



Anomalous mechanisms of the loss of observability in non-Hermitian quantum models

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Abstract

Via several toy-model quantum Hamiltonians $H(\lambda)$ of a non-tridiagonal low-dimensional matrix form the existence of unusual observability horizons is revealed. At the corresponding limiting values of parameter $\lambda = \lambda^{(critical)}$ these new types of quantum phase transitions are interpreted as the points of confluence of several decoupled Kato's exceptional points of equal or different orders. Such a phenomenon of degeneracy of non-Hermitian degeneracies seems to ask for a reclassification of the possible topologies of the complex energy Riemann surfaces in the vicinity of branch points.

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1. Introduction

The questions of stability and instability belong to the most fundamental features of reality solved and resolved by quantum theory [1]. From historical perspective the formalism succeeded in establishing a paradigmatic correspondence between the norm-preserving evolution and the observability of the energy-operators *alias* Hamiltonians. In the language of mathematics one

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may recall the so called Stone theorem [2] which connects the unitarity of evolution with the self-adjointness property of Hamiltonians $\mathfrak{h} = \mathfrak{h}^\dagger$ in a preselected Hilbert space of states \mathcal{L} .

In 1998 Bender with Boettcher [3] turned the attention of physics community to a half-forgotten possibility [4] of having the *unitary* quantum evolution controlled, in a less conventional Hilbert space (say, \mathcal{K}), by a less conventional Schrödinger equation

$$i \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle, \quad |\psi(t)\rangle \in \mathcal{K} \quad (1)$$

in which the Hamiltonian operator itself is *manifestly non-Hermitian*, $H \neq H^\dagger$. The related possibility of circumventing the apparent limitations attributed to the Stone theorem gave birth to a powerful theoretical paradigm. In its older applications in condensed matter physics [5] and in nuclear physics [6] the underlying, hiddenly Hermitian Hamiltonians were called quasi-Hermitian (cf. also [7]). Nowadays, the formalism is better known under the names of \mathcal{PT} -symmetric [8] *alias* pseudo-Hermitian [9] *alias* three-Hilbert-space (3HS, [10,11]) reformulation of quantum mechanics.

One of the main practical consequences of the application of the upgraded paradigm to the concrete physical systems (cf., e.g., the most recent summaries of the state of art in [12–14]) may be seen in the fact that the change of the language opened the way to a new analysis of the mechanisms of the loss of the quantum stability and observability. In essence, in contrast to the Hermitian theory (in which the stability is “robust” because it follows from the “obligatory” assumption that the Hamiltonian is, or must be, self-adjoint), the work with non-Hermitian Hamiltonians appeared open to *both* of the “robust” and “fragile” possibilities. Indeed, for the non-Hermitian Hamiltonians which depend on some real parameters, $H = H(a, b, \dots)$, the quantum system in question is, typically, found stable for parameters lying *strictly* inside a “physical” domain of parameters \mathcal{D} with a nontrivial boundary $\partial\mathcal{D}$ (see, e.g., [15] for illustration).

One of the most impressive real-system illustrations of the extended descriptive capacity of the 3HS formulation of quantum theory may be found in papers [16–19]. In these papers the authors generalized, to a non-Hermitian but \mathcal{PT} -symmetric (i.e., stable) version, the conventional Hubbard’s [20] multi-bosonic [21] Hermitian model. Whereas the traditional Hermitian version of the model only became truly popular after a discovery of its support of sophisticated, difficult to localize [22] phase transitions between superfluid and Mott-insulator quantum phases [23–25], the innovative non-Hermitian but \mathcal{PT} -symmetric Bose-Hubbard (BH) model opened, *inter alia*, the possibility of reaching the very boundary $\partial\mathcal{D}$ of the physical domain of its parameters. Even near this loss-of-observability boundary the evolution remains unitary in a way shown in [26,27].

Naturally, the observability of the system (i.e., the reality of the energies) becomes lost when the parameters happen to cross the boundary $\partial\mathcal{D}$. In the special cases exhibiting the parity-time symmetry (\mathcal{PT} -symmetry) one speaks about the spontaneous breakdown of this symmetry, of particular interest in quantum field theory [28]. In general, even the mere closeness of these boundaries (called, in mathematics, exceptional points, EPs [29]) opens, immediately, a lot of interesting new physics (cf., e.g., [30]). The theoretical scope and experimental realizations of these phenomena range from the quantum Bose-Einstein condensation [18,31] and from the action of spin-orbit interaction in coupled resonators inducing exceptional points of arbitrary order [32] up to some unusual effects in classical acoustics reflecting, in particular, the new mathematics of the emergence and coalescence of exceptional points occurring in multiplets [33].

In this context, our present study was motivated by a fairly puzzling empirical observation that virtually all of the phenomenologically meaningful EP-related loss-of-observability bound-

aries $\partial\mathcal{D}$ seem to belong to the same, specific subcategory of EPs at which the corresponding (by definition [29], non-diagonalizable) limits $H(a^{(EP)}, b^{(EP)}, \dots)$ of the Hamiltonians represent, in the words of one of the typical physics-oriented studies, “a peculiar type of non-Hermitian degeneracy where a macroscopic fraction of the states coalesce at a single point with a geometrical multiplicity of one” [30]. Equivalently, this means that matrices $H(a^{(EP)}, b^{(EP)}, \dots)$ prove mathematically user-friendly because, using the words of Ref. [18], they “are similar to a *single* N by N Jordan block”

$$J^{(N)}(\eta) = \begin{bmatrix} \eta & 1 & 0 & \dots & 0 \\ 0 & \eta & 1 & \ddots & \vdots \\ 0 & 0 & \eta & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 0 & \eta \end{bmatrix}. \quad (2)$$

In the latter text (which precedes equation Nr. 32 in [18]) we emphasized here the word *single* because we did not find, in the literature, *any* sufficiently realistic illustrative example of *any* less specific, generic EP degeneracy characterized by the geometric multiplicity greater than one. Equivalently, we did not find any illustrative model in which the Hamiltonian matrices $H(a^{(EP)}, b^{(EP)}, \dots)$ would *not* be similar, at the loss-of-observability boundary $\partial\mathcal{D}$, to a *single* N by N Jordan block (2).

