2.5), it is sufficient to consider $\mathbb{A}_+$ and focus on the sector $S^z \geq 0$. Thus, in the Bethe ansatz equations (2.10) we will always assume

$$M \leq \tfrac{1}{2}\, N \,. \tag{2.15}$$

Note that combining (2.13) with the operator relation (2.6) one finds

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\, \mathbb{T}(\zeta)\, \hat{\mathcal{C}}\hat{\mathcal{P}} = \zeta^N\, \mathbb{T}\big(\zeta^{-1}\big) \,. \tag{2.16}$$

The eigenvectors can be constructed within the framework of the algebraic Bethe ansatz [39]. To this end, introduce the following notation for the entries of the monodromy matrix

$$\boldsymbol{M}(\zeta) = \begin{pmatrix} \hat{\mathsf{A}}(\zeta) & a(\zeta)\, \hat{\mathsf{B}}(\zeta) \\ d(\zeta)\, \hat{\mathsf{C}}(\zeta) & \hat{\mathsf{D}}(\zeta) \end{pmatrix} , \tag{2.17}$$

where $\hat{\mathsf{A}}$, $\hat{\mathsf{B}}$, $\hat{\mathsf{C}}$, $\hat{\mathsf{D}}$ are operators acting in (2.1), while $a(\zeta)$, $d(\zeta)$ stand for the polynomials

$$a(\zeta) = -\mathrm{i}\,\omega^{+1}\, q^{+\frac{N+1}{2}} \prod_{J=1}^{N} \big(1 + q^{-1}\zeta/\eta_J\big) , \qquad d(\zeta) = +\mathrm{i}\,\omega^{-1}\, q^{-\frac{N+1}{2}} \prod_{J=1}^{N} \big(1 + q\,\zeta/\eta_J\big) \,. \tag{2.18}$$

Let $\boldsymbol{\Psi}^{(0)} \in \mathbb{C}_N^2 \otimes \mathbb{C}_{N-1}^2 \otimes \cdots \otimes \mathbb{C}_1^2$ be the pseudovacuum

$$\boldsymbol{\Psi}^{(0)} = \underbrace{|\uparrow\rangle \otimes |\uparrow\rangle \otimes \ldots \otimes |\uparrow\rangle}_{N} \,. \tag{2.19}$$

Then the state

$$\boldsymbol{\Psi}\big(\{\zeta_j\}\big) = \hat{\mathsf{B}}(\zeta_M) \cdots \hat{\mathsf{B}}(\zeta_2)\, \hat{\mathsf{B}}(\zeta_1)\, \boldsymbol{\Psi}^{(0)} \qquad\qquad (M = \tfrac{1}{2}\, N - S^z) \tag{2.20}$$

is a common eigenstate for the commuting family of operators provided that the set $\{\zeta_j\}_{j=1}^M$ satisfies the Bethe ansatz equations (2.10).

In this work we consider the case where q and ω are unimodular:

$$q^* = q^{-1} , \qquad \omega^* = \omega^{-1} \,. \tag{2.21}$$

If the inhomogeneities satisfying (2.12) are also taken to be unimodular,

$$\eta_J^* = \eta_J^{-1} = \eta_{N+1-J} \tag{2.22}$$

then the system possesses $\mathcal{T}$-invariance. The time reversal transformation is realized as an anti-unitary operator acting on an arbitrary state $\boldsymbol{\Xi} \in \mathscr{V}_N$ as

$$\hat{\mathcal{T}}\, \boldsymbol{\Xi} = \hat{\mathsf{U}}\, \boldsymbol{\Xi}^* \qquad \text{with} \qquad \hat{\mathsf{U}} = \prod_{m=1}^{N} \sigma_m^x \,. \tag{2.23}$$

Similar to the $\mathcal{CP}$ conjugation, it flips the sign of S^z so that $\mathcal{CPT}$ acts invariantly in the sector with given S^z. Moreover, for the state $\boldsymbol{\Psi}$ (2.20) corresponding to the set $\{\zeta_j\}_{j=1}^M$ solving (2.10), one has

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\, \boldsymbol{\Psi}\big(\{\zeta_j\}\big) = \boldsymbol{\Psi}\big(\{\zeta_j^*\}\big) \,. \tag{2.24}$$

The Bethe state in the r.h.s. of the above formula is built using the Bethe roots for the complex conjugated set $\{\zeta_j^*\}_{j=1}^M$, which is also a solution of the Bethe ansatz equations.

When further restrictions are placed on the inhomogeneities, additional global symmetries appear in the model. In particular, suppose that N is divisible by the integer r,

$$N = r L \,, \tag{2.25}$$

and the η_J are taken to satisfy the periodicity condition

$$\eta_{J+r} = \eta_J \qquad (J = 1, 2, \ldots, N)\,, \tag{2.26}$$

where $\eta_{J+N} \equiv \eta_J$. Then one can introduce the lattice translation operator

$$\mathbb{K}\ : \quad \big[\mathbb{K},\, \mathbb{T}(\zeta)\big] = \big[\mathbb{K},\, \mathbb{A}_\pm(\zeta)\big] = 0\,, \qquad \mathbb{K}^L = \mathrm{e}^{2\pi \mathrm{i} \mathtt{k}\, \mathbb{S}^z} \,. \tag{2.27}$$

Its matrix elements read explicitly as

$$\big(\mathbb{K}\big)_{a_N a_{N-1}\ldots a_1}^{b_N b_{N-1}\ldots b_1} = \mathrm{e}^{\mathrm{i}\pi \mathtt{k}\,(a_1 + a_2 + \ldots + a_r)}\, \delta_{a_N}^{b_{N-r}}\, \delta_{a_{N-1}}^{b_{N-r-1}} \ldots \delta_{a_1}^{b_{N-r+1}} \tag{2.28}$$

and its eigenvalue K corresponding to the Bethe state (2.20) is expressed in terms of $A_+(\zeta)$ (2.9) as

$$K = \prod_{\ell=1}^{r} \omega\, q^{-\frac{N}{2} + \mathbb{S}^z}\, \frac{A_+(-q^{+1}\eta_\ell)}{A_+(-q^{-1}\eta_\ell)} \,. \tag{2.29}$$

The transfer matrix and $\mathbb{A}_\pm(\zeta)$ are not Hermitian w.r.t. the standard matrix conjugation, $\hat{\mathsf{O}}^\dagger = (\hat{\mathsf{O}}^*)^T$ with $\hat{\mathsf{O}} \in \mathrm{End}(\mathscr{V}_N)$. Nevertheless it is possible to introduce the Hermitian structure in the 2^N dimensional linear space $\mathscr{V}_N$, which is consistent with the integrable structure of the inhomogeneous six - vertex model. Such Hermitian structures were discussed in the work [29]. A special rôle belongs to the one associated with the conjugation

$$\hat{\mathsf{O}}^\star = \hat{\mathsf{X}}_\star^{-1}\, \hat{\mathsf{O}}^\dagger\, \hat{\mathsf{X}}_\star \,. \tag{2.30}$$

Here $\hat{\mathsf{X}}_\star = \hat{\mathsf{X}}_\star^\dagger$ stands for the matrix

$$\hat{\mathsf{X}}_\star = \hat{\mathsf{X}}\, \mathrm{e}^{\mathrm{i}\pi(\mathbb{S}^z - \frac{N}{2})}\, \mathbb{A}_+^{(\infty)} \tag{2.31}$$

with $\mathbb{A}_+^{(\infty)}$ given in (2.14), while $\hat{\mathsf{X}}$ is defined through the ordered product

$$\hat{\mathsf{X}} = \Bigg(\prod_{J=1}^{N} (\eta_J)^{\frac{1}{2}\sigma_J^z}\Bigg) \overset{\curvearrowright}{\prod_{m=2}^{N}} \Bigg[\overset{\curvearrowright}{\prod_{n=N-m+1}^{N-1}} \check{\boldsymbol{R}}_{n+1,n}\big(\eta_{n+m-N}/\eta_m\big) \Bigg] . \tag{2.32}$$

In the above formula we use the notation

$$\check{\boldsymbol{R}}_{n+1,n}(\zeta) = \frac{1}{q - q^{-1}\zeta}\ \boldsymbol{R}_{n+1,n}(-\zeta)\, \boldsymbol{P}_{n+1,n} , \tag{2.33}$$

where $\boldsymbol{R}_{n+1,n}(-\zeta)$ is the matrix (2.3) acting on the $n+1$-st and n-th components of the tensor product (2.1), while $\boldsymbol{P}_{n+1,n}$ is the permutation matrix that interchanges the two components. Assuming the conditions (2.21), (2.22) it is possible to show that under the $\star$ - conjugation (2.30) the transfer matrix as well as $\mathbb{A}_\pm(\zeta)$ satisfy

$$\big[\mathbb{T}(\zeta)\big]^\star = \mathbb{T}(\zeta^*) , \qquad \big[\mathbb{A}_\pm(\zeta)\big]^\star = \mathbb{A}_\pm(\zeta^*) \tag{2.34}$$

(for details see sec. 5 in ref. [29]).

For the conjugation (2.30) there exists a unique sesquilinear form, which is defined through the relations

$$\big(\Xi_2, \hat{\mathsf{O}}\,\Xi_1\big)_\star = \big(\hat{\mathsf{O}}^\star\,\Xi_2, \Xi_1\big)_\star \qquad (\forall\ \Xi_1, \Xi_2 \in \mathscr{V}_N) \tag{2.35}$$

together with the overall normalization

$$(\boldsymbol{\Psi}^{(0)}, \boldsymbol{\Psi}^{(0)})_\star = 1 , \tag{2.36}$$

where $\boldsymbol{\Psi}^{(0)}$ (2.19) is the pseudovacuum. Then it follows from (2.34) as well as the relations

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\mathbb{T}(\zeta)\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}} = \mathbb{T}(\zeta^*) , \qquad \hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\mathbb{A}_\pm(\zeta)\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}} = \mathbb{A}_\pm(\zeta^*) \tag{2.37}$$

that w.r.t. the sesquilinear form the Bethe states satisfy the orthogonality condition

$$\big(\boldsymbol{\Psi}^{(2)}, \boldsymbol{\Psi}^{(1)}\big)_\star = 0 \quad \text{unless} \quad \boldsymbol{\Psi}^{(2)} = \hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{\Psi}^{(1)} . \tag{2.38}$$

The “norm” of the Bethe state (2.20), in terms of the corresponding set $\{\zeta_m\}$, is given by [40–42]

$$\begin{aligned}\big(\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\Psi}, \boldsymbol{\Psi}\big)_\star = {} & \big(q - q^{-1}\big)^{2M} \prod_{m\neq j}^{M} \frac{q\zeta_j - q^{-1}\zeta_m}{\zeta_m - \zeta_j} \\ & \times \det\Bigg[\delta_{j,m}\Bigg(\kappa(\zeta_j) + \sum_{l=1}^{M} \frac{(q+q^{-1})\,\zeta_j\zeta_l}{(q\zeta_l - q^{-1}\,\zeta_j)(q\zeta_j - q^{-1}\,\zeta_l)}\Bigg) \\ & - \frac{(q+q^{-1})\,\zeta_j\zeta_m}{(q\zeta_m - q^{-1}\,\zeta_j)(q\zeta_j - q^{-1}\,\zeta_m)}\Bigg]\end{aligned} \tag{2.39}$$

with

$$\kappa(\zeta) = -\sum_{J=1}^{N} \frac{\zeta}{\eta_J(1 + q^{-1}\zeta/\eta_J)(1 + q^{+1}\,\zeta/\eta_J)} .$$

Part I. Homogeneous six-vertex model

3. The Hamiltonian

The purpose of this work is the study of the scaling limit of the alternating six-vertex model, where the η_J are fixed to be $\eta_J = \mathrm{i}\,(-1)^{J-1}$. This is a special case of (2.22) and (2.26) (with $r = 2$). However, since many of our considerations are parallel to those for the homogeneous model, where all the $\eta_J = 1$, we'll begin our discussion with this more familiar example. In this case the transfer matrix $\mathbb{T}(\zeta)$ (2.4) and the translation operator $\mathbb{K}$ (2.27), (2.28) ($r = 1$, $L = N$) commute with the spin $\frac{1}{2}$ XXZ Hamiltonian

$$\mathbb{H}_{XXZ} = -\frac{1}{2\sin(\pi\beta^2)} \sum_{i=1}^{N} \Big(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cos(\pi\beta^2)\left(\sigma_i^z \sigma_{i+1}^z - \hat{\mathbf{1}}\right)\Big) \tag{3.1}$$

with

$$\sigma_{N+1}^x \pm \mathrm{i}\sigma_{N+1}^y = \mathrm{e}^{2\pi\mathrm{i}\mathtt{k}} \left(\sigma_1^x \pm \mathrm{i}\sigma_1^y\right), \qquad\qquad \sigma_{N+1}^z = \sigma_1^z\,. \tag{3.2}$$

Here we have parameterized the unimodular numbers q and ω as

$$q = \mathrm{e}^{\mathrm{i}\pi\beta^2}, \qquad\qquad \omega = \mathrm{e}^{\mathrm{i}\pi\mathtt{k}}, \tag{3.3}$$

where β and $\mathtt{k}$ lie in the domains

$$0 < \beta < 1\,, \qquad -\tfrac{1}{2} < \mathtt{k} \le \tfrac{1}{2}\,. \tag{3.4}$$

The eigenvalue of $\mathbb{H}_{XXZ}$ on the state $\boldsymbol{\Psi}$ (2.20) is given in terms of the Bethe roots by

$$\mathcal{E} = -\sum_{m=1}^{M} \frac{4\sin(\pi\beta^2)}{\zeta_m + \zeta_m^{-1} + 2\cos(\pi\beta^2)}\,, \tag{3.5}$$

while for the eigenvalue of $\mathbb{K}$, see eq. (2.29) with $r = 1$ and $\eta_\ell = 1$.

4. RG flow for the Bethe states

The scaling limit is a certain large N limit for a particular class of "low energy" states. The latter are defined w.r.t. a reference state – the vacuum. In the case of the homogeneous six-vertex model the reference state is the lowest energy state of the Hamiltonian (3.1). In turn, the class of states we'll be considering are those whose energy counted from the vacuum energy is sufficiently low. It is well known that, with the parameter β^2 (3.3) lying in the interval $0 < \beta^2 < 1$, the system is critical and as $N \to \infty$ the low energy part of the spectrum organizes into the conformal towers [43]. In a given tower, the eigenvalues of the Hamiltonian (3.1) and the lattice translation operator (2.28) are described by the formulae:

$$\mathcal{E} = e_\infty N + \frac{2\pi\,v_{\mathrm{F}}}{N}\left(P^2 + \bar{P}^2 - \tfrac{1}{12} + \mathtt{L} + \bar{\mathtt{L}}\right) + o\big(N^{-1}\big) \tag{4.1}$$

$$K = \sigma \exp\left(\frac{2\pi\mathrm{i}}{N}\left(P^2 - \bar{P}^2 + \mathtt{L} - \bar{\mathtt{L}}\right)\right).$$

Here e_∞ is the specific bulk energy, while v_{F} is usually referred to as the Fermi velocity and in our conventions for the Hamiltonian (3.1) they read explicitly as

$$e_\infty = -\frac{2v_\mathrm{F}}{\pi}\int_0^\infty \mathrm{d}t\ \frac{\sinh\big(\frac{\beta^2 t}{1-\beta^2}\big)}{\sinh\big(\frac{t}{1-\beta^2}\big)\cosh(t)} \tag{4.2}$$

$$v_\mathrm{F} = \frac{1}{1-\beta^2}\ .$$

Contrary to e_∞ and v_F, which are the same for all the low energy states, the factor $\sigma = \pm 1$ so that the low energy states are splitted into two sectors corresponding to different values of the sign. The pair $(\bar{P}, P)$ labels the different conformal towers, $\mathcal{V}_{\bar{P},P}$, and its admissible values are described by eq. (4.8) below. Each level subspace of the tower, $\mathcal{V}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{P},P}$, is specified by the non-negative integers $\mathtt{L}, \bar{\mathtt{L}} = 0, 1, 2, \ldots$ and has dimensions

$$\dim \mathcal{V}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{P},P} = \mathtt{par}_1(\bar{\mathtt{L}})\,\mathtt{par}_1(\mathtt{L})\,, \tag{4.3}$$

where $\mathtt{par}_1(\mathtt{L})$ and $\mathtt{par}_1(\bar{\mathtt{L}})$ are the number of integer partitions of $\mathtt{L}$ and $\bar{\mathtt{L}}$ respectively. Furthermore, it turns out that

$$\mathcal{V}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{P},P} = \bar{\mathcal{F}}^{(\bar{\mathtt{L}})}_{\bar{P}} \otimes \mathcal{F}^{(\mathtt{L})}_{P} \tag{4.4}$$

with $\mathcal{F}^{(\mathtt{L})}_{P}$ standing for the level subspace of the Fock space $\mathcal{F}_P$:

$$\mathcal{F}^{(\mathtt{L})}_{P} = \mathrm{span}\big\{a_{-n_1}\ldots a_{-n_j}\,|P\rangle\ :\ n_1+\ldots+n_j = \mathtt{L}\,,\ \ \forall n_j > 0\,\big\}\,,\qquad \dim\mathcal{F}^{(\mathtt{L})}_{P} = \mathtt{par}_1(\mathtt{L})\,. \tag{4.5}$$

We'll use the conventions that the Heisenberg algebra generators $\{a_n\}$ obey the commutation relations

$$[a_m, a_n] = \tfrac{m}{2}\,\delta_{m+n,0}\ , \tag{4.6}$$

while the highest weight vector is defined through the conditions

$$a_n\,|P\rangle = 0 \quad (\forall n > 0)\,,\qquad a_0\,|P\rangle = P\,|P\rangle\ . \tag{4.7}$$

The factor $\bar{\mathcal{F}}^{(\bar{\mathtt{L}})}_{\bar{P}}$ in the tensor product in the r.h.s. of eq. (4.4) denotes the level subspace of the highest weight representation of the Heisenberg algebra, generated by the operators $\bar{a}_m$ that commute with $\{a_m\}$ and satisfy the same commutation relations as in (4.6), with highest weight vector $|\bar{P}\rangle$.

The zero-mode momenta P, $\bar{P}$ labeling the conformal tower are not arbitrary, but take a certain discrete set of values. Namely, in the sector characterized by the eigenvalue S^z of the z-component of the total spin operator, they are given by

$$P = \tfrac{1}{2}\big(\beta S^z + \beta^{-1}(\mathtt{k}+\mathtt{w})\big)\,,\qquad \bar{P} = \tfrac{1}{2}\big(\beta S^z - \beta^{-1}(\mathtt{k}+\mathtt{w})\big)\,. \tag{4.8}$$

The integer $\mathtt{w} = 0, \pm1, \pm2, \ldots$ appearing in eq. (4.8) will be referred to below as the winding number. Together with the non-negative integers $\mathtt{L}$ and $\bar{\mathtt{L}}$, the winding number is an important characteristic of the low energy stationary states. In particular, the sign factor σ in the second line of eq. (4.1) coincides with the parity of $\mathtt{w}$, i.e.,

$$\sigma = (-1)^{\mathtt{w}}\ . \tag{4.9}$$

Note that formulae (4.1), (4.8) imply that the eigenvalues of the lattice translation operator $\mathbb{K}$ satisfy $K^N = (-1)^{\mathtt{w}(N-2S^z)}\, \mathrm{e}^{2\pi \mathrm{i}\mathtt{k} S^z}$. Since $N - 2S^z = 2M$ is always even it follows that $\mathbb{K}^N = \mathrm{e}^{2\pi\mathrm{i}\mathtt{k}\,\mathbb{S}^z}$ (see the last equality in (2.27) with $L = N$).

The natural question arises, for a *given* Bethe state $\boldsymbol{\Psi}$ what would be its scaling limit? In other words, what particular state in $\mathcal{V}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{P},P}$ (4.4), (4.8) would appear in the large N limit of $\boldsymbol{\Psi}$ (2.20). In fact, to formulate this question meaningfully, one should first organize the Bethe states for different N into a one parameter family, i.e., define an individual Renormalization Group (RG) trajectory $\boldsymbol{\Psi}_N$. This procedure could only make sense for the low energy part of the spectrum as the Hilbert spaces $\mathscr{V}_N$ (2.1) are not isomorphic for different lattice sizes.

For a general lattice system it is not clear how to assign the size dependence for individual low energy stationary states. However, in the case under consideration, one can exploit the integrability of the model for the construction of the RG trajectory $\boldsymbol{\Psi}_N$. For this end, first re-write the Bethe ansatz equations (2.10) with all the $\eta_J = 1$ in the logarithmic form:

$$L p(\zeta_m) = 2\pi\mathtt{k} - 2\pi\, I_m - \sum_{j=1}^{M} \Theta(\zeta_m, \zeta_j)\ , \tag{4.10}$$

where

$$p(\zeta) = -\mathrm{i}\,\log\left(\frac{1+q\,\zeta}{q+\zeta}\right), \qquad \Theta(\zeta,\zeta') = -\mathrm{i}\,\log\left(\frac{q\,\zeta' - q^{-1}\zeta}{q\,\zeta - q^{-1}\zeta'}\right) \tag{4.11}$$

and I_m are the so-called Bethe numbers which are integers or half-integers for M odd or even respectively. An unambiguous definition of I_m requires fixing the branches of the logarithms in (4.11). Although this is an important step in any practical calculation, we will not touch on it here and only mention that

$$I_m^{(\mathrm{vac})} = -\tfrac{1}{2}\,(M+1) + m \qquad \big(m = 1, \ldots, M = \tfrac{1}{2}\,N - S^z\big) \tag{4.12}$$

for the vacuum state in the sector with fixed value of S^z. For sufficiently large N the Bethe numbers corresponding to the low energy states are given by $I_m^{(\mathrm{vac})} + \delta I_m$, where the variation δI_m from the "vacuum" distribution (4.12) are nonzero only in the vicinity of the edges, i.e., for $m \ll M$ or $M - m \ll M$. The set $\{\delta I_m\}$ can be used to define the individual RG flow trajectories $\boldsymbol{\Psi}_N$ in the following way.

Starting with a spin chain for relatively small N one performs the numerical diagonalization of the Hamiltonian. Together with the energies (3.5), one should also compute the eigenvalues of $\mathbb{A}_+(\zeta)$. The explicit construction of the $2^N \times 2^N$ matrix $\mathbb{A}_+(\zeta)$ can be found in sec. 3 of [29]. Its eigenvalues are polynomials whose zeroes coincide with the corresponding Bethe roots (see eq. (2.9)). This allows one to extract the set $\{\zeta_m\}_{m=1}^M$ for a particular Bethe state $\boldsymbol{\Psi}_N$ and, using (4.10), also the set of $\{\delta I_m\}$. For the state $\boldsymbol{\Psi}_{N+2}$, the Bethe ansatz equations are taken to have the same $\{\delta I_m\}$ in the vicinity of the edges. Moreover, for their iterative solution the initial approximation can be constructed using the Bethe roots for $\boldsymbol{\Psi}_N$. This procedure provides a way for defining the RG flow of an individual low energy Bethe state. Having at hand the RG trajectory $\boldsymbol{\Psi}_N$ and taking its large N limit, our previous discussion means that

$$\boldsymbol{\Psi}_N \asymp \Omega_N\ \bar{\boldsymbol{\psi}}_{\bar{P}}(\bar{\boldsymbol{v}}) \otimes \boldsymbol{\psi}_P(\boldsymbol{v}) \qquad \text{as} \qquad N \to \infty\ . \tag{4.13}$$

Here the limiting state $\bar{\boldsymbol{\psi}}_{\bar{P}}(\bar{\boldsymbol{v}}) \otimes \boldsymbol{\psi}_P(\boldsymbol{v})$ does not depend on N and belongs to the subspace $\mathcal{V}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{P},P}$ (4.4), while the constant Ω_N (in fact a functional, $\Omega_N = \Omega[\boldsymbol{\Psi}_N]$, whose value depends on the Bethe state) diverges in the large N limit and will be discussed in details later. In the next subsection we'll describe the state $\bar{\boldsymbol{\psi}}_{\bar{P}}(\bar{\boldsymbol{v}}) \otimes \boldsymbol{\psi}_P(\boldsymbol{v})$ in (4.13). It is sufficient to focus on the "right" vector $\boldsymbol{\psi}_P(\boldsymbol{v})$ since there is only a notational difference between the left and right component in the tensor product.

5. Identification of the RG trajectory with a state in the conformal tower

5.1. The sum rules for the scaled Bethe roots

For finite N the Bethe states are unambiguously characterized by the eigenvalues of $\mathbb{A}_+(\zeta)$. Therefore the chiral state $\boldsymbol{\psi}_P(\boldsymbol{v})$ may be determined through the study of the large N behaviour of the eigenvalue (2.9) corresponding to $\boldsymbol{\Psi}_N$. The scaling limit for $A_+(\zeta)$ was discussed in the series of papers [30–32,35–37]. Below we present the results relevant to this work.

Let $\boldsymbol{\Psi}_N$ be the RG trajectory formed by the low energy Bethe states, whose energy and momentum are described by the asymptotic formula (4.1), and consider the eigenvalue $A_+(\zeta)$ (2.9) computed on this family. Its logarithm can be expanded in the infinite series,

$$\log A_+(\zeta) = -\sum_{j=1}^{\infty} h_j^{(N)}\,\zeta^j\,, \tag{5.1}$$

where the coefficients are given by the finite sums

$$h_j^{(N)} = j^{-1} \sum_{m=1}^{\frac{N}{2}-S^z} (\zeta_m)^{-j}\,. \tag{5.2}$$

Keeping $S^z \geq 0$ fixed, consider the large N limit of $h_j^{(N)}$ with given $j = 1, 2, 3, \ldots$. Despite that the r.h.s of eq. (5.2) is a symmetric function of the Bethe roots, for analysing this limit it is useful to impose an ordering for the set $\{\zeta_m\}$. For the case of the vacuum states all ζ_m are real and, with the Bethe numbers given by eq. (4.12), they are ordered as $\zeta_1 < \zeta_2 < \ldots < \zeta_M$. In general, the Bethe roots are complex numbers and we can order them w.r.t. to their real part

$$\Re e(\zeta_1) \leq \Re e(\zeta_2) \leq \ldots \leq \Re e(\zeta_M)$$

(the ordering prescription for the Bethe roots with coinciding real parts is not essential for our purposes). As was discussed in the work [37] for fixed S^z and $m = 1, 2, \ldots$ the following limits exist:

$$s_m = \lim_{N\to\infty} \big(\tfrac{N}{\pi}\big)^{2(1-\beta^2)}\,\zeta_m\,. \tag{5.3}$$

Furthermore the numbers s_m grow according to

$$(s_m)^{\frac{1}{2(1-\beta^2)}} = m + O(1) \qquad \text{as} \qquad m \to +\infty\,. \tag{5.4}$$

For $0 < \beta^2 < \frac{1}{2}$ the above formulae imply the existence of the limit

$$h_j^{(\infty)} = \lim_{N\to\infty} N^{-2j(1-\beta^2)}\,h_j^{(N)} = \pi^{-2j(1-\beta^2)}\;j^{-1}\sum_{m=1}^{\infty}(s_m)^{-j} \tag{5.5}$$

with fixed $j = 1, 2, \ldots$. Combining (5.1) and (5.5) one arrives at

$$\lim_{N\to\infty} \log A_+\big(N^{-2(1-\beta^2)}\,\tilde{\zeta}\,\big) = -\sum_{j=1}^{\infty} h_j^{(\infty)}\,\tilde{\zeta}^{\,j}\,, \tag{5.6}$$

where the r.h.s. is understood as a formal power series expansion in $\tilde{\zeta}$ without any reference to its convergence. It turns out there exists a mutually commuting set of operators $\mathbf{H}_j^{(+)}$ that act in the Fock space $\mathcal{F}_P$ and whose eigenvalues for a certain common eigenvector $\boldsymbol{\psi}_P(\boldsymbol{v}) \in \mathcal{F}_P^{(\mathrm{L})}$ coincide with $h_j^{(\infty)}$ up to an overall multiplicative factor. The construction of these operators was discussed in the works [31,32] and goes along the following line.

Consider the chiral Bose field

$$\varphi(u) = \varphi_0 + a_0\,u + \mathrm{i}\sum_{m\neq 0} \frac{a_m}{m}\,\mathrm{e}^{-\mathrm{i}mu}\,, \tag{5.7}$$

where a_m are the Heisenberg generators satisfying (4.6) and the additional operator φ_0 obeys the commutation relations

$$[\varphi_0, a_m] = \tfrac{\mathrm{i}}{2}\,\delta_{m,0}\,. \tag{5.8}$$

Introduce the path-ordered exponent

$$\boldsymbol{L}_\pm(\lambda) = \mathrm{e}^{\pm\mathrm{i}\pi\beta a_0\mathcal{H}}\,\overleftarrow{\mathcal{P}}\exp\bigg(\int_0^{2\pi}\mathrm{d}u\,\Big(V_-(u)\,q^{\pm\frac{\mathcal{H}}{2}}\,\mathcal{E}_\pm + \lambda^2\,V_+(u)\,q^{\mp\frac{\mathcal{H}}{2}}\,\mathcal{E}_\mp\Big)\bigg) \tag{5.9}$$

involving the vertex operators

$$V_\pm(u) = \mathrm{e}^{\pm 2\mathrm{i}\beta\varphi}(u) \tag{5.10}$$

as well as the generators of the q-oscillator algebra $\mathcal{E}_\pm$ and $\mathcal{H}$:

$$[\mathcal{H}, \mathcal{E}_\pm] = \pm 2\,\mathcal{E}_\pm\,, \qquad q\,\mathcal{E}_+\mathcal{E}_- - q^{-1}\,\mathcal{E}_-\mathcal{E}_+ = \frac{1}{q-q^{-1}}\,. \tag{5.11}$$

Let $\rho_\pm$ be representations of this algebra such that the traces

$$\mathrm{Tr}_{\rho_\pm}\big[\mathrm{e}^{\pm 2\mathrm{i}\pi\beta P\mathcal{H}}\big] \neq 0 \qquad \text{with} \qquad \Im m(P) < 0 \tag{5.12}$$

exist and are non-vanishing. Then one may introduce the operators $\mathbb{A}_\pm(\lambda)$ as

$$\mathbb{A}_\pm(\lambda) = \frac{\mathrm{Tr}_{\rho_\pm}\big[\,\mathrm{e}^{\pm\mathrm{i}\pi\beta a_0\mathcal{H}}\boldsymbol{L}_\pm(\lambda)\,\big]}{\mathrm{Tr}_{\rho_\pm}\big[\,\mathrm{e}^{\pm 2\mathrm{i}\pi\beta a_0\mathcal{H}}\,\big]}\,. \tag{5.13}$$

Formula (5.13) defines a power series in λ^2. Since $\mathbb{A}_\pm(0) = \mathbf{1}$, its logarithm obeys the formal power series expansion

$$\log\mathbb{A}_\pm(\lambda) = -\sum_{j=1}^{\infty}\mathbf{H}_j^{(\pm)}\,\lambda^{2j}\,. \tag{5.14}$$

Each of the coefficients $\mathbf{H}_j^{(\pm)}$ is expressed in terms of ordered integrals over the vertex operators (5.10). A simple analysis gives that for $0<\beta^2<\frac{1}{2}$ all these integrals converge and each term in the power series expansion is well-defined. It is possible to show [32] that $\mathbf{H}_j^{(\pm)}$ act invariantly in the level subspaces $\mathcal{F}_P^{(\mathrm{L})}$ and mutually commute,

$$\mathbf{H}_j^{(\pm)}\ :\ \mathcal{F}_P^{(\mathtt{L})}\mapsto\mathcal{F}_P^{(\mathtt{L})}\,,\qquad \big[\,\mathbf{H}_j^{(\pm)},\,\mathbf{H}_{j'}^{(\pm)}\,\big]=\big[\,\mathbf{H}_j^{(\pm)},\,\mathbf{H}_{j'}^{(\mp)}\,\big]=0\,. \tag{5.15}$$

Note that, although to take the trace in eq. (5.13) it is required that $\Im m(P)<0$, the matrix elements of $\mathbf{H}_j^{(\pm)}$ restricted to $\mathcal{F}_P^{(\mathtt{L})}$ may be analytically continued to any complex P, except for

$$P=\mp\tfrac{1}{2}\left(m\beta^{-1}+j\beta\right),\qquad \text{where}\qquad m=0,1,2,\ldots\,. \tag{5.16}$$

The simultaneous diagonalization of the mutually commuting operators $\mathbf{H}_j^{(\pm)}$, being restricted to $\mathcal{F}_P^{(\mathtt{L})}$, becomes a diagonalization problem of finite $\mathtt{par}_1(\mathtt{L})\times\mathtt{par}_1(\mathtt{L})$ dimensional matrices. Then for an RG trajectory $\boldsymbol{\Psi}_N$ with given P, $\mathtt{L}$ and characterized by the set $\big\{h_j^{(N)}\big\}_{j=1}^{\infty}$, there exists a common eigenvector $\boldsymbol{\psi}_P(\boldsymbol{v})\in\mathcal{F}_P^{(\mathtt{L})}$ such that the eigenvalues of the operators $\mathbf{H}_j^{(+)}$,

$$\mathbf{H}_j^{(+)}\,\boldsymbol{\psi}_P(\boldsymbol{v})=H_j(\boldsymbol{v})\,\boldsymbol{\psi}_P(\boldsymbol{v})\,, \tag{5.17}$$

are related to $h_j^{(\infty)}$ (5.5) as

$$h_j^{(\infty)}=\left[\,\Gamma(1-\beta^2)\left(\frac{\sqrt{\pi}\,\Gamma\big(\frac{\beta^2}{2-2\beta^2}\big)}{\Gamma\big(\frac{1}{2-2\beta^2}\big)}\right)^{1-\beta^2}\,\right]^{-2j}\ H_j(\boldsymbol{v})\,. \tag{5.18}$$

In writing the above it is assumed that the exponential operators (5.10) are normalized in the following way

$$q\,\mathrm{e}^{\pm2\mathrm{i}\beta\varphi}(u_1)\,\mathrm{e}^{\mp2\mathrm{i}\beta\varphi}(u_2)\big|_{(u_1-u_2)\to0^+}\to(u_1-u_2)^{-2\beta^2}>0\,, \tag{5.19}$$

i.e., we set the coefficient of the most singular term in the operator product expansion to be one. It is clear that the eigenvector $\boldsymbol{\psi}_P(\boldsymbol{v})$ should be identified with the chiral state, which appears in the r.h.s. of (4.13). A similar analysis can be repeated to specify the barred state $\bar{\boldsymbol{\psi}}_{\bar{P}}(\bar{\boldsymbol{v}})\in\bar{\mathcal{F}}_{\bar{P}}^{(\bar{\mathtt{L}})}$ in that relation.

The l.h.s. of (5.18) is an infinite sum over inverse powers of the scaled Bethe roots (5.3). Since the eigenvalues $H_j(\boldsymbol{v})$ may be calculated independently using the definition (5.13), (5.14), the relation (5.18) provides a set of sum rules for s_m. It is instructive to consider the explicit formulae for $H_j(\boldsymbol{v})$ corresponding to the Fock vacuum $|P\rangle$. As explained in [31] they are expressed in terms of the $2m$ - fold integrals

$$\begin{aligned}Q_m(h,g)=&\int_0^{2\pi}\mathrm{d}u_1\int_0^{u_1}\mathrm{d}v_1\int_0^{v_1}\mathrm{d}u_2\int_0^{u_2}\mathrm{d}v_2\,\ldots\int_0^{v_{m-1}}\mathrm{d}u_m\int_0^{u_m}\mathrm{d}v_m\\ &\times\prod_{j>i}^{m}\left[\Big(4\sin\big(\tfrac{u_i-u_j}{2}\big)\,\sin\big(\tfrac{v_i-v_j}{2}\big)\Big)^{2g}\right]\ \prod_{j\geq i}^{m}\Big(2\sin\big(\tfrac{u_i-v_j}{2}\big)\Big)^{-2g}\\ &\times\prod_{j>i}^{m}\Big(2\sin\big(\tfrac{v_i-u_j}{2}\big)\Big)^{-2g}\ 2\cos\Big(2h\,\big(\pi+\sum_{i=1}^{m}(v_i-u_i)\big)\Big)\end{aligned} \tag{5.20}$$

with $m\leq j$. For instance

$$H_1^{(\mathrm{vac})} = \frac{Q_1(\beta P, \beta^2)}{4\sin(\pi\beta^2)\,\sin\big(\pi\beta(\beta+2P)\big)} \tag{5.21}$$

$$H_2^{(\mathrm{vac})} = \frac{Q_2(\beta P, \beta^2)}{4\sin(2\pi\beta^2)\sin\big(2\pi\beta(\beta+P)\big)} + \frac{\cos\big(2\pi\beta(\beta+P)\big)\big(Q_1(\beta P,\beta^2)\big)^2}{16\sin(2\pi\beta^2)\,\sin^2\big(\pi\beta(\beta+2P)\big)\,\sin\big(2\pi\beta(\beta+P)\big)}$$

Taking a brief look at eq. (5.20), one finds that the integrals converge as $0 < g < \frac{1}{2}$ and, in this parametric domain, $Q_m(h,g) = Q_m(-h,g)$ are entire functions of h^2. However the multifold integrals (5.20) are not well suited for numerical purposes. In the Appendix of ref. [44] a technique is developed which brings $H_j^{(\mathrm{vac})}$ to a form that is convenient for computation. Following that work, introduce the functions

$$f_1(h,g) = \frac{\pi\,\Gamma(1-2g)\,\Gamma(g+2h)}{\sin(\pi g)\,\Gamma(1-g+2h)} \tag{5.22}$$

$$f_2(h,g) = 2^{1-4g}\,\frac{\Gamma^2(1-g)}{\Gamma^2(\frac{1}{2}+g)}\,\frac{\Gamma(2g+2h)}{\Gamma(1-2g+2h)}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{2\pi}\,\frac{S_1(x)}{x+\mathrm{i}h} \qquad \big(0<g<\tfrac{1}{2},\ \Re e(h)>0\big)\,,$$

where

$$S_1(x) = \sinh(2\pi x)\Gamma(1-2g+2\mathrm{i}x)\Gamma(1-2g-2\mathrm{i}x)\,\big(\Gamma(g+2\mathrm{i}x)\Gamma(g-2\mathrm{i}x)\big)^2\,. \tag{5.23}$$

Then for the first two eigenvalues, one has

$$H_j^{(\mathrm{vac})} = f_j\big(\beta P, \beta^2\big)\,. \tag{5.24}$$

Notice that, although the ordered integral $Q_1(h,g)$ converges only for $0<g<\frac{1}{2}$, the expression (5.22), (5.24) gives an analytic continuation of $H_1^{(\mathrm{vac})}$ to the domain $\frac{1}{2}<\beta^2<1$, which possesses a simple pole at $\beta^2=\frac{1}{2}$. The analytic continuation of $f_2(h,g)$ yields [44]

$$f_2(h,g) = 2^{1-4g}\,\frac{\Gamma^2(1-g)}{\Gamma^2(\frac{1}{2}+g)}\,\frac{\Gamma(2g+2h)}{\Gamma(1-2g+2h)}\left(\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{2\pi}\,\frac{S_1(x)}{x+\mathrm{i}h} - \frac{\sin(2\pi g)\Gamma(3-4g)\Gamma^2(1-g)\Gamma^2(3g-1)}{(2h+1-2g)(2h-1+2g)}\right) \qquad \big(\tfrac{1}{2}<g<1,\ \Re e(h)>0\big)\,. \tag{5.25}$$

The above expression shows that $H_2^{(\mathrm{vac})}$ possesses a simple pole at $\beta^2=\frac{3}{4}$. Similarly, by means of analytical continuation in β^2, the functions $H_j^{(\mathrm{vac})}$ with given $j=1,2,3,\ldots$ may be defined for any $0<\beta^2<1$ except $\beta^2 = 1-\frac{1}{2j}$.

A natural question arises, is it possible to extend the definition, not only of the vacuum eigenvalues $H_j^{(\mathrm{vac})}$, but of the operators $\mathbf{H}_j^{(\pm)}$ themselves to the domain $0<\beta^2<1$? Let us reiterate that for given $j=1,2,\ldots$ formulae (5.9), (5.13) and (5.14) define $\mathbf{H}_j^{(\pm)}$ in terms of the ordered integrals over the vertex operators, which can be taken literally for $0<\beta^2<\frac{1}{2}$ only. As discussed in the work [32] (see also [46]), by re-expressing the ordered integrals in terms of contour

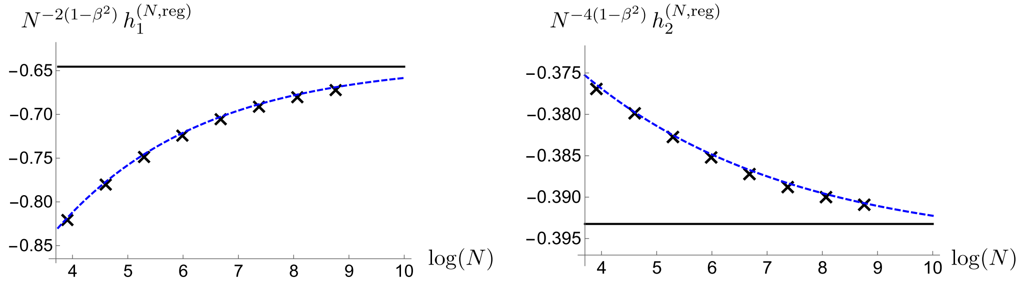

Fig. 1. The crosses come from numerical data that was obtained from the solution of the Bethe ansatz equations with $N = 50, 100, 200, \ldots$ corresponding to an RG trajectory $\boldsymbol{\Psi}_N$. The latter is characterized by $\mathtt{L} = \bar{\mathtt{L}} = 0$ and P, $\bar{P}$ given by eq. (4.8) with $S^z = 0$, $\mathtt{w} = 1$. The parameters are taken to be $\beta^2 = \frac{9}{10}$ and $\mathtt{k} = -\frac{1}{20}$. Notice that, since β^2 is close to one, it is necessary to perform the subtraction $h_j^{(N,\mathrm{reg})} \equiv h_j^{(N)} + \frac{(-1)^{j+1} N}{2j\cos(\pi j\beta^2)}$ as in eq. (5.27), so that $N^{-2j(1-\beta^2)} h_j^{(N,\mathrm{reg})}$ with $j = 1, 2$ tends to a finite number in the large N limit. The solid line represents the limiting value given by the r.h.s. of eq. (5.18), where $H_1^{(\mathrm{vac})}$, $H_2^{(\mathrm{vac})}$ were computed using (5.24) with f_1 as in (5.22) and f_2 from (5.25). The blue dashed line comes from fitting the numerical data and was included for visualization.

integrals, it is possible to extend the applicability of these formulae to any $0 < \beta^2 < 1$ except the points $\beta^2 = 1 - \frac{1}{2k}$ with $k = 1, 2, \ldots$. At $\beta^2 = 1 - \frac{1}{2k}$ all the operators $\mathbf{H}_j^{(\pm)}$ with $j \neq k$ remain non-singular. However $\mathbf{H}_k^{(\pm)}$ possesses a simple pole, whose residue is proportional to the identity operator:

$$\mathbf{H}_k^{(\pm)} = -\frac{\Gamma(\frac{1}{2}+k)\,\Gamma^{2k}(\frac{1}{2k})}{\sqrt{\pi}\,(2k-1)k\,\Gamma(1+k)}\;\frac{\mathbf{1}}{\beta^2 - 1 + \frac{1}{2k}} + O(1) \qquad (k = 1, 2, 3, \ldots)\,. \tag{5.26}$$

By subtracting the singular term in (5.26) from $\mathbf{H}_k^{(\pm)}$ one may introduce the regularized operator $\mathbf{H}_k^{(\pm,\mathrm{reg})}$. The latter is defined up to an overall additive constant, which should be fixed by imposing some normalization condition.

In the parametric domain $\frac{1}{2} \leq \beta^2 < 1$ both (5.3) and (5.4) continue to hold. However the large N limit (5.5) involving the coefficients $h_j^{(N)}$ (5.2) no longer exists when $1 \leq j \leq \frac{1}{2(1-\beta^2)}$. To properly define $h_j^{(\infty)}$, a certain subtraction needs to be made from $N^{-2j(1-\beta^2)}\, h_j^{(N)}$ so that its large N limit can be taken. Namely,

$$h_j^{(\infty)} = \lim_{N\to\infty} N^{-2j(1-\beta^2)} \left[h_j^{(N)} + \frac{(-1)^{j+1}\, N}{2j\cos(\pi j\beta^2)} \right], \qquad j = 1, 2, \ldots < \frac{1}{2(1-\beta^2)}\,. \tag{5.27}$$

Without going into details, we just mention that the existence of the above limit follows from the Bethe ansatz equations (2.10) with $\eta_J = 1$. When $\beta^2 = 1 - \frac{1}{2k}$ with $k = 1, 2, \ldots$, not only $h_j^{(N)}$ with $j = 1, \ldots, k-1$ but also $h_k^{(N)}$ requires regularization:

$$h_k^{(\infty)} = \lim_{N\to\infty} \left[N^{-1}\, h_k^{(N)} - \frac{1}{\pi k}\log(N B_k) \right] \qquad \left(\beta^2 = 1 - \tfrac{1}{2k}\right), \tag{5.28}$$

where B_k is an arbitrary (k - dependent) constant.

The validity of the relation (5.18) may be extended to the domain $0 < \beta^2 < 1$ provided that for $\beta^2 > \frac{1}{2}$ the coefficients $h_j^{(\infty)}$ are defined as in (5.27), while the eigenvalues $H_j(\boldsymbol{v})$ are understood via analytic continuation. Note that in the case when $\beta^2 = 1 - \frac{1}{2k}$, eq. (5.18) with $j = k$ becomes a relation between $h_k^{(\infty)}$ (5.28) and the eigenvalues of the regularized operator $\mathbf{H}_k^{(+,\mathrm{reg})}$. The arbitrary constant B_k in (5.28) is related to the ambiguity in the definition of $\mathbf{H}_k^{(+,\mathrm{reg})}$ discussed above. In what follows we define the regularized operator as

$$\mathbf{H}_k^{(\pm,\mathrm{reg})} = \lim_{\beta^2 \to 1-\frac{1}{2k}} \left[\mathbf{H}_k^{(\pm)} + \frac{\Gamma(\frac{1}{2}+k)\,\Gamma^{2k}(\frac{1}{2k})}{\sqrt{\pi}\,(2k-1)k\,\Gamma(1+k)}\ \frac{1}{\beta^2 - 1 + \frac{1}{2k}} \right]. \tag{5.29}$$

Then it is not difficult to show that

$$B_1 = \frac{\mathrm{e}^{\gamma_{\mathrm{E}}}}{\pi}\ , \tag{5.30}$$

where γ_{E} stands for the Euler constant. The analytical expression for B_k with $k \geq 2$ is not known. Numerical calculations yield

$$\log(B_2/B_1) = 3.57079634\ . \tag{5.31}$$

Also, in Fig. 1 some numerical data illustrating (5.18), (5.27) is presented.

5.2. Scaling limit of $A_+(\zeta)$

The set of “scaled” Bethe roots $\{s_m\}_{m=1}^{\infty}$ (5.3) admits a remarkable interpretation. Namely, following the works [36,37], consider the Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) - E \right) \Phi = 0 \tag{5.32}$$

with the so-called Monster potentials of the form

$$V(x) = \frac{16(\alpha+1)P^2 - 1}{4x^2} + x^{2\alpha} - 2\,\frac{\mathrm{d}^2}{\mathrm{d}x^2} \sum_{b=1}^{\mathtt{L}} \log\left(x^{2\alpha+2} - \tfrac{\alpha+1}{\alpha}\, v_b \right). \tag{5.33}$$

Here the set of complex numbers $\{v_a\}_{a=1}^{\mathtt{L}}$ obeys the system of $\mathtt{L}$ algebraic equations:

$$\sum_{b\neq a} \frac{v_a\,(v_a^2 + (3+\alpha)(1+2\alpha)v_a v_b + \alpha(1+2\alpha)\,v_b^2)}{(v_a - v_b)^3} - \frac{v_a}{4} + P^2 - \frac{\alpha^2}{4(\alpha+1)} = 0\ . \tag{5.34}$$

With these constraints imposed on the positions of the singularities any solution of the Schrödinger equation is monodromy free everywhere except for $x = 0$ and $x = \infty$ for any value of E. In other words the solutions remain single-valued in the vicinity of each singularity specified by v_a. For this reason the complex numbers $\{v_a\}$ are referred to as apparent singularities. Assuming that $\alpha > 0$, one can consider the standard spectral problem for the ODE defined on the ray $x > 0$. This leads to a discrete spectral set $\{E_m\}_{m=1}^{\infty}$. Then for an RG trajectory with given P and $\mathtt{L}$ and characterized by the set of scaled Bethe numbers $\{s_m\}_{m=1}^{\infty}$ (5.3), there exists a Monster potential of the form (5.33), (5.34) such that

$$E_m = \left(N_0/\pi\right)^{-2(1-\beta^2)}\, s_m\ . \tag{5.35}$$

Here

$$N_0 = \frac{\sqrt{\pi}\,\Gamma\big(1+\frac{\beta^2}{2-2\beta^2}\big)}{2\Gamma\big(\frac{3}{2}+\frac{\beta^2}{2-2\beta^2}\big)} \tag{5.36}$$

and the parameter α is related to β as

$$\alpha = \beta^{-2} - 1 > 0\,. \tag{5.37}$$

It was conjectured in the work [36] and proven by Conti and Masoero [45] that for generic (complex) values of P and α the number of distinct, up to the action of the symmetric group S_N, solutions of (5.34) coincides with $\mathtt{par}_1(\mathtt{L})$. In other words, for given $\mathtt{L}$, the number of Monster potentials is equal to the number of states in the level subspace $\mathcal{F}_P^{(\mathtt{L})}$. This allows one to label the chiral state $\boldsymbol{\psi}_P(\boldsymbol{v})$ entering in the scaling limit (4.13) by the unordered set of solutions of the system (5.34),

$$\boldsymbol{\psi}_P(\boldsymbol{v}) \quad : \quad \boldsymbol{v} = \{v_a\}_{a=1}^{\mathtt{L}}\,. \tag{5.38}$$

Introduce the spectral determinant

$$D_+(E\,|\,\boldsymbol{v}) = \prod_{m=1}^{\infty}\bigg(1-\frac{E}{E_m}\bigg)\,. \tag{5.39}$$

Due to (5.35), (5.4) $E_m \sim m^{2(1-\beta^2)}$ as $m\to\infty$, so that the infinite product in the r.h.s. converges when $0<\beta^2<\frac{1}{2}$ for any value of E. Hence in this domain (5.39) defines an entire function. Expanding $\log D_+(E)$ in an infinite series, it follows from eqs. (5.5) and (5.35) that the resulting series expansion coincides term by term with the r.h.s. of (5.6) where $\tilde{\zeta} = N_0^{2(1-\beta^2)}\,E$. This immediately yields that the series in (5.6) has a non-vanishing radius of convergence and

$$\lim_{N\to\infty} A_+\big((N/N_0)^{2(\beta^2-1)}E\big) = D_+(E\,|\,\boldsymbol{v}) \qquad\qquad (0<\beta^2<\tfrac{1}{2})\,. \tag{5.40}$$

To introduce $D_+(E\,|\,\boldsymbol{v})$ for $\frac{1}{2}\le\beta^2<1$, Weierstrass factors must be included in the infinite product (5.39) in order to ensure its convergence. Alternatively one can define the spectral determinant for $0<\beta^2<1$ through the set of conditions

(i) $D_+(E\,|\,\boldsymbol{v})$ is an entire function whose zeroes coincide with $\{E_m\}_{m=1}^{\infty}$.
(ii) $D_+(E\,|\,\boldsymbol{v})$ satisfies the normalization condition:

$$D_+(0\,|\,\boldsymbol{v}) = 1\,. \tag{5.41}$$

(iii) It possesses the asymptotic behaviour as $E\to\infty$, $|\arg(-E)|<\pi$:

$$\log D_+(E\,|\,\boldsymbol{v}) \asymp \begin{cases} \dfrac{N_0}{\cos\big(\frac{\pi\beta^2}{2-2\beta^2}\big)}\,(-E)^{\frac{1}{2-2\beta^2}} + o\big(E\big) & \text{for } \beta^2\neq 1-\frac{1}{2k} \\[2ex] \dfrac{\Gamma(\frac{1}{2}+k)}{2\sqrt{\pi}\,\Gamma(1+k)}\,E^k\,\big(\log(-E)+c_k\big)+o\big(E\big) & \text{for } \beta^2 = 1-\frac{1}{2k} \end{cases} \tag{5.42}$$

with $k=1,2,3,\ldots$ and

$$c_k = \psi\Big(\frac{1}{2}+k\Big) - \psi(1+k) - \frac{1}{k} + \frac{1}{k}\bigg[\,\log\Big(\frac{2k-1}{4k}\Big) - \frac{1}{2k-1} - \psi\Big(\frac{1}{2k}\Big)\bigg]\,. \tag{5.43}$$

The function $D_+(E\,|\,\boldsymbol{v})$, thus defined, coincides with the eigenvalue of the operator $\mathcal{a}_+(\lambda)$ for the vector $\boldsymbol{\psi}_P(\boldsymbol{v})$ (5.38)

$$\mathcal{a}_+(\lambda)\,\boldsymbol{\psi}_P(\boldsymbol{v}) = D_+(E\,|\,\boldsymbol{v})\,\boldsymbol{\psi}_P(\boldsymbol{v})\ . \tag{5.44}$$

As it follows from eqs. (5.6), (5.14), (5.18), (5.40) E and λ are related as

$$\lambda^2 = \frac{(\beta^2/2)^{2-2\beta^2}}{\Gamma^2(1-\beta^2)}\ E\ . \tag{5.45}$$

The above conditions (i) – (iii) fully specify $D_+(E\,|\,\boldsymbol{v})$, i.e., all the eigenvalues of $\mathcal{a}_+(\lambda)$. Notice that with the choice of the constant c_k as in (5.43), for $\beta^2 = 1-\frac{1}{2k}$ with $k=1,2,3,\dots$ the operator $\mathcal{a}_+(\lambda)$ is defined as

$$\begin{aligned}\mathcal{a}_+\big|_{\beta^2=1-\frac{1}{2k}} &= \lim_{\substack{\beta^2\to 1-\frac{1}{2k}\\ E-\text{fixed}}} \mathcal{a}_+\big(\lambda(E)\big)\ \exp\bigg[-\frac{\Gamma(\frac{1}{2}+k)\,\Gamma^{2k}(\frac{1}{2k})}{\sqrt{\pi}\,(2k-1)k\,\Gamma(1+k)}\ \frac{\big(\lambda(E)\big)^{2k}}{\beta^2-1+\frac{1}{2k}}\bigg] \\ &= \exp\bigg(-\mathbf{H}_k^{(+,\mathrm{reg})}\,\lambda^{2k} - \sum_{j\neq k}\mathbf{H}_j^{(+)}\,\lambda^{2j}\bigg)\,,\end{aligned} \tag{5.46}$$

where $\mathbf{H}_k^{(+,\mathrm{reg})}$ is given by eq. (5.29).

Now we can describe the scaling limit of the eigenvalue $A_+(\zeta)$ corresponding to the RG trajectory $\boldsymbol{\Psi}_N$ for any $0<\beta^2<1$. Namely,

$$\lim_{N\to\infty} G^{(N)}\big(E\,|\,\beta^2\big)\ A_+\big((N/N_0)^{2(\beta^2-1)}E\big) = D_+(E\,|\,\boldsymbol{v}) \qquad \big(0<\beta^2<1\big) \tag{5.47}$$

with

$$G^{(N)}(E\,|\,g) = \begin{cases} \exp\bigg(\displaystyle\sum_{m=1}^{[\frac{1}{2(1-g)}]} \frac{(-1)^m\,N}{2m\cos(\pi m g)}\,(N/N_0)^{2m(g-1)}\ E^m\bigg) \\ \quad \text{for}\quad g\neq 1-\frac{1}{2k} \\ \exp\bigg(\dfrac{N_0E^k}{\pi k}\ \log(NB_k) + \displaystyle\sum_{m=1}^{k-1}\frac{N}{2m\cos(\frac{\pi m}{2k})}\ (N/N_0)^{-\frac{m}{k}}\ E^m\bigg) \\ \quad \text{for}\quad g = 1-\frac{1}{2k}\end{cases} \tag{5.48}$$

Here $k=1,2,\dots$ and [...] stands for the integer part.

The operator $\mathcal{a}_-(\lambda)$ (5.13) appears in the scaling limit of $\mathbb{A}_-$, which was briefly mentioned in the Preliminaries (for a further discussion, including its definition, see sec. 3 of [29]). Similar to (5.44), its eigenvalues are related to the spectral determinant $D_-(E)$ corresponding to another spectral problem for the same Schrödinger equation.

There is an efficient way of computing $D_\pm(E)$. To describe it, first introduce two solutions of (5.32), (5.33) satisfying the asymptotic condition

$$\Phi_{\pm P}(x) \to \frac{1}{\sqrt{\pi}}\,(\beta^2/2)^{\frac{1}{2}\pm 2\beta P}\ \Gamma(\mp 2\beta P)\,x^{\frac{1}{2}\pm\frac{2}{\beta}P} \qquad \text{as}\qquad x\to 0 \qquad \big(0<\Re e(2P)<\beta\big) \tag{5.49}$$

where $\beta = (\alpha+1)^{-\frac{1}{2}} > 0$. This unambiguously defines the solutions in the strip in the complex P plane. It turns out that through analytic continuation in P it is possible to introduce the solutions $\Phi_{\pm P}$ for any complex values of P, except for the set $P = \frac{1}{2}\,(m\beta^{-1} + n\beta)$ with m, n integers. Let Ξ be another solution that decays at large positive x according to the asymptotic formula

$$\Xi(x) \asymp x^{-\frac{1}{2\beta^2}(1-\beta^2)} \exp\Big(-\beta^2 x^{\frac{1}{\beta^2}} + o(1)\Big) \qquad \text{as} \qquad x \to +\infty\,. \tag{5.50}$$

This condition specifies $\Xi(x)$ for $0 < \beta^2 < \frac{1}{2}$. In the parametric domain $\frac{1}{2} < \beta^2 < 1$ a more accurate description of the large - x asymptotic is required. Namely, the argument in the exponent in (5.50) should be replaced by $-\beta^2 x^{1/\beta^2} + x^{1/\beta^2} \sum_{m\geq 1} d_m\, x^{2m(1-1/\beta^2)}$, where the coefficients d_m are easily obtained using the standard WKB technique. This way the solution $\Xi(x)$ may be introduced for any $0 < \beta^2 < 1$ except the points $\beta^2 \neq 1 - \frac{1}{2k}$ with $k = 1, 2, 3, \ldots$ by means of the asymptotic condition

$$\Xi(x) \asymp x^{-\frac{1}{2\beta^2}(1-\beta^2)} \exp\Big[-\beta^2 x^{\beta^{-2}}\, {}_2F_1\big(-\tfrac{1}{2}, -\tfrac{1}{2(1-\beta^2)}, 1 - \tfrac{1}{2(1-\beta^2)}\,\big|\, E\, x^{2(1-\beta^{-2})}\big) + o(1)\Big], \tag{5.51}$$

where ${}_2F_1$ is the Gauss hypergeometric function. The case $\beta^2 = 1 - \frac{1}{2k}$ requires further attention and will not be discussed here. It turns out that the spectral determinant $D_+(E\,|\,\boldsymbol{v})$ defined by (i) – (iii) above as well as $D_-(E\,|\,\boldsymbol{v})$, which can be introduced by a similar set of conditions, are given by

$$D_\pm(E\,|\,\boldsymbol{v}) = \mp \sin(2\pi\beta P)\; W[\,\Phi_{\pm P}, \Xi\,] \tag{5.52}$$

with $W[\,\Phi_{\pm P}, \Xi\,] = \Xi\,\partial_x \Phi_{\pm P} - \Phi_{\pm P}\,\partial_x \Xi$ being the Wronskian. Indeed, using basic facts from the analytic theory of differential equations, it is easy to show that (5.52) defines entire functions of E. When $D_+(E\,|\,\boldsymbol{v})$ vanishes, the solutions ψ_+ and χ are linearly dependent, so that $\{E_m\}_{m=1}^{\infty} : D_+(E_m\,|\,\boldsymbol{v}) = 0$, is the spectral set for the corresponding spectral problem. Similarly, the zeroes of $D_-(E\,|\,\boldsymbol{v})$ form the spectral set for the problem, where Ξ becomes proportional to Φ_-. The normalization of the solutions $\Phi_\pm$ (5.49) and the overall factor in (5.52) ensure that $D_\pm(0\,|\,\boldsymbol{v}) = 1$. Using the WKB technique one can check that the functions (5.52) satisfy the large - E asymptotic (5.42). Finally we note

$$\Xi(x) = D_+(E\,|\,\boldsymbol{v})\; \Phi_{-P}(x) + D_-(E\,|\,\boldsymbol{v})\; \Phi_{+P}(x) \tag{5.53}$$

so that $D_\pm(E\,|\,\boldsymbol{v})$ are the connection coefficients in the expansion of Ξ in terms of the fundamental set of solutions $\{\Phi_{\pm P}\}$.

5.3. *Scaling limit of the transfer matrix*

Let $U_q(\mathfrak{sl}_2)$ be the quantum universal enveloping algebra whose generators satisfy the commutation relations

$$[\,\mathtt{h}, \mathtt{e}_\pm\,] = \pm 2\,\mathtt{e}_\pm\,, \qquad [\,\mathtt{e}_+, \mathtt{e}_-\,] = \frac{q^{\mathtt{h}} - q^{-\mathtt{h}}}{q - q^{-1}}\,. \tag{5.54}$$

Following ref. [30] consider the formal path ordered exponent built out of the vertex operators (5.10),

$$\boldsymbol{L}(\lambda) = \lambda^{+\frac{1}{2}\mathtt{h}}\,\mathrm{e}^{\mathrm{i}\pi\beta a_0\mathtt{h}}\,\overleftarrow{\mathcal{P}}\exp\bigg(\int_0^{2\pi}\mathrm{d}u\,\Big(V_-(u)\,q^{+\frac{\mathtt{h}}{2}}\,\mathtt{e}_+ + \lambda^2\,V_+(u)\,q^{-\frac{\mathtt{h}}{2}}\,\mathtt{e}_-\Big)\bigg)\lambda^{-\frac{1}{2}\mathtt{h}}\,. \tag{5.55}$$

In the fundamental representation of $U_q(\mathfrak{sl}_2)$, such that $\pi_{\frac{1}{2}}(\mathtt{e}_\pm) = \sigma^\pm$ and $\pi_{\frac{1}{2}}(\mathtt{h}) = \sigma^z$, $\boldsymbol{L}(\lambda)$ becomes an operator valued 2×2 matrix

$$\boldsymbol{L}_{\frac{1}{2}}(\lambda) = \pi_{\frac{1}{2}}\big(\boldsymbol{L}(\lambda)\big)\,. \tag{5.56}$$

As was shown in the work [32], the trace

$$\boldsymbol{\tau}(\lambda) = \mathrm{Tr}\Big[\mathrm{e}^{\mathrm{i}\pi\beta a_0\sigma^3}\,\boldsymbol{L}_{\frac{1}{2}}(\lambda)\Big] \tag{5.57}$$

commutes with the operator (5.13), $\big[\,\boldsymbol{\tau}(\lambda),\,\boldsymbol{a}_+(\lambda')\,\big] = 0$, and furthermore satisfies the relation

$$\boldsymbol{\tau}(\lambda)\,\boldsymbol{a}_+(\lambda) = \mathrm{e}^{+2\mathrm{i}\pi\beta a_0}\,\boldsymbol{a}_+(q\lambda) + \mathrm{e}^{-2\mathrm{i}\pi\beta a_0}\,\boldsymbol{a}_+(q^{-1}\lambda)\,. \tag{5.58}$$

It should be clear that (5.58) is the scaling counterpart of (2.6), where $\boldsymbol{\tau}(\lambda)$ appears in the scaling limit of the transfer matrix $\mathbb{T}(\zeta)$. One can obtain a formula that describes the scaling limit of the eigenvalues of $\mathbb{T}(\zeta)$ via a comparison of these two relations. Let $T^{(N)}(\zeta)$ be the eigenvalue of the transfer matrix for an RG trajectory $\boldsymbol{\Psi}_N$ and consider eq. (2.6) specialized to that common eigenvector. Substituting the parameter ζ by $(N/N_0)^{2(\beta^2-1)}\,E$ and then using formulae (5.47) and (5.44) one finds

$$\lim_{N\to\infty} G^{(N)}\big(q^2E\,|\,\beta^2\big)\,G^{(N)}\big(q^{-2}E\,|\,\beta^2\big)\,T^{(N)}\big(\,(N/N_0)^{2(\beta^2-1)}\,E\big) = (-1)^{\mathtt{w}}\,\tau(\lambda)\,. \tag{5.59}$$

Here we take into account that in the large N limit,

$$f\big((N/N_0)^{2(\beta^2-1)}\,E\,\big) = G^{(N)}\big(\,qE\,|\,\beta^2\,\big)\,G^{(N)}\big(\,q^{-1}E\,|\,\beta^2\,\big)\,\big(\,1+o(1)\,\big)\,, \tag{5.60}$$

where $f(\zeta)$ is the function (2.7) with all the inhomogeneities set to one. Also recall that $q = \mathrm{e}^{\mathrm{i}\pi\beta^2}$ and $E \propto \lambda^2$ as in (5.45).

The sign factor in the r.h.s. of (5.59) appears for the following reason. According to (4.8), the eigenvalues of the operators $\mathrm{e}^{+\mathrm{i}\pi\beta a_0}$ entering into eq. (5.58) are given by $\mathrm{e}^{\mathrm{i}\pi(\mathtt{k}+\mathtt{w})+\pi\beta^2 S^z}$. This differs from the eigenvalues of the corresponding factors $\omega\,q^{+\mathbb{S}^z}$ in (2.6) by $(-1)^{\mathtt{w}}$. Notice that the same sign factor enters into the asymptotic formula (4.1) for the eigenvalues of the lattice translation operator $\mathbb{K}$, where it is denoted by $\sigma = (-1)^{\mathtt{w}}$. Thus eq. (5.59) can be rewritten in the operator form as:

$$\lim_{N\to\infty} G^{(N)}\big(q^2E\,|\,\beta^2\big)G^{(N)}\big(q^{-2}E\,|\,\beta^2\big)\mathbb{T}\big(\,(N/N_0)^{2(\beta^2-1)}\,E\big)\,\mathbb{K} = \boldsymbol{\tau}(\lambda)\,. \tag{5.61}$$

Since both $\boldsymbol{\tau}$ and $\boldsymbol{a}_\pm$ admit a regular power series expansion in λ^2, eq. (5.58) allows one to express the operators $\mathbf{H}_j^{(\pm)}$ (5.14) in terms of

$$\mathbf{Q}_j\ :\qquad \boldsymbol{\tau}(\lambda) = \sum_{j=0}^{\infty}\mathbf{Q}_j\,\lambda^{2j}\,. \tag{5.62}$$

This leads to relations between the corresponding eigenvalues $H_j^{(\pm)}(\boldsymbol{v})$ and $Q_j(\boldsymbol{v})$. For the case of the Fock vacuum, formula (5.21) gives the first few eigenvalues $H_j^{(\mathrm{vac})}$ in terms of the $2m$ fold integrals $Q_m(h,g)$ (5.20) with $h = \beta P$ and $g = \beta^2$, which coincide with the vacuum eigenvalues of $\mathbf{Q}_m$.

5.4. Chiral states $\psi_P(v)$

We have yet to discuss an important practical problem: having at hand the Bethe roots corresponding to the family $\boldsymbol{\Psi}_N$ for a few values of N, how to identify the states $\boldsymbol{\psi}_P(\boldsymbol{v})$ and $\bar{\boldsymbol{\psi}}_{\bar{P}}(\bar{\boldsymbol{v}})$ appearing in the r.h.s. of eq. (4.13).

To obtain the set $\boldsymbol{v} = \{v_a\}_{a=1}^{\mathtt{L}}$ labeling the state $\boldsymbol{\psi}_P(\boldsymbol{v})$ one can in principle compute the connection coefficient $D_+(E\,|\,\boldsymbol{v})$ from the Bethe roots for $\boldsymbol{\Psi}_N$ using eq. (5.47). In practice, however, instead of using the full spectral determinant it is sufficient to focus on its large - E asymptotic expansion. Eq. (5.42) describes just the leading large - E behaviour. A more detailed description involves the asymptotic coefficient R_P, which depends on the set of apparent singularities:

$$D_+(E|\boldsymbol{v}) = R_P(\boldsymbol{v})\ (-E)^{-P/\beta}\ \exp\left(\frac{N_0}{\cos(\frac{\pi\beta^2}{2-2\beta^2})}\ (-E)^{\frac{1}{2-2\beta^2}} + o(1)\right) \tag{5.63}$$

(here $\beta^2 \neq 1 - \frac{1}{2k}$). In the recent work [47] a closed expression for $R_P(\boldsymbol{v})$ was obtained for the Schrödinger equation with Monster potentials involving an arbitrary number of apparent singularities $\mathtt{L} = 0, 1, 2, \ldots$. It takes the form

$$R_P(\boldsymbol{v}) = R_P^{(0)}\ \check{R}_P(\boldsymbol{v}) \tag{5.64}$$

with

$$R_P^{(0)} = \beta^{1+4P\beta}\ 2^{2P(\beta^{-1}-\beta)}\ \frac{\Gamma(1+\frac{2P}{\beta})}{\Gamma(1+2P\beta)} \tag{5.65}$$

and $\check{R}_P(\boldsymbol{v})$ is given in eq. (5.19) in [47]. On the other hand, formula (5.47) implies an important relation which allows one to extract the asymptotic coefficient $R_P(\boldsymbol{v})$ numerically from the Bethe roots for sufficiently large N [37]:

$$\prod_{m=1}^{M}\big(\zeta_m^{-1}+q\big)\big(\zeta_m^{-1}+q^{-1}\big) \asymp \big(R_P(\boldsymbol{v})\big)^2\ \big(N/N_0\big)^{-4(\beta^{-1}-\beta)P}\ \big(4(1-\beta^2)\big)^N\ \big(1+o(1)\big)\,. \tag{5.66}$$

Together with the similar relation

$$\prod_{m=1}^{M}\big(\zeta_m+q\big)\big(\zeta_m+q^{-1}\big) \asymp \big(R_{\bar{P}}(\bar{\boldsymbol{v}})\big)^2\ \big(N/N_0\big)^{-4(\beta^{-1}-\beta)\bar{P}}\ \big(4(1-\beta^2)\big)^N\ \big(1+o(1)\big)\,, \tag{5.67}$$

this provides a way of identifying the sets $\boldsymbol{v} = \{v_a\}_{a=1}^{\mathtt{L}}$ and $\bar{\boldsymbol{v}} = \{\bar{v}_a\}_{a=1}^{\bar{\mathtt{L}}}$, which label the state $\bar{\boldsymbol{\psi}}_{\bar{P}}(\bar{\boldsymbol{v}}) \otimes \boldsymbol{\psi}_P(\boldsymbol{v}) \in \bar{\mathcal{F}}_{\bar{P}}^{(\bar{\mathtt{L}})} \otimes \mathcal{F}_P^{(\mathtt{L})}$ that occurs in the scaling limit of $\boldsymbol{\Psi}_N$ (4.13). In practice we found this to be an effective procedure for small $\mathtt{L}$ and $\bar{\mathtt{L}}$ (≤ 5).

The state $\boldsymbol{\psi}_P(\boldsymbol{v}) \in \mathcal{F}_P^{(\mathtt{L})}$ can be constructed, in principle, through the diagonalization problem of the operator $\mathcal{a}_+$. However the computation of its matrix elements using eqs. (5.9), (5.13) is an unduly complicated task. It turns out that in practice the most effective way of determining the states $\boldsymbol{\psi}_P(\boldsymbol{v}) \in \mathcal{F}_P^{(\mathtt{L})}$ for small values of $\mathtt{L}$ is based on the diagonalization of the so-called reflection operator. The latter commutes with $\mathcal{a}_\pm(\lambda)$ and its eigenvalues coincide with the subleading coefficients $R_P(\boldsymbol{v})$ entering into the asymptotic formula (5.63). There is a simple algebraic procedure for constructing the reflection operator restricted to a level subspace $\mathcal{F}_P^{(\mathtt{L})}$ with given $\mathtt{L}$. For $\mathtt{L} = 1, 2, 3$ some explicit formulae can be found in the Appendix of ref. [47].

As an illustration here we quote the explicit expression for the states $\boldsymbol{\psi}_P(\boldsymbol{v}) \in \mathcal{F}_P^{(\mathtt{L})}$ for the first two levels. For $\mathtt{L}=1$, when the Monster potential contains only one apparent singularity, the system (5.34) dramatically simplifies. Its solution is

$$v_1 = (2P-\rho)(2P+\rho) \qquad \text{with} \qquad \rho = \beta^{-1} - \beta \tag{5.68}$$

and β is related to α as in (5.37). Since $\dim \mathcal{F}_P^{(1)} = 1$ one has

$$\boldsymbol{\psi}_P\big(\boldsymbol{v}^{(1)}\big) = \frac{1}{2P+\rho}\ a_{-1}\,|\,P\rangle\ . \tag{5.69}$$

For $\mathtt{L}=2$ there are two solutions of (5.34), which we denote as $\boldsymbol{v}^{(2,+)} = (v_1^+, v_2^+)$ and $\boldsymbol{v}^{(2,-)} = (v_1^-, v_2^-)$. They read explicitly as

$$\begin{aligned} v_1^{\pm} &= 2\,\omega_\pm\big(\omega_\pm + \beta^{-1}\big)\big(\omega_\pm^2 + \beta^2 - \beta^{-2}\big) \\ v_2^{\pm} &= 2\,\omega_\pm\big(\omega_\pm - \beta^{-1}\big)\big(\omega_\pm^2 + \beta^2 - \beta^{-2}\big)\ , \end{aligned} \tag{5.70}$$

where

$$\omega_\pm = \frac{1}{2}\sqrt{(1+2\beta^2)(2\beta^{-2}-1) \pm B}\,, \qquad B = \sqrt{(2\rho^2-1)^2 + 32P^2} > 0\ . \tag{5.71}$$

The corresponding basis states $|\boldsymbol{v}^{(2,\pm)}\rangle \in \mathcal{F}_P^{(2)}$ are given by

$$\boldsymbol{\psi}_P\big(\boldsymbol{v}^{(2,\pm)}\big) = \big((4P+2\rho)^2 - 2\rho^2 - 1 \pm B\big)\Big(\frac{1}{4}\,a_{-1}^2 - \frac{P}{1-2\rho^2 \mp B}\,a_{-2}\Big)|P\rangle\ . \tag{5.72}$$

The normalization of the states (5.69) and (5.72) will be explained in the next section.

6. Scaling limit of the Bethe state norms

The chiral states $\boldsymbol{\psi}_P(\boldsymbol{v})$ appearing in the scaling limit of $\boldsymbol{\Psi}_N$ (4.13) have been identified as eigenstates of the operators $\mathbb{A}_\pm(\zeta)$ that act in the Fock space $\mathcal{F}_P$. Of course $\bar{\boldsymbol{\psi}}_{\bar{P}}(\bar{\boldsymbol{v}}) \in \bar{\mathcal{F}}_{\bar{P}}$ may be specified similarly. On the other hand, for a given N, $\boldsymbol{\Psi}_N$ is a state in the finite dimensional space $\mathscr{V}_N$ (2.1). In order to assign a precise meaning to the asymptotic formula (4.13) we should equip $\mathscr{V}_N$ and the Fock spaces with suitable Hermitian structures. For $\mathscr{V}_N$ we take the Hermitian structure to be one, which is consistent with the integrable structure in the model. As was already mentioned in sec. 2, this means that the sesquilinear form is such that the condition (2.38) is obeyed. An important feature of the homogeneous six-vertex model is that any set of solutions to the Bethe ansatz equations coincides with the complex conjugated set so that eq. (2.24) becomes

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{\Psi} = \boldsymbol{\Psi}\ . \tag{6.1}$$

Hence, for the homogeneous case, consistency between the integrable and Hermitian structures implies that different Bethe states are orthogonal to each other. The corresponding Hermitian structure in the chiral Fock space should be chosen so that $\big(\boldsymbol{\psi}_P(\boldsymbol{v}'), \boldsymbol{\psi}_P(\boldsymbol{v})\big) = 0$ for $\boldsymbol{v}' \neq \boldsymbol{v}$. Then specifying the norms of the Bethe states as well as the norms of $\boldsymbol{\psi}_P(\boldsymbol{v})$ and $\bar{\boldsymbol{\psi}}_{\bar{P}}(\bar{\boldsymbol{v}})$, one may obtain the constant Ω_N as the ratio of the norms of the states appearing on both sides of eq. (4.13).

The Fock space $\mathcal{F}_P$ admits an inner product that is consistent with the natural conjugation condition for the Heisenberg generators:

$$a_m^\dagger = a_{-m} \qquad (\forall m) \ . \tag{6.2}$$

Using the definitions (5.13), (5.57) one can show that for real λ^2 the operators $\mathbb{A}_\pm(\lambda)$ and $\boldsymbol{\tau}(\lambda)$ are Hermitian:

$$\big[\mathbb{A}_\pm(\lambda)\big]^\dagger = \mathbb{A}_\pm(\lambda^*) \ , \qquad \big[\boldsymbol{\tau}(\lambda)\big]^\dagger = \boldsymbol{\tau}(\lambda^*) \ . \tag{6.3}$$

Assuming that the spectrum of $\mathbb{A}_\pm(\lambda)$ is non-degenerate, one concludes that different states $\boldsymbol{\psi}_P(\boldsymbol{v})$ and $\boldsymbol{\psi}_P(\boldsymbol{v}')$ are orthogonal w.r.t. the inner product associated with this conjugation, i.e.,

$$\big(\boldsymbol{\psi}_P(\boldsymbol{v}'), \boldsymbol{\psi}_P(\boldsymbol{v})\big)_\dagger = F_P(\boldsymbol{v})\, \delta_{\boldsymbol{v}',\boldsymbol{v}} \ . \tag{6.4}$$

Here the "†" subscript is used to emphasize that the inner product corresponds to the conjugation (6.2) that is consistent with the Heisenberg algebra commutation relations.

It is possible to introduce another natural inner product in $\mathcal{F}_P$ such that the orthogonality condition similar to (6.4) is satisfied. To describe it, we'll use the fact that the Fock space admits the structure of the highest weight representation of the Virasoro algebra. Consider the composite field $T(u)$ built from $\partial\varphi$ (5.7)

$$T(u) = (\partial\varphi)^2 - \mathrm{i}\rho\, \partial^2\varphi \ , \tag{6.5}$$

where ρ is a real parameter and $\partial \equiv \frac{\partial}{\partial u}$. The Fourier coefficients

$$T(u) = -\frac{c}{24} + \sum_{m=-\infty}^{\infty} L_m\, \mathrm{e}^{-\mathrm{i}mu} \ , \tag{6.6}$$

are generators of the Virasoro algebra

$$[L_n, L_m] = (n-m)\, L_{n+m} + \tfrac{c}{12}\, n(n^2-1)\, \delta_{n+m,0} \tag{6.7}$$

with central charge

$$c = 1 - 6\rho^2 \ . \tag{6.8}$$

The above relations define the structure of the Verma module for the Virasoro algebra $\mathcal{V}_\Delta$ on $\mathcal{F}_P$ with highest weight

$$\Delta = P^2 - \tfrac{1}{4}\rho^2 \ . \tag{6.9}$$

One can introduce the inner product $(\cdot,\cdot)_\star$ in $\mathcal{F}_P \cong \mathcal{V}_\Delta$ that is consistent with the natural conjugation condition for the Virasoro algebra generators:

$$L_m^\star = L_{-m} \ . \tag{6.10}$$

Although the $\star$ - conjugation, as it follows from eqs. (5.7), (6.5) and (6.6), acts non-trivially on the Heisenberg generators $\{a_m\}$, it turns out that $\mathbb{A}_\pm(\lambda)$ and $\tau(\lambda)$ satisfy the Hermiticity conditions

$$\big[\mathbb{A}_\pm(\lambda)\big]^\star = \mathbb{A}_\pm(\lambda^*) \ , \qquad \big[\boldsymbol{\tau}(\lambda)\big]^\star = \boldsymbol{\tau}(\lambda^*) \tag{6.11}$$

provided that

$$\rho = \beta^{-1} - \beta \ . \tag{6.12}$$

Thus one has

$$\big(\boldsymbol{\psi}_P(\boldsymbol{v}'), \boldsymbol{\psi}_P(\boldsymbol{v})\big)_\star = V_P(\boldsymbol{v})\, \delta_{\boldsymbol{v}',\boldsymbol{v}} \ . \tag{6.13}$$

Here the "Virasoro norm" $V_P(\boldsymbol{v})$ is of course different from the Heisenberg one $F_P(\boldsymbol{v})$ from eq. (6.4).

The states $\boldsymbol{\psi}_P(\boldsymbol{v})$ have been defined as eigenvectors of $\mathbb{a}_+(\lambda)$ which specifies them up to an overall factor. It will be convenient for us to fix this last ambiguity by imposing

$$\boldsymbol{\psi}_P(\boldsymbol{v}) = \big((L_{-1})^{\mathtt{L}} + \ldots\big)\,|P\rangle\,, \qquad\qquad \boldsymbol{v} = \{v_a\}_{a=1}^{\mathtt{L}}\,, \tag{6.14}$$

where the dots denote the terms involving L_{-m} with $2 \le m \le \mathtt{L}$. Let us emphasize that, though the condition (6.14) is written in terms of the Virasoro algebra generators, the vector $\boldsymbol{\psi}_P(\boldsymbol{v})$ is considered as a state in the Fock space $\mathcal{F}_P$ with the operators $\{L_{-m}\}$ expressed in terms of the Heisenberg generators via eqs. (5.7), (6.5) and (6.6). For instance, the formulae (5.69) and (5.72) give the states $\boldsymbol{\psi}_P(\boldsymbol{v}) \in \mathcal{F}_P^{(\mathtt{L})}$ with $\mathtt{L} = 1$ and $\mathtt{L} = 2$, respectively, that are normalized according to (6.14). Having imposed a normalization condition for $\boldsymbol{\psi}_P(\boldsymbol{v})$, each of the norms $F_P(\boldsymbol{v})$ (6.4) and $V_P(\boldsymbol{v})$ (6.13) is determined up to an overall multiplicative factor that does not depend on the state in $\mathcal{F}_P$. The latter may be fixed by specifying the value of the norms of the Fock vacuum $\boldsymbol{\psi}_P^{(\mathrm{vac})} \equiv |P\rangle$, i.e., $(\boldsymbol{\psi}_P^{(\mathrm{vac})}, \boldsymbol{\psi}_P^{(\mathrm{vac})})_\dagger$ and $(\boldsymbol{\psi}_P^{(\mathrm{vac})}, \boldsymbol{\psi}_P^{(\mathrm{vac})})_\star$, respectively. For the states of the other chirality $\bar{\boldsymbol{\psi}}_{\bar{P}}(\bar{\boldsymbol{v}}) \in \bar{\mathcal{F}}_{\bar{P}}^{(\bar{\mathtt{L}})}$ such that $\bar{\boldsymbol{\psi}}_{\bar{P}}(\bar{\boldsymbol{v}}) = \big((\bar{L}_{-1})^{\bar{\mathtt{L}}} + \ldots\big)|\bar{P}\rangle$, the Heisenberg and Virasoro norms can be introduced similarly and will be uniquely defined up to the choice of the factors $(\bar{\boldsymbol{\psi}}_{\bar{P}}^{(\mathrm{vac})}, \bar{\boldsymbol{\psi}}_{\bar{P}}^{(\mathrm{vac})})_\dagger$ and $(\bar{\boldsymbol{\psi}}_{\bar{P}}^{(\mathrm{vac})}, \bar{\boldsymbol{\psi}}_{\bar{P}}^{(\mathrm{vac})})_\star$.

The large N limit of the norms of the low energy Bethe states for the homogeneous six-vertex model was studied in the work [37]. The results imply that in the scaling limit the Hermitian form

$$\big(\boldsymbol{\Psi}^{(2)}, \boldsymbol{\Psi}^{(1)}\big)_\star = \big(\boldsymbol{\Psi}^{(1)}, \boldsymbol{\Psi}^{(1)}\big)_\star\, \delta_{\boldsymbol{\Psi}^{(2)},\boldsymbol{\Psi}^{(1)}}\,, \tag{6.15}$$

where $(\boldsymbol{\Psi}^{(1)}, \boldsymbol{\Psi}^{(1)})_\star = (\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\Psi}^{(1)}, \boldsymbol{\Psi}^{(1)})_\star$ is given by (2.39) with $\eta_J = 1$, induces the Hermitian form in the space $\bar{\mathcal{F}}_{\bar{P}} \otimes \mathcal{F}_P$ defined by the conditions (6.13), (6.14) and the similar relations for the barred counterpart. Furthermore, the natural choice for the norms of the Fock vacua turns out to be

$$\big(\boldsymbol{\psi}_P^{(\mathrm{vac})}, \boldsymbol{\psi}_P^{(\mathrm{vac})}\big)_\star = Z_+(P\,|\,\beta)\,, \qquad \big(\bar{\boldsymbol{\psi}}_{\bar{P}}^{(\mathrm{vac})}, \bar{\boldsymbol{\psi}}_{\bar{P}}^{(\mathrm{vac})}\big)_\star = Z_+(\bar{P}\,|\,\beta)\,, \tag{6.16}$$

where

$$\begin{aligned} Z_+(P\,|\,\beta) = (A_{\mathrm{G}})^{-2\beta^2}\,(2\pi)^{\frac{1}{2}-2P\beta}\,\beta^{h(P)+4P\beta+1}\,\frac{\mathrm{e}^{-(\frac{2P}{\beta}+h(P)+\frac{1}{2}-\frac{1}{6}\beta^2)\gamma_{\mathrm{E}}}}{\Gamma(1+\frac{2P}{\beta})\,\Gamma(1+2P\beta)} \\ \times \prod_{m=1}^{\infty} \frac{2\pi\,(m\beta^2)^{2m\beta^2+4P\beta+1}\,\mathrm{e}^{-2m\beta^2+\frac{1}{m}(\frac{2P}{\beta}+h(P)+\frac{1}{2}-\frac{1}{6}\beta^2)}}{\Gamma^2(1+2P\beta+m\beta^2)}\,. \end{aligned} \tag{6.17}$$

Here we use the notation

$$h(P) = 4P^2 + \tfrac{1}{6}\,(\beta^2 + \beta^{-2} - 3) \tag{6.18}$$

and A_{G}, γ_{E} stand for the Glaisher and Euler constants, respectively. Now that the norms of the states on both sides of the asymptotic formula (4.13) are unambiguously specified, the constant Ω_N can be obtained through the study of the large N behaviour of the ratio $(\boldsymbol{\Psi}, \boldsymbol{\Psi})_\star / \big(V_{\bar{P}}(\bar{\boldsymbol{v}})\, V_P(\boldsymbol{v})\big)$. Numerical work from ref. [37] suggests

$$|\,\Omega_N\,|^2 = C_0^2(\beta)\, N^{\frac{1}{6}}\,\big(N/C(\beta)\big)^{-h(P)-h(\bar{P})-4\mathtt{L}-4\bar{\mathtt{L}}}\, \mathrm{e}^{\mathcal{A}_2 N^2}\,, \tag{6.19}$$

where $\mathcal{A}_2$, $C_0(\beta)$ and $C(\beta)$ are the same for all the low energy states and only depend on β. The constant $\mathcal{A}_2$ is given by the integral

$$\mathcal{A}_2 = \int_0^\infty \frac{\mathrm{d}t}{t}\ \frac{\sinh(\frac{\beta^2 t}{1-\beta^2})\ \sinh(t)}{2\sinh(\frac{t}{1-\beta^2})\ \cosh^2(t)}\ . \tag{6.20}$$

For $C_0(\beta)$ and $C(\beta)$ the explicit analytical form is currently unknown. Numerical data for these constants is presented in Appendix A.

Formula (6.19) specifies Ω_N up to an overall phase factor. As usual, this ambiguity can be fixed by using global $\mathcal{CPT}$ - symmetry. The generators $\hat{\mathcal{C}}$, $\hat{\mathcal{P}}$ and $\hat{\mathcal{T}}$ acting in the tensor product $\bar{\mathcal{F}}_{\bar{P}} \otimes \mathcal{F}_P$ may be introduced in such a way that the combination $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}$ commutes with all of the Heisenberg modes,

$$\big[\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}},\, a_m\big] = \big[\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}},\, \bar{a}_m\big] = 0 \qquad\qquad (\forall m)\,, \tag{6.21}$$

and acts identically on the vacuum

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,|\bar{P}\rangle \otimes |P\rangle = |\bar{P}\rangle \otimes |P\rangle\ . \tag{6.22}$$

Then the $\mathcal{CPT}$ conjugation acts as the identity operator on any state $\bar{\boldsymbol{\psi}}_{\bar{P}}(\bar{\boldsymbol{v}}) \otimes \boldsymbol{\psi}_P(\boldsymbol{v})$, where $\boldsymbol{\psi}_P(\boldsymbol{v})$ is normalized by the condition (6.14) and similarly for $\bar{\boldsymbol{\psi}}_{\bar{P}}(\bar{\boldsymbol{v}})$. In other words all the coefficients of the states $\boldsymbol{\psi}_P(\boldsymbol{v})$ expanded in the basis $\{a_{-i_1}\ldots a_{-i_m}|P\rangle\ :\ 1\le i_1\le i_2\le\ldots\le i_m,\ P-\text{real}\}$ are real numbers (for an illustration see eqs. (5.69)-(5.72)). Since the $\mathcal{CPT}$ conjugation also acts trivially on the Bethe states (6.1), it follows that the constant Ω_N must be real, and without loss of generality we can take it to be positive:

$$\Omega_N = \sqrt{\mathcal{K}_N^{(\mathrm{L})}(P)\,\mathcal{K}_N^{(\bar{\mathrm{L}})}(\bar{P})} \quad \text{with} \quad \mathcal{K}_N^{(\mathrm{L})}(P) = C_0(\beta)\,N^{\frac{1}{12}}\,\big(N/C(\beta)\big)^{-h(P)-4\mathrm{L}}\ \mathrm{e}^{\frac{1}{2}\mathcal{A}_2 N^2}\ . \tag{6.23}$$

The following comment is in order here. Together with the $\mathcal{CPT}$ - invariance the system possesses global $\mathcal{CP}$ and $\mathcal{T}$ symmetry separately. The action of the $\mathcal{CP}$ transformation intertwines the spaces $\bar{\mathcal{F}}_{\bar{P}} \otimes \mathcal{F}_P \mapsto \bar{\mathcal{F}}_{-P} \otimes \mathcal{F}_{-\bar{P}}$ and is defined by the following relations

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\,a_m = \bar{a}_m\,\hat{\mathcal{C}}\hat{\mathcal{P}} \qquad \text{and} \qquad \hat{\mathcal{C}}\hat{\mathcal{P}}\,|\bar{P}\rangle \otimes |P\rangle = |-P\rangle \otimes |-\bar{P}\rangle\ . \tag{6.24}$$

It was already mentioned that for the lattice model, the $\mathcal{CP}$ and $\mathcal{T}$ transformations acting in $\mathscr{V}_N$ intertwine the sectors with $+S^z$ and $-S^z$, while we only focus on the Bethe states (2.20) in the sector $S^z \ge 0$ (see sec. 4 in [29] for the explicit formulae for the action of the $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ conjugations in $\mathscr{V}_N$). The state $\hat{\mathcal{C}}\hat{\mathcal{P}}\,\boldsymbol{\Psi} \in \mathscr{V}_N$ can be written in the form similar to (2.20), but with the set $\{\zeta_m\}$ being the zeroes of the corresponding eigenvalue $A_-(\zeta)$ of the operator $\mathbb{A}_-(\zeta)$. Recall that the solutions sets of the Bethe ansatz equations (2.10) are roots of $A_+(\zeta)$.

