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Quantum Field Theory and Statistical Systems

Hom-Lie-Virasoro symmetries in Bloch electron systems and quantum plane in tight binding models

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Abstract

We discuss the Curtright-Zachos (CZ) deformation of the Virasoro algebra and its extensions in terms of magnetic translation (MT) group in a discrete Bloch electron system, so-called the tight binding model (TBM), as well as in its continuous system. We verify that the CZ generators are essentially composed of a specific combination of MT operators representing deformed and undeformed $U(1)$ translational groups, which determine phase factors for a $*$-bracket commutator. The phase factors can be formulated as a $*$-ordered product of the commutable $U(1)$ operators by interpreting the AB phase factor of discrete MT action as fluctuation parameter q of a quantum plane. We also show that some sequences of TBM Hamiltonians are described by the CZ generators.

1. Introduction

1.1. FFZ and Moyal deformation

The Moyal sine algebra (FFZ algebra) [1] is related to the Moyal bracket deformation, which is well-known as a Lie-algebraic deformation of the Poisson brackets and has been applied to

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describe noncommutative phenomena in various regions such as noncommutative geometry in string theories and quantum Hall physics.

The Moyal bracket and its star product are defined by [2]

$$\{f(x,p), g(x,p)\}_* = \frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\theta^{ab}\partial_1^a\partial_2^b\right) f(x,p)g(x,p)\,, \tag{1.1}$$

$$f * g = \exp\left\{i\frac{\hbar}{2}\theta^{ab}\partial_1^a\partial_2^b\right\} f(x,p)g(x,p)\,, \tag{1.2}$$

where $\theta^{xp} = -\theta^{px} = \theta$, and ∂_1 and ∂_2 denote forward (left) and backward (right) derivative operations respectively. The Moyal commutator is related to the Moyal bracket

$$[f,g]_* = f*g - g*f = i\hbar\{f,g\}_* \tag{1.3}$$

and is often mentioned as Moyal deformation or Moyal quantization which can be regarded as a quantum deformation of the Dirac bracket

$$i\hbar\{f,g\}_* \to [f,g]\,. \tag{1.4}$$

It is known that this quantization leads to the $SU(\infty)$ Lie algebra, so-called the Moyal sine algebra, if one takes the bases $\tau_{n,k} = e^{i(nx+kp)}$ of T^2 phase space [1]:

$$[\tau_{n,k}, \tau_{m,l}] = 2i\sin\left(\frac{\hbar\theta}{2}(nl-mk)\right)\tau_{n+m,k+l}\,. \tag{1.5}$$

In the context of deformation quantization one may keep the star product structure

$$[\tau_{n,k}, \tau_{m,l}]_* = 2i\sin\left(\frac{\hbar\theta}{2}(nl-mk)\right)\tau_{n+m,k+l}\,. \tag{1.6}$$

What we call the FFZ algebra in this note is originally given by (1.5) with the introduction of deformation parameter q and the q-bracket for an arbitrary object A

$$q = e^{i\hbar\theta}\,, \quad \text{and} \quad [A] = \frac{q^A - q^{-A}}{q-q^{-1}}\,. \tag{1.7}$$

Changing the normalization

$$T_{(n,k)} = \frac{1}{q-q^{-1}}\tau_{n,k} \tag{1.8}$$

we have the FFZ algebra

$$[T_{(n,k)}, T_{(m,l)}] = \left[\frac{nl-mk}{2}\right]T_{(n+m,k+l)}\,, \tag{1.9}$$

where we assume that $\tau_{n,k}$ is generalized to an arbitrary operator behaving like the Moyal star products [1]

$$\tau_{n,k}\tau_{m,l} = q^{\frac{nl-mk}{2}}\tau_{n+m,k+l}\,. \tag{1.10}$$

It is well known that magnetic translation (MT) operators satisfy this fusion relation [3].

1.1.1. Noncommutative geometry

Quantum field theory on noncommutative spaces [4] is one of many applications of the Moyal brackets. The coordinates of the endpoints of open strings constrained to a D-brane in the presence of a constant Neveu-Schwarz B field are reported to be relevant to a noncommutative algebra [5]. The field theory on noncommutative boundary space (known as noncommutative field theory) has attracted much attention in string and M-theories, and the noncommutativity is expressed in the Moyal star bracket $[x^\mu, x^\nu]_* = i\hbar\theta^{\mu\nu}$. This noncommutative feature is understood to be originated from the same idea as noncommutative magnetic translations in two-dimensional quantum mechanics in a constant magnetic field [3].

There is also an interesting topic in quantum gravity in a relevance to infinite dimensional symmetries. Moyal deformations of self-dual gravity have recently been studied in the context of noncommutativity and $W_{1+\infty}$ algebra [6], while the classical counterpart $w_{1+\infty}$ algebra [7] is related to soft graviton symmetries in asymptotically flat 4D quantum gravity [8]

$$[w_n^p, w_m^q] = (n(q-1) - m(p-1))w_{n+m}^{p+q-2} . \tag{1.11}$$

1.1.2. Quantum Hall effect

On the other hand, another type of W_∞ algebra has been examined in quantum Hall physics [9, 10] and in the conformal field theory of edge excitations as well as in bulk physics extension [11]. It is also known that W_∞ is associated to the area-preserving diffeomorphisms (for a review see [12]) of incompressible fluids. The generators $\tilde{w}_n^k$ satisfy [13]

$$[\tilde{w}_n^k, \tilde{w}_m^l] = ((k+1)(m+1) - (n+1)(l+1))\tilde{w}_{n+m}^{k+l} , \tag{1.12}$$

and their generating functions $\rho(k, \bar{k})$ consisting of $\tilde{w}_n^k$ as Fourier components [14] obey the Girvin-MacDonald-Platzman algebra [10]

$$[\rho(k, \bar{k}), \rho(p, \bar{p})] = (e^{p\bar{k}/4} - e^{\bar{p}k/4})\rho(k+p, \bar{k}+\bar{p}) . \tag{1.13}$$