In such a context we originally sought for any nontrivial EP loss-of-observability model in which one would not end up with the extreme geometric multiplicity equal to one. In parallel, we felt motivated by the suspicion that there may exist a hidden mathematical correspondence between the extreme multiplicity-one degeneracy and the most manipulation-friendly [34] tridiagonality of all of the known EP-supporting Hamiltonians, with the above-mentioned BH model being the prominent, though not the only one [35] exactly solvable example.

This led us to the formulation of the project. We sought for N by N Hamiltonian-operator matrices $H^{(N)}$ for which the EP loss-of-observability mechanism would be “anomalous”, i.e., characterized by the geometric multiplicity greater than one. As long as we were aware that such a project may, presumably, require the study of non-tridiagonal matrices, we restricted our attention just to the Hamiltonians which remain tractable non-numerically. Thus, being also aware of the fact that in the experimental setting the matrix dimensions N need not be too large, we decided to direct our attention to the tractable but already nontrivial family of models with $N = 6$.

Under this constraint we will be able to complement the “common” choice of the non-degenerate Jordan block limit (2) by its two alternatives, viz., by the degenerate Jordan-block direct sums

$$J^{(4+2)}(\eta) = \begin{bmatrix} \eta & 1 & 0 & 0 & 0 & 0 \\ 0 & \eta & 1 & 0 & 0 & 0 \\ 0 & 0 & \eta & 1 & 0 & 0 \\ 0 & 0 & 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta & 1 \\ 0 & 0 & 0 & 0 & 0 & \eta \end{bmatrix}, \quad (3)$$

and

$$J^{(2+2+2)}(\eta) = \begin{bmatrix} \eta & 1 & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & 1 & 0 & 0 \\ 0 & 0 & 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta & 1 \\ 0 & 0 & 0 & 0 & 0 & \eta \end{bmatrix} \quad (4)$$

corresponding to the geometric multiplicities equal to two and three, respectively.

2. EP-related mechanisms of the loss of observability

As we already emphasized, one of the best realistic simulations of the process of the loss of observability is provided by the three-parametric N by N matrix \mathcal{PT} -symmetric Bose-Hubbard Hamiltonian $H^{(N)}(\gamma, v, c)$ of Ref. [18]. Indeed, after an arbitrary choice of the dimension N (related to the number of bosons $\tilde{N} = N - 1$ in the quantum system in question) this Hamiltonian offers a non-numerical, exactly solvable toy model exhibiting the non-Hermitian N -tuple EP-conditioned degeneracy of the whole N -plet of the real-energy observable eigenvalues with the geometric multiplicity equal to one. For this class of models the EP-related mechanisms of the loss of observability is already known (see, e.g., Refs. [27,36] and a brief description of their results in Appendix A below). It seems also worth adding that all of the similar tridiagonal-matrix models admit also a straightforward and recurrently constructive [37] unitary-evolution probabilistic interpretation (for a concise outline of the corresponding version of abstract quantum theory see Appendix B below).

2.1. Conventional, non-degenerate-EP structure

For parameter-dependent Hamiltonians $H^{(N)} = H^{(N)}(\lambda)$, any set of eigenvectors (cf. Eq. (16) in Appendix A below) ceases to form a complete basis at $\lambda = \lambda^{(EP)}$. In such a limit the degeneracy of eigenvalues is accompanied by the degeneracy of eigenvectors [29]. In the non-degenerate, maximally non-diagonalizable extreme with $\lambda \rightarrow \lambda^{(EP)} = \lambda^{(EPN)}$ one has

$$\lim_{\lambda \rightarrow \lambda^{(EPN)}} E_n(\lambda) = \eta, \quad \lim_{\lambda \rightarrow \lambda^{(EPN)}} |\psi_n(\lambda)\rangle = |\Phi\rangle, \quad n = 0, 1, \dots, N - 1 \quad (5)$$

(see, e.g., [35]) so that the conventional time-independent bound-state Schrödinger equation $H^{(N)}(\lambda) |\psi\rangle = E |\psi\rangle$ ceases to be solvable. It can only be replaced by an alternative relation

$$H^{(N)}(\lambda^{(EP)}) Q = Q \mathcal{J}^{(N)}(\eta) \quad (6)$$

with the so called transition matrix Q and with a suitable, purpose-dependent non-diagonal, one-parametric matrix $\mathcal{J}^{(N)}(\eta)$ as sampled here by Eqs. (2), (3) and (4).

In different contexts and applications, various versions of the latter, “canonical representation” matrix can be found in the literature (cf., e.g., [38]). Most often they are being chosen in the form of Jordan block (2). Such a matrix remains non-diagonalizable so that it cannot be treated as a quantum observable anymore. It can only play the role of an EP-related substitute for the spectrum. In parallel, the matrices of associated solutions Q are called transition matrices. They may be perceived as a formal EP analogue of the set of conventional wave functions.

2.2. Unconventional, degenerate-EP structure EP2+EP2+EP2

For the sake of definiteness let us consider the “next-to-trivial” Hilbert space with dimension $N = 6$. Let us also assume that our matrix toy-model Hamiltonian is real, asymmetric and non-tridiagonal. Moreover, in Eq. (6) the elementary Jordan block of Eq. (2) will be excluded as “the known case”. Under these specifications the usual EP-related degeneracy of the bound state energies

$$\lim_{\lambda \rightarrow \lambda^{(EP)}} E_n(\lambda) = \eta, \quad n = 0, 1, 2, 3, 4, 5 \tag{7}$$

may be accompanied by the following nontrivial decoupling of the EP limits of the wave-function solutions of our Schrödinger Eq. (16),

$$\lim_{\lambda \rightarrow \lambda^{(EP)}} |\psi_{k_j}(\lambda)\rangle = |\Phi_j\rangle, \quad k_j = 2j - 2, 2j - 1, \quad j = 1, 2, 3. \tag{8}$$

Thus, the “known”, EP6-related loss-of-observability pattern (5) becomes replaced by its anomalous EP2+EP2+EP2 alternative.