The Hermitian form (6.15) in the finite dimensional space $\mathscr{V}_N$ is not positive definite. At the same time this space can be equipped with a positive definite inner product, which is consistent with the integrable structure. This is a special property of the homogeneous case. The positive definite Hermitian form in $\mathscr{V}_N = \mathbb{C}_N^2 \otimes \mathbb{C}_{N-1}^2 \otimes \ldots \otimes \mathbb{C}_1^2$ is induced by that of each two-dimensional component in the tensor product. The latter is defined as $\langle\sigma|\sigma'\rangle = \delta_{\sigma,\sigma'}$, where $|\pm\rangle \in \mathbb{C}^2$ stand for the two basis vectors such that $\sigma^z|\pm\rangle = \pm|\pm\rangle$. To make connection with the works [41,42] let's change the overall normalization of the Bethe state (2.20) and introduce

$$\boldsymbol{\Psi}' = \alpha(\zeta_1,\ldots,\zeta_M)\,\boldsymbol{\Psi}\big(\{\zeta_j\}\big)\ , \tag{6.25}$$

where $\alpha(\zeta_1,\ldots,\zeta_M)$ is given by

$$\alpha(\zeta_1,\ldots,\zeta_M)=\big(-\mathrm{i}q^{-\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}\pi\mathrm{k}}\,(q-q^{-1})\big)^{-M}\ A_+(-q^{-1}) \tag{6.26}$$

with $A_+(-q^{-1})=\prod_{m=1}^{M}\big(1+1/(q\zeta_m)\big)$. Then the wavefunction

$$\Psi'(x_M,\ldots,x_1)\ :\qquad \boldsymbol{\Psi}'=\sum_{1\le x_1<x_2<\ldots<x_M\le N}\Psi'(x_M,\ldots,x_1)\,\sigma^-_{x_M}\cdots\sigma^-_{x_1}\,|\,0\,\rangle \tag{6.27}$$

can be written in the form

$$\Psi'(x_M,\ldots,x_1)=\sum_{\hat{P}}A_{\hat{P}}\ \mathrm{e}^{\mathrm{i}\sum_{m=1}^{M}p_{\hat{P}m}x_m}\ . \tag{6.28}$$

Here the summation is taken over all $M!$ permutations $\hat{P}$ of the integers $(1,2,\ldots,M)$, and we use the notation

$$A_{\hat{P}}=\prod_{1\le j<m\le M}\frac{q\zeta_{\hat{P}j}-q^{-1}\zeta_{\hat{P}m}}{\zeta_{\hat{P}j}-\zeta_{\hat{P}m}}\,, \tag{6.29}$$

while $p_m=p(\zeta_m)$ which was defined in eq. (4.11). The norm of the Bethe state $\boldsymbol{\Psi}'$ (6.27) w.r.t. the positive definite inner product reads as

$$\|\ \boldsymbol{\Psi}'\ \|^2=\sum_{1\le x_1<x_2<\ldots<x_M\le N}\big|\Psi'(x_1,\ldots,x_M)\big|^2\ . \tag{6.30}$$

There exists a remarkable formula for this norm, which was originally conjectured by Gaudin, McCoy and Wu in ref. [41] and proven by Korepin in [42]. In our notation it reads as

$$\|\ \boldsymbol{\Psi}'\ \|^2=\big|\alpha(\zeta_1,\ldots,\zeta_M)\big|^2\ (\boldsymbol{\Psi},\boldsymbol{\Psi})_\star\ \prod_{m=1}^{M}\zeta_m\ , \tag{6.31}$$

where $(\boldsymbol{\Psi},\boldsymbol{\Psi})_\star=(\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\Psi},\boldsymbol{\Psi})_\star$ is given by (2.39) with $\eta_J=1$ and $\alpha(\zeta_1,\ldots,\zeta_M)$ is as in (6.26).

The scaling limit of the norm (6.31) for the RG trajectory $\boldsymbol{\Psi}_N$ was studied in ref. [37]. It was found that the positive definite inner product in the space $\mathscr{V}_N$ becomes the positive definite Hermitian form in $\bar{\mathcal{F}}_{\bar{P}}\otimes\mathcal{F}_P$ consistent with the conjugation conditions $a_m^\dagger=a_{-m}$ and $\bar{a}_m^\dagger=\bar{a}_{-m}$. In this case it is convenient to fix the norms of the highest states in the Fock spaces as

$$\begin{aligned}\big(\boldsymbol{\psi}_P^{(\mathrm{vac})},\boldsymbol{\psi}_P^{(\mathrm{vac})}\big)_\dagger&=(2/\beta)^{2P(\beta^{-1}-\beta)}\ Z^2(P\,|\,\beta)\\ \big(\bar{\boldsymbol{\psi}}_{\bar{P}}^{(\mathrm{vac})},\bar{\boldsymbol{\psi}}_{\bar{P}}^{(\mathrm{vac})}\big)_\dagger&=(2/\beta)^{2\bar{P}(\beta^{-1}-\beta)}\ Z^2(\bar{P}\,|\,\beta)\end{aligned}\ , \tag{6.32}$$

where we use the special function $Z(P\,|\,\beta)$ from ref. [37]. The latter can be represented through the convergent product similar to (6.17):

$$\begin{aligned}Z(P\,|\,\beta)=\,&(A_{\mathrm{G}})^{-\beta^2}\ (2\pi)^{\frac{1}{4}-P\beta}\ \beta^{\frac{1}{2}h(P)+1+P(\beta^{-1}+3\beta)}\ \frac{\mathrm{e}^{-\frac{1}{2}(\frac{2P}{\beta}+h(P)+\frac{1}{2}-\frac{1}{6}\beta^2)\gamma_{\mathrm{E}}}}{\Gamma(1+2P\beta)}\\ &\times\prod_{m=1}^{\infty}\frac{\sqrt{2\pi}\ (m\beta^2)^{m\beta^2+2P\beta+\frac{1}{2}}\ \mathrm{e}^{-m\beta^2+\frac{1}{2m}(\frac{2P}{\beta}+h(P)+\frac{1}{2}-\frac{1}{6}\beta^2)}}{\Gamma(1+2P\beta+m\beta^2)}\ .\end{aligned} \tag{6.33}$$

The norms $F_{\bar{P}}(\bar{\boldsymbol{v}})F_P(\boldsymbol{v})$ of the states $\bar{\boldsymbol{\psi}}_{\bar{P}}(\bar{\boldsymbol{v}})\otimes\boldsymbol{\psi}_P(\boldsymbol{v})\in\bar{\mathcal{F}}_{\bar{P}}\otimes\mathcal{F}_P$ are fully determined by eqs. (6.4) and (6.14), their barred counterparts and formula (6.32). As was pointed out in [37] the Heisenberg $F_P(\boldsymbol{v})$ (6.4) and Virasoro $V_P(\boldsymbol{v})$ (6.13) norms are related to each other through the eigenvalues of the reflection operator (5.63)-(5.65). With the norms of the Fock vacua $\big(\boldsymbol{\psi}_P^{(\mathrm{vac})},\boldsymbol{\psi}_P^{(\mathrm{vac})}\big)_\star$ and $\big(\boldsymbol{\psi}_P^{(\mathrm{vac})},\boldsymbol{\psi}_P^{(\mathrm{vac})}\big)_\dagger$ fixed as in (6.16) and (6.32), respectively, one has the relation

$$\frac{F_P(\boldsymbol{v})}{V_P(\boldsymbol{v})}=R_P(\boldsymbol{v})\ . \tag{6.34}$$

A numerical study leads to the following asymptotic formula describing the large N behaviour of the positive definite norm (6.30):

$$\begin{aligned}\parallel\boldsymbol{\Psi}'_N\parallel^2\asymp F_{\bar{P}}(\bar{\boldsymbol{v}})F_P(\boldsymbol{v})\,\big(2\sin(\pi\beta^2)\big)^{\frac{2}{\beta}(P+\bar{P})}\\ \times(N/N_0)^{-2(\beta^{-1}-\beta)(P+\bar{P})}\,\mathcal{K}_N^{(\mathrm{L})}(P)\,\mathcal{K}_N^{(\bar{\mathrm{L}})}(\bar{P})\,\mathrm{e}^{\mathcal{A}_1N}\ .\end{aligned} \tag{6.35}$$

Here $\mathcal{K}_N^{(\mathrm{L})}(P)$ is defined in eq. (6.23), the constant N_0 is given by eq. (5.36), while

$$\mathcal{A}_1=\log\bigg(\frac{2(1-\beta^2)}{\sin(\pi\beta^2)}\bigg)\ . \tag{6.36}$$

Part II. Inhomogeneous six-vertex model with global $\mathcal{Z}_2$ symmetry

7. Introduction

As was mentioned in the Preliminaries, additional global symmetries in the inhomogeneous six-vertex model arise when certain constraints are imposed on the inhomogeneities. For example, the restrictions $\eta_{N+1-J}=\eta_J^{-1}$ and $(\eta_J)^*=\eta_J^{-1}$ lead to $\mathcal{CP}$ and $\mathcal{T}$ invariance, while imposing the condition $\eta_{J+r}=\eta_J$ with $N=rL$ gives rise to the lattice translation symmetry. Proceeding further and completely fixing the inhomogeneities as

$$\eta_J=(-1)^r\,\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2J-1)}\qquad(J=1,\dots,rL) \tag{7.1}$$

one arrives at a model possessing global $\mathcal{Z}_r$ invariance (for further details see sec. 7 of ref. [29]). It turns out that, like in the homogeneous case, which formally corresponds to $r=1$, the model is critical when q and ω are unimodular (2.21). However for $r\geq2$, different types of critical behaviour occur depending on the value of $\arg(q^2)$. For instance, in the case of the $\mathcal{Z}_2$ invariant model there are two such domains with $\arg(q^2)\in(0,\pi)$ and $\arg(q^2)\in(\pi,2\pi)$.

This work is devoted to the study of the critical behaviour of the $\mathcal{Z}_2$ invariant six-vertex model with $\arg(q^2)\in(0,\pi)$. The generator of the extra symmetry $\hat{\mathcal{D}}\in\mathrm{End}(\mathscr{V}_N)$ is built out of the matrices (2.33) as

$$\hat{\mathcal{D}}=\prod_{m=1}^{N/2}\check{\boldsymbol{R}}_{2m,2m-1}(-1)\ :\qquad\qquad \hat{\mathcal{D}}^2=1\ . \tag{7.2}$$

Its adjoint action on the local spin operators is given by

$$\hat{\mathcal{D}}\,\sigma_m^\pm\,\hat{\mathcal{D}}=\frac{1}{q+q^{-1}}\,\Big(2\,\sigma_{m+1}^\pm-(q-q^{-1})\,\sigma_{m+1}^z\sigma_m^\pm\Big) \tag{7.3a}$$

$$\hat{\mathcal{D}}\,\sigma_m^z\,\hat{\mathcal{D}}=\frac{1}{(q+q^{-1})^2}\,\Big(4\,\sigma_{m+1}^z+(q-q^{-1})^2\,\sigma_m^z+4\,(q-q^{-1})\,\big(\sigma_{m+1}^+\sigma_m^-+\sigma_{m+1}^-\sigma_m^+\big)\Big)$$

for odd m and

$$\hat{\mathcal{D}}\,\sigma_m^{\pm}\,\hat{\mathcal{D}} = \frac{1}{q+q^{-1}}\,\Big(2\,\sigma_{m-1}^{\pm} + (q-q^{-1})\,\sigma_m^{\pm}\sigma_{m-1}^{z}\Big) \tag{7.3b}$$

$$\hat{\mathcal{D}}\,\sigma_m^{z}\,\hat{\mathcal{D}} = \frac{1}{(q+q^{-1})^2}\,\Big(4\,\sigma_{m-1}^{z} + (q-q^{-1})^2\,\sigma_m^{z} - 4\,(q-q^{-1})\,\big(\sigma_m^{-}\,\sigma_{m-1}^{+} + \sigma_m^{+}\,\sigma_{m-1}^{-}\big)\Big)$$

for even m. On the transfer matrix and the operators $\mathbb{A}_{\pm}(\zeta)$, the adjoint action of $\hat{\mathcal{D}}$ reads as

$$\hat{\mathcal{D}}\,\mathbb{T}(\zeta)\,\hat{\mathcal{D}} = \mathbb{T}(-\zeta)\,, \qquad \hat{\mathcal{D}}\,\mathbb{A}_{\pm}(\zeta)\,\hat{\mathcal{D}} = \mathbb{A}_{\pm}(-\zeta)\,. \tag{7.4}$$

Note that the above equation implies that for the Bethe state (2.20) corresponding to the solution set $\{\zeta_j\}$ of the Bethe ansatz equations,

$$\hat{\mathcal{D}}\,\boldsymbol{\Psi}\big(\{\zeta_j\}\big) = \boldsymbol{\Psi}\big(\{-\zeta_j\}\big)\,. \tag{7.5}$$

Since the system (2.10) with $\eta_J = \mathrm{i}\,(-1)^{J-1}$ is invariant under the substitution $\zeta_j \mapsto -\zeta_j$, the set $\{-\zeta_j\}$ also solves the Bethe ansatz equations.

Despite that $\hat{\mathcal{D}}$ does not commute with the transfer matrix, it is a symmetry of the model in the following sense. The transfer matrix commutes with the Hamiltonian[1]

$$\mathbb{H} = -\frac{\mathrm{i}}{q^2-q^{-2}}\,\sum_{m=1}^{N}\Big((q-q^{-1})^2\,\sigma_m^z\,\sigma_{m+1}^z + 2\,\big(\sigma_m^x\,\sigma_{m+2}^x + \sigma_m^y\,\sigma_{m+2}^y + \sigma_m^z\,\sigma_{m+2}^z\big)$$

$$+\,(q-q^{-1})\big(\sigma_m^x\sigma_{m+1}^x + \sigma_m^y\sigma_{m+1}^y\big)\big(\sigma_{m-1}^z - \sigma_{m+2}^z\big)\Big) + \mathrm{i}\,N\,\frac{q^2+q^{-2}}{q^2-q^{-2}}\,\hat{\mathbf{1}}\,, \tag{7.6}$$

where

$$\sigma_{N+\ell}^x \pm \mathrm{i}\sigma_{N+\ell}^y = \mathrm{e}^{\pm 2\pi \mathrm{i}\mathtt{k}}\,\big(\sigma_\ell^x \pm \mathrm{i}\sigma_\ell^y\big)\,, \qquad \sigma_{N+\ell}^z = \sigma_\ell^z \qquad (\ell = 1,2) \tag{7.7}$$

and this Hamiltonian commutes with the generator of the $\mathcal{Z}_2$ - symmetry

$$\big[\,\hat{\mathcal{D}},\,\mathbb{H}\,\big] = 0\,. \tag{7.8}$$

Recall that the parameter $\mathtt{k}$ entering into the boundary conditions (7.7) is related to ω from (2.4) as $\omega^2 = \mathrm{e}^{2\pi\mathrm{i}\mathtt{k}}$. The eigenvalue of the Hamiltonian (7.6) for the state $\boldsymbol{\Psi}$ is given in terms of the Bethe roots by

$$\mathcal{E} = \sum_{m=1}^{M}\,\frac{4\mathrm{i}\,(q^2-q^{-2})}{\zeta_m^2+\zeta_m^{-2}+q^2+q^{-2}}\,. \tag{7.9}$$

In this work we'll use the parameterization

$$q = \mathrm{e}^{\frac{\mathrm{i}\pi}{n+2}} \qquad \text{with} \qquad n > 0\,. \tag{7.10}$$

[1] This form for the Hamiltonian, up to an overall multiplicative factor and an additive constant, appeared in ref. [16]. The one defined by eq. (2) in the work [15] coincides with $\hat{\mathbb{V}}\,\mathbb{H}\,\hat{\mathbb{V}}^{-1}$, where $\mathbb{H}$ is as in (7.6), while $\hat{\mathbb{V}} = \prod_{m=1}^{N/2}\exp\big(\frac{\mathrm{i}\pi}{4}\,\sigma_{2m-1}^z\big)$.

8. The low energy Bethe states

The class of states for which we'll be considering the large N limit is taken to be the low energy states for the Hamiltonian (7.6). Similar to what was discussed in the homogeneous case, the Bethe ansatz equations allow one to organize the low energy Bethe states for different N into the RG trajectories $\boldsymbol{\Psi}_N$. For technical details of the construction of these trajectories and some specific examples see the work [15]. In the large N limit the low energy Bethe states form the conformal towers similar to those in the XXZ spin chain. In particular, each such tower is characterized by a set of quantum numbers which includes the value of S^z and the winding number $\mathtt{w} = 0, \pm1, \pm2, \ldots$. Following ref. [15] we will employ the notation

$$p = \tfrac{1}{2}\left(S^z + (\mathtt{k}+\mathtt{w})(n+2)\right), \qquad \bar{p} = \tfrac{1}{2}\left(S^z - (\mathtt{k}+\mathtt{w})(n+2)\right). \tag{8.1}$$

Any state in the conformal tower can be assigned a pair of non-negative integers – the chiral levels $(\bar{\mathtt{L}}, \mathtt{L})$. The extensive numerical work performed in refs. [9–15] suggests that the large N behaviour of the eigenvalues of $\mathbb{H}$ (7.6) and the lattice translation operator $\mathbb{K}$ (2.27), (2.28) with $r = 2$, for the RG trajectory with given p, $\bar{p}$, $\mathtt{L}$ and $\bar{\mathtt{L}}$, are described by the formulae

$$\mathcal{E} = e_\infty N + \frac{4\pi v_{\mathrm{F}}}{N}\left(\frac{p^2+\bar{p}^2}{n+2} + \frac{2b^2}{n} - \frac{1}{6} + \mathtt{L} + \bar{\mathtt{L}}\right) + o\big(N^{-1-\epsilon}\big) \tag{8.2a}$$

$$K = \exp\left(\frac{4\pi\mathrm{i}}{N}\left(\frac{p^2-\bar{p}^2}{n+2} + \mathtt{L} - \bar{\mathtt{L}}\right)\right). \tag{8.2b}$$

Here

$$e_\infty = -\frac{2v_{\mathrm{F}}}{\pi}\int_0^\infty \mathrm{d}t\, \frac{\sinh\left(\frac{2t}{n}\right)}{\sinh\left(\frac{(n+2)t}{n}\right)\cosh(t)}, \qquad v_{\mathrm{F}} = \frac{2(n+2)}{n}, \tag{8.3}$$

while the correction term $o\big(N^{-1-\epsilon}\big)$ contains an infinitesimally small positive $\epsilon > 0$ (for a more detailed description of the correction term see ref. [15]). An important difference of eq. (8.2a) compared with the homogeneous case (4.1) is the presence of the additional N - dependent term $\propto b^2$ with $b = b(N)$. It turns out that $b(N)$ is related to the eigenvalue of the so-called quasi-shift operator, that was introduced in ref. [11]. The latter is expressed in terms of the transfer matrix (2.4) as

$$\mathbb{B} = \mathbb{T}(-\mathrm{i}q^{-1})\big[\mathbb{T}(+\mathrm{i}q^{-1})\big]^{-1} \tag{8.4}$$

and its eigenvalues are given by

$$B = \frac{A_+(-\mathrm{i}q)A_+(+\mathrm{i}q^{-1})}{A_+(+\mathrm{i}q)A_+(-\mathrm{i}q^{-1})}. \tag{8.5}$$

Then b entering into eq. (8.2a) and B are related as

$$b(N) = \frac{n}{4\pi}\log(B), \tag{8.6}$$

where $B = B(N)$ denotes the eigenvalue of the quasi-shift operator corresponding to $\boldsymbol{\Psi}_N$.

Since B is in general a complex number, the definition (8.6) requires the specification of the branch of the logarithm. It turns out that fixing the branch such that $b(N)$ is real whenever $B > 0$ ensures that (8.6) is consistent with formula (8.2a) that describes the low energy spectrum. Thus we define $b(N)$ for all the low energy Bethe states with $|\arg(B)| < \pi$ by supplementing (8.6) with the condition

$$-\frac{n}{4} < \Im m\big(b(N)\big) < \frac{n}{4}\ . \tag{8.7}$$

Special attention is needed for the Bethe states with $|\arg(B)| = \pi$. The explicit diagonalization of the commuting families of operators for small N reveals the existence of states with $B = -1$, see Fig. 2. Although for such states $\delta\mathcal{E} \equiv \frac{N}{4\pi v_{\rm F}}\,(\mathcal{E} - N\,e_\infty)$ is of order one, we found that computing $\delta\mathcal{E}$ for increasing N through the solution of the Bethe ansatz equations, $|\delta\mathcal{E}|$ grows logarithmically with N and hence the states are not counted as low energy ones. In addition, there are states for which B is a complex number that tends to -1 in the large N limit. For most of these, $|\delta\mathcal{E}|$ goes to infinity similar as with the states where $B = -1$. However, there do exist the RG trajectories for which the energy follows eq. (8.2a), while $\lim_{N\to\infty} b(N) = \pm\frac{\mathrm{i}n}{4}$ (see Fig. 4).

It is worth mentioning how the value of $b(N)$ transforms under the action of the $\mathcal{CPT}$ and $\mathcal{D}$ conjugations on the Bethe state $\boldsymbol{\Psi}_N$. The quasi-shift operator satisfies the following relations with their generators

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\ \mathbb{B}\ \hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}} = \mathbb{B}\,, \qquad\qquad \hat{\mathcal{D}}\,\mathbb{B}\,\hat{\mathcal{D}} = \mathbb{B}^{-1}\ . \tag{8.8}$$

The first equation implies that the eigenvalue of $\mathbb{B}$ for the Bethe state and the $\mathcal{CPT}$ - transformed state (2.24) are complex conjugate of each other. In turn,

$$\mathcal{CPT}\ :\quad b(N) \mapsto b^*(N)\ . \tag{8.9}$$

The last equation in (8.8) combined with (8.6) yields that under the $\mathcal{Z}_2$ symmetry transformation

$$\mathcal{D}\ :\quad b(N) \mapsto -b(N)\ . \tag{8.10}$$

Note that both the $\mathcal{CPT}$ and $\mathcal{D}$ conjugations preserve the strip (8.7).

To summarize this section, let us emphasize that the definition of the low energy states for the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model is far from evident. In what follows we'll use the "working" definition that a low energy Bethe state $\boldsymbol{\Psi}_N$ is one, whose energy and momentum is described by eqs. (8.1)-(8.3), with some $\mathtt{w} = 0, \pm1, \pm2$ and non-negative integers $\mathtt{L}$, $\bar{\mathtt{L}}$, while $b(N)$ is defined through eq. (8.6) along with the condition (8.7). Also it is important to note that the case of periodic boundary conditions $\mathtt{k} = 0$ requires special attention. In our analysis, unless explicitly stated, it will always be assumed that $(n+2)\,\mathtt{k} \notin \mathbb{Z}$. The results for periodic boundary conditions may be obtained through taking the limit $\mathtt{k} \to 0$. If necessary, we'll include comments regarding this limit separately.

9. The RG invariant *s*

The specification of the RG trajectory for the $\mathcal{Z}_2$ invariant six-vertex model has some essential differences to the homogeneous case. In particular, for the "primary" Bethe states where $\mathtt{L} = \bar{\mathtt{L}} = 0$, there exist many RG trajectories $\boldsymbol{\Psi}_N$, which correspond to the same values of the RG invariants p and $\bar{p}$ (8.1) and are distinguished by the eigenvalue of the quasi-shift operator (8.5). Following ref. [10], let's illustrate this on a class of Bethe states which occur when $|\mathtt{k}| < \frac{2}{n+2}$. Fixing N and S^z, the corresponding Bethe roots $\{\zeta_m\}_{m=1}^M$ are real, while the states are distinguished by the integers $M_- - M_+$, where M_- stands for the number of negative roots, $\zeta_m < 0$, while M_+ is the number of positive ones, $\zeta_m > 0$. An example of such a pattern is depicted in the left panel of Fig. 5 in the complex β plane with $\beta = -\frac{1}{2}\log(\zeta)$. Although in principle one can construct a

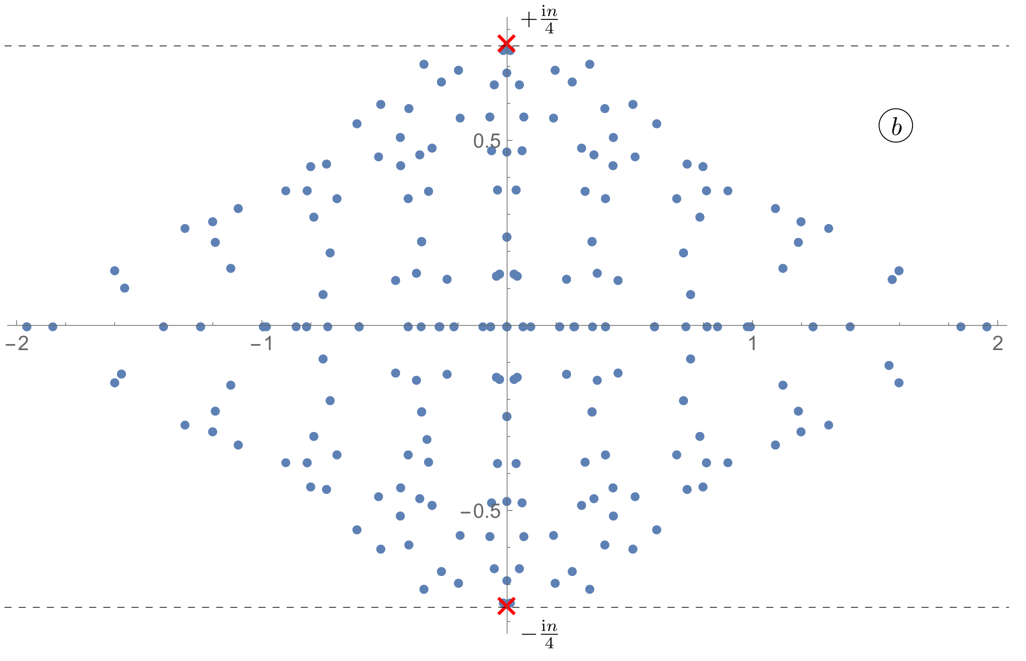

Fig. 2. The blue dots mark the values of $b = \frac{n}{4\pi}\log(B)$ in the complex plane for the first 400 lowest energy states of the Hamiltonian (7.6), (7.7) with $N = 24$ in the sector $S^z = 0$. The branch of the logarithm of B is chosen such that $|\Im m\big(\log B\big)| \le \pi$. The parameters were taken to be $n = 3$, $\pi\mathrm{k} = 18/100$. There are four states with $B = -1$, which are represented by the red crosses in the figure. Two of them have the same complex energy and they are related to each other through the $\mathcal{Z}_2$ transformation (7.5). The other two have the complex conjugated energy, they may be obtained from the $\mathcal{Z}_2$ doublet by means of the $\mathcal{CPT}$ conjugation (2.24). The typical pattern of Bethe roots for one of these states is depicted in the left panel of Fig. 3. The right panel of that figure plots the absolute value of $\delta\mathcal{E} \equiv \frac{N}{4\pi v_{\mathrm{F}}}\,(\mathcal{E} - N\, e_\infty)$ as a function of N. Clearly the energy is not described by (8.2a). Moreover, the eigenvalue of the lattice translation operator $\mathbb{K}$ remains fixed at $K = -1$ for any N, which does not follow (8.2b). Due to this we do not count these states as low energy ones.

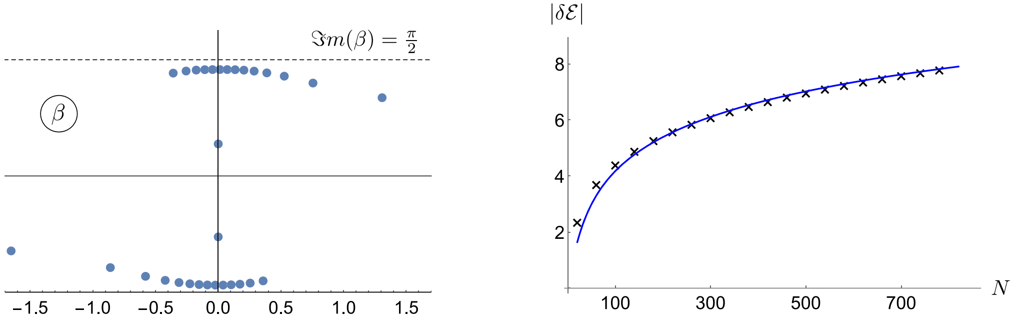

Fig. 3. The typical pattern of Bethe roots $\beta_j = -\frac{1}{2}\log\zeta_j$ for one of the four states having $B = K = -1$, with $n = 3$, $\pi\mathrm{k} = 18/100$ and $N = 60$ (left panel). The scaled energy $\delta\mathcal{E} = \frac{N}{4\pi v_{\mathrm{F}}}\,(\mathcal{E} - N\, e_\infty)$ for these states grows logarithmically for large N. The crosses in the right figure depict the numerical values of $|\delta\mathcal{E}|$ found via the solution of the Bethe ansatz equations. The solid line comes from the fit $\delta\mathcal{E} \approx -4.5702 - 0.2272\,\mathrm{i} + (1.7724 - 0.4110\,\mathrm{i})\log(N)$.

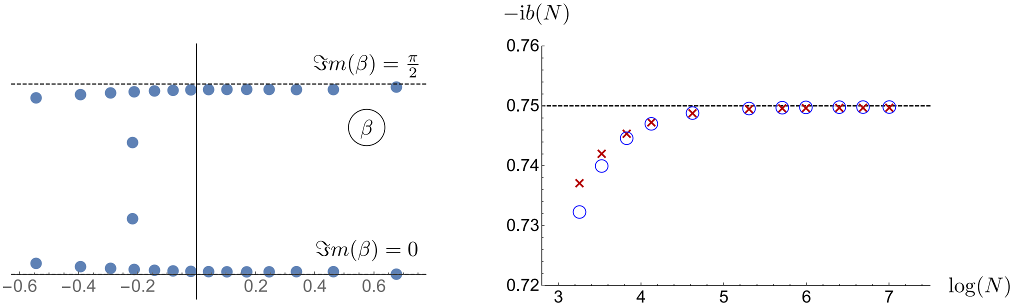

Fig. 4. The left panel depicts the pattern of Bethe roots $\beta_j=-\frac{1}{2}\log(\zeta_j)$ for a low energy state $\mathbf{\Psi}_N$ for which $\lim_{N\to\infty}b(N)=\frac{in}{4}$, i.e., belongs to the boundary of the strip (8.7). The parameters are $n=3$, $\mathtt{k}=\frac{1}{10}$, while the state belongs to the sector $S^z=\mathtt{w}=0$ and $\mathtt{L}=\bar{\mathtt{L}}=1$. On the right panel $-ib(N)$ (which turns out to be real) is plotted as a function of $\log(N)$ for this RG trajectory. The red crosses correspond to $b(N)$ calculated from the eigenvalue of the quasi-shift operator using eq. (8.6), while the dashed line shows the limiting value $\lim_{N\to\infty}b(N)=0.75\,\mathrm{i}$. To illustrate that the energy for this state obeys eq. (8.2a), we depict via the open circles the values of $b(N)$ computed from the numerical data for the energy by inverting (8.2a) for $b(N)$ with the correction terms ignored.

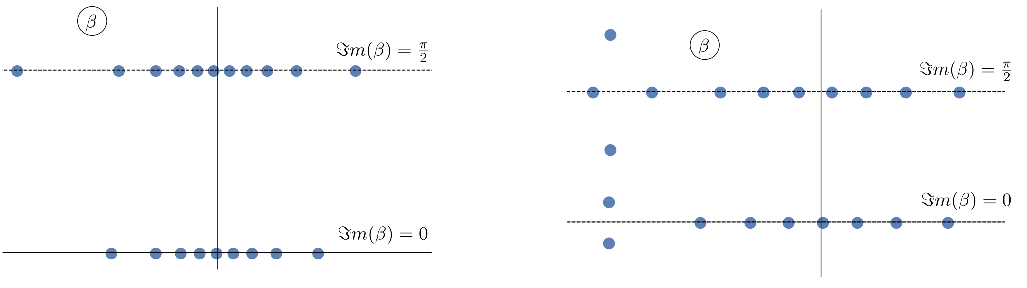

Fig. 5. The left panel shows the pattern of Bethe roots in the complex β plane with $\beta=-\frac{1}{2}\log(\zeta)$ for the primary Bethe state with $N=40$, $n=3$, $\mathtt{k}=\frac{1}{10}$ and $S^z=\mathtt{w}=0$. The value of b obtained from (8.5), (8.6) is consistent with the asymptotic relation (9.2) with $\mathtt{m}=2$. The right panel depicts the pattern of Bethe roots for the eigenstate of the Hamiltonian (7.6), (7.7) with $n=3$, $\mathtt{k}=\frac{1}{25}$ and $N=40$ characterized by $\mathtt{w}=1$, $S^z=0$, $\mathtt{L}=\bar{\mathtt{L}}=0$. For the RG trajectory continued from this state, $\lim_{N\to\infty}b(N)=-\frac{\mathrm{i}}{10}$, for further details see fig. 8 in ref. [15].

state with M_--M_+ being any integer from $-\frac{N}{2}+S^z$ to $\frac{N}{2}-S^z$, it should be emphasized that the states will only be low energy ones provided that $|M_--M_+|\ll N$. With this restriction they turn out to be primary Bethe states all having the same p, $\bar{p}$ given by eq. (8.1) with $\mathtt{w}=0$. The value of $b(N)$ (8.6) is always real and possesses the following leading large N behaviour

$$b(N)\asymp\frac{\pi\mathtt{m}}{4\log(N)}\,,\qquad N\to\infty\quad\text{with}\quad\mathtt{m}-\text{fixed}\qquad(\mathtt{L}=\bar{\mathtt{L}}=0)\,,\tag{9.1}$$

where $\mathtt{m}=M_--M_+$.

Formula (9.1) resembles the quantization condition of a quantum mechanical particle in a potential well of length $\propto\log(N)$. A more accurate quantization condition is achieved by taking into account the phase shift that the particle picks up in the vicinity of the turning points.[2] In

[2] The analogy with the potential well may seem rather artificial here. However in the third part of this work, in sec. 21.3, it will arise naturally in the discussion of the CFT underlying the scaling behaviour of the $\mathcal{Z}_2$ invariant six-vertex model.

ref. [11], based on a numerical analysis, the following remarkable formula was proposed for describing the large N behaviour of $b(N)$ for an RG trajectory with $\mathtt{L}=\bar{\mathtt{L}}=0$:

$$8\,b(N)\,\log\left(\frac{N}{2N_0}\right)+\delta\,\Big|_{s=b(N)}=2\pi\,\mathtt{m}+O\big((\log N)^{-\infty}\big)\,. \tag{9.2}$$

Here the phase shift δ is a continuous function of $s\in(-\infty,+\infty)$ such that $\delta|_{s=0}=0$ and

$$\mathrm{e}^{\frac{\mathrm{i}}{2}\delta}=2^{\frac{4\mathrm{i}s(n+2)}{n}}\,\frac{\Gamma(\frac{1}{2}+p-\mathrm{i}s)\,\Gamma(\frac{1}{2}+\bar{p}-\mathrm{i}s)}{\Gamma(\frac{1}{2}+p+\mathrm{i}s)\,\Gamma(\frac{1}{2}+\bar{p}+\mathrm{i}s)}\qquad\qquad(\mathtt{L}=\bar{\mathtt{L}}=0)\,. \tag{9.3}$$

The integer $\mathtt{m}$ takes even values if $\frac{N}{2}-S^z$ is even and odd values otherwise, so that

$$(-1)^{\mathtt{m}}=(-1)^{\frac{N}{2}-S^z}\,, \tag{9.4}$$

while the symbol $O\big((\log N)^{-\infty}\big)$ indicates that (9.2) holds true up to power law corrections in N. The explicit formula for the n dependent constant N_0 was found in the later work [15] and reads as

$$N_0=\frac{\sqrt{\pi}\,\Gamma\big(1+\frac{1}{n}\big)}{2\Gamma\big(\frac{3}{2}+\frac{1}{n}\big)}\,. \tag{9.5}$$

Since the above expression coincides with N_0 from (5.36) upon the substitution $\beta^2\mapsto\frac{2}{n+2}$, with some abuse of notation we use the same symbol for these two constants.

For a primary Bethe state with $|\mathtt{k}|>\frac{2}{n+2}$ or non-zero $\mathtt{w}$ some of the Bethe roots ζ_j become complex. Nevertheless, numerical work shows that eq. (9.2) holds true for the primary Bethe states for any generic value of the twist parameter $-\frac{1}{2}<\mathtt{k}<\frac{1}{2}$, the positive integer $S^z=0,1,2,\ldots$ as well as the winding number $\mathtt{w}=0,\pm1,\pm2\ldots$. However there is a possibility that there could be multiple primary Bethe states having distinct $b(N)$, which satisfy eq. (9.2) with the same integer $\mathtt{m}$. We observed that for sufficiently large N for one of these states $b(N)$ is always real, while for the rest it is pure imaginary. This is tied to the fact that for $N\gg1$ the l.h.s. of (9.2) becomes a monotonic continuous function of real b. Thus for given $N\gg1$, p and $\bar{p}$, one can distinguish the primary Bethe states $\boldsymbol{\Psi}_N$ having *real* $b(N)$ via the integer $\mathtt{m}$ from eq. (9.2). Moreover $b=b_{\mathtt{m}}(N)$ obeys the ordering

$$b_{\mathtt{m}}(N)<b_{\mathtt{m}'}(N)\qquad\text{whenever}\qquad\mathtt{m}<\mathtt{m}'\,. \tag{9.6}$$

For $\mathtt{m}$ in (9.2) to correspond to a low energy state, it should be bounded as $|\mathtt{m}|\le\mathtt{m}_{\max}$ with some positive integer $\mathtt{m}_{\max}=\mathtt{m}_{\max}(N)\ll N$. This is similar to the case with $\mathtt{w}=0$ and $|\mathtt{k}|<\frac{2}{n+2}$ discussed above. Again numerical work suggests that it is possible to construct a Bethe state for any $\mathtt{m}=-\mathtt{m}_{\max},-\mathtt{m}_{\max}+2,\ldots,\mathtt{m}_{\max}-2,\,\mathtt{m}_{\max}$. Eq. (9.1) implies that $b_{\mathtt{m}+1}(N)-b_{\mathtt{m}}(N)\propto 1/\log(N)$ and hence for $N\gg1$ the $b_{\mathtt{m}}(N)$ become densely distributed within the segment $(-b_{\max}(N),+b_{\max}(N))$ where $b_{\max}(N)=b_{\mathtt{m}}(N)$ with $\mathtt{m}=\mathtt{m}_{\max}(N)$. Though it is difficult to give an accurate estimate of $b_{\max}(N)$, one may expect that

$$\lim_{N\to\infty}\frac{\log(N)}{\mathtt{m}_{\max}(N)}=0 \tag{9.7}$$

and hence $\lim_{N\to\infty}b_{\max}(N)=\infty$. This way we conclude that in the scaling limit the spectrum develops a continuous component, which we will label by the parameter $-\infty<s<+\infty$. As it follows from (9.2), for $N\gg1$ the number of primary Bethe states with real $b(N)$ lying in the segment $(s,s+\Delta s)$ is approximated by $\rho^{(0,0)}_{\bar{p},p}(s)\,\Delta s$ with

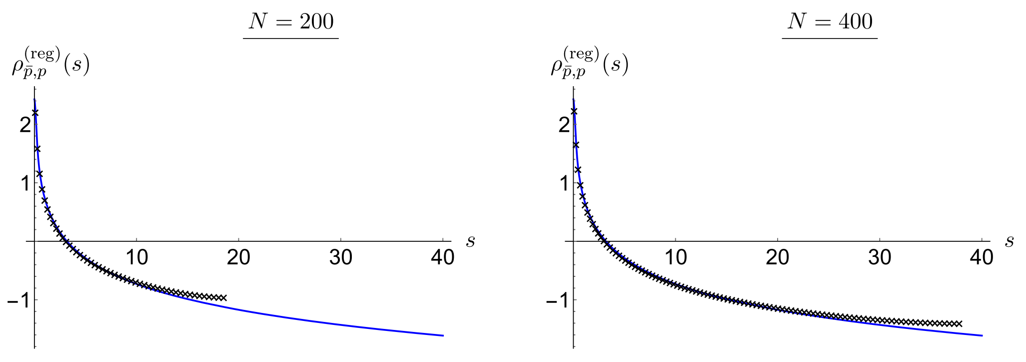

Fig. 6. For the two cases $N = 200, 400$ with the parameters taken to be $S^z = 0$, $\mathtt{k} = \frac{1}{10}$, $n = 3$, the Bethe states were constructed for which the Bethe roots are real with M_+ of them being positive and M_- being negative, i.e., $\{\zeta_m\}_{m=1}^{M} = \{\zeta_m^{(+)} > 0\}_{m=1}^{M_+} \cup \{\zeta_m^{(-)} < 0\}_{m=1}^{M_-}$. Note that when $|M_- - M_+| \ll N$ these states are part of the low energy spectrum with $\mathtt{L} = \bar{\mathtt{L}} = \mathtt{w} = 0$. The corresponding values of $b = b_{\mathtt{m}}$ were computed for $\mathtt{m} \equiv M_- - M_+ = 0, 2, 4, \ldots \frac{N}{2}$ and the crosses on the plots represent $(b_{\mathtt{m}+1} - b_{\mathtt{m}})^{-1} - \frac{2}{\pi}\log\big(\frac{N}{2N_0}\big)$ as a function of $s = (b_{\mathtt{m}+1} + b_{\mathtt{m}})/2$. The blue line depicts $\rho^{(\mathrm{reg})}_{\bar{p},p} = \rho^{(0,0)}_{\bar{p},p}(s) - \frac{2}{\pi}\log\big(\frac{N}{2N_0}\big)$, where $\rho^{(0,0)}_{\bar{p},p}(s)$ is the density of primary Bethe states given in eq. (9.8).

$$\rho^{(0,0)}_{\bar{p},p}(s) = \frac{2}{\pi}\,\log\left(\frac{N}{2N_0}\right) + \frac{1}{2\pi\mathrm{i}}\,\partial_s \log\left[2^{\frac{4\mathrm{i}s(n+2)}{n}}\,\frac{\Gamma(\frac{1}{2}+p-\mathrm{i}s)\,\Gamma(\frac{1}{2}+\bar{p}-\mathrm{i}s)}{\Gamma(\frac{1}{2}+p+\mathrm{i}s)\,\Gamma(\frac{1}{2}+\bar{p}+\mathrm{i}s)}\right]. \tag{9.8}$$

For an illustration see Fig. 6. The parameter s can be understood as an RG invariant along with p and $\bar{p}$. Then the scaling limit of a family of primary Bethe states $\boldsymbol{\Psi}_N$ corresponding to a given value of s can be achieved by assigning an N dependence to the integer $\mathtt{m}$ via the formula

$$8\,s\,\log\left(\frac{N}{2N_0}\right) + \delta = 2\pi\,\mathtt{m}(N) + O\big((\log N)^{-\infty}\big)\,. \tag{9.9}$$

With this understanding of the scaling limit it follows from (9.9), (9.2) that $\operatorname{slim}_{N\to\infty} b(N) = s$ for a RG trajectory labeled by s and hence

$$\operatorname*{slim}_{N\to\infty}\ \frac{N}{4\pi\, v_{\mathrm{F}}}\,\Big(\mathcal{E} - e_\infty\, N\Big) = \frac{p^2+\bar{p}^2}{n+2} + \frac{2s^2}{n} - \frac{1}{6} \qquad\qquad (\mathtt{L} = \bar{\mathtt{L}} = 0)\,. \tag{9.10}$$

Numerical studies show that there exist the RG trajectories $\boldsymbol{\Psi}_N$ with $b(N)$ tending to a pure imaginary number in the large N limit. The pattern of Bethe roots for one such trajectory with $\mathtt{L} = \bar{\mathtt{L}} = 0$ is shown in the right panel of Fig. 5. In this case the quantum number $\mathtt{m}$ is not well defined (besides its parity). However, $b(N)$ still satisfies the exponential form of eq. (9.2),

$$\left(\frac{N}{2N_0}\right)^{4\mathrm{i}b(N)}\ \mathrm{e}^{\frac{\mathrm{i}}{2}\delta}\,\Big|_{s=b(N)} = \sigma + O\big((\log N)^{-\infty}\big) \tag{9.11}$$

with $\mathrm{e}^{\frac{\mathrm{i}}{2}\delta}$ as in (9.3) and σ is a sign factor, which coincides with the parity of $\frac{N}{2} - S^z$:

$$\sigma = (-1)^{\frac{N}{2}-S^z}\,. \tag{9.12}$$

In constructing a RG trajectory the value of σ should be kept fixed.

Let's consider a trajectory with $\lim_{N\to\infty} b(N) = s$ such that $\Im m(s) \neq 0$. The factor $N^{4\mathrm{i}b(N)}$ in the l.h.s. of eq. (9.11) goes to zero as $N \to \infty$ if $\Im m(s) > 0$ or tends to infinity when $\Im m(s) < 0$. Hence in order for (9.11) to be obeyed, one must have that

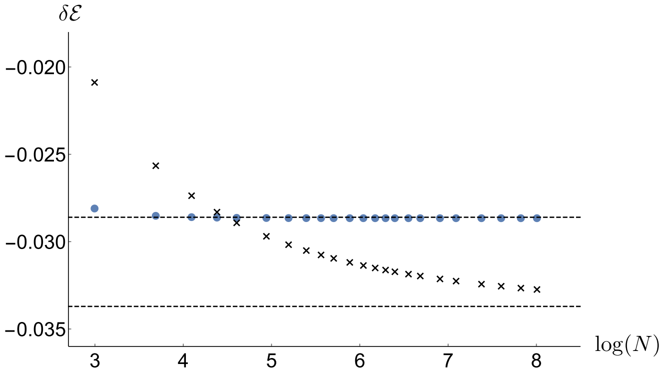

Fig. 7. The value of $\delta\mathcal{E} = \frac{N}{4\pi v_{\mathrm{F}}}\,(\mathcal{E} - e_\infty N)$ is plotted as a function of $\log(N)$ for two RG trajectories in the case $n = 3$ and $\mathtt{k} = 0.235 > \frac{1}{n+2}$. The blue dots depict $\delta\mathcal{E}$ for the trajectory having $\mathtt{L} = \bar{\mathtt{L}} = S^z = \mathtt{w} = 0$ and with $s = 0$. If $\log(N) \lessapprox 4.5$ the corresponding Bethe state is the ground state of the Hamiltonian $\mathbb{H}$. For the second RG trajectory (crosses) the RG invariants $\mathtt{L}$, $\bar{\mathtt{L}}$, S^z, $\mathtt{w}$ are also zero but $s = \frac{\mathrm{i}}{2}\big((n+2)\,\mathtt{k} - 1\big) = \frac{7\mathrm{i}}{80}$ is a pure imaginary number. The dashed lines denote the limiting values $\delta\mathcal{E} = -0.0286\ldots$ for the RG trajectory with zero s and $\delta\mathcal{E} = -0.0337\ldots$ for the other one.

$$\mathrm{e}^{-\frac{\mathrm{i}}{2}\delta} = 0 \qquad \text{for} \qquad \Im m(s) > 0 \tag{9.13}$$
$$\mathrm{e}^{+\frac{\mathrm{i}}{2}\delta} = 0 \qquad \text{for} \qquad \Im m(s) < 0\,.$$

This condition, combined with the explicit formula for $\mathrm{e}^{\frac{\mathrm{i}}{2}\delta}$ (9.3), yields that the limiting values $s = \lim_{N\to\infty} b(N)$ with $\Im m(s) \neq 0$ must be of the form

$$s = \pm\mathrm{i}\left(-\,p_{\min} - \tfrac{1}{2} - a\right), \tag{9.14}$$

where $p_{\min} = \min(p, \bar{p})$ and a may be any non-negative integer provided that

$$a\ :\ a \geq 0\ \ \text{and}\ \ \ -p_{\min} - \tfrac{n+2}{4} \leq a < -p_{\min} - \tfrac{1}{2}\ . \tag{9.15}$$

In writing the above inequalities on a we've taken into account the restriction (8.7). Note that $-p_{\min} - \frac{1}{2} - a$ must be a positive number. Hence such values (9.14) are only possible if either $p < -\frac{1}{2}$, in which case $\bar{p} > \frac{1}{2}$ or the other way around: $\bar{p} < -\frac{1}{2}$ and $p > \frac{1}{2}$.

The RG trajectory for which $b(N)$ tends to a pure imaginary number can be labeled by the limiting value of $b(N)$, i.e., s from (9.14), (9.15). The latter should be treated as an RG invariant along with p and $\bar{p}$. The scaling limit of the energy for such states is still described by eq. (9.10). It should be mentioned that when $\frac{1}{n+2} < |\mathtt{k}| < \frac{1}{2}$, the $\mathcal{Z}_2$ doublet of the primary Bethe states with $\mathtt{w} = S^z = 0$ and $s = \pm\frac{\mathrm{i}}{2}\big((n+2)\,|\mathtt{k}| - 1\big)$ turn out to be the lowest energy states of the lattice Hamiltonian $\mathbb{H}$ for $N \gg 1$. Their energy is lower than that of the primary Bethe state with $\mathtt{w} = S^z = s = 0$, which is the ground state in the case $|\mathtt{k}| \leq \frac{1}{n+2}$. An example is provided in Fig. 7.

10. Summary of numerical work: basic conjectures

From the study of the primary Bethe states we've found that, with a proper understanding of the scaling limit, the RG trajectories with $\mathtt{L} = \bar{\mathtt{L}} = 0$ are labeled by p, $\bar{p}$ and s. The last

RG invariant may take any real values $s \in (-\infty, +\infty)$ as well as a finite discrete set of pure imaginary numbers given by eqs. (9.14) and (9.15). For the low energy Bethe states with $\mathtt{L}+\bar{\mathtt{L}} > 0$ the same qualitative picture is expected to hold true as well. In particular, these trajectories may be assigned the number $s = \operatorname{slim}_{N\to\infty} b(N)$, subject to the constraint

$$-\frac{n}{4} \leq \Im m(s) \leq \frac{n}{4} \tag{10.1}$$

(to be compared with (8.7)). This way the conformal towers appearing in the scaling limit are labeled by $\bar{p}$, p, as well as s, whose set of admissible values contains both a continuous and a discrete component. An explicit description of both these components is given in this section. Since our analysis involves many assumptions, which are mostly justified through the numerical work, we formulate our findings regarding the conformal towers as a series of conjectures.

Clearly, when $\mathtt{L}+\bar{\mathtt{L}} > 0$, the RG invariants p, $\bar{p}$ and s are insufficient for the unambiguous specialization of a RG trajectory or, equivalently, a state in the level subspace of the conformal tower. The latter is achieved by means of the two non-ordered sets

$$\boldsymbol{w} = \{w_a\}_{a=1}^{\mathtt{L}} , \qquad\qquad \bar{\boldsymbol{w}} = \{\bar{w}_a\}_{a=1}^{\bar{\mathtt{L}}} , \tag{10.2}$$

which play a rôle similar to $\boldsymbol{v} = \{v_a\}_{a=1}^{\mathtt{L}}$ and $\bar{\boldsymbol{v}} = \{\bar{v}_a\}_{a=1}^{\bar{\mathtt{L}}}$ in the homogeneous case. For the $\mathcal{Z}_2$ invariant six-vertex model, the algebraic systems satisfied by $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$ read, respectively, as

$$\begin{aligned} &4n\, w_a^2 + 8\mathrm{i}s\,(n+1)\, w_a - (n+2)\,\big((n+1)^2-4p^2\big) \\ &\qquad +4\sum_{b\neq a}^{\mathtt{L}} \frac{w_a\,(\,(n+2)^2\, w_a^2 - n(2n+5)\, w_a w_b + n(n+1)\, w_b^2\,)}{(w_a-w_b)^3} = 0 \\ &\qquad (a=1,\ldots,\mathtt{L})\,, \end{aligned} \tag{10.3a}$$

$$\begin{aligned} &4n\, \bar{w}_a^2 + 8\mathrm{i}s\,(n+1)\, \bar{w}_a - (n+2)\,\big((n+1)^2-4\bar{p}^2\big) \\ &\qquad +4\sum_{b\neq a}^{\bar{\mathtt{L}}} \frac{\bar{w}_a\,(\,(n+2)^2\, \bar{w}_a^2 - n(2n+5)\, \bar{w}_a \bar{w}_b + n(n+1)\, \bar{w}_b^2\,)}{(\bar{w}_a-\bar{w}_b)^3} = 0 \\ &\qquad (a=1,\ldots,\bar{\mathtt{L}})\,. \end{aligned} \tag{10.3b}$$

It was conjectured in the work [15] that for fixed $\mathtt{L}$ and for generic values of p, n the number of solutions of (10.3a), up to the action of the permutation group, is equal to $\mathtt{par}_2(\mathtt{L})$ – the number of bipartitions of $\mathtt{L}$,

$$\sum_{\mathtt{L}=0}^{\infty} \mathtt{par}_2(\mathtt{L})\, \mathtt{q}^{\mathtt{L}} = \frac{1}{(\mathtt{q},\mathtt{q})_\infty^2} = 1+2\,\mathtt{q}+5\,\mathtt{q}^2+10\,\mathtt{q}^3+20\,\mathtt{q}^4+36\,\mathtt{q}^5+\ldots\,. \tag{10.4}$$

Here and below we use the notation

$$(z,\mathtt{q})_\infty = \prod_{m=0}^{\infty} (1-z\,\mathtt{q}^m)\,. \tag{10.5}$$

In turn, the number of solutions of (10.3b) is expected to be $\mathrm{par}_2(\bar{\mathtt{L}})$.

In the description of the scaling limit for the primary Bethe states a key rôle was played by eq. (9.11). This relation was extended to the RG trajectories with any values of the non-negative integers $\mathtt{L}$ and $\bar{\mathtt{L}}$ in the work [15]. All that is required is a modification of the phase shift δ, which is now a function of the sets $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$ solving eqs. (10.3), as well as $\bar{p}$, p and s so that $\delta = \delta(\bar{\boldsymbol{w}}, \boldsymbol{w}\,|\,\bar{p}, p, s)$. In that same work a formula was proposed, which expresses $\mathrm{e}^{\frac{\mathrm{i}}{2}\delta}$ in terms of the connection coefficients of a certain ODE (see also the next section and, in particular, eq. (11.22) below). It reads as

$$\mathrm{e}^{\frac{\mathrm{i}}{2}\delta(\bar{\boldsymbol{w}},\boldsymbol{w}|\bar{p},p,s)} = D_{\bar{p},s}(\bar{\boldsymbol{w}})\, D_{p,s}(\boldsymbol{w})\,, \tag{10.6}$$

where

$$D_{p,s}(\boldsymbol{w}) = 2^{\frac{2\mathrm{i}(n+2)s}{n}}\ \frac{\Gamma(\frac{1}{2}+p-\mathrm{i}s)}{\Gamma(\frac{1}{2}+p+\mathrm{i}s)}\ \check{D}_{p,s}(\boldsymbol{w}) \tag{10.7}$$

and $\check{D}_{p,s}(\boldsymbol{w})$ are normalized to be one for $\mathtt{L}=0$. For general $\mathtt{L}$, the explicit expression for $\check{D}_{p,s}(\boldsymbol{w})$ as a function of p, s and the set $\boldsymbol{w}$ was derived in ref. [47]. It's quoted in formula (B.2) in Appendix B.

10.1. Continuous spectrum

In general (9.11), regarded as an equation determining the large N dependence of $b(N)$, has complex solutions. Nevertheless there exists a class of them such that $\lim_{N\to\infty}\Im m\big(b(N)\big)=0$. For their description it is useful to take the logarithm of both sides of (9.11) and bring it to the form (9.2). The phase shift δ entering into that equation depends on s both explicitly and implicitly through the solution sets $\boldsymbol{w}$, $\bar{\boldsymbol{w}}$ of (10.3). One should choose $\boldsymbol{w}$, $\bar{\boldsymbol{w}}$ in such a way so that they are continuous functions of s. It will be argued later that for real p, $\bar{p}$ the product $D_{\bar{p},s}(\bar{\boldsymbol{w}})\, D_{p,s}(\boldsymbol{w})$ in (10.6) is never zero or infinity for any $s\in(-\infty,+\infty)$. Due to this δ can be made to be a uniformly bounded continuous function of real s.

Suppose that the term $\propto \log(N)$ in the l.h.s. of eq. (9.2) dominates. Then an iterative solution yields

$$b_{\mathtt{m}}(N) \asymp \frac{\pi\mathtt{m}-\frac{1}{2}\,\delta_0}{4\log\big(N\mathrm{e}^{\frac{1}{8}\delta_0'}/(2N_0)\big)} + O\big((\log N)^{-3}\big)\,, \qquad N\to\infty \quad \text{with} \quad \mathtt{m}-\text{fixed}\,, \tag{10.8}$$

where

$$\delta_0 = \delta\big|_{s=0}\,, \qquad\qquad \delta_0' = \partial_s\,\delta\big|_{s=0}\,. \tag{10.9}$$

These last two numbers are typically complex so that $\Im m\big(b_{\mathtt{m}}(N)\big) = O\big(1/\log(N)\big)$. The solutions of this class can be labeled by the integer $\mathtt{m}$ and, in addition, obey the ordering

$$\Re e\big(b_{\mathtt{m}}(N)\big) < \Re e\big(b_{\mathtt{m}'}(N)\big)\,, \quad \text{for} \quad \mathtt{m}<\mathtt{m}' \qquad (N\gg 1)\,. \tag{10.10}$$

Let $\mathcal{H}^{(\mathrm{cont})}_{N|S^z}$ be the set of low energy Bethe states in the sector with given S^z such that $\Im m\big(b(N)\big)\to 0$ as $N\to\infty$. Appealing to numerical work, we expect that for fixed $N\gg 1$, and given $\mathtt{L}$, $\bar{\mathtt{L}}$, $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$, the states from $\mathcal{H}^{(\mathrm{cont})}_{N|S^z}$ can be labeled by the integer $\mathtt{m}$, which is defined through eq. (9.2). This integer takes the values $\mathtt{m} = -\mathtt{m}_{\max}, -\mathtt{m}_{\max}+2, \ldots, \mathtt{m}_{\max}-2, \mathtt{m}_{\max}$

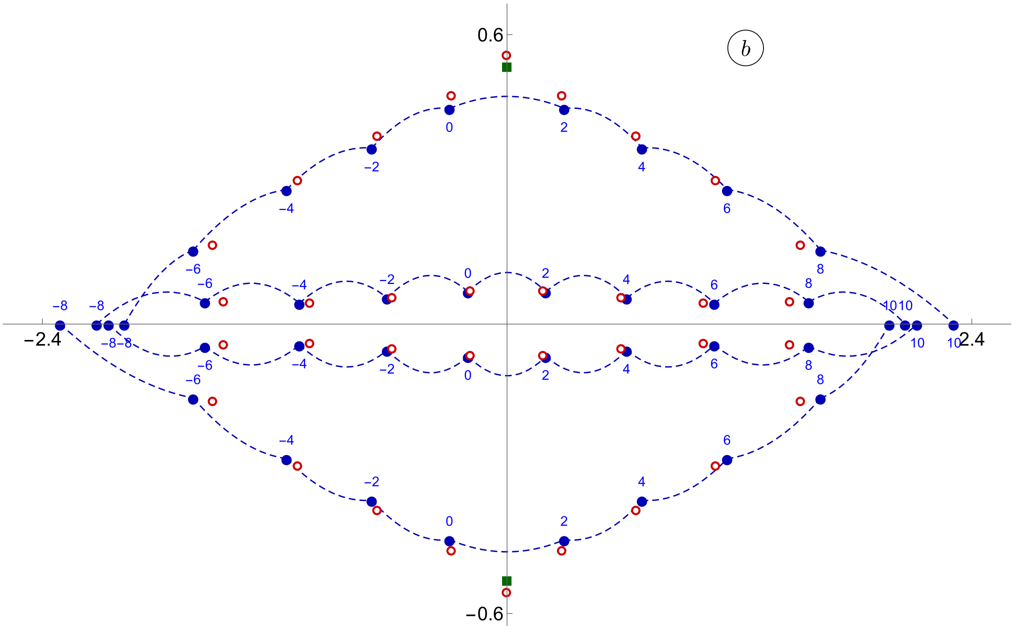

Fig. 8. The figure depicts all the solutions of (9.11), regarded as an equation for $b(N)$ with the correction term ignored, in the rectangle $-0.6 < \Im m\big(b(N)\big) < 0.6$ and $-2.4 < \Re e\big(b(N)\big) < 2.4$. The case being considered is $N = 22$ and $\mathtt{L} = \bar{\mathtt{L}} = 1$, while the parameters are $n = 3$, $\mathtt{k} = -0.18$, $S^z = 1$ and $\mathtt{w} = 0$ so that $p = \frac{1}{20}$, $\bar{p} = \frac{19}{20}$. The algebraic system (10.3a) for $\mathtt{L} = 1$ becomes a quadratic equation on $w \equiv w_1$, whose two solutions are given by $w_\pm = -\frac{n+1}{2n}\big(2\mathrm{i}\, s \pm \sqrt{C}\big)$, where $C = n(n+2)\big(1 - \frac{4p^2}{(n+1)^2} - \frac{4s^2}{n(n+2)}\big)$. Here the branch of the square root is taken so that $\sqrt{C}$ is positive when $C > 0$. In the case when $C < 0$, we set $w_\pm = -\mathrm{i}\,\frac{n+1}{2n}\big(2s \mp \sqrt{-C}\big)$. Similar formulae, with p substituted by $\bar{p}$, are used to define $\bar{w}_\pm$ which solve (10.3b) with $\bar{\mathtt{L}} = 1$. The phase shift entering into (9.11) may be any one of the four functions $\delta(\bar{w}_\sigma, w_{\sigma'}) \equiv \delta(\bar{w}_\sigma, w_{\sigma'} \,|\, \bar{p}, p, s)$ with $\sigma, \sigma' = \pm 1$. The filled circles corresponding to the solutions of (9.11) with the same function δ are grouped together by the dashed line for visualization. Those from the top set represent solutions of (9.2) with $\delta = \delta(\bar{w}_+, w_+)$. The integer $\mathtt{m}$ entering into that equation is indicated by the label beside each circle. The lower set of connected filled circles corresponds to $\delta = \delta(\bar{w}_-, w_+)$, the next lowest (just below the real axis) to $\delta = \delta(\bar{w}_+, w_-)$, while for the bottom most set of circles $\delta = \delta(\bar{w}_-, w_-)$. The green boxes also depict solutions of (9.11) with $\delta = \delta(\bar{w}_+, w_+)$ for the top box and $\delta = \delta(\bar{w}_-, w_-)$ for the bottom one. However, whereas all the filled circles correspond to the RG trajectories with $\lim_{N\to\infty} \Im m\big(b(N)\big) = 0$, for the green boxes $b(N)$ tends to a non-vanishing pure imaginary value $\lim_{N\to\infty} \Im m\big(b(N)\big) = \pm \mathrm{i}\,(p + \frac{1}{2}) = \pm 0.55\,\mathrm{i}$. For these solutions the definition of the integer $\mathtt{m}$ from (9.2) is ambiguous, since δ turns out to have a logarithmic branch point at $s = \pm \mathrm{i}\,(p + \frac{1}{2})$. Finally, the empty circles represent the value of $b(N)$ (8.6) that was obtained by means of direct diagonalization of the quasi-shift operator (8.4) within the sector $\mathtt{L} = \bar{\mathtt{L}} = 1$, $S^z = 1$ and $\mathtt{w} = 0$. Note that the states corresponding to $\mathtt{m} = -8$ and $\mathtt{m} = 10$ were not observed among the first 700 lowest energy states.

with some $\mathtt{m}_{\max} = \mathtt{m}_{\max}(N) \ll N$. For an illustration see Fig. 8. The asymptotic condition (10.8) implies that $b_{\mathtt{m}+1}(N) - b_{\mathtt{m}}(N) \propto 1/\log(N)$ so that the set $\{b_{\mathtt{m}}(N)\}$ becomes densely distributed within the segment $\big(-b_{\max}(N), +b_{\max}(N)\big)$. As N tends to infinity we suppose that $b_{\max}(N) \to +\infty$. All the above properties are analogous to those of the primary Bethe states with real $b(N)$ discussed before, except that now $\Im m\big(b(N)\big)$ vanishes only in the limit $N \to \infty$. This way we come to the conjecture:

(I) For fixed $N \gg 1$ let $\Delta\mathcal{N}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p,s}$ be the number of Bethe states from the set $\mathcal{H}^{(\mathrm{cont})}_{N|S^z}$ with given $\mathtt{L}$, $\bar{\mathtt{L}}$, p, $\bar{p}$ such that $\Re e\big(b(N)\big)$ lies in the interval $(s, s+\Delta s) \subset \big(-b_{\max}(N), +b_{\max}(N)\big)$. Then

$$\Delta\mathcal{N}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p,s} \approx \rho^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p}(s)\,\Delta s \qquad (\Delta s \ll 1) \tag{10.11}$$

with the density

$$\rho^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p}(s) = \mathrm{par}_2(\mathtt{L})\,\mathrm{par}_2(\bar{\mathtt{L}})\,\rho^{(0,0)}_{\bar{p},p}(s) + \tag{10.12}$$

$$\frac{1}{2\pi\mathrm{i}}\,\partial_s\Bigg(\mathrm{par}_2(\mathtt{L})\log\Bigg(\prod_{\substack{\bar{\boldsymbol{w}}\\ \bar{\mathtt{L}}-\text{fixed}}}\check{D}_{\bar{p},s}(\bar{\boldsymbol{w}})\Bigg) + \mathrm{par}_2(\bar{\mathtt{L}})\log\Bigg(\prod_{\substack{\boldsymbol{w}\\ \mathtt{L}-\text{fixed}}}\check{D}_{p,s}(\boldsymbol{w})\Bigg)\Bigg)\,.$$

The density of primary Bethe states $\rho^{(0,0)}_{\bar{p},p}(s)$ is quoted in eq. (9.8). Also the product in $\prod_{\boldsymbol{w}}\check{D}_{p,s}(\boldsymbol{w})$ goes over all the $\mathrm{par}_2(\mathtt{L})$ solutions $\boldsymbol{w}$ of eq. (10.3a) with fixed $\mathtt{L}$ and similarly for $\prod_{\bar{\boldsymbol{w}}}\check{D}_{\bar{p},s}(\bar{\boldsymbol{w}})$.

In ref. [47] the following explicit formula for the product $\prod_{\boldsymbol{w}}\check{D}_{p,s}(\boldsymbol{w})$ was obtained:

$$\prod_{\substack{\boldsymbol{w}\\ \mathtt{L}-\text{fixed}}}\check{D}_{p,s}(\boldsymbol{w})$$

$$= \prod_{m=1}^{\mathtt{L}}\prod_{\substack{1\le j,k\\ jk\le m}}\left[\frac{(2p-2\mathrm{i}s+2k-j)\,(2p+2\mathrm{i}s-2k+j)}{(2p+2\mathrm{i}s+2k-j)\,(2p-2\mathrm{i}s-2k+j)}\right]^{\mathtt{par}_1(m-kj)\,\mathtt{par}_1(\mathtt{L}-m)} \tag{10.13}$$

Since $\rho^{(0,0)}_{\bar{p},p}(s) = O\big(\log(N)\big)$, see eq. (9.8), Conjecture (I) implies that

$$\frac{\Delta\mathcal{N}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p,s}}{\Delta\mathcal{N}^{(0,0)}_{\bar{p},p,s}} = \mathrm{par}_2(\mathtt{L})\,\mathrm{par}_2(\bar{\mathtt{L}}) + O\big(1/\log(N)\big)\,. \tag{10.14}$$

Like for the primary Bethe states, the scaling limit of any state from $\mathcal{H}^{(\mathrm{cont})}_{N|S^z}$ can be defined by assigning the integer $\mathtt{m}$ an N dependence via eq. (9.9) so that $\mathrm{slim}_{N\to\infty} b(N) = s$ with some real s. Although $\mathcal{H}^{(\mathrm{cont})}_{N|S^z}$ has been so far regarded merely as the formal set of all the low energy Bethe states with $\lim_{N\to\infty}\Im m\big(b(N)\big) = 0$, it is natural to introduce the structure of the linear space on $\mathcal{H}^{(\mathrm{cont})}_{N|S^z}$. Then, with the above understanding of the scaling limit, it follows that for given $S^z = 0, 1, 2, \ldots,$ the linear space

$$\mathcal{H}^{(\mathrm{cont})}_{S^z} = \underset{N\to\infty}{\mathrm{slim}}\,\mathcal{H}^{(\mathrm{cont})}_{N|S^z} \tag{10.15}$$

admits the decomposition:

$$\mathcal{H}^{(\mathrm{cont})}_{S^z} = \bigoplus_{\mathtt{w}\in\mathbb{Z}}\int_{\mathbb{R}}^{\oplus}\mathrm{d}s\left[\bigoplus_{\mathtt{L},\bar{\mathtt{L}}=0}^{\infty}\mathcal{H}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p,s}\right], \qquad \text{where} \qquad \begin{aligned} p &= \tfrac{1}{2}\,S^z + \tfrac{1}{2}\,(n+2)\,(\mathtt{k}+\mathtt{w}) \\ \bar{p} &= \tfrac{1}{2}\,S^z - \tfrac{1}{2}\,(n+2)\,(\mathtt{k}+\mathtt{w}) \end{aligned} \tag{10.16}$$

and each level subspace has dimensions

$$\dim\big(\mathcal{H}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p,s}\big)=\mathrm{par}_2(\mathtt{L})\,\mathrm{par}_2(\bar{\mathtt{L}})\ . \tag{10.17}$$

In the scaling limit, the low energy Bethe states in $\mathcal{H}^{(\mathrm{cont})}_{N|S^z}$ with given $\bar{p}$, p, $\bar{\mathtt{L}}$, $\mathtt{L}$ and s form a basis in $\mathcal{H}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p,s}$. For all of them the scaled energy is

$$E=\underset{N\to\infty}{\mathrm{slim}}\ \frac{N}{4\pi\,v_{\mathrm{F}}}\,\Big(\mathcal{E}-e_\infty\,N\Big)=\frac{p^2+\bar{p}^2}{n+2}+\frac{2s^2}{n}-\frac{1}{6}+\mathtt{L}+\bar{\mathtt{L}}\,. \tag{10.18}$$

Thus, one can introduce the CFT Hamiltonian $\hat{H}\in\mathrm{End}\big(\mathcal{H}^{(\mathrm{cont})}_{S^z}\big)$ such that any state from $\mathcal{H}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p,s}$ is an eigenstate of this operator with energy given by the r.h.s. of (10.18). Symbolically,

$$\hat{H}=\underset{N\to\infty}{\mathrm{slim}}\ \frac{N}{4\pi\,v_{\mathrm{F}}}\,\Big(\mathbb{H}-e_\infty\,N\Big)\,. \tag{10.19}$$

10.2. Discrete spectrum

The states from $\mathcal{H}^{(\mathrm{cont})}_{N|S^z}$ do not cover all the low energy states in the lattice model. As we saw previously, there exist RG trajectories such that $\lim_{N\to\infty}b(N)=s$ with $\Im m(s)\neq 0$. The possible limiting values of $b(N)$ are still determined by the condition (9.13), which already appeared in our discussion of the primary Bethe states. In view of eq. (10.6), for a RG trajectory $\boldsymbol{\Psi}_N$ labeled by $\mathtt{L}$, $\bar{\mathtt{L}}$, p, $\bar{p}$ as well as the solutions sets $\boldsymbol{w}$, $\bar{\boldsymbol{w}}$ of the algebraic system (10.3), this condition can be rewritten as

$$\begin{aligned}&\big(D_{\bar{p},s}(\bar{\boldsymbol{w}})\,D_{p,s}(\boldsymbol{w})\big)^{-1}=0\qquad\text{for}\qquad \Im m(s)>0\\ &\big(D_{\bar{p},s}(\bar{\boldsymbol{w}})\,D_{p,s}(\boldsymbol{w})\big)^{+1}=0\qquad\text{for}\qquad \Im m(s)<0\,.\end{aligned} \tag{10.20}$$

The formula for $D_{p,s}(\boldsymbol{w})$ follows from eq. (10.7) as well as (B.2) from Appendix B. Notice that the algebraic system (10.3) is invariant w.r.t. the substitution $s\mapsto -s$, $\boldsymbol{w}\mapsto-\boldsymbol{w}$ and $\bar{\boldsymbol{w}}\mapsto-\bar{\boldsymbol{w}}$. Furthermore it turns out that

$$D_{p,s}(\boldsymbol{w})=\big(D_{p,-s}(-\boldsymbol{w})\big)^{-1}\,,\qquad D_{\bar{p},s}(\bar{\boldsymbol{w}})=\big(D_{\bar{p},-s}(-\bar{\boldsymbol{w}})\big)^{-1}\,. \tag{10.21}$$

Thus, without loss of generality, one can always focus on the domain $\Im m(s)>0$ and consider only the first line of (10.20).

The classification of the possible values of non-real s appearing in the scaling limit reduces to a study of the condition (10.20), where $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$ are solution sets of the joint algebraic system (10.3) with given non-negative integers $\mathtt{L}$ and $\bar{\mathtt{L}}$. Due to eqs. (8.1), (10.1) the parameters p, $\bar{p}$ and s will be restricted to the domain

$$\Im m(p)=\Im m(\bar{p})=0\,,\qquad p+\bar{p}=S^z=0,1,2,\dots\,;\qquad 0<\Im m(s)\leq\frac{n}{4}\,, \tag{10.22}$$

while n is a generic positive number. The position of the singularities of $D_{p,s}(\boldsymbol{w})$ as a function of s can in principle be found using the explicit formulae (10.7) and (B.2). However since s enters into the algebraic system (10.3a) that is solved by $\boldsymbol{w}$, such an analysis of $D_{p,s}(\boldsymbol{w})$ is rather difficult except for the first few levels. Nevertheless, one can make use of (10.13), which provides a simple expression for the product of $D_{p,s}(\boldsymbol{w})$ over all the $\mathrm{par}_2(\mathtt{L})$ solutions of (10.3a) with fixed $\mathtt{L}$. It is possible to re-write it in the form

$$\prod_{\substack{\boldsymbol{w}\\ \mathtt{L}-\text{fixed}}} D_{p,s}(\boldsymbol{w}) = \left(2^{\frac{2\mathrm{i}(n+2)s}{n}}\,\frac{\Gamma(\frac{1}{2}+p-\mathrm{i}s)}{\Gamma(\frac{1}{2}+p+\mathrm{i}s)}\right)^{\mathtt{par}_2(\mathtt{L})}\prod_{\substack{\boldsymbol{w}\\ \mathtt{L}-\text{fixed}}} \check{D}_{p,s}(\boldsymbol{w})\,, \tag{10.23}$$

where

$$\prod_{\substack{\boldsymbol{w}\\ \mathtt{L}-\text{fixed}}} \check{D}_{p,s}(\boldsymbol{w}) = \prod_{a=0}^{\mathtt{L}-1}\left[\frac{\left(\frac{1}{2}+a+p-\mathrm{i}s\right)\left(\frac{1}{2}+a-p-\mathrm{i}s\right)}{\left(\frac{1}{2}+a+p+\mathrm{i}s\right)\left(\frac{1}{2}+a-p+\mathrm{i}s\right)}\right]^{\mathrm{par}_2(\mathtt{L})-d_a(\mathtt{L})}\,. \tag{10.24}$$

Here the generating function for the integers $0 \le d_a(\mathtt{L}) \le \mathrm{par}_2(\mathtt{L})$ reads as

$$\chi_a(\mathfrak{q}) \equiv (\mathfrak{q},\mathfrak{q})_\infty^{-2}\sum_{m=0}^{\infty}(-1)^m\,\mathfrak{q}^{ma+\frac{m(m+1)}{2}} = \sum_{\mathtt{L}=0}^{\infty} d_a(\mathtt{L})\,\mathfrak{q}^{\mathtt{L}} \tag{10.25}$$

(for details see Appendix B). Assuming that all the singularities of $D_{p,s}(\boldsymbol{w})$ are poles and that there is no mutual cancellation of poles and zeroes in the product $\prod_{\boldsymbol{w}} \check{D}_{p,s}(\boldsymbol{w})$ in the r.h.s. of (10.23) one concludes that the poles of $D_{p,s}(\boldsymbol{w})$ may only be at $s = \pm\mathrm{i}\,(p+\frac{1}{2}+a)$ with a an integer. Provided some further assumptions are made (again see Appendix B), an analysis of eq. (10.23) together with the analogous formula for $\prod_{\bar{\boldsymbol{w}}} D_{\bar{p},s}(\bar{\boldsymbol{w}})$ leads one to the following conjectures:

(A) Let the parameters be such that $0 < \Im m(s) \le \frac{n}{4}$, $\Im m(p) = \Im m(\bar{p}) = 0$ with $p+\bar{p} = S^z = 0,1,2,\ldots$ and n a generic positive number. For any sets $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$, the values of s at which (10.20) is satisfied must be of the form $s = \mathrm{i}\mathfrak{q}_a,\ \mathrm{i}\bar{\mathfrak{q}}_a$ with

$$\mathfrak{q}_a = -p-\tfrac{1}{2}-a\,, \qquad \bar{\mathfrak{q}}_a = -\bar{p}-\tfrac{1}{2}-a\,. \tag{10.26}$$

Here a is an integer such that

$$\begin{array}{lll} -p-\frac{n+2}{4} \le a < -\frac{1}{2}-p & \text{for} & s=\mathrm{i}\mathfrak{q}_a \\ -\bar{p}-\frac{n+2}{4} \le a < -\frac{1}{2}-\bar{p} & \text{for} & s=\mathrm{i}\bar{\mathfrak{q}}_a\,. \end{array} \tag{10.27}$$

With the last restriction $0 < \mathfrak{q}_a, \bar{\mathfrak{q}}_a \le \frac{n}{4}$.

(B) There are $\mathrm{par}_2(\mathtt{L}) \times \mathrm{par}_2(\bar{\mathtt{L}})$ solutions of the joint system (10.3). Let $\mathcal{N}_a^{(\bar{\mathtt{L}},\mathtt{L})}$ and $\bar{\mathcal{N}}_a^{(\bar{\mathtt{L}},\mathtt{L})}$ denote the number of them for which (10.20) is obeyed at $s = \mathrm{i}\mathfrak{q}_a$ and $s = \mathrm{i}\bar{\mathfrak{q}}_a$, respectively. Then

$$\mathcal{N}_a^{(\bar{\mathtt{L}},\mathtt{L})} = d_{S^z+a}(\bar{\mathtt{L}})\,d_a(\mathtt{L})\,, \qquad \bar{\mathcal{N}}_a^{(\bar{\mathtt{L}},\mathtt{L})} = d_a(\bar{\mathtt{L}})\,d_{S^z+a}(\mathtt{L})\,, \tag{10.28}$$

where the integers $d_a(\mathtt{L})$ are defined via (10.25).[3]

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[3] The generating function $\chi_a(\mathfrak{q})$ obeys the identity

$$\chi_a(\mathfrak{q}) + \chi_{-1-a}(\mathfrak{q}) = (\mathfrak{q},\mathfrak{q})_\infty^{-2}\,,$$

which in turn implies that

$$d_a(\mathtt{L}) + d_{-1-a}(\mathtt{L}) = \mathrm{par}_2(\mathtt{L})\,.$$

This makes the definition of $0 \le d_a(\mathtt{L}) \le \mathrm{par}_2(\mathtt{L})$ (10.25) applicable for the case of negative a. Also there exists the following integral representation for $\chi_a(\mathfrak{q})$:

$$\chi_a(\mathfrak{q}) = \oint_{|z|<1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\,\frac{z^{-a-1}}{(z,\mathfrak{q})_\infty(z^{-1}\,\mathfrak{q},\mathfrak{q})_\infty}\,.$$

Since the condition (10.20) as well as the algebraic equations satisfied by $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$ are invariant w.r.t. the substitutions $s \mapsto -s$, $\boldsymbol{w} \mapsto -\boldsymbol{w}$ and $\bar{\boldsymbol{w}} \mapsto -\bar{\boldsymbol{w}}$, all the above follows through essentially unchanged for $-\frac{n}{4} \leq \Im m(s) < 0$. Thus, to take into account the full domain $|\Im m(s)| \leq \frac{n}{4}$, one just needs to replace $s = \mathrm{i}\mathfrak{q}_a, \mathrm{i}\bar{\mathfrak{q}}_a$ appearing in the above two conjectures with $s = \pm\mathrm{i}\mathfrak{q}_a, \pm\mathrm{i}\bar{\mathfrak{q}}_a$.

Let $\mathcal{H}^{(\mathrm{disc})}_{N|S^z}$ be the set of low energy states such that in the $N \to \infty$ limit $b(N)$ tends to a pure imaginary number $\pm\mathrm{i}\mathfrak{q}_a$ or $\pm\mathrm{i}\bar{\mathfrak{q}}_a$ defined through eqs. (10.26) and (10.27). In our investigations of the low energy states of the lattice model we found only the RG trajectories belonging to $\mathcal{H}^{(\mathrm{disc})}_{N|S^z}$, or those with $\lim_{N\to\infty} \Im m\big(b(N)\big) = 0$ which are members of $\mathcal{H}^{(\mathrm{cont})}_{N|S^z}$. For instance, we performed a numerical check by explicitly constructing the RG trajectories for all the low energy states in the lattice model with $N = 22$ sites in the sector $S^z = 1$, $\mathtt{w} = 0$ and $\mathtt{L} + \bar{\mathtt{L}} = 1$. For those states having both $\Im m\big(b(N)\big) \geq 0$ and $\Re e\big(b(N)\big) \geq 0$, a plot of b as a function of N is provided in Fig. 9. A systematic exposition of our numerical work is given in sec. 18 below. This way we come to expect

(II) For any low energy Bethe state $\boldsymbol{\Psi}_N$ belonging to the sector with $S^z = 0, 1, 2, 3\ldots$ either $\lim_{N\to\infty} \Im m\big(b(N)\big) = 0$ and $\boldsymbol{\Psi}_N \in \mathcal{H}^{(\mathrm{cont})}_{N|S^z}$, or else $\lim_{N\to\infty} \Im m\big(b(N)\big) = s$ with $s = \pm\mathrm{i}\mathfrak{q}_a,\ \pm\mathrm{i}\bar{\mathfrak{q}}_a$ (10.26), (10.27). In the latter case $\boldsymbol{\Psi}_N \in \mathcal{H}^{(\mathrm{disc})}_{N|S^z}$.

It should be kept in mind that eq. (10.20) is a necessary condition for the existence of an RG trajectory belonging to the set $\mathcal{H}^{(\mathrm{disc})}_{N|S^z}$. Establishing that such a trajectory actually exists can not be done based on the formal analysis of this equation alone. Moreover, one may imagine that there could be multiple RG trajectories, labeled by the identical sets $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$, whose $b(N)$ tends to the same pure imaginary value of s. Nevertheless in our numerical work we have always observed that for every $\boldsymbol{w}$, $\bar{\boldsymbol{w}}$ and $s = \pm\mathrm{i}\mathfrak{q}_a, \pm\mathrm{i}\bar{\mathfrak{q}}_a$ at which (10.20) is obeyed, there exists one and only one RG trajectory with $\lim_{N\to\infty} b(N) = s$.[4] For instance, the right panel of Fig. 5 depicts the typical pattern of Bethe roots for $\boldsymbol{\Psi}_N$ with $\lim_{N\to\infty} b(N) = -\mathrm{i}\bar{\mathfrak{q}}_2$. Among others, Fig. 9 presents numerical data for an RG trajectory with $\mathtt{L} = 0$, $\bar{\mathtt{L}} = 1$ and for which $b(N) \to \mathrm{i}\mathfrak{q}_{-1}$. This leads us to the conjecture:

(III) For sufficiently large N and fixed $\mathtt{L}$, $\bar{\mathtt{L}}$, p, $\bar{p}$, the number of low energy Bethe states $\boldsymbol{\Psi}_N$ such that $\lim_{N\to\infty} b(N) = \pm\mathrm{i}\mathfrak{q}_a$ is given by $\mathcal{N}_a^{(\bar{\mathtt{L}},\mathtt{L})} = d_{S^z+a}(\bar{\mathtt{L}})\ d_a(\mathtt{L})$, where $d_a(\mathtt{L})$ are defined through eq. (10.25). Similarly, there are $\bar{\mathcal{N}}_a^{(\bar{\mathtt{L}},\mathtt{L})} = d_a(\bar{\mathtt{L}})\ d_{S^z+a}(\mathtt{L})$ trajectories with $\lim_{N\to\infty} b(N) = \pm\mathrm{i}\bar{\mathfrak{q}}_a$.

It should be pointed out that the integers $\mathcal{N}_a^{(\bar{\mathtt{L}},\mathtt{L})}$ satisfy the condition

$$\mathcal{N}_a^{(0,0)} = \begin{cases} 1 & \text{for} \quad a \geq 0 \\ 0 & \text{for} \quad a < 0 \end{cases} \tag{10.29}$$

and in describing the conformal towers corresponding to pure imaginary s, it is necessary to distinguish the cases $a \geq 0$ and $a < 0$. For this purpose we denote by $\mathcal{H}^{(\mathrm{disc},+)}_{N|S^z}$ the set of RG

[4] Recall that in constructing an RG trajectory $\boldsymbol{\Psi}_N$ with pure imaginary $s = \lim_{N\to\infty} b(N)$ the parity of $\frac{N}{2} - S^z$ must be kept fixed. There exist the RG trajectories with even $\frac{N}{2} - S^z$ and odd $\frac{N}{2} - S^z$ which have the same value of s. However since for fixed parity of $\frac{N}{2} - S^z$ only one of these trajectories is present, we do not count this as a degeneracy.

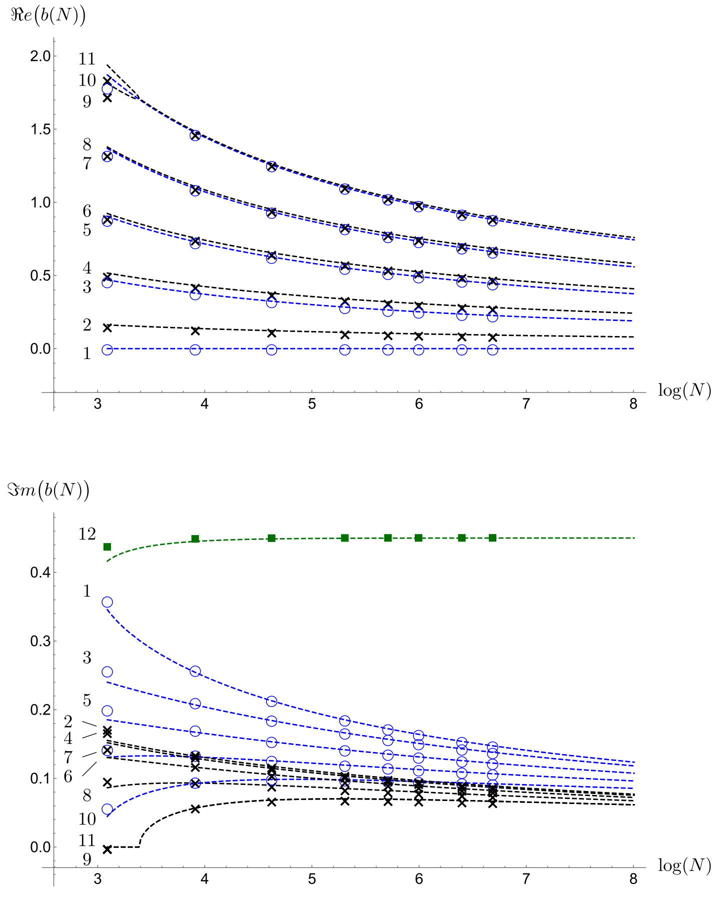

Fig. 9. The plots present the real and imaginary parts of $b(N)$, as functions of $\log(N)$, for twelve RG trajectories. The initial points of the trajectories correspond to all the low energy states in the sector $S^z = 1$, $\mathtt{w} = 0$, $\mathtt{L} + \bar{\mathtt{L}} = 1$ and with $\Re e\big(b(N)\big) \geq 0$, $\Im m\big(b(N)\big) \geq 0$ (see also Fig. 20). These were found from the numerical diagonalization of the Hamiltonian and quasi-shift operator for the spin chain of length $N = 22$ and with $n = 3$, $\mathtt{k} = -0.18$. The points for $N > 22$ were obtained via the solution of the Bethe ansatz equations. The dashed lines represent $b(N)$, which solves (9.11) regarded as an equation for $b(N)$ with the correction terms ignored. For the first eleven trajectories $\lim_{N\to\infty} \Im m\big(b(N)\big) = 0$ (the different symbols crosses/circles and colours black/blue are used only to improve the readability of the plot). Note that for trajectories 9 and 11, $b(N)$ is real for $N \leq 26$ and forms a complex conjugated pair as $N \geq 30$ (the trajectory with $\Im m\big(b(N)\big) < 0$ is not depicted for $N \geq 30$). For the 12-th trajectory (green squares and dashed line) $b(N)$ is pure imaginary for any $N \geq 22$ and $\lim_{N\to\infty} b(N) = 0.45\,\mathrm{i}$.

trajectories with $b(N) \to \pm\mathrm{i}\mathfrak{q}_a, \pm\mathrm{i}\bar{\mathfrak{q}}_a$ and $a \geq 0$, while $\mathcal{H}^{(\mathrm{disc},-)}_{N|S^z}$ is the set of trajectories labeled by $s = \pm\mathrm{i}\mathfrak{q}_a, \pm\mathrm{i}\bar{\mathfrak{q}}_a$ with $a < 0$. As before the sets $\mathcal{H}^{(\mathrm{disc},\pm)}_{N|S^z}$ may be equipped with the structure of a linear space. Their scaling limits,

$$\mathcal{H}^{(\mathrm{disc},\pm)}_{S^z} = \operatorname*{slim}_{N\to\infty} \mathcal{H}^{(\mathrm{disc},\pm)}_{N|S^z}\,, \tag{10.30}$$

are decomposed into a direct sum over finite dimensional subspaces, similar to eq. (10.16), but with the direct integral replaced by a sum over the admissible values of the RG invariant $s =$

$\pm \mathrm{i}\mathfrak{q}_a$, $\pm \mathrm{i}\bar{\mathfrak{q}}_a$. We'll postpone a detailed account of this decomposition to sec. 12 as it requires a discussion of the scaling limit of the eigenvalues of the lattice operators $\mathbb{A}_\pm(\zeta)$.