Interestingly, this relation can be identified with the FFZ commutation relation (1.9) by changing the normalization

$$\rho(k, \bar{k}) = e^{-\frac{k\bar{k}}{8}} W_{k,\bar{k}} , \tag{1.14}$$

which leads to [14]

$$[W_{k,\bar{k}}, W_{p,\bar{p}}] = 2\sinh\left(\frac{p\bar{k} - \bar{p}k}{8}\right) W_{k+p,\bar{k}+\bar{p}} . \tag{1.15}$$

1.2. Quantum algebra and Hall effect

Another interesting aspect of quantum Hall effect is a certain relevance to integrable models in two-dimensional lattice systems. The algebraic approach of dynamical symmetries is useful when the Hamiltonian can be written in terms of the symmetry generators. The problem of Bloch electrons in a constant magnetic field can be solved by making use of a relation between the group of magnetic translations and $U_q(sl_2)$ quantum group [15] (strictly speaking the quantum algebra, which is a universal enveloping algebra with non-cocommutative Hopf algebra structure [15–18]). The Hamiltonian of a particle on a two-dimensional square lattice in the magnetic field (tight binding model) is composed of $U_q(sl_2)$ raising and lowering operators, which are expressed in N-dimensional Weyl bases X and Y with the commutation relation $qXY = YX$ [19].

The same approach on a triangular lattice is shown in [20] adding the 3rd basis Z satisfying $qZX = XZ$ and $qYZ = ZY$. Spectra of these models can be represented by means of solutions of Bethe ansatz type algebraic equations [19–21]. The magnetic translations in the discretized systems are given by products of the Weyl base matrices, and their commutation relations are accompanied by a global phase factor such as $q^2 T_x T_y = T_y T_x$ owing to the Weyl commutation relations.

While in a continuous coordinate system, magnetic translations are made of differential operators, and their phase factors are no longer global but rather comprising local parameters as seen in (1.6), and the Hamiltonian cannot be written in terms of those quantum group generators. Nevertheless it is interesting that there exist the generators of quantum algebra $U_q(sl_2)$ expressed by magnetic translations [22] in somewhat parallel form to the cases of the lattice systems. Furnished with the local parameters of Moyal type (1.6), the generators can further be extended to q-deformed Virasoro (super)algebras [23,24] which are studied in field theory context [25–30]. Although these deformed algebras are not a quantum algebra but infinite dimensional Lie algebras, an interesting point is definitely an appearance of Moyal noncommutative feature of magnetic translations. Despite the clear correspondence of the tight binding Hamiltonian to the continuous system in a continuum limit, the inheritance of properties of magnetic translations such as quantum groups and the Weyl bases is still unclear. In other words, they are considered to disappear or not to be in the limit, regardless of having the similar characteristics mentioned above.

Related to the quantum Hall effects and q-deformed algebras, there is also an interesting approach using Tsallis statistics [31] to thermodynamic calculations [32–34].

1.3. Quantum space and CZ algebra

In order to reveal the truth and resolve such a conflicting situation, quantum groups and planes may help. Quantum groups [35] are another notion to describe noncommutative geometry, and they are formulated to be dual objects to quantum algebras [36,37]. They are deformations of matrix groups whose matrix entries obey certain commutation relations depending on a deformation parameter q. Quantum space possesses noncommuting coordinates X^i and their differentials $\partial_i = \dfrac{\partial}{\partial X^i}$ which satisfy the relations

$$X^i X^j = B^{ij}_{kl} X^k X^l\,, \quad \partial_i \partial_j = F^{lk}_{ji} \partial_k \partial_l\,, \quad \partial_j X^i = \delta^i_j + C^{ik}_{jl} X^l \partial_k\,, \tag{1.16}$$

where the coefficient matrices B, C and F satisfy certain Yang-Baxter relations [38]. This ensures that the coordinates and their derivatives behave covariantly under the action of quantum group matrix. One of intriguing and the simplest example is the bosonic part of $GL_q(1,1)$ quantum superspace

$$\partial_x x = 1 + q^{-2} x \partial_x\,, \tag{1.17}$$

since the Virasoro operators in this quantum space

$$L_n = -q^{-1} x^{n+1} \partial_x \tag{1.18}$$

satisfy another version of q-deformed Virasoro algebra called the Curtright-Zachos (CZ) algebra [39]

$$[\, L_n\,, L_m\,]_{(m-n)} = [n-m] L_{n+m}\,, \tag{1.19}$$

where the symbol $[x]$ is defined in (1.7) and the bracket on LHS denotes the deformed commutator $[A, B]_{(x)} = q^x AB - q^{-x} BA$. There are many results concerning representations of the CZ algebra such as: q-harmonic oscillators [40], central extensions [41,42] and operator product formula (OPE) [42], matrix representation [43], quantum space differential calculi [44–46], and fractional spin representation [47]. Multi-parameter deformations [48,49] and supersymmetric extensions [44,45,50] are also studied. Motivated by its application to physical systems, several investigations have been made: deformation of soliton equations [51,52], transformation to commutator form [52], Jacobi consistency conditions [42,53], and so on.

1.4. Contents

The purpose of this paper is three fold: first one is to clarify a dynamical origin of the phase factors used in the commutator deformation of the algebra (1.19). Second is to clarify a relation between the phase factors and quantum plane picture which is relevant to physical models that possess a quantum algebra symmetry. Third is to present various ways of constructing the CZ generators. We investigate some properties of the CZ generators based on the algebras of MT and discrete magnetic translation (DMT), and discuss their relations to the Hamiltonian systems relevant to a tight binding model (TBM), which is a discrete model of a particle on a two-dimensional lattice in constant magnetic field. We expect to catch a glimpse of quantization of space, or rather the quantum plane structure, as a natural effect of space discretization in TBM, in addition to which is known to possess the quantum algebra symmetry $U_q(sl_2)$ (see Appendix A).