Formally, the upgrade may be characterized by the replacement of Eq. (2) by another elementary option (4). In this matrix (mimicking a triply degenerate exceptional point at the observability horizon) the auxiliary parameter η (i.e., the value of the energy in Eq. (7)) may be arbitrary.

3. Elementary models with three degenerate EP2s

3.1. Toy model EP Hamiltonian with two diagonals

Our first illustrative non-tridiagonal toy model has the most elementary cross-diagonal form

$$H^{(2+2+2)}(\varepsilon) = \begin{bmatrix} \varepsilon - 5 & 0 & 0 & 0 & 0 & 5 \\ 0 & \varepsilon - 3 & 0 & 0 & -3 & 0 \\ 0 & 0 & \varepsilon - 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & \varepsilon + 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & \varepsilon + 3 & 0 \\ -5 & 0 & 0 & 0 & 0 & \varepsilon + 5 \end{bmatrix}. \tag{9}$$

Having chosen $\varepsilon = 0$ (this value represents, after all, just an inessential shift of the scale) the insertion in Eq. (6) yielded the solution with transition matrix

$$Q = \begin{bmatrix} -5 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & -3 & 0 & -3 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 3 & 1 & 3 & 1 & 3 & 1 \\ -5 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

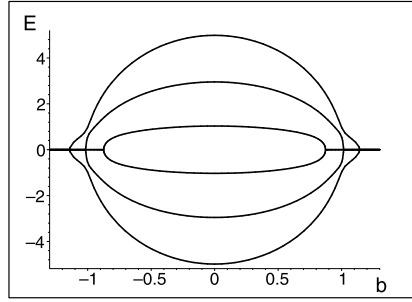


Fig. 1. Real parts of the eigenenergies of Hamiltonian $H^{(2+2+2)}(0, 0.3, b, 5b)$ of Eq. (10).

and with the block-diagonal, three times degenerate EP2+EP2+EP2 structure of the Jordan matrix of Eq. (4) with $\eta = \varepsilon = 0$. This confirmed our expectations that the search for non-numerical six by six real matrix models with degenerate EPs should employ the non-tridiagonal candidates for the Hamiltonians.

3.2. A tentative inclusion of perturbations

Elementary structure of the anomalous EP-limit matrix (9) inspires a move towards its three-parametric perturbed partner

$$H^{(2+2+2)}(a, b, c) = \begin{bmatrix} -5 & 0 & 0 & -a & 0 & c \\ 0 & -3 & a & 0 & -3b & 0 \\ 0 & -a & -1 & b & 0 & -a \\ a & 0 & -b & +1 & a & 0 \\ 0 & 3b & 0 & -a & +3 & 0 \\ -c & 0 & a & 0 & 0 & +5 \end{bmatrix}. \quad (10)$$

The bound-state spectrum can be obtained non-numerically, in the well known closed form called Cardano formulae. The existence of such a solution facilitates our analysis, indeed. At the same time, the fully general results of such a type already become long, non-transparent and unsuitable for an explicit printed display. Graphical samples offer a more appropriate form of presentation of the key features of the dependence of the energy levels E_n , real or complex, on our three variable parameters. Thus, the graphical sample of the spectrum as provided by Fig. 1 offers a nice example of the unfolding of the anomalous EP degeneracies of the preceding subsection. At a not too small fixed value of $a = 3/10$, the six real energy levels remain real in a fairly large interval of b in the regime where $c = 5b$. The picture still keeps traces of the unfolding of the degeneracy of the energies near the two EP2+EP2+EP2 singularities which are localized at the critical values of $b = \pm 1$. Far from these singularities the spectrum looks robust and not too sensitive to the perturbations controlled by the second-diagonal couplings b and c .

Fig. 2 complements the latter visualization of the spectrum by a cross section through the energy surface at $b = 1/10$, i.e., far from the critical value. This picture makes it clear that even in the regular dynamical regime the stability of the inner four excited states may be fragile. In particular, the stability appears sensitive to larger perturbations controlled by parameter a .

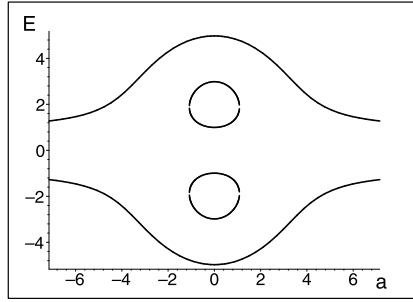


Fig. 2. Real spectrum of Hamiltonian $H^{(2+2+2)}(a, b, 5b)$ of Eq. (10) at a fixed value of $b = 1/10$.

3.3. Another, modified model of the loss of observability

The more or less accidental choice of the parameter-dependence of the perturbations should be complemented by the study of its alternatives. Most obviously, the a -dependence of the energies could be modified and/or combined with an independent variation of c . Thus, we decided to analyze a modified, tilded form of Hamiltonian

$$\widetilde{H}^{(2+2+2)}(a, b) = \begin{bmatrix} -5 & 0 & 0 & -5a & 0 & 5b \\ 0 & -3 & 3a & 0 & -3b & 0 \\ 0 & -3a & -1 & b & 0 & -5a \\ 5a & 0 & -b & 1 & 3a & 0 \\ 0 & 3b & 0 & -3a & 3 & 0 \\ -5b & 0 & 5a & 0 & 0 & 5 \end{bmatrix}. \quad (11)$$

At a fixed b the spectral locus appeared to be entirely different, composed of a vertical triplet of slightly deformed circles (presentation of such an elementary picture would be redundant). At a variable b the real energies appeared to form three vertically arranged separate tubes. They were found to merge, not too surprisingly, at the EP2+EP2+EP2 ends with $b = b_{\pm}^{(EP)} = \pm 1$.