11. Scaling limit of $A_\pm(\zeta)$

In ref. [15] it was proposed that the scaling limit of the eigenvalues of $\mathbb{A}_\pm(\zeta)$ for the low energy Bethe states is given in terms of the connection coefficients of a certain ODE. Let $A_\pm(\zeta)$ be the eigenvalue corresponding to $\boldsymbol{\Psi}_N$, labeled by the full set of RG invariants p, $\bar{p}$, $\boldsymbol{w}$, $\bar{\boldsymbol{w}}$ and s. Then

$$\underset{\substack{N\to\infty \\ b(N)\to s}}{\mathrm{slim}} \; G^{(N/2)}\big(-\mu^2\,|\,\tfrac{2}{n+2}\big)\, A_\pm\big(\mathrm{i}\,\big(N/(2N_0)\big)^{-\frac{n}{n+2}}\,\mu\big) = D_\pm(\mu\,|\,\boldsymbol{w},p,s)\,, \tag{11.1}$$

where $G^{(N)}(E\,|\,g)$ and N_0 are given in eqs. (5.48) and (9.5), respectively. The functions $D_\pm(\mu\,|\,\boldsymbol{w},p,s)$ coincide with the connection coefficients for the linear differential equation:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{p^2-\frac{1}{4}}{z^2}+\frac{2\mathrm{i}s}{z}+1+\sum_{a=1}^{\mathtt{L}}\left(\frac{2}{(z-w_a)^2}+\frac{n}{z(z-w_a)}\right)+\mu^{-2-n}\,z^n\right]\Phi=0\,. \tag{11.2}$$

The fact that the set $\boldsymbol{w}\equiv\{w_a\}_{a=1}^{\mathtt{L}}$ satisfies the algebraic system (10.3a) ensures that any solution of this ODE is monodromy free in the vicinity of each apparent singularity at $z=w_a$. To specify the connection coefficients introduce the two basis solutions, $\Phi_{\pm p}(z)$, of (11.2) such that

$$\Phi_{\pm p}(z)\to\frac{1}{\sqrt{\pi}}\,(n+2)^{\mp\frac{2p}{n+2}-\frac{1}{2}}\,\mu^{\mp p-\frac{1}{2}}\,\Gamma(\mp\tfrac{2p}{n+2})\;z^{\frac{1}{2}\pm p}\quad\text{as }\;z\to 0\qquad\big(0<\Re e(p)<1\big)\,. \tag{11.3}$$

For large z the term $\mu^{-2-n}\,z^n$ in (11.2) becomes dominant and one can define another solution through the $z\to+\infty$ asymptotic (to be compared with (5.51))

$$\Xi(z)\asymp\Big(\frac{z}{\mu}\Big)^{-\frac{n}{4}}\exp\left[-\frac{2}{n+2}\Big(\frac{z}{\mu}\Big)^{\frac{n}{2}+1}{}_2F_1\big(-\tfrac{1}{2},-\tfrac{n+2}{2n},\tfrac{n-2}{2n}\,\big|-\mu^{n+2}\,z^{-n}\big)\;+\;o(1)\right]. \tag{11.4}$$

Here we make the technical assumption that $\mu>0$ and $n\neq\frac{2}{2k-1}$ with $k=1,2,\ldots$. The connection coefficients $D_\pm(\mu\,|\,\boldsymbol{w},p,s)$ are given by

$$D_\pm(\mu\,|\,\boldsymbol{w},p,s)=\mp\mu\,\sin(\tfrac{2\pi p}{n+2})\,W[\,\Phi_{\pm p},\Xi\,] \tag{11.5}$$

with $W[\,\Phi_{\pm p},\Xi\,]=\Xi\partial_z\Phi_{\pm p}-\Phi_{\pm p}\partial_z\Xi$ being the Wronskian. The overall factor in (11.5) has been chosen so that, for generic values of p,

$$D_\pm(0\,|\,\boldsymbol{w},p,s)=1\,. \tag{11.6}$$

It can be shown that when $n>0$ and p is a generic complex number, $D_\pm(\mu\,|\,\boldsymbol{w},p,s)$ are entire functions of μ.

Unfortunately the formula (11.1), where the r.h.s. coincides with the connection coefficient (11.5), at the current moment remains a conjecture. Below we'll discuss some possible ways of checking this relation. However, before doing so let us explain at the formal level the link between the ODE (11.2), (10.3a), and the one appearing in the homogeneous case (5.32)-(5.34). When $\mathtt{L}$

is even and $s=0$, the algebraic system (10.3a) admits solutions such that $w_{\mathtt{L}+1-a}=-w_a$. Then the substitution

$$s\mapsto 0\,,\qquad w_a=-w_{\mathtt{L}+1-a}\mapsto \mathrm{i}\sqrt{\tfrac{v_a}{\alpha(\alpha+1)}}\,,\qquad n\mapsto -\tfrac{2\alpha}{\alpha+1}\,,\qquad p^2\mapsto \tfrac{4}{\alpha+1}\,P^2 \tag{11.7}$$

brings (10.3a) to the form (5.34). With these specializations and upon the change of variables $\Phi(z)\mapsto x^{\frac{\alpha}{2}}\,\Phi(x)$, $z\mapsto \frac{x^{\alpha+1}}{\alpha+1}$, the ODE (11.2) is transformed to the one given by (5.32), (5.33) with $E=(1+\alpha)^{\frac{2\alpha}{1+\alpha}}\,\mu^{-\frac{2}{1+\alpha}}$.

Expanding both sides of the relation (11.1) in a Taylor series in μ leads to an infinite set of sum rules for the Bethe roots. In particular, the series expansion of $A_+(\zeta)$ involves the finite sums

$$h_j^{(N)}=j^{-1}\sum_{m=1}^{\frac{N}{2}-S^z}(\zeta_m)^{-j} \tag{11.8}$$

computed on the corresponding RG trajectory $\boldsymbol{\Psi}_N$. On the other hand, in view of eq. (11.6) and that the connection coefficients are entire functions of μ, one has

$$\log D_\pm(\mu\,|\,\boldsymbol{w},p,s)=-\sum_{j=1}^{\infty}J_j^{(\pm)}(\boldsymbol{w},p,s)\,\lambda^j\,. \tag{11.9}$$

Here, for future convenience, we swap μ for the parameter λ defined via the relation

$$\mu=-\mathrm{i}\,(n+2)^{-\frac{2}{n+2}}\,\Gamma^2\big(-\tfrac{1}{n+2}\big)\,\lambda\,. \tag{11.10}$$

Through the perturbation theory for the differential equation (11.2), one can in principle derive an explicit expression for the expansion coefficients $J_j^{(\pm)}(\boldsymbol{w},p,s)$. The computations turn out to be quite cumbersome, however, for the case $\mathtt{L}=0$ when there are no apparent singularities, the first two $J_j^{(\mathrm{vac})}(p,s)\equiv J_j^{(+)}(\boldsymbol{w},p,s)\big|_{\mathtt{L}=0}$ are given by

$$J_1^{(\mathrm{vac})}(p,s)=-2s\;f_1\big(\tfrac{p}{n+2},\tfrac{1}{n+2}\big) \tag{11.11}$$

$$J_2^{(\mathrm{vac})}(p,s)=2^{\frac{4}{n+2}}\,\frac{\pi\,\Gamma^2(-\frac{1}{n+2})}{\Gamma^2(\frac12-\frac{1}{n+2})}\;f_1\big(\tfrac{p}{n+2},\tfrac{2}{n+2}\big)+4s^2\,f_2\big(\tfrac{p}{n+2},\tfrac{1}{n+2}\big)\,,$$

where the functions f_j are defined in eqs. (5.22), (5.25). The analogous expression for $J_{1,2}^{(-)}(\boldsymbol{w},p,s)\big|_{\mathtt{L}=0}$ may be obtained from (11.11) through the substitution $p\to -p$. Note that for general $j=1,2,\ldots$ the coefficients $J_j^{(\mathrm{vac})}(p,s)$ turn out to be polynomials in s of order j. A quick inspection of (11.11) as well as the formula (5.22) for f_1 shows that $J_2^{(\mathrm{vac})}(p,s)$ contains a simple pole when $\frac{2}{n+2}$ is equal to $\frac12$. In fact, similar to the homogeneous case, the coefficients $J_{2k}^{(\pm)}(\boldsymbol{w},p,s)$ with $k=1,2,3,\ldots$ possess a simple pole if $\frac{2}{n+2}=1-\frac{1}{2k}$. In this case one can define the regularized coefficient through the subtraction

$$J_{2k}^{(\pm,\mathrm{reg})}=\lim_{\frac{2}{n+2}\to 1-\frac{1}{2k}}\bigg[\,J_{2k}^{(\pm)}+\bigg(\frac{\Gamma^2(-\frac{1}{n+2})\,\Gamma(\frac{1}{2k})}{(n+2)\,\Gamma(1-\frac{2}{n+2})}\bigg)^{2k}\,\frac{\Gamma(-\frac12+k)}{2\sqrt{\pi}\,k\,\Gamma(1+k)}\;\frac{1}{\frac{2}{n+2}-1+\frac{1}{2k}}\,\bigg]\,. \tag{11.12}$$

Formulae (11.1) and (11.9) imply the infinite set of relations for $h_j^{(N)}$ (11.8):

$$\underset{\substack{N\to\infty\\ b(N)\to s}}{\mathrm{slim}}\ N^{-\frac{jn}{n+2}}\ h_j^{(N)} = \Big[(2N_0)^{\frac{n}{n+2}}\ (n+2)^{-\frac{2}{n+2}}\ \Gamma^2\big(-\tfrac{1}{n+2}\big)\Big]^{-j}\ J_j(\boldsymbol{w},p,s)\ , \tag{11.13}$$

which hold true for any odd $j=1,3,\ldots$ and for all even $j>1+\frac{2}{n}$. When j is even and $j<1+\frac{2}{n}$ (11.13) should be replaced by

$$\begin{aligned}&\underset{\substack{N\to\infty\\ b(N)\to s}}{\mathrm{slim}}\ N^{-\frac{jn}{n+2}}\left[h_j^{(N)}+\frac{(-1)^{\frac{j}{2}+1}N}{2j\cos(\frac{\pi j}{n+2})}\right]\\ &\qquad=\Big[(2N_0)^{\frac{n}{n+2}}(n+2)^{-\frac{2}{n+2}}\ \Gamma^2\big(-\tfrac{1}{n+2}\big)\Big]^{-j}\ J_j(\boldsymbol{w},p,s)\ .\end{aligned} \tag{11.14}$$

Finally in the case $n=\frac{2}{2k-1}$

$$\begin{aligned}&\underset{\substack{N\to\infty\\ b(N)\to s}}{\mathrm{slim}}\left[N^{-1}\,h_{2k}^{(N)}-\frac{1}{2\pi k}\log(NB_k/2)\right]\\ &\qquad=(2N_0)^{-1}\Big[(n+2)^{-\frac{2}{n+2}}\ \Gamma^2\big(-\tfrac{1}{n+2}\big)\Big]^{-2k}\ J_{2k}(\boldsymbol{w},p,s)\,.\end{aligned} \tag{11.15}$$

Here B_k are the constants that enter in to the function $G^{(N)}(E\,|\,g)$ (5.48), which appears in the relation (11.1). The first two of them are given by eqs. (5.30) and (5.31).

We performed extensive numerical checks of (11.13)-(11.15) considering both the Bethe states belonging to the space $\mathcal{H}_{N|S^z}^{(\mathrm{cont})}$ and $\mathcal{H}_{N|S^z}^{(\mathrm{disc},\pm)}$, where the RG invariant s is real and pure imaginary, respectively. Some results concerning the primary Bethe states for which $\mathtt{L}=\bar{\mathtt{L}}=0$ are shown in Fig. 10.

It is worth mentioning that formula (11.1) recovers the universal properties of the eigenvalues $A_\pm(\zeta)$ in the vicinity of $\zeta=0$. By organizing the scaling limit differently one can describe the universal behaviour of $A_\pm(\zeta)$ near the point $\zeta=\infty$. The latter involves the connection coefficients for an ODE similar to (11.2), but with p and $\{w_a\}_{a=1}^{\mathtt{L}}$ replaced by their barred counterparts $\bar{p}$ and $\{\bar{w}_a\}_{a=1}^{\bar{\mathtt{L}}}$, respectively. All the above could, of course, be repeated for this case as well. In the scaling limit the low energy Bethe states take the form $\bar{\boldsymbol{\psi}}_{\bar{p},s}(\bar{\boldsymbol{w}})\otimes\boldsymbol{\psi}_{p,s}(\boldsymbol{w})$, where the chiral states $\bar{\boldsymbol{\psi}}_{\bar{p},s}(\bar{\boldsymbol{w}})$ and $\boldsymbol{\psi}_{p,s}(\boldsymbol{w})$ are specified by the connection coefficients $D_\pm(\mu\,|\,\bar{\boldsymbol{w}},\bar{p},s)$ and $D_\pm(\mu\,|\,\boldsymbol{w},p,s)$, respectively.

Apart from the sum rules there is another way of checking the key relation (11.1) (and its barred counterpart), which turns out to be more convenient for the case of the non-primary Bethe states. It involves the coefficients, which occur in the large μ asymptotic expansion of $D_\pm(\mu\,|\,\boldsymbol{w},p,s)$. For $D_+\equiv D_+(\mu\,|\,\boldsymbol{w},p,s)$, the latter takes the form

$$D_+\asymp\mathfrak{C}_{p,s}^{(\pm)}(\boldsymbol{w})\,\big(\pm\mu\big)^{\pm\frac{\mathrm{i}(n+2)s}{n}-p}\ \exp\left(\frac{N_0}{\cos(\frac{\pi}{n})}\ \big(\pm\mu\big)^{\frac{n+2}{n}}+o(1)\right)\quad\text{for}\quad\Re e(\pm\mu)>0\,. \tag{11.16}$$

At the special values $n=\frac{2}{2k-1}$ with $k=1,2,\ldots$:

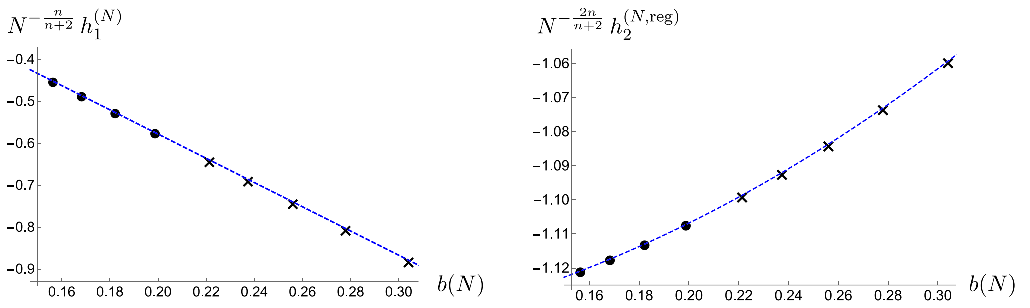

Fig. 10. The sums $h_1^{(N)}$ and $h_2^{(N,\text{reg})} \equiv h_2^{(N)} + \frac{N}{4\cos(\frac{2\pi}{n+2})}$ were computed by solving the Bethe ansatz equations with the value of the parameters $n = \frac{3}{2}$, $\mathtt{k} = \frac{1}{10}$ and $S^z = 0$. The solution sets were taken to be the ones corresponding to the primary Bethe states, which were discussed at the beginning of sec. 9. For the black dots, the integer $\mathtt{m} = M_- - M_+$ was set to $\mathtt{m} = 2$, while $N = 100, 200, 400, 800$. For the crosses $\mathtt{m} = 3$ and $N = 102, 202, 402, 802, 1602$. The numerical data is plotted versus $b(N)$. The latter was computed from the Bethe roots via eqs. (8.5) and (8.6). The dashed blue lines come from the predictions (11.13), (11.14) with the vacuum eigenvalues $J_{1,2}^{(\text{vac})}(p, s)$ calculated through eq. (11.11). Note that the relative error between the numerical data and the analytical formula is of the order of $10^{-4} - 10^{-6}$.

$$D_+ \asymp \mathfrak{C}_{p,s}^{(\pm)}(\boldsymbol{w})\,\big(\pm\mu\big)^{\pm 2\mathrm{i}ks - p}\,\exp\bigg(\frac{(-1)^k\,\Gamma(\frac{1}{2}+k)}{\sqrt{\pi}\,\Gamma(1+k)}\,\mu^{2k}\,\big(\log(\pm\mu) + \tfrac{1}{2}\,c_k\big) + o(1)\bigg) \tag{11.17}$$

for $\Re e(\pm\mu) > 0$ and c_k are the same as in eq. (5.43). Note that the large μ behaviour of $D_-(\mu\,|\,\boldsymbol{w}, p, s)$ can be obtained from that of D_+ by means of the substitution $p \mapsto -p$. It is possible to compute the coefficients $\mathfrak{C}_{p,s}^{(\pm)}(\boldsymbol{w})$ explicitly through an analysis of the ODE (11.2). In particular, when there are no apparent singularities,

$$\mathfrak{C}_{p,s}^{(0,\pm)} = \sqrt{\frac{2\pi}{n+2}}\;\; 2^{-p \pm \frac{\mathrm{i}(n+2)s}{n}}\;(n+2)^{-\frac{2p}{n+2}}\;\frac{\Gamma(1+2p)}{\Gamma(1+\frac{2p}{n+2})\,\Gamma(\frac{1}{2}+p\pm\mathrm{i}s)}\;. \tag{11.18}$$

For a general set $\boldsymbol{w}$ satisfying the algebraic system (10.3), the expression for $\mathfrak{C}_{p,s}^{(\pm)}(\boldsymbol{w})$ is provided by eq. (3.7) in ref. [47].

The coefficients $\mathfrak{C}_{p,s}^{(\pm)}(\boldsymbol{w})$, and the similarly defined $\mathfrak{C}_{\bar{p},s}^{(\pm)}(\bar{\boldsymbol{w}})$, occur in the large N asymptotic formulae for the products over the Bethe roots which resemble the relation (5.66) for the homogeneous case:

$$\prod_{m=1}^{M} q\,\big(\zeta_m \mp \mathrm{i}q^{-1}\big)\big(\zeta_m^{-1} \mp \mathrm{i}q^{-1}\big) \asymp \mathrm{e}^{\pm\frac{2\pi s}{n}}\;\mathfrak{C}_{\bar{p},s}^{(\pm)}(\bar{\boldsymbol{w}})\,\mathfrak{C}_{p,s}^{(\pm)}(\boldsymbol{w})\,\bigg(\frac{N}{2N_0}\bigg)^{-\frac{n(\bar{p}+p)}{n+2}\pm 2\mathrm{i}s}\bigg|_{s=b(N)}$$
$$\times\,\bigg(\frac{4n}{n+2}\bigg)^{N/2}\,\big(1+o(1)\big) \tag{11.19}$$

as well as

$$\prod_{m=1}^{M}\zeta_m^2 \asymp \frac{\mathfrak{C}_{\bar{p},s}^{(+)}(\bar{\boldsymbol{w}})\,\mathfrak{C}_{\bar{p},s}^{(-)}(\bar{\boldsymbol{w}})}{\mathfrak{C}_{p,s}^{(+)}(\boldsymbol{w})\,\mathfrak{C}_{p,s}^{(-)}(\boldsymbol{w})}\bigg|_{s=b(N)}\;\bigg(\frac{N}{2N_0}\bigg)^{\frac{2n(p-\bar{p})}{n+2}}\;\big(1+o(1)\big)\,. \tag{11.20}$$

Note that in these formulae we substitute the RG invariant s by the "running coupling" $b(N) = \frac{n}{4\pi}\log(B)$ with B being the eigenvalue of the quasi shift operator (8.5). This significantly improves their accuracy.

There are many consequences of (11.19) and (11.20). An important one follows from taking the ratio of the asymptotic relation (11.19) corresponding to "+" with that corresponding to "−". Keeping in mind that the eigenvalues of the quasi-shift operator are given by (8.5), it is easy to see that the l.h.s. of the ratio coincides with $(-1)^{\frac{N}{2}-S^z}\, B$. Then since $B = \mathrm{e}^{\frac{4\pi}{n}b(N)}$ one finds

$$\left(\frac{N}{2N_0}\right)^{4\mathrm{i}s} \frac{\mathfrak{C}^{(+)}_{\bar{p},s}(\bar{\boldsymbol{w}})\,\mathfrak{C}^{(+)}_{p,s}(\boldsymbol{w})}{\mathfrak{C}^{(-)}_{\bar{p},s}(\bar{\boldsymbol{w}})\,\mathfrak{C}^{(-)}_{p,s}(\boldsymbol{w})}\bigg|_{s=b(N)} \asymp (-1)^{\frac{N}{2}-S^z}\big(1+o(1)\big)\,. \tag{11.21}$$

Upon the identification

$$D_{p,s}(\boldsymbol{w}) = \frac{\mathfrak{C}^{(+)}_{p,s}(\boldsymbol{w})}{\mathfrak{C}^{(-)}_{p,s}(\boldsymbol{w})}\,, \qquad\qquad D_{\bar{p},s}(\bar{\boldsymbol{w}}) = \frac{\mathfrak{C}^{(+)}_{\bar{p},s}(\bar{\boldsymbol{w}})}{\mathfrak{C}^{(-)}_{\bar{p},s}(\bar{\boldsymbol{w}})} \tag{11.22}$$

this is nothing but the asymptotic formula (9.11), where $\mathrm{e}^{\frac{\mathrm{i}}{2}\delta} = D_{p,s}(\boldsymbol{w})\,D_{\bar{p},s}(\bar{\boldsymbol{w}})$.

Another important outcome of the relations (11.19) and (11.20) involves the products

$$R_{p,s}(\boldsymbol{w}) = \mathfrak{C}^{(+)}_{p,s}(\boldsymbol{w})\,\mathfrak{C}^{(-)}_{p,s}(\boldsymbol{w})\,, \qquad\qquad R_{\bar{p},s}(\bar{\boldsymbol{w}}) = \mathfrak{C}^{(+)}_{\bar{p},s}(\bar{\boldsymbol{w}})\,\mathfrak{C}^{(-)}_{\bar{p},s}(\bar{\boldsymbol{w}})\,. \tag{11.23}$$

Namely,

$$\prod_{m=1}^{M}\big(\zeta_m^{-2}+q^2\big)\big(\zeta_m^{-2}+q^{-2}\big) \asymp \big(R_{p,s}(\boldsymbol{w})\big)^2\Big|_{s=b(N)} \left(\frac{N}{2N_0}\right)^{-\frac{4np}{n+2}} \left(\frac{4n}{n+2}\right)^{N}\big(1+o(1)\big) \tag{11.24}$$

and

$$\prod_{m=1}^{M}\big(\zeta_m^{+2}+q^2\big)\big(\zeta_m^{+2}+q^{-2}\big) \asymp \big(R_{\bar{p},s}(\bar{\boldsymbol{w}})\big)^2\Big|_{s=b(N)} \left(\frac{N}{2N_0}\right)^{-\frac{4n\bar{p}}{n+2}} \left(\frac{4n}{n+2}\right)^{N}\big(1+o(1)\big)\,. \tag{11.25}$$

The advantage of (11.24) compared with the sum rules (11.13)-(11.15) is that, contrary to the expansion coefficients $J_j^{(\pm)}(\boldsymbol{w},p,s)$ from the Taylor series (11.9), there exists a closed expression for $R_{p,s}(\boldsymbol{w})$ in terms of the set $\boldsymbol{w}$. In the case with no apparent singularities, eq. (11.18) implies that

$$R^{(0)}_{p,s} = 2^{1+2p}\,(n+2)^{-1-\frac{4p}{n+2}}\left[\frac{\Gamma(1+p)}{\Gamma(1+\frac{2p}{n+2})}\right]^2 \frac{\Gamma^2(\frac{1}{2}+p)}{\Gamma(\frac{1}{2}+p+\mathrm{i}s)\,\Gamma(\frac{1}{2}+p-\mathrm{i}s)}\,. \tag{11.26}$$

For $\mathtt{L}\geq 0$,

$$R_{p,s}(\boldsymbol{w}) = R^{(0)}_{p,s}\,\check{R}_{p,s}(\boldsymbol{w}) \tag{11.27}$$

and eq. (B.4) in Appendix B gives $\check{R}_{p,s}(\boldsymbol{w})$ in terms of p, s and $\boldsymbol{w}$ (the latter, up to notation, coincides with (3.11) from ref. [47]).

It should be pointed out that for the RG trajectories with pure imaginary s, the large N asymptotic formulae (11.19), (11.20) as well as their derivatives require some special attention.

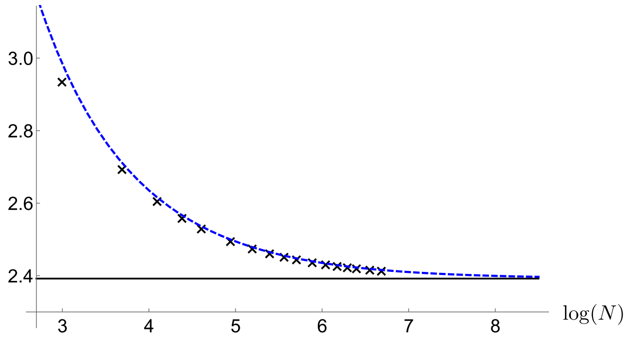

Fig. 11. Numerical data for an RG trajectory $\boldsymbol{\Psi}_N$ with $\mathtt{L}=\bar{\mathtt{L}}=S^z=\mathtt{w}=0$ and labeled by pure imaginary $s=\mathrm{i}\,(-p-\frac{1}{2})=\frac{\mathrm{i}}{4}$ is used to illustrate the asymptotic formula (11.29). The parameters have been set to be $n=3$, $\mathtt{k}=-\frac{3}{10}$. Depicted by the crosses is the r.h.s. of (11.29), calculated from the solution of the Bethe ansatz equations corresponding to $\boldsymbol{\Psi}_N$ and then divided by the leading and subleading large N asymptotic that is given in the second line of that relation, i.e., $\big(\frac{N}{2N_0}\big)^{-\frac{4np}{n+2}-8|s|}\,\big(\frac{4n}{n+2}\big)^{-N}\prod(\zeta_m^{-2}+q^2)(\zeta_m^{-2}+q^{-2})$. The predicted limiting value of the last quantity, $2^{-\frac{1}{3}}\,5^{-\frac{4}{5}}\,\pi^3/\,\Gamma^4(\frac{7}{10})=2.392\ldots$, is represented by the solid line. The dashed line corresponds to $\big(\frac{N}{2N_0}\big)^{-8|b|}\,\big(R^{(0)}_{p,b}\big)^2$, where $b=b(N)$ was obtained by solving eq. (9.11) with the correction terms ignored, $\sigma=+1$ and $\mathrm{e}^{\frac{\mathrm{i}}{2}\delta}$ is given by (9.3).

As an illustration, let's consider (11.24) applied to such a trajectory with $\mathtt{L}=\bar{\mathtt{L}}=0$. In this case the admissible values of s are described by eqs. (9.14), (9.15). When $p<-\frac{1}{2}$ this gives $s=\pm\mathrm{i}\,(-p-\frac{1}{2}-a)$ and a is a non-negative integer such that $-p-\frac{n+2}{4}\leq a<-p-\frac{1}{2}$. At these values of s the vacuum eigenvalue $R^{(0)}_{p,s}$ vanishes. Nevertheless (11.24) continues to hold if one follows the prescription of replacing s by $b(N)$ computed from the Bethe roots for $\boldsymbol{\Psi}_N$. Relation (9.11) with $\mathrm{e}^{\frac{\mathrm{i}}{2}\delta}$ as in (9.3) implies that

$$b(N)=\pm\mathrm{i}\bigg[\mathfrak{q}_a+\frac{\sigma\,(-1)^a}{a!\,(S^z+a)!}\;2^{-\frac{4}{n}(n+2)\mathfrak{q}_a}\;\frac{\Gamma(\bar{p}-p-a)}{\Gamma(1+2p+a)}\,\bigg(\frac{N}{2N_0}\bigg)^{-4\mathfrak{q}_a}+o\big(N^{-4\mathfrak{q}_a}\big)\bigg]\,, \tag{11.28}$$

where $\sigma=(-1)^{\frac{N}{2}-S^z}$. Then one finds

$$\begin{aligned}&\prod_{m=1}^{M}\big(\zeta_m^{-2}+q^2\big)\big(\zeta_m^{-2}+q^{-2}\big)\\ &\asymp\bigg[\frac{2^{-\frac{4}{n}\,(n+2)\,\mathfrak{q}_a}\;\Gamma(\bar{p}-p-a)}{(n+2)^{1+\frac{4p}{n+2}}\;(S^z+a)!}\bigg]^2\,\bigg[\frac{2^{\frac{1}{2}+p}\,\Gamma(1+p)\,\Gamma(\frac{1}{2}+p)}{\Gamma(1+\frac{2p}{n+2})\,\Gamma(1+a+2p)}\bigg]^4\\ &\times\bigg(\frac{N}{2N_0}\bigg)^{-\frac{4np}{n+2}-8\mathfrak{q}_a}\bigg(\frac{4n}{n+2}\bigg)^N\big(1+o(1)\big)\qquad\qquad\big(\mathtt{L}=\bar{\mathtt{L}}=0,\ s=\pm\mathrm{i}\mathfrak{q}_a\big)\,.\end{aligned} \tag{11.29}$$

In Fig. 11 the prediction coming from the last relation is compared with the numerical data.

A second comment regarding (11.19), (11.20) concerns the remainder term denoted by $o(1)$. It is expected to be a double series of the form $\sum_{i,j}C_{i,j}\,N^{-i-jn}$. The expansion coefficients $C_{i,j}$ may become singular when s belongs to the discrete set of admissible values, $s=\pm\mathrm{i}\mathfrak{q}_a,\pm\mathrm{i}\bar{\mathfrak{q}}_a$, which results in a change of the leading large N behaviour of the products over the Bethe roots.

12. Conformal towers for pure imaginary s

The low energy Bethe states characterized by pure imaginary values of the RG invariant $s = \sigma \mathrm{i}\mathfrak{q}_a, \sigma \mathrm{i}\bar{\mathfrak{q}}_a$ (σ is a sign factor) were split into the two sectors $\mathcal{H}_{N|S^z}^{(\mathrm{disc},+)}$ and $\mathcal{H}_{N|S^z}^{(\mathrm{disc},-)}$ according to whether a was a non-negative or a negative integer. The reason for doing so was motivated by the following observation. Together with the integer a the states are labeled by the pair of non-negative integers $(\bar{\mathtt{L}}, \mathtt{L})$. For the case of $\mathcal{H}_{S^z}^{(\mathrm{disc},+)} = \mathrm{slim}_{N\to\infty} \mathcal{H}_{N|S^z}^{(\mathrm{disc},+)}$ there exists at least one state for any value of $\mathtt{L}$ and $\bar{\mathtt{L}}$. Contrary to this, there are no states belonging to $\mathcal{H}_{S^z}^{(\mathrm{disc},-)}$ at the levels $\mathtt{L} = 0, 1, 2, \ldots, |a| - 1$ for $s = \sigma \mathrm{i}\mathfrak{q}_a$ and $\bar{\mathtt{L}} = 0, 1, 2, \ldots, |a| - 1$ when $s = \sigma \mathrm{i}\bar{\mathfrak{q}}_a$. To be more precise, introduce the notation $\mathcal{H}_{\bar{p},p,\sigma \mathrm{i}\mathfrak{q}_a}^{(\bar{\mathtt{L}},\mathtt{L},\pm)}$ and $\mathcal{H}_{\bar{p},p,\sigma \mathrm{i}\bar{\mathfrak{q}}_a}^{(\bar{\mathtt{L}},\mathtt{L},\pm)}$, for the level subspaces of the corresponding conformal towers belonging to $\mathcal{H}_{S^z}^{(\mathrm{disc},\pm)}$. According to conjecture (III) from sec. 10.2 the dimensions of these level subspaces are given by $\mathcal{N}_a^{(\bar{\mathtt{L}},\mathtt{L})} = d_{S^z+a}(\bar{\mathtt{L}})\, d_a(\mathtt{L})$ and $\bar{\mathcal{N}}_a^{(\bar{\mathtt{L}},\mathtt{L})} = d_a(\bar{\mathtt{L}})\, d_{S^z+a}(\mathtt{L})$, respectively. It follows from the definition (10.25) of the integers $d_a(\mathtt{L})$ that $\dim\big(\mathcal{H}_{\bar{p},p,\sigma \mathrm{i}\mathfrak{q}_a}^{(\bar{\mathtt{L}},\mathtt{L},+)}\big)$ is always greater or equal to one, while

$$\dim\big(\mathcal{H}_{\bar{p},p,\sigma \mathrm{i}\mathfrak{q}_a}^{(\bar{\mathtt{L}},\mathtt{L},-)}\big) = \begin{cases} 0 & \text{for } \mathtt{L} < |a| \text{ or } \bar{\mathtt{L}} < |a| - S^z \\ 1 & \text{for } \mathtt{L} = |a| \text{ and } \bar{\mathtt{L}} = \max(0, |a| - S^z) \qquad (a = -1, -2, \ldots)\,. \\ \geq 1 & \text{otherwise} \end{cases} \tag{12.1}$$

For $\dim\big(\mathcal{H}_{\bar{p},p,\sigma \mathrm{i}\bar{\mathfrak{q}}_a}^{(\bar{\mathtt{L}},\mathtt{L},-)}\big)$ similar conditions hold true with $\mathtt{L}$ and $\bar{\mathtt{L}}$ interchanged.

To avoid excessive technical details and cumbersome notation, let's first focus on the conformal tower with $s = +\mathrm{i}\mathfrak{q}_a$ and $|a| \leq S^z$. Then the lowest energy state would occur at $\mathtt{L} = |a|$, $\bar{\mathtt{L}} = 0$ and be characterized by the set $\boldsymbol{w}$ solving (10.3a) and subject to the extra constraint $\big(D_{p,\mathrm{i}\mathfrak{q}_a}(\boldsymbol{w})\big)^{-1} = 0$ (see eq. (10.20)). Since the space $\mathcal{H}_{\bar{p},p,\mathrm{i}\mathfrak{q}_a}^{(0,|a|,-)}$ is one dimensional, these conditions must uniquely determine $\boldsymbol{w} = \{w_j\}_{j=1}^{|a|}$. It turns out to be possible to give an explicit description of this set by showing that the $|a|$ numbers $2w_j$ are roots of the generalized Laguerre polynomial[5]

$$L_{|a|}^{(-2p-n-2)}(2w_j) = 0\,. \tag{12.2}$$

As will be explained shortly, the connection coefficients of the ODE (11.2) with the apparent singularities as in (12.2) coincide with the connection coefficients of a similar ODE having no apparent singularities:

$$D_{\pm}(\mu\,|\,\boldsymbol{w}, p, \mathrm{i}\mathfrak{q}_a) = D_{\pm}^{(\mathrm{vac})}(\mu\,|\,p', \mathrm{i}\mathfrak{q}'_a)\,, \tag{12.3}$$

where

$$p' = p + \tfrac{1}{2}(n+2)\,, \qquad \mathfrak{q}'_a = \mathfrak{q}_a - \tfrac{n}{2}\,. \tag{12.4}$$

---

[5] Recall the definition of the generalized Laguerre polynomials:

$$L_m^{(\alpha)}(x) = \frac{x^{-\alpha}\,\mathrm{e}^{x}}{m!}\,\frac{\mathrm{d}^m}{\mathrm{d}x^m}\big(\mathrm{e}^{-x}\,x^{m+\alpha}\big)\,.$$

This suggests that the lowest energy states in the conformal towers from $\mathcal{H}^{(\text{disc},-)}_{S^z}$ can be described by the differential equations without apparent singularities, similar to the primary states of the conformal towers from $\mathcal{H}^{(\text{disc},+)}_{S^z}$.

Relation (12.3) between the connection coefficients and its generalization arises from a simple relation for the corresponding ODEs. Let's consider the case where the level $\mathtt{L}$, or equivalently the number of apparent singularities, is greater or equal to $|a|$, i.e., can be written in the form $\mathtt{L}=|a|+\mathtt{L}'$ with $\mathtt{L}'$ a non-negative integer. If Φ is a solution of the ODE (11.2) with $s=\mathrm{i}\mathfrak{q}_a$ and $a<0$, one can show via a straightforward computation that the function

$$\Phi'=z^{-\frac{n}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}z}-1-\frac{p+\frac{1}{2}}{z}+\sum_{j=1}^{\mathtt{L}}\frac{1}{z-w_j}-\sum_{j=1}^{\mathtt{L}'}\frac{1}{z-w'_j}\right)\Phi \tag{12.5}$$

satisfies the differential equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{(p')^2-\frac{1}{4}}{z^2}+\frac{2\mathrm{i}s'}{z}+1+\sum_{j=1}^{\mathtt{L}'}\left(\frac{2}{(z-w'_j)^2}+\frac{n}{z\,(z-w'_j)}\right)+\mu^{-2-n}\,z^n\right]\Phi'=0\,. \tag{12.6}$$

Here $s'=\mathrm{i}\mathfrak{q}'_a$ (12.4), and the sets $\boldsymbol{w}=\{w_j\}_{j=1}^{\mathtt{L}}$, $\boldsymbol{w}'=\{w'_j\}_{j=1}^{\mathtt{L}'}$ must obey the coupled algebraic system

$$\begin{aligned}&\frac{1+n+2p}{2w_l}+1-\sum_{j\neq l}^{\mathtt{L}}\frac{1}{w_l-w_j}+\sum_{j=1}^{\mathtt{L}'}\frac{1}{w_l-w'_j}=0\,, \qquad l=1,2,\ldots\mathtt{L}\\ &\frac{1+2p}{2w'_l}+1+\sum_{j\neq l}^{\mathtt{L}'}\frac{1}{w'_l-w'_j}-\sum_{j=1}^{\mathtt{L}}\frac{1}{w'_l-w_j}=0\,, \qquad l=1,2,\ldots\mathtt{L}'\,.\end{aligned} \tag{12.7}$$

In the case $\mathtt{L}'=0$ the above equations simplify to

$$\frac{1+n+2p}{2w_l}+1-\sum_{j\neq l}^{\mathtt{L}}\frac{1}{w_l-w_j}=0\,, \qquad l=1,2,\ldots\mathtt{L}\,. \tag{12.8}$$

It turns out that their solution is unique (up to permutation of the w_j) and that $2w_j$ coincide with the roots of the generalized Laguerre polynomial as prescribed by (12.2). For $\mathtt{L}'>0$ the rigorous analysis of the solutions of the algebraic system (12.7) is an interesting mathematical problem. However, it would take us well beyond the original aim of describing the states in the level subspaces $\mathcal{H}^{(\bar{\mathtt{L}},\mathtt{L},-)}_{\bar{p},p,\mathrm{i}\mathfrak{q}_a}$. Our intuition, supported by a numerical study, leads us to the following picture. If the set $\boldsymbol{w}$ obeys (10.3a) as well as the extra condition $\big(D_{p,\mathrm{i}\mathfrak{q}_a}(\boldsymbol{w})\big)^{-1}=0$, then (12.7) reduces to just $\mathtt{L}'$ independent equations that uniquely determine the set $\boldsymbol{w}'$, which satisfies

$$\begin{aligned}&4n\,(w'_a)^2+8\mathrm{i}s'\,(n+1)\,w'_a-(n+2)\,\big((n+1)^2-4(p')^2\big)\\ &\qquad+4\sum_{b\neq a}^{\mathtt{L}'}\frac{w'_a\,(\,(n+2)^2\,(w'_a)^2-n(2n+5)\,w'_aw'_b+n(n+1)\,(w'_b)^2\,)}{(w'_a-w'_b)^3}=0\end{aligned} \tag{12.9}$$

along with $\big(D_{p',\mathrm{i}\mathfrak{q}'_a}(\boldsymbol{w}')\big)^{-1}=0$. It is clear from (12.5) that the functions ψ and $z^{\frac{n}{2}}\psi'$ possess the same monodromy properties and hence

$$D_\pm(\mu\,|\,\boldsymbol{w},p,\mathrm{i}\mathtt{q}_a)=D_\pm(\mu\,|\,\boldsymbol{w}',p',\mathrm{i}\mathtt{q}'_a) \tag{12.10}$$

for any two sets $\boldsymbol{w}$, $\boldsymbol{w}'$ satisfying (12.7).

We are now in a position to describe the conformal towers from $\mathcal{H}^{(\mathrm{disc},+)}_{S^z}$ and $\mathcal{H}^{(\mathrm{disc},-)}_{S^z}$ in a uniform way. For the case of $\mathcal{H}^{(\mathrm{disc},+)}_{S^z}$ each conformal tower is decomposed into the level subspaces, which themselves can be expressed as a tensor product of the form

$$\mathcal{H}^{(\bar{\mathtt{L}},\mathtt{L},+)}_{\bar{p},p,\sigma\mathrm{i}\mathtt{q}_a}=\bar{\mathcal{V}}^{(\bar{\mathtt{L}})}_{\bar{p},\sigma\mathrm{i}\mathtt{q}_a}\otimes\mathcal{V}^{(\mathtt{L})}_{p,\sigma\mathrm{i}\mathtt{q}_a}\,,\qquad\qquad \mathcal{H}^{(\bar{\mathtt{L}},\mathtt{L},+)}_{\bar{p},p,\sigma\mathrm{i}\bar{\mathtt{q}}_a}=\bar{\mathcal{V}}^{(\bar{\mathtt{L}})}_{\bar{p},\sigma\mathrm{i}\bar{\mathtt{q}}_a}\otimes\mathcal{V}^{(\mathtt{L})}_{p,\sigma\mathrm{i}\bar{\mathtt{q}}_a}\,. \tag{12.11}$$

Here $\sigma=\pm$ while $\mathtt{q}_a=-p-\frac{1}{2}-a$, $\bar{\mathtt{q}}_a=-\bar{p}-\frac{1}{2}-a$ and a is a non-negative integer subject to the restrictions (10.27) which ensure that

$$0<\mathtt{q}_a,\bar{\mathtt{q}}_a\le\tfrac{n}{4}\,. \tag{12.12}$$

The chiral components in (12.11) are finite dimensional linear spaces whose dimensions are given by

$$\begin{aligned}
&\dim\big(\bar{\mathcal{V}}^{(\bar{\mathtt{L}})}_{\bar{p},\sigma\mathrm{i}\mathtt{q}_a}\big)=d_{S^z+a}(\bar{\mathtt{L}})\,, &&\dim\big(\mathcal{V}^{(\mathtt{L})}_{p,\sigma\mathrm{i}\mathtt{q}_a}\big)=d_a(\mathtt{L})\\
&\dim\big(\bar{\mathcal{V}}^{(\bar{\mathtt{L}})}_{\bar{p},\sigma\mathrm{i}\bar{\mathtt{q}}_a}\big)=d_a(\bar{\mathtt{L}})\,, &&\dim\big(\mathcal{V}^{(\mathtt{L})}_{p,\sigma\mathrm{i}\bar{\mathtt{q}}_a}\big)=d_{S^z+a}(\mathtt{L})\,.
\end{aligned} \tag{12.13}$$

To describe the level subspaces of the conformal towers from $\mathcal{H}^{(\mathrm{disc},-)}_{S^z}$ one should distinguish the two cases $s=\sigma\mathrm{i}\mathtt{q}_a$ and $s=\sigma\mathrm{i}\bar{\mathtt{q}}_a$. When $s=\sigma\mathrm{i}\mathtt{q}_a$ it is useful to introduce the notation p_+, $\bar{p}_+$, $\mathtt{L}_+$, $\bar{\mathtt{L}}_+$ and $\mathtt{q}'_a$ as

$$\begin{aligned}
&p_+=p+\tfrac{1}{2}\,(n+2)\,, &&\mathtt{L}_+=\mathtt{L}-|a| &&\text{and} &&\mathtt{q}'_a=\mathtt{q}_a-\tfrac{n}{2}\,,\\
&\bar{p}_+=\bar{p}-\tfrac{1}{2}\,(n+2)\,, &&\bar{\mathtt{L}}_+=\bar{\mathtt{L}}-|a|+S^z
\end{aligned} \tag{12.14}$$

where a is negative and obeys the same constraints (10.27). Then the analogue of (12.11) is given by

$$\mathcal{H}^{(\bar{\mathtt{L}},\mathtt{L},-)}_{\bar{p},p,\sigma\mathrm{i}\mathtt{q}_a}=\begin{cases}\bar{\mathcal{V}}^{(\bar{\mathtt{L}})}_{\bar{p},\sigma\mathrm{i}\mathtt{q}_a}\otimes\mathcal{V}^{(\mathtt{L}_+)}_{p_+,\sigma\mathrm{i}\mathtt{q}'_a} & \text{for}\quad |a|\le S^z\\ \bar{\mathcal{V}}^{(\bar{\mathtt{L}}_+)}_{\bar{p}_+,\sigma\mathrm{i}\mathtt{q}'_a}\otimes\mathcal{V}^{(\mathtt{L}_+)}_{p_+,\sigma\mathrm{i}\mathtt{q}'_a} & \text{for}\quad |a|>S^z\end{cases}\,. \tag{12.15}$$

Here the dimension of each chiral component reads as

$$\begin{aligned}
&\dim\big(\bar{\mathcal{V}}^{(\bar{\mathtt{L}})}_{\bar{p},\sigma\mathrm{i}\mathtt{q}_a}\big)=d_{|S^z+a|}(\bar{\mathtt{L}})\,, &&\dim\big(\bar{\mathcal{V}}^{(\bar{\mathtt{L}}_+)}_{\bar{p}_+,\sigma\mathrm{i}\mathtt{q}'_a}\big)=d_{|S^z+a|}(\bar{\mathtt{L}}_+)\\
&\dim\big(\mathcal{V}^{(\mathtt{L}_+)}_{p_+,\sigma\mathrm{i}\bar{\mathtt{q}}'_a}\big)=d_{|a|}(\mathtt{L}_+)\,.
\end{aligned} \tag{12.16}$$

For the case $s=\sigma\mathrm{i}\bar{\mathtt{q}}_a$ we define p_-, $\bar{p}_-$, $\mathtt{L}_-$, $\bar{\mathtt{L}}_-$ and $\bar{\mathtt{q}}'_a$, through the formulae

$$\begin{aligned}
&p_-=p-\tfrac{1}{2}\,(n+2)\,, &&\mathtt{L}_-=\mathtt{L}-|a|+S^z &&\text{and} &&\bar{\mathtt{q}}'_a=\bar{\mathtt{q}}_a-\tfrac{n}{2}\,.\\
&\bar{p}_-=\bar{p}+\tfrac{1}{2}\,(n+2)\,, &&\bar{\mathtt{L}}_-=\bar{\mathtt{L}}-|a|
\end{aligned} \tag{12.17}$$

With this notation, the decomposition of the level subspace looks similar to that for $s=\sigma\mathrm{i}\mathtt{q}_a$. Namely,

$$\mathcal{H}^{(\bar{\mathtt{L}},\mathtt{L},-)}_{\bar{p},p,\sigma\mathrm{i}\bar{\mathtt{q}}_a} = \begin{cases} \bar{\mathcal{V}}^{(\bar{\mathtt{L}}_-)}_{\bar{p}_-,\sigma\mathrm{i}\bar{\mathtt{q}}'_a} \otimes \mathcal{V}^{(\mathtt{L})}_{p,\sigma\mathrm{i}\bar{\mathtt{q}}_a} & \text{for} \quad |a| \le S^z \\ \bar{\mathcal{V}}^{(\bar{\mathtt{L}}_-)}_{\bar{p}_-,\sigma\mathrm{i}\bar{\mathtt{q}}'_a} \otimes \mathcal{V}^{(\mathtt{L}_-)}_{p_-,\sigma\mathrm{i}\bar{\mathtt{q}}'_a} & \text{for} \quad |a| > S^z \end{cases} \tag{12.18}$$

and the dimensions of the chiral subspaces are given by

$$\dim\big(\mathcal{V}^{(\mathtt{L})}_{p,\sigma\mathrm{i}\bar{\mathtt{q}}_a}\big) = d_{|S^z+a|}(\mathtt{L})\,, \qquad \dim\big(\mathcal{V}^{(\mathtt{L}_-)}_{p_-,\sigma\mathrm{i}\bar{\mathtt{q}}'_a}\big) = d_{|S^z+a|}(\mathtt{L}_-)$$

$$\dim\big(\bar{\mathcal{V}}^{(\bar{\mathtt{L}}_-)}_{\bar{p}_-,\sigma\mathrm{i}\bar{\mathtt{q}}'_a}\big) = d_{|a|}(\bar{\mathtt{L}}_-)\,. \tag{12.19}$$

The following comment is in order here. It is simple to check the identities

$$\frac{p^2}{n+2} - \frac{\mathtt{q}_a^2}{n} + \mathtt{L} = \frac{(p_+)^2}{n+2} - \frac{(\mathtt{q}'_a)^2}{n} + \mathtt{L}_+\,, \qquad \frac{\bar{p}^2}{n+2} - \frac{\mathtt{q}_a^2}{n} + \bar{\mathtt{L}} = \frac{(\bar{p}_+)^2}{n+2} - \frac{(\mathtt{q}'_a)^2}{n} + \bar{\mathtt{L}}_+\,, \tag{12.20}$$

where p_+, $\bar{p}_+$, $\mathtt{L}_+$, $\bar{\mathtt{L}}_+$ and $\mathtt{q}'_a$ are given in (12.14). This makes it possible to re-write the scaled energy, defined by eq. (10.18), for the level subspaces $\mathcal{H}^{(\bar{\mathtt{L}},\mathtt{L},-)}_{\bar{p},p,\sigma\mathrm{i}\mathtt{q}_a}$ in terms of the numbers labeling the chiral components $\bar{\mathcal{V}}$ and $\mathcal{V}$ in the r.h.s. of eq. (12.15). For example, for the case $s = \sigma\mathrm{i}\mathtt{q}_a$ and $|a| > S^z$ one has

$$E = \frac{(p_+)^2 + (\bar{p}_+)^2}{n+2} - \frac{2(\mathtt{q}'_a)^2}{n} - \frac{1}{6} + \mathtt{L}_+ + \bar{\mathtt{L}}_+\,. \tag{12.21}$$

The same can be done for $\mathcal{H}^{(\bar{\mathtt{L}},\mathtt{L},-)}_{\bar{p},p,\sigma\mathrm{i}\bar{\mathtt{q}}_a}$ using the similar relations to (12.20) involving p_-, $\bar{p}_-$, $\mathtt{L}_-$, $\bar{\mathtt{L}}_-$ and $\bar{\mathtt{q}}'_a$. It should be emphasized that $\mathtt{q}'_a$ and $\bar{\mathtt{q}}'_a$ do not lie in the strip from (12.12), but rather

$$-\tfrac{n}{2} < \mathtt{q}'_a, \bar{\mathtt{q}}'_a \le -\tfrac{n}{4}\,. \tag{12.22}$$

As a result $s' = \pm\mathrm{i}\mathtt{q}'_a, \pm\mathrm{i}\bar{\mathtt{q}}'_a$ does not obey the constraint (10.1).

13. Scaling limit of the lattice operators $\mathbb{A}_\pm(\zeta)$

The key rôle in the description of the scaling limit of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model is played by the relation (11.1). Much of our numerical work, which was outlined in sec. 11, was devoted to its verification. Accepting that (11.1) holds true, it can be interpreted as an operator relation

$$\operatorname*{slim}_{\substack{N\to\infty\\ b(N)\to s}} G^{(N/2)}\big(-\mu^2\,|\,\tfrac{2}{n+2}\big)\,\mathbb{A}_\pm\big(\mathrm{i}\,\big(N/(2N_0)\big)^{-\frac{n}{n+2}}\,\mu\big) = \boldsymbol{a}_\pm(\lambda) \tag{13.1}$$

with $\boldsymbol{a}_\pm(\lambda)$ acting invariantly in the right chiral component of the level subspaces of the conformal towers. In view of (11.9), the operators $\log\boldsymbol{a}_\pm(\lambda)$ possess the series expansion

$$\log\boldsymbol{a}_\pm(\lambda) = -\sum_{j=1}^{\infty}\mathbf{J}^{(\pm)}_j\,\lambda^j \tag{13.2}$$

and the eigenvalues of $\mathbf{J}^{(\pm)}_j$ coincide with $J^{(\pm)}_j(\boldsymbol{w}, p, s)$. Recall that the parameters λ and μ are proportional to each other as in eq. (11.10). The dimensions of the level subspaces of the conformal towers have already been described in sections 10.1 and 12. In particular (10.17) suggests that for real s,

$$\mathcal{H}^{(\bar{\mathrm{L}},\mathrm{L})}_{\bar{p},p,s} = \bar{\mathcal{V}}^{(\bar{\mathrm{L}})}_{\bar{p},s} \otimes \mathcal{V}^{(\mathrm{L})}_{p,s} \qquad \text{with} \qquad \dim\big(\mathcal{V}^{(\mathrm{L})}_{p,s}\big) = \mathrm{par}_2(\mathrm{L}) , \quad \dim\big(\bar{\mathcal{V}}^{(\bar{\mathrm{L}})}_{\bar{p},s}\big) = \mathrm{par}_2(\bar{\mathrm{L}}) . \tag{13.3}$$

As this is simpler than for the case of pure imaginary s, where the corresponding dimensions are given by eqs. (12.13), (12.16) and (12.19), we start by describing the operators $\mathbb{a}_\pm(\lambda) : \mathcal{V}^{(\mathrm{L})}_{p,s} \mapsto \mathcal{V}^{(\mathrm{L})}_{p,s}$ with s being a real number.

13.1. The case of real s

For real s the dimensions of $\mathcal{V}^{(\mathrm{L})}_{p,s}$ coincide with those of the level subspace of the Fock space generated by two independent copies of the Heisenberg algebra. Hence one can identify them as linear spaces:

$$\mathcal{V}^{(\mathrm{L})}_{p,s} = \mathcal{F}^{(\mathrm{L})}_{\mathbf{P}} \qquad\qquad (s \in \mathbb{R}) . \tag{13.4}$$

We take the commutation relations for the Heisenberg algebra generators to be

$$[a_m, a_j] = \tfrac{m}{2}\, \delta_{m+j,0} , \quad [b_m, b_j] = \tfrac{m}{2}\, \delta_{m+j,0} , \quad [a_m, b_j] = 0 , \tag{13.5}$$

while $\mathbf{P}$ stands for the highest weight, i.e., the values of a_0 and b_0 in $\mathcal{F}_{\mathbf{P}}$. The dimensions of $\mathcal{F}^{(\mathrm{L})}_{\mathbf{P}}$ of course do not depend on the value of the highest weight. We'll set

$$\mathbf{P} = \big(\tfrac{p}{\sqrt{n+2}}, \tfrac{s}{\sqrt{n}}\big) . \tag{13.6}$$

The construction of the operators $\mathbb{a}_\pm(\lambda)$ parallels that for the homogeneous case. Formulae (5.9), (5.13) remain essentially unchanged, but the vertex operators are now given by

$$V_+(u) = \mathrm{e}^{+\frac{2\mathrm{i}\varphi}{\sqrt{n+2}}}(u) , \qquad\qquad V_-(u) = -2\sqrt{n}\, \partial\vartheta\, \mathrm{e}^{-\frac{2\mathrm{i}\varphi}{\sqrt{n+2}}}(u) . \tag{13.7}$$

Here $\varphi(u)$ is the same as in eq. (5.7) and the additional chiral field

$$\partial\vartheta(u) = \sum_{m=-\infty}^{\infty} b_m\, \mathrm{e}^{-\mathrm{i}mu} \tag{13.8}$$

involves the Heisenberg generators $\{b_m\}$ (13.5). Then it turns out that

$$\mathbb{a}_\pm(\lambda) = \frac{\mathrm{Tr}_{\rho_\pm}\big[\mathrm{e}^{\pm\frac{\mathrm{i}\pi}{\sqrt{n+2}}a_0\mathcal{H}}\, \boldsymbol{L}_\pm(\lambda)\big]}{\mathrm{Tr}_{\rho_\pm}\big[\mathrm{e}^{\pm\frac{2\mathrm{i}\pi}{\sqrt{n+2}}a_0\mathcal{H}}\big]} \tag{13.9}$$

with

$$\boldsymbol{L}_\pm(\lambda) = \mathrm{e}^{\pm\frac{\mathrm{i}\pi}{\sqrt{n+2}}a_0\mathcal{H}}\, \overleftarrow{\mathcal{P}}\exp\bigg(\int_0^{2\pi}\mathrm{d}u\,\Big(V_-(u)\, q^{\pm\frac{\mathcal{H}}{2}}\, \mathcal{E}_\pm + \lambda\, V_+(u)\, q^{\mp\frac{\mathcal{H}}{2}}\, \mathcal{E}_\mp\Big)\bigg) . \tag{13.10}$$

As before $\mathcal{E}_\pm$ and $\mathcal{H}$ stand for the generators of the q-oscillator algebra (5.11) and $\rho_\pm$ are representations of this algebra – the same as in (5.13). Since $\mathbb{a}_\pm(0) = \mathbf{1}$ the formal power series (13.9) can be rewritten as the Taylor series (13.2) for $\log\mathbb{a}_\pm(\lambda)$. The expansion coefficients $\mathbf{J}^{(\pm)}_j$ involve the ordered multifold integrals. Like in the homogeneous case, expressing these in terms of the contour integrals makes the operators $\mathbf{J}^{(\pm)}_j$ well defined for any $n > 0$ except $n = \frac{2}{2k-1}$

with $k = 1, 2, \ldots$. In the latter case $\mathbf{J}_{2k}^{(\pm)}$ requires regularization and we define $\mathbf{J}_{2k}^{(\pm,\mathrm{reg})}$ through a subtraction of the counterterm of the unit operator, similar to eq. (5.29).

Following the lines of ref. [32] one can prove that $\mathcal{a}_\pm(\lambda)$, defined as above, act invariantly in $\mathcal{F}_{\mathbf{P}}^{(\mathtt{L})}$ and form a commuting family

$$\big[\mathcal{a}_\pm(\lambda),\, \mathcal{a}_\pm(\lambda')\big] = 0\,. \tag{13.11}$$

In addition, it is possible to derive a set of operator relations for $\mathcal{a}_\pm(\lambda)$, which in turn become functional relations for their eigenvalues. The latter are identical to those satisfied by the connection coefficients, which follow from the basic properties of the ODE (11.2). Among these is the so-called quantum Wronskian relation

$$q^{2p}\, D_+(q^{+1}\mu)\, D_-(q^{-1}\mu) - q^{-2p}\, D_-(q^{+1}\mu)\, D_+(q^{-1}\mu) = 2\mathrm{i}\sin\big(\tfrac{2\pi p}{n+2}\big)\,, \tag{13.12}$$

where $D_\pm(\mu) \equiv D_\pm(\mu\,|\,\boldsymbol{w}, p, s)$. The fact that the functional relations coincide is not sufficient to prove that each of the $\mathrm{par}_2(\mathtt{L})$ eigenvalues of $\mathcal{a}_\pm(\lambda)$ in $\mathcal{F}_{\mathbf{P}}^{(\mathtt{L})}$ is given by a connection coefficient $D_\pm(\mu)$ for one of the $\mathrm{par}_2(\mathtt{L})$ solution sets $\boldsymbol{w}$ of the algebraic system (10.3a). Nevertheless we confirmed, for instance, that the vacuum eigenvalues of $\mathbf{J}_1^{(\pm)}$ and $\mathbf{J}_2^{(\pm)}$, computed from the definition (13.9), (13.10), coincide with $J_1^{(\mathrm{vac})}(p,s)$ and $J_2^{(\mathrm{vac})}(p,s)$ from (11.11), which were obtained via the perturbation theory of the ODE (11.2). This strongly suggests that

$$\mathcal{a}_\pm(\lambda)\, \boldsymbol{\psi}_{p,s}(\boldsymbol{w}) = D_\pm(\mu\,|\,\boldsymbol{w}, p, s)\, \boldsymbol{\psi}_{p,s}(\boldsymbol{w})\,, \tag{13.13}$$

where $\boldsymbol{\psi}_{p,s}(\boldsymbol{w}) \in \mathcal{F}_{\mathbf{P}}^{(\mathtt{L})}$ stands for the corresponding eigenvector and with the λ-μ relation as in (11.10).

13.2. The case of pure imaginary s

It should be pointed out that formulae (13.9) and (13.10) define the operators $\mathcal{a}_\pm(\lambda)$, acting invariantly in the level subspace of the Fock space,

$$\mathcal{a}_\pm(\lambda):\quad \mathcal{F}_{\mathbf{P}}^{(\mathtt{L})} \mapsto \mathcal{F}_{\mathbf{P}}^{(\mathtt{L})}\,, \tag{13.14}$$

for any value of the highest weight $\mathbf{P} \equiv (P_1, P_2)$ except when $q^{2\sqrt{n+2}\,P_1} = \pm q^m$ and $m = 0, \pm1, \pm2 \ldots$. In the latter case $\mathcal{a}_\pm(\lambda)$ may still be introduced, though some special treatment is required. The same holds true for the connection coefficients and it is expected that (13.13) is valid for any complex P_1 and P_2. On the other hand, the operators which appear in the scaling limit (13.1) act in the chiral components of the conformal towers. For the discrete spectrum, where P_1 and P_2 are related as

$$\sqrt{n+2}\; P_1 + \tfrac{1}{2} \pm \mathrm{i}\sqrt{n}\; P_2 \in \mathbb{Z}\,, \tag{13.15}$$

these have dimensions that are typically less than $\dim\big(\mathcal{F}_{\mathbf{P}}^{(\mathtt{L})}\big)$. In this case the operators from (13.1) should be understood to be the ones in (13.14) restricted to a certain subspace of $\mathcal{F}_{\mathbf{P}}^{(\mathtt{L})}$. Since a rigorous treatment of these restrictions involves many technical details, here we just give a sketch of the underlying ideas.

Let's take P_1 and P_2, satisfying (13.15), in the form

$$P_1 = \tfrac{1}{\sqrt{n+2}}\big(p + \tfrac{1}{2}(n+2)\,\ell\big)\,, \qquad P_2 = -\tfrac{\mathrm{i}\sigma}{\sqrt{n}}\,(p + \tfrac{1}{2} + a + \tfrac{n}{2}\,\ell)\,, \tag{13.16}$$

where a, ℓ are integers, $\sigma=\pm 1$, while p can be arbitrary. The Fock space $\mathcal{F}_{\mathbf{P}}$ corresponding to this value of the highest weight $\mathbf{P}=(P_1,P_2)$ will be denoted as $\mathcal{F}\big[{}^{\sigma,\ell}_{p,a}\big]$. Consider the so-called "screening charge" built from the chiral fields φ and ϑ [49] (see also [53,54])

$$\hat{\mathsf{Q}}_\sigma=\int_0^{2\pi}\mathrm{d}u\ \mathrm{e}^{\mathrm{i}\sqrt{n+2}\,\varphi+\sigma\sqrt{n}\,\vartheta}(u)\ . \tag{13.17}$$

The field ϑ is defined via a formula similar to eq. (5.7) and involves the mode ϑ_0 conjugated to b_0 such that $[\vartheta_0,b_m]=\frac{\mathrm{i}}{2}\delta_{m,0}$. It turns out that for a fixed choice of the sign factor $\sigma=\pm 1$ the following holds true:

(a) The screening charge is a well defined operator in the direct sum of the Fock spaces

$$\hat{\mathsf{Q}}_\sigma\in\mathrm{End}\Big(\bigoplus_{\ell=-\infty}^{\infty}\mathcal{F}\big[{}^{\sigma,\ell}_{p,a}\big]\Big) \tag{13.18}$$

and acts as the intertwiner

$$\hat{\mathsf{Q}}_\sigma\ :\quad\mathcal{F}\big[{}^{\sigma,\ell}_{p,a}\big]\mapsto\mathcal{F}\big[{}^{\sigma,\ell+1}_{p,\ a}\big]\ . \tag{13.19}$$

(b) The operator $\hat{\mathsf{Q}}_\sigma$ is nilpotent

$$\hat{\mathsf{Q}}_\sigma^2=0\ . \tag{13.20}$$

(c) The action of $\hat{\mathsf{Q}}_\sigma$ commutes with the action of $\boldsymbol{a}_\pm(\lambda)$:

$$\hat{\mathsf{Q}}_\sigma\,\boldsymbol{a}_\pm(\lambda)=\boldsymbol{a}_\pm(\lambda)\,\hat{\mathsf{Q}}_\sigma\ . \tag{13.21}$$

Introduce the notation

$$\mathcal{K}\big[{}^{\sigma,\ell}_{p,a}\big]=\mathrm{Ker}(\hat{\mathsf{Q}}_\sigma)\cap\mathcal{F}\big[{}^{\sigma,\ell}_{p,a}\big]\ ,\qquad\mathcal{I}\big[{}^{\sigma,\ell}_{p,a}\big]=\mathrm{Im}(\hat{\mathsf{Q}}_\sigma)\cap\mathcal{F}\big[{}^{\sigma,\ell}_{p,a}\big]\ . \tag{13.22}$$

Property (b) implies $\mathcal{I}\big[{}^{\sigma,\ell}_{p,a}\big]\subseteq\mathcal{K}\big[{}^{\sigma,\ell}_{p,a}\big]$, while (c) gives that (13.22) are invariant subspaces of $\mathcal{F}\big[{}^{\sigma,\ell}_{p,a}\big]$ w.r.t. the action of $\boldsymbol{a}_\pm(\lambda)$. Moreover one expects that

$$\mathcal{K}\big[{}^{\sigma,\ell}_{p,a}\big]=\mathcal{F}\big[{}^{\sigma,\ell}_{p,a}\big]\ ,\qquad\mathcal{I}\big[{}^{\sigma,\ell}_{p,a}\big]=\emptyset\qquad(\ell>a+1) \tag{13.23}$$

$$\mathcal{K}\big[{}^{\sigma,a+1}_{p,\ a}\big]=\mathcal{I}\big[{}^{\sigma,a+1}_{p,\ a}\big]=\mathcal{F}\big[{}^{\sigma,a+1}_{p,\ a}\big]\ .$$

In the case $\ell\le a$ the subspaces (13.22) coincide,

$$\mathcal{K}\big[{}^{\sigma,\ell}_{p,a}\big]=\mathcal{I}\big[{}^{\sigma,\ell}_{p,a}\big]\qquad(\ell\le a) \tag{13.24}$$

and are proper subspaces in the sense that they are neither the empty set nor equal to the Fock space itself. In particular, it is easy to check that the highest state in $\mathcal{F}\big[{}^{\sigma,\ell}_{p,a}\big]$ does not belong to the kernel of $\hat{\mathsf{Q}}_\sigma$, while the $\hat{\mathsf{Q}}_\sigma$-image of the highest state from $\mathcal{F}\big[{}^{\sigma,\ell-1}_{p,\ a}\big]$ takes the form

$$\left[\left(\frac{\partial}{\partial z}\right)^{a+1-\ell}\exp\left(\sum_{m=1}^{\infty}\big(\sqrt{n+2}\,a_{-m}-\sigma\mathrm{i}\sqrt{n}\,b_{-m}\big)\,\frac{z^m}{m}\right)\right]_{z=0}|\mathbf{P}\rangle\in\mathcal{I}\big[{}^{\sigma,\ell}_{p,a}\big]\ . \tag{13.25}$$

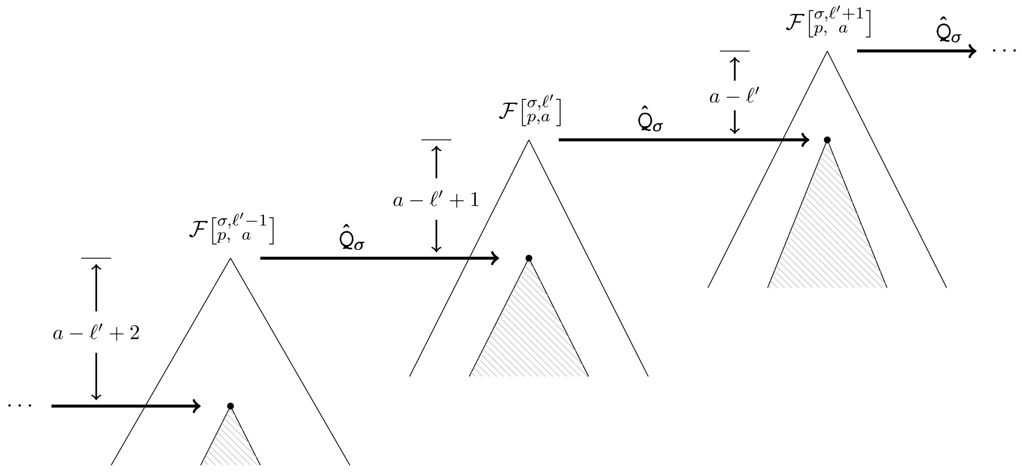

Fig. 12. A depiction of a fragment of the half-infinite chain complex for the action of the screening charges (13.18) with $\ell' \le a$. The shaded regions represent the proper subspaces $\mathcal{K}\big[{}^{\sigma,\ell}_{p,a}\big]=\mathcal{I}\big[{}^{\sigma,\ell}_{p,a}\big]\subset\mathcal{F}\big[{}^{\sigma,\ell}_{p,a}\big]$ with $\ell=\ell'-1,\ell',\ell'+1$ and the bullet at each vertex corresponds to the state (13.25). The chain is infinitely extended to the left. If $\ell'=a+1$, the chain terminates since, according to eq. (13.23), the whole Fock space $\mathcal{F}\big[{}^{\sigma,a+1}_{p,\ \ a}\big]$ lies in the kernel of $\hat{\mathsf{Q}}_\sigma$.

Fig. 12 provides a visualization of the action of the screening charges on the Fock spaces. With the above properties one can show (see, e.g., [53,54]) that the dimensions of the level subspaces of the factor space $\mathcal{V}\big[{}^{\sigma,\ell}_{p,a}\big]\equiv\mathcal{F}\big[{}^{\sigma,\ell}_{p,a}\big]/\mathcal{I}\big[{}^{\sigma,\ell}_{p,a}\big]$ are given by

$$\dim\Big(\mathcal{V}^{(L)}\big[{}^{\sigma,\ell}_{p,a}\big]\Big)=d_{a-\ell}(\mathtt{L})\qquad(\ell\le a)\tag{13.26}$$

with $d_a(\mathtt{L})$ as in (10.25).

From here on out, without loss of generality, we set the parameter ℓ in (13.16) to be zero. In fact, ℓ is a fake parameter that was introduced only for convenience. Then using our previous notation, $\mathfrak{q}_a=-p-\frac{1}{2}-a$, one has $P_1=\frac{p}{\sqrt{n+2}}$, $P_2=\frac{\sigma\mathrm{i}\mathfrak{q}_a}{\sqrt{n}}$ so that

$$\sqrt{n+2}\;P_1+\tfrac{1}{2}-\sigma\,\mathrm{i}\sqrt{n}\;P_2=-a\ .\tag{13.27}$$

In view of (13.23) and (13.24) one should distinguish the cases $a<0$ and $a\ge0$. Let's first take a to be a non-negative integer. The eigenstates $\boldsymbol{\psi}_{p,\sigma\mathrm{i}\mathfrak{q}_a}(\boldsymbol{w})$ (13.13) form a basis in the level subspace of $\mathcal{F}\big[{}^{\sigma,0}_{p,a}\big]$. Those which are annihilated by the screening charge, $\hat{\mathsf{Q}}_\sigma\boldsymbol{\psi}_{p,\sigma\mathrm{i}\mathfrak{q}_a}(\boldsymbol{w})=0$, are a basis in the level subspace of $\mathcal{K}\big[{}^{\sigma,0}_{p,a}\big]$. Among them is the state (13.25) occurring at the level $\mathtt{L}=a+1$. The eigenstates $\boldsymbol{\psi}_{p,\sigma\mathrm{i}\mathfrak{q}_a}(\boldsymbol{w})$ which do not belong to the kernel of $\hat{\mathsf{Q}}_\sigma$ provide a basis for the level subspace $\mathcal{V}^{(\mathtt{L})}_{p,\sigma\mathrm{i}\mathfrak{q}_a}$ of the space

$$\mathcal{V}_{p,\sigma\mathrm{i}\mathfrak{q}_a}=\big(\hat{\mathbf{1}}-\hat{\Pi}_{\mathcal{K}_a}\big)\big(\mathcal{F}\big[{}^{\sigma,0}_{p,a}\big]\big)\ ,\qquad\qquad \mathcal{V}_{p,\sigma\mathrm{i}\mathfrak{q}_a}=\bigoplus_{\mathtt{L}\ge0}\mathcal{V}^{(\mathtt{L})}_{p,\sigma\mathrm{i}\mathfrak{q}_a}\qquad\quad(a\ge0)\ .\tag{13.28}$$

Here $\hat{\Pi}_{\mathcal{K}_a}$ stands for the projector onto $\mathcal{K}\big[{}^{\sigma,0}_{p,a}\big]$. The dimensions of $\mathcal{V}^{(\mathtt{L})}_{p,\sigma\mathrm{i}\mathfrak{q}_a}$ are given by $d_a(\mathtt{L})$ and, moreover, they can be identified with the chiral components appearing in the decomposition of the conformal tower (12.11).

When a from (13.27) is a negative integer, the above analysis does not follow through literally since the full Fock space $\mathcal{F}\big[{\sigma,0 \atop p,a}\big]$ belongs to the kernel of the screening charge. Nevertheless, one can use the fact that the algebraic equations (10.3a), obeyed by the sets $\boldsymbol{w}$ labeling the eigenbasis of $\mathbb{a}_\pm(\lambda)$, do not depend on the sign of p. This allows one to introduce the operator $\hat{\mathrm{C}}_{\mathrm{R}}$ via the formula

$$\hat{\mathrm{C}}_{\mathrm{R}}\, \boldsymbol{\psi}_{p,s}(\boldsymbol{w}) = \boldsymbol{\psi}_{-p,s}(\boldsymbol{w}) \, . \tag{13.29}$$

The precise specification of $\hat{\mathrm{C}}_{\mathrm{R}}$ requires fixing the normalization of the states $\boldsymbol{\psi}_{\pm p,s}(\boldsymbol{w})$. However, this is not important for our purposes as all that is needed is that $\hat{\mathrm{C}}_{\mathrm{R}}$ intertwines the Fock spaces,

$$\hat{\mathrm{C}}_{\mathrm{R}}\big(\mathcal{F}_{(\pm P_1, P_2)}\big) = \mathcal{F}_{(\mp P_1, P_2)} \tag{13.30}$$

and obeys the following commutation relations

$$\hat{\mathrm{C}}_{\mathrm{R}}\, \mathbb{a}_\pm(\lambda) = \mathbb{a}_\mp(\lambda)\, \hat{\mathrm{C}}_{\mathrm{R}} \, . \tag{13.31}$$

It is easy to see that for P_2 as in eq. (13.16) with $\ell = 0$, a change in the sign of p is equivalent to the substitutions $\sigma \mapsto -\sigma$ and $a \mapsto -a-1$. Hence

$$\hat{\mathrm{C}}_{\mathrm{R}}\Big(\mathcal{F}\big[{\sigma,0 \atop \pm p,\, a}\big]\Big) = \mathcal{F}\big[{-\sigma,\ \ 0 \atop \mp p\,,-a-1}\big] \, . \tag{13.32}$$

The transformed space contains the proper subspace $\mathcal{K}\big[{-\sigma,\ \ 0 \atop \mp p\,,-a-1}\big]$, which is invariant w.r.t. the action of $\mathbb{a}_\pm(\lambda)$. Then instead of (13.28) one should introduce $\mathcal{V}_{p,\sigma\mathrm{i}\mathfrak{q}_a}$ for negative a as

$$\mathcal{V}_{p,\sigma\mathrm{i}\mathfrak{q}_a} = \hat{\mathrm{C}}_{\mathrm{R}}\,\big(\hat{\mathbf{1}} - \hat{\Pi}_{\mathcal{K}_{-a-1}}\big)\,\big(\mathcal{F}\big[{-\sigma,\ \ 0 \atop -p\,,-a-1}\big]\big) \qquad \mathcal{V}_{p,\sigma\mathrm{i}\mathfrak{q}_a} = \bigoplus_{\mathtt{L}\geq 0} \mathcal{V}^{(\mathtt{L})}_{p,\sigma\mathrm{i}\mathfrak{q}_a} \qquad (a<0)\,, \tag{13.33}$$

where $\hat{\Pi}_{\mathcal{K}_{-a-1}}$ stands for the projector onto the subspace $\mathcal{K}\big[{-\sigma,\ \ 0 \atop -p\,,-a-1}\big]$. The dimensions of the level subspaces are given by $\dim\big(\mathcal{V}^{(\mathtt{L})}_{p,\sigma\mathrm{i}\mathfrak{q}_a}\big) = d_{|a+1|}(\mathtt{L})$.

The right chiral level subspaces of the conformal towers from $\mathcal{H}^{(\mathrm{disc},\pm)}_{S^z}$, appearing in (12.11), (12.15) and (12.18), may be organized into the graded linear spaces

$$\begin{array}{ll} \mathcal{V}_{p,\sigma\mathrm{i}\mathfrak{q}_a} = \bigoplus\limits_{\mathtt{L}\geq 0} \mathcal{V}^{(\mathtt{L})}_{p,\sigma\mathrm{i}\mathfrak{q}_a}\,, & \mathcal{V}_{p_+,\sigma\mathrm{i}\mathfrak{q}'_a} = \bigoplus\limits_{\mathtt{L}_+\geq 0} \mathcal{V}^{(\mathtt{L}_+)}_{p_+,\sigma\mathrm{i}\mathfrak{q}'_a} \\ \mathcal{V}_{p,\sigma\mathrm{i}\bar{\mathfrak{q}}_a} = \bigoplus\limits_{\mathtt{L}\geq 0} \mathcal{V}^{(\mathtt{L})}_{p,\sigma\mathrm{i}\bar{\mathfrak{q}}_a}\,, & \mathcal{V}_{p_-,\sigma\mathrm{i}\bar{\mathfrak{q}}'_a} = \bigoplus\limits_{\mathtt{L}_-\geq 0} \mathcal{V}^{(\mathtt{L}_-)}_{p_-,\sigma\mathrm{i}\bar{\mathfrak{q}}'_a} \end{array} . \tag{13.34}$$

In all four cases the corresponding combination $\sqrt{n+2}\;P_1 + \frac{1}{2} \pm \mathrm{i}\sqrt{n}\;P_2 \in \mathbb{Z}$, for some choice of the sign $\pm$, so that formulae (13.28) and (13.33) provide a description of these spaces in terms of the Fock spaces. In turn, this defines the action of $\mathbb{a}_\pm(\lambda)$ in the chiral components of the conformal towers for pure imaginary s.

Finally we note that the description of $\mathcal{V}_{p,\sigma\mathrm{i}\mathfrak{q}_a}$ with $a<0$ requires, in addition to the screening charges, the intertwiner $\hat{\mathrm{C}}_{\mathrm{R}}$. The latter was defined rather formally through the eigenbasis of $\mathbb{a}_\pm(\lambda)$. As will be explained in sec. 16, the operator $\hat{\mathrm{C}}_{\mathrm{R}}$ may be introduced in an invariant way that does not require the choice of a specific basis.

14. Scaling limit of the transfer matrix

In our previous discussion the scaling limit was taken in such a way that resulted in the operators acting in the chiral Fock space $\mathcal{F}_{\mathbf{P}}$. Of course it is possible to organize the scaling limit of $\mathbb{A}_{\pm}(\zeta)$ and the transfer matrix $\mathbb{T}(\zeta)$ that yields the operators acting in the barred chiral space $\bar{\mathcal{F}}_{\bar{\mathbf{P}}}$. Since the corresponding formulae are very similar they were omitted up to this point. However here, for future references, we will need to describe both $\boldsymbol{\tau}(\lambda) \in \mathrm{End}(\mathcal{F}_{\mathbf{P}})$ and $\bar{\boldsymbol{\tau}}(\bar{\lambda}) \in \mathrm{End}(\bar{\mathcal{F}}_{\bar{\mathbf{P}}})$ that appear in the scaling limit of the transfer matrix. For this purpose let us introduce the barred counterpart of the vertex operators from (13.7):

$$\bar{V}_{+}(\bar{u}) = \mathrm{e}^{+\frac{2\mathrm{i}\bar{\varphi}}{\sqrt{n+2}}}(\bar{u}) \, , \qquad \bar{V}_{-}(\bar{u}) = -2\sqrt{n}\; \bar{\partial}\bar{\vartheta}\; \mathrm{e}^{-\frac{2\mathrm{i}\bar{\varphi}}{\sqrt{n+2}}}(\bar{u}) \, , \tag{14.1}$$

where

$$\bar{\varphi}(\bar{u}) = \bar{\varphi}_0 + \bar{a}_0\,\bar{u} + \mathrm{i}\sum_{m\neq 0}\frac{\bar{a}_m}{m}\,\mathrm{e}^{-\mathrm{i}m\bar{u}} \qquad \big([\bar{\varphi}_0, \bar{a}_m] = \tfrac{\mathrm{i}}{2}\,\delta_{m,0}\big) \tag{14.2}$$

$$\bar{\vartheta}(\bar{u}) = \bar{\vartheta}_0 + \bar{b}_0\,\bar{u} + \mathrm{i}\sum_{m\neq 0}\frac{\bar{b}_m}{m}\,\mathrm{e}^{-\mathrm{i}m\bar{u}} \qquad \big([\bar{\vartheta}_0, \bar{b}_m] = \tfrac{\mathrm{i}}{2}\,\delta_{m,0}\big)\, .$$

Then consider the two formal path ordered exponents, which are defined similarly as in the homogeneous case (see sec. 5.3):

$$\boldsymbol{L}(\lambda) = \lambda^{+\frac{\mathrm{h}}{4}}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{\sqrt{n+2}}a_0\mathrm{h}}\,\overleftarrow{\mathcal{P}}\exp\bigg(\int_0^{2\pi}\mathrm{d}u\,\Big(V_{-}(u)\,q^{+\frac{\mathrm{h}}{2}}\,\mathrm{e}_{+} + \lambda\,V_{+}(u)\,q^{-\frac{\mathrm{h}}{2}}\,\mathrm{e}_{-}\Big)\bigg)\,\lambda^{-\frac{\mathrm{h}}{4}} \tag{14.3}$$

$$\bar{\boldsymbol{L}}(\bar{\lambda}) = \bar{\lambda}^{+\frac{\mathrm{h}}{4}}\,\overrightarrow{\mathcal{P}}\exp\bigg(\int_0^{2\pi}\mathrm{d}\bar{u}\,\Big(\bar{V}_{-}(\bar{u})\,q^{+\frac{\mathrm{h}}{2}}\,\mathrm{e}_{+} + \bar{\lambda}\,\bar{V}_{+}(\bar{u})\,q^{-\frac{\mathrm{h}}{2}}\,\mathrm{e}_{-}\Big)\bigg)\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{\sqrt{n+2}}\bar{a}_0\mathrm{h}}\,\bar{\lambda}^{-\frac{\mathrm{h}}{4}}\, .$$

The universal enveloping algebra $U_q(\mathfrak{sl}_2)$ admits a $2j+1$ dimensional representation ($j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$), so that

$$\boldsymbol{L}_j(\lambda) = \pi_j\big(\boldsymbol{L}(\lambda)\big)\, , \qquad \bar{\boldsymbol{L}}_j(\bar{\lambda}) = \pi_j\big(\bar{\boldsymbol{L}}(\bar{\lambda})\big) \tag{14.4}$$

are $(2j+1)\times(2j+1)$ operator valued matrices. Following the same line of arguments as in ref. [32], one can show that these satisfy the Yang-Baxter algebra of the form

$$\begin{aligned}
&R_{jj'}\big(\sqrt{\lambda_1/\lambda_2}\big)\,\big(\boldsymbol{L}_j(\lambda_1)\otimes\mathbf{1}\big)\big(\mathbf{1}\otimes\boldsymbol{L}_{j'}(\lambda_2)\big)\\
&\quad = \big(\mathbf{1}\otimes\boldsymbol{L}_{j'}(\lambda_2)\big)\big(\boldsymbol{L}_j(\lambda_1)\otimes\mathbf{1}\big)\,R_{jj'}\big(\sqrt{\lambda_1/\lambda_2}\big)\\
&R_{jj'}\Big(\sqrt{\bar{\lambda}_2/\bar{\lambda}_1}\Big)\,\big(\bar{\boldsymbol{L}}_j(\bar{\lambda}_1)\otimes\mathbf{1}\big)\big(\mathbf{1}\otimes\bar{\boldsymbol{L}}_{j'}(\bar{\lambda}_2)\big)\\
&\quad = \big(\mathbf{1}\otimes\bar{\boldsymbol{L}}_{j'}(\bar{\lambda}_2)\big)\big(\bar{\boldsymbol{L}}_j(\bar{\lambda}_1)\otimes\mathbf{1}\big)\,R_{jj'}\Big(\sqrt{\bar{\lambda}_2/\bar{\lambda}_1}\Big)\\
&\big(\boldsymbol{L}_j(\lambda)\otimes\mathbf{1}\big)\,\big(\mathbf{1}\otimes\bar{\boldsymbol{L}}_{j'}(\bar{\lambda})\big) = \big(\mathbf{1}\otimes\bar{\boldsymbol{L}}_{j'}(\bar{\lambda})\big)\,\big(\boldsymbol{L}_j(\lambda)\otimes\mathbf{1}\big)\, .
\end{aligned} \tag{14.5}$$

Here $R_{jj'}(\lambda)$ is the trigonometric solution to the Yang-Baxter equation which acts in the tensor product $\pi_j\otimes\pi_{j'}$ and in particular

$$R_{½½}(\lambda) = \begin{pmatrix} q^{-1}\lambda - q\lambda^{-1} & 0 & 0 & 0 \\ 0 & \lambda-\lambda^{-1} & q^{-1}-q & 0 \\ 0 & q^{-1}-q & \lambda-\lambda^{-1} & 0 \\ 0 & 0 & 0 & q^{-1}\lambda - q\lambda^{-1} \end{pmatrix} . \tag{14.6}$$

Notice that in the first line of (14.5) the R-matrix depends on the ratio λ_1/λ_2, while in the second line it depends on $\bar{\lambda}_2/\bar{\lambda}_1$. This is because in the definition (14.3), the path ordering for $\boldsymbol{L}(\lambda)$ is opposite to that of $\bar{\boldsymbol{L}}(\bar{\lambda})$. An immediate consequence of the algebraic relations (14.5) is that the operators

$$\boldsymbol{\tau}(\lambda) = \mathrm{Tr}\Big[\mathrm{e}^{\frac{\mathrm{i}\pi}{\sqrt{n+2}}\, a_0\sigma^3}\, \boldsymbol{L}_{\frac{1}{2}}(\lambda)\Big], \qquad \bar{\boldsymbol{\tau}}(\bar{\lambda}) = \mathrm{Tr}\Big[\bar{\boldsymbol{L}}_{\frac{1}{2}}(\bar{\lambda})\, \mathrm{e}^{-\frac{\mathrm{i}\pi}{\sqrt{n+2}}\, \bar{a}_0\sigma^3}\Big] \tag{14.7}$$

obey the commutativity conditions

$$\big[\boldsymbol{\tau}(\lambda),\, \boldsymbol{\tau}(\lambda')\big] = \big[\bar{\boldsymbol{\tau}}(\bar{\lambda}),\, \bar{\boldsymbol{\tau}}(\bar{\lambda}')\big] = \big[\boldsymbol{\tau}(\lambda),\, \bar{\boldsymbol{\tau}}(\bar{\lambda})\big] = 0\, . \tag{14.8}$$

As in the homogeneous case, $\boldsymbol{\tau}(\lambda)$ commutes with $\mathcal{A}_{\pm}(\lambda')$ and satisfies the relation

$$\boldsymbol{\tau}(\lambda)\mathcal{A}_+(\lambda) = \mathrm{e}^{+\frac{2\mathrm{i}\pi}{\sqrt{n+2}}\, a_0}\, \mathcal{A}_+(q^{+2}\lambda) + \mathrm{e}^{-\frac{2\mathrm{i}\pi}{\sqrt{n+2}}\, a_0}\, \mathcal{A}_+(q^{-2}\lambda)\, . \tag{14.9}$$

However, it deserves to be mentioned that now $\boldsymbol{\tau}(\lambda)$ and $\mathcal{A}_{\pm}(\lambda)$ possess a power series expansion in λ rather than λ^2. The similar statements hold true for the barred counterparts $\bar{\boldsymbol{\tau}}(\bar{\lambda})$ and $\bar{\mathcal{A}}_{\pm}(\bar{\lambda})$. The latter is defined by the formulae analogous to (13.7)-(13.10).