In Section 2, reviewing some properties of CZ algebra, we explain mathematical settings and MT algebras as our basic tool. The magnetic translations in TBM are reviewed in Section 5. In Section 3, we consider MT realizations of CZ algebras starting from a q-derivative representation of the algebra. We present a generalized algebra CZ^*, which includes CZ algebra as a subalgebra of it. The generators of these algebras are composed of certain combinations of specific MT operators, and all these algebras as well as MT algebra can be encapsulated into a single expression of deformed commutators with a $*$-product structure.[1] The structure of CZ^* generators consists of commutative MT and noncommutative MT parts. The commutative part plays the role of the fundamental CZ algebraic relation (structure constant of the algebra), and the noncommutative part plays the role of a nonlocal operator of $*$-commutative translation (deformed $U(1)$) and the role of defining a weight for $*$-product phase factors as well.

In Section 4, we construct a matrix representation of CZ^* and verify the property of the representation given in Section 3 (we call the representation in Section 3 "commutative representation"). We investigate a mechanism how quantum plane structure and the $*$-bracket structure arise in CZ^* algebra. CZ^* contains two subalgebras $CZ^{\pm}$, and they correspond to two orthogonal directions on quantum plane in TBM. Section 4.2 outlines the role and significance of commutative representations. Section 4.3 defines the algebra family of CZ^* representations that provide the TBM Hamiltonian sequences, in preliminary for the specific verification explained in Section 5. Section 4.4 presents another definition of our $*$-product. We first show that the commutative representations are composed of composite operators of DMT units in two directions. Then making use of the commutative representations and introducing a concept of $*$-ordering product, we formulate the appearance of the phase factor of the CZ^* commutators on basis of

[1] It is not clear that our $*$-product is the same as the Moyal $*$-product, however for convenience of terminology we adopt the same symbol without causing any serious confusion.

the quantum plane picture of DMT in TBM. The derivation of the matrix representations of DMT is shown in Appendix B, where confirmation of their MT algebra is made as well.

In Section 5, we discuss the TBM Hamiltonian series universally represented by the matrix representations of CZ^* algebras. In Section 5.1, we derive the DMT algebra (exchange, fusion and circulation rules) and observe the correspondence between the AB phase and the quantum plane fluctuations accompanied by the DMT movement. In Section 5.2, we describe the TBM Hamiltonian sequence $\check{H}_k$ using the $n = \pm 1$ modes of CZ^* family generators. Section 5.3 discusses extensions to general modes of CZ^* family in accord to the Hamiltonian systems $\hat{H}_n$ with the effective spacing of magnetic lattice extended from 1 to n. In Section 5.4, more generic sequence $\hat{H}_{(n,k)}$ with the quantum plane fluctuation (power of q) extended from 1 to k is represented by the genuine CZ^* generators. As an aid to understanding some formulae used in Section 5, we review the quantum group symmetry of the original TBM Hamiltonian, and add some remarks on q-inversion symmetry of the Hamiltonian in Appendix A.

2. Curtright-Zachos (CZ) algebra

We make a brief review of mathematical settings on the CZ algebra. There are two types of bracket deformation, and they differ in how they introduce their phase factors. The way of introducing the phase factors depends on explicit realizations of the generators of CZ algebra. Namely, there is no a priori way to determine how to introduce the phase factors and we have to start with a realization of the generators. However, the only known realization with physical implications is the deformation of harmonic oscillators, and we hence need other realizations to explore physical and mathematical properties of the CZ algebra. To this end, in Section 2.2, we introduce magnetic translation (MT) algebra, which is a typical realization of the Moyal sine algebra (FFZ algebra) in physical system. The property of generating phase factors when MT operators commute is compatible with the deformed commutation relation of the CZ algebra, and it is very convenient to investigate various features of the CZ algebra. Using MT representations in a later section, we will explain that there exists a larger algebra containing the CZ algebra, and investigate $*$-bracket formulation for these algebras altogether.

2.1. Bracket deformation

The CZ algebra is neither a standard Lie algebra nor a quantum algebra, but a Hom-Lie algebra [54–56], which is a deformed Lie algebra satisfying skew symmetry and Hom-Jacobi conditions [56]. The Hom-Lie algebras were originally introduced in [54] as motivated by examples of deformed Lie algebras derived from twisted discretization of vector fields. The skew symmetry is provided by the relation

$$[L_n, L_m]_{(m-n)} = -[L_m, L_n]_{(n-m)} \,. \tag{2.1}$$

The Hom-Jacobi condition of the CZ algebra is expressed as

$$(q^n + q^{-n})[L_n, [L_m, L_\ell]_{(\ell-m)}]_{(m+\ell-n)} + \text{cyc. perm.} = 0\,, \tag{2.2}$$

and this condition consists of two parts. One is the Yang-Baxter associativity relation

$$[L_n, [L_m, L_\ell]_{(\ell-m)}]_{(m+\ell-2n)} + \text{cyc. perm.} = 0\,, \tag{2.3}$$

and the other is the consistency condition [42]

$$[L_n, [L_m, L_\ell]_{(\ell-m)}]_{(m+\ell)} + \text{cyc. perm.} = 0\,. \tag{2.4}$$

Representations in q-harmonic oscillator and quantum superspace as well as (1.18) are known to satisfy these conditions [44,45,52].

Another feature that distinguishes it from the usual Virasoro algebra is the existence of the central element S_0 [42]

$$S_0 = 1 + (q - q^{-1})L_0\,, \tag{2.5}$$

satisfying the commutation relations

$$S_0^k L_n = q^{-2nk} L_n S_0^k\,, \quad \text{or} \quad [S_0^k, L_n]_{(nk,-nk)} = 0\,. \tag{2.6}$$

Unlike the standard quantum and Lie algebras, the CZ algebra is defined in terms of the deformed bracket. We have two options to express the CZ algebra (1.19) according to the way of defining deformed commutators: one is the exterior type

$$[\,A\,,\,B\,]_{(x,y)} = q^x AB - q^y BA\,, \tag{2.7}$$

$$[A, B]_{(x)} = [A, B]_{(x,-x)} = q^x AB - q^{-x} BA\,, \tag{2.8}$$

and the other is the intrinsic type

$$[A, B]_* = A * B - B * A\,. \tag{2.9}$$

The phase factors q^x attached to the products AB in Eqs. (2.7) and (2.8) are given by hand. On the other hand in (2.9) the factors are implicitly included in the definition of $A * B$, which is supposed to give rise to phase factors according to a certain product mechanism determined based on a phase space structure[2] similarly to the Moyal commutators. This does not necessarily mean to supply the Moyal star products.