4. Models with the geometric multiplicity equal to two

Obviously, our two illustrative examples demonstrated that our trial and error search of degenerate EPs was successful when based on the transition from tridiagonal to more-diagonal sparse matrices of Hamiltonians. Now, let us apply the same model-building strategy to some more complicated matrix-element arrangements.

4.1. The first two-block illustration: EP4 + EP2

After we skipped the not too interesting EP3+EP3 scenario we decided to search for the confluence EP4+EP2, i.e., for a merger of the two exceptional points of different orders using, this time, the direct sum (3) of two Jordan-block submatrices. With the most comfortable choice of shift $\varepsilon = 0$ our search resulted in the following non-tridiagonal and manifestly non-Hermitian limiting (i.e., unphysical, non-diagonalizable) EP4+EP2 Hamiltonian

$$H^{(4+2)} = \begin{bmatrix} -9 & 3\sqrt{3} & 0 & 0 & 0 & 0 \\ -3\sqrt{3} & -3 & 0 & 0 & 6 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -6 & 0 & 0 & 3 & 3\sqrt{3} \\ 0 & 0 & 0 & 0 & -3\sqrt{3} & 9 \end{bmatrix}. \quad (12)$$

This Hamiltonian may be assigned the following well-behaved transition matrix

$$Q = \begin{bmatrix} -162 & 54 & -9 & 1 & 0 & 0 \\ -162\sqrt{3} & 36\sqrt{3} & -3\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ -162\sqrt{3} & 18\sqrt{3} & 0 & 0 & 0 & 0 \\ -162 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with non-vanishing determinant $\det Q = 26244$. Again, our conclusion is that even the rather naive trial and error search for the anomalous EP candidates may succeed. Moreover, even in the more complicated EP4+EP2 arrangement the search reveals the existence of very sparse candidate matrices, provided only that they are chosen more-than-tridiagonal.

What should follow would be an explicit construction of some perturbations compatible with the reality of the spectrum, and describing the path-of-unitarity trajectories of the loss of observability. Nevertheless, the present elementary six by six matrix form of our unperturbed Hamiltonian would certainly make these steps an elementary, redundant and boring exercise in linear algebra.

Naturally, the task would be mathematically more difficult at the higher matrix dimensions. Incidentally, in the light of the well known related experimental-simulation challenges [31] there is no really urgent need of any extensive and exhaustive generalization. Just the display of a few characteristic illustrative examples seems sufficient for the inspiration of experimentalists at present.

4.2. Another EP4 + EP2 model

In the same spirit as above let us now introduce the following two-parametric perturbed Hamiltonian of a minimal-coupling type,

$$H^{(6)}(\tau, \beta) = \begin{bmatrix} -9 & 3\sqrt{3-3\tau} & 0 & 0 & 0 & 0 \\ -3\sqrt{3-3\tau} & -3 & \sqrt{\beta} & 0 & -6\sqrt{1-\tau} & 0 \\ 0 & -\sqrt{\beta} & -1 & \sqrt{1-\tau} & 0 & 0 \\ 0 & 0 & -\sqrt{1-\tau} & 1 & \sqrt{\beta} & 0 \\ 0 & 6\sqrt{1-\tau} & 0 & -\sqrt{\beta} & 3 & 3\sqrt{3-3\tau} \\ 0 & 0 & 0 & 0 & -3\sqrt{3-3\tau} & 9 \end{bmatrix}. \quad (13)$$

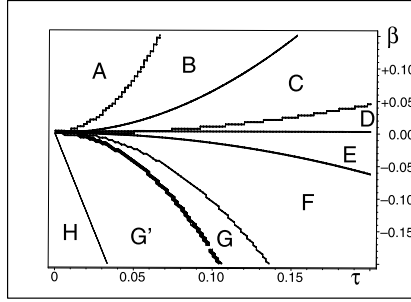


Fig. 3. Numerically determined curves of zeros of the real and/or imaginary parts of the roots s_n of secular equation (14). An auxiliary line of $\beta = 0$ has been added to guide the eye, and the curve separating the domains G and G' should be ignored as a spurious artifact of the numerics.

In the decoupled case with $\beta = 0$ the secular equation $E^6 - 91 \tau E^4 + 819 \tau^2 E^2 - 729 \tau^3 = 0$ becomes solvable exactly yielding the parabolic τ -dependence of the six real bound state energies

$$E(\tau) = \pm\sqrt{\tau}, \pm 3\sqrt{\tau}, \pm 9\sqrt{\tau}.$$

They merge, as required, in the EP=EP4+EP2 limit of vanishing $\tau \rightarrow 0$. In the general regular dynamical regime with $\beta \neq 0$, the secular equation for energy $E = \pm\sqrt{s}$ is more complicated but still cubic,

$$s^3 + (-91 \tau + 2 \beta) s^2 + (-114 \beta + 819 \tau^2 - 42 \tau \beta + \beta^2) s - 729 \tau^3 - 486 \tau \beta - 81 \beta^2 = 0. \tag{14}$$

This equation is Cardano-solvable so that it may be again expected to determine all of the relevant properties of the energy spectrum, in principle at least. In practice, however, the similar answers of a complicated algebraic form remain as clumsy as their above-studied predecessors. Fortunately, the remedy remains the same as before. Our graphical description of the energy-level curves $E_n(\tau, \beta)$ with $n = 0, 1, 2, 3,$ and 5 may be made available as fully two-parametric and, in this sense, exhaustive.