It is instructive to consider explicitly the first few terms in the Taylor series for the vacuum eigenvalues of $\boldsymbol{\tau}(\lambda)$. From the definition (14.7) it follows that

$$\begin{aligned} \tau^{(\mathrm{vac})}(\lambda\,|\,p,s) &= 2\,\cos\big(\tfrac{2\pi p}{n+2}\big) - 2s\; Q_1\big(\tfrac{p}{n+2},\, \tfrac{1}{n+2}\big)\,\lambda \\ &\quad + \Big(4s^2\; Q_2\big(\tfrac{p}{n+2},\, \tfrac{1}{n+2}\big) - 2n\; \tilde{Q}_2\big(\tfrac{p}{n+2},\, \tfrac{1}{n+2}\big)\Big)\, \lambda^2 + O(\lambda^3)\, , \end{aligned} \tag{14.10}$$

where $Q_{1,2}(h,g)$ are given in eq. (5.20), while

$$\begin{aligned} \tilde{Q}_2(h,g) = &\int_0^{2\pi} \mathrm{d}u_1 \int_0^{u_1} \mathrm{d}v_1 \int_0^{v_1} \mathrm{d}u_2 \int_0^{u_2} \mathrm{d}v_2\, \Big(2\sin\big(\tfrac{u_1-u_2}{2}\big)\Big)^{2g-2} \Big(2\sin\big(\tfrac{v_1-v_2}{2}\big)\Big)^{2g} \\ &\times \Big(2\sin\big(\tfrac{u_1-v_1}{2}\big)\Big)^{-2g} \Big(2\sin\big(\tfrac{u_1-v_2}{2}\big)\Big)^{-2g} \Big(2\sin\big(\tfrac{v_1-u_2}{2}\big)\Big)^{-2g} \Big(2\sin\big(\tfrac{u_2-v_2}{2}\big)\Big)^{-2g} \\ &\times 2\,\cos\big(2h(\pi - u_1 - u_2 + v_1 + v_2)\big)\, . \end{aligned} \tag{14.11}$$

Remarkably, the four-fold integral $\tilde{Q}_2(h,g)$ may be computed analytically. Indeed, using eqs. (14.10) and (13.2) to expand both sides of (14.9) in λ, and comparing the coefficient of λ^2 from both sides of that equation, one can express $\tilde{Q}_2(h,g)$ in terms of the vacuum eigenvalues of $\mathbf{J}_1^{(+)}$ and $\mathbf{J}_2^{(+)}$. Then with eq. (11.11) at hand one finds

$$\tilde{Q}_2(h,g) = \frac{g}{2(2g-1)}\, \frac{\pi^2\, \Gamma(1-4g)}{\Gamma(1-2g+2h)\Gamma(1-2g-2h)}\, \frac{\Gamma^4(-g)}{\Gamma^2(-2g)}\, . \tag{14.12}$$

The last formula shows that the integral in the r.h.s. of (14.11) converges only in the left half plane $\Re e(g) < 0$. Nevertheless, (14.12) provides an analytic continuation of this multi-fold integral to the whole complex plane. Note that $Q_1(h,g)$ and $Q_2(h,g)$ (5.20) converge for any $\Re e(g) < \frac{1}{2}$.

The analysis of the vacuum eigenvalues leads one to conclude that the definition (14.7), understood as a series expansion involving ordered integrals over the vertex operators, can not be taken literally for any $n > 0$. This is an important difference to the homogeneous case, where the expression (5.57) makes sense in the domain $0 < \beta^2 < \frac{1}{2}$. For the $\mathcal{Z}_2$ invariant model, the formulae for $\boldsymbol{\tau}(\lambda)$ and $\bar{\boldsymbol{\tau}}(\bar{\lambda})$ (14.7) as well as eq. (13.9) that defines $\boldsymbol{a}_{\pm}(\lambda)$ may only be understood via analytic continuation in complex n. The latter is achieved by re-writing the ordered integrals in terms of the contour integrals following the procedure explained in the work [32].

The scaling limit of the eigenvalue of the transfer matrix corresponding to the RG trajectory $\boldsymbol{\Psi}_N$ may be obtained through a comparison of eqs. (2.6), (2.7) where $\eta_J = \mathrm{i}\,(-1)^{J+1}$ and the scaling counterpart (14.9). Keeping in mind the formula (11.1) describing the scaling limit of $A_+(\zeta)$ as well as (13.13), one finds

$$\begin{aligned}&\operatorname*{slim}_{\substack{N\to\infty\\ b(N)\to s}}\; G^{(N/2)}\big(-q^2\mu^2\,|\,\tfrac{2}{n+2}\big)G^{(N/2)}\big(-q^{-2}\mu^2\,|\,\tfrac{2}{n+2}\big)\,T^{(N)}\Big(\big(N/(2N_0)\big)^{-\frac{n}{n+2}}\,\mathrm{i}\mu\Big)\\ &\quad=(-1)^{\mathtt{w}}\,\tau(\lambda)\,,\end{aligned}\tag{14.13}$$

where $\tau(\lambda)=\tau(\lambda\,|\,\boldsymbol{w},p,s)$ stands for the eigenvalue of the operator $\boldsymbol{\tau}(\lambda)$ on the state $\boldsymbol{\psi}_{p,s}(\boldsymbol{w})$. Recall that $q=\mathrm{e}^{\frac{\mathrm{i}\pi}{n+2}}$ while (11.10) provides the relation between λ and μ. Contrary to the homogeneous case the sign factor $(-1)^{\mathtt{w}}$ does not show up in the formula (8.2b) for the eigenvalues of the lattice translation operator $\mathbb{K}$ on $\boldsymbol{\Psi}_N$. However, in the sector of low energy states, one can define the operator $\sqrt{\mathbb{K}}$, which belongs to the commuting family of operators and whose eigenvalues on $\boldsymbol{\Psi}_N$ are given by[6]

$$\sqrt{K}=(-1)^{\mathtt{w}}\exp\bigg(\frac{2\pi\mathrm{i}}{N}\bigg(\frac{p^2-\bar{p}^2}{n+2}+\mathtt{L}-\bar{\mathtt{L}}\bigg)\bigg)\,.\tag{14.14}$$

Then eq. (14.13) may be rewritten in the operator form

$$\begin{aligned}&\operatorname*{slim}_{\substack{N\to\infty\\ b(N)\to s}}\; G^{(N/2)}\big(-q^2\mu^2\,|\,\tfrac{2}{n+2}\big)\,G^{(N/2)}\big(-q^{-2}\mu^2\,|\,\tfrac{2}{n+2}\big)\,\mathbb{T}\Big(\big(N/(2N_0)\big)^{-\frac{n}{n+2}}\,\mathrm{i}\mu\Big)\sqrt{\mathbb{K}}\\ &\quad=\boldsymbol{\tau}(\lambda)\,.\end{aligned}\tag{14.15}$$

The "barred" version of the above relation reads as

$$\begin{aligned}&\operatorname*{slim}_{\substack{N\to\infty\\ b(N)\to s}}\; G^{(N/2)}\big(-q^2\bar{\mu}^{-2}\,|\,\tfrac{2}{n+2}\big)\,G^{(N/2)}\big(-q^{-2}\bar{\mu}^{-2}\,|\,\tfrac{2}{n+2}\big)\,\mathbb{T}\Big(\big(N/(2N_0)\big)^{+\frac{n}{n+2}}\\ &\quad\times(\mathrm{i}\bar{\mu})^{-1}\Big)\sqrt{\mathbb{K}}=\bar{\boldsymbol{\tau}}(\bar{\lambda})\,,\end{aligned}\tag{14.16}$$

where $\bar{\mu}$ is given in terms of $\bar{\lambda}$ similar to (11.10):

$$\bar{\mu}=-\mathrm{i}\,(n+2)^{-\frac{2}{n+2}}\,\Gamma^2\big(-\tfrac{1}{n+2}\big)\,\bar{\lambda}\,.\tag{14.17}$$

[6] Notice that $\sqrt{\mathbb{K}}$ does not coincide with the one - site lattice translation operator $\mathcal{K}$, whose matrix elements are

$$(\mathcal{K})^{b_N b_{N-1}\ldots b_1}_{a_N a_{N-1}\ldots a_1}=\mathrm{e}^{\mathrm{i}\pi\mathtt{k}\,a_1}\,\delta^{b_{N-1}}_{a_N}\,\delta^{b_{N-2}}_{a_{N-1}}\ldots\delta^{b_N}_{a_1}\,.$$

Despite that $\mathcal{K}^2=\mathbb{K}$, the one - site translation does not commute with the transfer matrix. As is discussed in sec. 7 of ref. [29],

$$\mathcal{K}^{-1}\,\mathbb{T}(\zeta)\,\mathcal{K}=\hat{\mathcal{D}}\,\mathbb{T}(\zeta)\,\hat{\mathcal{D}}=\mathbb{T}(-\zeta)\,.$$

There is an alternative way to define $\sqrt{\mathbb{K}}$. To this end, consider the operators

$$\mathbb{K}^{(\pm)} = \mathrm{e}^{\mathrm{i}\pi \mathrm{k}}\, q^{-\frac{N}{2}+\mathbb{S}^z}\ \mathbb{A}_+(\mp \mathrm{i}q^{+1})\,\big[\mathbb{A}_+(\mp\mathrm{i}q^{-1})\big]^{-1}\,. \tag{14.18}$$

As it follows from eqs. (2.29), (8.5) the lattice translation operator $\mathbb{K}$ and the quasi-shift $\mathbb{B}$ are expressed in terms of (14.18) as follows

$$\mathbb{B} = \mathbb{K}^{(+)}\big(\mathbb{K}^{(-)}\big)^{-1}\,, \qquad \mathbb{K} = \mathbb{K}^{(+)}\,\mathbb{K}^{(-)}\,. \tag{14.19}$$

It was discussed in sec. 8 that for the low energy states one can unambiguously introduce the operator $\frac{n}{4\pi}\log\mathbb{B}$. Its eigenvalues are equal to $b(N)$ that appears in eq. (8.2a) describing the low energy spectrum of the lattice Hamiltonian and lie in the strip $|\Im m\big(b(N)\big)| < \frac{n}{4}$ (see (8.7)). This allows one to define the operator $\sqrt{\mathbb{B}}$, acting on the low energy states, with eigenvalues given by $\mathrm{e}^{\frac{2\pi}{n}b}$. Our numerical work confirms the relation

$$\mathbb{K}^{(\pm)} = \sqrt{\mathbb{K}}\,\big(\sqrt{\mathbb{B}}\big)^{\pm 1}\,. \tag{14.20}$$

The latter, instead of (14.14), may be used to introduce the operator $\sqrt{\mathbb{K}}$.

15. Local integrals of motion and the chiral states $\boldsymbol{\psi}_{p,s}(\boldsymbol{w})$

In the scaling limit the low energy Bethe states take the form $\bar{\boldsymbol{\psi}}_{\bar{p},s}(\bar{\boldsymbol{w}})\otimes\boldsymbol{\psi}_{p,s}(\boldsymbol{w})$. The chiral states may be interpreted as states in the Fock spaces, based on the diagonalization problem of $\boldsymbol{a}_\pm(\lambda)$ and $\bar{\boldsymbol{a}}_\pm(\bar{\lambda})$. The latter, being defined in terms of a path-ordered exponential, are difficult to work with for any practical calculations. As in the homogeneous case, for the explicit construction of $\boldsymbol{\psi}_{p,s}(\boldsymbol{w})\in\mathcal{F}_{\mathbf{P}}$ it turns out to be most convenient to diagonalize the operators which occur in the large λ asymptotic of $\boldsymbol{a}_\pm(\lambda)$ and/or $\boldsymbol{\tau}(\lambda)$. Among these are the so-called local Integrals of Motion (IM). For $n>2$ they are the only operators that appear in the large λ expansion of $\boldsymbol{\tau}(\lambda)$. It follows from the results of the work [63] that as $\lambda\to\infty$

$$\log\boldsymbol{\tau}(\lambda)\asymp\begin{cases} -2\pi\sum\limits_{m=-1}^{\infty} c_m\,\mathbf{I}_m\,\big(+(n+2)\lambda\big)^{-\frac{(n+2)m}{n}} & \Re e(\lambda)>0\\ +2\pi\sum\limits_{m=-1}^{\infty} c_m\,(-1)^m\,\mathbf{I}_m\,\big(-(n+2)\lambda\big)^{-\frac{(n+2)m}{n}} & \Re e(\lambda)<0\end{cases}\qquad (n>2)\,. \tag{15.1}$$

Here $\mathbf{I}_{-1}=1$, $\mathbf{I}_0=\int_0^{2\pi}\frac{\mathrm{d}u}{2\pi}\,\partial\vartheta=b_0$, while the non-trivial local IM $\{\mathbf{I}_m\}_{m=1}^{\infty}$ have the form

$$\mathbf{I}_m=\int_0^{2\pi}\frac{\mathrm{d}u}{2\pi}\,T_{m+1}(u) \tag{15.2}$$

with $T_{m+1}(u)$ being a chiral local density, i.e., a differential polynomial in $\partial\varphi(u)$ and $\partial\vartheta(u)$, of Lorentz spin $m+1$. Another way to formulate the last condition is to assign a grade 1 to $\partial\vartheta(u)$, $\partial\varphi(u)$ as well as the derivative. Then T_{m+1} is a homogeneous polynomial in the chiral fields and their derivatives of grade $m+1$. The first few densities read explicitly as [62,63]

$$\begin{aligned} T_2 &= (\partial\vartheta)^2+(\partial\varphi)^2\\ T_3 &= (\partial\vartheta)^3+\frac{3(n+2)}{3n+4}\,(\partial\varphi)^2\partial\vartheta+\frac{3\mathrm{i}(n+1)\sqrt{n+2}}{3n+4}\,\partial^2\varphi\,\partial\vartheta \end{aligned} \tag{15.3}$$

$$T_4 = (\partial\vartheta)^4 - \frac{n^2-2}{5n+6}\,(\partial^2\vartheta)^2 + \frac{6\,(n+2)}{5n+6}\,(\partial\vartheta)^2\,(\partial\varphi)^2 + \frac{6\mathrm{i}\,(n+1)\sqrt{n+2}}{5n+6}\,(\partial\vartheta)^2\,\partial^2\varphi$$
$$- \frac{(n+1)^2}{5n+6}\,(\partial^2\varphi)^2 + \frac{n+2}{5n+6}\,(\partial\varphi)^4 \,.$$

Notice that the local IM are defined up to an overall normalization. If we take $T_{m+1} = (\partial\vartheta)^{m+1} + \ldots$, where the "…" denote terms containing lower powers of $\partial\vartheta$, then the numerical coefficients c_m in (15.1) are given by

$$c_m = \frac{2^m\,\Gamma(\frac{1}{2}+\frac{n+1}{n}m)}{\sqrt{\pi}\,(m+1)!\,\Gamma(1+\frac{m}{n})}\,\big(1+\tfrac{2}{n}\big)^{-m}\,\Big[\Gamma\big(1-\tfrac{1}{n+2}\big)\Big]^{-\frac{2(n+2)m}{n}}\,n^{-\frac{m+1}{2}}\,. \tag{15.4}$$

The local IM act invariantly in the level subspace of the Fock space. Restricted to $\mathcal{F}^{(\mathtt{L})}_{\mathbf{P}}$ they are given by a sum of a finite number of terms involving the Heisenberg generators (13.5). This makes the computation of the matrix elements of $\mathbf{I}_m$ and, in turn, the diagonalization problem

$$\mathbf{I}_m\,\boldsymbol{\psi}_{p,s}(\boldsymbol{w}) = I_m(\boldsymbol{w},p,s)\,\boldsymbol{\psi}_{p,s}(\boldsymbol{w}) \tag{15.5}$$

a straightforward task – much simpler than the spectral problem of $\mathbb{A}_\pm(\lambda)$. The Fock highest state is, of course, an eigenstate and the corresponding eigenvalues for the first few IM can be easily extracted from the explicit formulae (15.3):

$$\begin{aligned}
I_1^{(\mathrm{vac})}(p,s) &= \frac{p^2}{n+2} + \frac{s^2}{n} - \frac{1}{12}\\
I_2^{(\mathrm{vac})}(p,s) &= \frac{s}{\sqrt{n}}\,\bigg(\frac{3p^2}{3n+4} + \frac{s^2}{n} - \frac{(2n+3)}{4(3n+4)}\bigg)\\
I_3^{(\mathrm{vac})}(p,s) &= \frac{p^4}{(5n+6)(n+2)} - \frac{p^2}{2(5n+6)} + \frac{6p^2\,s^2}{n(5n+6)} + \frac{s^4}{n^2} - \frac{(3n+4)\,s^2}{2n(5n+6)}\\
&\quad - \frac{(n-6)\,(2n+3)}{240\,(5n+6)}\,.
\end{aligned} \tag{15.6}$$

The vacuum and higher level eigenvalues may be alternatively obtained through a WKB analysis of the ODE (11.2). It turns out that $I_m(\boldsymbol{w},p,s)$ is a symmetric polynomial in $\boldsymbol{w} = \{w_j\}_{j=1}^{\mathtt{L}}$ of degree $m-1$. For instance,

$$\begin{aligned}
I_1(\boldsymbol{w},p,s) &= I_1^{(\mathrm{vac})}\big(\sqrt{p^2+(n+2)\,\mathtt{L}},\,s\big)\\
I_2(\boldsymbol{w},p,s) &= I_2^{(\mathrm{vac})}\big(\sqrt{p^2+(n+2)\,\mathtt{L}},\,s\big) + \frac{3\mathrm{i}\sqrt{n}}{3n+4}\,\sum_{j=1}^{\mathtt{L}} w_j\\
I_3(\boldsymbol{w},p,s) &= I_3^{(\mathrm{vac})}\big(\sqrt{p^2+(n+2)\,\mathtt{L}},\,s\big)\\
&\quad - \frac{4}{(5n+6)(n+2)}\,\bigg(n\sum_{j=1}^{\mathtt{L}} w_j^2 - \mathrm{i}s\,(n+4)\,\sum_{j=1}^{\mathtt{L}} w_j\bigg)\,.
\end{aligned} \tag{15.7}$$

It is expected that the joint spectrum of the local IM lifts all the degeneracies in $\mathcal{F}^{(\mathtt{L})}_{\mathbf{P}}$ so that the state $\boldsymbol{\psi}_{p,s}(\boldsymbol{w})$ is uniquely specified by the eigenvalues $I_m(\boldsymbol{w},p,s)$. For $\mathtt{L}\le 5$ we found it sufficient to use just the first three IM, along with formulae (15.7), to obtain $\boldsymbol{\psi}_{p,s}(\boldsymbol{w})$ for some given set $\boldsymbol{w}$.

The local IM also appear in the large λ asymptotic expansion for $\mathcal{A}_\pm(\lambda)$. However, unlike eq. (15.1) for $\boldsymbol{\tau}(\lambda)$, the latter involves the so-called dual non-local IM as well. The simplest of these are the operators $\check{\mathbf{C}}^{(\pm)}$:

$$\check{\mathbf{C}}^{(\pm)}\,\boldsymbol{\psi}_{p,s}(\boldsymbol{w}) = \check{C}^{(\pm)}_{p,s}(\boldsymbol{w})\,\boldsymbol{\psi}_{p,s}(\boldsymbol{w}) \qquad \text{with} \qquad \check{C}^{(\pm)}_{p,s}(\boldsymbol{w}) = \mathfrak{C}^{(\pm)}_{p,s}(\boldsymbol{w})/\mathfrak{C}^{(0,\pm)}_{p,s}\,. \tag{15.8}$$

Here $\mathfrak{C}^{(\pm)}_{p,s}(\boldsymbol{w})$ are the coefficients that enter into the asymptotic formula (11.16), while $\mathfrak{C}^{(0,\pm)}_{p,s} = \mathfrak{C}^{(\pm)}_{p,s}(\boldsymbol{w})\big|_{\mathrm{L}=0}$ are given in (11.18). This way, as indicated by the "check" symbol, the operators are normalized so that their eigenvalue on the Fock highest state is one. Earlier, we used the functions $\check{D}_{p,s}(\boldsymbol{w})$ and $\check{R}_{p,s}(\boldsymbol{w})$ (see, e.g., eqs. (10.7) and (11.27)), which coincide with the eigenvalues of the reflection operators

$$\check{\mathbf{D}} = \check{\mathbf{C}}^{(+)}\,\big(\check{\mathbf{C}}^{(-)}\big)^{-1}\,, \qquad\qquad \check{\mathbf{R}} = \check{\mathbf{C}}^{(+)}\,\big(\check{\mathbf{C}}^{(-)}\big)^{-1}\,. \tag{15.9}$$

The construction of $\check{\mathbf{C}}^{(\pm)}$, $\check{\mathbf{D}}$ and $\check{\mathbf{R}}$ as operators acting in the Fock space, as well as a closed analytic expression for their eigenvalues in terms of the sets $\boldsymbol{w}$ is given in sec. 3 of ref. [47]. For the reader's convenience, we quote the formulae for $\check{D}_{p,s}(\boldsymbol{w})$ and $\check{R}_{p,s}(\boldsymbol{w})$ in Appendix B. We found the diagonalization problem of the reflection operators useful for the construction of the chiral states $\boldsymbol{\psi}_{p,s}(\boldsymbol{w})$.

16. Extended conformal symmetry

16.1. The W_∞ - algebra

The graded linear space $\mathcal{V}_{p,s} = \bigoplus_{\mathrm{L}} \mathcal{V}^{(\mathrm{L})}_{p,s}$ (13.4) for real s as well as the spaces $\mathcal{V}_{p,s}$ with $s = \pm\mathrm{i}\mathrm{q}_a, \pm\mathrm{i}\bar{\mathrm{q}}_a$ and $\mathcal{V}_{\rho,\nu}$ with $(\rho,\nu) = (p_+ \pm \mathrm{i}\mathrm{q}'_a)$, $(p_- \pm \mathrm{i}\bar{\mathrm{q}}'_a)$ (13.34) are the building blocks of the right chiral components of the conformal towers in $\mathcal{H}^{(\mathrm{cont})}_{S^z}$ and $\mathcal{H}^{(\mathrm{disc},\pm)}_{S^z}$. The operators from the commuting family generated by $\mathcal{A}_\pm(\lambda)$, including the local IM, act invariantly inside these spaces. However the local densities $T_{m+1}(u)$, which occur in the definition of the local IM (15.2) do not, in general, act invariantly in $\mathcal{V}_{p,s}$ for pure imaginary s when the graded space does not coincide with the Fock space. Since the local densities are defined up to a total derivative one could try to choose them in such a way so that the chiral spaces $\mathcal{V}_{p,s}$ are invariant w.r.t. their action both for real or pure imaginary s. The most general form of the spin 2 density is $W_2 = T_2 + \alpha_1\,\partial^2\vartheta + \alpha_2\,\partial^2\varphi$, where T_2 is given in (15.3) and α_1, α_2 are arbitrary constants. This local field would leave $\mathcal{V}_{p,s}$ invariant provided it commutes with the two screening charges (13.17), i.e.,

$$W_2(u)\,\mathrm{Q}_\sigma = \mathrm{Q}_\sigma\,W_2(u) \qquad\qquad (\sigma = \pm)\,. \tag{16.1}$$

The commutativity condition fixes $W_2(u)$ to be

$$W_2 = (\partial\vartheta)^2 + (\partial\varphi)^2 + \frac{\mathrm{i}}{\sqrt{n+2}}\,\partial^2\varphi\,. \tag{16.2}$$

Then a simple calculation shows that $W_2(u)$ satisfies the Operator Product Expansion (OPE)

$$W_2(u)\,W_2(0) = \frac{c}{2u^4} - \frac{2}{u^2}\,W_2(0) - \frac{1}{u}\,\partial W_2(0) + O(1) \tag{16.3}$$

with

$$c = \frac{2\,(n-1)}{n+2}\,. \tag{16.4}$$

In turn the modes $\widetilde{W}_2(m)$, defined through the Fourier series

$$W_2(u) = -\frac{c}{24} + \sum_{m=-\infty}^{\infty} \widetilde{W}_2(m)\, \mathrm{e}^{-\mathrm{i}mu}\,, \tag{16.5}$$

form the Virasoro algebra with the central charge (16.4). Thus the chiral spaces $\mathcal{V}_{p,s}$ can be classified according to the irreps of this conformal symmetry algebra. Note that the local IM $\mathbf{I}_1$ coincides with the zero mode $\widetilde{W}_2(0)$ up to an additive constant and its eigenvalue is related to the conformal dimension of a state as

$$I_1 = \Delta - \frac{c}{24}\,. \tag{16.6}$$

The conformal algebra admits a natural extension. Clearly a local field defined through the commutator $\big[W_2(u),\, \mathbf{I}_2\big]$ acts invariantly in $\mathcal{V}_{p,s}$ for any values of real or pure imaginary s. An explicit calculation shows that

$$\partial W_3(u) = \frac{3n+4}{3\mathrm{i}(n+2)}\,\big[W_2(u),\, \mathbf{I}_2\big]\,, \tag{16.7}$$

for[7]

$$W_3 = \frac{6n+8}{3n+6}\,(\partial\vartheta)^3 + 2\,(\partial\varphi)^2\partial\vartheta + \mathrm{i}\sqrt{n+2}\;\partial^2\varphi\,\partial\vartheta - \frac{\mathrm{i}n}{\sqrt{n+2}}\,\partial\varphi\,\partial^2\vartheta + \frac{n}{6(n+2)}\,\partial^3\vartheta\,. \tag{16.8}$$

The choice of the overall factor in the definition of $W_3(u)$ is somewhat arbitrary and we take it to be $\frac{6n+8}{3n+6}$ for future convenience. Computing the OPE of W_2 and W_3 yields

$$W_2(u)\,W_3(0) = -\frac{3}{u^2}\,W_3(0) - \frac{1}{u}\,\partial W_3(0) + O(1)\,, \tag{16.9}$$

which means that $W_3(u)$ is a primary chiral field of spin 3. Similar to (16.5) it can be expanded in the Fourier series

$$W_3(u) = \sum_{m=-\infty}^{\infty} \widetilde{W}_3(m)\, \mathrm{e}^{-\mathrm{i}mu}\,. \tag{16.10}$$

Notice that the zero mode $\widetilde{W}_3(0)$ coincides with the local IM $\mathbf{I}_2$ (15.2), (15.3) up to an overall factor,

$$\mathbf{I}_2 = \frac{3n+6}{6n+8}\,\widetilde{W}_3(0)\,, \tag{16.11}$$

so that the operators $\widetilde{W}_2(0)$ and $\widetilde{W}_3(0)$ can be diagonalized simultaneously.

The linear space of local spin 3 fields, invariantly acting in $\mathcal{V}_{p,s}$, is generated by W_3 and ∂W_2. As for the spin 4 fields acting in $\mathcal{V}_{p,s}$, they include the derivatives $\partial^2 W_2$, ∂W_3 as well as the

[7] The currents W_2 (16.2) and W_3 (16.8), along with the screening charge $\mathcal{Q}_\sigma$ (13.17), were originally obtained in the work [49]. See the footnote on page 649 therein.

composite field W_2^2, which is defined as the first regular term in the OPE (16.3). There is one more linearly independent spin 4 field $W_4(u)$, which we introduce through the OPE:

$$\begin{aligned} &W_3(u)\, W_3(0) \\ &\quad = -\frac{c(c+7)(2c-1)}{9(c-2)u^6} + \frac{(c+7)(2c-1)}{3(c-2)u^4}\,\big(W_2(u)+W_2(0)\big) - \frac{1}{u^2}\,\Big(W_4(u)+W_4(0) \\ &\qquad + W_2^2(u) + W_2^2(0) + \frac{2c^2+22c-25}{30(c-2)}\,\big(\partial^2 W_2(u) + \partial^2 W_2(0)\big)\Big) + O(1)\,. \end{aligned} \tag{16.12}$$

The definition of $W_4(u)$ is not unique and it is fixed as in (16.12) for the following reason. A priori, it would be natural to have $W_4(u)$ be a spin 4 primary field. However it turns out that this is impossible to achieve for any linear combination of W_4, $\partial^2 W_2$, ∂W_3 and W_2^2. With W_4 defined through (16.12), the OPE of W_4 and W_2,

$$W_2(u)\, W_4(0) = \frac{(c+10)(17c+2)}{15(c-2)\,u^4}\, W_2(0) - \frac{4}{u^2}\, W_4(0) - \frac{1}{u}\, \partial W_4(0) + O(1)\,, \tag{16.13}$$

does not contain the singular terms $\propto u^{-6}$ and u^{-3}. Since the densities for the local IM are defined up to a total derivative, $\mathbf{I}_3$ (15.2), (15.3) must be expressible as an integral over a linear combination of $W_4(u)$ and $W_2^2(u)$. A straightforward calculation yields

$$\mathbf{I}_3 = \frac{n+2}{(2n+3)(5n+6)} \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\, \Big((n+2)\, W_4 + (2n+3)\, W_2^2\Big)\,. \tag{16.14}$$

Continuing the process one can describe the linear space of local spin $j=2,3,4,\ldots$ fields that act invariantly in $\mathcal{V}_{p,s}$. It turns out that a basis would consist of composite fields built from the W fields of lower spin and their derivatives as well as one extra field W_j. The latter, of course, is not uniquely defined and can be generated through the OPE of the W fields of lower spin, similar to how W_4 is generated in the OPE (16.12). For generic values of $n>0$ the total number of linearly independent spin j fields is $N(j) = d_0(j) - d_1(j-1) = 1, 2, 4, 6, 11, \ldots$, where the integers $d_a(\mathtt{L})$ are described by eq. (10.25).[8] In turn the densities for the local IM (15.2) are expressible as a linear combination of such fields. Since T_{m+1} is defined up to a total derivative, it can be written as a sum of $N(m+1) - N(m)$ terms. The corresponding coefficients may be fixed through the commutativity condition $[\mathbf{I}_m, \mathbf{I}_2] = 0$. One of them would remain undetermined, which manifests the freedom in the overall normalization of $\mathbf{I}_m$.

There exists a simple way of obtaining the "independent" set of local fields $\{W_j(u)\}_{j=2}^{\infty}$, which is based on the following observation. Consider the pair of chiral non-local fields of Lorentz spin $1-\frac{1}{n}$,

$$\xi_\pm(u) = n^{-1}\,\Big(\sqrt{n}\,\partial\vartheta \pm \mathrm{i}\sqrt{n+2}\,\partial\varphi\Big)\, \mathrm{e}^{\pm\frac{2\vartheta}{\sqrt{n}}}(u)\,. \tag{16.15}$$

It is simple to check that they commute with the screening charges (13.18)

$$\xi_+(u)\, \mathbb{Q}_\sigma = \mathbb{Q}_\sigma\, \xi_+(u)\,, \qquad \xi_-(u)\, \mathbb{Q}_\sigma = \mathbb{Q}_\sigma\, \xi_-(u) \qquad (\sigma=\pm)\,. \tag{16.16}$$

Hence the local fields occurring in the OPE

[8] The expression for $N(j)$ follows from the formula for the character of the W_∞ - algebra (16.31) specialized to the case $|\rho| = \frac{1}{2}$, $\nu = 0$.

$$\xi_+(u)\,\xi_-(0) = -n^{-1}\,u^{-2(1-\frac{1}{n})}$$
$$\times\Big[1-\frac{n+2}{2n}\,\big(W_2(u)+W_2(0)\big)\,u^2-\frac{n+2}{2n\sqrt{n}}\,\big(W_3(u)+W_3(0)\big)\,u^3$$
$$-\frac{(n+2)^2}{4n^2\,(2n+3)}\,\Big(W_4(u)+W_4(0)-\frac{n\,(2n+3)}{5\,(n+2)}\,\big(\partial^2\,W_2(u)+\partial^2\,W_2(0)\big)\Big)\,u^4+\ldots\Big] \tag{16.17}$$

would also commute with Q_σ. A straightforward calculation shows that the coefficients of u^2 and u^3 involve the fields W_2 from (16.2) and W_3 in (16.8). The spin 4 field W_4 appearing in the coefficient of u^4 is the same as the one defined via the OPE (16.13). One can choose the fields W_j with $j\geq 5$ in such a way that the remaining terms in the expansion in (16.17), denoted by the ellipsis, coincide with $\sum_{j\geq 5}\big(W_j(u)+W_j(0)\big)\,u^j$. Notice that (16.15) is the well known bosonization formula [50–54], which extends the notion of the Fateev-Zamolodchikov $\mathcal{Z}_n$ parafermions [48] to the case when n is non-integer.

The infinite set of local chiral fields $\{W_j(u)\}_{j=2}^\infty$ form a closed operator algebra, in the sense that the singular part of the OPE of $W_j(u)W_{j'}(0)$ is expressible in terms of composite fields built out of the W fields and their derivatives. This algebra was discussed in the work [55], and we will refer to it as the W_∞ - algebra. Repeating all the above for the left chirality one would arrive at a barred copy of the algebra, $\overline{W}_\infty$, for the currents $\{\overline{W}_j(\bar{u})\}_{j=2}^\infty$. Thus the algebra of extended conformal symmetry underlying the critical behaviour of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model is $\overline{W}_\infty\otimes W_\infty$.

Let's make some comments regarding our terminology, which was borrowed from CFT [61]. In the description of a Lorentz invariant $1+1$D quantum field theory one employs the space-time co-ordinates $x^\mu=(t,x)$. For a CFT in finite volume, the space co-ordinate can be rescaled so that x belongs to the segment of length 2π. Moreover it is always possible to choose the unit measurement of time such that the "speed of light" is one. Then it is convenient to use the light cone co-ordinates

$$u=t+x\,,\qquad\qquad \bar{u}=t-x\qquad\qquad (0\leq x\leq 2\pi)\,. \tag{16.18}$$

A theory with extended conformal symmetry possesses chiral currents, which are local fields such that $W_j(t,x)=W_j(u)$ and $\overline{W}_j(t,x)=\overline{W}_j(\bar{u})$ as a consequence of the equations of motion. We use the convention that the (half-)integer j coincides with the Lorentz spin in the case of $W_j(t,x)$ and minus the spin for $\overline{W}_j(t,x)$. The theory with $\overline{W}_\infty\otimes W_\infty$ extended conformal symmetry contains, among the local fields, two infinite sets currents with $j=2,3,4,\ldots$, which are independent in the sense that no one current can be expressed as a differential polynomial in the others. The fields $W_2(u)$ and $\overline{W}_2(\bar{u})$ are naturally identified with the holomorphic and anti-holomorphic components of the energy momentum tensor, respectively. Assuming the boundary conditions of the theory are such that the chiral currents are periodic, as is the case here,

$$W_j(t,x)=W_j(t,x+2\pi)\,,\qquad\qquad \overline{W}_j(t,x)=\overline{W}_j(t,x+2\pi)\,, \tag{16.19}$$

they can be expanded in the Fourier series:

$$W_j=-\frac{c}{24}\,\delta_{j,2}+\sum_{m=-\infty}^{\infty}\widetilde{W}_j(m)\,\mathrm{e}^{-\mathrm{i}mu}\,,\qquad \overline{W}_j=-\frac{c}{24}\,\delta_{j,2}+\sum_{m=-\infty}^{\infty}\widetilde{\overline{W}}_j(m)\,\mathrm{e}^{-\mathrm{i}m\bar{u}}\,. \tag{16.20}$$

As usual, the modes $\widetilde{W}_2(m)$ and $\widetilde{\overline{W}}_2(m)$ generate two independent copies of the Virasoro algebra with central charge c and the CFT Hamiltonian is given by

$$\hat{H} = \widetilde{W}_2(0) + \widetilde{\overline{W}}_2(0) - \frac{c}{12} \ . \tag{16.21}$$

The states in a 2D CFT can be chosen to have a definite value of the pair of conformal dimensions $(\bar{\Delta}, \Delta)$. The corresponding CFT energy reads as

$$E = \Delta + \bar{\Delta} - \frac{c}{12} \ , \tag{16.22}$$

while the Lorentz spin coincides with the difference $\Delta - \bar{\Delta}$.

In the case of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model, with the anisotropy parameter $q = \mathrm{e}^{\frac{\mathrm{i}\pi}{n+2}}$ and $n > 0$, the central charge is given by $c = \frac{2(n-1)}{n+2}$ and lies in the interval

$$-1 < c < 2 \ . \tag{16.23}$$

The CFT energy appears in the large N asymptotics of the eigenvalues of the lattice Hamiltonian while the Lorentz spin is related to the eigenvalue of the lattice translation operator. Namely (8.2) can be written as [43]

$$\mathcal{E} = N e_\infty + \frac{4\pi v_{\mathrm{F}}}{N}\left(\Delta + \bar{\Delta} - \frac{c}{12}\right) + o(N^{-1})$$
$$K = \exp\left(\frac{4\pi \mathrm{i}}{N}\left(\Delta - \bar{\Delta}\right)\right) . \tag{16.24}$$

16.2. Highest weight irreps of the W_∞ - algebra

For a theory possessing extended conformal symmetry, the space of states is naturally classified according to the highest weight irreps of the symmetry algebra. It is convenient to describe the latter in terms of the Verma module. The Verma module of the W_∞ - algebra contains the highest state which is defined by the conditions

$$\widetilde{W}_j(m)\,|\boldsymbol{\omega}\rangle = 0 \qquad\qquad (\forall m > 0) \, , \qquad\qquad \widetilde{W}_j(0)\,|\boldsymbol{\omega}\rangle = \omega_j\,|\boldsymbol{\omega}\rangle \, , \tag{16.25}$$

where $\boldsymbol{\omega} = (\omega_2, \omega_3)$ is the highest weight. The component ω_2 is equal to the conformal dimension of the highest state and is simply related to the eigenvalue of the local IM $\mathbf{I}_1$ (16.6), while ω_3 coincides up to an overall factor with the eigenvalue of $\mathbf{I}_2$ (16.11). It turns out that the highest state $|\boldsymbol{\omega}\rangle$ is fully specified by the relations (16.25) with $j = 2, 3$. Moreover, the Verma module is spanned by the states of the form

$$\widetilde{W}_2(-l_1)\ldots\widetilde{W}_2(-l_m)\,\widetilde{W}_3(-l'_1)\ldots\widetilde{W}_3(-l'_{m'})|\boldsymbol{\omega}\rangle \tag{16.26}$$

with $1 \le l_1 \le l_2 \le \ldots \le l_m$ and $1 \le l'_1 \le l'_2 \le \ldots \le l'_{m'}$, which contain the Fourier modes of the spin 2 and spin 3 currents only. It is a naturally graded linear space and the dimensions of its level subspace with $\ell = \sum_i l_i + \sum_{i'} l'_{i'}$ is given by $\mathrm{par}_2(\ell)$. Formulae (16.2) and (16.8) introduce the structure of the W_∞ Verma module in the Fock space $\mathcal{F}_{\mathbf{P}}$ with the highest weight $\boldsymbol{\omega} = (\omega_2, \omega_3)$ related to $\mathbf{P} = (\frac{\rho}{\sqrt{n+2}}, \frac{\nu}{\sqrt{n}})$ as

$$\omega_2 = \frac{\rho^2 - \frac{1}{4}}{n+2} + \frac{\nu^2}{n} \tag{16.27}$$

$$\omega_3 = \frac{2\nu}{\sqrt{n}}\left(\frac{\rho^2}{n+2} + \frac{(3n+4)\,\nu^2}{3n\,(n+2)} - \frac{2n+3}{12\,(n+2)}\right).$$

In fact, it is convenient to use ρ and ν to parameterize $\boldsymbol{\omega}$ without necessarily any reference to the Fock space. Note that in this parameterization the highest weight depends only on ρ^2 so that ρ should be identified with $-\rho$. The highest weight irrep of the W_∞ - algebra, with $\boldsymbol{\omega}$ parameterized by the pair (ρ,ν) as in (16.27), will be denoted by $\mathcal{W}_{\rho,\nu} \equiv \mathcal{W}_{-\rho,\nu}$.

For generic complex values of ρ and ν the Verma module is an irrep of the W_∞ - algebra. Its character,

$$\mathrm{ch}_{\rho,\nu}(\mathtt{q}) \equiv \mathrm{Tr}_{\mathcal{W}_{\rho,\nu}}\Big[\,\mathtt{q}^{\widetilde{W}_2(0)-\frac{c}{24}}\,\Big], \tag{16.28}$$

with $c = \frac{2(n-1)}{n+2}$ is given by

$$\mathrm{ch}_{\rho,\nu}(\mathtt{q}) = \frac{\mathtt{q}^{-\frac{1}{12}+\frac{\nu^2}{n}+\frac{\rho^2}{n+2}}}{(\mathtt{q},\mathtt{q})_\infty^2} \qquad \big(\rho,\ \nu\ \text{ generic}\big)\,. \tag{16.29}$$

When certain constraints are imposed on ρ and ν, the Verma module contains null vectors – highest states occurring at non-zero levels. In this case the highest weight irrep can be obtained from the Verma module by factoring out all of the invariant subspace(s) generated by the null vector(s). As was demonstrated in sec. 13.2 using the Fock space realization of the Verma module, when $\rho+\frac{1}{2}+\mathrm{i}\nu = -a_+ = 0, \pm1, \pm2, \ldots$ there is a null vector $|\chi_+\rangle$ at the level $\big|a_+ + \frac{1}{2}\big| + \frac{1}{2}$. Similarly if $-\rho+\frac{1}{2}+\mathrm{i}\nu = -a_- = 0, \pm1, \pm2, \ldots$, a null-vector $|\chi_-\rangle$ occurs at the level $\big|a_- + \frac{1}{2}\big| + \frac{1}{2}$. Such Verma modules are usually referred to as degenerate. It turns out that if either $\rho+\frac{1}{2}+\mathrm{i}\nu = -a \in \mathbb{Z}$ or $\rho+\frac{1}{2}-\mathrm{i}\nu = -a \in \mathbb{Z}$ and $2\rho \notin \mathbb{Z}$, the character (16.28) is given by [53]

$$\mathrm{ch}_{\rho,\nu}(\mathtt{q}) = \frac{\mathtt{q}^{-\frac{1}{12}+\frac{\nu^2}{n}+\frac{\rho^2}{n+2}}}{(\mathtt{q},\mathtt{q})_\infty^2}\ \sum_{m=0}^{\infty}(-1)^m\ \mathtt{q}^{m|a+\frac{1}{2}|+\frac{m^2}{2}} \qquad \begin{array}{l}\rho+\frac{1}{2}\pm\mathrm{i}\nu\in\mathbb{Z}\\ \rho\ \ \text{generic}\end{array}\,. \tag{16.30}$$

Note that when $2\rho, 2\mathrm{i}\nu \in \mathbb{Z}$, while $2(\rho+\mathrm{i}\nu)$ is an odd integer then the Verma module contains both null-vectors $|\chi_\pm\rangle$. In this case, assuming n is irrational,[9]

$$\mathrm{ch}_{\rho,\nu}(\mathtt{q}) = \frac{\mathtt{q}^{-\frac{1}{12}+\frac{\nu^2}{n}+\frac{\rho^2}{n+2}}}{(\mathtt{q},\mathtt{q})_\infty^2}\ \sum_{m=0}^{\infty}(-1)^m\ \mathtt{q}^{\frac{m^2}{2}}\left(\mathtt{q}^{m\,||\rho|-|\nu||} - \mathtt{q}^{(m+1)(|\rho|+|\nu|+1)-\frac{1}{2}}\right), \tag{16.31}$$

where $\Im m(\rho) = \Re e(\nu) = 0$ such that

$$|\rho| \pm |\nu| \in \tfrac{1}{2} + \mathbb{Z}\,. \tag{16.32}$$

The chiral subspaces $\mathcal{V}_{p,\sigma\mathrm{i}\mathtt{q}_a}$, $\mathcal{V}_{p,\sigma\mathrm{i}\bar{\mathtt{q}}_a}$, $\mathcal{V}_{p_+,\sigma\mathrm{i}\mathtt{q}'_a}$ and $\mathcal{V}_{p_-,\sigma\mathrm{i}\bar{\mathtt{q}}'_a}$ (13.34) of the conformal towers in $\mathcal{H}^{(\mathrm{disc},\pm)}_{S^z}$, that were discussed in sec. 13.2, are highest weight irreps of the W_∞ - algebra. Namely $\mathcal{V}_{\rho,\nu} \cong \mathcal{W}_{\rho,\nu}$, where for the four spaces (ρ,ν) should be replaced by $(p,\sigma\mathrm{i}\mathtt{q}_a)$, $(p,\sigma\mathrm{i}\bar{\mathtt{q}}_a)$, $(p_+,\sigma\mathrm{i}\mathtt{q}'_a)$ and $(p_-,\sigma\mathrm{i}\bar{\mathtt{q}}'_a)$, respectively. The admissible values of $\rho =$

[9] For integer $n = 2, 3, \ldots$ the corresponding formula for the character was first obtained in ref. [56] (see also [53]). In addition note that eqs. (16.29)-(16.32), which assume that $c = 2 - \frac{6}{n+2} < 2$, can be applied to the case $c > 2$ if one makes the formal substitutions $n \to -n-2$, $\rho \to \mathrm{i}s$, $\nu \to \mathrm{i}p$. The central charge and highest weight of the irrep would be parameterized as in (22.6) and (22.7) below, see refs. [54,55].

p, $p_\pm$ for the lattice model has the form $2\rho = m_1 + (n+2)(\mathtt{k} + m_2)$, where m_1, m_2 are integers. We will mainly focus on the case when the twist parameter $\mathtt{k}$ and/or anisotropy parameter n are generic and assume that $2\rho \notin \mathbb{Z}$. When ν is real and $2\rho = m_1 + (n+2)(\mathtt{k} + m_2)$ with $(n+2)\,\mathtt{k} \notin \mathbb{Z}$ the chiral subspace $\mathcal{V}_{\rho,\nu} = \mathcal{F}_{\mathbf{P}}$ (see eq. (13.4)) is an irreducible representation of the W_∞ - algebra, i.e., $\mathcal{V}_{\rho,\nu} \cong \mathcal{W}_{\rho,\nu}$.

In the case of generic ν but with $2\rho = m_1 + (n+2)\,m_2$, i.e., $\mathtt{k} = 0$, the Verma module may become degenerate. This is related to the existence of the "bosonic" screening charge (see, e.g., [52–54]):

$$\hat{\mathrm{Q}} = \int_{u_0}^{u_0+2\pi} \mathrm{d}u\; \partial\vartheta\; \mathrm{e}^{-\frac{2\mathrm{i}\varphi}{\sqrt{n+2}}}(u) \tag{16.33}$$

(in the physical slang the formal operators $\hat{\mathrm{Q}}_\sigma$ (13.17) are referred to as "fermionic" screening charges). Similar to $\hat{\mathrm{Q}}_\sigma$, the integrand here has conformal dimensions $\Delta = 1$ w.r.t. the chiral component of the energy momentum tensor $W_2(u)$ (16.2). Thus, being a 1-form, the screening charge density can be integrated so that the action of $\hat{\mathrm{Q}}$ is formally defined on any Fock space $\mathcal{F}_{\mathbf{P}}$. For generic values of $\mathbf{P}$ the integration contour in (16.33) is not closed, i.e., $\hat{\mathrm{Q}}$ depends on the arbitrarily chosen initial integration point u_0. However, when restricted to the Fock space $\mathcal{F}_{\mathbf{P}}$ with $P_1 = \frac{1}{2}\,(m(n+2)+r)$ and arbitrary P_2, one can show that the action of the r-th power of $\hat{\mathrm{Q}}$ is well defined and does not depend on the choice of u_0. It is not difficult to see that for positive integers m and r the state

$$|\chi\rangle = \lim_{\Im m(v)\to+\infty} \mathrm{e}^{-\mathrm{i}v\Delta_\chi}\; \hat{\mathrm{Q}}^r\; \mathrm{e}^{\mathrm{i}(m\sqrt{n+2}+\frac{r-1}{\sqrt{n+2}})\varphi}(v)\,|\mathbf{P}_0\rangle\,, \quad \text{where} \quad \mathbf{P}_0 = \Big(\frac{1}{2\sqrt{n+2}},\, P_2\Big) \tag{16.34}$$

and $\Delta_\chi = \frac{(m(n+2)+r)^2-1}{4(n+2)}$ is non-trivial and belongs to the level subspace $\mathcal{F}_{\mathbf{P}}^{(\mathtt{L})}$ with $\mathbf{P} = \big(\frac{m(n+2)-r}{2\sqrt{n+2}},\, P_2\big)$ at level $\mathtt{L} = mr$. Furthermore it turns out to be a highest state of the W_∞ - algebra. Once the invariant subspace generated by this null vector is factored out one obtains an irrep whose character is given by

$$\mathrm{ch}_{\rho,\nu}(\mathrm{q}) = \mathrm{q}^{-\frac{1}{12}+\frac{\nu^2}{n}+\frac{\rho^2}{n+2}}\;\frac{1-\mathrm{q}^{mr}}{(\mathrm{q},\mathrm{q})_\infty^2}\;, \qquad \begin{array}{l} \rho = \pm\frac{1}{2}\big(m(n+2)-r\big)\,, \quad m,r = 1,2,\ldots \\ \nu,\ n\ \ \text{generic} \end{array} \tag{16.35}$$

A final comment is in order regarding the intertwiner $\hat{\mathrm{C}}_{\mathrm{R}} : \mathcal{F}_{(\pm P_1, P_2)} \mapsto \mathcal{F}_{(\mp P_1, P_2)}$, which was used in the description of the irreps of the W_∞ - algebra in terms of the Fock spaces for some cases with pure imaginary ν. This operator was introduced through the eigenbasis of $\mathcal{a}_\pm(\lambda)$ (13.31). An alternative definition is based on the fact that, as it follows from (16.27), the Fock spaces $\mathcal{F}_{(+P_1,P_2)}$ and $\mathcal{F}_{(-P_1,P_2)}$ are equivalent highest weight representations of the W_∞ - algebra. Then the intertwiner $\hat{\mathrm{C}}_{\mathrm{R}}$ can be unambiguously defined by the commutativity condition with the W currents

$$\hat{\mathrm{C}}_{\mathrm{R}}\, W_j(u) = W_j(u)\, \hat{\mathrm{C}}_{\mathrm{R}} \qquad\qquad (j = 2, 3) \tag{16.36}$$

supplemented by its action on the highest weight: $\hat{\mathrm{C}}_{\mathrm{R}}|(\pm P_1, P_2)\rangle = |(\mp P_1, P_2)\rangle$.

17. The space of states in the scaling limit

We are now ready to synthesize the analyses of the previous sections and describe the linear space of states occurring in the scaling limit of the low energy sector of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model. Some of the formulae presented here constitute the main results of our study of the lattice model and will be referred back to in the later part of the paper.

17.1. The sectors with $S^z = 0, 1, 2, \ldots$ and $(n+2)\,\mathtt{k} \notin \mathbb{Z}$

Recall our working definition of a low energy state – a state which can be assigned the quantum numbers S^z, $\mathtt{w}$, $\mathtt{L}$ and $\bar{\mathtt{L}}$, such that the energy and eigenvalue of the lattice translation operator follow the large N asymptotics (8.1)-(8.3), where $b = b(N)$ is defined by (8.6) along with the condition $|\Im m\big(b(N)\big)| < \frac{n}{4}$. In the scaling limit the states with fixed value of S^z were organized into the three sectors $\mathcal{H}^{(\mathrm{cont})}_{S^z}$, $\mathcal{H}^{(\mathrm{disc},+)}_{S^z}$ and $\mathcal{H}^{(\mathrm{disc},-)}_{S^z}$. Each of these is further split into the subsectors labeled by the winding number $\mathtt{w} = 0, \pm 1, \pm 2, \ldots$:

$$\mathcal{H}^{(\mathrm{cont})}_{S^z} = \bigoplus_{\mathtt{w}\in\mathbb{Z}} \mathcal{H}^{(\mathrm{cont})}_{S^z,\mathtt{w}}\,, \qquad\qquad \mathcal{H}^{(\mathrm{disc},\pm)}_{S^z} = \bigoplus_{\mathtt{w}\in\mathbb{Z}} \mathcal{H}^{(\mathrm{disc},\pm)}_{S^z,\mathtt{w}}\,. \tag{17.1}$$

The subsector $\mathcal{H}^{(\mathrm{cont})}_{S^z,\mathtt{w}}$ is described through a direct integral as

$$\mathcal{H}^{(\mathrm{cont})}_{S^z,\mathtt{w}} = \int_{\mathbb{R}}^{\oplus} \mathrm{d}s\ \bar{\mathcal{V}}_{\bar{p},s} \otimes \mathcal{V}_{p,s}\,, \qquad \text{where} \qquad \begin{aligned} p &= \tfrac{1}{2}\,S^z + \tfrac{1}{2}\,(n+2)\,(\mathtt{k}+\mathtt{w}) \\ \bar{p} &= \tfrac{1}{2}\,S^z - \tfrac{1}{2}\,(n+2)\,(\mathtt{k}+\mathtt{w}) \end{aligned} \tag{17.2}$$

and $\bar{\mathcal{V}}_{\bar{p},s} \otimes \mathcal{V}_{p,s}$ is isomorphic to a highest weight irrep of the $\overline{W}_\infty \otimes W_\infty$ - algebra. Contrary to $\mathcal{H}^{(\mathrm{cont})}_{S^z,\mathtt{w}}$, the decomposition of the linear space $\mathcal{H}^{(\mathrm{disc},+)}_{S^z,\mathtt{w}}$ involves a direct sum over the discrete set of pure imaginary admissible values of s. It reads as

$$\mathcal{H}^{(\mathrm{disc},+)}_{S^z,\mathtt{w}} = \bigoplus_{\sigma=\pm 1} \Big(\mathcal{H}^{(1,+)}_{S^z,\mathtt{w},\sigma} \oplus \mathcal{H}^{(2,+)}_{S^z,\mathtt{w},\sigma}\Big) \tag{17.3}$$

with

$$\mathcal{H}^{(1,+)}_{S^z,\mathtt{w},\sigma} = \bigoplus_{a\in\Sigma(p)} \bar{\mathcal{V}}_{\bar{p},\sigma\mathrm{i}\mathfrak{q}_a} \otimes \mathcal{V}_{p,\sigma\mathrm{i}\mathfrak{q}_a}\,, \qquad\qquad \mathcal{H}^{(2,+)}_{S^z,\mathtt{w},\sigma} = \bigoplus_{a\in\Sigma(\bar{p})} \bar{\mathcal{V}}_{\bar{p},\sigma\mathrm{i}\bar{\mathfrak{q}}_a} \otimes \mathcal{V}_{p,\sigma\mathrm{i}\bar{\mathfrak{q}}_a}\,. \tag{17.4}$$

Here

$$\mathfrak{q}_a = -p - \tfrac{1}{2} - a, \qquad\qquad \bar{\mathfrak{q}}_a = -\bar{p} - \tfrac{1}{2} - a \tag{17.5}$$

and the summation is taken over the non-negative integer a restricted to the sets

$$\Sigma(p) = \Big\{a:\ a\in\mathbb{Z}_+,\ -p-\tfrac{n+2}{4} \le a < -\tfrac{1}{2} - p\Big\} \tag{17.6}$$

as well as $\Sigma(\bar{p})$, which is given by the same formula with p substituted by $\bar{p}$. Each of the components $\bar{\mathcal{V}}_{\bar{p},s} \otimes \mathcal{V}_{p,s}$ from (17.2), (17.4), being a highest weight irrep of the $\overline{W}_\infty \otimes W_\infty$ - algebra, is a naturally graded linear space. The pair of non-negative quantum numbers $(\bar{\mathtt{L}}, \mathtt{L})$ for a state coincides with its level in the highest weight irrep.

The subsector $\mathcal{H}^{(\mathrm{disc},-)}_{S^z,\mathtt{w}}$ is also decomposed into the irreps of the $\overline{W}_\infty \otimes W_\infty$ - algebra. However an important difference from the cases $\mathcal{H}^{(\mathrm{cont})}_{S^z,\mathtt{w}}$ and $\mathcal{H}^{(\mathrm{disc},+)}_{S^z,\mathtt{w}}$ is that the pair $(\bar{\mathtt{L}}, \mathtt{L})$ does not coincide with the level of the state in the highest weight irrep. The linear structure of $\mathcal{H}^{(\text{disc},-)}_{S^z,\mathtt{w}}$ is more involved. To describe it, in addition to p, $\bar{p}$, $\mathfrak{q}_a$ and $\bar{\mathfrak{q}}_a$, we use the notation

$$\begin{aligned} p_+ &= \tfrac{1}{2}\,S^z + \tfrac{1}{2}\,(n+2)(\mathtt{k}+\mathtt{w}+1)\,, &\quad \bar{p}_+ &= \tfrac{1}{2}\,S^z - \tfrac{1}{2}\,(n+2)(\mathtt{k}+\mathtt{w}+1)\\ p_- &= \tfrac{1}{2}\,S^z + \tfrac{1}{2}\,(n+2)(\mathtt{k}+\mathtt{w}-1)\,, & \bar{p}_- &= \tfrac{1}{2}\,S^z - \tfrac{1}{2}\,(n+2)(\mathtt{k}+\mathtt{w}-1)\\ \mathfrak{q}'_a &= -p - \tfrac{n+1}{2} - a\,, & \bar{\mathfrak{q}}'_a &= -\bar{p} - \tfrac{n+1}{2} - a\,. \end{aligned} \tag{17.7}$$

Then

$$\mathcal{H}^{(\text{disc},-)}_{S^z,\mathtt{w}} = \bigoplus_{\sigma=\pm 1}\Big(\mathcal{H}^{(1,-)}_{S^z,\mathtt{w},\sigma}\oplus\mathcal{H}^{(2,-)}_{S^z,\mathtt{w},\sigma}\oplus\mathcal{H}^{(3,-)}_{S^z,\mathtt{w},\sigma}\oplus\mathcal{H}^{(4,-)}_{S^z,\mathtt{w},\sigma}\Big) \tag{17.8}$$

and the decomposition of each of the four spaces $\mathcal{H}^{(i,-)}_{S^z,\mathtt{w},\sigma}$ into irreps of the $\overline{W}_\infty\otimes W_\infty$ - algebra reads explicitly as

$$\begin{aligned} \mathcal{H}^{(1,-)}_{S^z,\mathtt{w},\sigma} &= \bigoplus_{a\in\Sigma_1(p)}\bar{\mathcal{V}}_{\bar{p}_+,\sigma\mathrm{i}\mathfrak{q}'_a}\otimes\mathcal{V}_{p_+,\sigma\mathrm{i}\mathfrak{q}'_a}\,, &\quad \mathcal{H}^{(2,-)}_{S^z,\mathtt{w},\sigma} &= \bigoplus_{a\in\Sigma_2(p)}\bar{\mathcal{V}}_{\bar{p},\sigma\mathrm{i}\mathfrak{q}_a}\otimes\mathcal{V}_{p_+,\sigma\mathrm{i}\mathfrak{q}'_a}\\ \mathcal{H}^{(3,-)}_{S^z,\mathtt{w},\sigma} &= \bigoplus_{a\in\Sigma_2(\bar{p})}\bar{\mathcal{V}}_{\bar{p}_-,\sigma\mathrm{i}\bar{\mathfrak{q}}'_a}\otimes\mathcal{V}_{p,\sigma\mathrm{i}\bar{\mathfrak{q}}_a}\,, & \mathcal{H}^{(4,-)}_{S^z,\mathtt{w},\sigma} &= \bigoplus_{a\in\Sigma_1(\bar{p})}\bar{\mathcal{V}}_{\bar{p}_-,\sigma\mathrm{i}\bar{\mathfrak{q}}'_a}\otimes\mathcal{V}_{p_-,\sigma\mathrm{i}\bar{\mathfrak{q}}'_a} \end{aligned} \tag{17.9}$$

Here the summation index a takes negative integer values and runs over the sets

$$\begin{aligned} \Sigma_1(p) &= \Big\{a:\ a\in\mathbb{Z}_-,\ -p-\tfrac{n+2}{4}\le a< -\tfrac{1}{2}-p\ \&\ a<-S^z\Big\}\\ \Sigma_2(p) &= \Big\{a:\ a\in\mathbb{Z}_-,\ -p-\tfrac{n+2}{4}\le a< -\tfrac{1}{2}-p\ \&\ a\ge -S^z\Big\} \end{aligned} \tag{17.10}$$

and $\Sigma_1(\bar{p})$, $\Sigma_2(\bar{p})$ which are defined by the analogous formulae. The levels w.r.t. the $\overline{W}_\infty\otimes W_\infty$ - algebra of the components in the r.h.s. of (17.9) do not coincide with $(\bar{\mathtt{L}},\mathtt{L})$. The relation between them depends on the case being considered, and can be read off from eqs. (12.15) and (12.18). For example, for the right chiral component $\mathcal{V}_{p_+,\sigma\mathrm{i}\mathfrak{q}'_a}$ the level w.r.t. the W_∞ - algebra, denoted by $\mathtt{L}_+$, is expressed in terms of $\mathtt{L}$ as $\mathtt{L}_+=\mathtt{L}-|a|$.

In the linear decompositions (17.2), (17.4) and (17.9), each of the chiral components $\mathcal{V}_{p,s}$, $\mathcal{V}_{p,\mathrm{i}\mathfrak{q}_a},\ldots$ is isomorphic to $\mathcal{W}_{\rho,\nu}$, the highest weight irrep of the W_∞ - algebra, whose highest weight is given by (16.27) with $(\rho,\nu)=(p,s),\ (p,\mathrm{i}\mathfrak{q}_a),\ldots$, respectively.

17.2. Global symmetries

The $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model possesses global $\mathcal{CPT}$ and $\mathcal{Z}_2$ symmetry. Since their generators $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}$ and $\hat{\mathcal{D}}$ commute with the lattice Hamiltonian, they preserve the low energy sector of the model. The action of the symmetry transformations on the low energy states in the scaling limit can be deduced from eqs. (2.37) and (7.4), which describe the commutation relations of $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}$ and $\hat{\mathcal{D}}$ with the lattice operators $\mathbb{A}_\pm(\zeta)$, $\mathbb{T}(\zeta)$. Combining them with the scaling relations (13.1), (14.15) yields

$$\begin{aligned} \hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{a}_\pm(\lambda)\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}} &= \boldsymbol{a}_\pm(\lambda^*)\,, &\quad \hat{\mathcal{D}}\,\boldsymbol{a}_\pm(\lambda)\,\hat{\mathcal{D}} &= \boldsymbol{a}_\pm(-\lambda)\\ \hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{\tau}(\lambda)\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}} &= \boldsymbol{\tau}(\lambda^*)\,, & \hat{\mathcal{D}}\,\boldsymbol{\tau}(\lambda)\,\hat{\mathcal{D}} &= \boldsymbol{\tau}(-\lambda) \end{aligned}\ . \tag{17.11}$$

The similar formulae also hold true for $\bar{\boldsymbol{a}}_\pm(\bar{\lambda})$ and $\bar{\boldsymbol{\tau}}(\bar{\lambda})$. For our purposes it is sufficient to focus on the commutation relations of the global symmetry generators with $\boldsymbol{\tau}(\lambda)$ and $\bar{\boldsymbol{\tau}}(\bar{\lambda})$. Keeping in mind that the local IM $\mathbf{I}_m$ $\big(\bar{\mathbf{I}}_m\big)$ occur in the large λ $(\bar{\lambda})$ asymptotic expansion for $\boldsymbol{\tau}(\lambda)$ $\big(\bar{\boldsymbol{\tau}}(\bar{\lambda})\big)$ as in eq. (15.1), one concludes that

$$\begin{aligned} &\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\mathbf{I}_m\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}=\mathbf{I}_m\,, &&\hat{\mathcal{D}}\,\mathbf{I}_m\,\hat{\mathcal{D}}=(-1)^{m+1}\,\mathbf{I}_m \\ &\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\bar{\mathbf{I}}_m\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}=\bar{\mathbf{I}}_m\,, &&\hat{\mathcal{D}}\,\bar{\mathbf{I}}_m\,\hat{\mathcal{D}}=(-1)^{m+1}\,\bar{\mathbf{I}}_m \end{aligned}\ . \tag{17.12}$$

The densities for the local IM can be expressed in terms of the W currents, so that the above relations would follow from

$$\begin{aligned} &\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,W_j(u)\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}=W_j(-u^*)\,, &&\hat{\mathcal{D}}\,W_j(u)\,\hat{\mathcal{D}}=(-1)^j\,W_j(u) \\ &\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\overline{W}_j(\bar{u})\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}=\overline{W}_j(-\bar{u}^*)\,, &&\hat{\mathcal{D}}\,\overline{W}_j(\bar{u})\,\hat{\mathcal{D}}=(-1)^j\,\overline{W}_j(\bar{u}) \end{aligned}\ . \tag{17.13}$$

These immediately imply that the symmetry transformations act, in general, as the intertwiners between the highest weight irreps appearing in the decompositions (17.2), (17.4) and (17.9). Namely,

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\ :\ \mathcal{V}_{\bar{\rho},\bar{\nu}}\otimes\mathcal{V}_{\rho,\nu}\mapsto\mathcal{V}_{\bar{\rho},\bar{\nu}^*}\otimes\mathcal{V}_{\rho,\nu^*}\,,\qquad\qquad \hat{\mathcal{D}}\ :\ \mathcal{V}_{\bar{\rho},\bar{\nu}}\otimes\mathcal{V}_{\rho,\nu}\mapsto\mathcal{V}_{\bar{\rho},-\bar{\nu}}\otimes\mathcal{V}_{\rho,-\nu}\,, \tag{17.14}$$

where $(\rho,\nu)=(p,s),\ (p,\sigma\mathrm{i}\mathtt{q}_a),\ldots$ and $(\bar{\rho},\bar{\nu})=(\bar{p},s),\ (\bar{p},\sigma\mathrm{i}\mathtt{q}_a),\ldots$. Notice that each of the subsectors $\mathcal{H}^{(\mathrm{cont})}_{S^z,\mathrm{w}}$ and $\mathcal{H}^{(\mathrm{disc},\pm)}_{S^z,\mathrm{w}}$ turn out to be invariant under the $\mathcal{CPT}$ and $\mathcal{D}$ transformations. In order to specify the action of the global symmetries on the states from the irrep $\mathcal{V}_{\bar{\rho},\bar{\nu}}\otimes\mathcal{V}_{\rho,\nu}$ one should return to the lattice system. The scaling limit of the low energy Bethe states yields the basis states

$$\boldsymbol{\psi}_{\bar{\rho},\rho,\bar{\nu},\nu}(\bar{\boldsymbol{w}},\boldsymbol{w})\equiv\bar{\boldsymbol{\psi}}_{\bar{\rho},\bar{\nu}}(\bar{\boldsymbol{w}})\otimes\boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w})\in\bar{\mathcal{V}}_{\bar{\rho},\bar{\nu}}\otimes\mathcal{V}_{\rho,\nu}\ . \tag{17.15}$$

Formulae (2.24) and (7.5), that describe the action of the $\mathcal{CPT}$ and $\mathcal{D}$ conjugations on $\boldsymbol{\Psi}_N$, allow one to deduce how the global symmetries act on $\boldsymbol{\psi}_{\bar{\rho},\rho,\bar{\nu},\nu}$. In particular, for the $\overline{W}_\infty\otimes W_\infty$ primary states

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{\psi}^{(\mathrm{vac})}_{\bar{\rho},\rho,\bar{\nu},\nu}=+\,\boldsymbol{\psi}^{(\mathrm{vac})}_{\bar{\rho},\rho,\bar{\nu}^*,\nu^*}\,,\qquad\qquad \hat{\mathcal{D}}\,\boldsymbol{\psi}^{(\mathrm{vac})}_{\bar{\rho},\rho,\bar{\nu},\nu}=+\,\boldsymbol{\psi}^{(\mathrm{vac})}_{\bar{\rho},\rho,-\bar{\nu},-\nu}\ . \tag{17.16}$$

The latter, combined with the commutation relations (17.13), unambiguously defines the symmetry transformations for any state in $\mathcal{V}_{\bar{\rho},\bar{\nu}}\otimes\mathcal{V}_{\rho,\nu}$.

Formula (17.13) involves the left and right W currents separately, so that the action of the $\mathcal{CPT}$ and $\mathcal{Z}_2$ symmetries may be naturally defined for each chiral component of the $\overline{W}_\infty\otimes W_\infty$ irrep. For instance, for the right chiral component:

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\ :\ \mathcal{V}_{\rho,\nu}\mapsto\mathcal{V}_{\rho,\nu^*}\,,\qquad\qquad \hat{\mathcal{D}}\ :\ \mathcal{V}_{\rho,\nu}\mapsto\mathcal{V}_{\rho,-\nu}\ . \tag{17.17}$$

For the chiral primary state $\boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,\nu}\in\mathcal{V}_{\rho,\nu}$, by choosing a proper normalization including the phase assignment, one can arrange that

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,\nu}=\boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,\nu^*}\,,\qquad\qquad \hat{\mathcal{D}}\,\boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,\nu}=\boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,-\nu}\ . \tag{17.18}$$

Recall that the chiral state $\boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w})\in\mathcal{V}_{\rho,\nu}$ in eq. (17.15) is an eigenvector of $\mathcal{A}_\pm(\lambda)$ with eigenvalue $D_\pm(\mu\,|\,\boldsymbol{w},\rho,\nu)$ (13.13) and similarly for $\bar{\boldsymbol{\psi}}_{\bar{\rho},\bar{\nu}}(\bar{\boldsymbol{w}})$. Again, with a proper choice of the normalization, the action of the $\mathcal{CPT}$ and $\mathcal{Z}_2$ conjugations on the eigenstates can be taken to be

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w}) = \boldsymbol{\psi}_{\rho,\nu^*}(-\boldsymbol{w}^*)\,, \qquad\qquad \hat{\mathcal{D}}\,\boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w}) = \boldsymbol{\psi}_{\rho,-\nu}(-\boldsymbol{w})\,. \tag{17.19}$$

The above is motivated through an examination of the algebraic system satisfied by the set $\boldsymbol{w} = \{w_a\}_{a=1}^{\mathtt{L}}$ (10.3a). Given a solution, the set $-\boldsymbol{w}^* \equiv \{-w_a^*\}_{a=1}^{\mathtt{L}}$ solves the same equations with the parameter s substituted by its complex conjugate, while $-\boldsymbol{w} \equiv \{-w_a\}_{a=1}^{\mathtt{L}}$ is a solution of (10.3a) with s replaced by $-s$. In turn the eigenvalues of $\mathbb{A}_\pm(\lambda)$ corresponding to $\boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w})$ obey

$$\big(D_\pm(\mu\,|\,\boldsymbol{w},\rho,\nu)\big)^* = D_\pm(-\mu^*\,|\,-\boldsymbol{w}^*,\rho,\nu^*)\,, \quad D_\pm(\mu\,|\,\boldsymbol{w},\rho,\nu) = D_\pm(-\mu\,|\,-\boldsymbol{w},\rho,-\nu) \tag{17.20}$$

where we take into account the imaginary unit entering into the λ-μ relation (11.10).

The lattice model also possesses $\mathcal{CP}$ invariance which, in turn, becomes a symmetry that acts in the space of states occurring in the scaling limit. The key relation for defining the $\mathcal{CP}$ conjugation is

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\,W_j(u) = \overline{W}_j(u)\,\hat{\mathcal{C}}\hat{\mathcal{P}}\,. \tag{17.21}$$

It may be advocated for using the similar arguments that led to (17.13). Namely, one should start with the commutation relation of $\hat{\mathcal{C}}\hat{\mathcal{P}}$ with the lattice transfer matrix,

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\,\mathbb{T}(\zeta)\,\hat{\mathcal{C}}\hat{\mathcal{P}} = \zeta^N\,\mathbb{T}(\zeta^{-1})\,, \tag{17.22}$$

which was already quoted in the Preliminaries. This, in view of eqs. (14.15) and (14.16), in the scaling limit becomes

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\,\boldsymbol{\tau}(\lambda)\,\hat{\mathcal{C}}\hat{\mathcal{P}} = \bar{\boldsymbol{\tau}}(\bar{\lambda})\,. \tag{17.23}$$

The latter, combined with the large λ asymptotic formula (15.1) and the similar one for $\bar{\boldsymbol{\tau}}(\bar{\lambda})$, results in $\hat{\mathcal{C}}\hat{\mathcal{P}}\,\mathbf{I}_m\,\hat{\mathcal{C}}\hat{\mathcal{P}} = \bar{\mathbf{I}}_m$, which is clearly consistent with (17.21).

Contrary to the other global symmetries, the $\mathcal{CP}$ conjugation does not commute with the lattice total spin operator $\mathbb{S}^z$. As a result, it acts invariantly only in the subsectors $\mathcal{H}^{(\mathrm{cont})}_{S^z,\mathtt{w}}$ and $\mathcal{H}^{(\mathrm{disc},\pm)}_{S^z,\mathtt{w}}$ with $S^z = 0$. In this case, the action of $\mathcal{CP}$ on the $\overline{W}_\infty \otimes W_\infty$ irreps is described by

$$\mathcal{CP}\,:\,\bar{\mathcal{V}}_{\bar{\rho},\bar{\nu}} \otimes \mathcal{V}_{\rho,\nu} \mapsto \bar{\mathcal{V}}_{-\rho,\nu} \otimes \mathcal{V}_{-\bar{\rho},\bar{\nu}}\,, \tag{17.24}$$

where again $(\rho,\nu) = (p,s), (p,\sigma\mathrm{i}\mathfrak{q}_a),\ldots$ and $(\bar{\rho},\bar{\nu}) = (\bar{p},s), (\bar{p},\sigma\mathrm{i}\mathfrak{q}_a),\ldots$. Recall that the space $\mathcal{V}_{\rho,\nu}$, being considered as a highest weight irrep of the W_∞ - algebra, is isomorphic to $\mathcal{V}_{-\rho,\nu}$ as the highest weight is not sensitive to a flip of the sign of ρ, see eq. (16.27). This makes (17.24) consistent with the relations (17.21).

The components $\bar{\mathcal{V}}_{\bar{\rho},\bar{\nu}} \otimes \mathcal{V}_{\rho,\nu}$ occurring in the decomposition of $\mathcal{H}^{(\mathrm{cont})}_{0,\mathtt{w}}$ (17.2) and $\mathcal{H}^{(\mathrm{disc},+)}_{0,\mathtt{w}}$ (17.3), are always such that $\rho + \bar{\rho} = 0$ and $\nu = \bar{\nu}$ so that the $\mathcal{CP}$ conjugation acts invariantly in each of them. At first glance, this property does not seem to hold true for the case of $\mathcal{H}^{(\mathrm{disc},-)}_{0,\mathtt{w}}$. The direct sum (17.8) for the subsector $\mathcal{H}^{(\mathrm{disc},-)}_{0,\mathtt{w}}$ in general contains eight terms. However, when $S^z = 0$ the sets $\Sigma_2(p)$ and $\Sigma_2(\bar{p})$ are empty and the linear spaces $\mathcal{H}^{(2,-)}_{S^z,\mathtt{w},\sigma}$ and $\mathcal{H}^{(3,-)}_{S^z,\mathtt{w},\sigma}$ (17.9) become trivial. In addition $\bar{p}_\pm + p_\pm = 0$ so that the components $\bar{\mathcal{V}}_{\bar{p}_+,\sigma\mathrm{i}\mathfrak{q}'_a} \otimes \mathcal{V}_{p_+,\sigma\mathrm{i}\mathfrak{q}'_a}$ and $\bar{\mathcal{V}}_{\bar{p}_-,\sigma\mathrm{i}\bar{\mathfrak{q}}'_a} \otimes \mathcal{V}_{p_-,\sigma\mathrm{i}\bar{\mathfrak{q}}'_a}$, appearing in the decomposition of the remaining four spaces $\mathcal{H}^{(1,-)}_{S^z,\mathtt{w},\sigma}$ and $\mathcal{H}^{(4,-)}_{S^z,\mathtt{w},\sigma}$, respectively, are preserved under the $\mathcal{CP}$ conjugation.

Similar as for the other global symmetries discussed above, the $\mathcal{CP}$ conjugation in $\mathcal{H}^{(\mathrm{cont})}_{0,\mathrm{w}}$ and $\mathcal{H}^{(\mathrm{disc},\pm)}_{0,\mathrm{w}}$ may be determined by considering its action on the low energy Bethe states of the finite lattice system. A numerical analysis suggests that for the $\overline{W}_\infty \otimes W_\infty$ primary states

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\, \boldsymbol{\psi}^{(\mathrm{vac})}_{\bar\rho,\rho,\bar\nu,\nu} = +\,\boldsymbol{\psi}^{(\mathrm{vac})}_{-\rho,-\bar\rho,\nu,\bar\nu} \qquad\qquad (\rho+\bar\rho=0)\ . \tag{17.25}$$

Together with the relation (17.21), this unambiguously defines the action of the $\mathcal{CP}$ transformation for any state from $\mathcal{H}^{(\mathrm{cont})}_{S^z,\mathrm{w}}$ and $\mathcal{H}^{(\mathrm{disc},\pm)}_{S^z,\mathrm{w}}$ with $S^z=0$.

In our study of the scaling limit we have focused on the case with $S^z \geq 0$. Since $\hat{\mathcal{C}}\hat{\mathcal{P}}\,\mathbb{S}^z = -\mathbb{S}^z\,\hat{\mathcal{C}}\hat{\mathcal{P}}$, one can make use of $\mathcal{CP}$ invariance to describe the scaling limit of the low energy states with $S^z<0$. These would organize into the subsectors $\mathcal{H}^{(\mathrm{cont})}_{S^z,\mathrm{w}}$ and $\mathcal{H}^{(\mathrm{disc},\pm)}_{S^z,\mathrm{w}}$, which are the $\mathcal{CP}$ image of the corresponding spaces having the opposite sign of S^z:

$$\mathcal{H}^{(\mathrm{cont})}_{S^z,\mathrm{w}} \equiv \hat{\mathcal{C}}\hat{\mathcal{P}}\big(\mathcal{H}^{(\mathrm{cont})}_{-S^z,\mathrm{w}}\big)\,, \qquad\qquad \mathcal{H}^{(\mathrm{disc},\pm)}_{S^z,\mathrm{w}} \equiv \hat{\mathcal{C}}\hat{\mathcal{P}}\big(\mathcal{H}^{(\mathrm{disc},\pm)}_{-S^z,\mathrm{w}}\big) \qquad\qquad (S^z<0)\ . \tag{17.26}$$

Supplementing the $\overline{W}_\infty \otimes W_\infty$ decomposition of $\mathcal{H}^{(\mathrm{cont})}_{S^z,\mathrm{w}}$ and $\mathcal{H}^{(\mathrm{disc},\pm)}_{S^z,\mathrm{w}}$ given in the previous subsection with eq. (17.21) provides a classification of the states from (17.26) w.r.t. the irreps of the conformal symmetry algebra. Note that formula (17.25) for $\rho+\bar\rho>0$ can be taken as the definition of $\boldsymbol{\psi}^{(\mathrm{vac})}_{-\rho,-\bar\rho,\nu,\bar\nu}$, which are the primary $\overline{W}_\infty \otimes W_\infty$ states in the irreps with $S^z<0$. This way the full space of states occurring in the scaling limit of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model is split into the continuous and discrete components of the form

$$\mathcal{H}^{(\mathrm{cont})} = \bigoplus_{S^z,\mathrm{w}\in\mathbb{Z}} \mathcal{H}^{(\mathrm{cont})}_{S^z,\mathrm{w}}\,, \qquad\qquad \mathcal{H}^{(\mathrm{disc},\pm)} = \bigoplus_{S^z,\mathrm{w}\in\mathbb{Z}} \mathcal{H}^{(\mathrm{disc},\pm)}_{S^z,\mathrm{w}}\ . \tag{17.27}$$

17.3. *Partition function in the scaling limit*

The linear decomposition of the spaces $\mathcal{H}^{(\mathrm{cont})}$ and $\mathcal{H}^{(\mathrm{disc},\pm)}$ described above allows one to study the scaling behaviour of the lattice partition function. For a lattice with N horizontal sites, we define the partition function associated with the Hamiltonian $\mathbb{H}$ (7.6) and the shift operator $\mathbb{K}$, given by (2.28) with $r=2$, via the formula

$$Z^{(\mathrm{lattice})}_N(M_1,M_2) = \mathrm{Tr}_{\mathscr{V}_N}\Big[\mathrm{e}^{-M_1\mathbb{H}}\ \mathbb{K}^{M_2}\Big] \tag{17.28}$$

with the trace being taken over the 2^N dimensional space $\mathscr{V}_N = \mathbb{C}^2_N\otimes\mathbb{C}^2_{N-1}\otimes\cdots\otimes\mathbb{C}^2_1$. Keeping fixed the ratios

$$\tau = \frac{2\mathrm{i}}{N}\big(v_{\mathrm{F}}\,M_1 - \mathrm{i}M_2\big)\,, \qquad \bar\tau = \frac{2\mathrm{i}}{N}\big(v_{\mathrm{F}}\,M_1 + \mathrm{i}M_2\big) \tag{17.29}$$

the large N behaviour of the lattice partition function is described as

$$Z^{(\mathrm{lattice})}_N(M_1,M_2) \asymp \mathrm{e}^{-M_1Ne_\infty}\,Z^{(\mathrm{scl})}\ . \tag{17.30}$$

Here $Z^{(\mathrm{scl})}$ is given in terms of a trace over the full space of states occurring in the scaling limit of the lattice model $\mathcal{H} = \mathcal{H}^{(\mathrm{cont})}\oplus\mathcal{H}^{(\mathrm{disc},+)}\oplus\mathcal{H}^{(\mathrm{disc},-)}$. Namely,

$$Z^{(\mathrm{scl})} = \mathrm{Tr}_{\mathcal{H}}\Big[\,\bar{\mathrm{q}}^{\overline{\widetilde{W}}_2(0)-\frac{c}{24}}\ \mathrm{q}^{\widetilde{W}_2(0)-\frac{c}{24}}\Big] \qquad \text{with} \qquad \mathrm{q}=\mathrm{e}^{2\pi\mathrm{i}\tau}\,,\ \bar{\mathrm{q}}=\mathrm{e}^{2\pi\mathrm{i}\bar\tau}\ . \tag{17.31}$$

The trace in (17.31) is naturally split into the contributions of the states from the continuous and discrete components:

$$Z^{(\mathrm{scl})} = Z^{(\mathrm{cont})} + Z^{(\mathrm{disc})} \,. \tag{17.32}$$

It is straightforward to calculate $Z^{(\mathrm{disc})}$ using the formulae (17.3)-(17.10), as well as the explicit expression (16.30) for the character of the highest weight irrep of the W_∞ - algebra. To write the result in a compact way we borrow the notation $\chi^d_{(\mathrm{j},a-\mathrm{j})}(\mathrm{q})$ from ref. [26]. Up to a simple factor, this function coincides with $\chi_a(\mathrm{q})$ defined in eq. (10.25) (see also footnote 3):

$$\chi^d_{(\mathrm{j},a-\mathrm{j})}(\mathrm{q}) \equiv \mathrm{q}^{-\frac{1}{12}-\frac{(\mathrm{j}+\frac{1}{2})^2}{n}+\frac{(\mathrm{j}-a)^2}{n+2}}\, \chi_a(\mathrm{q}) \qquad (a\in\mathbb{Z})\,. \tag{17.33}$$

It is related to the character of the irrep as

$$\chi^d_{(\mathrm{j},a-\mathrm{j})}(\mathrm{q}) = \mathrm{ch}_{a-\mathrm{j},\mathrm{i}(\mathrm{j}+\frac{1}{2})}(\mathrm{q}) \times \begin{cases} 1 & \text{for} \quad a\geq 0 \\ \mathrm{q}^{-a} & \text{for} \quad a<0 \end{cases} . \tag{17.34}$$

Also introduce the notation $\mathfrak{J}(\mathrm{v},\mathrm{u})$ for the finite set of all real numbers belonging to the half-open segment $[-\frac{n+1}{2}, -\frac{1}{2})$ such that

$$\mathfrak{J}(\mathrm{v},\mathrm{u}) \equiv \Big\{ \mathrm{j}:\ \mathrm{j}\in\big[-\tfrac{n+1}{2},-\tfrac{1}{2}\big)\ \&\ \mathrm{j}-\tfrac{1}{2}\,\mathrm{v}-\tfrac{1}{2}\,(n+2)(\mathrm{k}+\mathrm{u})\in\mathbb{Z}\Big\}\,. \tag{17.35}$$

Then the calculation of the trace over the space $\mathcal{H}^{(\mathrm{disc})} = \mathcal{H}^{(\mathrm{disc},+)} \oplus \mathcal{H}^{(\mathrm{disc},-)}$ yields

$$Z^{(\mathrm{disc})} = 2 \sum_{\mathrm{v},\mathrm{u}\in\mathbb{Z}} \ \sum_{\mathrm{j}\in\mathfrak{J}(\mathrm{v},\mathrm{u})} \chi^d_{(\mathrm{j},\bar{\mathfrak{p}})}(\bar{\mathrm{q}})\, \chi^d_{(\mathrm{j},-\mathfrak{p})}(\mathrm{q})\,, \tag{17.36}$$

where[10]

$$\bar{\mathfrak{p}} = \tfrac{1}{2}\,\mathrm{v} - \tfrac{1}{2}\,(n+2)\,(\mathrm{k}+\mathrm{u})\,, \qquad\qquad \mathfrak{p} = \tfrac{1}{2}\,\mathrm{v} + \tfrac{1}{2}\,(n+2)\,(\mathrm{k}+\mathrm{u})\,. \tag{17.37}$$

The overall factor of 2 in the formula for $Z^{(\mathrm{disc})}$ occurs due to the global $\mathcal{Z}_2$ invariance of the model.

The following comment is in order here. For arbitrary values of k, the inclusion of the end-points into the interval for j in (17.35) has no effect on the set $\mathfrak{J}(\mathrm{v},\mathrm{u})$. However for $\mathrm{k}=0$ and with n generic, which is of special interest, j may coincide with $-\frac{n+1}{2}$ or $-\frac{1}{2}$. Taking the limit $\mathrm{k}\to 0$ of $Z^{(\mathrm{disc})}$ one finds that in order for (17.36) to correctly describe the contribution of the discrete spectrum to the partition function $Z^{(\mathrm{scl})}$ for the model with periodic boundary conditions, one of the endpoints in (17.35) must be included. The choice of whether to include $\mathrm{j}=-\frac{n+1}{2}$ or $\mathrm{j}=-\frac{1}{2}$ does not matter, since they correspond to the contribution of the same states to $Z^{(\mathrm{disc})}$.

The contribution of the continuous spectrum to the partition function $Z^{(\mathrm{scl})}$ is simply obtained by combining the decomposition of $\mathcal{H}^{(\mathrm{cont})}$ into the direct integral (17.2) with the density of states (10.12). For future reference, we write it in the form

[10] In the formula (17.36) for $Z^{(\mathrm{disc})}$, the integers v and u are formal summation variables, which can not be identified with the eigenvalue of $\mathbb{S}^z$ and the winding number w. In turn the notation $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ in (17.37) should not be confused with p and $\bar{p}$ from (8.1).

$$Z^{(\mathrm{cont})}=\sqrt{\frac{n}{\Im m(\tau)}}\ \frac{\log\big(2^{\frac{2}{n}}\,N/N_0\big)}{\pi\,(\bar{\mathtt{q}},\bar{\mathtt{q}})^2_\infty(\mathtt{q},\mathtt{q})^2_\infty}\sum_{S^z,\mathtt{w}=-\infty}^{\infty}\bar{\mathtt{q}}^{-\frac{1}{12}+\frac{\bar{p}^2}{n+2}}\,\mathtt{q}^{-\frac{1}{12}+\frac{p^2}{n+2}}$$
$$+\sum_{S^z,\mathtt{w}=-\infty}^{\infty}\int_{-\infty}^{+\infty}\mathrm{d}s\sum_{\mathtt{L},\bar{\mathtt{L}}\geq 0}\tilde{\rho}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p}(s)\ \bar{\mathtt{q}}^{-\frac{1}{12}+\frac{s^2}{n}+\frac{\bar{p}^2}{n+2}+\bar{\mathtt{L}}}\ \mathtt{q}^{-\frac{1}{12}+\frac{s^2}{n}+\frac{p^2}{n+2}+\mathtt{L}}\,. \tag{17.38}$$

Here we take into account that $\bar{\tau}=-\tau^*$ so that

$$\bar{\mathtt{q}}\mathtt{q}=\mathrm{e}^{-4\pi\Im m(\tau)}\,. \tag{17.39}$$

The summand in the second line of (17.38) is naturally interpreted as the regularized matrix elements of a certain density matrix and the expansion coefficients $\tilde{\rho}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p}$ read explicitly as

$$\tilde{\rho}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p}(s)=\frac{1}{2\pi\mathrm{i}}\,\partial_s\log\Big[\big(\mathfrak{D}^{(\bar{\mathtt{L}})}_{\bar{p}}(s)\big)^{\mathrm{par}_2(\mathtt{L})}\ \big(\mathfrak{D}^{(\mathtt{L})}_{p}(s)\big)^{\mathrm{par}_2(\bar{\mathtt{L}})}\Big] \tag{17.40}$$

with

$$\mathfrak{D}^{(\mathtt{L})}_p(s)=\bigg(\frac{\Gamma(\frac{1}{2}+p-\mathrm{i}s)}{\Gamma(\frac{1}{2}+p+\mathrm{i}s)}\bigg)^{\mathtt{par}_2(\mathtt{L})}$$
$$\times\prod_{a=0}^{\mathtt{L}-1}\left[\frac{\big(\frac{1}{2}+a+p-\mathrm{i}s\big)\big(\frac{1}{2}+a-p-\mathrm{i}s\big)}{\big(\frac{1}{2}+a+p+\mathrm{i}s\big)\big(\frac{1}{2}+a-p+\mathrm{i}s\big)}\right]^{\mathrm{par}_2(\mathtt{L})-d_a(\mathtt{L})}\,. \tag{17.41}$$

The integers $d_a(\mathtt{L})$, appearing in the exponent, are defined in (10.25). Due to the property $d_a(\mathtt{L})=\mathrm{par}_2(\mathtt{L})$ for $a\geq\mathtt{L}$, the upper limit in the product in (17.41) may be set to infinity. This allows one to perform the sum over $\mathtt{L}$ and $\bar{\mathtt{L}}$ in the second line of (17.38) and bring it to the form, which is convenient for numerical calculations:

$$\sum_{\mathtt{L},\bar{\mathtt{L}}\geq 0}\tilde{\rho}^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p}(s)\ \bar{\mathtt{q}}^{\bar{\mathtt{L}}}\mathtt{q}^{\mathtt{L}}=-\frac{r_{\bar{p}}(s,\bar{\mathtt{q}})+r_p(s,\mathtt{q})}{\pi\,(\bar{\mathtt{q}},\bar{\mathtt{q}})^2_\infty(\mathtt{q},\mathtt{q})^2_\infty} \tag{17.42}$$

with

$$r_p(s,\mathtt{q})=\frac{1}{2}\sum_{\sigma=\pm}\psi\big(\tfrac{1}{2}+p+\mathrm{i}\sigma s\big)$$
$$+\oint_{|z|<1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\ \frac{(\mathtt{q},\mathtt{q})^2_\infty}{(z,\mathtt{q})_\infty(z^{-1}\mathtt{q},\mathtt{q})_\infty}\ \frac{1}{2}\sum_{\sigma,\sigma'=\pm}\Phi(z,1,\tfrac{1}{2}+\sigma' p+\mathrm{i}\sigma s)\,. \tag{17.43}$$

Here $\psi(\alpha)=\partial_\alpha\log\Gamma(\alpha)$, while $\Phi(z,1,\alpha)$ stands for the Lerch transcendent,

$$\Phi(z,s,\alpha)=\sum_{m=0}^{\infty}\frac{z^m}{(m+\alpha)^s}\,. \tag{17.44}$$

Similar to the contribution of the discrete spectrum to the partition function, the formula for $Z^{(\mathrm{cont})}$ requires special attention for the case of periodic boundary conditions. This is because at $\mathtt{k}=0$, p ($\bar{p}$) can take half integer values for which the function $r_p(s,\mathtt{q})$ ($r_{\bar{p}}(s,\mathtt{q})$) (17.43) contains simple poles at $s=0$. These poles $\propto\frac{\sigma'}{s}$ formally cancel out after summation over the sign factor σ'. However, the naive cancellation does not take into account the possibility of contact terms proportional to the Dirac delta function $\delta(s)$ which would give a finite contribution to the integral in (17.38). To resolve the ambiguity one should start with $Z^{(\text{cont})}$ for non-vanishing $\mathtt{k}$ and then perform the limit $\mathtt{k} \to 0$ using the Sokhotski-Plemelj formula.