For the time being leaving the question open why the products are given in the following form and what its essential origin is, we suppose that

$$L_n * L_m = q^{m-n} L_n L_m\,, \tag{2.10}$$

$$S_0^k * L_n = q^{nk} S_0^k L_n\,, \tag{2.11}$$

and we then have

$$\begin{aligned}[L_n, L_m]_* &= [\,L_n\,,\,L_m\,]_{(m-n)} \\ &= q^{m-n} L_n L_m - q^{n-m} L_m L_n\,. \end{aligned} \tag{2.12}$$

In order to search for a certain origin of (2.10), we later investigate MT representations of CZ algebras in Section 3. In Section 4.4, we further explore mathematical features (quantum plane and $*$-ordered product) of discrete magnetic translations (DMTs) in tight binding model (TBM) on a two-dimensional square lattice as well.

[2] We introduced the product as $(A_n B_m)_q = q^{m-n} A_n B_m$ in the original paper [42].

2.2. Magnetic translation algebra (MTA)

Magnetic translations $\hat{\tau}_{\boldsymbol{R}}$ of a charged particle in a constant magnetic field on a continuous coordinate surface are written in terms of differential operators in the following nonlocal way (choice of unit $\phi_0 = hc/e = 1$)

$$\hat{\tau}_{\boldsymbol{R}} = e^{2\pi i \xi(\boldsymbol{x},\boldsymbol{R})} T_{\boldsymbol{R}} , \tag{2.13}$$

where $T_{\boldsymbol{R}}$ is a translation by $\boldsymbol{R}$, and ξ is a gauge function defined by

$$\boldsymbol{A}(\boldsymbol{x}+\boldsymbol{R}) - \boldsymbol{A}(\boldsymbol{x}) = \nabla \xi(\boldsymbol{x},\boldsymbol{R}) . \tag{2.14}$$

They satisfy the following algebraic relations (exchange and fusion rules):

$$\hat{\tau}_{\boldsymbol{R}_1} \hat{\tau}_{\boldsymbol{R}_2} = e^{2\pi i \phi} \hat{\tau}_{\boldsymbol{R}_2} \hat{\tau}_{\boldsymbol{R}_1} , \tag{2.15}$$

$$\hat{\tau}_{\boldsymbol{R}_1} \hat{\tau}_{\boldsymbol{R}_2} = e^{2\pi i \xi(\boldsymbol{R_1},\boldsymbol{R_2})} \hat{\tau}_{\boldsymbol{R_2}+\boldsymbol{R_1}} , \tag{2.16}$$

from which the circulation algebra is derived [3]

$$\hat{\tau}^{-1}_{\boldsymbol{R}_1} \hat{\tau}^{-1}_{\boldsymbol{R}_2} \hat{\tau}_{\boldsymbol{R}_1} \hat{\tau}_{\boldsymbol{R}_2} = e^{2\pi i \phi} , \tag{2.17}$$

where ϕ is a magnetic flux (proportional to the area $|\boldsymbol{R}_1 \times \boldsymbol{R}_2|$ enclosed by the end points of circular operations). It is given by the differences of gauge function between two points

$$\phi = \xi(\boldsymbol{R_1}, \boldsymbol{R_2}) - \xi(\boldsymbol{R_2}, \boldsymbol{R_1}) , \tag{2.18}$$

where we denote

$$\boldsymbol{R_1} = (n_1, m_1) = (n, k) , \quad \boldsymbol{R_2} = (n_2, m_2) = (m, l) . \tag{2.19}$$

In this paper we refer the set of exchange, fusion and circulation rules as MTA (magnetic translation algebra).

Introducing the q parameter with magnetic length $l_B = \sqrt{\hbar c/eB}$, unit length a and magnetic field B,

$$q = \exp\left(2\pi i \frac{Ba^2}{\phi_0}\right) = \exp\left(i a^2 l_B^{-2}\right) , \tag{2.20}$$

under the choice of symmetric gauge we have

$$\hat{\tau}_{\boldsymbol{R}_1} \hat{\tau}_{\boldsymbol{R}_2} = q^{nl-mk} \hat{\tau}_{\boldsymbol{R}_2} \hat{\tau}_{\boldsymbol{R}_1} , \tag{2.21}$$

$$\hat{\tau}_{\boldsymbol{R}_1} \hat{\tau}_{\boldsymbol{R}_2} = q^{\frac{nl-mk}{2}} \hat{\tau}_{\boldsymbol{R_1}+\boldsymbol{R_2}} . \tag{2.22}$$

$$\hat{\tau}^{-1}_{\boldsymbol{R}_1} \hat{\tau}^{-1}_{\boldsymbol{R}_2} \hat{\tau}_{\boldsymbol{R}_1} \hat{\tau}_{\boldsymbol{R}_2} = q^{nl-mk} . \tag{2.23}$$

Applying the change of normalization to (2.22)

$$\hat{\tau}_{\boldsymbol{R}_i} = \hat{\tau}^{(m_i)}_{n_i} = (q-q^{-1}) \hat{T}^{(m_i)}_{n_i} , \tag{2.24}$$

we have the realization (1.9) of FFZ algebra [1] by means of MTA

$$[\hat{T}^{(k)}_n , \hat{T}^{(l)}_m] = [\frac{nl-mk}{2}] \hat{T}^{(l+k)}_{n+m} . \tag{2.25}$$