The price to pay is that our results must be now presented in a combination of Fig. 3 with Table 1. Indeed, the Table is needed to endow the candidates for the energy curves with the physical meaning as well as with the phenomenological acceptability. One can conclude that the quantum system in question remains unitary (i.e., stable *alias*, in the sense of Ref. [6], quasi-Hermitian) only inside the subdomains D and E of the whole plane of parameters. Only inside the “corridor” D + E our secular equation (14) has the full triplet of the real and positive roots s_n yielding the six acceptable (i.e., real) bound-state energies $E_n^{(\pm)} = \pm\sqrt{s_n}$.

Let us add that the picture contains a “redundant”, $\beta = 0$ line which was inserted by hands. Guiding the eye, this line separates the two β -sign-different but, otherwise, fully unitarity-supporting access-to-EP subcorridors D and E. We may summarize:

- the picture and table clearly demonstrate the existence and uniqueness of a non-empty unitarity-compatible corridor $\mathcal{D} = D + E$ of quantum stability;

Table 1
Conditions of unitarity of evolution for Hamiltonian (13).

domain (see Fig. 3)	condition (levels $n = 0, 1, 2$):	
	$\text{Im } s_n(\beta, \tau) = 0,$	$\text{Re } s_n(\beta, \tau) > 0$
A	satisfied	not satisfied
B	not satisfied	not satisfied
C	not satisfied	satisfied
D	satisfied	satisfied
E	satisfied	satisfied
F	not satisfied	not satisfied
G/G'	not satisfied	not satisfied
H	not satisfied	not satisfied

- this corridor \mathcal{D} connects the interior of physical domain of parameters with its EP extreme of maximal-non-Hermiticity;
- the boundary between D and E is artificial, marking the change of sign of β in parameter $\sqrt{\beta}$. In this sense the line $\beta = 0$ separates the non-Hermitian and Hermitian versions of the Hamiltonian.

4.3. Robust degeneracies and non-perturbative phenomena

In our paper we reminded the readers that whenever we modify the Hamiltonians, what follows is a change of the operator of the metric defining the correct physical Hilbert space of states \mathcal{H} . This means that the standard constructive recipes of perturbation theory have to be modified [36]. In the single-Jordan-block models a truly instructive illustration of some slightly counter-intuitive consequences may be found in Refs. [18] and [39]. Also in the present, “anomalous” context of the degenerate-Jordan-block models, the intuitive and/or hand-waving arguments may often fail as well.

In order to illustrate the danger, let us endow the above-mentioned EP-limit model (12) with a particularly minimalistic perturbation,

$$H^{(4+2)} \rightarrow \widetilde{H^{(4+2)}}(\gamma) = \begin{bmatrix} -9 & 3\sqrt{3} & 0 & 0 & 0 & 0 \\ -3\sqrt{3} & -3 & 0 & 0 & 6 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -6 & 0 & -3\gamma & 3 & 3\sqrt{3} \\ 0 & 0 & 0 & 0 & -3\sqrt{3} & 9 \end{bmatrix}. \tag{15}$$

To our great surprise such a perturbation does not remove the degeneracy. It even changes, unexpectedly, the type of the EP singularity. Indeed, our new matrix may be assigned the perturbation-independent canonical Jordan-block representation $J^{(6)}$ as well as the perturbation-dependent transition matrix

$$Q = \begin{bmatrix} -486\sqrt{3}\gamma & 54\sqrt{3}\gamma & 9\sqrt{3}\gamma & -3\sqrt{3}\gamma & 1/2\sqrt{3}\gamma & -1/18\sqrt{3}\gamma \\ -1458\gamma & 0 & 45\gamma & -6\gamma & 1/2\gamma & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1458\gamma & -162\gamma & 36\gamma & 0 & 0 & 0 \\ -486\sqrt{3}\gamma & -108\sqrt{3}\gamma & 0 & 0 & 0 & 0 \end{bmatrix}$$

with determinant $\det Q = 19131876\gamma^4$. This determinant is γ -dependent and non-vanishing at any non-vanishing $\gamma \neq 0$. At the same time, the transition matrix ceases to be invertible (i.e., applicable) in the zero-perturbation limit $\gamma \rightarrow 0$. In this limit the system keeps staying on the loss-of-observability horizon but this horizon encounters an abrupt change of its type, EP6 \rightarrow EP4+EP2! Naturally (cf., e.g., Ref. [40]), the detection of similar qualitative features of non-diagonalizable matrices often lies beyond the capability of the conventional numerical algorithms.

5. Discussion

The detailed constructive study of behavior of quantum systems near the EP horizon is a purely numerical task in general. The dedicated literature abounds with the specific phenomenological implementations of the idea. They range from the abstract concepts of the quantum phase transitions and quantum chaos (cf., e.g., review papers [41,42]) up to many innovative experiments in optics and photonic [43] (note, e.g., that the light can stop at an analogue of quantum horizon [44]), or in optomechanics [45], etc. Using some elementary toy models one could even contemplate a purely schematic conceptual application of the non-Hermitian Heisenberg picture near horizons in quantum cosmology [46].

The study of horizons remains basically numerical even if the Hamiltonian itself is represented by an effective finite-dimensional N by N matrix $H^{(N)}(\lambda)$ with $N \geq 4$ [47]. The only non-numerical exception admitting all finite matrix dimensions $N < \infty$ seems to have emerged in the context of tridiagonal Hamiltonian matrices (cf., e.g., [18,35] or [48]). In such a class of user-friendlier models the horizons (occurring at a finite coupling constant $\lambda = \lambda^{(critical)}$) have successfully been identified with elementary Kato's [29] exceptional points, $\lambda^{(critical)} = \lambda^{(EP)}$.

Naturally, the deeply dynamical character of the latter singularities (in review [41] they were called "ubiquitous") converted them recently in one of the central subjects in theoretical efforts as well as in a variety of experimental simulations. In the latter setting let us just recall, *pars pro toto*, the fact that in the one-dimensional photonic crystals one can construct an EP horizon at which "two EPs ... coalesce ... and create a singularity of higher order" [49].