17.4. The case of $\mathtt{k} = 0$ with generic $n > 0$

In view of applications to local quantum field theory, of special interest is when the spectrum of the Lorentz spin in $\mathcal{H}^{(\text{cont})}$ and $\mathcal{H}^{(\text{disc},\pm)}$ consists of (half-)integers. Since the Lorentz spin of the states, characterized by the quantum numbers S^z, $\mathtt{w}$, $\mathtt{L}$ and $\bar{\mathtt{L}}$, reads as

$$\Delta - \bar{\Delta} = S^z\,(\mathtt{k} + \mathtt{w}) + \mathtt{L} - \bar{\mathtt{L}} \tag{17.45}$$

this motivates a detailed study of a few special cases. Among them is the $\mathcal{CP}$ invariant sector of the model, where $S^z = 0$. If in addition one sets $\mathtt{k} = \pm\frac{1}{n+2}$, then the space of states contains a $\mathcal{CP}$ and $\mathcal{Z}_2$ invariant $\overline{W}_\infty \otimes W_\infty$ primary state with conformal dimensions $\Delta = \bar{\Delta} = 0$. Thus the sector $S^z = 0$ with $\mathtt{k} = \pm\frac{1}{n+2}$ (and, perhaps, with n a positive integer) is interesting to study in the context of the RSOS reductions of the inhomogeneous six-vertex model.[11] However, this will not be considered here. Instead we'll focus on another situation when the Lorentz spin (17.45) takes integer values, namely, when $\mathtt{k} = 0$.

The case $\mathtt{k} = 0$, i.e., periodic boundary conditions for σ^a_m entering into the Hamiltonian $\mathbb{H}$ (7.6), (7.7), has a special feature. As discussed in ref. [29] for arbitrary $\mathtt{k}$ the matrix

$$\hat{\mathcal{C}} = c_N \prod_{J=1}^{N} (\eta_J)^{\frac{1}{2}\sigma^z_J}\,\sigma^x_J \qquad\qquad \big(\eta_J = (-1)^{J+1}\,\mathrm{i}\big)\,, \tag{17.46}$$

where $c_N^2 = 1$, satisfies the following commutation relations with $\mathbb{A}_\pm(\zeta)$ and the transfer matrix:

$$\hat{\mathcal{C}}\,\mathbb{A}_\pm(\zeta\,|\,\mathtt{k})\,\hat{\mathcal{C}} = \mathbb{A}_\mp(\zeta\,|-\mathtt{k})\,, \qquad \hat{\mathcal{C}}\,\mathbb{T}(\zeta\,|\,\mathtt{k})\,\hat{\mathcal{C}} = \mathbb{T}(\zeta\,|-\mathtt{k})\,. \tag{17.47}$$

In turn the Hamiltonian $\mathbb{H}$ does not commute with $\hat{\mathcal{C}}$ when the twist is non-trivial. However, for $\mathtt{k} = 0$ the system possesses an additional global symmetry – $\mathcal{C}$ invariance. The space of states $\mathscr{V}_N$ (2.1) can be split into two components distinguished by their $\mathcal{C}$ parity. Numerical work shows that for $\mathtt{k} = 0$ and *generic* values of the anisotropy parameter $n > 0$ the transfer matrix resolves all the degeneracies in the energy spectrum in each component. This implies that one can introduce a basis in the finite dimensional space $\mathscr{V}_N$, which diagonalizes $\mathbb{T}(\zeta)$ and $\hat{\mathcal{C}}$ simultaneously. Though the latter commutes with the transfer matrix for $\mathtt{k} = 0$, it anti-commutes with the total spin operator $\mathbb{S}^z = \frac{1}{2}\sum_J \sigma^z_J$. Hence each basis state would no longer have a definite value of S^z, except for the states with $S^z = 0$. Note that the matrices $\mathbb{A}_\pm(\zeta)$ restricted to this sector coincide, so that

$$A_+(\zeta) = A_-(\zeta) \qquad\qquad (\mathtt{k} = S^z = 0)\,. \tag{17.48}$$

This follows from the first equation in (17.47), specialized to $\mathtt{k} = 0$, and the fact that the transfer matrix, which commutes with $\mathbb{A}_\pm(\zeta)$, by itself lifts all the degeneracies in the $S^z = 0$ sector. Also it turns out that in this sector the generator $\hat{\mathcal{C}}$, up to a sign factor, coincides with $\mathbb{A}_+^{(\infty)}$ (2.14). Namely, one can show that

[11] In fact, RSOS reductions of the inhomogeneous six-vertex model for various boundary conditions have been already considered in refs. [57,58].

$$\hat{\mathcal{C}}\,\boldsymbol{\Psi} = \mathcal{C}_{\boldsymbol{\Psi}}\,\boldsymbol{\Psi}\,, \qquad \mathcal{C}_{\boldsymbol{\Psi}} = c_N \prod_{m=1}^{N/2} \zeta_m^{-1} \qquad \big(\mathtt{k} = S^z = 0\big)\,. \tag{17.49}$$

Since $\hat{\mathcal{C}}$ anticommutes with $\mathbb{S}^z$, the $\mathcal{C}$ even and odd components of $\mathscr{V}_N$ do not possess the U(1) symmetry. Nevertheless, these sectors are still invariant w.r.t. the subgroup of U(1), whose generator corresponds to a 180° rotation and may be chosen to be

$$\hat{\mathcal{U}} = (-1)^{N/2}\,\mathrm{e}^{\mathrm{i}\pi\mathbb{S}^z}\,, \qquad \hat{\mathcal{U}}^2 = 1\,. \tag{17.50}$$

The extra factor $(-1)^{N/2}$ has been included so that the eigenvalues of $\hat{\mathcal{U}}$ coincide with the sign factor σ (9.12) entering into the asymptotic relation (9.11). Recall that $\hat{\mathcal{C}}\hat{\mathcal{P}}$ commutes with the generator $\hat{\mathcal{D}}$ of the $\mathcal{Z}_2$ symmetry for arbitrary values of the twist parameter $\mathtt{k}$. However the matrix $\hat{\mathcal{C}}$ satisfies the commutation relations

$$\hat{\mathcal{C}}\,\hat{\mathcal{D}} = \hat{\mathcal{U}}\,\hat{\mathcal{D}}\,\hat{\mathcal{C}}\,, \qquad \big[\,\hat{\mathcal{C}}\,,\hat{\mathcal{U}}\,\big] = \big[\,\hat{\mathcal{D}}\,,\hat{\mathcal{U}}\,\big] = 0\,. \tag{17.51}$$

The following comment is in order here. The definition of the $\mathcal{C}$ conjugation (17.46) contains the sign factor $c_N = \pm 1$, which may depend on the number of sites. We found it convenient to set

$$c_N = \begin{cases} (-1)^{N/4} & N/2 - \text{even} \\ 1 & N/2 - \text{odd} \end{cases}\,. \tag{17.52}$$

For $N/2$ even the ground state (the state with the lowest possible energy) of the lattice Hamiltonian with periodic boundary conditions is non-degenerate. With the choice of the sign factor as in (17.52), its $\mathcal{C}$ parity is equal to $+1$. When $N/2$ is odd the ground state is a $\mathcal{Z}_2$ doublet and the $\mathcal{C}$ parity of the two states is $+1$ and -1.

In taking the scaling limit, one can apply the same arguments that lead to eqs. (17.13) and (17.21) for the previously discussed global symmetries. This way one finds

$$\hat{\mathcal{C}}\; W_j(u)\,\hat{\mathcal{C}} = W_j(u)\,, \qquad \hat{\mathcal{C}}\;\overline{W}_j(\bar{u})\,\hat{\mathcal{C}} = \overline{W}_j(\bar{u}) \qquad (\mathtt{k}=0)\,. \tag{17.53}$$

Hence $\hat{\mathcal{C}}$ maps a $\overline{W}_\infty \otimes W_\infty$ highest weight irrep to an equivalent representation, i.e., one that is characterized by the same highest weight. For the components $\mathcal{H}^{(\text{cont})}_{S^z,\mathrm{w}}$ and $\mathcal{H}^{(\text{disc},+)}_{S^z,\mathrm{w}}$ occurring in the linear decomposition (17.27) the $\mathcal{C}$ conjugation acts as

$$\hat{\mathcal{C}}\ :\ \begin{array}{l} \mathcal{H}^{(\text{cont})}_{+S^z,+\mathrm{w}} \mapsto \mathcal{H}^{(\text{cont})}_{-S^z,-\mathrm{w}} \\[2mm] \mathcal{H}^{(\text{disc},+)}_{+S^z,+\mathrm{w}} \mapsto \mathcal{H}^{(\text{disc},+)}_{-S^z,-\mathrm{w}} \end{array} \qquad \text{for} \qquad S^z > 0\,. \tag{17.54}$$

The case of $\mathcal{H}^{(\text{disc},-)}_{S^z,\mathrm{w}}$ is more involved. It turns out that the action of $\hat{\mathcal{C}}$ is described by the relations

$$\hat{\mathcal{C}}\ :\ \mathcal{H}^{(i,-)}_{+S^z,+\mathrm{w},\sigma} \mapsto \mathcal{H}^{(5-i,-)}_{-S^z,-\mathrm{w},\sigma} \qquad \text{for} \qquad S^z > 0\,. \tag{17.55}$$

Here $\mathcal{H}^{(i,-)}_{S^z,\mathrm{w},\sigma}$ with $i = 1,\ldots,4$ are given by (17.9) for $S^z > 0$, while

$$\mathcal{H}^{(i,-)}_{S^z,\mathrm{w},\sigma} \equiv \mathcal{C}\mathcal{P}\big(\mathcal{H}^{(i,-)}_{-S^z,\mathrm{w},\sigma}\big) \qquad \text{for} \qquad S^z < 0\,.$$

Special attention is required for the $\mathcal{C}\mathcal{P}$ invariant sector where $S^z = 0$. First we note that the scaling limit of the eigenvalues of $\mathbb{A}_+(\zeta)$ described by (11.1) involves the connection coefficients $D_+(\mu\,|\,\boldsymbol{w}, +\frac{1}{2}(n+2)\,\mathrm{w}, s)$ and $D_+(\bar{\mu}\,|\,\bar{\boldsymbol{w}}, -\frac{1}{2}(n+2)\,\mathrm{w}, s)$, which do not depend on the sign of w.

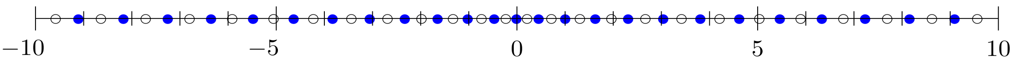

Fig. 13. For fixed $N=100$ and $S^z=0$ the value of real $b(N)$ is plotted on the number line for the class of Bethe states described in sec. 9. These states are such that all the corresponding Bethe roots are real, and are distinguished by the difference between the number of positive roots M_+ and the number of negative roots M_-, which for the 51 states used to produce the figure varies from $M_+-M_-=-50,-48,\ldots,-2,0,2,\ldots,50$. The parameter $\mathtt{k}$ was set to zero so that, since $S^z=0$, the states have a definite $\mathcal{C}$ parity. The latter, computed from the Bethe roots using formula (17.49), is indicated by the solid fill for the $\mathcal{C}$ even states and no fill for the $\mathcal{C}$ odd ones. The number of $\mathcal{C}$ even (odd) states with $b(N)\in(s,s+\Delta s)\subset(-b_{\max},+b_{\max})$ is approximately $\frac{1}{2}\,\rho^{(0,0)}_{0,0}(s)\,\Delta s$ with $\rho^{(0,0)}_{\bar{p},p}$ being the density of primary Bethe states given in eq. (9.8). The anisotropy parameter was taken to be $n=2.93$.

This can be seen from (17.48) and that $D_+(\mu\,|\,\boldsymbol{w},+\frac{1}{2}(n+2)\,\mathtt{w},s)=D_-(\mu\,|\,\boldsymbol{w},-\frac{1}{2}(n+2)\,\mathtt{w},s)$. Thus in our prescription, the sign of the winding number $\mathtt{w}\neq 0$ for an RG trajectory $\boldsymbol{\Psi}_N$ remains undetermined when $\mathtt{k}=S^z=0$. Nevertheless we found that the pair of low energy states which become indistinguishable in the scaling limit have different energies for finite N. This allows one to set, by definition, that the state with $-|\mathtt{w}|$ and $+|\mathtt{w}|$ has the lower and higher value of $|\mathcal{E}|$, respectively. Another way to resolve the ambiguity in the sign of the winding number is to start with the Bethe state with $\mathtt{k}\neq 0$ and consider the limit $\mathtt{k}\to 0$. It follows from the formula for the energy (8.1), (8.2a) that for small positive $\mathtt{k}$ and $S^z=0$ the state with $\mathtt{w}>0$ will be of higher energy than the corresponding state having the opposite sign of $\mathtt{w}$. In the limit $\mathtt{k}\to 0^+$ the Bethe states with $+|\mathtt{w}|$ and $-|\mathtt{w}|$ would become the states with higher and lower energy, respectively, and the two ways of specifying the sign of the winding number turn out to be equivalent. Having resolved the issue with the sign, each of the spaces $\mathcal{H}^{(\mathrm{cont})}_{0,\mathtt{w}}$, $\mathcal{H}^{(\mathrm{disc},+)}_{0,\mathtt{w}}$ and $\mathcal{H}^{(\mathrm{disc},-)}_{0,\mathtt{w}}$ occurring in the scaling limit become invariant w.r.t. the $\mathcal{C}$ conjugation.

Recall that the space $\mathcal{H}^{(\mathrm{cont})}_{0,\mathtt{w}}$ is formed by the scaling limit of the low energy Bethe states $\boldsymbol{\Psi}_N$ with $S^z=0$, such that $\lim_{N\to\infty}\Im m\big(b(N)\big)=0$. It turns out that for finite N the difference between the number of $\mathcal{C}$ even and $\mathcal{C}$ odd states that become part of $\mathcal{H}^{(\mathrm{cont})}_{0,\mathtt{w}}$ is an order one number as $N\to\infty$, see Fig. 13. For the low energy Bethe states with given $\mathcal{C}$ parity and fixed $(\bar{\mathtt{L}},\mathtt{L})$, the corresponding values of $\Re e\big(b(N)\big)$ become densely distributed within the segment $(-\Lambda_N,\Lambda_N)$ with $\lim_{N\to\infty}\Lambda_N=\infty$, and the density of states for $\Re e\big(b(N)\big)\in(s,s+\Delta s)$ turns out to be half the total density, $\rho^{(\bar{\mathtt{L}},\mathtt{L})}_{\bar{p},p}(s)$, from eq. (10.12).

In the case of $\mathcal{H}^{(\mathrm{disc},+)}_{0,\mathtt{w}}=\bigoplus_{\sigma=\pm1}\Big(\mathcal{H}^{(1,+)}_{0,\mathtt{w},\sigma}\oplus\mathcal{H}^{(2,+)}_{0,\mathtt{w},\sigma}\Big)$ our numerical work shows that the value of the $\mathcal{C}$ parity is the same for all the states in each component $\mathcal{H}^{(i,+)}_{0,\mathtt{w},\sigma}$ with $\mathtt{w}\neq 0$. However, it depends on whether the scaling limit with $N\to\infty$ is taken such that $N/2$ is kept fixed to be even or odd:

$$\hat{\mathcal{C}}\big(\mathcal{H}^{(i,+)}_{0,\mathtt{w},\sigma}\big)=-\,\sigma^{N/2}\,\mathrm{sgn}(\mathtt{w})\,\mathcal{H}^{(i,+)}_{0,\mathtt{w},\sigma}\qquad (i=1,2)\,. \tag{17.56}$$

Note that for $\mathtt{w}=0$ the space $\mathcal{H}^{(\mathrm{disc},+)}_{0,0}=0$ as follows from eqs. (17.4) and (17.6). For $\mathcal{H}^{(\mathrm{disc},-)}_{0,\mathtt{w}}$ the subspaces $\mathcal{H}^{(i,-)}_{0,\mathtt{w},\sigma}$ with $i=2,3$ are trivial since the summation index a runs over the empty sets $\Sigma_2(p)$ and $\Sigma_2(\bar{p})$ in (17.9). For similar reasons $\mathcal{H}^{(4,-)}_{0,\mathtt{w},\sigma}$ and $\mathcal{H}^{(1,-)}_{0,\mathtt{w},\sigma}$ are also trivial for $\mathtt{w}>0$ and $\mathtt{w}<0$, respectively. It is expected that

$$\begin{aligned}
&\hat{\mathcal{C}}\big(\mathcal{H}^{(1,-)}_{0,\mathtt{w},\sigma}\big)=+\,c^{(1)}_{\mathtt{w}}\,\sigma^{N/2}\,\mathcal{H}^{(1,-)}_{0,\mathtt{w},\sigma}\,, &&\mathcal{H}^{(4,-)}_{0,\mathtt{w},\sigma}=0 &&(\mathtt{w}>0)\\
&\hat{\mathcal{C}}\big(\mathcal{H}^{(4,-)}_{0,\mathtt{w},\sigma}\big)=-\,c^{(4)}_{\mathtt{w}}\,\sigma^{N/2}\,\mathcal{H}^{(4,-)}_{0,\mathtt{w},\sigma}\,, &&\mathcal{H}^{(1,-)}_{0,\mathtt{w},\sigma}=0 &&(\mathtt{w}<0)\,,
\end{aligned} \tag{17.57}$$

where $c_{\mathtt{w}}^{(1)}$ and $c_{\mathtt{w}}^{(4)}$ are some signs that could depend on $\mathtt{w}$. However the value of these sign factors is still unknown to us as their determination involves the analysis of the Bethe states, which are of rather high energy compared to the ground state. The subspace $\mathcal{H}_{0,0}^{(\text{disc},-)}$ contains the two non-trivial components $\mathcal{H}_{0,0,\sigma}^{(1,-)}$ and $\mathcal{H}_{0,0,\sigma}^{(4,-)}$, which are classified identically w.r.t. the algebra of extended conformal symmetry. In turn, there is an ambiguity in assigning a low energy Bethe state to either one of these components. However this can be resolved by making use of $\mathcal{C}$ parity for finite N and then taking the large N limit, so that by definition

$$\hat{\mathcal{C}}\big(\mathcal{H}_{0,0,\sigma}^{(1,-)}\big)=+\,\sigma^{N/2}\,\mathcal{H}_{0,0,\sigma}^{(1,-)}\,,\qquad\qquad \hat{\mathcal{C}}\big(\mathcal{H}_{0,0,\sigma}^{(4,-)}\big)=-\,\sigma^{N/2}\,\mathcal{H}_{0,0,\sigma}^{(4,-)}\,. \tag{17.58}$$

This way each of the spaces $\mathcal{H}^{(\text{cont})}$ and $\mathcal{H}^{(\text{disc},\pm)}$ is splitted into two sectors characterized by the value of the $\mathcal{C}$ parity. The decompositions of the even and odd components into the highest weight irreps of the $\overline{W}_\infty\otimes W_\infty$ - algebra are identical. We'll restrict our further discussion and only focus on the $\mathcal{C}$ even (or equivalently odd) sector of $\mathcal{H}^{(\text{cont})}$.

Let's turn to formula (17.2), which describes the decomposition of $\mathcal{H}_{S^z,\mathtt{w}}^{(\text{cont})}$ into the highest weight irreps for $(n+2)\,\mathtt{k}\notin\mathbb{Z}$. Each of the chiral components in the integrand therein coincides with the Verma module, which is an irreducible representation of the chiral W_∞ - algebra. However, as was discussed in sec. 16.2, for $\mathtt{k}=0$ some of these Verma modules become reducible. The degenerate Verma module, $\mathcal{V}er_{\rho,s}$, splits into the two irreps

$$\mathcal{V}er_{\rho,s}=\mathcal{W}_{\rho,s}\oplus\mathcal{W}_{\rho+m(n+2),s}\ ,\quad \text{where}\quad \rho=\tfrac{1}{2}\big(r-m\,(n+2)\big)\,,\quad m,r=1,2,\ldots \tag{17.59}$$

and s is an arbitrary real number. With this in mind, it is straightforward to obtain from eq. (17.2) the decomposition into the highest weight irreps of the $\mathcal{C}$ even sector of $\mathcal{H}^{(\text{cont})}$ for $\mathtt{k}=0$:

$$\mathcal{H}_{\text{even}}^{(\text{cont})}=\tilde{\mathcal{H}}_{\text{even}}^{(\text{cont})}\oplus\mathcal{H}^{(\text{null})}\ . \tag{17.60}$$

Here

$$\tilde{\mathcal{H}}_{\text{even}}^{(\text{cont})}=\bigoplus_{\mathtt{v}=0}^{\infty}\Bigg[\bigoplus_{\mathtt{w}=-\infty}^{\infty}\tilde{\mathcal{H}}_{\mathtt{v},\mathtt{w}}^{(\text{cont})}\Bigg] \tag{17.61}$$

with

$$\tilde{\mathcal{H}}_{\mathtt{v},\mathtt{w}}^{(\text{cont})}=\int_{\mathbb{R}}^{\oplus}\mathrm{d}s\;\overline{\mathcal{W}}_{\bar\rho,s}\otimes\mathcal{W}_{\rho,s}\qquad\text{and}\qquad \begin{matrix}\rho=\frac{1}{2}\,\mathtt{v}+\frac{1}{2}\,(n+2)\,\mathtt{w}\\ \bar\rho=\frac{1}{2}\,\mathtt{v}-\frac{1}{2}\,(n+2)\,\mathtt{w}\end{matrix}\ , \tag{17.62}$$

while the space $\mathcal{H}^{(\text{null})}$ is a direct sum of two components,

$$\mathcal{H}^{(\text{null})}=\mathcal{H}_{+}^{(\text{null})}\oplus\mathcal{H}_{-}^{(\text{null})}\,, \tag{17.63}$$

that are decomposed identically into the irreps of the algebra of extended conformal symmetry

$$\mathcal{H}_{\pm}^{(\text{null})}=\bigoplus_{\mathtt{v},\mathtt{w}=1}^{+\infty}\int_{\mathbb{R}}^{\oplus}\mathrm{d}s\;\overline{\mathcal{W}}_{\rho,s}\otimes\mathcal{W}_{\rho,s}\qquad\big(\rho=\tfrac{1}{2}\,\mathtt{v}+\tfrac{1}{2}\,(n+2)\,\mathtt{w}\big)\,. \tag{17.64}$$

The superscript "null" emphasizes that the highest state in either one of the chiral irreps occurring in the decomposition of $\mathcal{H}_{\pm}^{(\text{null})}$ coincides with the null vector in the original Verma module (see (17.59)).

Similar to $\mathcal{H}_{\pm}^{(\mathrm{null})}$ the subspaces $\tilde{\mathcal{H}}_{0,+\mathtt{w}}^{(\mathrm{cont})}$ and $\tilde{\mathcal{H}}_{0,-\mathtt{w}}^{(\mathrm{cont})}$ also possess identical decompositions w.r.t. the $\overline{W}_\infty \otimes W_\infty$ algebra. This way $\mathcal{H}_{\mathrm{even}}^{(\mathrm{cont})}$ contains degeneracies, which are not present in the $\mathcal{C}$ even sector of $\mathscr{V}_N$ for any finite N. As a result, at least at the formal level, one can introduce two extra $\mathcal{Z}_2$ symmetry transformations in $\mathcal{H}_{\mathrm{even}}^{(\mathrm{cont})}$, which commute with the algebra of extended conformal symmetry. The first one, $\hat{\mathcal{X}}^{(\mathtt{w})}$, acts as the identity operator on all the subspaces appearing in the linear decompositions (17.60) and (17.61) except for $\tilde{\mathcal{H}}_{0,\mathtt{w}}^{(\mathrm{cont})}$ with $\mathtt{w} \neq 0$. In the latter case, it intertwines the subspaces with opposite signs of $\mathtt{w}$:

$$
\begin{aligned}
&\hat{\mathcal{X}}^{(\mathtt{w})}\big(\tilde{\mathcal{H}}_{\mathtt{v},\mathtt{w}}^{(\mathrm{cont})}\big) = \tilde{\mathcal{H}}_{\mathtt{v},\mathtt{w}}^{(\mathrm{cont})} \qquad \text{for} \qquad \mathtt{v}=\mathtt{w}=0 \ \ \& \ \ \mathtt{v}\geq 1,\ \mathtt{w}\in\mathbb{Z}\\
&\hat{\mathcal{X}}^{(\mathtt{w})}\big(\mathcal{H}^{(\mathrm{null})}\big) = \mathcal{H}^{(\mathrm{null})} \\
&\hat{\mathcal{X}}^{(\mathtt{w})}:\ \ \tilde{\mathcal{H}}_{0,\mathtt{w}}^{(\mathrm{cont})} \mapsto \tilde{\mathcal{H}}_{0,-\mathtt{w}}^{(\mathrm{cont})} \qquad\qquad (\mathtt{w}\neq 0)\,.
\end{aligned}
\tag{17.65}
$$

The second $\mathcal{Z}_2$ transformation, $\hat{\mathcal{X}}^{(\mathrm{null})}$, acts between the "$\pm$" components of the space $\mathcal{H}^{(\mathrm{null})}$ (17.63),

$$
\hat{\mathcal{X}}^{(\mathrm{null})}\big(\tilde{\mathcal{H}}_{\mathrm{even}}^{(\mathrm{cont})}\big) = \tilde{\mathcal{H}}_{\mathrm{even}}^{(\mathrm{cont})}\,, \qquad\qquad \hat{\mathcal{X}}^{(\mathrm{null})}:\ \ \mathcal{H}_{\pm}^{(\mathrm{null})} \mapsto \mathcal{H}_{\mp}^{(\mathrm{null})}\,. \tag{17.66}
$$

For any values of the twist parameter $\mathtt{k}$ the lattice system possesses $\mathcal{CP}$ symmetry. Thus when $\mathtt{k}=0$ not only $\mathcal{C}$, but also the $\mathcal{P}$ conjugation becomes a global symmetry of the model. The generator $\hat{\mathcal{P}} \in \mathrm{End}(\mathscr{V}_N)$ can be chosen to be

$$
(\hat{\mathcal{P}})_{a_N a_{N-1}\ldots a_1}^{b_N b_{N-1}\ldots b_1} = c_N\, \delta_{a_N}^{b_1}\, \delta_{a_{N-1}}^{b_2} \ldots \delta_{a_1}^{b_N} \prod_{J=1}^{N} \eta_J^{a_J/2} \qquad\qquad \big(\eta_J = \mathrm{i}\,(-1)^{J-1}\big)\,, \tag{17.67}
$$

where $a_J, b_J = \pm 1$ and c_N is the same sign factor as in (17.52). Though $\hat{\mathcal{P}}$ commutes with the lattice Hamiltonian subject to periodic boundary conditions, in view of the relations (2.16) and (17.47), it does not commute with the transfer matrix. Instead,

$$
\hat{\mathcal{P}}\, \mathbb{T}(\zeta)\, \hat{\mathcal{P}} = \zeta^N\, \mathbb{T}\big(\zeta^{-1}\big) \qquad\qquad (\mathtt{k}=0)\,. \tag{17.68}
$$

Since $[\hat{\mathcal{C}},\, \hat{\mathcal{P}}] = 0$ the $\mathcal{C}$ even and odd components of the finite dimensional space $\mathscr{V}_N$ are $\mathcal{P}$ invariant. However it turns out that there are some subtleties in taking the scaling limit of the operator (17.67). Assuming that the limit exists, eqs. (17.21) and (17.53) would imply that

$$
\hat{\mathcal{P}}\; W_j(u) = \overline{W}_j(u)\, \hat{\mathcal{P}} \qquad\qquad (\mathtt{k}=0)\,. \tag{17.69}
$$

To determine the action of the parity conjugation in $\mathcal{H}_{\mathrm{even}}^{(\mathrm{cont})}$, all that remains is to find how it acts on the $\overline{W}_\infty \otimes W_\infty$ primary states in the decompositions (17.62), (17.64). Without loss of generality, one can always set

$$
\hat{\mathcal{P}}\, \boldsymbol{\psi}_{\bar{\rho},\rho,\nu}^{(\mathrm{vac})} = +\boldsymbol{\psi}_{\rho,\bar{\rho},\nu}^{(\mathrm{vac})} \qquad\qquad \text{for} \qquad\qquad \rho \neq \pm\bar{\rho}\,. \tag{17.70}
$$

Otherwise, when $|\bar{\rho}| = |\rho|$, the primary state $\boldsymbol{\psi}_{\bar{\rho},\rho,\nu}^{(\mathrm{vac})}$ is an eigenvector of $\hat{\mathcal{P}}$,[12]

[12] Recall that the low energy Bethe states, which become the primary states $\boldsymbol{\psi}_{\bar{\rho},\rho,\nu}^{(\mathrm{vac})} \in \tilde{\mathcal{H}}_{\mathrm{even}}^{(\mathrm{cont})}$ with $\mathtt{v}=0$ and having opposite signs of the winding number have different energies on the finite lattice. Thus, despite that such primary states correspond to equivalent irreps of the $\overline{W}_\infty \otimes W_\infty$ algebra, each of them is an eigenvector of $\hat{\mathcal{P}}$. Similarly the primary states $\boldsymbol{\psi}_{\rho,\rho,\nu}^{(\mathrm{vac},\pm)} \in \mathcal{H}_{\pm}^{(\mathrm{null})}$ are also eigenvectors of $\hat{\mathcal{P}}$.

$$\hat{\mathcal{P}}\,\boldsymbol{\psi}^{(\mathrm{vac})}_{\bar{\rho},\rho,\nu} = \sigma_{\boldsymbol{\psi}}\;\boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,\bar{\rho},\nu} \qquad\qquad (\rho=\pm\bar{\rho})\,. \tag{17.71}$$

Here the sign factor $\sigma_{\boldsymbol{\psi}}$ can not be eliminated by a change of the normalization of the state and its determination requires a numerical study of the lattice system. As it follows from the result quoted in (17.25), $\sigma_{\boldsymbol{\psi}}=+1$ for $\rho=-\bar{\rho}=\frac{1}{2}\,(n+2)\,\mathtt{w}$. In sec. 9 the primary Bethe states $\boldsymbol{\Psi}_N$ with vanishing winding number and $S^z\geq 0$ were discussed. The $\mathcal{C}$ even combination, $\boldsymbol{\Psi}_N+\hat{\mathcal{C}}\,\boldsymbol{\Psi}_N$, which has the same $\mathcal{P}$ parity as $\boldsymbol{\Psi}_N$ itself, in the scaling limit becomes the primary state in the sector $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{\mathrm{v},0}$. We found that the parity of $\boldsymbol{\Psi}_N$ is given by

$$\hat{\mathcal{P}}\,\boldsymbol{\Psi}_N = c_N\,(-1)^{\frac{1}{2}(N/2-S^z+\mathtt{m})}\,\boldsymbol{\Psi}_N \qquad\qquad (\mathtt{w}=0)\,, \tag{17.72}$$

where $\mathtt{m}$ is the integer that coincides with the difference between the number of negative and positive Bethe roots. This formula implies that as $N\gg 1$ the value of $b(N)$ for the $\mathcal{P}$ even and odd low energy primary Bethe states is densely distributed within the segment $(-\Lambda_N,\Lambda_N)$ with equal densities. A similar situation occurs for $\boldsymbol{\Psi}_N$, which become the primary states in $\mathcal{H}^{(\mathrm{null})}_{\pm}$.

To summarize the space $\mathcal{H}^{(\mathrm{cont})}_{\mathrm{even}}$ possesses $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{Z}_2$ global symmetries while $\mathcal{C}$, by definition, acts trivially inside it. Moreover, there is another $\mathcal{Z}_2$ symmetry $\mathcal{U}$, which comes from the invariance of the $\mathcal{C}$ even sector of $\mathscr{V}_N$ w.r.t. to the transformation (17.50). Being restricted to any irrep occurring in the decomposition of $\mathcal{H}^{(\mathrm{cont})}_{\mathrm{even}}$, the $\mathcal{U}$ transformation acts as the identity modulo a sign factor:

$$\hat{\mathcal{U}}\left(\overline{\mathcal{W}}_{\bar{\rho},s}\otimes\mathcal{W}_{\rho,s}\right)=\pm(-1)^{\rho+\bar{\rho}}\;\overline{\mathcal{W}}_{\bar{\rho},s}\otimes\mathcal{W}_{\rho,s}\subset\mathcal{H}^{(\mathrm{cont})}_{\mathrm{even}}\,. \tag{17.73}$$

Here "$\pm$" depends on whether, for the construction of the RG trajectories $\boldsymbol{\Psi}_N$, $N/2$ is kept to be an even or an odd integer. Finally, there are the two formal $\mathcal{Z}_2$ symmetries $\mathcal{X}^{(\mathrm{w})}$ and $\mathcal{X}^{(\mathrm{null})}$ acting in $\mathcal{H}^{(\mathrm{cont})}_{\mathrm{even}}$, which are broken in the lattice system.

18. Numerical work

Our analysis of the scaling limit is based on a definition of a low energy state which was referred to as a "working" one. This was to emphasize that it contains several non-trivial assumptions regarding the spectrum of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model. Among the strongest of them is that the pair of integers $(\bar{\mathtt{L}},\mathtt{L})$ in (8.2) may only take non-negative values. This would be natural to assume once that pair has been identified with the levels of the state in the highest weight irrep of the extended conformal symmetry algebra. However, as was pointed out in sec. 17.1, the space of states in the scaling limit contains the sector $\mathcal{H}^{(\mathrm{disc},-)}$, where this identification does not hold true. The condition $\bar{\mathtt{L}},\mathtt{L}\geq 0$ was motivated through a numerical study of the low energy spectrum of the lattice Hamiltonian.

A related question concerns the constraint $\big|\Im m\big(b(N)\big)\big|<\frac{n}{4}$. Recall that the latter was introduced so that $b(N)$, which is proportional to the logarithm of the eigenvalue of the quasi-shift operator (8.6), would be defined unambiguously and in a way that is consistent with the large N asymptotic formula for the low energy spectrum (8.2a). However it turns out that one can choose the branch of the logarithm in (8.6) differently, such that $b(N)$ is continuous in N and (8.2a) is still valid. These two ways of specifying the branch are equivalent in the case of the continuous spectrum where $\lim_{N\to\infty}\Im m\big(b(N)\big)=0$. However there exist RG trajectories with $\big|\Im m\big(b(N)\big)\big|<\frac{n}{4}$ for sufficiently small N, while if $b(N)$ is defined to be continuous in N its limiting value as $N\to\infty$ lies outside of this strip. An example is depicted in Fig. 14, where $n=3$,

$\mathtt{k} = 0.099$, $\mathtt{w} = -1$, $S^z = \mathtt{L} = \bar{\mathtt{L}} = 0$, while $\lim_{N\to\infty} \Im m\big(b(N)\big) = -\frac{3}{2} - p = 0.7525 > \frac{n}{4}$. Remarkably, the trajectory still fits within our working definition of the low energy state. This is due to the simple identity

$$\frac{p^2+\bar{p}^2}{n+2} - \frac{2\,(p+\frac{1}{2}+a)^2}{n} + \mathtt{L} + \bar{\mathtt{L}} = \frac{p_+^2+\bar{p}_+^2}{n+2} - \frac{2\,(p+\frac{1+n}{2}+a)^2}{n} + \mathtt{L}_+ + \bar{\mathtt{L}}_+ \tag{18.1}$$

with

$$p_+ = p + \tfrac{1}{2}\,(n+2)\,, \qquad \bar{p}_+ = \bar{p} - \tfrac{1}{2}\,(n+2)\,, \qquad \mathtt{L}_+ = \mathtt{L} + a\,, \qquad \bar{\mathtt{L}}_+ = \bar{\mathtt{L}} + a + S^z\,. \tag{18.2}$$

Thus the RG trajectory from Fig. 14 may be equivalently assigned $S^z = \mathtt{w} = 0$, $\mathtt{L} = \bar{\mathtt{L}} = 1$ and with the limiting value of $b(N)$ belonging to the strip, $\lim_{N\to\infty} b(N) = \big(-\frac{3}{2} - p - \frac{n}{2}\big)\,\mathrm{i} = -0.7475\,\mathrm{i}$. Notice that the levels $\mathtt{L}$ and $\bar{\mathtt{L}}$ have increased by one rather than becoming negative.

That every low energy state of the lattice Hamiltonian fits within the "working" definition is difficult to justify rigorously. The same can also be said for the conjectures from sec. 10 concerning the space of low energy states of the lattice model with $N \gg 1$. Nevertheless the assumptions turn out to be in full accordance with the results of a numerical investigation of the low energy spectrum of $\mathbb{H}$. The latter is based on the following procedure. First, for sufficiently small N we performed the numerical diagonalization of the Hamiltonian, the lattice translation, the quasi-shift and other operators belonging to the commuting family in a sector with given $0 \le S^z \ll N$. We consider only those states below a certain cutoff in the energy and lattice momentum. To each of them we tried to assign the winding number $\mathtt{w}$ and a pair of non-negative integers $\mathtt{L}$, $\bar{\mathtt{L}}$ such that eq. (8.2) with $|\Im m\big(b(N)\big)| < \frac{n}{4}$ approximately holds true. As will be discussed in a moment we found that practically the most effective way to make such an identification was using the subleading corrections to that formula. Once the main characteristics of a low energy state are specified, we try to match the value of $b(N)$ with some solution of (9.11), considered as an equation for $b(N)$ with the finite size correction terms ignored.[13] Having a solution to (9.11) for given N, it can be continued for increasing values of N without numerical diagonalization of the lattice operators and the numerical solutions of the Bethe ansatz equations. Thus the properties of the space of states in the scaling limit may be determined from an analysis of eq. (9.11) alone. This was the way in which we arrived at the conjectures in sec. 10.

Let's illustrate the above procedure on a concrete example. Among others we numerically analyzed the first 400 low energy eigenstates of the Hamiltonian $\mathbb{H}$ (7.6), (7.7) with

$$N = 22\,, \qquad q = \mathrm{e}^{\frac{\mathrm{i}\pi}{5}} \qquad (n=3)\,, \qquad \mathtt{k} = -9/50 \tag{18.3}$$

in the sector $S^z = 1$. The total number of states in this sector is over 6×10^5 so a brute force numerical diagonalization is not possible. However the Hamiltonian is a sparse matrix with a significant number of vanishing elements. This allows one to find the first few hundred low energy eigenvectors and eigenvalues using the Krylov-Arnoldi method [59,60] within a reasonable computer time. It turned out that among the first 400 eigenstates of $\mathbb{H}$, ordered according to the real part of the energy, there were only four non-degenerate eigenstates, while the remaining

[13] For given $p = \frac{1}{2}\big(S^z + (n+2)(\mathtt{k}+\mathtt{w})\big)$, $\bar{p} = \frac{1}{2}\big(S^z - (n+2)(\mathtt{k}+\mathtt{w})\big)$, $\mathtt{L}$ and $\bar{\mathtt{L}}$, the number of solution sets $\boldsymbol{w}$, $\bar{\boldsymbol{w}}$ of (10.3) (with s substituted by $b(N)$) is finite, so that there are a finite number of equations (9.11) with $\delta = \delta(\bar{\boldsymbol{w}}, \boldsymbol{w}\,|\,\bar{p}, p, s)$ to be checked. In fact, an analysis of the subleading corrections to the energy are typically enough to determine the sets $\bar{\boldsymbol{w}}$, $\boldsymbol{w}$ for a given low energy state.

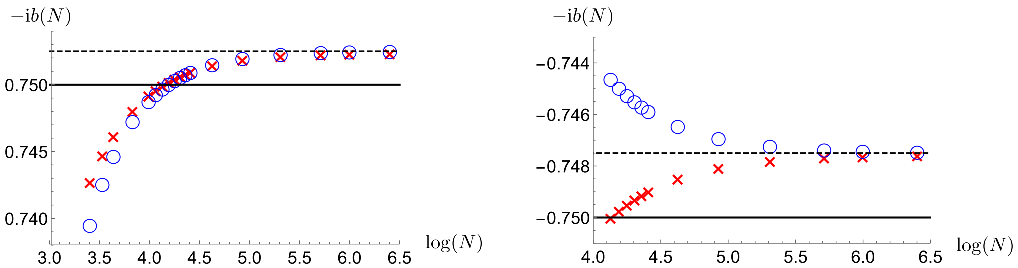

Fig. 14. The red crosses depict the imaginary part of $b(N)=\frac{n}{4\pi}\log(B)$, where B is the eigenvalue of the quasi-shift operator (8.4), for an RG trajectory $\boldsymbol{\Psi}_N$ with the parameters set to be $n=3$, $\mathtt{k}=0.099$ and $S^z=0$. For the left panel the logarithm was defined so that $b(N)$ is a continuous function of N. As N increases $b(N)$ leaves the strip $\big|\Im m\big(b(N)\big)\big|<\frac{n}{4}$, whose boundary is marked by the solid black line, and tends to the limiting value $\lim_{N\to\infty}b(N)=0.7525\mathrm{i}$ corresponding to the dashed line. It turns out that this way of specifying $b(N)$ is consistent with the asymptotic formula (8.2a) for the energy provided that one takes $\mathtt{w}=-1$ and $\mathtt{L}=\bar{\mathtt{L}}=0$. This is illustrated by the blue open circles, which correspond to $b(N)$ obtained from the energy $\mathcal{E}$ by inverting eq. (8.2a) with the correction terms ignored. For the right panel, the branch of the logarithm for $b(N)=\frac{n}{4\pi}\log(B)$ (red crosses) was taken such that $\big|\Im m\big(b(N)\big)\big|<\frac{n}{4}$. The blue circles depict $b(N)$ calculated from the energy $\mathcal{E}$, where in inverting eq. (8.2a) we set $\mathtt{w}=0$ and $\mathtt{L}=\bar{\mathtt{L}}=1$.

ones formed the $\mathcal{Z}_2$ doublets. Having at hand the eigenvectors of the Hamiltonian, the computation of the eigenvalues of $\mathbb{A}_+(\zeta)$ becomes a relatively easy task. For the case of a doublet one needs to calculate two rows of the matrix $\mathbb{A}_+(\zeta)$, contract them with the eigenvectors and then diagonalize the resulting 2×2 matrix. Note that, unlike the Hamiltonian, $\mathbb{A}_+(\zeta)$ is a dense matrix having no vanishing entries. Thus even the calculation of all of its 4.1×10^{11} matrix elements in the $S^z=1$ sector for $N=22$ would be simply impossible. The same also applies to the transfer matrix $\mathbb{T}(\zeta)$. For the lattice model with $N=22$ and $S^z=1$ the eigenvalues of $\mathbb{A}_+(\zeta)$ are tenth degree polynomials in ζ, whose zeroes solve the Bethe ansatz equations (2.10). This allows one to calculate the Bethe roots for all the 400 eigenstates. In turn, the eigenvalues of the lattice translation and quasi-shift operators are obtained from the Bethe roots using formulae (2.29) specialized to $r=2$, $\eta_J=(-1)^{J+1}\,\mathrm{i}$ and (8.5), respectively. Apart from these we also found it useful to consider the eigenvalues of the operators $\mathbb{H}^{(\pm)}$. They belong to the commuting family, are related to each other through the $\mathcal{Z}_2$ transformation $\mathcal{D}$ and their sum is equal to the Hamiltonian:

$$\mathbb{H}=\mathbb{H}^{(+)}+\mathbb{H}^{(-)}\,,\qquad \big[\mathbb{H}^{(\pm)},\,\mathbb{H}\big]=0\,,\qquad \mathbb{H}^{(\mp)}=\mathcal{D}\,\mathbb{H}^{(\pm)}\,\mathcal{D}\,. \tag{18.4}$$

The explicit formula for the matrices $\mathbb{H}^{(\pm)}$ is quoted in sec. 8.2 in the work [29]. Their eigenvalues are expressed in terms of the Bethe roots as

$$\mathbb{H}^{(\pm)}\,\boldsymbol{\Psi}\big(\{\zeta_m\}\big)=\mathcal{E}^{(\pm)}\,\boldsymbol{\Psi}\big(\{\zeta_m\}\big)\,,\qquad \mathcal{E}^{(\pm)}=\pm\sum_{m=1}^{N/2-S^z}\frac{2\,(q-q^{-1})}{\zeta_m-\zeta_m^{-1}\mp\mathrm{i}\,(q+q^{-1})}\,. \tag{18.5}$$

For the considered set of 400 eigenstates the absolute value of the scaled energy

$$\delta\mathcal{E}=\frac{N}{4\pi\,v_F}\,\big(\mathcal{E}-N\,e_\infty\big) \tag{18.6}$$

varies between 0 and $|\delta\mathcal{E}_{\max}|\approx 3.3$. Therefore, in view of the large N asymptotic formula (8.2a) our analysis could only apply to the case

$$0 \le \mathtt{L} + \bar{\mathtt{L}} \le 3\,, \qquad\qquad \mathtt{L}, \bar{\mathtt{L}} \ge 0\,. \tag{18.7}$$

For the low energy states the branch of the logarithm of the eigenvalues of the lattice translation operator (8.2b) can be chosen such that

$$\frac{N}{4\pi \mathrm{i}}\log(K) - S^z\,\mathtt{k} = \mathtt{L} - \bar{\mathtt{L}} + S^z\,\mathtt{w}\,. \tag{18.8}$$

The r.h.s. is an integer, which for $N = 22$ can take any values in the range $-5 \le \mathtt{L} - \bar{\mathtt{L}} + S^z\,\mathtt{w} \le 5$. However we imposed the momentum cut-off

$$\left|\frac{N}{4\pi \mathrm{i}}\log(K) - S^z\,\mathtt{k}\right| \le 3 \tag{18.9}$$

since otherwise this would require considerations of the states with $\mathtt{L}, \bar{\mathtt{L}} = 4, 5$ which are excluded by the relation (18.7). Among the original 400 states, 338 of them satisfy this condition.

We now come up against the problem of assigning each of the 338 states the non-negative integers $\mathtt{L}$ and $\bar{\mathtt{L}}$ as well as the winding number $\mathtt{w}$. For this purpose we used the finite size correction formulae to the eigenvalues of $\mathbb{H}^{(\pm)}$ presented in the work [15]. In the scaling limit the low energy Bethe states $\boldsymbol{\Psi}_N$ take the form $\bar{\boldsymbol{\psi}}_{\bar{p},s}(\bar{\boldsymbol{w}}) \otimes \boldsymbol{\psi}_{p,s}(\boldsymbol{w})$. As was already mentioned for $\mathtt{L}, \bar{\mathtt{L}} \le 5$ the chiral states $\boldsymbol{\psi}_{p,s}(\boldsymbol{w})$ and $\bar{\boldsymbol{\psi}}_{\bar{p},s}(\bar{\boldsymbol{w}})$ are completely determined by the eigenvalues of the local IM $I_m(\boldsymbol{w}, p, s)$ and $I_m(\bar{\boldsymbol{w}}, \bar{p}, s)$, respectively, with $m = 1, 2, 3$ (15.7). For finite N the subleading correction to the scaled energy $\delta\mathcal{E} = \frac{N}{4\pi v_F}\big(\mathcal{E} - N e_\infty\big)$, corresponding to $\boldsymbol{\Psi}_N$, is described by the formula

$$\delta\mathcal{E} = I_{1,N} + \bar{I}_{1,N} - \frac{4n^2}{N^2}\left(2\pi^2 g_1\, I_{1,N}\, \bar{I}_{1,N} + g_3\left(I_{3,N} + \bar{I}_{3,N}\right)\right) + O\big(N^{-4}, N^{-2n}\big)\,. \tag{18.10}$$

Here g_1, g_3 stand for the numerical constants

$$g_1 = -\frac{\cot(\frac{\pi}{n})}{2\pi\, n^2}\,, \qquad\qquad g_3 = \frac{\pi\,\Gamma(\frac{7}{2} + \frac{3}{n})\Gamma^3(1 + \frac{1}{n})}{18\,\Gamma(\frac{3}{n})\Gamma^3(\frac{3}{2} + \frac{1}{n})}\,, \tag{18.11}$$

while

$$I_{m,N} = I_m(\boldsymbol{w}, p, s)\big|_{s=b(N)}\,, \qquad\qquad \bar{I}_{m,N} = I_m(\bar{\boldsymbol{w}}, \bar{p}, s)\big|_{s=b(N)} \tag{18.12}$$

and the sets $\boldsymbol{w}$, $\bar{\boldsymbol{w}}$ solve the algebraic system (10.3) with s replaced by the "running coupling" $b(N)$. In (18.10) the notation $O\big(N^{-a}, N^{-b}\big)$ stands for $o(N^{-c})$, where $c = \min(a, b) - \epsilon$ for all $\epsilon > 0$. It should be pointed out that the large N asymptotic formula for $\delta\mathcal{E}$ is not literally applicable when $n \le 1$. In this case the description of the finite size corrections is more involved and includes a contribution from the so-called dual non-local IM (see ref. [15] for a further discussion). As it follows from (18.4) the energy $\mathcal{E}$ coincides with the sum of the eigenvalues of $\mathbb{H}^{(+)}$ and $\mathbb{H}^{(-)}$. The finite size corrections for the difference $\mathcal{E}^{(+)} - \mathcal{E}^{(-)}$ is expressed in terms of the eigenvalues of the local IM $\mathbf{I}_2$ and $\bar{\mathbf{I}}_2$:

$$\frac{\mathcal{E}^{(+)} - \mathcal{E}^{(-)}}{4\pi\, v_{\mathrm{F}}} = -\frac{2\mathrm{i}n^{3/2}}{N^2}\; g_2\left(I_{2,N} - \bar{I}_{2,N}\right) + o\big(N^{-2}\big)\,, \tag{18.13}$$

where

$$g_2 = \frac{\sqrt{\pi}\,\Gamma(\frac{5}{2} + \frac{2}{n})\Gamma^2(1 + \frac{1}{n})}{3\,\Gamma(\frac{2}{n})\Gamma^2(\frac{3}{2} + \frac{1}{n})}\,.$$

Table 1

A classification of the 338 lowest energy states, subject to the momentum cut-off (18.9), of the lattice Hamiltonian $\mathbb{H}$ with $N = 22$ in the sector $S^z = 1$. The states are assigned to $\mathcal{H}^{(\mathrm{cont})}_{N|S^z}$ or $\mathcal{H}^{(\mathrm{disc},\pm)}_{N|S^z}$ based on the predictions of the asymptotic formula (9.11), see also Figs. 8, 15 and those contained in Appendix C. In the case $\mathtt{w} = 1$, $\mathtt{L} = 2$, $\bar{\mathtt{L}} = 0$ the number of states that were delegated to $\mathcal{H}^{(\mathrm{disc},+)}_{N|S^z}$ is six, which is less than what is predicted by eq. (10.28) (ten). Note that for these states $|\delta\mathcal{E}|$ (18.6) is close to $|\delta\mathcal{E}_{\max}| \approx 3.3$. In all other cases the number of states in $\mathcal{H}^{(\mathrm{disc},\pm)}_{N|S^z}$ agrees with (10.28). The parameters entering into the Hamiltonian were taken to be $q = \mathrm{e}^{\frac{\mathrm{i}\pi}{5}}$ $(n = 3)$ and $\mathtt{k} = -0.18$.

$\mathtt{w} = 0$

$(\mathtt{L}, \bar{\mathtt{L}})$	$\mathcal{H}^{(\mathrm{cont})}_{N\|S^z}$	$\mathcal{H}^{(\mathrm{disc},+)}_{N\|S^z}$	$\mathcal{H}^{(\mathrm{disc},-)}_{N\|S^z}$
(0,0)	9	0	0
(1,0)	18	0	2
(0,1)	20	0	0
(1,1)	32	0	2
(2,0)	36	0	4
(0,2)	40	0	0
(1,2)	22	0	8
(2,1)	16	0	4
(3,0)	34	0	8
(0,3)	46	0	0

$\mathtt{w} = 1$

$(\mathtt{L}, \bar{\mathtt{L}})$	$\mathcal{H}^{(\mathrm{cont})}_{N\|S^z}$	$\mathcal{H}^{(\mathrm{disc},+)}_{N\|S^z}$	$\mathcal{H}^{(\mathrm{disc},-)}_{N\|S^z}$
(0,0)	5	2	0
(1,0)	6	4	0
(0,1)	8	4	0
(2,0)	2	6	0

The procedure that we used for assigning the full set of RG invariants to the low energy Bethe states is the following. For a given state $\boldsymbol{\Psi}_N$ the eigenvalues $\mathcal{E}^{(\pm)}$, K and B are calculated from the Bethe roots obtained via the diagonalization of $\mathbb{H}$ and $\mathbb{A}_+(\zeta)$ described above. Then $b(N) = \frac{n}{4\pi}\log(B)$ is used to compute the r.h.s. of (18.10) for all possible pairs $(\mathtt{L}, \bar{\mathtt{L}})$ satisfying (18.7), with $\mathtt{w}$ determined through the relation (18.8). This involves solving the algebraic system (10.3), where $p = \frac{1}{2}\big(S^z + (n+2)(\mathtt{k}+\mathtt{w})\big)$, $\bar{p} = \frac{1}{2}\big(S^z - (n+2)(\mathtt{k}+\mathtt{w})\big)$ and s is swapped for $b(N)$. The obtained values of the r.h.s. of (18.10) are then matched with $\delta\mathcal{E} = \frac{N}{4\pi v_{\mathrm{F}}}\big(\mathcal{E}^{(+)} + \mathcal{E}^{(-)} - N e_\infty\big)$. In almost all cases the procedure allows one to unambiguously determine the integers $\mathtt{L}$, $\bar{\mathtt{L}}$, $\mathtt{w}$ as well as the sets $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$ associated to the state $\boldsymbol{\Psi}_N$. It should be mentioned that we encountered about a half dozen cases, out of the 338, where we could not unambiguously identify the states with the help of (18.10) alone. In all these cases the issue was resolved by employing the relation (18.13) and the product rule (11.20). The results of the above procedure are summarized in Table 1.

The table also contains a classification of the states according to whether they become part of the continuous or discrete spectrum in the scaling limit. This is achieved by matching $b(N)$ with a certain solution of the equation

$$\left(\frac{N}{2N_0}\right)^{4\mathrm{i}b} \exp\big(\tfrac{\mathrm{i}}{2}\,\delta(\bar{\boldsymbol{w}}(b), \boldsymbol{w}(b)\,|\,\bar{p}, p, b)\big) = (-1)^{N/2-S^z}\,, \tag{18.14}$$

which comes from dropping the correction terms in the asymptotic formula (9.11). Here the phase shift depends on b explicitly as well as implicitly through the sets $\boldsymbol{w}(b)$ and $\bar{\boldsymbol{w}}(b)$ which solve (10.3) with s substituted by b. For fixed $\mathtt{L}$ and $\bar{\mathtt{L}}$ there are $\mathrm{par}_2(\mathtt{L}) \times \mathrm{par}_2(\bar{\mathtt{L}})$ pairs $(\boldsymbol{w}(b), \bar{\boldsymbol{w}}(b))$ so that $\mathrm{par}_2(\mathtt{L}) \times \mathrm{par}_2(\bar{\mathtt{L}})$ equations of the form (18.14) need to be considered. Among the solutions of all of these equations, one should choose the particular one, b_*, which is closest to the value of $b(N)$ corresponding to the Bethe state $\boldsymbol{\Psi}_N$. In practice this is not too difficult a task. The

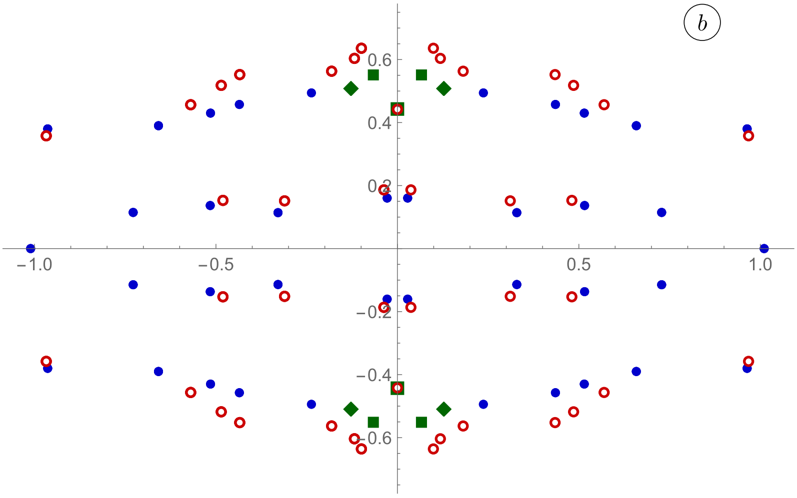

Fig. 15. The open circles depict the distribution of $b(N)=\frac{n}{4\pi}\log(B)$ in the complex plane, where B is the eigenvalue of the quasi-shift operator, for the $42=34+8$ states with $\mathtt{L}=3$, $\bar{\mathtt{L}}=0$ and $\mathtt{w}=0$ as indicated in Table 1. The filled circles, squares and diamonds correspond to the solutions b_* of eq. (18.14). For the circles $\lim_{N\to\infty}\Im m\big(b_*(N)\big)=0$, while for the squares, $b_*(N)\to\pm\frac{9\mathrm{i}}{20}$ as $N\to\infty$. The filled diamonds form two pairs which have the same value of $\Im m\big(b_*(N)\big)$ and opposite real part. At large but finite N the diamonds from the upper pair collide at the imaginary axis at which point for one of the diamonds $b_*(N)\to+\frac{9\mathrm{i}}{20}$ while for the other one $b_*(N)\to 0$. The N dependence of $b_*(N)$ for the lower pair of diamonds is obtained from that of the upper pair via complex conjugation.

initial approximation for finding the solution b_* may be taken to be $b_{\mathrm{in}}=b(N)=\frac{n}{4\pi}\log(B)$. The proper sets $\boldsymbol{w}_*(b)$, $\bar{\boldsymbol{w}}_*(b)$, are the ones which at $b=b_{\mathrm{in}}$ coincide with the $\boldsymbol{w}$, $\bar{\boldsymbol{w}}$ that were assigned to the Bethe state $\boldsymbol{\Psi}_N$ through the examination of the finite size corrections. Once b_* is determined its N dependence, with $b_*=b_*(N)$ being a continuous function, is obtained by means of varying N in eq. (18.14) with the sign factor $(-1)^{N/2-S^z}$ kept fixed. Then the state $\boldsymbol{\Psi}_N$ is delegated to $\mathcal{H}^{(\mathrm{cont})}_{N|S^z}$ or $\mathcal{H}^{(\mathrm{disc})}_{N|S^z}$ depending on whether or not $\lim_{N\to\infty}\Im m\big(b_*(N)\big)$ vanishes. Note that for any given Bethe state one can always verify through the explicit construction of the corresponding RG trajectory, by solving the Bethe ansatz equations, that $b(N)$ asymptotically approaches $b_*(N)$ as $N\to\infty$ see, e.g., Fig. 9.

For the 34 states in the sector $\mathtt{L}=\bar{\mathtt{L}}=1$ and $\mathtt{w}=0$ the correspondence between $b(N)$, obtained via the numerical diagonalization of the lattice operators, and the solutions b_* of eq. (18.14) has already been illustrated in Fig. 8. Of these states, 32 are predicted to form part of the continuous spectrum in the scaling limit, i.e., belong to $\mathcal{H}^{(\mathrm{cont})}_{N|S^z}$, while the two states which have been matched to b_* depicted by the solid squares in the figure are part of $\mathcal{H}^{(\mathrm{disc})}_{N|S^z}$. We also confirmed this by explicitly constructing the RG trajectories corresponding to these two states and verifying that $\lim_{N\to\infty}\Im m\big(b(N)\big)\neq 0$. The results of a similar analysis for the states in the sector with $\mathtt{L}=3$ and $\bar{\mathtt{L}}=\mathtt{w}=0$ are presented in Fig. 15. Note that in this case, since the sum $\mathtt{L}+\bar{\mathtt{L}}$ reaches the upper bound in (18.7), for many of the solutions b_* there is no corresponding lattice state among the 400 lowest energy states of $\mathbb{H}$. An additional six figures that cover the remaining cases listed in Table 1 are contained in Appendix C.

Analogous computations were performed for the low energy states of the lattice Hamiltonian for different sectors S^z as well as various values of the anisotropy parameter $q=\mathrm{e}^{\frac{\mathrm{i}\pi}{n+2}}$ and the twist parameter $\mathtt{k}$.

19. Hermitian structure of the space of states in the scaling limit

19.1. Hermitian versus integrable structure

Up till now we have been focused on describing the linear structure of the space of states occurring in the scaling limit of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model. This was achieved through the decomposition of $\mathcal{H}$ into the highest weight irreps of the $\overline{W}_\infty \otimes W_\infty$ - algebra, accounting for the superselection rules imposed by the global symmetries along the way. The space of states also possesses an integrable structure. It is inherited from the finite dimensional (pseudo-)Hilbert space of the lattice model, where there exists a basis of Bethe states diagonalizing the matrices $\mathbb{A}_\pm(\zeta)$. The latter, in the scaling limit, become the operators $\mathcal{a}_\pm(\lambda)$ and $\bar{\mathcal{a}}_\pm(\bar\lambda)$, while the scaling limit of the low energy Bethe states yields the states

$$\boldsymbol{\psi}_{\bar\rho,\rho,\bar\nu,\nu}(\bar{\boldsymbol{w}},\boldsymbol{w}) \equiv \bar{\boldsymbol{\psi}}_{\bar\rho,\bar\nu}(\bar{\boldsymbol{w}}) \otimes \boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w}) \in \bar{\mathcal{V}}_{\bar\rho,\bar\nu} \otimes \mathcal{V}_{\rho,\nu}\ . \tag{19.1}$$

In turn, these form a basis for $\mathcal{H}$ diagonalizing $\mathcal{a}_\pm(\lambda)$, $\bar{\mathcal{a}}_\pm(\bar\lambda)$. Here we discuss the Hermitian structures consistent with the integrable one.

We'll consider two types of sesquilinear forms in $\mathcal{H}$. The first one, labeled by the subscript "+", is such that the eigenstates $\boldsymbol{\psi}_{\bar\rho,\rho,\bar\nu,\nu}(\bar{\boldsymbol{w}},\boldsymbol{w})$ satisfy the condition[14]

$$\big(\boldsymbol{\psi}_{\bar\rho',\rho',\bar\nu',\nu'}(\bar{\boldsymbol{w}}',\boldsymbol{w}'),\boldsymbol{\psi}_{\bar\rho,\rho,\bar\nu,\nu}(\bar{\boldsymbol{w}},\boldsymbol{w})\big)_+ = 0 \qquad \text{unless} \qquad \boldsymbol{\psi}_{\bar\rho',\rho',\bar\nu',\nu'} = \hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{\psi}_{\bar\rho,\rho,\bar\nu,\nu}\ . \tag{19.2}$$

Recall that the $\mathcal{CPT}$ conjugation acts as

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}:\ \bar{\mathcal{V}}_{\bar\rho,\bar\nu} \otimes \mathcal{V}_{\rho,\nu} \mapsto \bar{\mathcal{V}}_{\bar\rho,\bar\nu^*} \otimes \mathcal{V}_{\rho,\nu^*}\ . \tag{19.3}$$

Hence for the continuous component of the space of states, $\mathcal{H}^{(\mathrm{cont})}$, where $\rho = p$, $\bar\rho = \bar p$ and $\nu = \bar\nu = s$ are all real the states $\boldsymbol{\psi}_{\bar\rho,\rho,\bar\nu,\nu}(\bar{\boldsymbol{w}},\boldsymbol{w})$ and $\boldsymbol{\psi}_{\bar\rho',\rho',\bar\nu',\nu'}(\bar{\boldsymbol{w}}',\boldsymbol{w}')$ in (19.2) belong to the same irrep of the $\overline{W}_\infty \otimes W_\infty$ - algebra. On the other hand for the irreps from $\mathcal{H}^{(\mathrm{disc})}$, where $\nu^* = -\nu$ and $\bar\nu^* = -\bar\nu$, the $\mathcal{CPT}$ conjugated space, $\mathcal{V}_{\bar\rho,-\bar\nu} \otimes \mathcal{V}_{\rho,-\nu}$, does not coincide with the initial one $\mathcal{V}_{\bar\rho,\bar\nu} \otimes \mathcal{V}_{\rho,\nu}$. This makes it natural to introduce another sesquilinear form, using the $\mathcal{Z}_2$ symmetry, such that

$$\big(\boldsymbol{\psi}_{\bar\rho',\rho',\bar\nu',\nu'}(\bar{\boldsymbol{w}}',\boldsymbol{w}'),\boldsymbol{\psi}_{\bar\rho,\rho,\bar\nu,\nu}(\bar{\boldsymbol{w}},\boldsymbol{w})\big)_- = 0 \qquad \text{unless} \qquad \boldsymbol{\psi}_{\bar\rho',\rho',\bar\nu',\nu'} = \hat{\mathcal{D}}\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{\psi}_{\bar\rho,\rho,\bar\nu,\nu}\ . \tag{19.4}$$

Since

$$\hat{\mathcal{D}}\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}:\ \bar{\mathcal{V}}_{\bar\rho,\bar\nu} \otimes \mathcal{V}_{\rho,\nu} \mapsto \bar{\mathcal{V}}_{\bar\rho,-\bar\nu^*} \otimes \mathcal{V}_{\rho,-\nu^*} \tag{19.5}$$

any irrep of the $\overline{W}_\infty \otimes W_\infty$ - algebra occurring in the decomposition of $\mathcal{H}^{(\mathrm{disc})}$ would coincide with its conjugate.

For each of the sesquilinear forms consistent with the integrable structure there is an evident candidate, which is defined through the conjugation conditions for the W currents:

[14] Although the normalization for the eigenstates has yet to be fixed, in writing formulae (19.2) and (19.4) we assume that $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{\psi}_{\bar\rho,\rho,\bar\nu,\nu}(\bar{\boldsymbol{w}},\boldsymbol{w}) = \boldsymbol{\psi}_{\bar\rho,\rho,\bar\nu^*,\nu^*}(-\bar{\boldsymbol{w}}^*,-\boldsymbol{w}^*)$ and $\hat{\mathcal{D}}\,\boldsymbol{\psi}_{\bar\rho,\rho,\bar\nu,\nu}(\bar{\boldsymbol{w}},\boldsymbol{w}) = \boldsymbol{\psi}_{\bar\rho,\rho,-\bar\nu,-\nu}(-\bar{\boldsymbol{w}},-\boldsymbol{w})$ (see eq. (17.19)).

$$\big(\chi_2, W_j(u)\chi_1\big)_\pm = (\pm 1)^j \,\big(W_j(u^*)\chi_2, \chi_1\big)_\pm \tag{19.6a}$$
$$\big(\chi_2, \overline{W}_j(\bar{u})\chi_1\big)_\pm = (\pm 1)^j \,\big(\overline{W}_j(\bar{u}^*)\chi_2, \chi_1\big)_\pm\,, \tag{19.6b}$$

where $\chi_1 \in \bar{\mathcal{V}}_{\bar{\rho},\bar{\nu}} \otimes \mathcal{V}_{\rho,\nu}$ and $\chi_2 \in \bar{\mathcal{V}}_{\bar{\rho},\pm\bar{\nu}^*} \otimes \mathcal{V}_{\rho,\pm\nu^*}$ are arbitrary states. Indeed since the local IM are expressed as integrals over the local densities built from the W currents, the above conjugation conditions imply

$$\big(\chi_2, \mathbf{I}_m\chi_1\big)_\pm = (\pm 1)^{m+1} \,\big(\mathbf{I}_m\chi_2, \chi_1\big)_\pm \tag{19.7a}$$
$$\big(\chi_2, \bar{\mathbf{I}}_m\chi_1\big)_\pm = (\pm 1)^{m+1} \,\big(\bar{\mathbf{I}}_m\chi_2, \chi_1\big)_\pm\,. \tag{19.7b}$$

For generic values of the twist and anisotropy parameters $\mathtt{k}$ and n it is expected that the set of eigenvalues $\{\mathbf{I}_m\}_{m=1}^\infty$ and $\{\bar{\mathbf{I}}_m\}_{m=1}^\infty$ unambiguously specifies the states in $\mathcal{H}$. Then (19.7) together with the commutation relations $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\mathbf{I}_m = \mathbf{I}_m\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}$, $\hat{\mathcal{D}}\,\mathbf{I}_m = (-1)^{m+1}\,\mathbf{I}_m\,\hat{\mathcal{D}}$ and similarly for $\bar{\mathbf{I}}_m$ (see (17.12)) leads to the orthogonality condition (19.2) or (19.4).

The relations (19.6) do not define the sesquilinear forms unambiguously. They should be supplemented by the value of the forms on the $\overline{W}_\infty \otimes W_\infty$ primary states. Independently of this choice, the orthogonality condition (19.2) along with the commutation relations of $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}$, $\hat{\mathcal{D}}$ with $\boldsymbol{a}_\pm(\lambda)$, $\boldsymbol{\tau}(\lambda)$ (17.11) and $\bar{\boldsymbol{a}}_\pm(\bar{\lambda})$, $\bar{\boldsymbol{\tau}}(\bar{\lambda})$ imply that for the "+" form

$$\big(\chi_2, \boldsymbol{a}_\pm(\lambda)\,\chi_1\big)_+ = \big(\boldsymbol{a}_\pm(\lambda^*)\,\chi_2, \chi_1\big)_+\,, \qquad \big(\chi_2, \boldsymbol{\tau}(\lambda)\,\chi_1\big)_+ = \big(\boldsymbol{\tau}(\lambda^*)\,\chi_2, \chi_1\big)_+ \tag{19.8a}$$
$$\big(\chi_2, \bar{\boldsymbol{a}}_\pm(\bar{\lambda})\,\chi_1\big)_+ = \big(\bar{\boldsymbol{a}}_\pm(\bar{\lambda}^*)\,\chi_2, \chi_1\big)_+\,, \qquad \big(\chi_2, \bar{\boldsymbol{\tau}}(\bar{\lambda})\,\chi_1\big)_+ = \big(\bar{\boldsymbol{\tau}}(\bar{\lambda}^*)\,\chi_2, \chi_1\big)_+\,. \tag{19.8b}$$

Similarly for the "−" sesquilinear form one has

$$\big(\chi_2, \boldsymbol{a}_\pm(\lambda)\,\chi_1\big)_- = \big(\boldsymbol{a}_\pm(-\lambda^*)\,\chi_2, \chi_1\big)_-\,, \qquad \big(\chi_2, \boldsymbol{\tau}(\lambda)\,\chi_1\big)_- = \big(\boldsymbol{\tau}(-\lambda^*)\,\chi_2, \chi_1\big)_- \tag{19.9a}$$

$$\big(\chi_2, \bar{\boldsymbol{a}}_\pm(\bar{\lambda})\,\chi_1\big)_- = \big(\bar{\boldsymbol{a}}_\pm(-\bar{\lambda}^*)\,\chi_2, \chi_1\big)_-\,, \qquad \big(\chi_2, \bar{\boldsymbol{\tau}}(\bar{\lambda})\,\chi_1\big)_- = \big(\bar{\boldsymbol{\tau}}(-\bar{\lambda}^*)\,\chi_2, \chi_1\big)_-\,. \tag{19.9b}$$

To establish the above formulae, it is sufficient to check them in the eigenbasis (19.1).

The sesquilinear forms allow one to introduce an inner product for the continuous and discrete components of $\mathcal{H}$. For the case of $\mathcal{H}^{(\mathrm{cont})}$ we take it to be

$$\langle\chi_1,\, \chi_2\rangle_{\mathrm{cont}} = (\chi_1,\, \chi_2)_+\,, \qquad\qquad \chi_1,\, \chi_2 \in \mathcal{H}^{(\mathrm{cont})}\,. \tag{19.10}$$

As it follows from (19.6) the Hermitian conjugation for the Fourier coefficients of the W currents (16.20) corresponding to such an inner product is

$$\big[\,\widetilde{W}_j(m)\,\big]^\star = \widetilde{W}_j(-m)\,, \qquad \big[\,\widetilde{\overline{W}}_j(m)\,\big]^\star = \widetilde{\overline{W}}_j(-m)\,. \tag{19.11}$$

Once the norms of the highest states are specified, this condition allows one to calculate the inner product for any given states from the same irrep of the $\overline{W}_\infty \otimes W_\infty$ - algebra. Two states which belong to irreps that are not isomorphic to each other, are orthogonal. In the case under consideration with the central charge $-1 < c < 2$ the inner product is not positive definite so that $\mathcal{H}^{(\mathrm{cont})}$ equipped with $\langle\cdot,\,\cdot\rangle_{\mathrm{cont}}$ becomes a pseudo-Hilbert space.

The structure of the pseudo-Hilbert space for $\mathcal{H}^{(\mathrm{disc})}$ can be introduced using the inner product

$$\langle\chi_1,\, \chi_2\rangle_{\mathrm{disc}} = (\chi_1,\, \chi_2)_-\,, \qquad\qquad \chi_1,\, \chi_2 \in \mathcal{H}^{(\mathrm{disc})}\,. \tag{19.12}$$

In this case the conjugation condition (19.11) is replaced by

$$\big[\,\widetilde{W}_j(m)\,\big]^* = (-1)^j\ \widetilde{W}_j(-m)\ , \qquad \big[\,\widetilde{\widetilde{W}}_j(m)\,\big]^* = (-1)^j\ \widetilde{\widetilde{W}}_j(-m)\ . \tag{19.13}$$

Likewise the inner product $\langle\cdot,\ \cdot\rangle_{\rm disc}$ is not positive definite when $c<2$.

19.2. Chiral sesquilinear forms

In the above discussion of the Hermitian structure for $\mathcal{H}$ it was sufficient to focus on an irrep occurring in the $\overline{W}_\infty\otimes W_\infty$ decomposition of this space. Recall that the $\mathcal{CPT}$ and $\mathcal{D}$ transformations, required for defining the conjugated irrep, can be introduced for each left and right chiral factor in $\bar{\mathcal{V}}_{\bar\rho,\bar\nu}\otimes\mathcal{V}_{\rho,\nu}$ separately (see sec. 17.2). This makes it possible to restrict the sesquilinear forms to the chiral components. For instance, for the "+" sesquilinear form on the right chiral spaces, relations (19.7a) and (19.8a) would continue to hold true with $\boldsymbol{\chi}_1\in\mathcal{V}_{\rho,\nu}$, $\boldsymbol{\chi}_2\in\mathcal{V}_{\rho,\nu^*}=\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\big(\mathcal{V}_{\rho,\nu}\big)$ and similarly for the left chiral ones.

The spaces $\mathcal{V}_{\rho,\nu}$ were originally realized in terms of the Fock spaces. For real ρ and ν, this irrep of the W_∞ - algebra coincides as a linear space with $\mathcal{F}_{\mathbf{P}}$, where $\mathbf{P}=\big(\frac{\rho}{\sqrt{n+2}},\frac{\nu}{\sqrt{n}}\big)$. When ν is pure imaginary and $\rho+\frac{1}{2}\pm \mathrm{i}\nu\in\mathbb{Z}$, the corresponding Fock space becomes reducible w.r.t. the W_∞ - algebra and $\mathcal{V}_{\rho,\nu}$ is obtained by factoring $\mathcal{F}_{\mathbf{P}}$ over the invariant subspace generated by the null vector as in eqs. (13.28) and (13.33). Despite this, the conjugation conditions (19.6a), combined with the bosonization formulae for the W currents (16.2), (16.8), allow one to lift the sesquilinear forms to the complex bilinear maps $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\big(\mathcal{F}_{\mathbf{P}}\big)\times\mathcal{F}_{\mathbf{P}}\mapsto\mathbb{C}$ for the "+" form and $\hat{\mathcal{D}}\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\big(\mathcal{F}_{\mathbf{P}}\big)\times\mathcal{F}_{\mathbf{P}}\mapsto\mathbb{C}$ in the case of the "−" one, for any $\mathbf{P}$. Note that the action of the $\mathcal{CPT}$ and $\mathcal{D}$ transformations in the Fock space can be defined through the relations

$$\begin{array}{lll}\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}:\mathcal{F}_{\mathbf{P}}\mapsto\mathcal{F}_{\mathbf{P}^*}\,, & \hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,a_m=a_m\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,, & \hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,b_m=b_m\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\\ \hat{\mathcal{D}}:\mathcal{F}_{\mathbf{P}}\mapsto\mathcal{F}_{\mathbf{P}'}\,, & \hat{\mathcal{D}}\,a_m=a_m\,\hat{\mathcal{D}}\,, & \hat{\mathcal{D}}\,b_m=-b_m\,\hat{\mathcal{D}}\end{array} \tag{19.14}$$

along with

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,|\mathbf{P}\rangle=|\mathbf{P}^*\rangle\,,\qquad \hat{\mathcal{D}}\,|\mathbf{P}\rangle=|\mathbf{P}'\rangle\,,\qquad\text{where}\qquad \begin{array}{l}\mathbf{P}^*=\big(\frac{\rho^*}{\sqrt{n+2}},+\frac{\nu^*}{\sqrt{n}}\big)\\ \mathbf{P}'=\big(\frac{\rho}{\sqrt{n+2}},-\frac{\nu}{\sqrt{n}}\big)\end{array}. \tag{19.15}$$

It is straightforward to check that $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{a}_\pm(\lambda)=\boldsymbol{a}_\pm(\lambda^*)\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}$ and $\hat{\mathcal{D}}\,\boldsymbol{a}_\pm(\lambda)=\boldsymbol{a}_\pm(-\lambda)\,\hat{\mathcal{D}}$ from the definition (13.9)-(13.10) of $\boldsymbol{a}_\pm(\lambda)$ as an operator acting in the Fock space as well as $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,W_j(u)=W_j(-u^*)\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}$, $\hat{\mathcal{D}}\,W_j(u)=(-1)^j\,W_j(u)\,\hat{\mathcal{D}}$ using the bosonization formulae for $W_j(u)$.

Remarkably there exists another pair of chiral sesquilinear forms for which relations (19.7a) and (19.8a) remain true, while the conjugation condition for the W currents (19.6a) is no longer valid. This may be motivated through the following observation. A computation based on the explicit formulae (15.2) and (15.3) shows that the first three local IM can be written as [47]

$$\mathbf{I}_1=\int_0^{2\pi}\frac{\mathrm{d}u}{2\pi}\,\Big((\partial\vartheta)^2+T\Big)$$

$$\mathbf{I}_2=\int_0^{2\pi}\frac{\mathrm{d}u}{2\pi}\,\Big((\partial\vartheta)^3+\frac{3\,(n+2)}{3n+4}\,(\partial\vartheta)\,T\Big) \tag{19.16}$$

$$\mathbf{I}_3 = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \left((\partial\vartheta)^4 - \frac{n^2-2}{5n+6} (\partial^2\vartheta)^2 + \frac{6\,(n+2)}{5n+6} (\partial\vartheta)^2\, T + \frac{n+2}{5n+6}\, T^2 \right),$$

where $T(u)$ stands for the chiral field

$$T(u) = (\partial\varphi)^2 + \mathrm{i}\,\frac{n+1}{\sqrt{n+2}}\,\partial^2\varphi\,. \tag{19.17}$$

It turns out to be possible to choose the densities for all the $\mathbf{I}_m$ to be a local field built from $\partial\vartheta(u)$ and $T(u)$. Since the latter satisfy the commutation relations

$$\begin{aligned} &\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,T(u) = T(-u^*)\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,, && \hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\partial\vartheta(u) = +\partial\vartheta(-u^*)\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}} \\ &\hat{\mathcal{D}}\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,T(u) = T(-u^*)\,\hat{\mathcal{D}}\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,, && \hat{\mathcal{D}}\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\partial\vartheta(u) = -\partial\vartheta(-u^*)\,\hat{\mathcal{D}}\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}} \end{aligned} \tag{19.18}$$

with the $\mathcal{CPT}$ and $\mathcal{D}$ transformations defined as in (19.14), one can introduce the sesquilinear forms through the relations

$$\begin{aligned} \big(\!\big(\chi_2, T(u)\,\chi_1\big)\!\big)_\pm &= \big(\!\big(T(u^*)\,\chi_2, \chi_1\big)\!\big)_\pm\,, \\ \big(\!\big(\chi_2, \partial\vartheta(u)\,\chi_1\big)\!\big)_\pm &= \pm\big(\!\big(\partial\vartheta(u^*)\,\chi_2, \chi_1\big)\!\big)_\pm\,. \end{aligned} \tag{19.19}$$

Here χ_1,χ_2 are arbitrary states such that $\chi_1 \in \mathcal{F}_{\mathbf{P}}$, while $\chi_2 \in \mathcal{F}_{\mathbf{P}^*}$ for the "+" case and $\chi_2 \in \mathcal{F}_{(\mathbf{P}')^*}$ for the "−" one with $(\mathbf{P}')^* = \big(\frac{\rho}{\sqrt{n+2}}, -\frac{\nu^*}{\sqrt{n}}\big)$.

To see that (19.19) indeed defines the sesquilinear forms on the Fock spaces let's introduce a suitable basis for $\mathcal{F}_{\mathbf{P}}$. The coefficients $\{L_m\}$ occurring in the expansion of $T(u)$ in the Fourier series (6.6) generate the Virasoro algebra with central charge $c = 1 - \frac{6(n+1)^2}{n+2}$. The number of states of the form

$$\begin{aligned} &L_{-m_1}\ldots L_{-m_j}\, b_{-m'_1}\ldots b_{-m'_{j'}}\,|\mathbf{P}\rangle\,, \\ &1\le m_1 \le m_2 \le \ldots \le m_j\,,\quad 1\le m'_1\le m'_2\le\ldots\le m'_{j'} \end{aligned} \tag{19.20}$$

with $\sum_j m_j + \sum_{j'} m'_{j'} = \ell$ is given by $\mathrm{par}_2(\ell)$, so that they provide a basis in the level subspace of the Fock space $\mathcal{F}^{(\ell)}_{\mathbf{P}}$. Then (19.19), together with the commutation relations for the Virasoro and Heisenberg algebra generators L_m and b_m, determine the sesquilinear form in the basis (19.20) up to an overall multiplicative constant. The latter is fixed by specifying the form on the Fock highest states $\boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,+\nu} \equiv |\mathbf{P}\rangle$. In view of what follows we'll take it to be

$$\big(\!\big(\boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,+\nu^*}, \boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,\nu}\big)\!\big)_+ = \big(\!\big(\boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,-\nu^*}, \boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,\nu}\big)\!\big)_- = Z_+\big(\tfrac{\rho}{\sqrt{n+2}}\,\big|\,\sqrt{n+2}\big)\,, \tag{19.21}$$

where the function $Z_+(P\,|\,\beta)$ is given in eq. (6.17).

Let's consider the chiral sesquilinear forms, defined through eqs. (19.19) and (19.21), in the eigenbasis of the operator $\mathcal{a}_+(\lambda) \in \mathrm{End}\big(\mathcal{F}_{\mathbf{P}}\big)$. The forms are constructed in such a way that the local IM satisfy the relations

$$\big(\!\big(\chi_2, \mathbf{I}_m\chi_1\big)\!\big)_\pm = (\pm 1)^{m+1}\big(\!\big(\mathbf{I}_m\chi_2, \chi_1\big)\!\big)_\pm\,. \tag{19.22}$$

Then it follows that the eigenstates $\boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w})$ obey the orthogonality conditions

$$\begin{aligned} \big(\!\big(\boldsymbol{\psi}_{\rho,+\nu^*}(\boldsymbol{w}_2), \boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w}_1)\big)\!\big)_+ &= U_{\rho,\nu}(\boldsymbol{w}_1)\,\delta_{\boldsymbol{w}_2,-\boldsymbol{w}_1^*} \\ \big(\!\big(\boldsymbol{\psi}_{\rho,-\nu^*}(\boldsymbol{w}_2), \boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w}_1)\big)\!\big)_- &= U_{\rho,\nu}(\boldsymbol{w}_1)\,\delta_{\boldsymbol{w}_2,+\boldsymbol{w}_1^*} \end{aligned}, \tag{19.23}$$

where we've taken into account formula (17.19) describing the action of the $\mathcal{CPT}$ and $\mathcal{D}$ transformation on the eigenstates. The function $U_{\rho,\nu}(\boldsymbol{w})$ depends on the normalization of $\boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w})$. The latter, up till now, have been considered as eigenstates of $\mathcal{A}_+(\lambda)$ without reference to their overall normalization. It turns out to be convenient to set this using the basis (19.20). Namely we'll take

$$\boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w}) = \big((L_{-1})^\ell + \ldots\big)\,\boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,\nu} \in \mathcal{F}^{(\ell)}_{\mathbf{P}} \qquad \big(\boldsymbol{w} = \{w_a\}_{a=1}^{\ell}\big)\,, \tag{19.24}$$

where the dots stand for the terms, which contain lower powers of L_{-1}.

With the normalization of the states $\boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w})$ fixed, the functions $U_{\rho,\nu}(\boldsymbol{w})$ in (19.23) are defined unambiguously. Here, as an illustration, we quote some explicit formulae for the basis states in the level subspace $\mathcal{F}^{(1)}_{\mathbf{P}}$. Normalized as in (19.24), they are given by

$$\boldsymbol{\psi}_{\rho,\nu}(w_\pm) = \bigg(L_{-1} + \frac{2\mathrm{i}\sqrt{n}}{n+2}\,w_\pm\,b_{-1}\bigg)\boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,\nu} \tag{19.25}$$

with $w_\pm$ being the two solutions of (10.3a), which for $\mathtt{L}=1$ becomes a quadratic equation,

$$w_\pm = -\frac{n+1}{2n}\left(2\mathrm{i}\,\nu \pm \sqrt{n(n+2)}\,\sqrt{1-\frac{4\rho^2}{(n+1)^2}-\frac{4\nu^2}{n(n+2)}}\,\right). \tag{19.26}$$

It is simple to check that the orthogonality conditions (19.23) are satisfied and find

$$U_{\rho,\nu}(\boldsymbol{w}) \;=\; Z_+\big(\tfrac{\rho}{\sqrt{n+2}}\,\big|\,\sqrt{n+2}\big)\;\frac{2n}{(n+2)^2}\;\times\;\begin{cases} w_+\,(w_- - w_+)\,, & \boldsymbol{w}=\{w_+\} \\ w_-\,(w_+ - w_-)\,, & \boldsymbol{w}=\{w_-\} \end{cases}. \tag{19.27}$$

As was pointed out in the work [47], the chiral sesquilinear forms $(\!(\cdot,\cdot)\!)_\pm$ and $(\cdot,\cdot)_\pm$ are related through the reflection operator:

$$(\!(\boldsymbol{\chi}_2,\boldsymbol{\chi}_1)\!)_\pm = f^{(\pm)}_{\rho,\nu}\;\times\;\big(\boldsymbol{\chi}_2,\check{\mathbf{R}}\,\boldsymbol{\chi}_1\big)_\pm\,. \tag{19.28}$$

Here $\boldsymbol{\chi}_1,\boldsymbol{\chi}_2$ are arbitrary states belonging to the Fock space and conjugated Fock space, respectively, while the factor $f^{(\pm)}_{\rho,\nu}$ is the same for all the states in $\mathcal{F}_{\mathbf{P}}$. The reflection operator $\check{\mathbf{R}}$ was already discussed at the end of sec. 15. Its explicit construction as an operator in the Fock space is given in ref. [47].