As a simple example of $\hat{\tau}^{(k)}_n$ let us have a look at angular momentum phase space (together with a spin value Δ) on a unit circle $z = e^{i\theta}$ of cylinder coordinate $w = \ln z$. Instead of the usual form $\hat{\tau}_{\boldsymbol{R}} = e^{\frac{i}{\hbar}\boldsymbol{R}\cdot\boldsymbol{\pi}}$ described by the gauge covariant derivatives $\hat{\pi}_i = \hat{p}_i + \frac{e}{c}A_i$, in this case we have

$$\hat{\tau}_n^{(k)} = \exp\left(\frac{i}{\hbar}\boldsymbol{R}\cdot\boldsymbol{\Theta}\right) = z^n q^{-k(z\partial+\frac{n}{2}+\Delta)}\,, \tag{2.26}$$

where

$$\boldsymbol{\Theta} = (\frac{\hbar}{l_B}\theta, -\frac{1}{l_B}\mathcal{J}_3 - \frac{\hbar}{l_B}\Delta) = (-i\frac{\hbar}{l_B}\ln z, -\frac{\hbar}{l_B}(z\partial+\Delta))\,, \tag{2.27}$$

$$\boldsymbol{R} = (nl_B, k\frac{a^2}{l_B})\,, \qquad \mathcal{J}_3 = -i\hbar\partial_\theta = -i\hbar(x\partial_y - y\partial_x)\,. \tag{2.28}$$

One can verify that (2.26) satisfies the MTA (2.21) (2.22) (2.23). (For simplicity, one may set $a = l_B = \hbar = 1$.)

3. $*$-bracket formulation and CZ^* algebra

Starting from a q-differential operator expression of CZ generators, we present the realization of CZ algebra by MT operators, which leads to the idea of $*$-bracket formulation. In this section we discuss three types of CZ algebraic system $CZ^\pm$ and CZ^*.

Section 3.1 sets out the principle of giving rise to phase factors such that MT appears commutative (i.e., deformed U(1)). Setting weights on MT operators and CZ generators (special combinations of MT), we show that the $*$-brackets are realized by certain rules. Two closed subalgebras of MT, $\hat{T}_n^{(0)}$ and $\hat{T}_n^{(2)}$, play an important role in the construction of the CZ operators. The ordinary commutative operator $\hat{T}_n^{(0)}$ defines the structure constant of CZ algebra, and the other $*$-bracket commutative one $\hat{T}_n^{(2)}$ carries the weight of the CZ operators.

In Section 3.2, we show another special combination CZ^-, which transforms into CZ^+ mutually with q-inversion. Both $CZ^\pm$ can be combined into one algebraic system CZ^* in the framework of $*$-brackets.

3.1. q-derivative representation of CZ algebra

Let us consider the q-analogue of differential operators ∂_q (called the q-derivatives) defined by

$$\partial_q f(z) = \frac{f(z) - f(zq^{-2})}{(q-q^{-1})z}\,. \tag{3.1}$$

This satisfies the following Leibniz rule and the formula

$$\partial_q(f(z)g(z)) = g(z)\partial_q f(z) + f(zq^{-2})\partial_q g(z)\,, \tag{3.2}$$

$$\partial_q z^n = q^{-n}[n]z^{n-1}\,. \tag{3.3}$$

Defining the analogue of $l_n = -z^{n+1}\partial_z$ by replacing ∂_z with ∂_q,

$$\hat{L}_n = -z^{n+1}\partial_q \tag{3.4}$$

we can verify that $\hat{L}_n$ satisfy the CZ algebra eq. (1.19).

$$[\,\hat{L}_n\,,\,\hat{L}_m\,]_{(m-n)} = [n-m]\hat{L}_{n+m}\,. \tag{3.5}$$

Noticing eqs. (3.1) and (3.4), the q-derivative can be rewritten by nonlocal expression of ordinary derivative ∂_z, and we have

$$\hat{L}_n = -z^n \frac{1-q^{-2z\partial}}{q-q^{-1}}\,. \tag{3.6}$$

This is related to the q-harmonic oscillators

$$a_q^\dagger = qz\,, \quad a_q = \frac{1}{qz}[z\partial]\,, \quad N = z\partial\,, \tag{3.7}$$

which satisfy the following relations

$$a_q a_q^\dagger - q a_q^\dagger a_q = q^{-N}\,, \quad [N] = a_q^\dagger a_q \tag{3.8}$$

$$[N, a_q] = -a_q\,, \quad [N, a_q^\dagger] = a_q^\dagger\,, \tag{3.9}$$

$$\hat{L}_n = -q^{-N}(a_q^\dagger)^{n+1} a_q\,. \tag{3.10}$$

Now, applying the magnetic translation (2.26) to (3.6) with the notation (2.24) we have another representation of the CZ generators

$$\hat{L}_n = -\hat{T}_n^{(0)} + q^{n+2\Delta}\hat{T}_n^{(2)}\,. \tag{3.11}$$

To make a contact with $*$-bracket formulation, we consider deformed commutators of $\hat{L}_n$ and $\hat{T}_m^{(k)}$. The basic algebraic relations (2.21) and (2.22) allow us to calculate various deformed commutators such as

$$[\,\hat{L}_n\,, \hat{\tau}_m^{(k)}\,]_{(2m,nk)} = -q^{m+\frac{nk}{2}}[m]\hat{\tau}_{n+m}^{(k)}\,, \tag{3.12}$$

$$[\,\hat{L}_n\,, \hat{T}_m^{(k)}\,]_{(m-\frac{nk}{2})} = -[m]\hat{T}_{n+m}^{(k)}\,, \tag{3.13}$$

$$[\hat{L}_n, \hat{T}_m^{(k)}] = -[\frac{nk}{2}]\hat{T}_{n+m}^{(k)} - [m-\frac{nk}{2}]\hat{T}_{n+m}^{(k+2)} \tag{3.14}$$

which are verified easily by utilizing the exchange relation between ∂_q and $q^{-kz\partial}$

$$q^{-kz\partial}\partial_q = q^k \partial_q q^{-kz\partial}\,. \tag{3.15}$$

At this stage, there is no guideline for choosing one out of them. We hence need a limiting condition or a principle to constrain the form of phase factors to be attached to the commutator deformations.