In our present paper we found all of the similar, EP-related pieces of information truly inspiring. Thus, we reopened the theoretical question of a generalization of the theory in which the EP horizons would be studied in more detail. Via a few examples we illustrated that such an analysis seems truly rewarding, especially when one starts to study the more-than-tridiagonal matrices $H^{(N)}$. By means of a detailed study of several toy-model matrices we demonstrated that in the more-than-tridiagonal cases there emerges an anomalous structure of the EPs. The Hamiltonians cease to be represented by the traditional, single canonical Jordan block of maximal dimension

N but rather by a block-diagonal matrix with the number of Jordan-block submatrices equal to the geometric multiplicity of the EP singularity in question.

The existence of a wealth of these anomalous EP structures has been revealed, tractable as a degenerate superposition of several independent exceptional points of lower orders. Such a result might prove inspiring for a continuing theoretical as well as experimental analysis.

5.1. Experimental realizations of access to EPs

The formal mathematical construction of a corridor \mathcal{D} explains the mechanism of the loss of the observability and of the fall of a unitary quantum system upon its EP singularity. For a real-world applicability of the theory these mechanisms must remain “robust”, i.e., reasonably insensitive to random perturbations. A brief complementary comment seems necessary because it is precisely this “robustness” assumption which was exposed to a harsh criticism by mathematicians recently (see, e.g., [50,51]).

In essence, this conflict of opinions is just an elementary terminological misunderstanding. The essence of the misunderstanding lies in the absence of a clear specification of the underlying physics. In brief, one could have said that the so called “closed,” unitary quantum systems remain “robust”, while the “fragility” exclusively applies just to the so called “open” quantum systems living, in our present terminology, in the “trivial-metric” Hilbert space \mathcal{K} . In this light the conflict is artificial: all of the rigorous proofs and numerous illustrative examples provided and discussed, e.g., in [50,51] are correct when applied to the non-unitary, open quantum systems (cf. also our older comment in [52]). Indeed, the authors of the criticism worked just with a simplified metric $\Theta = I$, i.e., in \mathcal{K} .

From the perspective of the closed, stable quantum systems the latter criticism did not take into account the correct metric $\Theta \neq I$. Far from EPs, in a weakly non-Hermitian dynamical regime (as studied, e.g., in [53]) it would still make good sense to analyze closed systems and replace, or rather approximate, the correct but complicated physical Hilbert space \mathcal{H} by its manifestly unphysical but much user-friendlier simplification \mathcal{K} . Then, indeed, a “sufficient smallness” of the perturbation specified via the correct norm in \mathcal{H} need not be too different from an approximate specification via its norm in \mathcal{K} .

5.2. Phenomenological applicability outlook

Let us now return to the generic, “non-pathological” quantum systems. The purpose of the analysis of the boundaries of their observability and stability is, in general, twofold. Firstly, it can lead to a basic intuitive understanding of the concept of an onset of instabilities. Secondly, it can clarify the range of applicability of mathematical methods. In this context, our present introduction of several schematic models of physical reality was motivated by both of these ambitions. In a concise comment on applicability in physics we may add that among the quantum phenomena explained via elementary models one may recall the phase transitions in Bose-Hubbard systems [40] as well as the Big Bang models in cosmology (for context see, e.g., an introductory report in [46], or section 5 in our recent review paper [54]).

In the methodical setting, in a direct continuation of our recent studies [52,55,56], just the elementary tools of linear algebra may be expected to be needed again. Still, even these not too complicated methods proved efficient enough to clarify that in any sufficiently ambitious theoretical description of physical reality an unexpectedly universal explanatory role proved played by the Kato’s notion of exceptional points.

Our present brief note opened several new EP-related directions of possible future research. More specifically, we managed to suppress, partially at least, the scepticism concerning the possibilities of an efficient non-numerical tractability of multi-diagonal models as expressed in paper [57]. We revealed that in a way guided by computer-assisted experimenting the non-numerical descriptive capability of the more-than-tridiagonal Hamiltonians H may be perceived more than promising.

Via our illustrative examples we demonstrated that in the future, a transition to non-tridiagonal matrix models might prove also productive from a purely phenomenological point of view. Rewarding, as we explained, due to the capability of non-tridiagonal Hamiltonian to mediate the mergers of individual EPs into multiplets, with all of the not yet explored observable consequences. Thus, the attention to the mergers could certainly further enrich our current understanding of the “unavoided crossing” phenomena [58], or of many other rather general realizations of quantum phase transitions [59].

6. Summary

The model-building strategy based on the exclusive use of tridiagonal matrices, Hermitian or not, can rely on several advantages as sampled, e.g., in [57]. Beyond the tridiagonal models, due care must be paid to details. Even at the not too large matrix dimensions N , one can sometimes be surprised by unexpected mathematical difficulties [34]. One of the obstacles appears particularly remarkable. It is characterized by a rather rarely considered possibility of a deviation of our choice of matrix $\mathcal{J}^{(N)}$ in Eq. (6) from its conventional, frequently used single-Jordan-block version as prescribed by Eq. (2). A few samples of such a deviation were presented also in our paper.

In our considerations we felt influenced by the recent interest in an experimental accessibility of a loss-of-observability quantum-catastrophic processes. In fact, such an accessibility of the EP-related horizon of observability is one of the most innovative and challenging consequences of the innovative use of non-Hermitian operators of observables with real spectra as studied in quasi-Hermitian [6] *alias* PT-symmetric [8] *alias* crypto-Hermitian [60] *alias* three-Hilbert-space [10] *alias* pseudo-Hermitian [9] representations of unitary quantum theory. In this context our present detailed study of the structure of these horizons revealed that the current preference of the closed quantum systems with non-degenerate EPs seems to be just an artifact which resulted from the bigger constructive comfort provided by Hamiltonians possessing a tridiagonal matrix representation. Thus, a broad new field of research seems to be open and waiting for a more systematic experimental as well as theoretical analysis.