Considering (19.28) in the eigenbasis of $\mathcal{A}_\pm(\lambda)$, one obtains

$$\big(\boldsymbol{\psi}_{\rho,\pm\nu^*}(\boldsymbol{w}_2),\boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w}_1)\big)_\pm = F^{(\pm)}_{\rho,\nu}(\boldsymbol{w}_1)\;\delta_{\boldsymbol{w}_2,\mp\boldsymbol{w}_1^*} \tag{19.29}$$

with

$$F^{(\pm)}_{\rho,\nu}(\boldsymbol{w}) = f^{(\pm)}_{\rho,\nu}\,U_{\rho,\nu}(\boldsymbol{w})/\check{R}_{\rho,\nu}(\boldsymbol{w})\,. \tag{19.30}$$

Taking into account that $\check{\mathbf{R}}$ acts as the identity on the Fock highest states, the above equations imply that

$$\big(\boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,\pm\nu^*},\boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,\nu}\big)_\pm = F^{(\pm,\mathrm{vac})}_{\rho,\nu} = Z_+\big(\tfrac{\rho}{\sqrt{n+2}}\,\big|\,\sqrt{n+2}\big)\;f^{(\pm)}_{\rho,\nu}\,. \tag{19.31}$$

This way the functions $f^{(\pm)}_{\rho,\nu}$ are determined once the value of the sesquilinear forms $(\cdot,\cdot)_\pm$ are fixed on the Fock vacua. We'll postpone making this choice till the next subsections.

In practice the calculation of the chiral sesquilinear forms $(\!(\cdot,\cdot)\!)_\pm$ on two given states is significantly simpler than that of $(\cdot,\cdot)_\pm$. For instance expressing $\boldsymbol{\psi}_{\rho,\nu}(w_\pm)$ (19.25) as states in the Verma module of the W_∞ - algebra results in the more cumbersome formulae

$$\boldsymbol{\psi}_{\rho,\nu}(w_\pm) = \frac{C_{2,\pm}\,\widetilde{W}_2(-1) + 4\sqrt{n}\,C_{3,\pm}\,\widetilde{W}_3(-1)}{n\,(1+n+2\rho)\,(1-2\rho+2\mathrm{i}\nu)\,(1-2\rho-2\mathrm{i}\nu)}\ \boldsymbol{\psi}^{(\mathrm{vac})}_{\rho,\nu} \tag{19.32}$$

with the coefficients

$$C_{2,\pm} = (1+n-2\rho)\big(n\,(1-2\rho)\,(1+n+2\rho) - 4(3n+4)\,\nu^2\big) + 4\mathrm{i}n\,\nu\,(n+2-4\rho)\,w_\pm$$
$$C_{3,\pm} = (n+2)(n+1-2\rho)\,\nu - \mathrm{i}n\,(1-2\rho)\,w_\pm\,. \tag{19.33}$$

The functions $F^{(\pm)}_{\rho,\nu}(\boldsymbol{w})$ in (19.29) may be computed directly from the definition (19.6a), (19.31) or obtained via (19.30):

$$F^{(\pm)}_{\rho,\nu}\big(\{w\}\big) = F^{(\pm,\mathrm{vac})}_{\rho,\nu}\ \frac{2n}{(n+2)^2}\ \frac{(n+1-2p-2w)(n+1-2p+2w)}{(n+1+2p-2w)(n+1+2p+2w)}$$
$$\times\begin{cases} w_+\,(w_- - w_+)\,, & w=w_+ \\ w_-\,(w_+ - w_-)\,, & w=w_- \end{cases}\,. \tag{19.34}$$

19.3. Scaling limit of the Bethe states with real s and $(n+2)\,\mathtt{k}\notin\mathbb{Z}$

The space of states of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model, i.e., the finite dimensional space $\mathscr{V}_N = \mathbb{C}^2_N\otimes\mathbb{C}^2_{N-1}\otimes\cdots\otimes\mathbb{C}^2_1$, admits a variety of Hermitian structures for which the Bethe states satisfy the orthogonality condition $\big(\boldsymbol{\Psi}^{(2)}_N,\,\boldsymbol{\Psi}^{(1)}_N\big)=0$ unless $\boldsymbol{\Psi}^{(2)}_N\propto\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{\Psi}^{(1)}_N$. These are distinguished by the value of the "norms" $(\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\Psi}_N,\boldsymbol{\Psi}_N)$. For the description of the Hermitian structures consistent with the integrable structure a fundamental rôle belongs to the sesquilinear form $(\cdot,\cdot)_\star$, which was mentioned in the Preliminaries (see eq. (2.35)). Here we present the results of our numerical study of the norm (2.39) for the RG trajectories characterized by the real RG invariant s. They enable one to establish a precise relation between the sesquilinear form $(\cdot,\cdot)_\star$ and those that are induced in the space $\mathcal{H}^{(\mathrm{cont})}$. This, in turn, completes our description of the scaling limit of the low energy Bethe states with real s.

We performed a numerical study of the norm $(\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\Psi}_N,\boldsymbol{\Psi}_N)_\star$ of the Bethe states (2.20) for a wide range of RG trajectories with $S^z=0,1,2,\ldots$, $\mathtt{w}=0,\pm1,\pm2,\ldots$, $\mathtt{L},\bar{\mathtt{L}}=0,1,2,\ldots$. It was found that the combination

$$G[\boldsymbol{\Psi}_N]\equiv\big(\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\Psi}_N,\boldsymbol{\Psi}_N\big)_\star\,(N/2)^{-\frac{1}{3}+f(p)+f(\bar{p})+4\mathtt{L}+4\bar{\mathtt{L}}}\,\mathrm{e}^{-\frac{1}{2}\mathcal{A}_2N^2}\,, \tag{19.35}$$

where p and $\bar{p}$ are given by (8.1), satisfies the asymptotic condition

$$G[\boldsymbol{\Psi}_N]=O\big(\log(N)\big)\qquad\text{as}\qquad N\to\infty\qquad\text{for real } s\,. \tag{19.36}$$

The constant $\mathcal{A}_2$ coincides with (6.20) upon the substitution $\beta^2\mapsto\frac{2}{n+2}$, i.e.,

$$\mathcal{A}_2=\int_0^\infty\frac{\mathrm{d}t}{t}\ \frac{\sinh(\frac{2t}{n})\,\sinh(t)}{2\sinh\big((1+\frac{2}{n})\,t\big)\cosh^2(t)}\,, \tag{19.37}$$

while

$$f(p)=\frac{4p^2}{n+2}+\frac{1}{6(n+2)}+\frac{n+2}{6}-\frac{1}{2}\,. \tag{19.38}$$

To provide a more precise description of the asymptotic behaviour (19.36), introduce

$$g_N\big(\bar{\boldsymbol{w}},\boldsymbol{w}\,|\,\bar{p},p,s\big)=\frac{1}{2\pi}\left[4\,\log\left(\frac{N}{2N_0}\right)-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}s}\,\log\Big(D_{\bar{p},s}(\bar{\boldsymbol{w}})\,D_{p,s}(\boldsymbol{w})\Big)\right] \tag{19.39}$$

with $D_{p,s}(\boldsymbol{w})$ being given by (10.7). Then for the RG trajectory $\boldsymbol{\Psi}_N$, which in the scaling limit becomes the state $\bar{\boldsymbol{\psi}}_{\bar{p},s}(\bar{\boldsymbol{w}})\otimes\boldsymbol{\psi}_{p,s}(\boldsymbol{w})$, our numerical study led us to the following asymptotic formula

$$G[\boldsymbol{\Psi}_N]\asymp\big(C_0^{(\mathrm{alt})}\big)^2\left(\frac{C}{\sqrt{2}}\right)^{\frac{8s^2}{n}+f(p)+f(\bar{p})+4\mathtt{L}+4\bar{\mathtt{L}}}\Big(2^{-1-\frac{2}{n}}\sqrt{n+2}\,N_0\Big)^{-\frac{8s^2}{n}}$$
$$\times\;U_{\bar{p},s}(\bar{\boldsymbol{w}})\,U_{p,s}(\boldsymbol{w})\,\Big(g_N(\bar{\boldsymbol{w}},\boldsymbol{w}\,|\,\bar{p},p,s)+o(1)\Big)\,. \tag{19.40}$$

Here $U_{p,s}(\boldsymbol{w})$ is the same function as in (19.23). As was previously discussed, it is unambiguously determined through relations (19.19), (19.21) specifying the chiral sesquilinear form $(\!(\cdot,\cdot)\!)_+$ and the normalization condition (19.24) for the chiral states $\boldsymbol{\psi}_{p,s}(\boldsymbol{w})$. Formula (19.40) also involves the positive constants $C_0^{(\mathrm{alt})}$ and C depending only on n. Their numerical values at different n are presented in Appendix A. Note that the constant C is the same as $C(\beta)$ from eq. (6.23) provided that β and n are identified as $\beta=\sqrt{\frac{2}{n+2}}$.

It is possible to give a natural explanation of the asymptotic formula (19.40) if we make the following assumptions concerning the scaling limit of the Bethe states.

(i) There exists the limit

$$\underset{\substack{N\to\infty\\ b(N)\to s}}{\mathrm{slim}}\;\Big(\mathcal{K}_N^{(\bar{\mathtt{L}})}(\bar{p},s)\,\mathcal{K}_N^{(\mathtt{L})}(p,s)\Big)^{-\frac{1}{2}}\;\boldsymbol{\Psi}_N$$
$$=\bar{\boldsymbol{\psi}}_{\bar{p},s}(\bar{\boldsymbol{w}})\otimes\boldsymbol{\psi}_{p,s}(\boldsymbol{w})\equiv\boldsymbol{\psi}_{\bar{p},p,s}(\bar{\boldsymbol{w}},\boldsymbol{w})\,, \tag{19.41}$$

where

$$\mathcal{K}_N^{(\mathtt{L})}(p,s)=C_0^{(\mathrm{alt})}\left(\frac{C}{\sqrt{2}}\right)^{\frac{4s^2}{n}+f(p)+4\mathtt{L}}\big(2^{-1-\frac{2}{n}}\sqrt{n+2}\,N_0\big)^{-\frac{4s^2}{n}}$$
$$\times\,(N/2)^{\frac{1}{6}-f(p)-4\mathtt{L}}\;\mathrm{e}^{\frac{1}{4}\mathcal{A}_2N^2}\,. \tag{19.42}$$

For given $S^z\geq 0$ the set of all possible states $\{\boldsymbol{\psi}_{\bar{p},p,s}(\bar{\boldsymbol{w}},\boldsymbol{w})\}$ form a basis in $\mathcal{H}_{S^z}^{(\mathrm{cont})}$.

(ii) The space $\mathcal{H}_{S^z}^{(\mathrm{cont})}$ is equipped with the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathrm{cont}}$, which in the basis $\boldsymbol{\psi}_{\bar{p},p,s}(\bar{\boldsymbol{w}},\boldsymbol{w})$ is given by

$$\big\langle\!\big\langle\boldsymbol{\psi}_{\bar{p}',p',s'}(\bar{\boldsymbol{w}}',\boldsymbol{w}'),\,\boldsymbol{\psi}_{\bar{p},p,s}(\bar{\boldsymbol{w}},\boldsymbol{w})\big\rangle\!\big\rangle_{\mathrm{cont}}=\delta_{\boldsymbol{w}',-\boldsymbol{w}^*}\,\delta_{\bar{\boldsymbol{w}}',-\bar{\boldsymbol{w}}^*}\,\delta_{p',p}\,\delta_{\bar{p}',\bar{p}}\,\delta(s'-s)$$
$$\times\,U_{\bar{p},s}(\bar{\boldsymbol{w}})\,U_{p,s}(\boldsymbol{w})\,. \tag{19.43}$$

To describe how (19.40) arises from (i)-(ii), let's for simplicity focus on the case of the primary Bethe states having $\mathtt{L}=\bar{\mathtt{L}}=0$. Since for fixed N and $\bar{p}$, p these states are distinguished by the integer $\mathtt{m}$ entering into (9.2), we'll denote them as $\boldsymbol{\Psi}_N^{(\mathtt{m})}$. Taking an arbitrary linear combination,

$$\Xi_N=\sum_{\mathtt{m}} C_{\mathtt{m}}\,\boldsymbol{\Psi}_N^{(\mathtt{m})}\,, \tag{19.44}$$

consider the sesquilinear form $(\cdot,\cdot)_\star$ of Ξ_N and the state $\boldsymbol{\Psi}_N^{(\mathtt{m}_0)}$ with given integer $\mathtt{m}_0$. The orthogonality condition (2.38) implies

$$\big(\Xi_N,\boldsymbol{\Psi}_N^{(\mathtt{m}_0)}\big)_\star=C_{\mathtt{m}_0}\,\big(\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\Psi}_N^{(\mathtt{m}_0)},\boldsymbol{\Psi}_N^{(\mathtt{m}_0)}\big)_\star\,. \tag{19.45}$$

At large N it follows from (i) that the state $\boldsymbol{\Psi}_N^{(\mathtt{m})}$ is approximated by

$$\boldsymbol{\Psi}_N^{(\mathtt{m})}\approx\Big(\mathcal{K}_N^{(0)}(\bar{p},s)\,\mathcal{K}_N^{(0)}(p,s)\Big)^{\frac{1}{2}}\,\boldsymbol{\psi}^{(\mathrm{vac})}_{\bar{p},p,s}\,. \tag{19.46}$$

Moreover the sum in (19.44) may be replaced by the integral

$$\sum_{\mathtt{m}}\mapsto\int\mathrm{d}s\,\rho(s)\,, \tag{19.47}$$

where the density of states $\rho(s)$ comes from eq. (9.8). The latter coincides with g_N (19.39) specialized to the primary Bethe states. Evaluating the l.h.s. of (19.45) using eqs. (19.46) and (19.47) as well as (19.43) leads to the asymptotic formula (19.40).

The similar arguments may be applied for the Bethe states with $\mathtt{L},\bar{\mathtt{L}}\geq 0$. However, it should be pointed out that g_N, in general, takes complex values. Therefore $g_N\big(\bar{\boldsymbol{w}},\boldsymbol{w}\,|\,\bar{p},\,p,\,s\big)\,\Delta s$ can not be interpreted as the number of low energy Bethe states $\boldsymbol{\Psi}_N$ with fixed p, $\bar{p}$, $\mathtt{L}$, $\bar{\mathtt{L}}$, $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$ having $\Re e\big(b(N)\big)$ belonging to the segment $(s,s+\Delta s)$. Nevertheless, the sum of $g_N\big(\bar{\boldsymbol{w}},\boldsymbol{w}\,|\,\bar{p},\,p,\,s\big)$ over all the $\mathrm{par}_2(\mathtt{L})\times\mathrm{par}_2(\bar{\mathtt{L}})$ solutions sets $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$ with fixed $\mathtt{L}$ and $\bar{\mathtt{L}}$ turns out to be a real positive function of s that coincides with the density (10.12):

$$\rho^{(\mathtt{L},\bar{\mathtt{L}})}_{\bar{p},p}(s)=\sum_{\substack{(\bar{\boldsymbol{w}},\boldsymbol{w})\\ \mathtt{L},\bar{\mathtt{L}}-\text{fixed}}} g_N(\bar{\boldsymbol{w}},\boldsymbol{w}\,|\,\bar{p},\,p,\,s)\,. \tag{19.48}$$

The above analysis suggests that the sesquilinear form $(\cdot,\cdot)_\star$ in the lattice model induces the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathrm{cont}}$ for the states in $\mathcal{H}^{(\mathrm{cont})}_{S^z}$ with $S^z\geq 0$. The latter is defined through (19.43) in the eigenbasis diagonalizing the operators $\mathbb{a}_\pm(\lambda)$ and $\bar{\mathbb{a}}_\pm(\bar{\lambda})$. A basis independent description is provided by the relations (19.19) for the "+" case and the similar ones involving $\overline{T}$, $\partial\bar{\vartheta}$, along with the value of the inner product for the highest states:

$$\langle\!\langle\boldsymbol{\psi}^{(\mathrm{vac})}_{\bar{p}',p',s'},\boldsymbol{\psi}^{(\mathrm{vac})}_{\bar{p},p,s}\rangle\!\rangle_{\mathrm{cont}}=\delta_{\bar{p}',\bar{p}}\,\delta_{p',p}\,\delta(s'-s)\;Z_+\big(\tfrac{\bar{p}}{\sqrt{n+2}}\,\big|\,\sqrt{n+2}\big)\,Z_+\big(\tfrac{p}{\sqrt{n+2}}\,\big|\,\sqrt{n+2}\big)\,. \tag{19.49}$$

For the sectors $\mathcal{H}^{(\mathrm{cont})}_{S^z}$ with $S^z<0$, the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathrm{cont}}$ is defined using the $\mathcal{CP}$ invariance of the model. With the same line of arguments that led to (17.21) one can show that $\hat{\mathcal{C}}\hat{\mathcal{P}}\,T(u)=\overline{T}(u)\,\hat{\mathcal{C}}\hat{\mathcal{P}}$ and $\hat{\mathcal{C}}\hat{\mathcal{P}}\,\partial\vartheta(u)=\partial\bar{\vartheta}(u)\,\hat{\mathcal{C}}\hat{\mathcal{P}}$. Thus for the sectors with $S^z<0$, the defining relations (19.19) for the case "+", its barred counterpart and (19.49) remain valid. In turn, formula (19.43) is applicable for any $S^z=0,\pm1,\pm2,\ldots$.

Eq. (19.41) describes a scaling limit of the Bethe states that leads to a Hermitian structure in the linear space $\mathcal{H}^{(\mathrm{cont})}$ that is consistent with the integrable structure. However the inner product

$\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\rm cont}$ (19.43) is not consistent with the natural conjugation (19.11) in the $\overline{W}_\infty\otimes W_\infty$ - algebra. It turns out to be possible to modify the definition of the scaling limit such that the inner product $\langle\cdot,\cdot\rangle_{\rm cont}$ (19.10) is induced in $\mathcal{H}^{(\rm cont)}$. This can be done in the following way.

Let's change the normalization of the Bethe states prescribed by eq. (2.20) and introduce

$$\boldsymbol{\Psi}'_N\big(\{\zeta_m\}\big)=\alpha(\zeta_1,\ldots,\zeta_M)\,\boldsymbol{\Psi}_N\big(\{\zeta_m\}\big) \tag{19.50}$$

with

$$\alpha(\zeta_1,\ldots,\zeta_M)=\mathrm{e}^{\mathrm{i}\pi\mathtt{k}}\,q^{-\frac{N}{2}+S^z}\,A_+^{(\infty)}\,\big[A_+(+\mathrm{i}q)A_+(-\mathrm{i}q)\big]^{-1}$$

and

$$A_+(\pm\mathrm{i}q)=\prod_{m=1}^M\big(1\mp\mathrm{i}q/\zeta_m\big)\,,\qquad\qquad A_+^{(\infty)}=\prod_{m=1}^M(-1/\zeta_m)\,.$$

Recall that for given N the Bethe states as defined by (2.20) satisfy the condition $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{\Psi}_N\big(\{\zeta_m\}\big)=\boldsymbol{\Psi}_N\big(\{\zeta_m^*\}\big)$. This is no longer true for $\boldsymbol{\Psi}'_N$. Instead, in view of formula (2.29) for the eigenvalues of the lattice translation operator,

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\Psi}'_N\big(\{\zeta_m\}\big)=\mathbb{K}^{-1}\,\boldsymbol{\Psi}'_N\big(\{\zeta_m^*\}\big)\,. \tag{19.51}$$

The proportionality factor α in (19.50) has been chosen so that as $N\to\infty$,

$$\alpha(\zeta_1,\ldots,\zeta_M)\,\big(\alpha(\zeta_1^*,\ldots,\zeta_M^*)\big)^*\asymp\big(R_{\bar{p},s}(\bar{\boldsymbol{w}})\,R_{p,s}(\boldsymbol{w})\big)^{-1} \times\bigg(\frac{N}{2N_0}\bigg)^{\frac{n(\bar{p}+p)}{n+2}}\bigg(\frac{n+2}{4n}\bigg)^N\big(1+o(1)\big)\,. \tag{19.52}$$

This follows by considering the product of the "+" and "−" cases in (11.19) and taking into account eq. (11.23). Then combining (19.52) with (19.40) one obtains

$$\big(\mathbb{K}\,\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\Psi}'_N,\boldsymbol{\Psi}'_N\big)_\star\asymp U_{\bar{p},s}(\bar{\boldsymbol{w}})\,U_{p,s}(\boldsymbol{w})\,\,\big(R_{\bar{p},s}(\bar{\boldsymbol{w}})\,R_{p,s}(\boldsymbol{w})\big)^{-1} \times\mathcal{N}_N^{(\bar{\mathtt{L}})}(\bar{p},s)\,\mathcal{N}_N^{(\mathtt{L})}(p,s)\,\Big(g_N\big(\bar{\boldsymbol{w}},\boldsymbol{w}\,|\,\bar{p},p,s\big)+o(1)\Big) \tag{19.53}$$

with

$$\mathcal{N}_N^{(\mathtt{L})}(p,s)=C_0^{(\rm alt)}\,\bigg(\frac{C}{\sqrt{2}}\bigg)^{\frac{4s^2}{n}+f(p)+4\mathtt{L}}\big(2^{-\frac{n+2}{n}}\sqrt{n+2}\,N_0\big)^{-\frac{4s^2}{n}} \times(N/2)^{\frac{1}{6}-f(p)+\frac{2np}{n+2}-4\mathtt{L}}\,\mathrm{e}^{\frac{1}{4}\mathcal{A}_2N^2}\,\bigg(\frac{n+2}{4n}\bigg)^{N/2}\,. \tag{19.54}$$

Similar arguments that lead us to (19.41) suggest that there exists the limit

$$\operatorname*{slim}_{\substack{N\to\infty\\ b(N)\to s}}\,\,\Big(\mathcal{N}_N^{(\bar{\mathtt{L}})}(\bar{p},s)\mathcal{N}_N^{(\mathtt{L})}(p,s)\Big)^{-\frac{1}{2}}\,\boldsymbol{\Psi}'_N\,=\,\boldsymbol{\psi}_{\bar{p},p,s}(\bar{\boldsymbol{w}},\boldsymbol{w}) \tag{19.55}$$

and the sesquilinear form for the lattice model induces an inner product in $\mathcal{H}^{(\rm cont)}$ such that

$$\big\langle \boldsymbol{\psi}_{\bar{p}',p',s'}(\bar{\boldsymbol{w}}',\boldsymbol{w}'),\ \boldsymbol{\psi}_{\bar{p},p,s}(\bar{\boldsymbol{w}},\boldsymbol{w})\big\rangle_{\rm cont}=\delta_{\bar{\boldsymbol{w}}',-\bar{\boldsymbol{w}}^*}\,\delta_{\boldsymbol{w}',-\boldsymbol{w}^*}\,\delta_{\bar{p}',\bar{p}}\,\delta_{p',p}\ \delta(s'-s)$$
$$\times\ F^{(+)}_{\bar{p},s}(\bar{\boldsymbol{w}})\ F^{(+)}_{p,s}(\boldsymbol{w})\ , \tag{19.56}$$

where

$$F^{(+)}_{\bar{p},s}(\bar{\boldsymbol{w}})=U_{\bar{p},s}(\bar{\boldsymbol{w}})/R_{\bar{p},s}(\bar{\boldsymbol{w}})\,,\qquad\qquad F^{(+)}_{p,s}(\boldsymbol{w})=U_{p,s}(\boldsymbol{w})/R_{p,s}(\boldsymbol{w})\ . \tag{19.57}$$

Notice that the states appearing in the scaling limit (19.55) satisfy the $\mathcal{CPT}$ conjugation condition $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{\psi}_{\bar{p},p,s}(\bar{\boldsymbol{w}},\boldsymbol{w})=\boldsymbol{\psi}_{\bar{p},p,s}(-\bar{\boldsymbol{w}}^*,-\boldsymbol{w}^*)$. Though the action of the $\mathcal{CPT}$ transformation on the Bethe states $\boldsymbol{\Psi}'_N$ involves a phase factor $K^{-1}=K^*$, as it follows from (8.2b), for any low energy Bethe state K tends to one as $N\to\infty$.

We now observe that, since $R_{p,s}(\boldsymbol{w})=R^{(0)}_{p,s}\,\check{R}_{p,s}(\boldsymbol{w})$, the function $F^{(+)}_{p,s}(\boldsymbol{w})$ can be expressed as in (19.30) with $f^{(+)}_{p,s}=1/R^{(0)}_{p,s}$. From the relation between the chiral sesquilinear forms $(\cdot,\cdot)_+$ and $(\!(\cdot,\cdot)\!)_+$, see eq. (19.28), one concludes that (19.56) defines an inner product in the space $\mathcal{H}^{(\rm cont)}$, which is consistent with the conjugation conditions (19.11). Furthermore, for the $\overline{W}_\infty\otimes W_\infty$ primary states, in view of the explicit formula (11.26) for $R^{(0)}_{p,s}$, one finds

$$\big\langle \boldsymbol{\psi}^{(\rm vac)}_{\bar{p}',p',s'},\ \boldsymbol{\psi}^{(\rm vac)}_{\bar{p},p,s}\big\rangle_{\rm cont}$$
$$=\delta_{p',p}\,\delta_{\bar{p}',\bar{p}}\ \delta(s'-s)\ \frac{\Gamma(1+\frac{2\bar{p}}{n+2})\,\Gamma(1+\frac{2p}{n+2})}{\Gamma(1+2\bar{p})\,\Gamma(1+2p)}\ \big|Z_{\bar{p},s}\,Z_{p,s}\big|^2 \tag{19.58}$$

where

$$Z_{p,s}=\frac{2^p}{\sqrt{2\pi}}\ (n+2)^{\frac{1}{4}-\frac{p\,(n-1)}{2\,(n+2)}}\ \Gamma(\tfrac{1}{2}+p+{\rm i}s)\ Z\big(\tfrac{p}{\sqrt{n+2}}\,\big|\,\sqrt{n+2}\big)\ . \tag{19.59}$$

Recall that $Z(P\,|\,\beta)$ is defined by (6.33).

Thus, with a proper taking of the scaling limit, the conjugation conditions for the W currents

$$\big[W_j(u)\big]^\star=W_j(u^*)\,,\qquad\qquad \big[\overline{W}_j(\bar{u})\big]^\star=\overline{W}_j(\bar{u}^*) \tag{19.60}$$

are induced by

$$\hat{\mathsf{O}}^\star=\hat{\mathsf{X}}_\star^{-1}\,\hat{\mathsf{O}}^\dagger\,\hat{\mathsf{X}}_\star\,,\qquad\quad \hat{\mathsf{O}}\in{\rm End}\big(\mathscr{V}_N\big)\,, \tag{19.61}$$

where $\hat{\mathsf{X}}_\star=\hat{\mathsf{X}}\ {\rm e}^{{\rm i}\pi(\mathbb{S}^z-N/2)}\ \mathbb{A}_+^{(\infty)}$ and "†" stands for the standard matrix Hermitian conjugation (see formulae (2.30)-(2.35) in the Preliminaries). The matrix $\mathbb{A}_+^{(\infty)}$ is diagonal in the basis of Bethe states and its eigenvalues are given by $\prod_{m=1}^M(-1/\zeta_m)$. For general values of the twist and anisotropy parameters $\Bbbk$ and n, $\mathbb{A}_+^{(\infty)}$ is invertible. However for certain values of the parameters, some of the Bethe states $\boldsymbol{\Psi}_N\big(\{\zeta_m\}\big)$ may be such that one of the Bethe roots become zero or infinity (in the last case the corresponding eigenvalue $A_+(\zeta)$ is a polynomial of order $M-1$). Then $\mathbb{A}_+^{(\infty)}$ is singular and special consideration is required to define the $\star$ - conjugation. Also the matrix $\hat{\mathsf{X}}$ (2.32) in the case of the $\mathcal{Z}_2$ invariant model may be expressed in terms of the generator of the $\mathcal{Z}_2$ symmetry (7.2), the total spin operator $\mathbb{S}^z$ and also $\Sigma^z=\sigma^z_N\,\sigma^z_{N-2}\ldots\sigma^z_2$ as $\hat{\mathsf{X}}=\Sigma^z\,\hat{\mathcal{D}}\,{\rm e}^{\frac{{\rm i}\pi}{2}(\mathbb{S}^z-N/2)}$. In turn $\hat{\mathsf{X}}_\star$ entering into the conjugation condition (19.61) can be written as

$$\hat{\mathsf{X}}_\star=\hat{\mathsf{X}}_\star^\dagger=\Sigma^z\,\hat{\mathcal{D}}\,{\rm e}^{\frac{{\rm i}\pi}{2}(N/2-\mathbb{S}^z)}\ \mathbb{A}_+^{(\infty)}\ . \tag{19.62}$$

Finally note that w.r.t. the $\star$ - conjugation the Hamiltonian $\mathbb{H}$ and lattice translation operator $\mathbb{K}$ satisfy the conditions

$$\mathbb{H}^\star = \mathbb{H}\,, \qquad\qquad \mathbb{K}^\star = \mathbb{K}^{-1}\,. \tag{19.63}$$

To avoid confusion, let's reiterate that this conjugation does not correspond to a positive definite inner product.

19.4. Comments on the case $\mathtt{k}=0$ with s real

Our considerations regarding the scaling limit of the Bethe states explicitly assumed that $(n+2)\,\mathtt{k}$ is not an integer. However, for the later parts of this work, the case $\mathtt{k}=0$ and s a real number is of special interest. As was already pointed out in sec. 17.4, some of the Verma modules of the chiral W_∞ - algebra, which appears in the decomposition of $\mathcal{H}^{(\mathrm{cont})}_{S^z,\mathtt{w}}$ (17.2), become reducible at $\mathtt{k}=0$. This way the space $\mathcal{H}^{(\mathrm{cont})}_{S^z,\mathtt{w}}$ splits into two sectors $\mathcal{H}^{(\mathrm{cont})}_{S^z,\mathtt{w}}=\tilde{\mathcal{H}}^{(\mathrm{cont})}_{S^z,\mathtt{w}}\oplus\mathcal{H}^{(\mathrm{null})}_{S^z,\mathtt{w}}$ similar to (17.60), where the $\overline{W}_\infty\otimes W_\infty$ decomposition of $\mathcal{H}^{(\mathrm{null})}_{S^z,\mathtt{w}}$ contains the irreps $\overline{\mathcal{W}}_{\rho,s}\otimes\mathcal{W}_{\rho,s}$, for which the highest state of either $\overline{\mathcal{W}}_{\rho,s}$ or $\mathcal{W}_{\rho,s}$ is a null vector in the Verma module. The two sectors $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{S^z,\mathtt{w}}$ and $\mathcal{H}^{(\mathrm{null})}_{S^z,\mathtt{w}}$ are orthogonal w.r.t. the inner product $\langle\cdot,\cdot\rangle_{\mathrm{cont}}$. Our analysis above is adapted most straightforwardly to those low energy Bethe states, which become part of $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{S^z,\mathtt{w}}$ in the scaling limit, and we'll only comment on this case.

Assuming that n is generic, the relations (19.55)-(19.59) are applicable for describing the scaling limit of the low energy Bethe states forming the sector $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{S^z,\mathtt{w}}$ for the case $S^z=1,2,3,\ldots$ as well as $S^z=\mathtt{w}=0$. However for $S^z=0$ the product of the two Γ-functions in the numerator in eq. (19.58) becomes $\frac{\pi\mathtt{w}}{\sin(\pi\mathtt{w})}$, i.e., is singular for non-zero integer $\mathtt{w}$. This is related to the fact, discussed in sec. 17.4, that the two states $\boldsymbol{\psi}_{\bar{p},p,s}(\bar{\boldsymbol{w}},\boldsymbol{w})$ with $p=-\bar{p}=\pm\frac{1}{2}\,(n+2)|\mathtt{w}|$ are indistinguishable. Nevertheless, similar to the Bethe states for finite N, one can resolve the ambiguity by starting with $\boldsymbol{\psi}_{\bar{p},p,s}(\bar{\boldsymbol{w}},\boldsymbol{w})$ with non-zero $\mathtt{k}>0$ and then setting $\mathtt{k}\to 0^+$. For taking this limit it is useful to change the normalization of the states and define $\widetilde{\boldsymbol{\psi}}^{(\mathtt{w})}_s(\bar{\boldsymbol{w}},\boldsymbol{w})=\sqrt{\mathtt{k}}\,\boldsymbol{\psi}_{\bar{p},p,s}(\bar{\boldsymbol{w}},\boldsymbol{w})$ where $p=-\bar{p}=\frac{1}{2}(n+2)(\mathtt{k}+\mathtt{w})$. Then any inner product involving the states $\widetilde{\boldsymbol{\psi}}^{(\mathtt{w})}_s(\bar{\boldsymbol{w}},\boldsymbol{w})$ remains well defined as $\mathtt{k}\to 0^+$. In particular for the primary $\overline{W}_\infty\otimes W_\infty$ states it follows from eq. (19.58) that

$$\begin{aligned}\big\langle\widetilde{\boldsymbol{\psi}}^{(\mathtt{w}',\mathrm{vac})}_{s'},\,\widetilde{\boldsymbol{\psi}}^{(\mathtt{w},\mathrm{vac})}_{s}\big\rangle_{\mathrm{cont}}&=\delta_{\mathtt{w}',\mathtt{w}}\,\delta(s'-s)\,(-1)^{\mathtt{w}}\,\frac{\sin\big(\pi\,(n+2)\,\mathtt{w}\big)}{\pi\,(n+2)}\\&\quad\times\big|Z_{p,s}\,Z_{-p,s}\big|^2\,\Big|_{p=\frac{1}{2}\,(n+2)\,\mathtt{w}}\qquad\quad (S^z=\mathtt{k}=0,\ \mathtt{w}\neq 0)\,.\end{aligned} \tag{19.64}$$

This way the subspaces $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{0,\mathtt{w}}$ with $\mathtt{w}\neq 0$ become equipped with the inner product determined through (19.64) as well as the conjugation conditions (19.60) for the W currents.

19.5. Scaling limit of the Bethe states with pure imaginary s and $(n+2)\,\mathtt{k}\notin\mathbb{Z}$

In sec. 19.1 we introduced two sesquilinear forms. For the form $(\cdot,\cdot)_+$ the irrep conjugated to $\mathcal{V}_{\bar{\rho},\bar{\nu}}\otimes\mathcal{V}_{\rho,\nu}$ coincides with $\mathcal{V}_{\bar{\rho},\bar{\nu}^*}\otimes\mathcal{V}_{\rho,\nu^*}$, while for $(\cdot,\cdot)_-$ the conjugated irrep is $\mathcal{V}_{\bar{\rho}-,\bar{\nu}^*}\otimes\mathcal{V}_{\rho,-\nu^*}$. Equipping the sector $\mathcal{H}^{(\mathrm{cont})}$ with the "plus" form and $\mathcal{H}^{(\mathrm{disc})}$ with the "minus" one, each of the irreps occurring in their decomposition w.r.t. the $\overline{W}_\infty \otimes W_\infty$ - algebra would be self-conjugated. The form $(\cdot,\cdot)_+$ is consistent with the formal anti-involution (19.60) in the $\overline{W}_\infty \otimes W_\infty$ - algebra. For the other form, the corresponding conjugation reads as

$$\big[W_j(u)\big]^* = (-1)^j\, W_j(u^*)\,, \qquad \big[\overline{W}_j(\bar{u})\big]^* = (-1)^j\, \overline{W}_j(\bar{u}^*)\,. \tag{19.65}$$

The two anti-involutions are related through the $\mathcal{Z}_2$ transformation:

$$\big[W_j(u)\big]^* = \hat{\mathcal{D}}\big[W_j(u)\big]^\star\,\hat{\mathcal{D}}\,, \qquad \big[\overline{W}_j(\bar{u})\big]^* = \hat{\mathcal{D}}\big[\overline{W}_j(\bar{u})\big]^\star\,\hat{\mathcal{D}}\,. \tag{19.66}$$

In the previous subsection it was pointed out that with a proper taking of the scaling limit the $\star$ - conjugation for the W currents is induced from the lattice one defined by eqs. (19.61) and (19.62). Therefore one might expect that the lattice version of (19.65) is given by

$$\hat{\mathsf{O}}^* = \hat{\mathcal{D}}\,\hat{\mathsf{O}}^\star\,\hat{\mathcal{D}} \tag{19.67}$$

for an arbitrary operator $\hat{\mathsf{O}}$ acting in the finite dimensional space $\mathscr{V}_N$. Since the lattice translation operator and Hamiltonian both commute with $\hat{\mathcal{D}}$, the relations (19.63) carry over to

$$\mathbb{H}^* = \mathbb{H}\,, \qquad \mathbb{K}^* = \mathbb{K}^{-1}\,. \tag{19.68}$$

Combining formulae (19.67) with (19.61), (19.62) and using that $\hat{\mathcal{D}}\,\mathbb{A}_+^{(\infty)}\,\hat{\mathcal{D}} = \mathrm{e}^{\mathrm{i}\pi(N/2-\mathbb{S}^z)}\,\mathbb{A}_+^{(\infty)}$ as well as $\hat{\mathcal{D}}^2 = 1$ one finds

$$\hat{\mathsf{O}}^* = \hat{\mathsf{X}}_*^{-1}\,\hat{\mathsf{O}}^\dagger\,\hat{\mathsf{X}}_*\,, \qquad \hat{\mathsf{O}} \in \mathrm{End}\big(\mathscr{V}_N\big) \tag{19.69}$$

with

$$\hat{\mathsf{X}}_* = \hat{\mathsf{X}}_*^\dagger = \Sigma^z\,\mathrm{e}^{\frac{\mathrm{i}\pi}{2}(\mathbb{S}^z-N/2)}\,\mathbb{A}_+^{(\infty)}\,. \tag{19.70}$$

Let $(\cdot,\cdot)_*$ be the sesquilinear form in the finite dimensional space $\mathscr{V}_N$ that is consistent with the conjugation (19.69) and such that its value on the pseudovacuum, $|\uparrow\rangle\otimes|\uparrow\rangle\otimes\ldots\otimes|\uparrow\rangle$, is one. It is easy to see that the form in the basis of Bethe states is described via the relations

$$\big(\boldsymbol{\Psi}^{(2)},\boldsymbol{\Psi}^{(1)}\big)_* = 0 \quad \text{unless} \quad \boldsymbol{\Psi}^{(2)} = \hat{\mathcal{D}}\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\boldsymbol{\Psi}^{(1)}\,, \tag{19.71}$$

and

$$\big(\hat{\mathcal{D}}\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\Psi},\boldsymbol{\Psi}\big)_* = \big(\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\Psi},\boldsymbol{\Psi}\big)_\star\,. \tag{19.72}$$

The r.h.s. in the last equation is given by (2.39).

The result of our numerical investigation of $\big(\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\Psi}_N,\boldsymbol{\Psi}_N\big)_\star$ for the low energy Bethe states is summarized by the formulae (19.35)-(19.40). However a literal attempt to apply them for the RG trajectories characterized by pure imaginary s meets an immediate problem. In view of the condition (10.20) the function g_N (19.39) develops a simple pole whenever s belongs to the admissible set of pure imaginary values. Nevertheless we checked that (19.40) remains valid for finite $N \gg 1$ provided s is substituted by the "running coupling" $b(N)$. Then combining the relation with formula (9.11) that describes the large N asymptotics of $b(N)$, one can obtain the large N asymptotic behaviour of (19.72) for a Bethe state that becomes part of the discrete spectrum in the scaling limit. The result may be formulated in the following way.

Let $\boldsymbol{\Psi}_N$ be the RG trajectory, which in the scaling limit becomes the state $\boldsymbol{\psi}_{\bar{\rho},\rho,\bar{\nu},\nu}(\bar{\boldsymbol{w}},\boldsymbol{w})$ from the space $\mathcal{H}^{(\mathrm{disc})}$. To be precise, suppose that state belongs to the level subspace $\mathcal{V}_{\bar{\rho},\bar{\nu}}^{(\bar{\ell})}\otimes\mathcal{V}_{\rho,\nu}^{(\ell)}$ of a highest weight irrep which occurs in the $\overline{W}_\infty \otimes W_\infty$ decomposition of $\mathcal{H}^{(\mathrm{disc},\pm)}$ described in sec. 17.1. The levels of the state in the irrep, $\bar{\ell}$ and ℓ, could be any one of $\mathtt{L}$, $\mathtt{L}_\pm$ and $\bar{\mathtt{L}}$, $\bar{\mathtt{L}}_\pm$, respectively, depending on the situation at hand, see eq. (13.34). Then a straightforward calculation shows that as $N \to \infty$,

$$\big(\hat{\mathcal{D}}\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\Psi}_N, \boldsymbol{\Psi}_N\big)_* \asymp \mathcal{M}_N^{(\bar{\ell})}(\bar{\rho},\bar{\nu})\, \mathcal{M}_N^{(\ell)}(\rho,\nu)\,\big(1+o(1)\big) \\ \times\, \sigma\, f^{(-)}_{\bar{\rho},\bar{\nu}}\, f^{(-)}_{\rho,\nu}\, \big(\check{C}^{(\bar{A})}_{\bar{\rho},\bar{\nu}}(\bar{\boldsymbol{w}})\big)^2\, \big(\check{C}^{(A)}_{\rho,\nu}(\boldsymbol{w})\big)^2\, \frac{U_{\bar{\rho},\bar{\nu}}(\bar{\boldsymbol{w}})}{\check{R}_{\bar{\rho},\bar{\nu}}(\bar{\boldsymbol{w}})}\, \frac{U_{\rho,\nu}(\boldsymbol{w})}{\check{R}_{\rho,\nu}(\boldsymbol{w})}\,, \tag{19.73}$$

where

$$A = \operatorname{sgn}(\mathrm{i}\nu)\,, \qquad \bar{A} = \operatorname{sgn}(\mathrm{i}\bar{\nu})\,. \tag{19.74}$$

All of the dependence on N is contained in the sign factor $\sigma = (-1)^{N/2-S^z}$, as well as the first line of this equation and

$$\mathcal{M}_N^{(\ell)}(\rho,\nu) = C_0^{(\mathrm{alt})}\left(\frac{N}{2}\right)^{\frac{1}{6}}\left(\frac{\sqrt{2}C}{N}\right)^{\frac{4\nu^2}{n}+f(\rho)+4\ell}\left(\frac{2^{\frac{2}{n}}N}{N_0}\right)^{\frac{2|\nu|}{n}(n-2|\nu|)} \mathrm{e}^{\frac{1}{4}\mathcal{A}_2 N^2}\,. \tag{19.75}$$

The functions $\check{C}^{(\pm)}_{\rho,\nu}(\boldsymbol{w})$ and $\check{R}_{\rho,\nu}(\boldsymbol{w})$ are the eigenvalues of the operator $\check{\mathbf{C}}^{(\pm)}$ (15.8) and the reflection operator $\check{\mathbf{R}}$ (15.9), respectively. Since the latter are operators acting in the Fock spaces, $\mathcal{V}_{\rho,\nu}$ should be understood as a subspace of the Fock space according to eqs. (13.28) and (13.33). Finally $f^{(-)}_{\rho,\nu}$ in (19.73) does not depend on the chiral state in the irrep $\mathcal{V}_{\rho,\nu}$ and reads explicitly as

$$f^{(-)}_{\rho,\nu} = \frac{\Gamma\big(\frac{1}{2}+\rho-|\nu|\big)}{2\pi\,(n+2)^{2\nu^2/n}} \times \begin{cases} (-1)^a\, a! & \text{if} \quad \frac{1}{2}+\rho+|\nu| = -a = 0,-1,-2,\ldots \\ \dfrac{2\pi}{\Gamma(\frac{1}{2}+\rho+|\nu|)} & \text{otherwise} \end{cases} \tag{19.76}$$

The same holds true for the barred counterparts.

The factor $\big(\check{C}^{(\bar{A})}_{\bar{\rho},\bar{\nu}}(\bar{\boldsymbol{w}})\big)^2\,\big(\check{C}^{(A)}_{\rho,\nu}(\boldsymbol{w})\big)^2$ prevents one from interpreting the second line of (19.73) as the inner product $\langle\cdot,\cdot\rangle_{\mathrm{disc}}$, consistent with the conjugation conditions (19.65) in the $\overline{W}_\infty \otimes W_\infty$ - algebra, evaluated on the eigenstate $\boldsymbol{\psi}_{\bar{\rho},\rho,\bar{\nu},\nu}(\bar{\boldsymbol{w}},\boldsymbol{w})$ and its $\mathcal{DCPT}$ conjugate. However, the eigenvalues of the operators $\check{\mathbf{C}}^{(\pm)}$ satisfy the relations

$$\big(\check{C}^{(-)}_{\rho,\nu}(\boldsymbol{w})\big)^* = \check{C}^{(+)}_{\rho,\nu^*}(-\boldsymbol{w}^*)\,, \qquad \check{C}^{(+)}_{\rho,\nu}(\boldsymbol{w}) = \check{C}^{(-)}_{\rho,-\nu}(-\boldsymbol{w})\,, \tag{19.77}$$

which follow from (17.20) and (11.16). Hence $\big(\check{C}^{(\pm)}_{\rho,\nu}(\boldsymbol{w})\big)^* = \check{C}^{(\pm)}_{\rho,\nu}(\boldsymbol{w}^*)$ for pure imaginary ν. Introduce the inner product in $\mathcal{H}^{(\mathrm{disc})}$ using the basis $\boldsymbol{\psi}_{\bar{\rho},\rho,\bar{\nu},\nu}(\bar{\boldsymbol{w}},\boldsymbol{w}) \equiv \bar{\boldsymbol{\psi}}_{\bar{\rho},\bar{\nu}}(\bar{\boldsymbol{w}}) \otimes \boldsymbol{\psi}_{\rho,\nu}(\boldsymbol{w})$ with the chiral eigenstates normalized as in (19.24), via the formula

$$\big\langle\boldsymbol{\psi}_{\bar{\rho}',\rho',\bar{\nu}',\nu'}(\bar{\boldsymbol{w}}',\boldsymbol{w}'),\, \boldsymbol{\psi}_{\bar{\rho},\rho,\bar{\nu},\nu}(\bar{\boldsymbol{w}},\boldsymbol{w})\big\rangle_{\mathrm{disc}} = \sigma\, f^{(-)}_{\bar{\rho},\bar{\nu}}\, f^{(-)}_{\rho,\nu}\, \frac{U_{\bar{\rho},\bar{\nu}}(\bar{\boldsymbol{w}})}{\check{R}_{\bar{\rho},\bar{\nu}}(\bar{\boldsymbol{w}})}\, \frac{U_{\rho,\nu}(\boldsymbol{w})}{\check{R}_{\rho,\nu}(\boldsymbol{w})} \\ \times\, \delta_{\bar{\boldsymbol{w}}',\bar{\boldsymbol{w}}^*}\, \delta_{\boldsymbol{w}',\boldsymbol{w}^*}\, \delta_{\bar{\rho}',\bar{\rho}}\, \delta_{\rho',\rho}\, \delta_{\bar{\nu}',\bar{\nu}}\, \delta_{\nu',\nu}\,. \tag{19.78}$$

Then it is easy to see that the second line in (19.73) coincides with $\big\langle\hat{\mathcal{D}}\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\boldsymbol{\psi}'_{\bar{\rho},\rho,\bar{\nu},\nu},\, \boldsymbol{\psi}'_{\bar{\rho},\rho,\bar{\nu},\nu}\big\rangle_{\mathrm{disc}}$ for the state

$$\boldsymbol{\psi}'_{\bar{\rho},\rho,\bar{\nu},\nu}(\bar{\boldsymbol{w}},\boldsymbol{w}) = \big(\check{\bar{\mathbf{C}}}^{(\bar{A})}\otimes\check{\mathbf{C}}^{(A)}\big)\,\boldsymbol{\psi}_{\bar{\rho},\rho,\bar{\nu},\nu}(\bar{\boldsymbol{w}},\boldsymbol{w}) \qquad \big(A=\mathrm{sgn}(\mathrm{i}\nu)\,,\ \bar{A}=\mathrm{sgn}(\mathrm{i}\bar{\nu})\big)\,. \tag{19.79}$$

This way we conclude that there exists the limit

$$\underset{N\to\infty}{\mathrm{slim}}\ \Big(\mathcal{M}_N^{(\bar{\ell})}(\bar{\rho},\bar{\nu})\,\mathcal{M}_N^{(\ell)}(\rho,\nu)\Big)^{-\frac{1}{2}}\ \boldsymbol{\Psi}_N = \boldsymbol{\psi}'_{\bar{\rho},\rho,\bar{\nu},\nu}(\bar{\boldsymbol{w}},\boldsymbol{w})\,. \tag{19.80}$$

As was already discussed, the inner product described in the eigenbasis by formula (19.78) may be equivalently introduced through the conjugation conditions (19.65) supplemented by its value on the primary $\overline{W}_\infty\otimes W_\infty$ states:

$$\big\langle\boldsymbol{\psi}^{(\mathrm{vac})}_{\bar{\rho}',\rho',\bar{\nu}',\nu'},\,\boldsymbol{\psi}^{(\mathrm{vac})}_{\bar{\rho},\rho,\bar{\nu},\nu}\big\rangle_{\mathrm{disc}} = \sigma\ f^{(-)}_{\bar{\rho},\bar{\nu}}\ f^{(-)}_{\rho,\nu}\ Z_+\big(\tfrac{\bar{\rho}}{\sqrt{n+2}}\,\big|\,\sqrt{n+2}\big)\,Z_+\big(\tfrac{\rho}{\sqrt{n+2}}\,\big|\,\sqrt{n+2}\big)$$
$$\times\,\delta_{\bar{\rho}',\bar{\rho}}\,\delta_{\rho',\rho}\,\delta_{\bar{\nu}',\bar{\nu}}\,\delta_{\nu',\nu}\,. \tag{19.81}$$

Here $f^{(-)}_{\rho,\nu}$ is given in (19.76), while $Z_+(P|\beta)$ was defined in (6.17). As for the sign factor $\sigma=(-1)^{N/2-S^z}$ it depends on whether, in constructing the RG trajectories, we keep $N/2-S^z$ to be an even or an odd number.

Let's highlight an important point to take away from our investigation. We found that the scaling limit should be defined differently for the low energy states, which become part of the spaces $\mathcal{H}^{(\mathrm{cont})}$ and $\mathcal{H}^{(\mathrm{disc})}$. These sectors are naturally equipped by different inner products, which are induced by different conjugation conditions for the operators in the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model. All this suggests that if a description of the critical behaviour of the lattice system within the framework of a local CFT exists, the states from $\mathcal{H}^{(\mathrm{cont})}$ and $\mathcal{H}^{(\mathrm{disc})}$ can not be interpreted simultaneously as normalizable states within a single field theory.

Part III. Towards the QFT

20. Integrable and Hermitian structures for $c\to 2^-$

The integrable structure which occurred in our study of the scaling limit of the inhomogeneous six - vertex model with $\mathcal{Z}_2$ symmetry, has a deep relation to the AKNS classical integrable hierarchy. The latter includes some famous classically integrable partial differential equations such as the non-linear Schrödinger and the Lund-Regge (complex sin(h)-Gordon I) equation. To explain this relation, one should consider the $n\to+\infty$ limit, which can be understood as a classical limit with

$$\hbar=\frac{2\pi}{n} \tag{20.1}$$

playing the rôle of the Planck constant.

Let us rescale the field φ (5.7) and introduce $\phi(u)=\frac{1}{\sqrt{n}}\,\varphi(u)$ as well as the similarly defined field

$$\theta(u)=\frac{1}{\sqrt{n}}\bigg(\vartheta_0+b_0\,u+\mathrm{i}\sum_{m\neq 0}\frac{b_m}{m}\,\mathrm{e}^{-\mathrm{i}mu}\bigg) \tag{20.2}$$

with $[\,\vartheta_0,b_m\,]=\frac{\mathrm{i}}{2}\,\delta_{m,0}$. A simple calculation shows that

$$[\,\phi(u_1),\phi(u_2)\,]=[\,\theta(u_1),\theta(u_2)\,]=-\mathrm{i}\hbar\,\tfrac{1}{4}\,\epsilon(u_1-u_2)\,,\qquad [\,\phi(u_1),\theta(u_2)\,]=0\,, \tag{20.3}$$

where $\epsilon(u) = 2m + 1$ for $2\pi m < u < 2\pi(m+1)$ $(m \in \mathbb{Z})$. Applying the correspondence principle, $\mathrm{i}\hbar^{-1}[\cdot,\cdot] \mapsto \{\cdot,\cdot\}$, one concludes that ϕ and θ become classical fields in the large n limit subject to the Poisson Bracket (PB) relations

$$\{\phi(u_1), \phi(u_2)\} = \{\theta(u_1), \theta(u_2)\} = \tfrac{1}{4}\,\epsilon(u_1 - u_2)\ , \qquad \{\phi(u_1), \theta(u_2)\} = 0\ . \tag{20.4}$$

For the W_j currents, the bosonization formulae (16.2), (16.8) imply that as $n \to \infty$ they become classical fields built form $\partial\phi$ and $\partial\theta$:

$$W_j \to n^{j/2}\ W_j^{(cl)}\ , \tag{20.5}$$

where explicitly

$$\begin{aligned} W_2^{(cl)} &= (\partial\phi)^2 + (\partial\theta)^2 \\ W_3^{(cl)} &= 2\,(\partial\theta)^3 + 2\,(\partial\phi)^2\partial\theta + \mathrm{i}\big(\partial^2\phi\,\partial\theta - \partial\phi\,\partial^2\theta\big)\ . \end{aligned} \tag{20.6}$$

Recall that all the W currents can be generated from the parafermion fields. In turn, the fields $W_j^{(cl)}$ are conveniently expressed in terms of the classical counterparts of (16.15)[15]:

$$\xi_\pm = (\partial\theta \pm \mathrm{i}\partial\phi)\,\mathrm{e}^{\pm 2\theta} \qquad (n \to \infty)\ . \tag{20.7}$$

In particular,

$$\begin{aligned} W_2^{(cl)} &= \xi_+\,\xi_-\ , \qquad W_3^{(cl)} = \tfrac{1}{2}\,\big(\xi_-\,\partial\xi_+ - \xi_+\,\partial\xi_-\big) \\ W_4^{(cl)} &= \tfrac{2}{5}\,\big(\xi_+\,\partial^2\xi_- + \xi_-\,\partial^2\xi_+\big) - \tfrac{6}{5}\,\partial\xi_+\partial\xi_-\ . \end{aligned} \tag{20.8}$$

Using eqs. (20.7) and (20.4), it is straightforward to compute the PBs involving ξ_+, ξ_- and show that

$$\begin{aligned} \big\{\xi_\pm(u_1), \xi_\pm(u_2)\big\} &= \epsilon(u_1 - u_2)\ \xi_\pm(u_1)\,\xi_\pm(u_2) \\ \big\{\xi_\pm(u_1), \xi_\mp(u_2)\big\} &= -\delta'(u_1 - u_2) - \epsilon(u_1 - u_2)\ \xi_\pm(u_1)\,\xi_\mp(u_2)\ . \end{aligned} \tag{20.9}$$

The above relations combined with the formulae expressing $W_j^{(cl)}$ in terms of $\xi_\pm$ such as (20.8) are sufficient for deriving the Poisson algebra for the classical W currents. They provide a short cut to this algebra automatically satisfying the Jacobi and skew symmetry conditions, that would otherwise need to be obtained from the $c = 2 - \frac{6}{n+2} \to 2^-$ limit of the OPEs such as (16.12). In particular, it is straightforward to show that

$$\begin{aligned} &\big\{W_2^{(cl)}(u_1), W_2^{(cl)}(u_2)\big\} = -\big(W_2^{(cl)}(u_1) + W_2^{(cl)}(u_2)\big)\delta'(u_1 - u_2) \\ &\big\{W_3^{(cl)}(u_1), W_2^{(cl)}(u_2)\big\} = -3\ W_3^{(cl)}(u_1)\ \delta'(u_1 - u_2) - \partial W_3^{(cl)}(u_1)\delta(u_1 - u_2) \\ &\big\{W_3^{(cl)}(u_1), W_3^{(cl)}(u_2)\big\} = -\tfrac{1}{4}\,\big(W_2^{(cl)}(u_1) + W_2^{(cl)}(u_2)\big)\delta'''(u_1 - u_2) - \delta'(u_1 - u_2)\,\times \\ &\Big(W_4^{(cl)}(u_1) + W_4^{(cl)}(u_2) + 2\,W_2^{(cl)}(u_1)\,W_2^{(cl)}(u_2) - \tfrac{3}{20}\big(\partial^2\,W_2^{(cl)}(u_1) + \partial^2\,W_2^{(cl)}(u_2)\big)\Big)\ . \end{aligned} \tag{20.10}$$

Taking into account that $u = t + x$ and $W_j^{(cl)}(t, x) = W_j^{(cl)}(t + x)$, the latter may be understood as an infinite system of equal-time PB relations for the classical W currents.

[15] We will use the same symbol for the quantum and classical fields $\xi_\pm$, similar as with ϕ and θ.

One should keep in mind that $\xi_\pm(u)$ are quasiperiodic fields contrary to the $W_j^{(cl)}(u)$, which are periodic:

$$W_j^{(cl)}(u+2\pi) = W_j^{(cl)}(u)\,, \qquad \xi_\pm(u+2\pi) = B^{\pm 1}\,\xi_\pm(u)\,. \tag{20.11}$$

Using the "bosonization" formulae (20.7) one finds

$$\{B, \xi_\pm(u)\} = \pm 2B\,\xi_\pm(u)\,, \tag{20.12}$$

while

$$\big\{B, W_j^{(cl)}(u)\big\} = 0\,. \tag{20.13}$$

From the definition of the quantum field θ (20.2) and eq. (20.7) it is easy to see that B is the classical counterpart of $\mathrm{e}^{\frac{4\pi}{\sqrt{n}}b_0}$, whose eigenvalues coincide with $\mathrm{e}^{\frac{4\pi s}{n}}$ (13.6). The latter is equal to the eigenvalues of the quasi-shift operator in the scaling limit (see eq. (8.6)). For this reason, with some abuse of notation, we use the same symbol B for the dynamical variable defined through (20.11) as the one denoting the eigenvalues of $\mathbb{B}$.

Let's turn to the classical limit of the local IM (16.11), (16.14). As it follows from (20.5),

$$\mathbf{I}_m \to n^{(m+1)/2}\, I_m^{(cl)}\,, \tag{20.14}$$

where the explicit formula for the first few $I_m^{(cl)}$, expressed in terms of $\xi_\pm$, may be obtained from (20.8)

$$\begin{aligned} I_1^{(cl)} &= \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\, \xi_+\xi_- \\ I_2^{(cl)} &= \frac{1}{4}\int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\big(\xi_-\,\partial\xi_+ - \xi_+\,\partial\xi_-\big) \\ I_3^{(cl)} &= \frac{1}{5}\int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\big((\xi_+\xi_-)^2 - \partial\xi_+\partial\xi_-\big)\,. \end{aligned} \tag{20.15}$$

In general $I_m^{(cl)}$ are given by an integral over a real local density built from $\xi_\pm$ and their derivatives. In this work, we always assumed that the quantum IM were normalized as $\mathbf{I}_m = n^{\frac{m+1}{2}} \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\big((\partial\theta)^{m+1} + \ldots\big)$. One can show that this translates to

$$\begin{aligned} I_m^{(cl)} &= \frac{m\,\Gamma^2(\frac{m}{2})}{2\sqrt{\pi}\,\Gamma(\frac{1}{2}+m)}\int_0^{2\pi}\frac{\mathrm{d}u}{2\pi}\Big(\big(\xi_+\xi_-\big)^{\frac{m+1}{2}} + \ldots\Big) \\ &\qquad (m-\text{odd}) \\ I_m^{(cl)} &= \frac{(m+1)\,\Gamma^2(\frac{1}{2}+\frac{m}{2})}{4\sqrt{\pi}\,\Gamma(\frac{1}{2}+m)}\int_0^{2\pi}\frac{\mathrm{d}u}{2\pi}\ \Big(\big(\xi_+\xi_-\big)^{\frac{m}{2}-1}\big(\xi_-\partial\xi_+ - \xi_+\,\partial\xi_-\big) + \ldots\Big) \\ &\qquad (m-\text{even})\,. \end{aligned} \tag{20.16}$$

Here the "…" stands for the monomials which are of lower power in $\xi_\pm$ and their derivatives. Of course, all the $I_m^{(cl)}$ mutually Poisson commute with each other. This set coincides with the commuting family of local IM for the AKNS integrable hierarchy.

The key ingredient in the theory of classically integrable partial differential equations is the zero curvature or Lax representation. In the case under consideration the auxiliary linear problem is given by

$$\big(\partial - \boldsymbol{A}(u\,|\,\lambda_c)\big)\boldsymbol{\Phi} = 0 \tag{20.17}$$

with

$$\boldsymbol{A}(u\,|\,\lambda_c) = \xi_-\,\mathtt{e}_- - \xi_+\,\mathtt{e}_+ + \lambda_c\,\mathtt{h}\,. \tag{20.18}$$

Here $\mathtt{e}_\pm$ and $\mathtt{h}$ are the generators of the $\mathfrak{sl}_2$ algebra, $[\mathtt{h},\,\mathtt{e}_\pm] = \pm 2\mathtt{e}_\pm$ and $[\mathtt{e}_+,\,\mathtt{e}_-] = \mathtt{h}$, while λ_c is the auxiliary spectral parameter. We define the classical transfer matrix as the trace

$$\tau^{(cl)}(\lambda_c) = \mathrm{Tr}_{\frac{1}{2}}\Big[\, B^{-\frac{\mathtt{h}}{2}}\ \overleftarrow{\mathcal{P}}\exp\Big(\int_0^{2\pi} \mathrm{d}u\, \boldsymbol{A}(u\,|\,\lambda_c)\Big)\Big] \tag{20.19}$$

taken over the fundamental representation, which is indicated by the subscript $\frac{1}{2}$. The factor $B^{-\frac{\mathtt{h}}{2}}$ is inserted to take into account that

$$\boldsymbol{A}(u+2\pi\,|\,\lambda_c) = B^{\frac{\mathtt{h}}{2}}\,\boldsymbol{A}(u\,|\,\lambda_c)\,B^{-\frac{\mathtt{h}}{2}}\ , \tag{20.20}$$

which is a consequence of the quasiperiodicity condition (20.11). The classical transfer matrix (20.19) appears in the $n\to\infty$ limit of $\boldsymbol{\tau}(\lambda)$ (14.7). The precise relation may be motivated via a comparison of their large λ asymptotic expansions. Representing $\tau^{(cl)}(\lambda_c)$ in the form

$$\tau^{(cl)}(\lambda_c) = 2\cos\big(\nu(\lambda_c)\big)\,, \tag{20.21}$$

one can show (see, e.g., [63,66])

$$\nu(\lambda_c) \asymp -2\pi\mathrm{i}\lambda_c + \tfrac{\mathrm{i}}{2}\,\log(B) + 2\pi\mathrm{i}\sum_{m=1}^{\infty}\frac{2^m\,\Gamma(\frac{1}{2}+m)}{\sqrt{\pi}\,(m+1)!}\ I_m^{(cl)}\,\lambda_c^{-m} \qquad (\lambda_c\to\infty)\ , \tag{20.22}$$

where $I_m^{(\mathrm{cl})}$ are the classical local IM (20.16). The similar expansion for the quantum transfer matrix (15.1) involves the quantum IM. In view of the relation between $\mathbf{I}_m$ and $I_m^{(cl)}$ (20.14), this suggests

$$\boldsymbol{\tau}(\lambda)\to\tau^{(cl)}(\lambda_c) \quad \text{as} \qquad n\to\infty \quad \text{with } \lambda_c = (n+2)\,\lambda \quad \text{fixed}\,. \tag{20.23}$$

It is possible to justify the relation (20.23) by explicitly calculating the classical limit of $\boldsymbol{\tau}(\lambda)$ order by order in λ following the lines of the work [46]. This requires a study of $\boldsymbol{L}_{\frac{1}{2}}(\lambda) \equiv \pi_{\frac{1}{2}}\big(\boldsymbol{L}(\lambda)\big)$, entering into the definition of the transfer matrix (14.7), in the large n limit. Eq. (14.3) gives $\boldsymbol{L}(\lambda)$ as a path ordered exponent, i.e., a series expansion in $\sqrt{\lambda}$ whose coefficients are ordered integrals over the vertex operators. However, as was already mentioned, the ordered integrals diverge for any $n>0$ and hence (14.3) is not literally applicable for taking the classical limit. Instead, each coefficient of the formal series $\boldsymbol{L}(\lambda)$ should be understood via analytic continuation in complex n, which may be achieved by re-writing the ordered integrals in terms of the contour integrals. In ref. [46] it was explained how to take the $n \to \infty$ limit of expressions involving the contour integrals. This results in the classical version of $\boldsymbol{L}(\lambda)$ as a series expansion in $\sqrt{\lambda}$ whose coefficients involve multifold integrals over the classical fields $\mathrm{e}^{\pm 2\mathrm{i}\phi}$ and $\partial\theta\mathrm{e}^{-2\mathrm{i}\phi}$. To compare the Taylor series for the classical limit of $\boldsymbol{\tau}(\lambda)$ obtained in this way with the r.h.s. of (20.23), one should apply a gauge transformation to the connection (20.18),

$$\boldsymbol{A} \mapsto \boldsymbol{G}^{-1}\boldsymbol{A}\,\boldsymbol{G} - \boldsymbol{G}^{-1}\partial\boldsymbol{G}\,, \tag{20.24}$$

and rewrite $\tau^{(cl)}(\lambda_c)$ in a way that is suitable for a small λ_c expansion. Using the matrix

$$\boldsymbol{G}(u) = \mathrm{e}^{\theta(u)\,\mathtt{h}}\,\mathrm{e}^{\frac{\pi}{4}(\mathtt{e}_+ - \mathtt{e}_-)}\,\mathrm{e}^{\mathrm{i}\phi(u)\,\mathtt{h}}\,, \tag{20.25}$$

a simple calculation shows that

$$B^{-\frac{\mathtt{h}}{2}}\,\overleftarrow{\mathcal{P}}\exp\left(\int_0^{2\pi}\mathrm{d}u\,\boldsymbol{A}(u\,|\,\lambda_c)\right) = \boldsymbol{G}(0)\,\lambda_c^{-\frac{\mathtt{h}}{4}}\,\left(\mathrm{e}^{\mathrm{i}\pi P\mathtt{h}}\,\boldsymbol{L}^{(cl)}(\lambda_c)\right)\,\lambda_c^{+\frac{\mathtt{h}}{4}}\,\boldsymbol{G}^{-1}(0) \tag{20.26}$$

with

$$\boldsymbol{L}^{(cl)}(\lambda_c) = \lambda_c^{+\frac{\mathtt{h}}{4}}\,\mathrm{e}^{\mathrm{i}\pi P\mathtt{h}}\,\overleftarrow{\mathcal{P}}\exp\left(\int_0^{2\pi}\mathrm{d}u\,\left(-2\partial\theta\,\mathrm{e}^{-2\mathrm{i}\phi}\,\mathtt{e}_+ + \lambda_c\left(\mathrm{e}^{+2\mathrm{i}\phi}\mathtt{e}_- + \mathrm{e}^{-2\mathrm{i}\phi}\mathtt{e}_+\right)\right)\right)\lambda_c^{-\frac{\mathtt{h}}{4}} \tag{20.27}$$

and P stands for the zero-mode momentum of the field ϕ:

$$P = \int_0^{2\pi}\frac{\mathrm{d}u}{2\pi}\,\partial\phi\,. \tag{20.28}$$

This way one obtains

$$\tau^{(cl)}(\lambda_c) = \mathrm{Tr}_{\frac{1}{2}}\left[\mathrm{e}^{\mathrm{i}\pi P\mathtt{h}}\,\boldsymbol{L}^{(cl)}(\lambda_c)\right]. \tag{20.29}$$

Notice that formally setting $n \to \infty$ into the path-ordered exponential (14.3) would reproduce the r.h.s. of (20.27) without the last term in the exponent, $\lambda_c\,\mathrm{e}^{-2\mathrm{i}\phi}\mathtt{e}_+$. However taking the classical limit of $\boldsymbol{L}(\lambda)$ as outlined above, with the ordered integrals being analytically regularized, we have checked that the first few terms in the Taylor series expansion of the classical limit of $\boldsymbol{\tau}(\lambda)$ reproduce the corresponding terms for $\tau^{(cl)}(\lambda_c)$ extracted from eqs. (20.29) and (20.27).

The monodromy matrix in the gauge (20.27) possesses a remarkable property. Namely, it obeys the Sklyanin exchange relations [65,66]

$$\left\{\boldsymbol{L}^{(cl)}(\lambda_c)\overset{\otimes}{,}\boldsymbol{L}^{(cl)}(\lambda_c')\right\} = \left[\boldsymbol{L}^{(cl)}(\lambda_c)\otimes\boldsymbol{L}^{(cl)}(\lambda_c'),\,\boldsymbol{r}\left(\sqrt{\lambda_c/\lambda_c'}\,\right)\right] \tag{20.30}$$

with the classical R-matrix

$$\boldsymbol{r}(\rho) = \frac{1}{\rho-\rho^{-1}}\left(\mathtt{e}_+\otimes\mathtt{e}_- + \mathtt{e}_-\otimes\mathtt{e}_+ + \tfrac{1}{4}(\rho+\rho^{-1})\,\mathtt{h}\otimes\mathtt{h}\right). \tag{20.31}$$

By expanding $\boldsymbol{L}^{(cl)}(\lambda_c)$ and $\boldsymbol{L}^{(cl)}(\lambda_c')$ as a series in $\sqrt{\lambda_c}$ and $\sqrt{\lambda_c'}$, respectively, and also $\boldsymbol{r}\left(\sqrt{\lambda_c/\lambda_c'}\right)$ say in the domain $|\lambda_c| < |\lambda_c'|$, eq. (20.30) can be checked order by order in these two variables. Note that (20.30) does not assume any choice of representation for the $\mathfrak{sl}_2$ generators. Specialized to the finite dimensional representation $\pi_j \otimes \pi_{j'}$, it becomes the classical counterpart of the Yang-Baxter algebra (14.5) with $\boldsymbol{L}^{(cl)}(\lambda_c)$ being the classical version of the operator (14.3), i.e.,

$$\boldsymbol{L}(\lambda) \to \boldsymbol{L}^{(cl)}(\lambda_c) \qquad (n \to \infty \quad \text{with} \quad \lambda_c = (n+2)\,\lambda \quad \text{fixed}) . \tag{20.32}$$

In the above discussion of the classical limit, $\xi_\pm$ have been treated as unrelated complex fields. There are two natural reality constraints which can be imposed on them that are consistent with the Poisson algebra (20.9). Namely,

$$\begin{aligned} &(\mathrm{I}): \quad \big(\xi_\pm(u)\big)^* = \xi_\pm(u^*)\,, \qquad B^* = B \\ &(\mathrm{II}): \quad \big(\xi_\pm(u)\big)^* = \xi_\mp(u^*)\,, \qquad B^* = B^{-1}\,. \end{aligned} \tag{20.33}$$

These imply the following reality conditions for the classical W-currents

$$\begin{aligned} &(\mathrm{I}): \quad \big(W_j^{(cl)}(u)\big)^* = W_j^{(cl)}(u^*) \\ &(\mathrm{II}): \quad \big(W_j^{(cl)}(u)\big)^* = (-1)^j\ W_j^{(cl)}(u^*) \end{aligned} \tag{20.34}$$

and for the classical transfer matrix

$$\begin{aligned} &(\mathrm{I}): \quad \big(\tau^{(cl)}(\lambda_c)\big)^* = \tau^{(cl)}(\lambda_c^*) \\ &(\mathrm{II}): \quad \big(\tau^{(cl)}(\lambda_c)\big)^* = \tau^{(cl)}(-\lambda_c^*)\,. \end{aligned} \tag{20.35}$$

Several comments are in order here. The conjugation (I) in (20.34) corresponds to the classical limit of the conjugation condition $\big[W_j(u)\big]^\star = W_j(u^*)$, which occurred in our study of the Hermitian structure for the space $\mathcal{H}^{(\mathrm{cont})}$ (see eq. (19.60)). Similarly, conjugation (II) is the classical version of $\big[W_j(u)\big]^* = (-1)^j\ W_j(u^*)$ (19.65) for the spaces $\mathcal{H}^{(\mathrm{disc},\pm)}$. It should also be pointed out that for both reality conditions $\tau^{(cl)}(0)$ is real. Furthermore, from (20.29) it follows that $\tau^{(cl)}(0) = 2\cos(2\pi P)$. In light of our previous discussion of the spectrum of the quantum transfer matrix $\boldsymbol{\tau}(\lambda)$ in the spaces $\mathcal{H}^{(\mathrm{cont})}$ and $\mathcal{H}^{(\mathrm{disc},\pm)}$ we'll take P to be real and assume that

$$-2 < \tau^{(cl)}(0) < 2\,. \tag{20.36}$$

The last comment serves to make a link to integrable partial differential equations. When reality condition (II) is imposed, i.e., $\xi_+ = \xi,\ \xi_- = \xi^*$, the Hamiltonian flow generated by the classical local IM $I_2^{(cl)} = -\big(I_2^{(cl)}\big)^*$ w.r.t. the Poisson structure (20.9) coincides with the nonlinear Schrödinger equation in the attractive (focusing) regime [64,66]:

$$\mathrm{i}\partial_\tau \xi = \{\xi, I_2^{(cl)}\} = -\partial^2\xi - |\xi|^2\,\xi\,. \tag{20.37}$$

The repulsive regime, where the sign in front of the non-linear term is flipped, is related to the $c \to 2^+$ limit of the W_∞ - algebra.

The classical limit of the $\overline{W}_\infty$ - algebra is described in the same way. In particular, for $\bar{\boldsymbol{\tau}}(\bar\lambda)$ (14.7), whose action is non-trivial on the left irrep of $\overline{W}_\infty \otimes W_\infty$, one has

$$\bar{\boldsymbol{\tau}}(\bar\lambda) \to \bar\tau^{(cl)}(\bar\lambda_c) \qquad (n \to \infty \quad \text{with} \quad \bar\lambda_c = (n+2)\,\bar\lambda \quad \text{fixed}) . \tag{20.38}$$

Here the classical transfer matrix reads as

$$\bar{\tau}^{(cl)}(\bar{\lambda}_c) = \mathrm{Tr}_{\frac{1}{2}}\Big[\bar{\boldsymbol{L}}^{(cl)}(\bar{\lambda}_c)\,\mathrm{e}^{-\mathrm{i}\pi\bar{P}\mathtt{h}}\Big] \qquad \text{with} \qquad \bar{P} = \int_0^{2\pi}\frac{\mathrm{d}\bar{u}}{2\pi}\,\bar{\partial}\bar{\phi}\ , \tag{20.39}$$

while $\bar{\boldsymbol{L}}^{(cl)}(\bar{\lambda}_c)$ stands for the path ordered exponent that appears in the classical limit of $\bar{\boldsymbol{L}}(\bar{\lambda})$ (14.3):

$$\bar{\boldsymbol{L}}^{(cl)}(\bar{\lambda}_c) = \bar{\lambda}_c^{+\frac{\mathtt{h}}{4}}\,\vec{\mathcal{P}}\exp\bigg(\int_0^{2\pi}\mathrm{d}\bar{u}\,\Big(-2\bar{\partial}\bar{\theta}\,\mathrm{e}^{-2\mathrm{i}\bar{\phi}}\,\mathtt{e}_+ + \bar{\lambda}_c\,\big(\mathrm{e}^{+2\mathrm{i}\bar{\phi}}\mathtt{e}_- + \mathrm{e}^{-2\mathrm{i}\bar{\phi}}\mathtt{e}_+\big)\Big)\bigg)\,\mathrm{e}^{-\mathrm{i}\pi\bar{P}\mathtt{h}}\ \bar{\lambda}_c^{-\frac{\mathtt{h}}{4}} \tag{20.40}$$

It follows from the Yang-Baxter algebra (14.5), that $\bar{\boldsymbol{L}}^{(cl)}(\bar{\lambda}_c)$ satisfies the Sklyanin exchange relations

$$\big\{\bar{\boldsymbol{L}}^{(cl)}(\bar{\lambda}_c)\,\overset{\otimes}{,}\,\bar{\boldsymbol{L}}^{(cl)}(\bar{\lambda}'_c)\big\} = -\Big[\,\bar{\boldsymbol{L}}^{(cl)}(\bar{\lambda}_c)\otimes\bar{\boldsymbol{L}}^{(cl)}(\bar{\lambda}'_c),\,\boldsymbol{r}\Big(\sqrt{\bar{\lambda}_c/\bar{\lambda}'_c}\,\Big)\Big] \tag{20.41}$$

with $\boldsymbol{r}(\rho)$ the same as in (20.31).

The classical transfer matrix may be expressed in terms of the quasiperiodic fields

$$\bar{\xi}_\pm = (\bar{\partial}\bar{\theta} \pm \mathrm{i}\bar{\partial}\bar{\phi})\,\mathrm{e}^{\pm 2\bar{\theta}}\ : \qquad \bar{\xi}_\pm(u+2\pi) = \bar{B}^{\pm 1}\,\bar{\xi}_\pm(u)\ . \tag{20.42}$$

This is achieved through the relation similar to (20.26). Namely,

$$\vec{\mathcal{P}}\exp\bigg(\int_0^{2\pi}\mathrm{d}\bar{u}\,\bar{\boldsymbol{A}}(\bar{u}\,|\,\bar{\lambda}_c)\bigg)\,\bar{B}^{+\frac{\mathtt{h}}{2}} = \bar{\boldsymbol{G}}(0)\,\bar{\lambda}_c^{-\frac{\mathtt{h}}{4}}\,\big(\bar{\boldsymbol{L}}^{(cl)}(\bar{\lambda}_c)\,\mathrm{e}^{-\mathrm{i}\pi\bar{P}\mathtt{h}}\big)\,\bar{\lambda}_c^{+\frac{\mathtt{h}}{4}}\,\bar{\boldsymbol{G}}^{-1}(0) \tag{20.43}$$

with $\bar{\boldsymbol{G}}(\bar{u}) = \mathrm{e}^{\bar{\theta}\mathtt{h}}\,\mathrm{e}^{-\frac{\pi}{4}(\mathtt{e}_+-\mathtt{e}_-)}\,\mathrm{e}^{\mathrm{i}\bar{\phi}\mathtt{h}}$ and

$$\bar{\boldsymbol{A}}(\bar{u}\,|\,\lambda_c) = \bar{\xi}_-\,\mathtt{e}_- - \bar{\xi}_+\,\mathtt{e}_+ - \bar{\lambda}_c\,\mathtt{h}\,. \tag{20.44}$$

The above equations, together with (20.39), yield

$$\bar{\tau}^{(cl)}(\bar{\lambda}_c) = \mathrm{Tr}_{\frac{1}{2}}\Big[\,\vec{\mathcal{P}}\exp\Big(\int_0^{2\pi}\mathrm{d}\bar{u}\,\bar{\boldsymbol{A}}(\bar{u}\,|\,\bar{\lambda}_c)\Big)\,\bar{B}^{+\frac{\mathtt{h}}{2}}\Big]\,. \tag{20.45}$$

Finally, for the left chirality the reality conditions are only notationally different from eqs. (20.33)-(20.35).

21. Lorentzian black hole NLSM

21.1. The classical field theory

Let's consider anew the Poisson algebra, whose first few PBs are given in eq. (20.10) with the $W_j^{(cl)}(u)$ being real classical fields. As it follows from the first two equations in (20.8) one can introduce, at least locally, the real fields $\xi_\pm$ through the relation

$$\big(\xi_\pm\big)^2 = W_2^{(cl)}\ \exp\bigg(\pm 2\int^u \mathrm{d}u\ W_3^{(cl)}/W_2^{(cl)}\bigg)\,. \tag{21.1}$$

Together with $W_2^{(cl)}$ and $W_3^{(cl)}$, the above formula involves a real integration constant. It can be interpreted as a dynamical variable conjugated to

$$\log(B) = \oint \mathrm{d}u \; W_3^{(cl)} \big/ W_2^{(cl)} \,. \tag{21.2}$$

The latter belongs to the center of the classical W_∞ algebra. Note that B must be real and, furthermore, we take it to be positive. Then the Poisson structure for the classical W currents is lifted to the Poisson algebra (20.9) for the real quasiperiodic fields $\xi_\pm$. The center of (20.9) is generated by the real constant

$$\tau^{(cl)}(0) = \mathrm{Tr}_{\frac{1}{2}}\Big[B^{-\frac{\mathfrak{h}}{2}}\; \mathbf{\Omega}(2\pi) \Big] \;: \quad -2 < \tau^{(cl)}(0) = 2\,\cos(2\pi P) < 2\,, \tag{21.3}$$

where we use the notation

$$\mathbf{\Omega}(u) = \overleftarrow{\mathcal{P}}\exp\bigg(\int_0^u \mathrm{d}u\, \big(\xi_-\, \mathsf{e}_- - \xi_+\, \mathsf{e}_+\big)\bigg)\,. \tag{21.4}$$

All the above holds true for the left chirality. In particular, the fields $\bar{\xi}_\pm$ satisfy the Poisson algebra similar to (20.9),

$$\begin{aligned}
&\big\{\bar{\xi}_\pm(\bar{u}_1), \bar{\xi}_\pm(\bar{u}_2)\big\} = \epsilon(\bar{u}_1 - \bar{u}_2)\, \bar{\xi}_\pm(\bar{u}_1)\, \bar{\xi}_\pm(\bar{u}_2)\,, \\
&\big\{\bar{\xi}_\pm(\bar{u}_1), \bar{\xi}_\mp(\bar{u}_2)\big\} = -\delta'(\bar{u}_1 - \bar{u}_2) - \epsilon(\bar{u}_1 - \bar{u}_2)\, \bar{\xi}_\pm(\bar{u}_1)\, \bar{\xi}_\mp(\bar{u}_2)
\end{aligned} \tag{21.5}$$

and Poisson commute with $\xi_\pm$:

$$\big\{\xi_\pm(u_1),\, \bar{\xi}_\pm(\bar{u}_2)\big\} = \{\xi_\pm(u_1),\, \bar{\xi}_\mp(\bar{u}_2)\} = 0\,. \tag{21.6}$$

The center of the Poisson algebra for $\xi_\pm$, $\bar{\xi}_\pm$ is generated by $\tau^{(cl)}(0)$ together with

$$\bar{\tau}^{(cl)}(0) = \mathrm{Tr}_{\frac{1}{2}}\Big[\bar{\mathbf{\Omega}}(2\pi)\; \bar{B}^{+\frac{\mathfrak{h}}{2}} \Big] \;: \quad -2 < \bar{\tau}^{(cl)}(0) = 2\,\cos(2\pi \bar{P}) < 2\,, \tag{21.7}$$

where

$$\bar{\mathbf{\Omega}}(\bar{u}) = \vec{\mathcal{P}}\exp\bigg(\int_0^{\bar{u}} \mathrm{d}\bar{u}\, \big(\bar{\xi}_-\, \mathsf{e}_- - \bar{\xi}_+\, \mathsf{e}_+\big)\bigg)\,. \tag{21.8}$$

As with B we assume that $\bar{B}$ is positive and, moreover, we'll impose the constraint

$$\bar{B} = B > 0\,. \tag{21.9}$$

The above gives a sketch of the basic properties of the phase space (more precisely the algebra of functions on the phase space) for a class of dynamical systems. Having in mind our purpose of identifying the CFT governing the critical behaviour of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model, we take the classical Hamiltonian as

$$H^{(cl)} = \int_0^{2\pi} \mathrm{d}x\, \Big(W_2^{(cl)}(x) + \overline{W}_2^{(cl)}(x) \Big)\,. \tag{21.10}$$

An immediate question arises as to the possibility of a Lagrangian description of such a dynamical system. In connection with this, it is useful to turn to the known classical Lagrangian field theory possessing the same type of Hamiltonian structure.

Consider the path ordered exponents $\boldsymbol{\Omega}(u)$ (21.4) and $\bar{\boldsymbol{\Omega}}(\bar{u})$ (21.8) with the $\mathfrak{sl}_2$ generators specialized to be in the fundamental representation $\pi_{\frac{1}{2}}(\mathtt{e}_\pm)=\sigma^\pm$ and $\pi_{\frac{1}{2}}(\mathtt{h})=\sigma^3$. Since $\xi_\pm$, $\bar{\xi}_\pm$ are real fields, the 2×2 matrices $\boldsymbol{\Omega}_{\frac{1}{2}}=\pi_{\frac{1}{2}}\big(\boldsymbol{\Omega}\big)$, $\bar{\boldsymbol{\Omega}}_{\frac{1}{2}}=\pi_{\frac{1}{2}}\big(\bar{\boldsymbol{\Omega}}\big)$ have real elements and their determinants are equal to one. With $\boldsymbol{g}_{\frac{1}{2}}(0)\in \mathrm{SL}(2,\mathbb{R})$ being an arbitrary constant matrix, introduce $\boldsymbol{g}_{\frac{1}{2}}(t,x)\in\mathrm{SL}(2,\mathbb{R})$:

$$\boldsymbol{g}_{\frac{1}{2}}(t,x)=\boldsymbol{\Omega}_{\frac{1}{2}}(t+x)\,\boldsymbol{g}_{\frac{1}{2}}(0)\,\bar{\boldsymbol{\Omega}}_{\frac{1}{2}}(t-x)\,. \tag{21.11}$$

Writing it in the form

$$\boldsymbol{g}_{\frac{1}{2}}=\begin{pmatrix} A & U\\ -V & D\end{pmatrix} \tag{21.12}$$

one finds via a straightforward computation that the real functions $U=U(t,x)$ and $V=V(t,x)$ satisfy the closed system of partial differential equations

$$(1-UV)\,\partial\bar{\partial}U=-V\,\partial U\bar{\partial}U\,,\qquad\qquad (1-UV)\,\partial\bar{\partial}V=-U\,\partial V\bar{\partial}V\,, \tag{21.13}$$

where $\partial=\frac{1}{2}(\partial_t+\partial_x)$ and $\bar{\partial}=\frac{1}{2}(\partial_t-\partial_x)$. The diagonal entries are determined through the relations

$$\partial\log\Big(\frac{A}{D}\Big)=\frac{U\partial V-V\partial U}{1-UV}\,,\qquad \bar{\partial}\log\Big(\frac{A}{D}\Big)=-\frac{U\bar{\partial}V-V\bar{\partial}U}{1-UV}\,,\quad AD=1-UV \tag{21.14}$$

The equations of motion (21.13) are the Euler-Lagrange equations corresponding to the Lagrangian density

$$\mathcal{L}=\frac{1}{2}\,\frac{\partial_t U\partial_t V-\partial_x V\partial_x U}{1-UV} \tag{21.15}$$

and the fields satisfy the reality conditions

$$\big(U(t,x)\big)^*=U(t,x)\,,\qquad\qquad \big(V(t,x)\big)^*=V(t,x)\,. \tag{21.16}$$

The latter is the Lagrangian density for the Non-Linear Sigma Model (NLSM) whose target space coincides with the so-called Lorentzian black hole. In the work [20] this model was obtained by gauging a non-compact one dimensional subgroup of the classical $\mathrm{SL}(2,\mathbb{R})$ WZW model.

Let's explain how the Lagrangian density (21.15) leads to the Poisson structure (20.9), (21.5) and (21.6). As it follows from eq. (21.11) the fields $\xi_\pm$ and $\bar{\xi}_\pm$ are given by

$$\xi_-\mathtt{e}_- -\xi_+\mathtt{e}_+=\partial\boldsymbol{g}\,\boldsymbol{g}^{-1}\,,\qquad \bar{\xi}_-\mathtt{e}_- -\bar{\xi}_+\mathtt{e}_+=\boldsymbol{g}^{-1}\,\bar{\partial}\boldsymbol{g}\,. \tag{21.17}$$

This allows one to express $\xi_\pm$, $\bar{\xi}_\pm$ in terms of U and V:

$$\begin{aligned}&\xi_+=U\,\partial A-A\,\partial U\,,\qquad \xi_-=V\,\partial D-D\,\partial V\\ &\bar{\xi}_+=U\,\bar{\partial}D-D\,\bar{\partial}U\,,\qquad \bar{\xi}_-=V\,\bar{\partial}A-A\,\bar{\partial}V\end{aligned} \tag{21.18}$$

and the equations of motion (21.13) as well as (21.14) imply that $\xi_\pm$, $\bar{\xi}_\pm$ are chiral fields:

$$\bar{\partial}\xi_\pm=0\,,\qquad \partial\bar{\xi}_\pm=0\,. \tag{21.19}$$

Note that in (21.17), we dropped the index $\frac{1}{2}$ denoting the fundamental representation of $\mathfrak{sl}_2$, since it remains valid as a relation in the Lie algebra without reference to a particular representation. The Lagrangian density (21.15) induces a canonical Poisson structure:

$$\big\{\Pi_U(t,x_1),\ U(t,x_2)\big\} = \delta(x_1-x_2)\,, \qquad \big\{\Pi_V(t,x_1),\ V(t,x_2)\big\} = \delta(x_1-x_2)\,, \tag{21.20}$$

where

$$\Pi_U = \frac{1}{2}\,\frac{\partial_t V}{1-UV}\,, \qquad \Pi_V = \frac{1}{2}\,\frac{\partial_t U}{1-UV}\,. \tag{21.21}$$

Combining this with (21.18) we indeed obtain the PB relations (20.9), (21.5), (21.6), where $u = t+x$, $\bar{u} = t-x$ and t is assumed to be fixed.

Having at hand the explicit formula (21.18) one can construct out of the fundamental fields U and V the classical W currents. Clearly they are local chiral fields:

$$W_j^{(cl)}(t,x) = W_j^{(cl)}(t+x)\,, \qquad \overline{W}_j^{(cl)}(t,x) = \overline{W}_j^{(cl)}(t-x)\,. \tag{21.22}$$

Since in all our previous discussions the $W_j^{(cl)}(u)$ were assumed to be periodic, we supplement (21.15) by the periodic boundary conditions

$$U(t,x+2\pi) = U(t,x)\,, \qquad V(t,x+2\pi) = V(t,x) \tag{21.23}$$

and take the classical action to be

$$S_{\text{LBH}} = \frac{1}{2\hbar}\int \mathrm{d}t \int_0^{2\pi} \mathrm{d}x\ \frac{\partial_t U\partial_t V - \partial_x V\partial_x U}{1-UV} \qquad \big(\hbar = 2\pi/n \to 0^+\big)\,. \tag{21.24}$$

The Hamiltonian is then given by (21.10). That the fields U and V are real is consistent with the reality condition $\big(\xi_\pm(u)\big)^* = \xi_\pm(u^*)$, $\big(\bar{\xi}_\pm(\bar{u})\big)^* = \bar{\xi}_\pm(\bar{u}^*)$ and, in turn,

$$\big(W_j^{(cl)}(t+x)\big)^* = W_j^{(cl)}(t+x)\,, \qquad \big(\overline{W}_j^{(cl)}(t-x)\big)^* = \overline{W}_j^{(cl)}(t-x)\,. \tag{21.25}$$

This way the classical field theory defined by the action (21.24) where U and V are real periodic fields reproduces the Poisson structure, Hamiltonian and reality conditions occurring in the scaling limit of the lattice model in the sector $\mathcal{H}^{(\text{cont})}$ and with the central charge $c = 2-\frac{6}{n+2} \to 2^-$ as $n \to +\infty$.