One possible strategy is to follow the fact that magnetic translations should reduce to the usual translations when magnetic field vanishes. This naturally leads to the idea that $\hat{T}_n^{(k)}$ should satisfy a deformation of translational group $U(1)$. The most likely deformation is hence to attach phase factors to (2.8) in a way to cancel the phase factors coming from the fusion rule (2.22), i.e.,

$$[\,\hat{T}_n^{(k)}\,, \hat{T}_m^{(l)}\,]_{(\frac{mk-nl}{2})} = 0\,. \tag{3.16}$$

Let us refer the upper index k in $\hat{T}_n^{(k)}$ to "weight" and define the weight of $\hat{L}_n$ to be 2. Considering the following set of operators for all integers n and k

$$\mathscr{M} = \{\,\hat{T}_n^{(k)}, \hat{L}_n\,\} \tag{3.17}$$

and allowing only the same phase factors as (3.16) for $\hat{L}_n$ as well as $\hat{T}_n^{(k)}$, we thereby find the only three types of deformed commutators allowed

$$[\,\hat{T}_n^{(k)}\,,\hat{T}_m^{(l)}\,]_{(\frac{mk-nl}{2})}\,,\quad [\,\hat{L}_n\,,\hat{T}_m^{(l)}\,]_{(m-\frac{nl}{2})}\,,\quad [\,\hat{L}_n\,,\hat{L}_m\,]_{(m-n)}\,, \tag{3.18}$$

where we have put $k=2$ in the 2nd bracket reducing to (3.13), and $k=l=2$ in the 3rd one reducing to (3.5). Note that we have omitted one type due to the skewness of the bracket (2.8)

$$[\,A\,,B\,]_{(x)} = -[\,B\,,A\,]_{(-x)}\,. \tag{3.19}$$

In this way we adopt eqs. (3.5), (3.13), (3.16) for the fundamental set of closed algebras of $\mathscr{M}$, of which constituent (3.5) is the CZ algebra as a deformation of Virasoro algebra, and (3.16) is a deformed $U(1)$ algebra of magnetic translation group as a deformation of $U(1)$ translation group.

In summary we confirm that $*$-bracket (2.9) can be defined for every element $X_n^{(k)} \in \mathscr{M}$ of weight k by

$$X_n^{(k)} * X_m^{(l)} = q^{-x} X_n^{(k)} X_m^{(l)}\,,\quad x = \frac{nl-mk}{2} \tag{3.20}$$

and we organize eqs. (3.5), (3.13), (3.16) into the single common bracket form

$$[\hat{T}_n^{(k)}, \hat{T}_m^{(l)}]_* = 0\,, \tag{3.21}$$

$$[\hat{L}_n, \hat{L}_m]_* = [n-m]\hat{L}_{n+m}\,, \tag{3.22}$$

$$[\hat{L}_n, \hat{T}_m^{(l)}]_* = -[m]\hat{T}_{n+m}^{(l)}\,. \tag{3.23}$$

Note that the algebra of central element (2.6) reads

$$[\hat{S}_0^k, \hat{L}_n]_* = 0\,,\quad \text{where}\quad \hat{S}_0 = 1 + (q-q^{-1})\hat{L}_0\,, \tag{3.24}$$

if we regard the weight of $\hat{S}_0^k$ as $2k$ in (3.20).

3.2. $CZ^{\pm}$ and CZ^{*} algebra

Inspecting the constitution of the CZ operators $\hat{L}_n$ in terms of subalgebras of $\hat{T}_n^{(k)}$, we consider q-inversion symmetry of $\mathscr{M}$, since the algebra set (3.21)-(3.23) is not invariant under the exchange of $q \leftrightarrow q^{-1}$.

In order to examine the structure of CZ algebra (3.22), let us set the same phase factor for $\hat{T}_n^{(k)}$ as for $\hat{L}_n$. Then we have

$$[\,\hat{T}_n^{(k)}\,,\hat{T}_m^{(l)}\,]_{(m-n)} = [\frac{n(l-2)-m(k-2)}{2}]\hat{T}_{n+m}^{(k+l)}\,. \tag{3.25}$$

If we put $k,l=0,2$ in this equation, we obtain the following closed subalgebras on $\hat{T}_n^{(k)}$ with weight 0 and 2, which are the parts of $\hat{L}_n$ defined in (3.11)

$$[\,\hat{T}_n^{(2)}\,,\hat{T}_m^{(2)}\,]_{(m-n)} = 0\,,\qquad [\,\hat{T}_n^{(0)}\,,\hat{T}_m^{(0)}\,]_{(m-n)} = [m-n]\hat{T}_{n+m}^{(0)}\,, \tag{3.26}$$

$$[\,\hat{T}_n^{(2)}\,,\hat{T}_m^{(0)}\,]_{(m-n)} = [-n]\hat{T}_{n+m}^{(2)}\,,\qquad [\,\hat{T}_n^{(0)}\,,\hat{T}_m^{(2)}\,]_{(m-n)} = [m]\hat{T}_{n+m}^{(2)}\,. \tag{3.27}$$

According to these algebras, $[\,\hat{L}_n\,,\hat{L}_m\,]_{(m-n)}$ is verified to be closed without participations of any other weights:

$$\begin{aligned}
[\,\hat{L}_n\,,\hat{L}_m\,]_{(m-n)} &= (q^n[n]-q^m[m])q^{2\Delta}\hat{T}_{n+m}^{(2)} + [m-n]\hat{T}_{n+m}^{(0)} \\
&= [n-m](q^{n+m+2\Delta}\hat{T}_{n+m}^{(2)} - \hat{T}_{n+m}^{(0)}) \\
&= [n-m]\hat{L}_{n+m}\,.
\end{aligned}$$

We should note that $\hat{T}_n^{(0)}$ is a commuting local operator which does not include a differential operator w.r.t. z, while $\hat{T}_n^{(2)}$ is a noncommuting nonlocal differential operator. However this non-commutativity feature is reversed each other in the view of deformed commutator world as can be seen in eq. (3.26). If we find a set of subalgebras similar to eqs. (3.26) and (3.27), it is possible to find another set of operators similar to $\hat{L}_n$.