In our present paper the first steps were made in the spirit of such a research project. Their constructive and computing part involves the tests (and the empirical confirmations) of the hypothesis of correspondence between the matrix tridiagonality of the Hamiltonian and the non-degenerate nature of the EP singularity. This confirmation was based on the use of the non-tridiagonal but still exactly solvable EP2+EP2+EP2 examples (9), (10) and (11) in section 3, and of their less elementary EP4+EP2 modifications (12) and (13) in section 4.

On a different, model-independent level of qualitative, serendipitous discoveries we pointed out that whenever one works with non-Hermitian matrices (which is, in the EP context and applications, necessary), one must always carefully check and test the possibilities of emergence of multiple not quite expected technical subtleties. In this sense, a word of warning was formulated and supported by an explicit “ill-behaved” illustrative matrix model (15).

Last but not least, due to the not entirely standard overall theoretical framework of our present paper, an enhancement of its clarity in combination with a comparatively self-contained form has been achieved by the inclusion of several review-like explanatory texts relocated to the two Appendices.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. The mechanism of the loss of observability in a small vicinity of a non-degenerate exceptional point

During most studies of various versions of N by N matrices $H^{(N)}(\lambda)$ which are tridiagonal, the localization of the Kato's exceptional points (of the maximal order N , with $\lambda = \lambda^{(EPN)}$) was achieved by non-numerical, purely algebraic methods [61]. In the context of our more ambitious and general present study, the current state of art should be summarized briefly.

A.1. Tridiagonal matrix models

First of all, let us remind the readers that the construction of a tridiagonal-matrix representation of a given realistic quantum Hamiltonian operator is encountered, as an intermediate step, in the majority of conventional numerical algorithms of solving the bound-state Schrödinger equations

$$H^{(N)} |\psi_n\rangle = E_n |\psi_n\rangle, \quad n = 0, 1, \dots, N, \quad N \leq \infty. \quad (16)$$

Thus, in the context of numerical quantum mechanics the tridiagonalization of matrices is a comparatively routine task. The situation is different in the non-numerical versions of applied linear algebra. In PT-symmetric quantum mechanics, in particular, the matrix-tridiagonality requirement is often appreciated as facilitating the constructive predictions of the results of measurements [57], mainly because these predictions require the evaluation of mean values of the operators of observables.

Under the assumption of the non-Hermiticity of Hamiltonians $H^{(N)} \neq [H^{(N)}]^\dagger$, tridiagonality is known to facilitate also the localization of the exceptional points forming, in the domains of parameters, the phenomenologically deeply relevant loss-of-the-observability boundaries [35,59]. Empirically, as we already mentioned, there seems to be a one-to-one correspondence between the tridiagonality of H and the non-degenerate nature of its EPs characterized by the use of the Jordan blocks (2) in Eq. (6). For this reason our present interest was concentrated on non-tridiagonal matrix models with the geometric multiplicities greater than one.

A.2. The losses of observability due to small perturbations

Our recent papers [27,36] offered a detailed explanation of the specific features of the processes of the loss of observability which, in certain specific systems, can be connected with the spontaneous breakdown of supersymmetry [15] or of \mathcal{PT} -symmetry [14]. Basically, these processes correspond to the passage of the system through an EP boundary $\partial\mathcal{D}$. Naturally, once the system starts living in a small vicinity of the EP singularity, it makes sense to recall the formal tools of perturbation theory [29].

In the conventional, non-degenerate-EP-related dynamical regime any Hamiltonian becomes close to its non-diagonalizable, manifestly unphysical EPN limit. Thus, we have to consider the perturbed forms

$$\tilde{H}(\lambda) = J^{(N)}(\eta) + g V^{(N)}(g), \quad g = g(\lambda) = \mathcal{O}(\lambda - \lambda^{(EP)}) \quad (17)$$

of its formal Jordan-block representations (2). The main conclusions of the analysis were summarized in Ref. [27]. At an arbitrary matrix dimension N the “tilded” model (17) is to be perceived as representing, under certain conditions, an admissible Hamiltonian of a unitary quantum system which remains stable under small perturbations. In this language every process of the loss of observability becomes realized inside a corridor which connects the weakly non-Hermitian dynamical regime inside \mathcal{D} with its EPN-related loss-of-observability boundary.

Lemma 1. [27] *For the class of the real perturbation matrices $V^{(N)}(g)$ with matrix elements which are uniformly bounded in \mathcal{K} , the eigenvalues of matrix $\tilde{H}(\lambda)$ of Eq. (17) have the generic leading-order form*

$$E_j(\lambda) \sim \text{a constant} + e^{2i\pi j/N} \sqrt[N]{\lambda - \lambda^{(EPN)}} + \dots, \quad j = 0, 1, \dots, N - 1. \quad (18)$$

Corollary 2. *For the generic perturbation of Lemma 1 the spectrum of energies can only be real at $N = 2$.*

For $N > 2$, the specification of the class of the admissible (i.e., “sufficiently small”) perturbations which would keep the system inside the unitarity corridor \mathcal{D} requires further, rather severe restrictions which may be found described and proved in [27].

Lemma 3. [27] *For a guarantee of the reality of the spectrum of matrix $\tilde{H}(\lambda)$ of Eq. (17) it is necessary to guarantee that $V_{j+k,j}^{(N)}(g) = \mathcal{O}(g^{(k-1)/2})$ at $k = 1, 2, \dots, N - 1$, at a sufficiently small difference $g(\lambda)$, and at all j .*

Corollary 4. [27,36] *Matrix $\tilde{H}(\lambda)$ of Eq. (17) may represent a unitary, closed quantum system if and only if the spectrum is real. At small λ the acceptable parameters are then confined to a comparatively narrow “stability corridor” forming an access to the EPN extreme.*

Appendix B. The 3HS formalism in nuce

The equivalence between the conventional and non-Hermitian-Hamiltonian descriptions of unitary (i.e., stable) quantum systems is based on an *ad hoc* Hermitization of the operators of observables (cf. [9,10]). Let us now briefly outline the essence of the idea.