Our qualitative discussion of the Poisson structure suggests that the phase space for the Lorentzian black hole NLSM (21.24) is made up of the symplectic leaves, $\Gamma_{\bar{P},P,B}$, labeled by the real numbers P, $\bar{P}$ and B. On each leaf the symplectic form is non-degenerate. The algebra of functions on the leaf, $\Gamma^\star_{\bar{P},P,B}$, is generated by the currents $W_j^{(cl)}(u)$ and $\overline{W}_j^{(cl)}(\bar{u})$, subject to the reality conditions (21.25), while the Poisson algebra on $\Gamma^\star_{\bar{P},P,B}$ is fully specified by the PBs (20.10) for the W currents, the similar relations for the left chiral currents as well as $\{W_j^{(cl)}(u), \overline{W}_{j'}^{(cl)}(\bar{u})\} = 0$. To get some insight into the global structure of the phase space of the model, it is useful to consider basic solutions of the classical equations of motion. These may be constructed using (21.11) and (21.12). First of all, in the “bosonization” formulae (20.7), (20.42) we set $\theta = \bar{\theta} = -\frac{\mathrm{i}\pi}{4}$ and $\partial\phi = P$, $\bar{\partial}\bar{\phi} = \bar{P}$. Then $\xi_\pm = P$ and $\bar{\xi}_\pm = \bar{P}$ become space-time independent real constants. Eqs. (21.11), (21.12) with $\boldsymbol{g}_{\frac{1}{2}}(0) = -\mathbf{1}$ yield

$$U(t,x) = V(t,x) = \sin\big((P+\bar{P})\,t + (P-\bar{P})\,x\big) \tag{21.26}$$

and one can easily see that the equations of motion (21.13) are indeed satisfied. The periodic boundary condition (21.23) requires that the difference $P - \bar{P}$ be an integer. It hints that the real numbers P and $\bar{P}$ labeling the symplectic leaves might not be arbitrary, but obey the condition:

$$P - \bar{P} = 0, \pm 1, \pm 2, \dots \,. \tag{21.27}$$

Other hints provided by the explicit solutions (21.26) concern the action of the global symmetries of the classical field theory on its phase space. There are two evident space-time symmetry transformations, $\mathcal{T}$ and $\mathcal{P}$, which are defined as

$$\begin{aligned} \mathcal{T} :\quad & U(t,x) \mapsto \ \ U(-t,x)\,, \qquad V(t,x) \mapsto \ \ V(-t,x) \\ \mathcal{P} :\quad & U(t,x) \mapsto -U(t,-x)\,, \qquad V(t,x) \mapsto -V(t,-x)\,. \end{aligned} \tag{21.28}$$

The extra sign in the definition of $\mathcal{P}$ is a matter of convention since the transformation

$$\mathcal{U} :\quad U \mapsto -U\,, \qquad V \mapsto -V \tag{21.29}$$

also leaves the action (21.24) invariant. The basic solutions (21.26) are unchanged under the $\mathcal{PT}$ transformation. More generally, we will assume that two solutions related via $\mathcal{PT}$ belong to the same symplectic leaf, i.e.,

$$\mathcal{PT} :\quad \Gamma_{\bar{P},P,B} \mapsto \Gamma_{\bar{P},P,B}\,. \tag{21.30}$$

Since (21.29) does not affect the W currents, it will likewise be assumed that

$$\mathcal{U} :\quad \Gamma_{\bar{P},P,B} \mapsto \Gamma_{\bar{P},P,B}\,. \tag{21.31}$$

The action of $\mathcal{P}$ and $\mathcal{T}$ on the fundamental fields, as described by formula (21.28), induces the action of these transformations on $\Gamma_{\bar{P},P,B}$. We make the assumption that two solutions related through $\mathcal{P}$ or $\mathcal{T}$ separately belong to different symplectic leaves. A brief examination of (21.26) motivates that

$$\begin{aligned} \mathcal{T} :\quad & \Gamma_{\bar{P},P,B} \mapsto \Gamma_{-P,-\bar{P},B} \\ \mathcal{P} :\quad & \Gamma_{\bar{P},P,B} \mapsto \Gamma_{P,\bar{P},B}\,. \end{aligned} \tag{21.32}$$

An immediate consequence is that $\mathcal{PT}$ maps $\Gamma_{\bar{P},P,B}$ to $\Gamma_{-\bar{P},-P,B}$. Consistency with the condition (21.30) requires the following identification to be made

$$\Gamma_{\bar{P},P,B} \equiv \Gamma_{-\bar{P},-P,B}\,. \tag{21.33}$$

With this important property one can always take

$$P + \bar{P} \geq 0 \tag{21.34}$$

without loss of generality. Formula (21.32) in addition implies that if the phase space contains the leaf $\Gamma_{\bar{P},P,B}$ it must also contain $\Gamma_{P,\bar{P},B}$.

Another global symmetry of the action is the $\mathcal{Z}_2$ transformation which interchanges the fields U and V:

$$\mathcal{D} :\quad U \mapsto V\,, \qquad V \mapsto U\,. \tag{21.35}$$

In turn, $\xi_\pm \mapsto \xi_\mp$ and taking into account eq. (20.11), its action on the symplectic leaves is given by

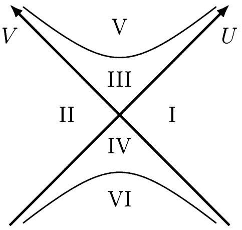

Fig. 16. Space-time diagram for the Lorentzian black hole (21.39). The cross defined by the equation $UV = 0$ is a horizon, while the metric possesses a physical singularity on the hyperbola $UV = 1$.

$$\mathcal{D}: \quad \Gamma_{\bar{P},P,B} \mapsto \Gamma_{\bar{P},P,B^{-1}} . \tag{21.36}$$

Finally, there is one more evident symmetry. The classical action (21.24) remains unchanged under the transformation

$$\mathcal{R}_a: \quad U \mapsto a^{+1}\, U\,, \qquad V \mapsto a^{-1}\, V \qquad \text{with} \qquad a > 0\,. \tag{21.37}$$

This acts on the non-local fields as $\xi_\pm \mapsto a^{\pm 1}\xi_\pm$, $\bar{\xi}_\pm \mapsto a^{\pm 1}\bar{\xi}_\pm$ and has no effect on the W currents. The symmetry is a continuous one and, in view of the PB relations (20.12) and the constraint $B = \bar{B}$ (see eq. (21.9)), the associated Noether charge may be identified with $\log(B)$.

Our intuition regarding the global properties of the field theory phase space was in a large part motivated through an examination of the basic solutions (21.26). These satisfy the inequality

$$0 \le UV < 1\ . \tag{21.38}$$

In all likelihood, for the phase space made up from the symplectic leaves $\Gamma_{\bar{P},P,B}$ with P, $\bar{P}$ and B subject to the conditions (21.27), (21.34) and $B > 0$ this constraint should be imposed on all the classical field configurations. In ref. [20] it was observed that the Lorentzian target space metric corresponding to the action (21.24),

$$(\mathrm{d}\sigma)^2 = \frac{\mathrm{d}U\mathrm{d}V}{1 - UV}\ , \tag{21.39}$$

exhibits the characteristic features of a black hole geometry. In particular, as depicted in the space-time diagram in Fig. 16, it possesses a horizon at $UV = 0$ as well as a curvature singularity at $UV = 1$ just as the Schwarzschild black hole in terms of Kruskal coordinates. An important property of the metric is that there is no globally defined time coordinate. There is, however, a non-trivial Killing vector which is time-like only in regions I and II of Fig. 16 and space-like in regions III and IV. The restriction (21.38) means that we are focusing on the Lorentzian NLSM with the fields U and V taking values in the domain, which is the union of regions III and IV in Fig. 16.

21.2. Quantization

One can proceed with the study of the quantum NLSM through the quantization of the algebra of functions on the symplectic leaves. Identifying the Planck constant as $\hbar = \frac{2\pi}{n}$, leads us to the quantum $\overline{W}_\infty \otimes W_\infty$ - algebra with central charge $c = 2 - \frac{6}{n+2} < 2$. The parameters $(\bar{P}, P, B)$ labeling the symplectic leaves are related to the highest weights of the irreducible representation of the quantum algebra. The highest weight of the W_∞ - algebra, $\boldsymbol{\omega} = (\omega_2, \omega_3)$, may be parameterized by (ρ, ν) as in (16.27) and similarly $\bar{\boldsymbol{\omega}}$ is swapped for $(\bar{\rho}, \bar{\nu})$. Based on the previous discussion, the following identification of the parameters can be made

$$\rho = (n+2)\,P\ , \qquad \bar{\rho} = (n+2)\,\bar{P}\ , \qquad \nu = \bar{\nu} = \frac{n}{4\pi}\log(B)\ . \tag{21.40}$$

Then formulae (21.27), (21.34) and $B > 0$ translate to the conditions

$$\rho + \bar{\rho} \geq 0, \qquad \rho - \bar{\rho} = (n+2)\,\mathrm{w}\,, \qquad -\infty < \nu = \bar{\nu} < +\infty \tag{21.41}$$

with $\mathrm{w} = 0, \pm 1, \pm 2, \ldots$. Recall that the components $\bar{\omega}_2$ and ω_2 of the highest weight labeling the $\overline{W}_\infty \otimes W_\infty$ irrep coincide with the conformal dimensions of the highest state, so that

$$\bar{\Delta}_{\bar{\rho},\nu} = \frac{\bar{\rho}^2 - \frac{1}{4}}{n+2} + \frac{\nu^2}{n}\,, \qquad \Delta_{\rho,\nu} = \frac{\rho^2 - \frac{1}{4}}{n+2} + \frac{\nu^2}{n}\ . \tag{21.42}$$

Assuming that $\rho + \bar{\rho}$ is a non-negative integer, the Lorentz spin of these states, $\Delta_{\rho,\nu} - \bar{\Delta}_{\bar{\rho},\nu}$, would only take integer values. Then ρ and $\bar{\rho}$ would have the form

$$\rho = \tfrac{1}{2}\,\mathrm{v} + \tfrac{1}{2}\,(n+2)\,\mathrm{w}\,, \qquad \bar{\rho} = \tfrac{1}{2}\,\mathrm{v} - \tfrac{1}{2}\,(n+2)\,\mathrm{w}\,, \tag{21.43}$$

where $\mathrm{v} = 0, 1, 2, \ldots$ and $\mathrm{w} \in \mathbb{Z}$. This way we come to expect that the space of states of the Lorentzian black hole NLSM, $\mathcal{H}_{\mathrm{LBH}}$, is decomposed into irreps of the $\overline{W}_\infty \otimes W_\infty$ algebra as

$$\mathcal{H}_{\mathrm{LBH}} = \bigoplus_{\mathrm{v}=0}^{\infty} \left[\ \bigoplus_{\mathrm{w}=-\infty}^{\infty} \int_{\mathbb{R}}^{\oplus} \mathrm{d}\nu\ \overline{\mathcal{W}}_{\bar{\rho},\nu} \otimes \mathcal{W}_{\rho,\nu} \right]. \tag{21.44}$$

The latter is identical to the linear decomposition (17.61) of the space $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{\mathrm{even}}$, which is a sector of $\mathcal{H}^{(\mathrm{cont})}_{\mathrm{even}}$ (17.60) – the space of states occurring in the scaling limit of the $\mathcal{C}$ even sector of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model subject to periodic boundary conditions ($\mathtt{k} = 0$).

The Hermitian structure in the space $\mathcal{H}_{\mathrm{LBH}}$ should be consistent with the Hermitian conjugation (19.60), which as was already mentioned is the quantum version of the classical reality condition (21.25) (see the comments below (20.35)). This leads us to propose that, not only the linear structure of the spaces $\mathcal{H}_{\mathrm{LBH}}$ and $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{\mathrm{even}}$ coincide, but also their Hermitian structures. In particular the inner product of the $\overline{W}_\infty \otimes W_\infty$ primary states $\Psi_{\mathrm{v},\mathrm{w},\nu}$ of the irreps appearing in the decomposition (21.44), up to real positive constants that depend on the overall normalization of $\Psi_{\mathrm{v},\mathrm{w},\nu}$, would be equal to $\langle\cdot,\cdot\rangle_{\mathrm{cont}}$ computed on the corresponding states from $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{\mathrm{even}}$. It follows from eqs. (19.58) and (19.64) that one can set

$$\big\langle \Psi_{\mathrm{v}',\mathrm{w}',\nu'}\,,\,\Psi_{\mathrm{v},\mathrm{w},\nu}\big\rangle = \delta_{\mathrm{v}',\mathrm{v}}\,\delta_{\mathrm{w}',\mathrm{w}}\,\delta(\nu'-\nu)\,N_{\mathrm{v},\mathrm{w}} \tag{21.45}$$

with

$$N_{0,0} = 1\,, \qquad N_{\mathrm{v},\mathrm{w}} = \begin{cases} (-1)^{\mathrm{w}}\ \dfrac{\sin(\pi\,(n+2)\mathrm{w})}{\pi\,(n+2)} & \mathrm{v} = 0,\ \mathrm{w} \neq 0 \\[3ex] \dfrac{\Gamma(1 - \mathrm{w} + \frac{\mathrm{v}}{n+2})\,\Gamma(1 + \mathrm{w} + \frac{\mathrm{v}}{n+2})}{\Gamma(1 + \mathrm{v} - (n+2)\,\mathrm{w})\,\Gamma(1 + \mathrm{v} + (n+2)\,\mathrm{w})} & \mathrm{v} \geq 1,\ \mathrm{w} \in \mathbb{Z} \end{cases} \tag{21.46}$$

Then $\mathcal{H}_{\mathrm{LBH}}$ is a pseudo-Hilbert space equipped with a non-positive definite inner product. This would reflect the fact that the target space for the NLSM (21.24) has Lorentzian signature.

The identification of the pseudo-Hilbert spaces $\mathcal{H}_{\mathrm{LBH}}$ and $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{\mathrm{even}}$ turns out to be consistent with all the global symmetries. First of all, as was discussed in sec. 17.4, the full linear space

$\mathcal{H}^{(\mathrm{cont})}_{\mathrm{even}}$ admits the two formal $\mathcal{Z}_2$ symmetries, $\mathcal{X}^{(\mathrm{w})}$ (17.65) and $\mathcal{X}^{(\mathrm{null})}$ (17.66), having no counterparts in the lattice system. The transformation $\mathcal{X}^{(\mathrm{null})}$ acts trivially in $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{\mathrm{even}}$, or equivalently, $\mathcal{H}_{\mathrm{LBH}}$ and thus should be ignored. The other global symmetry $\mathcal{X}^{(\mathrm{w})}$ arises due to the degeneracy in the decomposition (21.44) in which the irreps, whose highest states are $\Psi_{0,+\mathrm{w},\nu}$ and $\Psi_{0,-\mathrm{w},\nu}$ with $\mathrm{w}\neq 0$ are equivalent. However, formula (21.46) shows that the "norms" of these two primary states, $N_{0,\pm\mathrm{w}}$, differ in their sign, which can not be eliminated by a change of their normalization. Hence we conclude that the $\mathcal{Z}_2$ transformation $\mathcal{X}^{(\mathrm{w})}$, in spite that it commutes with the generators of the $\overline{W}_\infty\otimes W_\infty$ - algebra, is not actually a symmetry of the pseudo-Hilbert space $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{\mathrm{even}}\cong\mathcal{H}_{\mathrm{LBH}}$. In connection to this, let's mention that in all likelihood the states $\Psi_{0,\pm\mathrm{w},0}$ correspond to the solutions $U=V=\cos(\mathrm{w}x)$ and $U=V=\sin(\mathrm{w}x)$ of the classical equations of motion (21.13), (21.23). These are distinguished by their properties w.r.t. the parity transformation $x\to -x$.

The classical NLSM possesses $\mathcal{P}$ invariance as well as the time-reversal symmetry. The action of the $\mathcal{P}$ and $\mathcal{T}$ transformations on the fundamental fields U and V is described by eq. (21.28). It is natural to expect that these global symmetries are present in the quantum NLSM as well. The corresponding generators would satisfy the following commutation relations with the W currents

$$\hat{\mathcal{P}}_{\mathrm{LBH}}\,W_j(u)=\overline{W}_j(u)\,\hat{\mathcal{P}}_{\mathrm{LBH}}\,,\qquad\qquad \hat{\mathcal{T}}_{\mathrm{LBH}}\,W_j(u)=\overline{W}_j(u)\,\hat{\mathcal{T}}_{\mathrm{LBH}}\,. \tag{21.47}$$

Their action on the primary states, without loss of generality, can be chosen to be

$$\hat{\mathcal{P}}_{\mathrm{LBH}}\,\Psi_{\mathrm{v},\mathrm{w},\nu}=\hat{\mathcal{T}}_{\mathrm{LBH}}\,\Psi_{\mathrm{v},\mathrm{w},\nu}=\sigma_{\mathrm{v},\mathrm{w}}\;\Psi_{\mathrm{v},-\mathrm{w},\nu} \tag{21.48}$$

with the sign factor being such that $\sigma_{\mathrm{v},\mathrm{w}}=+1$ for $\mathrm{v}=\mathrm{w}=0$ and $\mathrm{v}\cdot\mathrm{w}\neq 0$. For the case when $\mathrm{v}=0$ and $\mathrm{w}\neq 0$ or $\mathrm{v}\geq 1$ and $\mathrm{w}=0$, the states $\Psi_{\mathrm{v},\mathrm{w},\nu}$ are eigenstates of the parity generator and the corresponding sign factors $\sigma_{0,\mathrm{w}}$ and $\sigma_{\mathrm{v},0}$ are as yet undetermined.

Finally it remains to discuss the $\mathcal{Z}_2$ symmetry $\mathcal{D}$ as well as the symmetry generated by $\hat{\mathcal{U}}$ (17.50), which is a remnant of the broken U(1). In the previous subsection we described the manifestation of these global symmetries in the classical field theory, where they were denoted by the same symbols. Note that for the quantum NLSM, the generator $\hat{\mathcal{U}}$ acts on any state in the irrep $\overline{\mathcal{W}}_{\bar{\rho},\nu}\otimes\mathcal{W}_{\rho,\nu}$ via multiplication by the sign factor $\pm(-1)^{\mathrm{v}}$. The extra sign "$\pm$" may be chosen at will as it remains the same for all the states in $\mathcal{H}_{\mathrm{LBH}}$. The action of the global symmetries in $\mathcal{H}_{\mathrm{LBH}}$ is defined through the commutation relations of their generators with the W currents supplemented by the action of the symmetries on the $\overline{W}_\infty\otimes W_\infty$ primary states. For the $\mathcal{Z}_2$ symmetry transformation $\mathcal{D}$ the relevant formulae are (17.13) and

$$\hat{\mathcal{D}}\,\Psi_{\mathrm{v},\mathrm{w},\nu}=\Psi_{\mathrm{v},\mathrm{w},-\nu}\,, \tag{21.49}$$

while $\hat{\mathcal{U}}$ commutes with the W currents and

$$\hat{\mathcal{U}}\,\Psi_{\mathrm{v},\mathrm{w},\nu}=\sigma\;\Psi_{\mathrm{v},\mathrm{w},\nu}\qquad\text{with}\qquad \sigma=\pm\,(-1)^{\mathrm{v}}\,. \tag{21.50}$$

There is no apparent candidate for a non-trivial $\mathcal{C}$ conjugation for the classical action (21.24), which is consistent with the fact that, by construction, $\hat{\mathcal{C}}=\mathrm{id}$ for the space $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{\mathrm{even}}$.

21.3. Minisuperspace approximation

For a better qualitative understanding of the quantum NLSM it is useful to consider the model within the so-called minisuperspace approximation. This entails taking into account only those field configurations that do not depend on the space co-ordinate x, such as the classical solutions

(21.26) with $P = \bar{P}$. We still take U and V to satisfy the constraint $0 \le UV < 1$ corresponding to the union of regions III and IV in Fig. 16. For a preliminary analysis it is convenient to parameterize U, V from this domain as

$$U = \mathrm{e}^{\Theta}\ \sin(\Phi)\ , \qquad V = \mathrm{e}^{-\Theta}\ \sin(\Phi)\ ; \qquad \Phi \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})\ , \quad \Theta \in (-\infty, \infty)\ . \tag{21.51}$$

Then the minisuperspace version of the classical action (21.24) reads as

$$S^{(\mathrm{ms})}_{\mathrm{LBH}} = \frac{\pi}{\hbar} \int \mathrm{d}t\ \big(\dot{\Phi}^2 - \tan^2(\Phi)\ \dot{\Theta}^2 \big)\ . \tag{21.52}$$

Since the generalized coordinate Θ is cyclic, its conjugate momentum $\Pi_\Theta = -\tan^2(\Phi)\,\dot{\Theta}$ is an integral of motion. The effective Lagrangian (the Routhian) for the non-cyclic degree of freedom is given by

$$L_{\mathrm{eff}} = \frac{1}{2}\ \big(\dot{\Phi}^2 - V_{\mathrm{eff}}(\Phi)\big)\ , \qquad\qquad V_{\mathrm{eff}}(\Phi) = -\Pi_\Theta^2\ \cot^2(\Phi)\ . \tag{21.53}$$

The latter describes a 1D particle falling to the origin $\Phi = 0$. An elementary calculation shows that for any value $\Pi_\Theta \neq 0$ the particle, starting its motion at $t = 0$, reaches the origin in a finite amount of time $t_{\mathrm{fall}} < +\infty$. For $t > t_{\mathrm{fall}}$ the motion remains undetermined. Thus the action (21.52) specifies the time evolution of the mechanical system only within a finite time interval (except for the trajectories with $\Pi_\Theta = 0$). To continue the classical trajectories for $t > t_{\mathrm{fall}}$ the unbounded effective potential should be somehow regularized. There are of course numerous ways of doing this. A simple minded one is to replace $V_{\mathrm{eff}}(\Phi) = -\Pi_\Theta^2\ \cot^2(\Phi)$ by a smooth potential $V^{(\mathrm{reg})}_{\mathrm{eff}}(\Phi)$, which together with its derivative is bounded from below within the infinitesimal interval $\Phi \in (-\epsilon, \epsilon)$. Outside this interval $V^{(\mathrm{reg})}_{\mathrm{eff}}(\Phi) = V_{\mathrm{eff}}(\Phi)$. To keep the original symmetry of the potential we assume that the regularized one is an even function:

$$V^{(\mathrm{reg})}_{\mathrm{eff}}(\Phi) = V^{(\mathrm{reg})}_{\mathrm{eff}}(-\Phi)\ . \tag{21.54}$$

Then the motion of Φ becomes globally defined and periodic for any values of $\Pi_\Theta \neq 0$.

With basic intuition from quantum mechanics, we can predict the symmetry properties of the minisuperspace stationary wave functions. First of all, that the regularized potential is an even function of Φ implies that the stationary states may be assigned a parity $\sigma = \pm 1$,

$$\hat{\mathcal{U}}\,\Psi^{(\sigma)}(U, V) \equiv \Psi^{(\sigma)}(-U, -V) = \sigma\ \Psi^{(\sigma)}(U, V)\ , \tag{21.55}$$

where we now switch to the original target space coordinates (U, V). This relates the values of the wave function in the domains III and IV from Fig. 16. Next, $\Psi^{(\sigma)}$ can be chosen to be an eigenfunction of the operator $\hat{\Pi}_\Theta = \frac{\hbar}{\mathrm{i}}\,\partial_\Theta = \frac{\hbar}{\mathrm{i}}\,(U\partial_U - V\partial_V)$. Since $\hat{\Pi}_\Theta$ is the infinitesimal generator of the continuous symmetry (21.37), its eigenvalue is related to the conserved charge ν (21.40):

$$\hat{\Pi}_\Theta\,\Psi^{(\sigma)}_\nu = 2\hbar\nu\ \Psi^{(\sigma)}_\nu\ . \tag{21.56}$$

It follows that

$$\Psi^{(\sigma)}_\nu(U, V) = \left(\frac{U}{V}\right)^{\mathrm{i}\nu}\ F^{(\sigma)}_\nu(UV)\ . \tag{21.57}$$

The minisuperspace approximation ignores the presence of the oscillatory modes so that the wave functions $\Psi^{(\sigma)}_\nu$ are expected to correspond to $\overline{W}_\infty \otimes W_\infty$ primary states, characterized by

$\rho = \bar{\rho}$ and ν. In turn the minisuperspace energy becomes $\Delta_{\rho,\nu} + \Delta_{\bar{\rho},\nu} = 2\Delta_{\rho,\nu}$ in the leading non-vanishing order of $\hbar = \frac{2\pi}{n}$ (the approximation is reliable only in the limit $n \to \infty$). Namely,

$$E^{(\mathrm{ms})} = \frac{\hbar}{\pi}\,\big(\rho^2 + \nu^2 - \tfrac{1}{4}\big)\,. \tag{21.58}$$

At this point, ρ can be thought of as a real number parameterizing the minisuperspace energy $E^{(\mathrm{ms})}$ and the corresponding wavefunction $\Psi^{(\sigma)}_{\rho,\nu}$. Since the highest weight is an even function of ρ,

$$\Psi^{(\sigma)}_{-\rho,\nu}(U,V) = \Psi^{(\sigma)}_{\rho,\nu}(U,V)\,. \tag{21.59}$$

As we mentioned before, one can assume that $\rho = \bar{\rho} \geq 0$.

Though the highest weight of the W_∞ irrep $\boldsymbol{\omega} = (\omega_2, \omega_3)$ is not sensitive to the sign of ρ, as follows from (16.27) it does depend on the sign of ν: $\omega_3(\rho,-\nu) = -\omega_3(\rho,\nu)$. Thus the primary states characterized by (ρ,ν) and $(\rho,-\nu)$ are distinguishable. They are related through the $\mathcal{Z}_2$ transformation, so that

$$\hat{\mathcal{D}}\,\Psi^{(\sigma)}_{\rho,\nu}(U,V) = \Psi^{(\sigma)}_{\rho,-\nu}(U,V)\,. \tag{21.60}$$

On the other hand, by definition, this symmetry interchanges U and V:

$$\hat{\mathcal{D}}\,\Psi^{(\sigma)}_{\rho,\nu}(U,V) \equiv \Psi^{(\sigma)}_{\rho,\nu}(V,U)\,. \tag{21.61}$$

Combining the above two relations with (21.57) one concludes that

$$\Psi^{(\sigma)}_{\rho,\nu}(U,V) = \left(\frac{U}{V}\right)^{\mathrm{i}\nu} F^{(\sigma)}_{\rho,\nu}(UV)\,, \qquad \text{where} \quad F^{(\sigma)}_{\rho,\nu}(z) = F^{(\sigma)}_{-\rho,\nu}(z) = F^{(\sigma)}_{\rho,-\nu}(z)\,. \tag{21.62}$$

Having described the symmetry properties of the stationary wave functions, we turn to deriving them explicitly. In the work [21], a minisuperspace analysis was performed for the NLSM (21.24) with the fields U, V belonging to region I from Fig. 16 (or equivalently II). Though this is not the domain of interest, we can still follow the same line of arguments of that paper. In particular, up to a trivial factor, the minisuperspace Hamiltonian coincides with the "dilatonic" Laplacian:

$$\hat{H}^{(\mathrm{ms})} = -\frac{\hbar}{4\pi}\,\triangle_D\,, \qquad \triangle_D = \frac{1}{\mathrm{e}^D\sqrt{-G}}\,\partial_i\big(\mathrm{e}^D\sqrt{-G}\,G^{ij}\,\partial_j\big)\,, \tag{21.63}$$

where the metric is the one in (21.39) and the dilaton field is given by

$$D = \log(1-UV)\,. \tag{21.64}$$

The stationary Schrödinger equation $\hat{H}^{(\mathrm{ms})}\,\Psi = E^{(\mathrm{ms})}\,\Psi$ reads explicitly as

$$-\Big((1-UV)\,\partial_U\partial_V - \tfrac{1}{2}\,(U\partial_U + V\partial_V)\Big)\,\Psi = \tfrac{\pi}{\hbar}\,E^{(\mathrm{ms})}\,\Psi\,. \tag{21.65}$$

Using the general form (21.62) for the stationary wave functions and parameterizing the energy as in (21.58), it is straightforward to show that $F(z) = z^{-\mathrm{i}\nu}\,F^{(\sigma)}_{\rho,\nu}(z)$ obeys the Gauss hypergeometric equation

$$z\,(1-z)\,F'' + \big(1 + 2\mathrm{i}\nu - 2\,(1+\mathrm{i}\nu)\,z\big)\,F' - \big(\tfrac{1}{2} + \mathrm{i}\nu + \rho\big)\big(\tfrac{1}{2} + \mathrm{i}\nu - \rho\big)\,F = 0\,. \tag{21.66}$$

Keeping in mind our preliminary analysis, the ODE (21.66) is applicable only in the domain $\epsilon^2 < z < 1$ with a small regularization parameter $\epsilon \ll 1$ (recall that $z = UV = \sin^2(\Phi)$).

The function $F^{(\sigma)}_{\rho,\nu}(z)$ is a certain linear combination of $z^{\pm i\nu}\ {}_2F_1\big(\frac{1}{2}\pm i\nu+\rho, \frac{1}{2}\pm i\nu-\rho, 1\pm 2i\nu, z\big)$, which can be specified as follows. Applying the elementary identity

$$\begin{aligned}
&e^{D}\sqrt{-G}\,\big(\Psi_1^*\,\hat{H}^{(\mathrm{ms})}\,\Psi_2-\Psi_2\,\hat{H}^{(\mathrm{ms})}\,\Psi_1^*\big)\\
&\quad=\frac{\hbar}{4\pi}\,\partial_i\Big(e^{D}\sqrt{-G}\,G^{ij}\big(\Psi_2\partial_j\Psi_1^*-\Psi_1^*\partial_j\Psi_2\big)\Big)
\end{aligned}\tag{21.67}$$

to the pair of stationary wave functions Ψ_1, Ψ_2 corresponding to the energies $E_1^{(\mathrm{ms})}$, $E_2^{(\mathrm{ms})}$, and then integrating the result over the domain $\mathrm{B}_\epsilon:\ \epsilon^2<UV<1$, one obtains

$$\begin{aligned}
&\big(E_2^{(\mathrm{ms})}-E_1^{(\mathrm{ms})}\big)\int_{\mathrm{B}_\epsilon}\mathrm{d}U\mathrm{d}V\ e^{D}\sqrt{-G}\ \Psi_1^*\Psi_2\\
&\quad=\frac{\hbar}{4\pi}\int_{\partial\mathrm{B}_\epsilon}\mathrm{d}\ell\, e^{D}\big(\Psi_2\partial_n\Psi_1^*-\Psi_1^*\partial_n\Psi_2\big)\,.
\end{aligned}\tag{21.68}$$

Here the integral in the r.h.s. is taken over the boundary of B_ϵ, which is the union of $UV=\epsilon^2$ and $UV=1$. Also, ∂_n stands for the normal derivative to $\partial\mathrm{B}_\epsilon$. As was discussed before, the wave functions possess a definite parity. Due to this either the wave function or its normal derivative vanishes at $UV=0$. Hence as $\epsilon\to 0$ the horizon $UV=0$ does not contribute to the r.h.s. of eq. (21.68). Further, since the dilaton factor e^{D} vanishes at the black hole singularity $UV=1$ one could make the whole boundary integral vanish by imposing that both the eigenfunctions and their normal derivatives remain finite at $UV=1$. In this case the wave functions corresponding to different energies would be orthogonal w.r.t. the inner product

$$\big\langle\Psi_1,\Psi_2\big\rangle=\int_{0<UV<1}\mathrm{d}U\mathrm{d}V\ e^{D}\sqrt{-G}\ \Psi_1^*\Psi_2\,.\tag{21.69}$$

This suggests to take $F^{(\sigma)}_{\rho,\nu}(z)$ in (21.62) as

$$F^{(\sigma)}_{\rho,\nu}(z)=z^{i\nu}\ {}_2F_1\big(\tfrac{1}{2}+i\nu+\rho,\tfrac{1}{2}+i\nu-\rho,1;1-z\big)\qquad(\epsilon^2<z<1)\tag{21.70}$$

or, equivalently,

$$\begin{aligned}
F^{(\sigma)}_{\rho,\nu}(z)&=A_{\rho,+\nu}\,z^{+i\nu}\ {}_2F_1\big(\tfrac{1}{2}+i\nu+\rho,\tfrac{1}{2}+i\nu-\rho,1+2i\nu;z\big)\\
&\quad+A_{\rho,-\nu}\,z^{-i\nu}\ {}_2F_1\big(\tfrac{1}{2}-i\nu+\rho,\tfrac{1}{2}-i\nu-\rho,1-2i\nu;z\big)\,,
\end{aligned}\tag{21.71}$$

where

$$A_{\rho,\nu}=\frac{\Gamma(-2i\nu)}{\Gamma(\frac{1}{2}-i\nu-\rho)\Gamma(\frac{1}{2}-i\nu+\rho)}\,.\tag{21.72}$$

For $z\ll 1$ it is convenient to use the variable y such that $z=e^{y}$. Then $F^{(\sigma)}_{\rho,\nu}$ asymptotically approaches to a superposition of two plane waves

$$F^{(\sigma)}_{\rho,\nu}\asymp A_{\rho,+\nu}\,e^{+i\nu y}+A_{\rho,-\nu}\,e^{-i\nu y}\qquad\big(1\ll(-y)<2\log(1/\epsilon)\big)\,.\tag{21.73}$$

The regularized interaction discussed before in the domain $(-y)>2\log(1/\epsilon)$ results in a quantization condition for ν

$$\epsilon^{-4i\nu}\,e^{\frac{i}{2}\delta^{(\mathrm{ms})}(\rho,\nu)}\asymp\sigma\,.\tag{21.74}$$

The phase shift $\delta^{(\mathrm{ms})}$ here depends on the precise form of the regularized potential. As $\epsilon \to 0$, the spectrum of ν becomes continuous and is characterized by the density of states

$$\rho^{(\mathrm{ms})}(\nu) = \tfrac{2}{\pi}\ \log(1/\epsilon) + \tfrac{1}{4\pi}\ \partial_\nu\, \delta^{(\mathrm{ms})}(\rho, \nu)\ . \tag{21.75}$$

The corresponding minisuperspace wave functions would be orthogonal w.r.t. the inner product (21.69):

$$\big\langle \Psi^{(\sigma')}_{\rho',\nu'}, \Psi^{(\sigma)}_{\rho,\nu} \big\rangle \propto \delta_{\rho',\rho}\ \delta_{\sigma',\sigma}\ \delta(\nu' - \nu)\ . \tag{21.76}$$

Here we use the Dirac δ-function for ν since the latter can be any real number. At the same time the Kronecker symbol indicates that ρ belongs to some discrete set. The quantization of ρ seems rather natural once we note that the term $\frac{\hbar}{\pi}\,(\rho^2 - \frac{1}{4})$ in the formula for the minisuperspace energy (21.58) can be interpreted as the contribution of the non-cyclic degree of freedom Φ, which executes periodic motion in the regularized effective potential. This is consistent with our general discussion of the quantization of the Lorentzian black hole NLSM. Setting $\mathtt{w} = 0$ in formula (21.43) for the admissible values of ρ and $\bar{\rho}$, one has $2\rho = 2\bar{\rho} = \mathtt{v} = 0, 1, 2, \ldots$. Also $\delta_{\sigma',\sigma}$ in (21.76) can be ignored – the sign factor σ is not an independent quantum number and is related to the parity of the integer $\mathtt{v}$ (see (21.50)).

Our analysis within the minisuperspace approximation is in agreement with the conjectured link between the Lorentzian black hole NLSM and the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model in the scaling limit. Moreover it elucidates the occurrence of the "quantization condition" (9.11), which plays a central rôle in the description of the critical behaviour of the lattice system. Namely, the density of states is not an immanent property of the CFT, but rather, a result of the regularization of the theory. The lattice model provides a particular integrable regularization, which yields a density of states as in eqs. (10.12), (9.8) with $\epsilon \propto N^{-1}$ being the regularization parameter. Note that the spectrum of states with pure imaginary s depends on the precise form of the phase shift δ. We interpret them as non-normalizable virtual states appearing as a result of the regularization and not belonging to the set of normalizable states from the pseudo-Hilbert space of the Lorentzian black hole NLSM.

22. Partition function for the Euclidean black hole NLSM

In the work [11] the authors put forward the pioneering conjecture that the Euclidean black hole NLSM is the CFT governing the scaling limit of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model in the domain of the anisotropy parameter $\arg(q^2) \in (0, \pi)$. This is not quite in line with the results of our study. Here we'd like to critically re-examine the arguments from ref. [11].

Let's recall the definition of the Euclidean black hole NLSM. The corresponding target space metric has Euclidean signature and is given by

$$(\mathrm{d}\sigma_{\mathrm{EBH}})^2 = \frac{\mathrm{d}U\mathrm{d}U^*}{1 + UU^*}\ . \tag{22.1}$$

It may be obtained from the metric $(\mathrm{d}\sigma)^2$ (21.39) in the following way. For the co-ordinates U and V taking values in region I from Fig. 16, one performs the "Wick rotation" in the target space, which makes them satisfy the reality condition

$$V = -U^*\ . \tag{22.2}$$

Then ignoring the overall negative sign, the metric (21.39) becomes $(\mathrm{d}\sigma_{\mathrm{EBH}})^2$. The classical action for the NLSM is now given by

$$S_{\text{EBH}} = \frac{1}{2\hbar}\int \mathrm{d}t \int_0^{2\pi} \mathrm{d}x \ \frac{\partial_t U \partial_t U^* - \partial_x U \partial_x U^*}{1+UU^*} \qquad \big(\hbar \to 0^+\big)\,. \tag{22.3}$$

In this case instead of imposing periodic boundary conditions, it is useful to consider the more general quasiperiodic ones

$$U(t, x+2\pi) = \mathrm{e}^{2\pi \mathrm{i} \mathtt{k}}\, U(t,x)\,. \tag{22.4}$$

The model (22.3) possesses $\mathrm{U}(1)$ symmetry and the Noether current is given by

$$I_\mu = \frac{1}{2\mathrm{i}}\ \frac{U^*\partial_\mu U - U^*\partial_\mu U}{1+UU^*}\,. \tag{22.5}$$

The Euclidean black hole NLSM has been well studied [18–27]. In particular, the classical field theory (22.3) still possesses an infinite set of chiral currents, which form the classical $\overline{W}_\infty \otimes W_\infty$ Poisson algebra. The quantization of the latter leads to the algebra of extended conformal symmetry with central charge $c>2$. The Hilbert space can be classified according to the highest weight irreps of the $\overline{W}_\infty \otimes W_\infty$ - algebra. It is convenient to parameterize the central charge and the highest weight of the irreps $\boldsymbol{\omega} = (\omega_2, \omega_3)$ using n, s and p as

$$c = 2 + \frac{6}{n} > 2 \tag{22.6}$$

and

$$\omega_2 = \frac{s^2+\frac{1}{4}}{n} + \frac{p^2}{n+2} \tag{22.7}$$
$$\omega_3 = \frac{2p}{\sqrt{n+2}}\left(\frac{s^2}{n} + \frac{(2+3n)\,p^2}{3n\,(n+2)} - \frac{2n+1}{12n}\right).$$

To avoid confusion let us emphasize that in these relations $n>0$, s and p are formal parameters, without the meaning that was assigned to them in the previous sections. The Hilbert space of the NLSM contains both a continuous $\mathcal{H}^{(\text{cont})}_{\text{EBH}}$ and a discrete component $\mathcal{H}^{(\text{disc})}_{\text{EBH}}$. Let's first focus on the continuous one. Its linear decomposition into the irreps of the $\overline{W}_\infty \otimes W_\infty$ - algebra is given by [21–27]

$$\mathcal{H}^{(\text{cont})}_{\text{EBH}} = \bigoplus_{\mathtt{v},\mathtt{w}=-\infty}^{+\infty} \int_{s>0}^{\oplus} \mathrm{d}s\ \overline{\mathcal{W}}^{(c>2)}_{\bar{p},s} \otimes \mathcal{W}^{(c>2)}_{p,s}\,,$$
$$\text{where}\quad \begin{aligned} p &= \tfrac{1}{2}\,\mathtt{v} + \tfrac{1}{2}\,(n+2)\,(\mathtt{k}+\mathtt{w}) \\ \bar{p} &= \tfrac{1}{2}\,\mathtt{v} - \tfrac{1}{2}\,(n+2)\,(\mathtt{k}+\mathtt{w}) \end{aligned} \tag{22.8}$$

Here $\mathtt{v}$ is the eigenvalue of the $\mathrm{U}(1)$ conserved charge $\hbar^{-1}\oint \mathrm{d}x\, I_0$ associated with the Noether current (22.5). It takes integer values provided that the Planck constant is identified with n as

$$\hbar = \frac{2\pi}{n+2}\,. \tag{22.9}$$

The integer $\mathtt{w}$ may be interpreted as a winding number related to the fact that the boundary condition (22.4) is invariant w.r.t. the substitution $\mathtt{k} \mapsto \mathtt{k}+\mathtt{w}$ with $\mathtt{w} \in \mathbb{Z}$. Let us note that the highest weight (22.7) is not sensitive to the sign of s. Due to this the direct integral in (22.8) is restricted to positive values of s. For the states at the level $\bar{\mathtt{L}}$ and $\mathtt{L}$ in the irrep $\overline{\mathcal{W}}^{(c>2)}_{\bar{p},s} \otimes \mathcal{W}^{(c>2)}_{p,s}$, the corresponding energy $E = \Delta + \bar{\Delta} - \frac{c}{12}$ in terms of the parameters n, s, p and $\bar{p}$ reads as

$$E = -\frac{1}{6} + \frac{2s^2}{n} + \frac{p^2 + \bar{p}^2}{n+2} + \mathtt{L} + \bar{\mathtt{L}}\,. \tag{22.10}$$

The study of the low energy spectrum of the Hamiltonian $\mathbb{H}$ (7.6) with $\arg(q^2) \in (0, \pi)$ was initiated in the work [9]. Within the Bethe ansatz approach, the leading $1/N$ correction to the energy was considered. Formula (8.2a) was obtained, where p and $\bar{p}$ are given in (8.1), while $b(N)$ is the eigenvalue of the quasi-shift operator (8.6). Then it was understood in [10] that the scaling limit of the low energy states could be organized so that $b(N)$ is replaced by the RG invariant s, which becomes a continuous parameter in the scaling limit (see also the discussion in sec. 9 from this paper). The observation that the universal correction term in (8.2a) coincides with (22.10) was among the original arguments that the critical behaviour of the lattice system is described by the Euclidean black hole NLSM. With such an identification the U(1) symmetry of the action (22.3) is interpreted as the counterpart of the lattice U(1) symmetry, so that the quantum number $\mathtt{v}$ coincides with S^z. Needless to say that $\mathtt{k}$ in the twisted boundary conditions (22.4) corresponds to the twist parameter $\mathtt{k}$ from (7.7) [14].

There are two immediate concerns to the above identification. The first is regarding the $\mathcal{Z}_2$ symmetry of the lattice model. In the scaling limit, the states related through the $\mathcal{Z}_2$ transformation are characterized by the RG invariant $+s$ and $-s$ and should be considered as distinct states in the Hilbert space of the CFT. On the other hand, the highest weight irrep $\overline{\mathcal{W}}^{(c>2)}_{\bar{p},s} \otimes \mathcal{W}^{(c>2)}_{p,s}$ is identical to $\overline{\mathcal{W}}^{(c>2)}_{\bar{p},-s} \otimes \mathcal{W}^{(c>2)}_{p,-s}$ and in the Euclidean black hole NLSM the states with $\pm s$ must be identified. For this reason the domain of integration in (22.8) is $s > 0$. The second concern is that the NLSM is a unitary field theory. Its Hilbert is equipped with a positive definite inner product [28] such that the Fourier modes of the W and $\overline{W}$ currents, generating the $\overline{W}_\infty \otimes W_\infty$ - algebra, satisfy the conjugation conditions

$$\big[\widetilde{W}_j(m)\big]^\dagger = \widetilde{W}_j(-m)\,, \qquad \big[\widetilde{\overline{W}}_j(m)\big]^\dagger = \widetilde{\overline{W}}_j(-m)\,. \tag{22.11}$$

Contrary to this, since the spectrum of the lattice Hamiltonian $\mathbb{H}$ is not real, there does not exist a positive definite inner product for the lattice system w.r.t. which the matrix $\mathbb{H}$ would be Hermitian.

In principle, the objections may be addressed as follows. Instead of considering the full Hilbert space $\mathcal{H}$ occurring in the scaling limit of the spin chain, one could focus on its $\mathcal{Z}_2$ invariant sector and identify this with the space of states of the Euclidean black hole NLSM. Also, it is a rather common situation when the lattice (regularized) system is equipped with a non-positive definite inner product, but unitary is restored in the scaling limit.

In the consequent work [11] an additional argument was presented in support of the relation between the lattice model and the quantum Euclidean black hole NLSM. It uses the results of refs. [24,25]. In these papers an explicit formula was presented for the partition function of the black hole NLSM with periodic boundary conditions ($\mathtt{k} = 0$) and the Euclidean world-sheet compactified on the torus. It was argued that the contribution of the continuous spectrum to the partition function takes the from (see formula (4.17) from ref. [25])

$$Z^{(\mathtt{k}=0)}_{\rm EBH} = \sum_{\mathtt{v},\mathtt{w}=-\infty}^{\infty} \int_0^\infty \mathrm{d}s\, \rho(s)\, \mathrm{ch}_{\bar{p},s}(\bar{\mathtt{q}})\, \mathrm{ch}_{p,s}(\mathtt{q}) \;+\; \ldots\;, \tag{22.12}$$

where the contribution of the discrete spectrum is denoted by the ellipsis. The product $\mathrm{ch}_{\bar{p},s}(\bar{\mathtt{q}})\,\mathrm{ch}_{p,s}(\mathtt{q})$ is the character of the highest weight irrep $\overline{\mathcal{W}}^{(c>2)}_{\bar{p},s}\otimes\mathcal{W}^{(c>2)}_{p,s}$ appearing in the decomposition (22.8) and explicitly

$$\mathrm{ch}_{p,s}(\mathtt{q})=\mathtt{q}^{-\frac{1}{12}+\frac{s^2}{n}+\frac{p^2}{n+2}}\,(\mathtt{q},\mathtt{q})_\infty^{-2}\,. \tag{22.13}$$

The density of states reads as

$$\rho(s)=\frac{2}{\pi}\,\log(1/\epsilon)+\frac{1}{2\pi\mathrm{i}}\,\partial_s\log\left[\frac{\Gamma(\frac{1}{2}+p-\mathrm{i}s)\,\Gamma(\frac{1}{2}+\bar{p}-\mathrm{i}s)}{\Gamma(\frac{1}{2}+p+\mathrm{i}s)\,\Gamma(\frac{1}{2}+\bar{p}+\mathrm{i}s)}\right]+o(1)\,, \tag{22.14}$$

where $\epsilon^{-1}\gg 1$ is a regularization parameter (for an explanation see the original work [24]).

Based on a numerical study of the Bethe ansatz equations, the observation was made in ref. [11] that with a proper understanding of the scaling limit, the density of primary Bethe states is given by the function (22.14). The rôle of the regularization parameter is played by the number of lattice sites, i.e., $\epsilon^{-1}\propto N$. Our interest in the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model was inspired by this remarkable observation.

The arguments for formula (22.12) rely essentially on the minisuperspace approximation. We didn't find compelling reasons in the papers [24,25] as to why the density of states remains the same for the excited states with $\mathtt{L},\bar{\mathtt{L}}>0$. At best, one might expect that formula (22.12) should be replaced by

$$Z_{\mathrm{EBH}}=\frac{1}{2}\sqrt{\frac{n}{\Im m(\tau)}}\ \frac{\log(1/\epsilon)}{\pi(\bar{\mathtt{q}},\bar{\mathtt{q}})^2_\infty(\mathtt{q},\mathtt{q})^2_\infty}\sum_{\mathtt{v},\mathtt{w}=-\infty}^{\infty}\bar{\mathtt{q}}^{-\frac{1}{12}+\frac{\bar{p}^2}{n+2}}\,\mathtt{q}^{-\frac{1}{12}+\frac{p^2}{n+2}} \tag{22.15}$$

$$+\sum_{\mathtt{v},\mathtt{w}\in\mathbb{Z}}\int_0^\infty\mathrm{d}s\sum_{\mathtt{L},\bar{\mathtt{L}}\geq 0}\rho^{(\bar{\mathtt{L}},\mathtt{L})}_{\mathrm{EBH}}(s\,|\,\bar{p},p)\ \bar{\mathtt{q}}^{-\frac{1}{12}+\frac{s^2}{n}+\frac{\bar{p}^2}{n+2}+\bar{\mathtt{L}}}\,\mathtt{q}^{-\frac{1}{12}+\frac{s^2}{n}+\frac{p^2}{n+2}+\mathtt{L}}+Z^{(\mathrm{disc})}_{\mathrm{EBH}}\,.$$

The divergent term in (22.15) admits a simple interpretation that has to do with the geometry of the target space of the NLSM. The manifold equipped with the metric $(\mathrm{d}\sigma_{\mathrm{EBH}})^2$ (22.1) may be embedded into three dimensional Euclidean space and visualized as a half-infinite cigar. The tip is located at $U=0$ while in the domain $|U|\gg 1$, where the metric becomes flat, the target manifold resembles a half-infinite cylinder. The first term in (22.15) is the partition function of two free bosons. One of them, $\arg(U)$, takes values in the interval $[-\pi,\pi]$, and satisfies the quasiperiodic (if $\mathtt{k}\neq 0$) boundary conditions. The other Bose field, $\log|U|$, takes values in a segment of length $\propto\log(1/\epsilon)\to\infty$ as $\epsilon\to 0$.

While the divergent term is somewhat universal the density matrix, whose matrix elements essentially coincide with $\rho^{(\bar{\mathtt{L}},\mathtt{L})}_{\mathrm{EBH}}(s\,|\,\bar{p},p)$ in (22.15), depends on the IR regularization of the target manifold. In the works [23–25] the Euclidean black hole NLSM occurs in the context of bosonic string theory on AdS_3. This provides a particular "integrable" IR regularization for which the NLSM partition function in the case of periodic boundary conditions reads explicitly as[16]

[16] Formula (3.9) in [25] for the partition function contains an additional factor of 2. This is related to the fact that the corresponding NLSM was obtained by gauging the U(1) symmetry, $\mathbf{g}\mapsto\mathbf{h}\,\mathbf{g}\,\mathbf{h}$ ($\mathbf{h}=\mathrm{e}^{\frac{\mathrm{i}\alpha}{2}\sigma^y}$), of the $\mathrm{SL}(2,\mathbb{R})$ WZW model. This results in two copies of the Euclidean black hole NLSM (see also the discussion in sec. 23.1).

$$Z_{\mathrm{EBH}}^{(\mathrm{k}=0)} = \frac{\sqrt{n(n+2)}}{\Im m(\tau)} \sum_{\mathrm{a},\mathrm{b}\in\mathbb{Z}} \int_{D_\epsilon} \mathrm{d}^2 z\ \mathrm{e}^{-\frac{\pi(n+2)}{\Im m(\tau)}|z+\mathrm{a}+\mathrm{b}\tau|^2+\frac{2\pi}{\Im m(\tau)}(\Im m(z))^2} \left|\frac{\eta(\tau)}{\vartheta_1(z|\tau)}\right|^2 . \tag{22.16}$$

Here ϑ_1 and η are the standard elliptic theta and Dedekind eta functions:

$$\vartheta_1(u|\tau) = 2\mathrm{q}^{\frac{1}{8}}\ \sin(\pi u)\ (\mathrm{e}^{2\pi\mathrm{i}u}\,\mathrm{q},\mathrm{q})_\infty\ (\mathrm{e}^{-2\pi\mathrm{i}u}\,\mathrm{q},\mathrm{q})_\infty\ (\mathrm{q},\mathrm{q})_\infty \tag{22.17}$$

$$\eta(\tau) = \mathrm{q}^{\frac{1}{24}}\,(\mathrm{q},\mathrm{q})_\infty \qquad\qquad (\mathrm{q}=\mathrm{e}^{2\pi\mathrm{i}\tau}) .$$

The integral in (22.16) is taken over the parallelogram D in the complex z plane with vertices at $z=\pm\frac{1}{2}\pm\frac{1}{2}\tau$. However since the integrand is singular at $z=0$, a small neighbourhood around the origin, whose size is controlled by the parameter ϵ, should be excluded from the integration domain. For instance if one chooses

$$D_\epsilon = D/\{z:\ |z|<\tfrac{1}{2\pi}\,\mathrm{e}^{-\gamma_{\mathrm{E}}}\,\epsilon\} , \tag{22.18}$$

where γ_{E} denotes the Euler constant, then for $|\mathrm{q}|\to 0$

$$Z_{\mathrm{EBH}}^{(\mathrm{k}=0)} = \frac{1}{2\pi}\sqrt{\frac{n}{\Im m(\tau)}}\ |\mathrm{q}|^{-\frac{1}{6}}\ \Big(\log(4\mathrm{e}^{\gamma_{\mathrm{E}}}/\epsilon)+o\big(|\mathrm{q}|^0\big)\Big) . \tag{22.19}$$

This is consistent with formulae (22.12) and (22.14). We define the regularized partition function of the Euclidean black hole NLSM as

$$Z_{\mathrm{EBH,reg}}^{(\mathrm{k}=0)} = \lim_{\epsilon\to 0}\ \big(Z_{\mathrm{EBH}}^{(\mathrm{k}=0)} - Z_\epsilon^{(\mathrm{sing})}\big) \tag{22.20}$$

with

$$Z_\epsilon^{(\mathrm{sing})} = \sqrt{\frac{n}{\Im m(\tau)}}\ \frac{\log(4\mathrm{e}^{\gamma_{\mathrm{E}}}/\epsilon)+\frac{1}{2}\log\big(\Im m(\tau)\big)}{2\pi\ (\bar{\mathrm{q}},\bar{\mathrm{q}})_\infty^2(\mathrm{q},\mathrm{q})_\infty^2} \sum_{\mathrm{v},\mathrm{w}\in\mathbb{Z}} \bar{\mathrm{q}}^{-\frac{1}{12}+\frac{\bar{p}^2}{n+2}}\ \mathrm{q}^{-\frac{1}{12}+\frac{p^2}{n+2}} \tag{22.21}$$

and recall that $\mathrm{q}=\mathrm{e}^{2\pi\mathrm{i}\tau}$, $\bar{\mathrm{q}}=\mathrm{e}^{-2\pi\mathrm{i}\tau^*}$. Here an extra term $\propto\log\big(\Im m(\tau)\big)$ was included into the definition of $Z_\epsilon^{(\mathrm{sing})}$ in order to ensure that the regularized partition function, $Z_{\mathrm{EBH,reg}}^{(\mathrm{k}=0)}$, is modular invariant.

Now that the partition function has been specified, the finite part in (22.15) is defined unambiguously. The explicit formula for $Z_{\mathrm{EBH}}^{(\mathrm{disc})}$, which accounts for the contribution of the discrete spectrum in the black hole NLSM, was presented in ref. [26]. It appears to be identical with $\frac{1}{2}\,Z^{(\mathrm{disc})}$ from (17.36) specialized to $\mathrm{k}=0$.[17] Then one may guess that

$$Z_{\mathrm{EBH}} = \tfrac{1}{2}\,Z^{(\mathrm{scl})} \tag{22.22}$$

provided the regularization parameter ϵ is related to N as

$$\epsilon^{-1} = \frac{2^{\frac{n+2}{n}}\,\Gamma\big(\frac{3}{2}+\frac{1}{n}\big)}{\sqrt{\pi}\,\Gamma\big(1+\frac{1}{n}\big)}\ N \tag{22.23}$$

(here we use the explicit expression (9.5) for the constant N_0). The relation (22.22) was confirmed numerically. Table 2 compares numerical data for $2Z_{\mathrm{EBH,reg}}^{(\mathrm{k}=0)}$ with $Z_{\mathrm{reg}}^{(\mathrm{cont})}+Z^{(\mathrm{disc})}$, where the NLSM partition function is regularized as in (22.20), while

[17] Apart from an obvious typo, formulae (2.5) and (2.10) from ref. [26] do not quite correctly take into account the contribution of the states to $Z_{\mathrm{EBH}}^{(\mathrm{disc})}$ with $\mathrm{j}=-\frac{n+1}{2},\ -\frac{1}{2}$ corresponding to the boundary of the interval in the set $\mathfrak{J}(\mathrm{v},\mathrm{u})$ (17.35).

Table 2
A comparison of the numerical data for $n=3$ of twice the regularized partition function of the Euclidean black hole NLSM (22.20) with $Z^{(\mathrm{cont})}_{\mathrm{reg}}+Z^{(\mathrm{disc})}$ for the lattice model with periodic boundary conditions ($\mathtt{k}=0$). Here $Z^{(\mathrm{disc})}$ is given by eqs. (17.35)-(17.37), while $Z^{(\mathrm{cont})}_{\mathrm{reg}}$ is defined by (22.24). The table also illustrates modular invariance of the regularized partition function for $\mathtt{k}=0$. Note that in order to achieve good accuracy for decreasing values of $\Im m(\tau)$ one must take into account an increasing number of terms in the sum over S^z, $\mathtt{w}$ for $Z^{(\mathrm{cont})}$ as well as $\mathtt{a}$, $\mathtt{b}$ in eq. (22.16). This significantly increases the computer time.

τ	$Z^{(\mathrm{cont})}_{\mathrm{reg}}$	$Z^{(\mathrm{disc})}$	$Z^{(\mathrm{cont})}_{\mathrm{reg}}+Z^{(\mathrm{disc})}$	$2Z^{(\mathtt{k}=0)}_{\mathrm{EBH,reg}}$
$\tau=.9\mathrm{i}$	-3.9509313	0.0210525	-3.9298787	-3.9298786
$-1/\tau$	-3.9358543	0.0059766	-3.9298776	-3.9298787
$\tau+1$	-3.9509313	0.0210525	-3.9298787	-3.9298786
$\tau=.2+.9\mathrm{i}$	-3.8983544	0.0065418	-3.8918125	-3.8918125
$-1/\tau$	-3.8925978	0.0007853	-3.8918125	-3.8918124
$\tau+1$	-3.8983544	0.0065418	-3.8918125	-3.8918124
$\tau=.66\mathrm{i}$	-4.4682528	0.0943594	-4.3738934	-4.3738934
$-1/\tau$	-4.3744476	0.0005542	-4.3738934	-4.3738933
$\tau+1$	-4.4682528	0.0943594	-4.3738934	-4.3738933
$\tau=.5\mathrm{i}$	-5.7668560	0.2960118	-5.4708441	-5.4708421
$-1/\tau$	-5.4708761	0.0000322	-5.4708439	-5.4708437
$\tau+1$	-5.7668560	0.2960118	-5.4708441	-5.4708421
$\tau=.33\mathrm{i}$	-12.070612	1.5569389	-10.513673	-10.5129976
$-1/\tau$	-10.513561	$7.662\cdot 10^{-8}$	-10.513561	-10.5135606
$\tau+1$	-12.070612	1.5569389	-10.513673	-10.5129975

$$Z^{(\mathrm{cont})}_{\mathrm{reg}} = \text{second line of eq. (17.38)} \tag{22.24}$$

$$-\sqrt{\frac{n}{\Im m(\tau)}}\ \frac{\log(4\mathrm{e}^{\gamma_{\mathrm{E}}})+\frac{1}{2}\log\big(\Im m(\tau)\big)}{\pi\ (\bar{\mathtt{q}},\bar{\mathtt{q}})^2_\infty(\mathtt{q},\mathtt{q})^2_\infty}\sum_{S^z,\mathtt{w}\in\mathbb{Z}}\bar{\mathtt{q}}^{-\frac{1}{12}+\frac{\bar{p}^2}{n+2}}\,\mathtt{q}^{-\frac{1}{12}+\frac{p^2}{n+2}}\,.$$

We also performed a numerical study of $Z^{(\mathrm{scl})}$ for $\mathtt{k}\neq 0$. It was found that the relation similar to (22.22) holds, but with a simple modification of the formula (22.16):

$$Z^{(\mathtt{k})}=\frac{\sqrt{n(n+2)}}{\Im m(\tau)}\sum_{\mathtt{a},\mathtt{b}\in\mathbb{Z}}\int_{D_\epsilon}\mathrm{d}^2z\ \mathrm{e}^{-\frac{\pi(n+2)}{\Im m(\tau)}|z+\mathtt{a}+(\mathtt{k}+\mathtt{b})\,\tau|^2+\frac{2\pi}{\Im m(\tau)}(\Im m(z))^2}\left|\frac{\eta(\tau)}{\vartheta_1(z|\tau)}\right|^2\,. \tag{22.25}$$

This is illustrated in Table 3. Thus we expect that with a proper regularization of the Euclidean black hole NLSM, (22.22) holds true for any values of the twist parameter $\mathtt{k}$ and furthermore

$$Z_{\mathrm{EBH}}=Z^{(\mathtt{k})}=\tfrac{1}{2}\,Z^{(\mathrm{scl})}\,. \tag{22.26}$$

A one - loop calculation, which is almost identical to that from [24], supports that Z_{EBH} for twisted boundary conditions indeed coincides with (22.25).

The highly non-trivial formula (22.22) is in full agreement with the original observation of ref. [11]. However, let's emphasize that in order to state that the Euclidean black hole NLSM governs the critical behaviour of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model, this relation is insufficient. Our numerical study of the finite size corrections to the CFT Hamiltonian, which are controlled by irrelevant perturbations (see eqs. (18.10) and (18.13)), shows that the extended

Table 3
The last column contains numerical data for $2Z^{(\mathrm{k})}_{\mathrm{reg}}$, where $Z^{(\mathrm{k})}_{\mathrm{reg}} = \lim_{\epsilon\to 0}\big(Z^{(\mathrm{k})} - Z^{(\mathrm{sing})}_{\epsilon}\big)$. Here $Z^{(\mathrm{k})}$ is given by (22.25), while $Z^{(\mathrm{sing})}_{\epsilon}$ is defined in (22.21) with p, $\bar{p}$ taken to be as in (22.8). This is compared to the numerical values for $Z^{(\mathrm{cont})}_{\mathrm{reg}} + Z^{(\mathrm{disc})}$, where $Z^{(\mathrm{disc})}$ was computed using eqs. (17.35)-(17.37) and $Z^{(\mathrm{cont})}_{\mathrm{reg}}$ via (22.24). The parameters were set to be $\mathrm{k} = -0.1$ and $n = 3$.

τ	$Z^{(\mathrm{cont})}_{\mathrm{reg}}$	$Z^{(\mathrm{disc})}$	$Z^{(\mathrm{cont})}_{\mathrm{reg}} + Z^{(\mathrm{disc})}$	$2Z^{(\mathrm{k})}_{\mathrm{reg}}$
$0.9\,\mathrm{i}$	-3.1430392	0.0233941	-3.1196452	-3.1196450
$0.2+0.9\,\mathrm{i}$	-3.0646040	0.0099983	-3.0546057	-3.0546064
$0.66\,\mathrm{i}$	-3.7836669	0.1033699	-3.6802970	-3.6802972
$0.2+0.66\,\mathrm{i}$	-3.5074556	0.0418838	-3.4655718	-3.4655717
$0.50\,\mathrm{i}$	-5.1054421	0.3209649	-4.7844771	-4.7844724
$0.33\,\mathrm{i}$	-11.2855973	1.6391928	-9.6464045	-9.646289
$0.25\,\mathrm{i}$	-26.5761236	5.4010183	-21.1751053	-21.171536

conformal symmetry algebra is the $\overline{W}_\infty \otimes W_\infty$ - algebra with $c < 2$. Accepting the latter also naturally resolves the issues with the $\mathcal{Z}_2$ symmetry and unitarity mentioned above.

23. Gauged SL(2, $\mathbb{R}$) WZW model

One should keep in mind the different status of the Euclidean and Lorentzian black hole NLSMs. The former is a well defined quantum theory, and there are many ways to check its consistency, including at the level of the conformal bootstrap [21–27]. Contrary to this the status of the quantum Lorentzian NLSM is rather tentative. Our conjecture is an attempt at assigning a meaning to the NLSM, which goes beyond the scope of the classical field theory. It also provides one with a handle on how to proceed with the quantization of some closely related models.

23.1. The classical field theory

As was already mentioned the Lorentzian black hole NLSM can be obtained by gauging a non-compact one dimensional subgroup of the classical SL(2, $\mathbb{R}$) WZW model [20,67]. Following the work [20] consider the classical action

$$\begin{aligned} S = \frac{1}{\hbar}\int \mathrm{d}t \int_0^{2\pi} \mathrm{d}x \Big[& \partial U \bar\partial V + \bar\partial U \partial V + \partial X \bar\partial Y + \bar\partial X \partial Y + \log(X/Y)\big(\partial U \bar\partial V - \bar\partial U \partial V\big) \\ & + \bar{a}\big(Y\partial X - X\partial Y - U\partial V + V\partial U\big) \\ & + a\big(Y\bar\partial X - X\bar\partial Y + U\bar\partial V - V\bar\partial U\big) - 2a\bar{a}\,(1-UV)\Big]. \end{aligned} \tag{23.1}$$

Here the integrand in the first line is just the classical Lagrangian density of the usual WZW model [68], $\mathcal{L}_{\mathrm{WZW}}[\mathbf{g}]$, expressed via the matrix entries of the fundamental WZW field

$$\mathbf{g} = \begin{pmatrix} X & U \\ -V & Y \end{pmatrix} . \tag{23.2}$$

Note that the term involving $\log(X/Y)$ comes from the Wess-Zumino term and, up to a total derivative, can be rewritten in various ways by employing the constraint

$$XY + UV = 1 . \tag{23.3}$$

The second line in (23.1) contains the fields a and $\bar{a}$, which are the chiral components of the gauge potential a_μ. The action is invariant w.r.t. the infinitesimal gauge transformation of the form

$$\delta X = \delta\omega\, X\ , \quad \delta Y = -\delta\omega\, Y\ , \quad \delta U = \delta V = 0\ ; \quad \delta a_\mu = \partial_\mu(\delta\omega)\ . \tag{23.4}$$

This can be seen by rewriting the Lagrangian density corresponding to the action (23.1) as

$$\mathcal{L} = \frac{1}{2}\left[\,\frac{\partial_\mu U \partial^\mu V}{1-UV} - (1-UV)\ f_\mu f^\mu + \epsilon^{\mu\nu}\partial_\mu C_\nu\right] \tag{23.5}$$

with

$$f_\mu = a_\mu - \tfrac{1}{2}\,\partial_\mu \log(X/Y) - \epsilon_{\mu\nu}\, J^\nu\ , \qquad C_\mu = \tfrac{1}{2}\,\log(X/Y)\,(U\partial_\mu V - V\partial_\mu U) \tag{23.6}$$

and

$$J_\mu = \frac{1}{2}\,\frac{U\partial_\mu V - V\partial_\mu U}{1-UV} \tag{23.7}$$

(here $J^0 = J_0,\ \ J^1 = -J_1$ while the Levi-Civita symbol $\epsilon_{\mu\nu} = -\epsilon^{\mu\nu}$ is defined to be $\epsilon_{01} = -\epsilon_{10} = 1$). The extremum condition $\frac{\delta}{\delta a_\mu} S = 0$ leads to the equation

$$a_\mu = \tfrac{1}{2}\,\partial_\mu \log(X/Y) + \epsilon_{\mu\nu}\, J^\nu\ . \tag{23.8}$$

The field strength corresponding to this vector potential is given by

$$\partial_\mu a_\nu - \partial_\nu a_\mu = \partial_\sigma J^\sigma\ \epsilon_{\mu\nu}\ . \tag{23.9}$$

It vanishes for any solution of the classical equations of motion, which includes the continuity equation $\partial_\mu J^\mu = 0$.

In the orthodox formulation of the gauged SL(2, $\mathbb{R}$) WZW model, the matrix valued field $\mathbf{g}$ is assumed to be periodic:

$$\mathbf{g}(t, x + 2\pi) = \mathbf{g}(t, x)\ . \tag{23.10}$$

If we take U and V from the domain

$$0 \le UV < 1\ , \tag{23.11}$$

it is natural to fix the gauge by setting $X = Y$ [20] which, in view of eq. (23.8), results in $a_\mu = \epsilon_{\mu\nu}\, J^\nu$. Then, after eliminating the auxiliary field a_μ, the action S (23.1) becomes that of the Lorentzian black hole NLSM (21.24). Note that, as was also pointed out in [20], if we take the SL(2, $\mathbb{R}$) picture literally the full target space of the Lorentzian black hole NLSM would contain two copies of the regions III and IV in Fig. 16 corresponding to the cases $X, Y > 0$ and $X, Y < 0$. In what follows we'll consider the same field theory, but with more general boundary conditions than (23.10). It is expected to be applicable for the description of the critical behaviour of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model with twisted boundary conditions.

The gauged SL(2, $\mathbb{R}$) WZW model possesses an alternative formulation [21,67]. Consider the Lagrange density which is just the sum of that of the WZW model and the massless Gaussian theory:

$$\tilde{\mathcal{L}} = \mathcal{L}_{\text{WZW}}[\mathbf{G}] + 2\,\partial\eta\bar{\partial}\eta\ . \tag{23.12}$$

The interaction is introduced through the constraints

$$\Upsilon \equiv \tfrac{1}{2}\,\mathrm{Tr}\big[\sigma^z\,\partial\mathbf{G}\,\mathbf{G}^{-1}\big] - \partial\eta = 0\,, \qquad \bar{\Upsilon} \equiv \tfrac{1}{2}\,\mathrm{Tr}\big[\sigma^z\,\mathbf{G}^{-1}\,\bar{\partial}\mathbf{G}\big] + \bar{\partial}\eta = 0\,. \tag{23.13}$$

If the infinitesimal gauge transformation of the WZW field and the massless Gaussian field is defined as

$$\delta\mathbf{G} = \tfrac{1}{2}\,\big(\sigma^z\mathbf{G} + \mathbf{G}\sigma^z\big)\,\delta\omega\,, \qquad \partial_\mu\delta\eta = \epsilon_{\mu\nu}\partial^\nu\,(\delta\omega)\,, \tag{23.14}$$

then $\delta\tilde{\mathcal{L}}$ turns out to be a total derivative provided the constraints (23.13) are imposed. The classical field theory, thus defined, is equivalent to the gauged WZW model governed by the action (23.1). In particular, for any field configuration satisfying the equations of motion for (23.12), (23.13),

$$\mathbf{g} = \mathrm{e}^{\frac{1}{2}\omega\sigma^z}\,\mathbf{G}\,\mathrm{e}^{\frac{1}{2}\omega\sigma^z}\,, \qquad a_\mu = -\epsilon_{\mu\nu}\partial^\nu\eta + \partial_\mu\omega \tag{23.15}$$

would be a solution of the Euler-Lagrange equations associated with the action (23.1). Here ω is an arbitrary periodic function $\omega(t, x+2\pi) = \omega(t,x)$, which appears as a manifestation of the gauge invariance of the model.

To specify the boundary conditions, let us first recall some basic facts concerning the phase space of the WZW model (see, e.g., [66,68,69]). The latter is conveniently described in terms of the left and right WZW currents,[18]

$$\partial\mathbf{G}\mathbf{G}^{-1} = L^A\mathbf{t}_A\,, \qquad \mathbf{G}^{-1}\bar{\partial}\mathbf{G} = \bar{R}^A\mathbf{t}_A\,, \tag{23.16}$$

which satisfy the closed set of equal-time Poisson bracket relations:

$$\begin{aligned}
&\big\{L^A(t,x),\,L^B(t,y)\big\} = -\tfrac{1}{2}\,\kappa^{AB}\,\delta'(x-y) - \tfrac{1}{2}\,f^{AB}{}_C\,L^C(t,x)\,\delta(x-y)\\
&\big\{\bar{R}^A(t,x),\,\bar{R}^B(t,y)\big\} = +\tfrac{1}{2}\,\kappa^{AB}\,\delta'(x-y) + \tfrac{1}{2}\,f^{AB}{}_C\,\bar{R}^C(t,x)\,\delta(x-y)\\
&\big\{L^A(t,x),\,R^B(t,y)\big\} = 0\,.
\end{aligned} \tag{23.17}$$

Assuming that the currents are periodic fields,

$$L^A(t, x+2\pi) = L^A(t,x)\,, \qquad \bar{R}^A(t,x+2\pi) = \bar{R}^A(t,x)\,, \tag{23.18}$$

the center of the Poisson algebra is generated by two elements

$$\mathfrak{C} = \mathrm{Tr}\Bigg[\overleftarrow{\mathcal{P}}\exp\bigg(+\int_{x_0}^{x_0+2\pi}\mathrm{d}x\,L^A\,\mathbf{t}_A\bigg)\Bigg]\,, \qquad \bar{\mathfrak{C}} = \mathrm{Tr}\Bigg[\overrightarrow{\mathcal{P}}\exp\bigg(-\int_{-x_0}^{-x_0-2\pi}\mathrm{d}x\,\bar{R}^A\,\mathbf{t}_A\bigg)\Bigg]\,. \tag{23.19}$$

We will focus on the field configurations such that the values of the central elements are restricted by the inequalities

[18] Here and below we use the notation $\mathbf{t}_A$ for the 2×2 real traceless matrices,

$$\mathbf{t}_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\,,\quad \mathbf{t}_+ = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\,,\quad \mathbf{t}_- = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\;: \qquad [\mathbf{t}_A,\mathbf{t}_B] = f_{AB}{}^C\,\mathbf{t}_C\,.$$

Indices are raised and lowered via the Killing form defined as

$$\kappa_{AB} = \tfrac{1}{2}\,\mathrm{Tr}[\,\mathbf{t}_A\mathbf{t}_B\,]\,, \qquad \kappa_{AC}\,\kappa^{CB} = \delta_A^B\,.$$

$$-2 < \mathfrak{C},\ \bar{\mathfrak{C}} < 2 \tag{23.20}$$

and use the parameterization

$$\mathfrak{C} = 2\cos(2\pi P)\ , \qquad \bar{\mathfrak{C}} = 2\cos(2\pi \bar{P}) \tag{23.21}$$

with real P and $\bar{P}$. In this case the path ordered exponentials inside the traces in (23.19) may be expressed as

$$\overleftarrow{\mathcal{P}}\exp\bigg(+\int_{x_0}^{x_0+2\pi} \mathrm{d}x\, L^A\, \mathbf{t}_A\bigg) = \mathbf{\Gamma}\, \mathrm{e}^{+2\pi \mathrm{i} P\sigma^y}\, \mathbf{\Gamma}^{-1} \tag{23.22}$$

$$\overrightarrow{\mathcal{P}}\exp\bigg(-\int_{-x_0}^{-x_0-2\pi} \mathrm{d}x\, \bar{R}^A\, \mathbf{t}_A\bigg) = \bar{\mathbf{\Gamma}}\, \mathrm{e}^{-2\pi \mathrm{i} \bar{P}\sigma^y}\, \bar{\mathbf{\Gamma}}^{-1}\ ,$$

where the 2×2 real non-degenerate matrices $\mathbf{\Gamma}$ and $\bar{\mathbf{\Gamma}}$ depend on the initial integration point x_0. If we require them to be $\mathrm{SL}(2,\mathbb{R})$ matrices, then $\mathrm{e}^{+2\pi\mathrm{i}P\sigma^y}$ and $\mathrm{e}^{-2\pi\mathrm{i}\bar{P}\sigma^y}$ are uniquely defined. At the same time there is an ambiguity in $\mathbf{\Gamma}$ and $\bar{\mathbf{\Gamma}}$ of the form $\mathbf{\Gamma}\mapsto \pm\mathbf{\Gamma}\,\mathrm{e}^{\mathrm{i}\gamma\sigma^y}$ and $\bar{\mathbf{\Gamma}}\mapsto \pm\bar{\mathbf{\Gamma}}\,\mathrm{e}^{\mathrm{i}\bar{\gamma}\sigma^y}$ with arbitrary real γ and $\bar{\gamma}$. This can be fixed using the Iwasawa decomposition for $\mathrm{SL}(2,\mathbb{R})$ matrices, which allows one to specify that

$$\mathbf{\Gamma} = \begin{pmatrix} d & 0 \\ 0 & d^{-1}\end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\ , \quad \bar{\mathbf{\Gamma}} = \begin{pmatrix} \bar{d} & 0 \\ 0 & \bar{d}^{-1}\end{pmatrix}\begin{pmatrix} 1 & \bar{b} \\ 0 & 1\end{pmatrix} \quad \text{with } d,\bar{d} > 0\ . \tag{23.23}$$

The values of the currents at $t = 0$ are not enough to fully define the time dependence of the matrix valued field $\mathbf{G}(t,x)$. Indeed the equations of motion in the WZW model are given by

$$\bar{\partial} L^A = 0\ , \qquad \partial \bar{R}^A = 0\ . \tag{23.24}$$

This implies that

$$\mathbf{G}(t,x) = \mathbf{\Omega}(t+x)\, \mathbf{G}(0,x_0)\, \bar{\mathbf{\Omega}}(t-x)\ , \tag{23.25}$$

where

$$\mathbf{\Omega}(u) = \overleftarrow{\mathcal{P}}\exp\bigg(+\int_{x_0}^{u} \mathrm{d}x\, L^A\, \mathbf{t}_A\bigg) \tag{23.26}$$

$$\bar{\mathbf{\Omega}}(\bar{u}) = \overrightarrow{\mathcal{P}}\exp\bigg(-\int_{-x_0}^{\bar{u}} \mathrm{d}x\, \bar{R}^A\, \mathbf{t}_A\bigg)\ ,$$

while $\mathbf{G}(0,x_0)$ is an arbitrary $\mathrm{SL}(2,\mathbb{R})$ matrix. Its entries, together with the initial values of the currents, constitute the full set of the initial data. We consider the field configurations at $t = 0$ to be such that

$$\mathbf{G}(0,x_0) = \mathbf{\Gamma}\, \mathrm{e}^{\mathrm{i}\alpha\sigma^y}\, \bar{\mathbf{\Gamma}}^{-1}\ , \tag{23.27}$$

where $\mathbf{\Gamma}$, $\bar{\mathbf{\Gamma}}$ are the same as in (23.22), (23.23) and α is some real number. This is motivated through the following arguments. Assuming L^A are given, the path ordered exponent $\mathbf{\Omega}(u)$ (23.26) solves the linear differential equation

$$\partial \boldsymbol{\Psi} = L^A \, \mathbf{t}_A \, \boldsymbol{\Psi} \, . \tag{23.28}$$

However $\boldsymbol{\Omega}(u)$, apart from the WZW currents, also depends on an arbitrarily chosen initial integration point x_0 at which it becomes the identity matrix. At the same time $\boldsymbol{\Psi}_P = \boldsymbol{\Omega}(u)\,\boldsymbol{\Gamma}$ is the Floquet solution of the matrix ODE (23.28), which is fixed unambiguously provided $\boldsymbol{\Gamma}$ is taken to be of the form (23.23). A change in the initial integration point x_0 to x_0' would result in the transformation $\boldsymbol{\Psi}_P \mapsto \boldsymbol{\Psi}_P \, \mathrm{e}^{\mathrm{i}\alpha_0\sigma^y}$, where $\alpha_0 = \alpha_0(x_0, x_0')$. The solutions of the ODE with periodic coefficients possess the band structure. Thus the parameter P labeling the Floquet solutions $\boldsymbol{\Psi}_P$ can be defined such that $P \in \mathbb{R}$ and $2P \notin \mathbb{Z}$, where the band number coincides with the greatest integer less than $P + \frac{1}{2}$. The above carries over to the Floquet solution $\overline{\boldsymbol{\Psi}}_{\bar{P}} = \bar{\boldsymbol{\Gamma}}^{-1}\, \bar{\boldsymbol{\Omega}}(\bar{u})$ of the barred counterpart of the ODE (23.28). This way the construction of the WZW field $\mathbf{G}(t, x)$ given by eqs. (23.25), (23.27) involves the Floquet solutions as well as an additional variable $\alpha \sim \alpha + 2\pi$. Thus the algebra of functions on the phase space of the WZW model, generated by the currents L^A and $\bar{R}^A$ subject to the periodic boundary conditions (23.18), should be extended by the inclusion of the compact generalized coordinate α. The latter can be viewed as a dynamical variable canonically conjugated to the sum $2\pi(P + \bar{P})$. As for their difference, having in mind the study of the lattice model, we assume that $\mathrm{e}^{2\pi\mathrm{i}(P-\bar{P})} = \mathrm{e}^{2\pi\mathrm{i}\mathrm{k}}$, with $\frac{1}{2} < \mathrm{k} \leq \frac{1}{2}$ being a fixed parameter. Equivalently,

$$P - \bar{P} = \mathrm{k} + \mathrm{w} \qquad (\mathrm{w} \in \mathbb{Z}) \tag{23.29}$$

and the integer w labels different disjoint components of the phase space.

The boundary values of the WZW field at $t = 0$, defined by the formulae (23.25) and (23.27), satisfy the relations

$$\mathbf{G}(0, x_0 + 2\pi) = \boldsymbol{\Gamma}\, \mathrm{e}^{2\pi\mathrm{i}\mathrm{k}\sigma^y}\, \boldsymbol{\Gamma}^{-1}\, \mathbf{G}(0, x_0) = \mathbf{G}(0, x_0)\, \bar{\boldsymbol{\Gamma}}\, \mathrm{e}^{2\pi\mathrm{i}\mathrm{k}\sigma^y}\, \bar{\boldsymbol{\Gamma}}^{-1} \, . \tag{23.30}$$

This implies

$$\mathrm{Tr}\Big[\, \mathbf{G}(t, x + 2\pi)\, \big(\mathbf{G}(t, x)\big)^{-1} \Big] = 2\, \cos(2\pi\mathrm{k}) \, , \tag{23.31}$$

which should be imposed along with the periodicity condition for the currents (23.18). In fact there is an extra condition that needs be taken into account. Substituting the matrix $\boldsymbol{\Gamma}$ (23.23) into eq. (23.30), one finds

$$\mathrm{Tr}\Big[\, (-\mathrm{i}\,\sigma^y)\, \mathbf{G}(0, x_0 + 2\pi)\, \big(\mathbf{G}(0, x_0)\big)^{-1} \Big] = \sin(2\pi\mathrm{k})\, \big(d^2 + d^{-2} + d^2 b^2\big) \, . \tag{23.32}$$

This results in the inequality

$$\mathrm{Tr}\Big[\, (-\mathrm{i}\,\sigma^y)\, \mathbf{G}(t, x + 2\pi)\, \big(\mathbf{G}(t, x)\big)^{-1} \Big] \Big/ \sin(2\pi\mathrm{k}) > 0 \, . \tag{23.33}$$

The constraints (23.13) will only make sense when both derivatives $\partial\eta$ and $\bar{\partial}\eta$ are periodic:

$$\partial\eta(t, x + 2\pi) = \partial\eta(t, x) \, , \qquad \bar{\partial}\eta(t, x + 2\pi) = \bar{\partial}\eta(t, x) \, . \tag{23.34}$$

In view of the relation (23.15), the gauge field $a_\mu(x, t)$ in the original formulation of the gauged WZW model is also periodic,

$$a_\mu(t, x + 2\pi) = a_\mu(t, x) \, , \tag{23.35}$$

as was implicitly assumed in our initial analysis of the model. The boundary condition (23.31) as well as the inequality (23.33) are invariant under the gauge transformation and therefore the field $\mathbf{g}$ satisfies the similar relations

$$\mathrm{Tr}\Big[\,\mathbf{g}(t,x+2\pi)\,\big(\mathbf{g}(t,x)\big)^{-1}\,\Big] = 2\,\cos(2\pi\mathtt{k}) \tag{23.36a}$$

$$\mathrm{Tr}\Big[\,(-\mathrm{i}\,\sigma^y)\,\mathbf{g}(t,x+2\pi)\,\big(\mathbf{g}(t,x)\big)^{-1}\,\Big]\Big/\sin(2\pi\mathtt{k}) > 0\,. \tag{23.36b}$$

Let us make the following important point. In the case of the gauged SL(2, $\mathbb{R}$) WZW model with $\mathtt{k}=0$, the condition (23.36) yields $\mathbf{g}(t,x+2\pi)=\mathbf{g}(t,x)$, i.e., periodicity of all the matrix elements X, Y, U, V. In turn one can use the gauge fixing condition $X=Y$. However for $\mathtt{k}\neq 0$, since X and Y are no longer periodic fields, the same gauge fixing condition is not applicable. This makes the model with $\mathtt{k}=0$ (which is equivalent to the Lorentzian black hole NLSM) a very special one that is not obtainable literally through a naive $\mathtt{k}\to 0$ limit.

The Poisson structure of the massless Gaussian model, whose Lagrange density is given by the second term in the r.h.s. of (23.12), reads as

$$\{\partial\eta(t,x),\partial\eta(t,y)\} = -\{\bar\partial\eta(t,x),\bar\partial\eta(t,y)\} = \tfrac{1}{2}\,\delta'(x-y)\,,\quad \{\partial\eta(t,x),\bar\partial\eta(t,y)\}=0\,. \tag{23.37}$$

With the boundary conditions (23.34) imposed, the center of this Poisson algebra is generated by

$$P_\eta = \int_0^{2\pi}\frac{\mathrm{d}x}{2\pi}\,\partial\eta\,,\qquad \bar P_\eta = \int_0^{2\pi}\frac{\mathrm{d}x}{2\pi}\,\bar\partial\eta\,. \tag{23.38}$$

The general solution of the equation of motion $\partial\bar\partial\eta=0$ is

$$\eta(t,x) = \tfrac{1}{2}\,\big(f(t+x)-\bar f(t-x)\big) \tag{23.39}$$

where, in view of the boundary conditions, the arbitrary functions f and $\bar f$ are quasiperiodic:

$$f(u+2\pi) = f(u)+P_\eta\,,\qquad \bar f(\bar u+2\pi)=\bar f(\bar u)+\bar P_\eta\,. \tag{23.40}$$

The constraints (23.13) imposed on the WZW field $\mathbf{G}$ and the Gaussian field, combined with (23.39), yield the relations

$$L^3 = -\tfrac{1}{2}\,\partial f\,,\qquad \bar R^3 = -\tfrac{1}{2}\,\bar\partial\bar f\,. \tag{23.41}$$

It is easy to see now that the matrix $\mathbf{G}$, satisfying the equations of motion, can be brought to the form

$$\mathbf{G}(t,x) = \mathrm{e}^{-\frac{1}{2}f(t+x)\sigma^z}\,\boldsymbol{g}_{\frac{1}{2}}(t,x)\,\mathrm{e}^{-\frac{1}{2}\bar f(t-x)\sigma^z}\,, \tag{23.42}$$

where $\boldsymbol{g}_{\frac{1}{2}}$ is defined as in eq. (21.17) specialized to the fundamental representation of $\mathfrak{sl}_2$. Namely,

$$\partial\boldsymbol{g}_{\frac{1}{2}}\,\boldsymbol{g}_{\frac{1}{2}}^{\;-1} = \xi_-\mathbf{t}_- - \xi_+\mathbf{t}_+\,,\qquad \boldsymbol{g}_{\frac{1}{2}}^{\;-1}\,\bar\partial\boldsymbol{g}_{\frac{1}{2}} = \bar\xi_-\mathbf{t}_- - \bar\xi_+\mathbf{t}_+ \tag{23.43}$$

with

$$\begin{aligned} \xi_- &= \mathrm{e}^{-f}\,L^-\,, &\qquad \xi_+ &= -\mathrm{e}^{+f}\,L^+\\ \bar\xi_- &= \mathrm{e}^{+\bar f}\,\bar R^-\,, &\qquad \bar\xi_+ &= -\mathrm{e}^{-\bar f}\,\bar R^+\,. \end{aligned} \tag{23.44}$$

The latter are real chiral fields, $\bar\partial\xi_\pm=\partial\bar\xi_\pm=0$, subject to the quasiperiodic boundary conditions

$$\xi_\pm(u+2\pi) = B^{\pm 1}\,\xi_\pm(u)\,, \qquad\qquad \bar\xi_\pm(\bar u+2\pi) = \bar B^{\pm 1}\,\bar\xi_\pm(\bar u)\,, \tag{23.45}$$

where $B = \mathrm{e}^{2\pi P_\eta}$ and $\bar B = \mathrm{e}^{2\pi \bar P_\eta}$. Making contact with our analysis of the lattice model, we set $B = \bar B$ or, equivalently, $P_\eta = \bar P_\eta$ (assuming that P_η and $\bar P_\eta$ are real). In this case, as it follows from eqs. (23.39) and (23.40), the field η is periodic:

$$\eta(t, x+2\pi) = \eta(t,x)\,. \tag{23.46}$$

Note that the on-shell gauge potential a_μ, entering into the initial formulation of the $\mathrm{SL}(2,\mathbb{R})$ gauged WZW model, satisfies the condition

$$B = \bar B = \exp\bigg(\oint \mathrm{d}x^\mu\, a_\mu\bigg)\,. \tag{23.47}$$

Consider now the classical W currents defined through the relations

$$\begin{matrix} W_2^{(cl)} = \xi_+\,\xi_-\,, & W_3^{(cl)} = \tfrac{1}{2}\,\big(\xi_-\,\partial\xi_+ - \xi_+\,\partial\xi_-\big)\,, & \ldots \\ \overline{W}_2^{(cl)} = \bar\xi_+\,\bar\xi_-\,, & \overline{W}_3^{(cl)} = \tfrac{1}{2}\,\big(\bar\xi_-\,\partial\bar\xi_+ - \bar\xi_+\,\partial\bar\xi_-\big)\,, & \ldots \end{matrix}\ . \tag{23.48}$$

Using (23.44) they can be rewritten in terms of the WZW currents along with $\partial\eta$ and $\bar\partial\eta$:

$$\begin{aligned} W_2^{(cl)} &= (\partial\eta)^2 - \big((L^3)^2 + L^+L^-\big)\,,\\ W_3^{(cl)} &= 2\,\partial\eta\, L^+L^- + \tfrac{1}{2}\,(L^+\partial L^- - L^-\partial L^+)\\ \overline{W}_2^{(cl)} &= (\bar\partial\eta)^2 - \big((\bar R^3)^2 + \bar R^+\bar R^-\big)\,,\\ \overline{W}_3^{(cl)} &= 2\,\bar\partial\eta\, \bar R^+\bar R^- + \tfrac{1}{2}\,(\bar R^+\partial\bar R^- - \bar R^-\bar\partial\bar R^+) \end{aligned}\ . \tag{23.49}$$

These formulae show that the W currents are real, chiral and periodic fields. Furthermore it is straightforward to check using the PB relations (23.17) and (23.37) that all the W currents Poisson commute (in a weak sense) with the constraints $\bar\Upsilon$ and Υ (23.13),

$$\begin{aligned} \big\{W_j^{(cl)}(t,x), \Upsilon(t,y)\big\}\big|_{\Upsilon=0} &= \big\{W_j^{(cl)}(t,x), \bar\Upsilon(t,y)\big\} = 0\\ \big\{\overline{W}_j^{(cl)}(t,x), \bar\Upsilon(t,y)\big\}\big|_{\bar\Upsilon=0} &= \big\{\overline{W}_j^{(cl)}(t,x), \Upsilon(t,y)\big\} = 0 \end{aligned}\ . \tag{23.50}$$

Since the fields $W_2^{(cl)}$ and $\overline{W}_2^{(cl)}$ coincide with the nonvanishing components of the stress energy tensor, the Hamiltonian Poisson commutes with $\bar\Upsilon$ and Υ. Also it is easy to see that

$$\big\{\Upsilon(t,x), \Upsilon(t,y)\big\} = \big\{\bar\Upsilon(t,x), \bar\Upsilon(t,y)\big\} = \big\{\Upsilon(t,x), \bar\Upsilon(t,y)\big\} = 0 \tag{23.51}$$

and, hence, the constraints (23.13) are of the first class. The W currents are "classical observables" and additional straightforward calculations show that they form the closed Poisson algebra which occurs in the $n\to\infty$ limit of the algebra of extended conformal symmetry of the lattice model.