It is in fact easy to find such a combination of $\hat{T}_n^{(k)}$ by considering q-inverted version of (3.25)

$$[\, \hat{T}_n^{(k)} \,, \hat{T}_m^{(l)} \,]_{(n-m)} = [\frac{n(l+2)-m(k+2)}{2}]\hat{T}_{n+m}^{(k+l)} \,, \tag{3.28}$$

and setting $k, l = 0, -2$ we find another subalgebras corresponding to eqs. (3.26) and (3.27)

$$[\, \hat{T}_n^{(-2)} \,, \hat{T}_m^{(-2)} \,]_{(n-m)} = 0 \,, \qquad [\, \hat{T}_n^{(0)} \,, \hat{T}_m^{(0)} \,]_{(n-m)} = [n-m]\hat{T}_{n+m}^{(0)} \,, \tag{3.29}$$

$$[\, \hat{T}_n^{(-2)} \,, \hat{T}_m^{(0)} \,]_{(n-m)} = [n]\hat{T}_{n+m}^{(-2)} \,, \quad [\, \hat{T}_n^{(0)} \,, \hat{T}_m^{(-2)} \,]_{(m-n)} = [-m]\hat{T}_{n+m}^{(-2)} \,. \tag{3.30}$$

Defining the q-inverted version of $\hat{L}_n$ with the new notation $\hat{L}_n^{\pm}$

$$\hat{L}_n^{+} = \hat{L}_n \,, \qquad \hat{L}_n^{-} = \hat{T}_n^{(0)} - q^{-n-2\Delta}\hat{T}_n^{(-2)} \,, \tag{3.31}$$

we obtain

$$[\, \hat{L}_n^{+} \,, \hat{L}_m^{+} \,]_{(m-n)} = [n-m]\hat{L}_{n+m}^{+} \,, \qquad [\, \hat{L}_n^{-} \,, \hat{L}_m^{-} \,]_{(n-m)} = [n-m]\hat{L}_{n+m}^{-} \tag{3.32}$$

$$[\, \hat{L}_n^{\pm} \,, \hat{T}_m^{(k)} \,]_{(\pm m - \frac{nk}{2})} = -[m]\hat{T}_{n+m}^{(k)} \,, \tag{3.33}$$

where we denote the two algebras in (3.32) as $CZ^{\pm}$ respectively. We notice that the signs of phase factors for $\hat{L}_n^{\pm}$ in (3.32) are completely opposite, while the phase factors in (3.33) are not symmetric w.r.t. the exchange of $\hat{L}_n^{\pm}$. This fact is originated in the q-inversion symmetry

$$\hat{\hat{T}}_n^{(k)} \leftrightarrow -\hat{T}_n^{(-k)} \,, \quad \hat{L}_n^{+} \leftrightarrow \hat{L}_n^{-} \,. \tag{3.34}$$

However, this clumsy combination perfectly disappears if we incorporate $\hat{L}_n^{-}$ into $\mathscr{M}$ assuming the weight of $\hat{L}_n^{-}$ to be -2. Thus changing its notation from $\mathscr{M}$ to

$$\mathscr{M}^{*} = \{\, \hat{L}_n^{\pm}, \hat{T}_n^{(k)} \,\} \,, \tag{3.35}$$

we can make the new elements $\hat{L}_n^{-}$ participate in the following extended algebras including (3.21)-(3.23) with (3.20):

$$[\hat{L}_n^{\pm}, \hat{L}_m^{\pm}]_{*} = [n-m]\hat{L}_{n+m}^{\pm} \,, \tag{3.36}$$

$$[\hat{L}_n^{\pm}, \hat{T}_m^{(l)}]_{*} = -[m]\hat{T}_{n+m}^{(l)} \,. \tag{3.37}$$

Although the two algebras $CZ^{\pm}$ are expressed in (3.36) respectively, the remaining intersecting algebra $[\hat{L}_n^{+}, \hat{L}_m^{-}]_{*}$ is yet open to incorporate into possible $\mathscr{M}^{*}$ algebra. Using (3.31) We then find

$$[\, \hat{L}_n^{+} \,, \hat{L}_m^{-} \,]_{(n+m)} = q^{-m}[n]\hat{L}_{n+m}^{+} - q^{n}[m]\hat{L}_{n+m}^{-} \,, \tag{3.38}$$

and its phase factors are verified to certainly be given by (3.20) with the weights $k = \pm 2$ applied to $X_n^{(k)} = \hat{L}_n^{\pm}$.

Now let us show that (3.36) and (3.38) can be organized into a closed algebra form. Introducing the notation ϵ, η to express the $\pm$ signs, (3.36) and (3.38) read

$$[\hat{L}_n^\epsilon, \hat{L}_m^\epsilon]_* = [n-m]\hat{L}_{n+m}^\epsilon\,, \tag{3.39}$$

$$[\hat{L}_n^\epsilon, \hat{L}_m^\eta]_* = q^{\eta m}[n]\hat{L}_{n+m}^\epsilon - q^{\epsilon n}[m]\hat{L}_{n+m}^\eta\,. \quad (\epsilon \neq \eta) \tag{3.40}$$

Furthermore using the formula

$$q^{\epsilon m}[n] - q^{\epsilon n}[m] = [n-m]\,, \tag{3.41}$$

we can arrange the $CZ^\pm$ and their mixing algebras (3.39) and (3.40) in one final compact form

$$[L_n^\epsilon, L_m^\eta]_* = q^{\eta m}[n]L_{n+m}^\epsilon - q^{\epsilon n}[m]L_{n+m}^\eta\,, \tag{3.42}$$

which we shall denote the CZ^* algebra.