B.1. Ad hoc constructions of the physical inner products

In a generic unitary quantum system let us assume that the Hamiltonian is a finite-dimensional, N by N matrix with real spectrum. Naturally, it is necessary to preserve a compatibility between the models with Hermitian Hamiltonians $\mathfrak{h}^{(N)} = (\mathfrak{h}^{(N)})^\dagger$, and those with non-Hermitian, quasi-Hermitian Hamiltonians $H^{(N)}$. In the latter case it is in fact sufficient to guarantee that there exists a positive definite and Hermitian N by N matrix $\Theta = \Theta^\dagger$ such that

$$\left(H^{(N)}\right)^\dagger \Theta = \Theta H^{(N)} \quad (19)$$

or, equivalently,

$$H^{(N)} = \left(H^{(N)}\right)^\ddagger := \Theta^{-1} \left(H^{(N)}\right)^\dagger \Theta \quad (20)$$

(see, e.g., review [6]).

The discovery of merits of building quantum phenomenology using quasi-Hermitian Hamiltonians is usually attributed to Dyson [5] but it was only Bender with Boettcher [3] who made the idea truly popular. They persuaded a broader physics community about the fairly deep theoretical appeal of quasi-Hermitian Hamiltonians (i.e., in their terminology, of the non-Hermitian but \mathcal{PT} -symmetric Hamiltonians – cf., e.g., detailed reviews [8,12,14]).

In the language of physics the concept of quasi-Hermiticity can be most easily interpreted after a factorization of the matrix called metric,

$$\Theta = \Omega^\dagger \Omega. \quad (21)$$

This enables one to treat the quasi-Hermitian Hamiltonians as the mere isospectral avatars of their standard textbook representations

$$\mathfrak{h} = \Omega H \Omega^{-1} = \mathfrak{h}^\dagger. \quad (22)$$

Thus, the mapping (22) may prove useful whenever the conventional Hamiltonian \mathfrak{h} (acting, say, in a conventional Hilbert space \mathcal{L}) happens to be more complicated than its partner H acting in another Hilbert space \mathcal{K} .

In the theoretical physics of unitary quantum systems the user-friendlier Hilbert space \mathcal{K} plays an important technical role. In spite of the obvious fact that the inner products in \mathcal{K} do not admit any immediate physical probabilistic interpretation, an efficient remedy is straightforward. One simply introduces the third, correct and physical Hilbert space \mathcal{H} which only differs from \mathcal{K} by an amended inner product,

$$\langle \psi_a | \psi_b \rangle_{(\mathcal{H})} = \langle \psi_a | \Theta | \psi_b \rangle_{(\mathcal{K})}. \quad (23)$$

Thus, our (by assumption, positive definite and nontrivial) matrix $\Theta \neq I$ is revealed to play the role of a non-trivial but physics-determining inner-product metric.

B.2. The concept of smallness of perturbations

One of the most decisive advantages mediated by the choice of quasi-Hermitian toy models may be seen in the fact that the information about dynamics (encoded, in conventional models,

into a single Hermitian Hamiltonian operator or matrix \mathfrak{h}) is now carried by the pair of operators (viz., by H and Θ). In the literature explaining such a methodical advantage (cf., e.g., [3,6]) people still often start the process of model-building directly from a single non-Hermitian operator H alone, especially when its real spectrum exhibits some phenomenologically desirable descriptive properties (cf., e.g., [8]).

The construction of partner operator Θ making the model consistent is not always paid the same attention. Often, the duty is only fulfilled formally, without recalling relation (19) [or, equivalently, (20) or (22)], and without treating it as a constraint which has to restrict the freedom in our choice of Θ . In fact, the technical difficulty of the construction of the metric redirected many physicists towards the manifestly non-unitary, effective-model setups in which the construction of the metric $\Theta(H)$ is not needed at all because, in this setting, the role of the physical space is played, directly, by the friendliest space, viz., by \mathcal{K} . Such a strategy is also preferred, due to its simplicity, by mathematicians [50].

In the alternative, technically more ambitious strategy aimed at the quasi-Hermitian-operator description of the stable, unitarily evolving quantum systems the key challenge lies in the proper treatment of the subtle differences between the auxiliary and physical Hilbert spaces \mathcal{K} and \mathcal{H} , respectively [52]. Such a challenge has two components. Firstly, one reveals that the correct physical metric is necessarily Hamiltonian-dependent. Even the physical Hilbert space itself must be, therefore, expected to vary with the variation of the dynamics, $\mathcal{H} = \mathcal{H}(H)$ [52]. Secondly, even at a fixed H the solution $\Theta = \Theta(H)$ of Eq. (19) remains ambiguous [62].

The full strength of the challenge emerges during the studies of stability, i.e., during the analysis of the influence of “small” perturbations upon the processes leading, potentially, to the loss of the observability of the system [36]. The problem has been clarified in [27] where we restricted our attention to the Hamiltonians near their non-degenerate EP singularity. Our detailed analysis of the spectra of the tilded model (17) revealed that one cannot define the acceptability of perturbations via a “sufficient smallness” of the mere single coupling constant g . The reason (discussed also in preceding Appendix A) lies in a strongly hierarchically anisotropic nature of *any available* metric $\Theta(H)$ at *any* sufficiently small g . Such an anisotropy implies, in a way illustrated in [27], that it is highly nontrivial to evaluate the “measurable size” of the perturbation in (17). In particular, this obstacle practically excludes a guarantee of the smallness of the relevant norm of a perturbation in \mathcal{H} using just its auxiliary, manifestly unphysical representation in \mathcal{K} , i.e., working without a supplementary reference to the structure of the metric [36].

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