23.2. BRST quantization

Once the gauged $\mathrm{SL}(2,\mathbb{R})$ WZW model is formulated as a classical dynamical system possessing constraints of the first class one can consider the problem of its quantization within the BRST approach. Here we briefly sketch the algebraic procedure for the construction of the chiral component of the space of states.

The chiral component of the energy momentum tensor of the quantum theory is split into three terms:

$$T_{\text{total}} = T_{\text{WZW}} + T_{\text{Gauss}} + T_{\text{ghost}} \ . \tag{23.52}$$

The first one is [70]

$$T_{\text{WZW}} = -\frac{n^2}{n + C_{\text{V}}} \, \kappa_{AB} \, L^A L^B \ . \tag{23.53}$$

It is built from the left currents of the WZW model

$$L^A(u) = n^{-1} \sum_{m=-\infty}^{\infty} j_m^A \, \mathrm{e}^{-\mathrm{i}mu} \qquad\qquad (A = 3, \pm) \tag{23.54}$$

whose Fourier coefficients obey the commutation relations

$$\big[\, j_m^A, j_r^B \,\big] = -n\, \kappa^{AB} \; \tfrac{m}{2}\, \delta_{m+r,0} - \tfrac{\mathrm{i}}{2}\, f^{AB}{}_C \; j_{m+r}^C \ . \tag{23.55}$$

Here the level (central element) of the Kac-Moody algebra has been denoted by n. It is related to the Plank constant as $\hbar = \frac{2\pi}{n}$. The constant C_{V} entering into (23.53) stands for the so-called dual Coxeter number:

$$C_{\text{V}}\, \kappa^{AB} = \tfrac{1}{4}\, f^{AC}{}_D \, f^{BD}{}_C \tag{23.56}$$

and in the case under consideration $C_{\text{V}} = 2$. The second term in (23.52) represents the contribution of the massless Gaussian field,

$$T_{\text{Gauss}} = n\, (\partial\eta)^2 \tag{23.57}$$

with

$$\partial\eta(u) = n^{-\frac{1}{2}} \sum_{m=-\infty}^{\infty} d_m \, \mathrm{e}^{-\mathrm{i}mu} \quad : \qquad \big[d_m, d_r\big] = \tfrac{m}{2}\, \delta_{m+r,0} \ . \tag{23.58}$$

Finally T_{ghost} is the chiral component of the energy momentum tensor for the bc - system:

$$T_{\text{ghost}} = \mathrm{i}\, b \partial c \ . \tag{23.59}$$

The ghost fields have conformal dimensions $(\Delta_b, \Delta_c) = (1, 0)$ and, as with the chiral fields L^A and $\partial\eta$, can also be expanded in the Fourier series

$$b(u) = \sum_{m=-\infty}^{\infty} b_m \, \mathrm{e}^{-\mathrm{i}mu} \ , \qquad\quad c(u) = \sum_{m=-\infty}^{\infty} c_m \, \mathrm{e}^{-\mathrm{i}mu} \ , \tag{23.60}$$

where

$$\big\{b_m, c_r\big\} = \delta_{m+r,0} \ , \qquad\qquad \big\{b_m, b_r\big\} = \big\{c_m, c_r\big\} = 0 \ .$$

The Virasoro central charge of the bc - system is equal to (-2), so that the total central charge associated with the energy momentum tensor (23.52) is given by

$$c_{\text{total}} = c_{\text{WZW}} + c_{\text{Gauss}} + c_{\text{ghost}} = \frac{3n}{n+2} + 1 - 2 = 2 - \frac{6}{n+2} \ . \tag{23.61}$$

The highest weight representation for the combined chiral algebra generated by the Fourier coefficients $j_m^A, \, d_m, b_m, c_m$ is constructed in the usual manner. First of all one requires that the highest state is annihilated by all the positive frequency modes with $m>0$. Since the zero modes of the WZW currents satisfy the commutation relations

$$\big[\, j_0^A\,,\, j_0^B\,\big]=-\tfrac{\mathrm{i}}{2}\, f^{AB}{}_C\ j_0^C\ , \tag{23.62}$$

the highest states form a representation of the $\mathfrak{sl}_2$ algebra. It makes sense to require that it is an irreducible one, characterized by the value of the Casimir operator

$$\hat{C}_{\mathbb{G}}=-\kappa_{AB}\ j_0^A\, j_0^B\ , \tag{23.63}$$

which in the $\mathfrak{sl}_2$ case is usually denoted as $\ell(\ell+1)$. In order to make a link with our previous notations we will employ the parameter $p=\ell-\frac{1}{2}$. Together with this quantum number the highest states can be labeled by the eigenvalues of the zero modes j_0^3 and d_0:

$$\begin{aligned}&\hat{C}_{\mathbb{G}}\,|p,\mu,s\rangle=\big(p^2-\tfrac{1}{4}\big)\ |p,\mu,s\rangle\,, \qquad j_0^3\,|p,\mu,s\rangle=\mu\ |p,\mu,s\rangle\,,\\ &d_0\,|p,\mu,s\rangle=\tfrac{s}{\sqrt{n}}\,|p,\mu,s\rangle\ .\end{aligned} \tag{23.64}$$

The highest states form a representation not only of the $\mathfrak{sl}_2$ algebra but also the simple fermionic one

$$\{b_0,c_0\}=1\ ,\quad b_0^2=c_0^2=0\ . \tag{23.65}$$

Thus we supplement the set of conditions defining them with

$$c_0\,|p,\mu,s\rangle_+=0\,, \qquad\qquad |p,\mu,s\rangle_-\equiv b_0\,|p,\mu,s\rangle_+\ . \tag{23.66}$$

The highest weight representation is built by the action of the negative frequency modes $j_m^A,\,d_m,\,b_m,\,c_m$ with $m<0$ on the highest state multiplet. The corresponding linear space will be denoted by $\mathcal{A}_{p,s}$. The latter possesses a grading induced by the Virasoro algebra generator $L_0^{(\text{total})}$. For given $\mathtt{L}=0,1,2,\ldots$, the level subspace $\mathcal{A}_{p,s}^{(\mathtt{L})}$ is finite dimensional and all its states have the same conformal dimension $\Delta_{p,s}+\mathtt{L}$ with

$$\Delta_{p,s}=\frac{p^2-\frac{1}{4}}{n+2}+\frac{s^2}{n}\ . \tag{23.67}$$

Note that the conformal dimensions of the primary states do not depend on the quantum number μ.

The parameter p and its barred counterpart $\bar{p}$ are related to the central elements (23.19)-(23.21) of the Poisson algebra of the WZW currents. In particular, the sum $p+\bar{p}$ can be identified with the eigenvalues of the operator $-\mathrm{i}\,\frac{\partial}{\partial\alpha}$ with α being the dynamical variable from (23.27). Then the compactness condition $\alpha\sim\alpha+2\pi$ yields the quantization rule $p+\bar{p}\in\mathbb{Z}$. This, in view of the classical relation (23.29), leads to

$$p=\tfrac{1}{2}\big(\mathtt{u}+(n+2)\,(\mathtt{k}+\mathtt{w})\big)\,,\qquad \bar{p}=\tfrac{1}{2}\big(\mathtt{u}-(n+2)\,(\mathtt{k}+\mathtt{w})\big)\quad \big(\mathtt{u},\mathtt{w}\in\mathbb{Z}\big)\ . \tag{23.68}$$

At the same time s may take any real value,

$$s\in\mathbb{R}\ . \tag{23.69}$$

The central rôle in the BRST approach belongs to the BRST charge and the ghost number operator. These obey the relations

$$\hat{Q}_{\text{BRST}}^2=0\ ,\qquad\quad \big[\hat{q}_{\text{ghost}},\,\hat{Q}_{\text{BRST}}\big]=\hat{Q}_{\text{BRST}}\ . \tag{23.70}$$

In the case at hand they read explicitly as

$$\hat{Q}_{\text{BRST}} = \frac{1}{\hbar}\int_0^{2\pi} \mathrm{d}u\, \big(L^3 - \partial\eta\big)\, c(u) = \big(j_0^3 - \sqrt{n}\, d_0\big)\, c_0 + \sum_{m\neq 0} \big(j_m^3 - \sqrt{n}\, d_m\big)\, c_{-m} \tag{23.71}$$

$$\hat{q}_{\text{ghost}} = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\, bc(u) = b_0 c_0 + \sum_{m=1}^{\infty} \big(b_{-m}c_m - c_{-m}b_m\big)\, .$$

It is easy to see that both operators commute with the zero mode of the current $L^3(u)$:

$$\big[j_0^3, \hat{Q}_{\text{BRST}}\big] = \big[j_0^3, \hat{q}_{\text{ghost}}\big] = 0\, . \tag{23.72}$$

The dimensions of the level subspaces $\widetilde{\mathcal{A}}^{(\mathtt{L})}_{p,\mu,s}$ depend essentially on whether or not $\mu - s$ vanishes. This difference is the coefficient in front of the ghost zero mode c_0 in (23.71) when the action of the BRST charge is restricted to the eigenspace $\mathcal{A}_{p,\mu,s}$. Consider the highest states $|p,\mu,s\rangle_\pm$. If $\mu \neq s$, then the state $|p,\mu,s\rangle_+$ is annihilated by the BRST charge. On the other hand $\hat{Q}_{\text{BRST}}|p,\mu,s\rangle_- \neq 0$ and is proportional to $|p,\mu,s\rangle_+$. This implies that the level subspace $\widetilde{\mathcal{A}}^{(0)}_{p,\mu,s}$ with $\mu \neq s$ is trivial. In the case when $\mu = s$ both highest states are annihilated by the BRST charge. However only $|p,s,s\rangle_+$ has zero ghost number so that $\dim\big(\widetilde{\mathcal{A}}^{(0)}_{p,s,s}\big) = 1$. Recall that $|p,s,s\rangle_+$ is a state from a $\mathfrak{sl}_2$ irrep characterized by p. The eigenvalues of j_0^3 for the other highest states from the multiplet are given by $\mu = s + \mathrm{i}\,m$, where m is a nonzero integer, and hence the difference $\mu - s$ for these states would be nonvanishing. Proceeding further, it is straightforward to check at least for small values of $\mathtt{L} = 0, 1, 2, \ldots$, that all the spaces $\widetilde{\mathcal{A}}^{(\mathtt{L})}_{p,\mu,s}$ are trivial for $\mu \neq s$, while the dimensions of $\widetilde{\mathcal{A}}^{(\mathtt{L})}_{p,s,s}$ with generic p is equal to the number of bipartitions of $\mathtt{L}$.

Perhaps the easiest way to explore the linear structure of the factor space $\widetilde{\mathcal{A}}_{p,\mu,s}$ is to bosonize the $\widehat{\mathfrak{sl}}(2,\mathbb{R})$ current algebra [50–52]. This allows one to isolate the physical states in $\mathcal{A}_{p,s}$ and to show that $\dim\big(\widetilde{\mathcal{A}}^{(\mathtt{L})}_{p,s,s}\big)$ coincides with the corresponding dimensions of the level subspaces of the highest weight representation of the W_∞ - algebra.

All the above leads us to the conjecture that the space

$$\mathcal{H}^{(\text{cont})} = \bigoplus_{\mathtt{u},\mathtt{w}\in\mathbb{Z}} \int_{\mathbb{R}}^{\oplus} \mathrm{d}s\; \overline{\mathcal{W}}_{\bar{p},s} \otimes \mathcal{W}_{p,s} \qquad \text{with} \qquad \begin{matrix} p = \frac{1}{2}\,\mathtt{u} + \frac{1}{2}\,(n+2)\,(\mathtt{k}+\mathtt{w}) \\ \bar{p} = \frac{1}{2}\,\mathtt{u} - \frac{1}{2}\,(n+2)\,(\mathtt{k}+\mathtt{w}) \end{matrix}\ , \tag{23.73}$$

that appears in the scaling limit of the lattice model with twisted boundary conditions, is the pseudo-Hilbert space which arises in the quantization of the classical field theory (23.12), (23.13) subject to the boundary conditions (23.18), (23.31), (23.33) and (23.46).

Our study of the Hermitian structures in the lattice model led us to conclude that the spaces $\mathcal{H}^{(\text{cont})}$ and $\mathcal{H}^{(\text{disc})}$ should be understood as a result of two differently defined scaling limits. In all likelihood the states from $\mathcal{H}^{(\text{disc})}$ and $\mathcal{H}^{(\text{cont})}$ can not be interpreted simultaneously as normalizable states within a single CFT. Perhaps the simplest idea for the field theory, whose quantization results in the pseudo-Hilbert space $\mathcal{H}^{(\text{disc})}$, is the model described by the same Lagrangian density and constraints (23.12), (23.13) as well as the boundary conditions (23.18) for the WZW currents and (23.34) for $\partial_\mu\eta$. However the fields now are subject to different reality conditions. The classical W currents should satisfy

$$\big(W_j^{(cl)}\big)^* = (-1)^j\ W_j^{(cl)}\,, \qquad \big(\overline{W}_j^{(cl)}\big)^* = (-1)^j\ \overline{W}_j^{(cl)}\,. \tag{23.74}$$

In view of eq. (23.49) this would follow from the reality conditions

$$\big(L^3\big)^* = -L^3\,, \qquad \big(L^{\pm}\big)^* = L^{\mp}\,, \qquad \big(R^3\big)^* = -R^3\,, \qquad \big(R^{\pm}\big)^* = R^{\mp} \tag{23.75}$$

imposed on the classical WZW currents and

$$\big(\partial\eta\big)^* = -\partial\eta\,, \qquad\qquad \big(\bar\partial\eta\big)^* = -\bar\partial\eta \tag{23.76}$$

for the Gaussian field. Furthermore $\mathrm{i}\eta$ is expected to be a real and compactified field,

$$\mathrm{i}\eta \sim \mathrm{i}\eta + 2\pi\ . \tag{23.77}$$

The latter implies that the zero mode momenta P_η and $\bar P_\eta$ (23.38) are no longer equal, but instead

$$\mathrm{i}\,(P_\eta - \bar P_\eta) \in \mathbb{Z}\ . \tag{23.78}$$

Notice that $B = \mathrm{e}^{2\pi P_\eta}$ and $\bar B = \mathrm{e}^{2\pi \bar P_\eta}$ appearing in the boundary conditions (23.45) still coincide. Such reality and boundary conditions for the currents correspond to the SU(2) WZW model gauged over the compact subgroup. However, they are not enough to fully specify the field theory. In the $\mathrm{SL}(2,\mathbb{R})$ case there were the additional constraints (23.31) and (23.33), whose motivation relied on the fact that the WZW field $\mathbf{G} \in \mathrm{SL}(2,\mathbb{R})$. At the moment, it is not clear to us what extra conditions need to imposed for the SU(2) case.

24. Lund-Regge model

The Yang-Baxter integrability of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model made possible a detailed numerical study of its critical behaviour. On the basis of this we formulated our central conjecture in sec. 21.2 regarding the space of states occurring in the scaling limit and made the identification of the sector $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{\mathrm{even}}$ with the pseudo-Hilbert space of the Lorentzian black hole NLSM. However the CFT itself does not assume any integrable structures and its space of states was described in terms of the representations of the algebra of extended conformal symmetry without reference to integrability. Nevertheless we believe that, finishing the paper, it would be instructive to discuss the integrable structure in the NLSM, inherited from the lattice system, within the context of the theory of partial differential equations solvable by the inverse scattering method.

24.1. Sklyanin exchange relations for the Lund-Regge model

Consider the Lagrangian density

$$\mathcal{L}_\varepsilon = \frac{1}{2}\,\frac{\partial_t U\partial_t V - \partial_x V\partial_x U}{1-UV} - \frac{\varepsilon^2}{2}\,UV\ , \tag{24.1}$$

which is a perturbation of that for the Lorentzian black hole NLSM (21.15) controlled by the parameter ε. We take the fields U and V to satisfy the periodic boundary conditions as in (21.23). The perturbation does not break the invariance w.r.t. the transformation (21.37) so that J_μ from (23.7) is still a Noether current. The continuity equation $\partial_\mu J^\mu = 0$ allows one to introduce the dual field

$$\tilde\Theta(\mathbf{x}) = -\int_{C_{\mathbf{x}}} \mathrm{d}x^\mu\,\epsilon_{\mu\nu}J^\nu\ . \tag{24.2}$$

Here $C_{\mathbf{x}}$ denotes an open integration contour which starts at an arbitrary chosen initial point and ends up at $\mathbf{x} = (t, x)$. Then in the co-ordinate frame (Φ, Θ) (21.51) the $\mathrm{SL}(2, \mathbb{R})$ matrix (21.12), whose matrix entries satisfy the conditions (21.14), has the form

$$\boldsymbol{g}_{\frac{1}{2}} = \begin{pmatrix} \pm\cos(\Phi)\,\mathrm{e}^{+\tilde{\Theta}} & \sin(\Phi)\,\mathrm{e}^{+\Theta} \\ -\sin(\Phi)\,\mathrm{e}^{-\Theta} & \pm\cos(\Phi)\,\mathrm{e}^{-\tilde{\Theta}} \end{pmatrix} . \tag{24.3}$$

Since U and V take values in regions III and IV from Fig. 16, the coordinate Φ was restricted to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. In this domain $\cos(\Phi)$ is positive so that the different signs "$\pm$" in (24.3) must be treated separately. However, one can take into account both cases at once if the domain of Φ is extended to the segment $[-\pi, \pi]$. Then the regions III and IV in Fig. 16 would be double covered. In fact, for the purposes of this subsection one can assume that the field Φ takes all possible real values, corresponding to a universal cover of $\mathrm{SL}(2, \mathbb{R})$.

One may still use eq. (21.17) to introduce the non-local fields $\xi_\pm$. The formal Lie group element, which appears in that relation, coincides with $\boldsymbol{g}_{\frac{1}{2}}$ when specialized to the fundamental representation. It can be written in the form of the Euler decomposition as

$$\boldsymbol{g} = \mathrm{e}^{\frac{1}{2}(\tilde{\Theta}+\Theta)\,\mathrm{h}}\;\mathrm{e}^{\Phi\,(\mathrm{e}_+-\mathrm{e}_-)}\;\mathrm{e}^{\frac{1}{2}(\tilde{\Theta}-\Theta)\,\mathrm{h}} . \tag{24.4}$$

The fields $\xi_\pm$ are no longer chiral. Instead, the Euler-Lagrange equations corresponding to $\mathcal{L}_\varepsilon$ imply

$$\bar{\partial}\xi_\pm = \frac{\varepsilon^2}{8}\,\sin(2\Phi)\,\mathrm{e}^{\pm(\Theta+\tilde{\Theta})} , \qquad \partial\bar{\xi}_\pm = \frac{\varepsilon^2}{8}\,\sin(2\Phi)\,\mathrm{e}^{\pm(\Theta-\tilde{\Theta})} . \tag{24.5}$$

The perturbation does not change the canonical momenta (21.21) and the Poisson bracket relations (21.20) continue to hold true. Hence $\xi_\pm$ and $\bar{\xi}_\pm$ still satisfy the PBs (20.9), (21.5) and (21.6) provided they are understood as equal-time relations.

Let $\boldsymbol{\mathcal{A}}_\mu$ be a space-time 1-form which takes values in the Lie algebra $\mathfrak{sl}_2$ and whose light-cone components, $\boldsymbol{\mathcal{A}} = \frac{1}{2}\,(\boldsymbol{\mathcal{A}}_0 + \boldsymbol{\mathcal{A}}_1)$ and $\bar{\boldsymbol{\mathcal{A}}} = \frac{1}{2}\,(\boldsymbol{\mathcal{A}}_0 - \boldsymbol{\mathcal{A}}_1)$, are defined as

$$\begin{aligned} \partial - \boldsymbol{\mathcal{A}} &= \quad \partial - \xi_-\,\mathrm{e}_- + \xi_+\,\mathrm{e}_+ - \lambda\,\mathrm{h} \\ \bar{\partial} - \bar{\boldsymbol{\mathcal{A}}} &= \boldsymbol{g}\,\big(\bar{\partial} + \bar{\xi}_-\,\mathrm{e}_- - \bar{\xi}_+\,\mathrm{e}_+ - \bar{\lambda}\,\mathrm{h}\big)\,\boldsymbol{g}^{-1} . \end{aligned} \tag{24.6}$$

It is straightforward to check using (24.5) that if the equations of motion are satisfied, then the connection $\partial_\mu - \boldsymbol{\mathcal{A}}_\mu$ is flat, i.e.,

$$[\,\partial - \boldsymbol{\mathcal{A}}, \bar{\partial} - \bar{\boldsymbol{\mathcal{A}}}\,] = 0 \tag{24.7}$$

provided the auxiliary spectral parameters λ and $\bar{\lambda}$ are related as

$$\lambda\,\bar{\lambda} = -\frac{\varepsilon^2}{16} . \tag{24.8}$$

The connection is not single valued on the space-time cylinder even when periodic boundary conditions are imposed on the fields U and V. Indeed as it follows from the definition of the dual field $\tilde{\Theta}$,

$$\exp\big(\tilde{\Theta}(t, x + 2\pi) - \tilde{\Theta}(t, x)\big) = \exp\bigg(\int_x^{x+2\pi} \mathrm{d}x'\,J_0\bigg) = B , \tag{24.9}$$

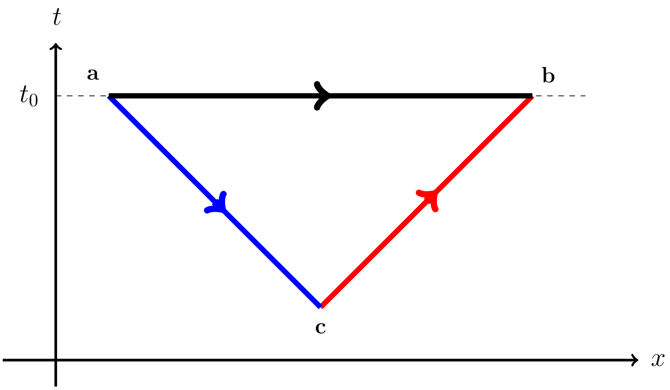

Fig. 17. The integration along the time slice $t = t_0$ (the segment $\vec{\mathbf{ab}}$) for the path ordered exponent $\boldsymbol{M}$ in eq. (24.12) can be replaced by an integration along the characteristics: $u = t_0$ with $t_0 < \bar{u} < t_0 - 2\pi$ ($\vec{\mathbf{ac}}$) and $\bar{u} = t_0 - 2\pi$ with $t_0 < u < t_0 + 2\pi$ ($\vec{\mathbf{cb}}$).

and therefore

$$\boldsymbol{g}(t, x + 2\pi) = B^{\frac{\mathrm{h}}{2}}\, \boldsymbol{g}(t, x)\, B^{\frac{\mathrm{h}}{2}} . \tag{24.10}$$

In turn, $\boldsymbol{\mathcal{A}}_\mu$ obeys the quasiperiodicity condition

$$\boldsymbol{\mathcal{A}}_\mu(t, x + 2\pi) = B^{\frac{\mathrm{h}}{2}}\, \boldsymbol{\mathcal{A}}_\mu(t, x)\, B^{-\frac{\mathrm{h}}{2}} . \tag{24.11}$$

Choosing some representation of $\mathfrak{sl}_2$, one can define the classical transfer matrix:

$$T_j = \mathrm{Tr}_j[\boldsymbol{M}] , \qquad \boldsymbol{M} = B^{-\frac{\mathrm{h}}{2}}\, \overleftarrow{\mathcal{P}}\exp\left(\int_0^{2\pi} \mathrm{d}x\, \boldsymbol{\mathcal{A}}_1\right) . \tag{24.12}$$

The zero curvature relation (24.7) implies that T_j does not depend on the time slice at which the path ordered integration is taken, i.e., it is an Integral of Motion. In fact, T_j is a one parameter family of IM, since the connection $\boldsymbol{\mathcal{A}}_\mu$ depends on λ, $\bar{\lambda}$ satisfying the constraint (24.8). The latter may be resolved by means of a single complex parameter β:

$$\lambda = \tfrac{\mathrm{i}}{4}\, \varepsilon\, \mathrm{e}^{2\beta} , \qquad \bar{\lambda} = \tfrac{\mathrm{i}}{4}\, \varepsilon\, \mathrm{e}^{-2\beta} . \tag{24.13}$$

An immediate question arises as to the mutual Poisson commutativity of $T_j(\beta)$ for different values of the spectral parameter. This may be addressed by directly following the line of arguments which were developed in the context of the sine-Gordon model in the work [71]. Since the connection is flat, the path ordered exponent in (24.12) is unchanged under deformations of the integration contour that keep the endpoints fixed. It is useful to swap the integration over the time slice to the one over the light-cone pieces as depicted in Fig. 17. The contribution of the integration along each of the characteristics $\bar{u} \equiv t - x = \overline{\text{const}}$ and $u \equiv t + x = \text{const}$ is taken into account by the two path ordered exponents $\boldsymbol{L}_\varepsilon^{(cl)}$ and $\bar{\boldsymbol{L}}_\varepsilon^{(cl)}$, respectively:

$$\boldsymbol{L}_\varepsilon^{(cl)}(\lambda) = \lambda^{-\frac{\mathrm{h}}{4}}\, \mathrm{e}^{\frac{\mathrm{i}}{2}\Phi_{\mathbf{cb}}\mathrm{h}}\, \boldsymbol{G}_{\mathbf{b}}\, \overleftarrow{\mathcal{P}}\exp\left(\int_{t_0}^{t_0+2\pi} \mathrm{d}u\, \big(\xi_-\mathrm{e}_- - \xi_+\mathrm{e}_+ + \lambda\,\mathrm{h}\big)\right)\Big|_{\bar{u}}\, \boldsymbol{G}_{\mathbf{c}}^{-1}\, \lambda^{+\frac{\mathrm{h}}{4}}$$

$$\bar{\boldsymbol{L}}_{\varepsilon}^{(cl)}(\bar{\lambda}) = \bar{\lambda}^{+\frac{\mathrm{h}}{4}}\, \bar{\boldsymbol{G}}_{\mathbf{c}}\, \vec{\mathcal{P}}\exp\bigg(\int_{t_0-2\pi}^{t_0} \mathrm{d}\bar{u}\,\big(\bar{\xi}_-\mathrm{e}_- - \bar{\xi}_+\mathrm{e}_+ - \bar{\lambda}\,\mathrm{h}\big)\bigg)\bigg|_u\, \bar{\boldsymbol{G}}_{\mathbf{a}}^{-1}\, \mathrm{e}^{\frac{\mathrm{i}}{2}\Phi_{\mathbf{ca}}\mathrm{h}}\, \bar{\lambda}^{-\frac{\mathrm{h}}{4}}\,. \tag{24.14}$$

Here $\Phi_{\mathbf{xy}} \equiv \Phi_{\mathbf{x}} - \Phi_{\mathbf{y}}$ and we use the shortcut notation $\Phi_{\mathbf{x}}$ to denote the value of the field $\Phi(\mathbf{x})$ at either one of the space-time points $\mathbf{x} = \mathbf{a}, \mathbf{b}, \mathbf{c}$ indicated in Fig. 17. Also $\boldsymbol{G}_{\mathbf{x}}$ and $\bar{\boldsymbol{G}}_{\mathbf{x}}$ read explicitly as

$$\begin{aligned} \boldsymbol{G}_{\mathbf{x}} &= \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}(\mathrm{e}_+ + \mathrm{e}_-)}\, \mathrm{e}^{-\frac{1}{2}(\tilde{\Theta}_{\mathbf{x}} + \Theta_{\mathbf{x}})\mathrm{h}}\,, \\ \bar{\boldsymbol{G}}_{\mathbf{x}} &= \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}(\mathrm{e}_+ + \mathrm{e}_-)}\, \mathrm{e}^{\frac{1}{2}(\tilde{\Theta}_{\mathbf{x}} - \Theta_{\mathbf{x}})\mathrm{h}} \qquad\qquad (\mathbf{x} = \mathbf{a}, \mathbf{b}, \mathbf{c})\,. \end{aligned} \tag{24.15}$$

The above formulae should be compared with eqs. (20.26) and (20.43) involving the path ordered exponents $\boldsymbol{L}^{(cl)}$ and $\bar{\boldsymbol{L}}^{(cl)}$. In fact, the definition (24.14) has been arranged in such a way so that $\boldsymbol{L}_{\varepsilon}^{(cl)}$ and $\bar{\boldsymbol{L}}_{\varepsilon}^{(cl)}$ satisfy the same Poisson algebra as in (20.30) and (20.41). Namely, one can show that

$$\begin{aligned} &\big\{\boldsymbol{L}_{\varepsilon}^{(cl)}(\lambda) \underset{,}{\otimes} \boldsymbol{L}_{\varepsilon}^{(cl)}(\lambda')\big\} = +\Big[\boldsymbol{L}_{\varepsilon}^{(cl)}(\lambda) \otimes \boldsymbol{L}_{\varepsilon}^{(cl)}(\lambda'),\, \boldsymbol{r}\big(\sqrt{\lambda/\lambda'}\,\big)\Big] \\ &\big\{\bar{\boldsymbol{L}}_{\varepsilon}^{(cl)}(\bar{\lambda}) \underset{,}{\otimes} \bar{\boldsymbol{L}}_{\varepsilon}^{(cl)}(\bar{\lambda}')\big\} = -\Big[\bar{\boldsymbol{L}}_{\varepsilon}^{(cl)}(\bar{\lambda}) \otimes \bar{\boldsymbol{L}}_{\varepsilon}^{(cl)}(\bar{\lambda}'),\, \boldsymbol{r}\big(\sqrt{\bar{\lambda}/\bar{\lambda}'}\,\big)\Big] \\ &\big\{\boldsymbol{L}_{\varepsilon}^{(cl)}(\lambda) \underset{,}{\otimes} \bar{\boldsymbol{L}}_{\varepsilon}^{(cl)}(\bar{\lambda})\big\} = 0 \end{aligned} \tag{24.16}$$

with $\boldsymbol{r}$ being the classical R-matrix (20.31).

A straightforward calculation yields that the monodromy matrix (24.12) may be brought to the form

$$\boldsymbol{M} = \boldsymbol{C}\,\tilde{\boldsymbol{M}}\,\boldsymbol{C}^{-1} \tag{24.17}$$

with

$$\tilde{\boldsymbol{M}} = \big(\tfrac{\mathrm{i}\varepsilon}{4}\big)^{\frac{\mathrm{h}}{2}}\; \boldsymbol{L}_{\varepsilon}^{(cl)}\; \big(\tfrac{\mathrm{i}\varepsilon}{4}\big)^{-\frac{\mathrm{h}}{2}}\; \mathrm{e}^{\mathrm{i}\Phi_{\mathbf{c}}\mathrm{h}}\; \bar{\boldsymbol{L}}_{\varepsilon}^{(cl)}\; \mathrm{e}^{-\mathrm{i}\Phi_{\mathbf{c}}\mathrm{h}}\,. \tag{24.18}$$

The matrix $\boldsymbol{C}$ has no effect on the value of the trace in (24.12) and is given by

$$\boldsymbol{C} = \mathrm{e}^{\frac{1}{2}(\tilde{\Theta}_{\mathbf{a}} + \Theta_{\mathbf{a}})\mathrm{h}}\, \mathrm{e}^{\frac{\mathrm{i}\pi}{4}(\mathrm{e}_+ + \mathrm{e}_-)}\, \mathrm{e}^{\frac{\mathrm{i}}{2}\Phi_{\mathbf{ac}}\mathrm{h}}\, \bar{\lambda}^{-\frac{\mathrm{h}}{4}}\,. \tag{24.19}$$

Generally speaking $\Phi_{\mathbf{c}}$, i.e., the value of the field Φ at the space-time point $\mathbf{c}$, as well as the difference $\Phi_{\mathbf{ab}} = \Phi_{\mathbf{a}} - \Phi_{\mathbf{b}}$ are dynamical variables. However, in order for the transfer matrix to be an IM, apart from the zero curvature relation (24.7), the connection must satisfy the quasiperiodic boundary condition (24.11) which, itself, requires Φ to be a periodic field. Due to this we need to impose $\Phi_{\mathbf{ab}} = 0$. The latter is a first class constraint and the corresponding gauge fixing condition can be achieved by assigning $\Phi_{\mathbf{c}}$ a certain value, say zero. Then in view of (24.18), (24.16) the matrix entries of $\tilde{\boldsymbol{M}}$, considered as a function of the spectral parameter β (24.13), turn out to satisfy the Sklyanin exchange relations

$$\big\{\tilde{\boldsymbol{M}}(\beta) \underset{,}{\otimes} \tilde{\boldsymbol{M}}(\beta')\big\} = \Big[\tilde{\boldsymbol{M}}(\beta) \otimes \tilde{\boldsymbol{M}}(\beta'),\, \boldsymbol{r}\big(\mathrm{e}^{\beta-\beta'}\big)\Big]. \tag{24.20}$$

Consequently,

$$\big\{T_{j'}(\beta'),\, T_j(\beta)\big\} = 0\,. \tag{24.21}$$

We have the following comments to make at this point. For the derivation of the Sklyanin exchange relations the model was considered, which is a deformation of the Lorentzian black hole NLSM. In this case the fundamental fields U and V are real. In fact, the reality conditions do not come into play and the same arguments may be applied when these fields are complex conjugate to each other, $V = U^*$ (the fields Θ and $\tilde{\Theta}$ are then pure imaginary so that $\boldsymbol{g}_{\frac{1}{2}}$ from (24.3) is a SU(2) matrix). With this reality condition imposed the model is usually referred to as the complex sine-Gordon I, or Lund-Regge model. The zero curvature relation, which is also insensitive to the reality conditions, was proposed in the works [72–74] for a connection in a gauge different to (24.6).

The Lund-Regge model attracted a great deal of attention in the context of the so-called non-ultralocality problem. In the ultralocal case the x-component of the flat connection satisfies the equal-time Poisson bracket relations of the form

$$\big\{\boldsymbol{A}_1(x|\lambda_1) \overset{\otimes}{,} \boldsymbol{A}_1(y|\lambda_2)\big\} = \big[\boldsymbol{A}_1(x|\lambda_1)\otimes\mathbf{1} + \mathbf{1}\otimes\boldsymbol{A}_1(y|\lambda_2), \boldsymbol{r}(\lambda_1/\lambda_2)\big]\,\delta(x-y)\ . \tag{24.22}$$

Then the Sklyanin exchange relations for the monodromy matrix are easily derived and the mutual Poisson commutativity condition for the transfer matrix for different values of the spectral parameter follows. The ultralocality condition, of course, depends on the gauge of $\boldsymbol{A}_\mu$. Considerable effort was made to find an "ultralocal" flat connection for the Lund-Regge model. However, as with many other interesting field theories admitting the zero curvature relation, the attempts were met with failure. This motivated the development of the so-called "canonical $r-s$ matrix" approach for integrable two dimensional models of non-ultralocal type [75].

Inspired by the observation that the quantum monodromy matrix is somehow better behaved than its classical limit, in the work [46] it was proposed to tackle the non-ultralocality problem by starting with the quantum counterpart of the Sklyanin exchange relations – the Yang-Baxter algebra. It was demonstrated on a specific example that by taking the classical limit of the quantum algebra one could recover the Sklyanin exchange relations for the classical monodromy matrix without reference to an ultralocal gauge. Here another illustration of this approach is given for the case of the Lund-Regge model. Indeed, the starting point was the Yang-Baxter algebra (14.5) which in the classical limit becomes the Sklyanin exchange relations for $\boldsymbol{L}^{(cl)}$ and $\bar{\boldsymbol{L}}^{(cl)}$. Then the derivation outlined above, which led to (24.20), is straightforward and almost identical to that of the work [71] for the sine-Gordon model.

24.2. UV limit of the quantum complex sinh-Gordon I model

Not much is known about the QFT corresponding to the classical Lagrangian density (24.1) with real fields U and V as well as the quantum complex sine-Gordon I model, where $U = V^*$. However the variant of the Lund-Regge model, which is sometimes referred to as the complex sinh-Gordon I model, is a well understood QFT now [62]. It is an integrable deformation of the Euclidean black hole NLSM (22.3), whose classical action is given by

$$S_{\rm cshG} = \frac{n+2}{4\pi}\int \mathrm{d}t \int_0^{2\pi} \mathrm{d}x\ \left[\frac{\partial_t U\partial_t U^* - \partial_x U\partial_x U^*}{1+UU^*} - \varepsilon^2\, UU^*\right] \qquad \big(n\to+\infty\big) \tag{24.23}$$

and the complex field U is subject to the quasiperiodic boundary conditions (22.4).

The IR behaviour of the QFT is described in terms of the factorizable scattering theory. The particle content of the complex sinh-Gordon I model consists of a doublet of the same mass m having opposite U(1) charges. The two particle S-matrix is diagonal (reflectionless). Notice that in the action (24.23), t and x are assumed to be dimensionless world-sheet co-ordinates, and x has been brought to the standard segment $x \in [0, 2\pi]$. If one were to restore the dimensions, x would belong to the interval $[0, R]$ with R having units of length. Then the parameter ε is identified as

$$\varepsilon = \frac{mR}{2\pi} \ . \tag{24.24}$$

To study the UV limit it is convenient to make use of the dual description of the model. The latter is based on the remarkable proposal originally put forward by Al. B. Zamolodchikov [22], that the so-called sine-Liouville theory provides a dual description of the quantum Euclidean black hole NLSM (see also [76]). The duality relation is easily extended to the deformed model (24.23) and the dual action reads as [62]

$$S_{\rm cshG}^{(\rm dual)} = \int {\rm d}t \int_0^R {\rm d}x\, \Big(\tfrac{1}{4\pi}\big[\,(\partial_\mu\varphi)^2 + (\partial_\mu\vartheta)^2\,\big] - 2g\,{\rm e}^{-\sqrt{n}\,\varphi}\, \cos\big(\sqrt{n+2}\,\vartheta\big) - g'\,{\rm e}^{\frac{2\varphi}{\sqrt{n}}}\Big) \tag{24.25}$$

(the fields φ and ϑ should not be confused with the chiral Bose fields used in the main body of this work). Despite that the action (24.25) formally depends on three parameters, one of these may be eliminated by a constant shift of the field φ. This way the relevant coupling constant in the theory is the combination $g^{\frac{2}{n}}g'$. It turns out that the particle mass m in (24.24) is related to the parameters in (24.25) as

$$(Cm)^{\frac{2(n+2)}{n}} = \Big(\frac{\pi}{n}\Big)^{\frac{2}{n}}\, \frac{\pi\,\Gamma(\frac{1}{n})}{\Gamma(1-\frac{1}{n})}\, g^{\frac{2}{n}}\, g' \,,$$
$$\text{where } C = (2n)^{-\frac{2}{n+2}}\, \frac{\Gamma(\frac{1}{n+2})\Gamma(\frac{3}{2}-\frac{1}{n+2})}{\sqrt{\pi}} \ . \tag{24.26}$$

The existence of the dual description (24.25) allows one to adapt the arguments of the work [77], which were applied for the quantum sinh-Gordon model. Namely, one starts with the Euclidean black hole NLSM, corresponding to the situation when the correlation length is infinite, i.e., $m = 0$. As was already mentioned, the space of states of this CFT is classified according to the highest weight irreps of the $\overline{W}_\infty \otimes W_\infty$ - algebra with the central charge $c > 2$. In particular, the decomposition of the continuous component of the Hilbert space is given in (22.8). Each state in the irrep is characterized by the values of $s,\ p,\ \bar{p}$, parameterizing the highest weight (22.7) as well as their levels $\mathtt{L}$, $\bar{\mathtt{L}}$. To resolve the degeneracy in the level subspaces, one can use the eigenbasis of the mutually commuting set of local IM. Then for fixed $s,\ p,\ \bar{p}$, $\mathtt{L}$ and $\bar{\mathtt{L}}$ the states would be specified by two sets $\boldsymbol{w} = \{w\}_{a=1}^{\mathtt{L}}$ and $\bar{\boldsymbol{w}} = \{\bar{w}\}_{a=1}^{\bar{\mathtt{L}}}$ solving a certain algebraic system. The latter coincides with (10.3), where n is substituted by $-n-2$, while the pair (s, p) is swapped for $({\rm i}p, {\rm i}s)$ in (10.3a) and $(s, \bar{p}) \mapsto ({\rm i}\bar{p}, {\rm i}s)$ in (10.3b). Following ref. [77], when the correlation length m^{-1} is much larger than R but finite, the UV behaviour of the energy of a state from the eigenbasis is described by the formula

$$E_{\rm cshG}(R) = \frac{2\pi}{R}\Big(-\frac{1}{6} + \frac{2s^2}{n} + \frac{p^2+\bar{p}^2}{n+2} + \mathtt{L} + \bar{\mathtt{L}} + O\big((\log\varepsilon)^{-\infty}\big)\Big) \ . \tag{24.27}$$

Here $s = s(R)$ and the R dependence, up to power law corrections, is determined through the quantization condition

$$(C\varepsilon)^{-\frac{4\mathrm{i}s}{n}\,(n+2)}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\delta_{\mathrm{cshG}}} = 1 + O\big((\log\varepsilon)^{-\infty}\big) \tag{24.28}$$

with $\delta_{\mathrm{cshG}} = \delta_{\mathrm{cshG}}(\bar{\boldsymbol{w}}, \boldsymbol{w}\,|\,\bar{p}, p, s)$. There are strong similarities between (24.27), (24.28) and the formulae (8.2a), (9.11) describing the scaling behaviour of the energy for the Bethe states in the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model. However in the lattice system the phase shift is related to the eigenvalue of the reflection operator $\check{\mathbf{D}}$ considered in the parametric domain corresponding to $c < 2$. Contrary to this, δ_{cshG} is expressed in terms of another reflection operator in the domain $c > 2$:

$$\mathrm{e}^{\frac{\mathrm{i}}{2}\delta_{\mathrm{schG}}} = \frac{\Gamma(\frac{1}{2}+p-\mathrm{i}s)\Gamma(\frac{1}{2}+\bar{p}-\mathrm{i}s)\Gamma^2(1+2\mathrm{i}s)\Gamma^2(1+\frac{2\mathrm{i}s}{n})}{\Gamma(\frac{1}{2}+p+\mathrm{i}s)\Gamma(\frac{1}{2}+\bar{p}+\mathrm{i}s)\Gamma^2(1-2\mathrm{i}s)\Gamma^2(1-\frac{2\mathrm{i}s}{n})}\;\check{R}^{(c>2)}_{\bar{p},s}(\bar{\boldsymbol{w}})\;\check{R}^{(c>2)}_{p,s}(\boldsymbol{w})\;, \tag{24.29}$$

where it is assumed that the constant C in (24.28) is the same as in (24.26). Recall, the check notation means the reflection operators have been normalized so that their eigenvalue for the highest state is one. An explicit formula for $\check{R}^{(c>2)}_{p,s}(\boldsymbol{w})$ was obtained in ref. [47].[19] In the case $p = \bar{p} = \mathtt{L} = \bar{\mathtt{L}} = 0$, the quantization condition (24.28) reduces to the one from [77], for the ground state of the sinh-Gordon model with the sinh-Gordon coupling constant $b = \sqrt{\frac{2}{n}}$. This is related to the fact that the system of coupled thermodynamic Bethe ansatz equations describing the ground state energy of the complex sinh-Gordon I model with periodic boundary conditions ($\mathtt{k} = 0$), boils down to a single integral equation which is identical to the one describing the vacuum energy of the sinh-Gordon model.

Formulae (24.27) and (24.28) afford an interpretation that sheds some light on an important previously made point. They define a particular integrable IR regularization for the target space manifold of the Euclidean black hole NLSM, which is different to the one discussed in refs. [24, 25]. To illustrate, consider the form of the dual action (24.25). Setting $g' = 0$ therein, one obtains the dual action for the NLSM. In this case, in the domain of the configuration space with $\varphi \to +\infty$ the dual Lagrange density becomes that of two non-interacting Bose fields. This corresponds to the asymptotically flat domain for the target space manifold. The addition of the Liouville wall potential $\propto g'\,\mathrm{e}^{\frac{2\varphi}{\sqrt{n}}}$ into (24.25) works as an IR regularization for the NLSM target space. It effectively restricts the value of the non-compact field φ to the finite interval $\propto \log(1/\varepsilon)$, which results in the quantization of its zero-mode momentum according to (24.29). Taking the limit $\varepsilon \to 0$ the continuous spectrum is restored but with a certain density of states, similar to as in the lattice model. The latter comes up, for instance, in the computation of the partition function for the Euclidean black hole NLSM, see eq. (22.15). The explicit formula for the density of states can be obtained using the identity

[19] The eigenvalues $\check{R}^{(c>2)}_{p,s}(\boldsymbol{w})$ coincide with those of the reflection operator $\check{\mathbb{R}}^{(\mathrm{AKNS})}$ defined by (3.36) in ref. [47] provided the parameters are identified as $P_1 = \frac{s}{\sqrt{n}}$, $P_2 = \frac{p}{\sqrt{n+2}}$ and $\sqrt{k} = \sqrt{n}$. Note that $\check{R}_{p,s}(\boldsymbol{w})$, which is given explicitly in Appendix B, corresponds to the case $c < 2$ and coincides with the eigenvalues of $\check{\mathbb{R}}^{(\mathrm{AKNS})}$ with the different identification of the parameters: $P_1 = \frac{p}{\sqrt{n+2}}$, $P_2 = \frac{s}{\sqrt{n}}$, $\sqrt{k} = -\mathrm{i}\sqrt{n+2}$.

$$\prod_{\substack{\boldsymbol{w} \\ \mathtt{L}-\text{fixed}}} \check{R}^{(c>2)}_{p,s}(\boldsymbol{w}) = \prod_{\substack{1\le j,m \\ jm\le \mathtt{L}}} \left[\frac{nj+m-2\mathrm{i}s}{nj+m+2\mathrm{i}s}\right]^{2\,\mathrm{par}_2(\mathtt{L}-mj)} \times \prod_{a=0}^{\mathtt{L}-1}\left[\frac{\left(\frac{1}{2}+a+p-\mathrm{i}s\right)\left(\frac{1}{2}+a-p-\mathrm{i}s\right)}{\left(\frac{1}{2}+a+p+\mathrm{i}s\right)\left(\frac{1}{2}+a-p+\mathrm{i}s\right)}\right]^{\mathrm{par}_2(\mathtt{L})-d_a(\mathtt{L})}, \tag{24.30}$$

where the integers $d_a(\mathtt{L})$ are defined via (10.25). The above relation is similar to (10.24), which was used in the derivation of the density of states (10.12). Thus apart from the regularization of the Euclidean black hole considered in [24,25], there is another integrable IR regularization of the target manifold, which yields a different density of states for the continuous spectrum. This illustrates the statement made at the end of sec. 21.3, that the density of states is not an intrinsic property of the CFT but depends on the (IR) regularization of the model.

25. Summary

The work contains a detailed study of the scaling limit of the critical $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model in the parametric domain where $\arg(q^2)\in(0,\pi)$ and subject to twisted boundary conditions. The Yang-Baxter integrability implies the set of Bethe ansatz equations characterizing the Bethe states. These form a basis in the 2^N dimensional space of states of the model defined on the lattice with N columns. On the one hand, our analysis was based on a numerical study of the Bethe ansatz equations at large N. On the other, we used the powerful analytical technique of the ODE/IQFT correspondence. The combination of numerical and analytical methods allows one to investigate in detail the scaling behaviour of not only the vacuum state in each sector with fixed value of S^z, but also the excited states as well. Below is a summary of the main outcomes of our work.

- The linear space of states $\mathcal{H}$ occurring in the scaling limit of the low energy states of the lattice system is classified w.r.t. the highest weight irreps of the $\overline{W}_\infty\otimes W_\infty$ - algebra with central charge $-1<c<2$. The space splits into two components $\mathcal{H}^{(\mathrm{cont})}$ and $\mathcal{H}^{(\mathrm{disc})}$ depending on whether the spectrum of the highest weights is continuous or discrete. The linear decomposition of both these components into the irreps is provided in sections 17.1 and 17.2 for generic values of the twist parameter $\mathtt{k}$.
- The scaling limit of the lattice model yields a certain density of states for the sector $\mathcal{H}^{(\mathrm{cont})}$. With this at hand, an expression was obtained in sec. 17.3 for the scaling limit of the partition function $Z^{(\mathrm{scl})}$ in the form of a series expansion in the modular nome(s), which is applicable for numerical study.
- We confirmed the remarkable proposal of the work [11], that one half of the partition function $Z^{(\mathrm{scl})}$ for the case of periodic boundary conditions ($\mathtt{k}=0$) coincides with the partition function Z_{EBH} of the Euclidean black hole Non-Linear Sigma Model, which was obtained in refs. [24,25] assuming a certain IR regularization of the target manifold. For generic $\mathtt{k}$ we checked that $\frac{1}{2}Z^{(\mathrm{scl})}$ likewise coincides with the partition function of the properly regularized Euclidean black hole NLSM with twisted boundary conditions.
- The finite dimensional space of states of the lattice possesses a variety of Hermitian structures with the inner product being such that the Bethe states obey a certain orthogonality condition. In sections 19.3 and 19.5 the Hermitian structures were identified, which in the scaling limit induces the inner products in $\mathcal{H}^{(\mathrm{cont})}$ and $\mathcal{H}^{(\mathrm{disc})}$ that are consistent with the nat-

ural conjugation conditions in the $\overline{W}_\infty \otimes W_\infty$ - algebra. Since the central charge $c < 2$, the inner products are not positive definite ones, so that $\mathcal{H}^{(\mathrm{cont})}$ and $\mathcal{H}^{(\mathrm{disc})}$ possess the structure of the pseudo-Hilbert space. We were led to conclude that the states from $\mathcal{H}^{(\mathrm{cont})}$ and $\mathcal{H}^{(\mathrm{disc})}$ can not be interpreted simultaneously as normalizable states within a single CFT. We believe that these two spaces should be considered as being the result of different scaling limits.

- The algebra of extended conformal symmetry of the Euclidean black hole NLSM is the $\overline{W}_\infty \otimes W_\infty$ - algebra but with the central charge $c > 2$. Moreover, the Hilbert space is equipped with a positive definite inner product and the QFT is unitary. For these reasons we reject the proposal that the Euclidean black hole NLSM underlies the critical behaviour of the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model, despite the spectacular coincidence of the partition functions Z_{EBH} and $\frac{1}{2}\, Z^{(\mathrm{scl})}$.
- We revised the original conjecture of [11] in the following way. In the case of periodic boundary conditions ($\mathtt{k} = 0$) the lattice model possesses an additional global symmetry, that of $\mathcal{C}$ conjugation which, in turn, is inherited by the space $\mathcal{H}$. The $\mathcal{C}$-even sector of $\mathcal{H}^{(\mathrm{cont})}$ contains the subspace $\tilde{\mathcal{H}}^{(\mathrm{cont})}_{\mathrm{even}}$ and we propose that it coincides with the pseudo-Hilbert space of the Lorentzian black hole NLSM for the space-like domain of the target manifold. Since the status of the Lorentzian black hole NLSM is tentative, our conjecture is essentially an attempt to assign a meaning to the QFT that goes beyond the classical limit and minisuperspace approximation. In sec. 23 some proposals are made concerning the field theory interpretation for the case of twisted boundary conditions with $\mathtt{k} \neq 0$. The local CFT, if it exists, whose pseudo-Hilbert space coincides with $\mathcal{H}^{(\mathrm{cont})}$, in the classical limit with $c \to 2^-$ is described by the gauged $\mathrm{SL}(2, \mathbb{R})$ WZW model subject to certain boundary conditions imposed on the fields.
- Among the numerous offshoots of our study, of special mention is the solution of the long standing non-ultralocality problem for the complex sine-Gordon I (Lund-Regge) model. Adapting the ideas developed in ref. [46], we traced the appearance of the classical Sklyanin exchange relations in this model (see sec. 24.1 for details).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

Formula (6.23) involved in the description of the norms of the Bethe states in the homogeneous six-vertex model contains the constants C and C_0. These depend only on the anisotropy parameter β^2 entering into the Hamiltonian $\mathbb{H}_{XXZ}$ (3.1). Though the analytical form of this dependence is not yet known, it is straightforward to obtain C and C_0 numerically for any $0 < \beta^2 < 1$. Here we present the corresponding numerical data, which was already quoted in sec.11.2 from ref. [37]. We also provide similar data for $C_0^{(\mathrm{alt})}$ appearing in eqs. (19.40), (19.42), (19.54) and (19.75), that enter into the description of the scaling limit of the Bethe states in the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model.

Using the fact that for $\beta^2 = \frac{1}{2}$ the homogeneous six-vertex model can be reformulated as a non-interacting system of 1D lattice fermions, it is possible to show that

$$\beta C\big|_{\beta^2=\frac{1}{2}} = \pi \qquad \text{and} \qquad C_0\big|_{\beta^2=\frac{1}{2}} = (\pi\,\mathrm{e})^{-\frac{1}{12}}\, A_{\mathrm{G}} = 1.07254 \tag{A.1}$$

with A_{G} being the Glaisher constant. From the numerical data, one expects

$$\lim_{\beta\to 0} \beta C = \mathrm{e}\,, \qquad \lim_{\beta\to 1} \beta C = \left(\frac{\pi}{2}\right)^3, \tag{A.2}$$

whereas

$$C_0\big|_{\beta=1} = C_0^2\big|_{\beta^2=\frac{1}{2}} = 1.15034\,. \tag{A.3}$$

A plot of $\log(\beta C)$ as a function of β^2 is given in Fig. 18 and some numerical values are provided in Table 4. As for C_0, following ref. [37], it is convenient to re-write it in the form

$$C_0 = \beta^{-\frac{1}{3}}\, \mathrm{e}^{-\frac{1}{6}(\beta^{-1}-\beta)^2} \left(\frac{\mathrm{e}^{\gamma_{\mathrm{E}}+1}}{4\pi}\right)^{\frac{1}{12}} \left(\mathrm{e}^{-\frac{1}{6}(\gamma_{\mathrm{E}}+1)}\, A_{\mathrm{G}}^2\right)^{\beta^2}\, \tilde{C}_0\,, \tag{A.4}$$

where for the free fermion case $\tilde{C}_0|_{\beta^2=\frac{1}{2}} = 1$. More generally $|\tilde{C}_0 - 1| < 0.003$ within the domain $0 \le \beta^2 \le 0.85$. Fig. 18 includes a plot of $\tilde{C}_0$, while some of its numerical values are quoted in Table 4.

The values of C entering into eqs. (19.40), (19.42), (19.54) and (19.75) for the description of the norms of the Bethe states in the $\mathcal{Z}_2$ invariant inhomogeneous six-vertex model, are the same as those given above provided one sets $\beta^2 = \frac{2}{n+2}$. These formulae in addition contain the n dependent constant $C_0^{(\mathrm{alt})}$. It turns out that X, defined through

$$\left(C_0^{(\mathrm{alt})}\right)^2 = 2X\,(\beta C)^{-1+\frac{1}{2}\beta^2}\, C_0^4 \qquad \left(\beta = \sqrt{\tfrac{2}{n+2}}\,\right) \tag{A.5}$$

is well approximated by a linear function of β^2, as shown in Fig. 19. Some numerics for X is contained in Table 5.

Appendix B

In this appendix we explain the assumptions and fill some gaps in the derivation that lead to conjectures (A) and (B) from sec. 10.2. As before we will always take $\mathtt{L}$ and $\bar{\mathtt{L}}$ to be some fixed non-negative integers; p and $\bar{p}$ to be real numbers such that $p + \bar{p} = S^z = 0, 1, 2, \ldots$; and s to belong to the strip $0 < \Im m(s) \le \frac{n}{4}$.

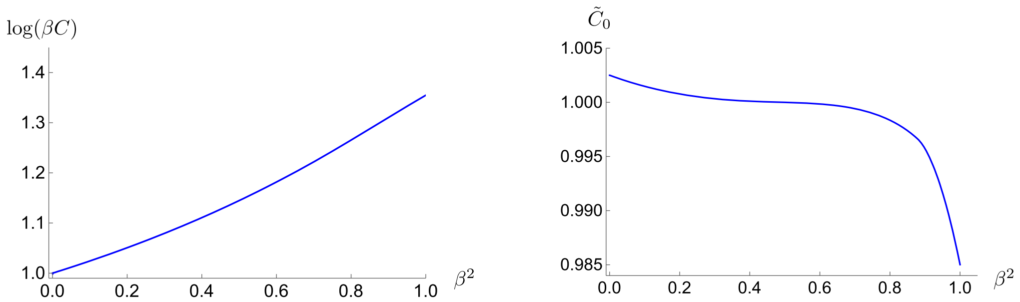

Fig. 18. Numerical data for $\log(\beta C)$ and $\tilde{C}_0$ (for $\tilde{C}_0$ see definition (A.4)) was interpolated and the result is plotted in the left and right panels, respectively, as a function of the parameter β^2. Some of the values from which the interpolation was obtained are listed in Table 4.

Table 4
Numerical data for $\log(\beta C)$ and $\tilde{C}_0$ (the latter is defined via (A.4)). The expected accuracy is indicated by the number of digits that are presented. Note that for $\beta^2 = 0.5$, corresponding to the free fermion case, $\tilde{C}_0 = 1$ and $\log(\beta C) = \log(\pi)$. The values of $\tilde{C}_0$ and $\log(\beta C)$ at $\beta^2 = 0$ given in the table are the result of interpolation, while for $\beta^2 = 1$ they follow from (A.2) and (A.3).

β^2	$\log(\beta C)$	$\tilde{C}_0$
0.0	1.00000	1.00250
0.1	1.02434	1.00148
0.2	1.05074	1.00077
0.3	1.07945	1.00033
0.4	1.11070	1.00011
0.5	1.14473	1.00000
0.6	1.18178	0.99985
0.7	1.22201	0.99942
0.8	1.26541	0.99834
0.9	$1.3102\overline{3}$	$0.9957\overline{2}$
1.0	1.35475	0.98501

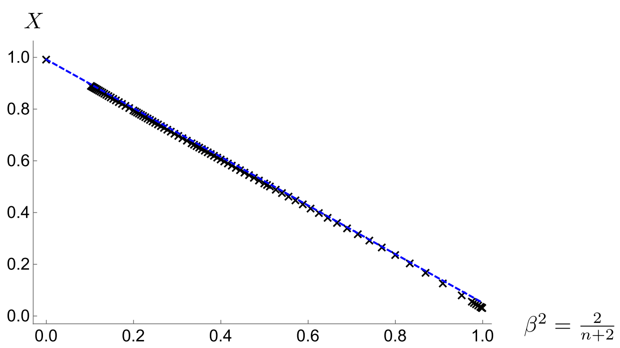

Fig. 19. A plot of X from (A.5) as a function of $\beta^2 = \frac{2}{n+2}$. The crosses correspond to the numerical data, a portion of which can be found in Table 5, while the dashed line represents the linear fit $0.991951 - 0.94084\,\beta^2$. Notice that $X\big|_{\beta^2=1} = 0.0391$.

Table 5
Some numerical values for the constant X, defined through (A.5).

n	X	n	X
2.0	0.521203985	4.0	0.676335763
2.5	0.572499215	5.0	0.721418356
3.0	0.613818210	6.0	0.755474688
3.5	0.647834794	7.0	0.782110959

The function $D_{p,s}(\boldsymbol{w})$ entering into (10.20) reads as

$$D_{p,s}(\boldsymbol{w}) = 2^{\frac{2\mathrm{i}(n+2)s}{n}} \, \frac{\Gamma(\frac{1}{2}+p-\mathrm{i}s)}{\Gamma(\frac{1}{2}+p+\mathrm{i}s)} \; \check{D}_{p,s}(\boldsymbol{w}) \,, \tag{B.1}$$

where the explicit analytical expression for $\check{D}_{p,s}(\boldsymbol{w})$ in terms of p, s and the solution set $\boldsymbol{w} = \{w_a\}_{a=1}^{\mathtt{L}}$ was obtained in ref. [47]. Formula (3.11) from that work gives

$$\check{D}_{p,s}(\boldsymbol{w}) = (-1)^{\mathtt{L}} \prod_{a=1}^{\mathtt{L}} \frac{p+a-\frac{1}{2}-\mathrm{i}s}{p+a-\frac{1}{2}+\mathrm{i}s} \; \frac{\det\big(w_a^{b-1}\, V_a^{(+)}(b)\big)}{\det\big(w_a^{b-1}\, V_a^{(-)}(b)\big)} \,, \tag{B.2}$$

where

$$\begin{aligned} V_a^{(\pm)}(D) = (D-1)^2 - \bigg(2p+2+n \mp 2w_a + \sum_{b\neq a}^{\mathtt{L}} \frac{4w_a}{w_a-w_b}\bigg)\,(D-1) \\ + \tfrac{1}{2}\,n^2 + \big(p+\tfrac{3}{2}\big)\, n \mp (n+1+2p \pm 2\mathrm{i}s)\; w_a + 2p+1 \\ + \bigg(\sum_{b\neq a}^{\mathtt{L}} \frac{2w_a}{w_a-w_b}\bigg)^2 + \big(4p+2\mp 4w_a+n\big) \sum_{b\neq a}^{\mathtt{L}} \frac{w_a}{w_a-w_b} \,. \end{aligned} \tag{B.3}$$

Note that there exists a similar expression for $R_{p,s}(\boldsymbol{w})$ – the coefficient which appears in the asymptotic formula (11.24) for a product over the Bethe roots. Namely $R_{p,s}(\boldsymbol{w}) = R_{p,s}^{(0)}\, \check{R}_{p,s}(\boldsymbol{w})$ with $R_{p,s}^{(0)}$ from (11.26) and $\check{R}_{p,s}(\boldsymbol{w})$, instead of a ratio of two determinants as in (B.2), is given by

$$\check{R}_{p,s}(\boldsymbol{w}) = \frac{(-1)^{\mathtt{L}}}{\prod_{a=1}^{\mathtt{L}} w_a^2} \; \frac{\det\big(w_a^{b-1}\, V_a^{(+)}(b)\big)\, \det\big(w_a^{b-1}\, V_a^{(-)}(b)\big)}{\prod_{b>a}(w_b-w_a)^2 \prod_{a=1}^{\mathtt{L}}(2p+2a-1+2\mathrm{i}s)\big(2p+2a-1-2\mathrm{i}s\big)} \,. \tag{B.4}$$

It was pointed out in ref. [47] that the product of $\check{D}_{p,s}(\boldsymbol{w})$ over all the $\mathrm{par}_2(\mathtt{L})$ solutions sets $\boldsymbol{w}$ of (10.3a) with fixed $\mathtt{L}$ admits a simple form. According to formula (5.23) from that work one has

$$\begin{aligned} &\prod_{\substack{\boldsymbol{w} \\ \mathtt{L}-\text{fixed}}} \check{D}_{p,s}(\boldsymbol{w}) \\ &= \prod_{m=1}^{\mathtt{L}} \prod_{\substack{1\leq j,k \\ jk\leq m}} \bigg[\frac{(2p-2\mathrm{i}s+2k-j)\,(2p+2\mathrm{i}s-2k+j)}{(2p-2\mathrm{i}s-2k+j)\,(2p+2\mathrm{i}s+2k-j)}\bigg]^{\mathtt{par}_1(m-kj)\,\mathtt{par}_1(\mathtt{L}-m)} \,. \end{aligned} \tag{B.5}$$

This expression can be rewritten in a way which is more convenient for an analysis of the condition (10.20). First one should split the r.h.s. of (B.5) into two terms, where the index j runs over even numbers $j=2\ell$ and odd numbers $j=2\ell-1$, respectively:

$$\prod_{\substack{\boldsymbol{w}\\ \mathtt{L}-\text{fixed}}} \check{D}_{p,s}(\boldsymbol{w}) = \prod_{m=1}^{\mathtt{L}} \prod_{\substack{1\le \ell,k\\ 2\ell k\le m}} \left[\frac{(2p-2\mathrm{i}s+2k-2\ell)\,(2p+2\mathrm{i}s-2k+2\ell)}{(2p-2\mathrm{i}s-2k+2\ell)\,(2p+2\mathrm{i}s+2k-2\ell)}\right]^{\mathtt{par}_1(m-2\ell k)\,\mathtt{par}_1(\mathtt{L}-m)}$$
$$\times \prod_{m=1}^{\mathtt{L}} \prod_{\substack{1\le \ell,k\\ (2\ell-1)k\le m}} \left[\frac{(2p-2\mathrm{i}s+2k-2\ell+1)\,(2p+2\mathrm{i}s-2k+2\ell-1)}{(2p-2\mathrm{i}s-2k+2\ell-1)\,(2p+2\mathrm{i}s+2k-2\ell+1)}\right]^{\mathtt{par}_1(m-(2\ell-1)k)\,\mathtt{par}_1(\mathtt{L}-m)} . \tag{B.6}$$

The first line in the r.h.s. of the above equation is one since the numerator coincides with the denominator when the dummy variables are swapped $k\leftrightarrow\ell$. As for the second line, it contains poles and zeroes at $s=\pm\mathrm{i}\,(p+\frac{1}{2}+a)$ with integer $a=-\mathtt{L},-\mathtt{L}+1,\ldots,\mathtt{L}$. To compute their multiplicity consider, for instance, the zero at $s=\mathrm{i}\,(p+\frac{1}{2}+a)$ with $a\ge 0$. The relevant terms are the first factor in the numerator with $k=a+\ell$ and the first factor of the denominator, where $\ell=a+k+1$. Counting the number of times they occur in the product leads to the following expression for the multiplicity of the zero

$$\sum_{m=1}^{\mathtt{L}} \mathtt{par}_1(\mathtt{L}-m)\sum_{j\ge 1}\mathtt{par}_1\big(m-(2j-1)(j+a)\big)-\mathtt{par}_1\big(m-j\,(2j+2a+1)\big)$$
$$=\sum_{m=1}^{\mathtt{L}} \mathtt{par}_1(\mathtt{L}-m)\sum_{j\ge 1}(-1)^{j+1}\,\mathtt{par}_1\big(m-j\,(a+\tfrac{j+1}{2})\big) .$$

This way, and using the identity $\mathrm{par}_2(\mathtt{L})=\sum_{m=0}^{\mathtt{L}}\mathrm{par}_1(\mathtt{L}-m)\,\mathrm{par}_1(m)$, one arrives at

$$\prod_{\substack{\boldsymbol{w}\\ \mathtt{L}-\text{fixed}}} \check{D}_{p,s}(\boldsymbol{w}) = \prod_{a=0}^{\mathtt{L}-1}\left[\frac{\big(\frac{1}{2}+a+p-\mathrm{i}s\big)\big(\frac{1}{2}+a-p-\mathrm{i}s\big)}{\big(\frac{1}{2}+a+p+\mathrm{i}s\big)\big(\frac{1}{2}+a-p+\mathrm{i}s\big)}\right]^{\mathtt{par}_2(\mathtt{L})-d_a(\mathtt{L})} \tag{B.7}$$

with

$$d_a(\mathtt{L})=\sum_{m=0}^{\mathtt{L}}\mathtt{par}_1(\mathtt{L}-m)\sum_{j\ge 0}(-1)^j\,\mathtt{par}_1\big(m-j\,(a+\tfrac{j+1}{2})\big) . \tag{B.8}$$

A direct computation yields that the generating function for the integers (B.8) is $\chi_a(\mathtt{q})$ from (10.25).

Conjecture (A) concerns the possible positions of the singularities of $D_{\bar{p},s}(\bar{\boldsymbol{w}})\,D_{p,s}(\boldsymbol{w})$ as a function of s, where $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$ solve the algebraic system (10.3) that contains s as a parameter. It is instructive to consider first the simplest case that is not the vacuum with $\mathtt{L}=1$ and $\bar{\mathtt{L}}=0$. When $\mathtt{L}$ is set to one eq. (10.3a) becomes a quadratic for $w\equiv w_1$, whose two solutions are:

$$w_\pm=-\frac{n+1}{2n}\left(2\mathrm{i}\,s\pm\sqrt{n(n+2)}\,\sqrt{1-\frac{4p^2}{(n+1)^2}-\frac{4s^2}{n(n+2)}}\right) . \tag{B.9}$$

In turn

$$D_{\bar{p},s}(\bar{\boldsymbol{w}})D_{p,s}(\boldsymbol{w})\big|_{\bar{\mathtt{L}}=0,\mathtt{L}=1}=2^{\frac{4\mathrm{i}(n+2)s}{n}}\ \frac{\Gamma(\frac{1}{2}+p-\mathrm{i}s)\,\Gamma(\frac{1}{2}+\bar{p}-\mathrm{i}s)}{\Gamma(\frac{1}{2}+p+\mathrm{i}s)\,\Gamma(\frac{1}{2}+\bar{p}+\mathrm{i}s)}\ \check{D}_{p,s}(w) \tag{B.10}$$

with

$$\check{D}_{p,s}(w)=\frac{(1+2p-2\mathrm{i}s)(1-2p-2\mathrm{i}s)}{(1+2p+2\mathrm{i}s)(1-2p+2\mathrm{i}s)}\ \frac{2nw-(n+2)(n-2\mathrm{i}s)}{2nw+(n+2)(n+2\mathrm{i}s)} \qquad (w=w_{\pm}) \tag{B.11}$$

(see also eq. (48) in [15]). One can check that the singularities of (B.11) are simple poles located at $s=\mathrm{i}\,(p+\frac{1}{2})$ and $s=-\mathrm{i}\,(p-\frac{1}{2})$, while its zeroes occur at $s=-\mathrm{i}\,(p+\frac{1}{2})$ and $s=\mathrm{i}\,(p-\frac{1}{2})$ also with multiplicity one. Conjecture (A) is based on the assumption that for given $\mathtt{L}=0,1,2,\ldots$ and any set $\boldsymbol{w}$ all the singularities of $\check{D}_{p,s}(\boldsymbol{w})$ as a function of s are poles and furthermore, no pole of $\check{D}_{p,s}(\boldsymbol{w})$ coincides with a zero of another $\check{D}_{p,s}(\boldsymbol{w}')$. Then eqs. (B.1) and (B.7) imply that the possible values of s at which the product $D_{\bar{p},s}(\bar{\boldsymbol{w}})\,D_{p,s}(\boldsymbol{w})$ is singular is given by

$$s=+\mathrm{i}\,(p+\tfrac{1}{2}+a),\ +\mathrm{i}\,(\bar{p}+\tfrac{1}{2}+a)\ \text{with}\ a\geq 0$$

(corresponding to the poles of the Γ - function) and

$$s=-\mathrm{i}\,(p+\tfrac{1}{2}+a),\ -\mathrm{i}\,(\bar{p}+\tfrac{1}{2}+a),$$

where a is any integer. Since the parameters p and $\bar{p}$ are related as $p+\bar{p}=S^z=0,1,2,\ldots$, one has $\mathrm{i}\,(p+\frac{1}{2}+a)=\mathrm{i}\,(-\bar{p}-\frac{1}{2}-a')$ and $\mathrm{i}\,(\bar{p}+\frac{1}{2}+a)=\mathrm{i}\,(-p-\frac{1}{2}-a')$ with $a'=-a-S^z-1$. Thus it is sufficient to focus on the cases

$$s=\mathrm{i}\mathfrak{q}_a\equiv\mathrm{i}\,(-p-\tfrac{1}{2}-a)$$

and

$$s=\mathrm{i}\bar{\mathfrak{q}}_a\equiv\mathrm{i}\,(-\bar{p}-\tfrac{1}{2}-a).$$

The additional requirement $0<\mathfrak{q}_a,\bar{\mathfrak{q}}_a\leq\frac{n}{4}$ yields the inequality (10.27) on the values of the integer a.

Making a further assumption that all the poles of $\check{D}_{p,s}(\boldsymbol{w})$ are simple, the number of solution sets $\boldsymbol{w}$ such that $\check{D}_{p,s}(\boldsymbol{w})$ is singular at $s=\mathrm{i}\mathfrak{q}_a$ or $\mathrm{i}\bar{\mathfrak{q}}_a$ may be determined by counting the multiplicity of that pole occurring in the product $\prod_{\boldsymbol{w}}\check{D}_{p,s}(\boldsymbol{w})$ (B.7). Notice that, due to the relation $\check{D}_{p,s}(\boldsymbol{w})=\big(\check{D}_{p,-s}(-\boldsymbol{w})\big)^{-1}$, it would also follow that the zeroes of $\check{D}_{p,s}(\boldsymbol{w})$ are simple as well. For the case $\mathtt{L}=1$ the assumption can be checked using the explicit formula (B.11), while for $\mathtt{L}=2,3$ it has been numerically verified. Once accepted, conjecture (B) in sec. 10.2, regarding the number of solutions $\mathcal{N}_a^{(\bar{\mathtt{L}},\mathtt{L})}$ and $\bar{\mathcal{N}}_a^{(\bar{\mathtt{L}},\mathtt{L})}$ of the joint system (10.3) for which $D_{\bar{p},s}(\bar{\boldsymbol{w}})\,D_{p,s}(\boldsymbol{w})$ is singular at $s=\mathrm{i}\mathfrak{q}_a$ and $s=\mathrm{i}\bar{\mathfrak{q}}_a$, respectively, follows.

As an illustration, let's consider the case $s=\mathrm{i}\mathfrak{q}_a=-\mathrm{i}\,(p+\frac{1}{2}+a)$, where $a\in\mathbb{Z}$. The multiplicity of the corresponding pole in the product $\prod_{\boldsymbol{w}}D_{p,s}(\boldsymbol{w})$ follows from (B.1) and (B.7). It is given by $d_a(\mathtt{L})$ provided that the definition (B.8) is extended to the case of negative a as

$$d_{-a-1}(\mathtt{L})=\mathtt{par}_2(\mathtt{L})-d_a(\mathtt{L}) \tag{B.12}$$

(see footnote 3). It is important to keep in mind that $D_{\bar{p},s}(\bar{\boldsymbol{w}})$ could possess a simple zero at $s = \mathrm{i}\mathtt{q}_a$ that would cancel the pole of $D_{p,s}(\boldsymbol{w})$ and render the product $D_{\bar{p},s}(\bar{\boldsymbol{w}})\, D_{p,s}(\boldsymbol{w})$ finite. The barred version of (B.1), (B.7) with p and $\mathtt{L}$ replaced by $\bar{p} = S^z - p$ and $\bar{\mathtt{L}}$, respectively, yields that the number of solution sets $\bar{\boldsymbol{w}}$ for which $D_{\bar{p},s}(\bar{\boldsymbol{w}})$ vanishes at $s = \mathrm{i}\mathtt{q}_a$ is $d_{-1-a-S^z}(\bar{\mathtt{L}}) = \mathtt{par}_2(\bar{\mathtt{L}}) - d_{a+S^z}(\bar{\mathtt{L}})$. This way one obtains $\mathcal{N}_a^{(\bar{\mathtt{L}},\mathtt{L})} = d_{a+S^z}(\bar{\mathtt{L}})\, d_a(\mathtt{L})$. For the case $s = \mathrm{i}\bar{\mathtt{q}}_a$ one should simply interchange $\mathtt{L} \leftrightarrow \bar{\mathtt{L}}$ in the r.h.s. of this relation, see eq. (10.28).

Appendix C. Supplementary figures

Figs. 20–26 below, together with Figs. 8 and 15 from the main body of the text, present numerical data for the low energy states of the Hamiltonian $\mathbb{H}$ (7.6), (7.7) with $N = 22$, $q = \mathrm{e}^{\frac{\mathrm{i}\pi}{5}}$ $(n = 3)$, $\mathtt{k} = -0.18$ and in the sector $S^z = 1$. This was used to perform the classification of the low energy states quoted in Table 1. The states are grouped according to their value of the winding number $\mathtt{w}$ as well as the levels $\mathtt{L}$ and $\bar{\mathtt{L}}$, which were assigned to them using the procedure described in sec. 18. In all the figures the open circles depict the distribution of $b(N) = \frac{n}{4\pi}\log(B)$ in the complex plane, where B is the eigenvalue of the quasi-shift operator $\mathbb{B}$ (8.4) that was obtained in the course of the numerical diagonalization of the Hamiltonian $\mathbb{H}$. The filled circles and squares correspond to the solutions b_* of eq. (18.14), where for the circles $\lim_{N\to\infty} \Im m\big(b_*(N)\big) = 0$, while for the squares $\lim_{N\to\infty} \Im m\big(b_*(N)\big) \neq 0$ and $\lim_{N\to\infty} \Re e\big(b_*(N)\big) = 0$.

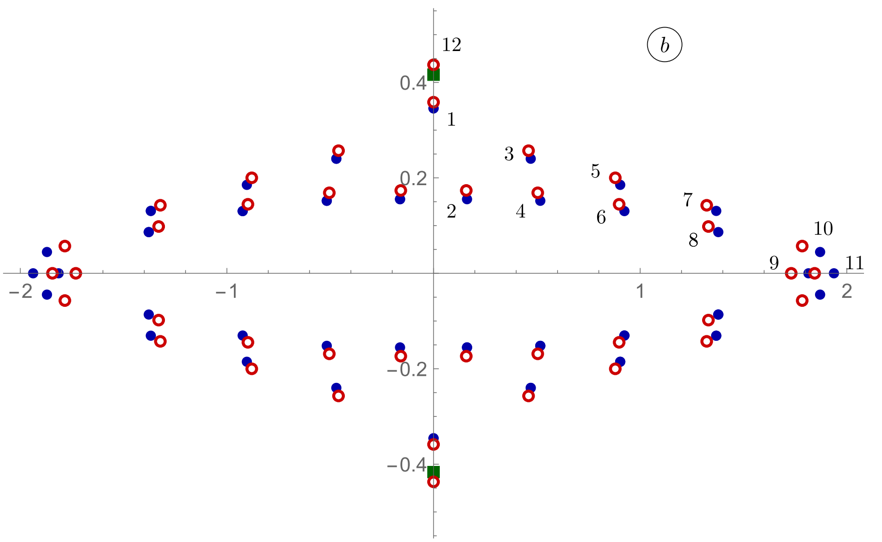

Fig. 20. The value of $b(N)$ for the $40 = 38 + 2$ states (open circles) having $\mathtt{w} = 0$ and $(\mathtt{L}, \bar{\mathtt{L}}) = (1, 0), (0, 1)$. The 12 states with $\Re e\big(b(N)\big) \geq 0$ and $\Im m\big(b(N)\big) \geq 0$ are numbered consistently with Fig. 9, where the N dependence of $b(N)$ for the corresponding RG trajectories is plotted.

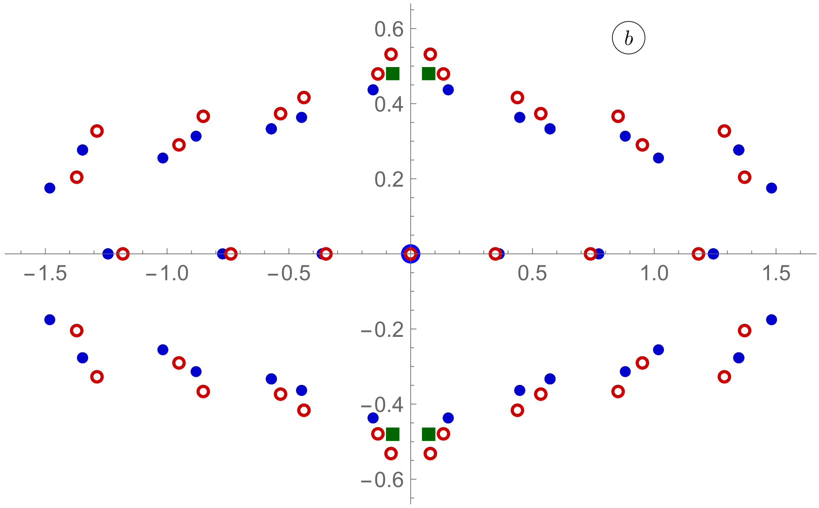

Fig. 21. The value of $b(N)$ for the $40 = 36 + 4$ states (open circles) having $\mathtt{w} = 0$ and $(\mathtt{L}, \bar{\mathtt{L}}) = (2, 0)$.

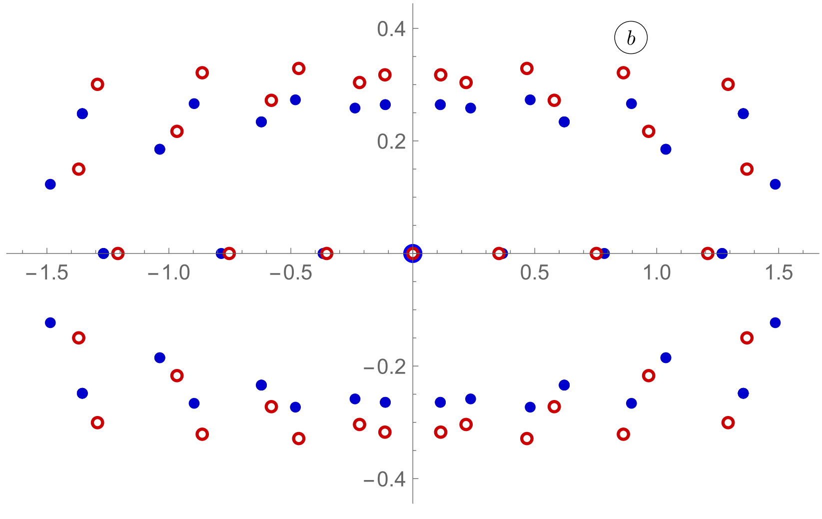

Fig. 22. The value of $b(N)$ for the $40 = 40 + 0$ states (open circles) having $\mathtt{w} = 0$ and $(\mathtt{L}, \bar{\mathtt{L}}) = (0, 2)$.

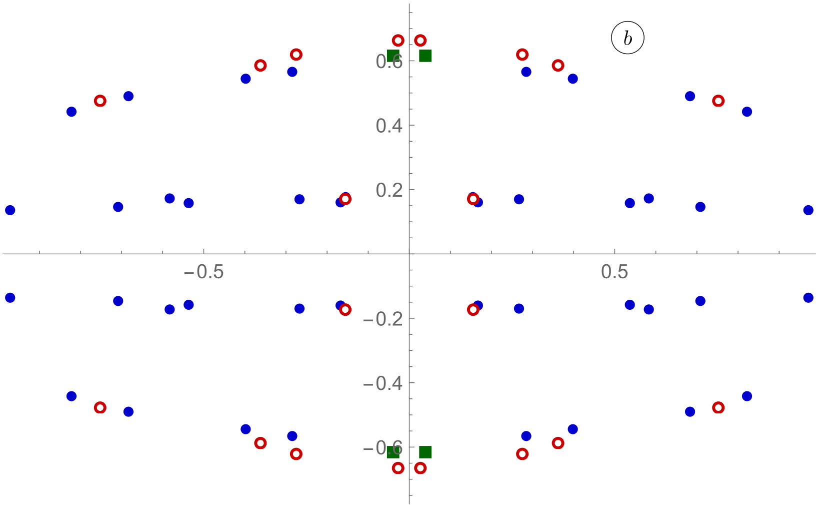

Fig. 23. The value of $b(N)$ for the $20 = 16 + 4$ states (open circles) having $\mathtt{w} = 0$ and $(\mathtt{L}, \bar{\mathtt{L}}) = (2, 1)$.

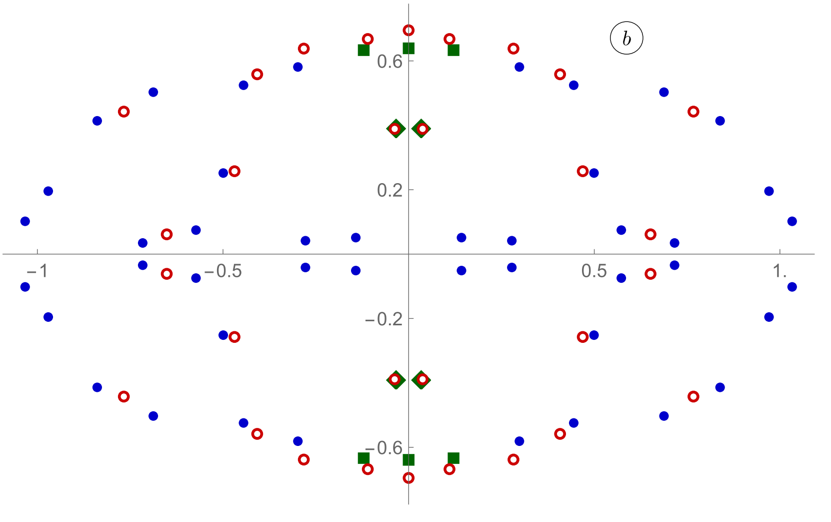

Fig. 24. The value of $b(N)$ for the $30 = 22 + 8$ states (open circles) having $\mathtt{w} = 0$ and $(\mathtt{L}, \bar{\mathtt{L}}) = (1, 2)$. As with the filled circles and squares, the filled diamonds correspond to solutions $b_*(N)$ of eq. (18.14). These form two pairs which have the same value of $\Im m\big(b_*(N)\big)$ and opposite real part. At large but finite N the diamonds from the upper pair collide at the imaginary axis at which point for one of the diamonds $b_*(N) \to +\frac{9\mathrm{i}}{20}$ while for the other one $b_*(N) \to 0$. The N dependence of $b_*(N)$ for the lower pair of diamonds is obtained from that of the upper pair via complex conjugation.

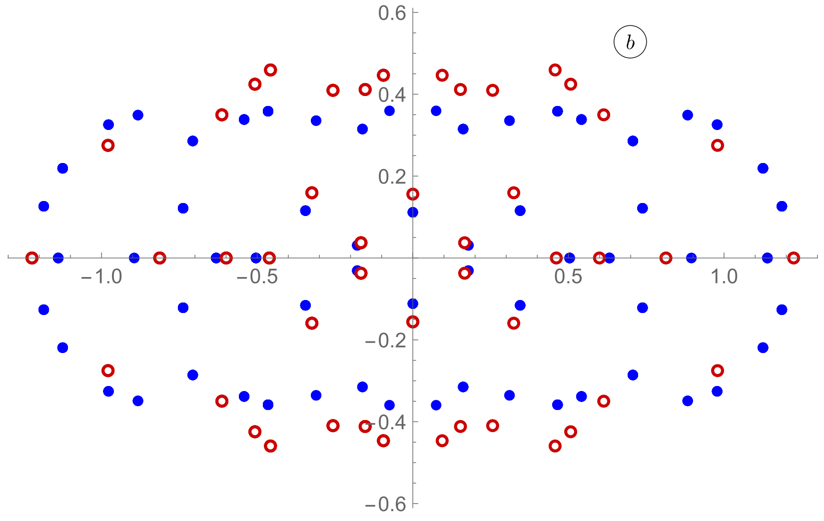

Fig. 25. The value of $b(N)$ for the $46 = 46 + 0$ states (open circles) having $\mathtt{w} = 0$ and $(\mathtt{L}, \bar{\mathtt{L}}) = (0, 3)$.

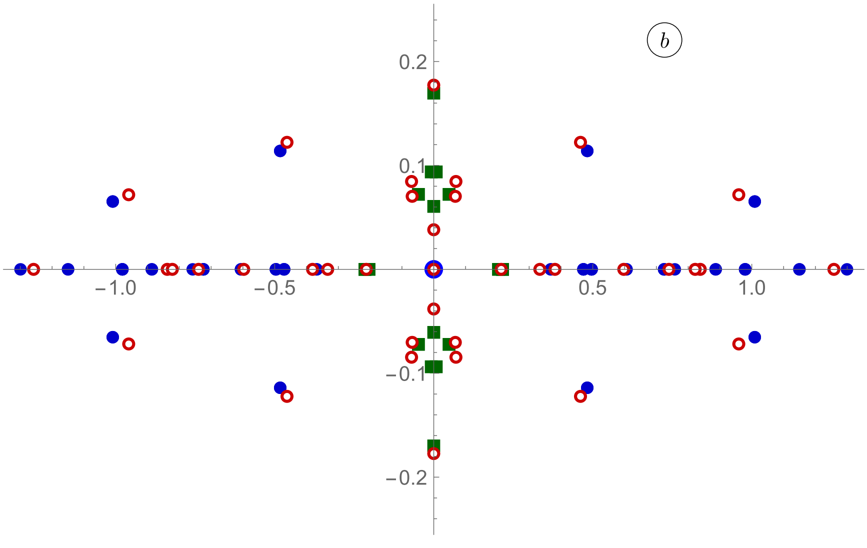

Fig. 26. The value of $b(N)$ for the $37 = 21 + 16$ states (open circles) with $\mathtt{w} = 1$ and any values of $\mathtt{L}$ and $\bar{\mathtt{L}}$.

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