In closing this section we put a brief remark on the q-derivative expression. (3.31) is equivalent to another q-derivative representation (similarly to the equivalence between (3.6) and (3.11))

$$\hat{L}_n^- = -z^{n+1}\partial_q^- = -z^n \frac{q^{2z\partial} - 1}{q - q^{-1}}\,, \tag{3.43}$$

where

$$\partial_q^- f(z) = \frac{f(zq^2) - f(z)}{z(q - q^{-1})}\,. \tag{3.44}$$

Together with (3.4), we have a $q \leftrightarrow q^{-1}$ symmetric form of the standard q-derivative (with q substituted by q^2)

$$\partial_{q^2} := \frac{\partial_q^+ + \partial_q^-}{q + q^{-1}} = -\frac{\hat{L}_{-1}^+ + \hat{L}_{-1}^-}{q + q^{-1}} = \frac{1}{z}[z\partial]_{q^2}\,. \tag{3.45}$$

This type of q-derivative is used in operator product expansion form of CZ^+ algebra [42,52].

4. Matrix representations of CZ^*

This section studies matrix representations of CZ^* and its related algebras. We discuss the role of $CZ^\pm$ commutative representations and their connection to quantum plane and $*$-bracket structure. We also present preliminary results about relationship of CZ^* related algebras with the TBM Hamiltonian series.

To begin with, we introduce the $CZ^\pm$ matrices in Section 4.1. The $CZ^\pm$ operator consists of a combination of commutative and non-commutative parts, as seen in the case of MT realization. Section 4.2 considers the meaning and role of commutative representations X^n and Y^n, which play the same part as the MT operator $\hat{T}_n^{(0)}$. The commutative representation shows its significance in connection to quantum plane picture and the $*$-brackets. Concrete verification is done in Section 4.4. The matrix expression for CZ^* is also given in Section 4.2. Section 4.3 defines the representation sequence CZ^* family, which is the sequence obtained by the replacement $q \to q^k$. We introduce some matrix representations of CZ^* family associated with the TBM Hamiltonian (concrete correspondence will be verified in Section 5).

In Section 4.4, considering a special combination of DMT that provides a commutative representation of $CZ^\pm$, we derive another definition of $*$-product based on the quantum plane picture of TBM. The AB phase (see Section 5.1) associated with the movement of particles by DMT corresponds to the fluctuation of the quantum plane, and the phase factor generated by the successive DMT operations is expressed by a certain ordered product to reproduce the $*$-bracket.

4.1. Cyclic representation of $CZ^{\pm}$ algebra

In order to examine the CZ^* algebra, we must first set up its generators. Matrix representations of CZ^- are already known [43], but those of the other algebras are not. Thus we have to find a general expression for CZ^+ generators satisfying (3.36). The general expression obtained in this subsection does not always satisfy the CZ^* relation (3.38). This problem is solved by using an automorphism of $CZ^{\pm}$ in Section 4.2.

Let us follow the basic idea given in [43]. The cyclic matrix representation of CZ generators are given if q is a root of unity

$$q = \exp\left(\frac{2\pi i}{N}\right), \quad N > 2 \tag{4.1}$$

The minimal set of L_n elements are as follows depending on whether N is even or odd:

$$\{L_{-m+1}, \cdots, L_{-1}, L_0, L_1, \cdots, L_m\} \quad \text{for} \quad N = 2m \tag{4.2}$$

$$\{L_{-m}, \cdots, L_{-1}, L_0, L_1, \cdots, L_m\} \quad \text{for} \quad N = 2m+1\,. \tag{4.3}$$

The minimal set called the "fundamental cell" [43] and the translational group $G_N = \{G_N^k : n \to n + kN; k \in Z\}$ on a one-dimensional lattice of period N (> 2) gives rise to an algebra automorphism of CZ algebra

$$L_n \to L_{n+kN}, \quad k = 0, \pm 1, \pm 2, \cdots \tag{4.4}$$

These correspond to a magnetic unit cell and the magnetic translation group in a Bloch electron system respectively.

Introducing the $N \times N$ Wyle base matrices X and Y satisfying $X^N = Y^N = 1$

$$X = \begin{pmatrix} 0 & 0 & 0 & \cdots 0 & 1 \\ 1 & 0 & 0 & \cdots 0 & 0 \\ 0 & 1 & 0 & \cdots 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} q & 0 & 0 & \cdots & 0 \\ 0 & q^2 & 0 & \cdots & 0 \\ 0 & 0 & q^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & q^N \end{pmatrix} \tag{4.5}$$

or rather the component representation

$$(X)_{jk} = \delta_{j,k+1}\,, \quad (Y)_{jk} = q^j \delta_{jk}\,, \quad \text{for} \quad j, k \in [1, N] \quad (\text{mod}\, N) \tag{4.6}$$

we find the cyclic matrix representation of $CZ^{\pm}$ algebra satisfying (3.36)

$$\mathrm{L}_n^{\pm} = \mp\left(\frac{1 - \mathrm{Q}^{\pm 2}}{q - q^{-1}} + A_n^{\pm}\mathrm{Q}^{\pm 2}\right)\mathrm{H}^n\,, \quad A_n^{\pm} = a_{\pm} + b(q^{\pm 2n} - 1)\,, \tag{4.7}$$

where $a_{\pm}$ and b are free parameters. H and Q are the matrices used in [43]. They are related to the Wyle base matrices through $\mathrm{H} = X^{-1} = X^T$ and $q\mathrm{Q} = Y$, where X and Y satisfy the relation

$$Y^m X^n = q^{mn} X^n Y^m\,. \tag{4.8}$$

4.2. Roles of trivial form and $CZ^{\pm}$

We here deal with $\mathrm{L}_n^{\pm}$ in parallel, in order to consider the extension of $CZ^{\pm}$ to CZ^* later (see bottom of the subsection). As discussed later (Section 4.4) the phase factors of deformed commutators for $\mathrm{L}_n^{\pm}$ correspond to translations accompanied by phases $q^{\pm 1}$ in two orthogonal directions on a quantum plane. This is one of the reasons why we extend the CZ algebra to $CZ^{\pm}$ in this paper.

Before discussing the CZ^* extension, some remarks are in order. The first one is that the role of Q^2 as a central element [43] can be extended to $Q^{\pm 2}$ for $CZ^{\pm}$ algebras as follows:

$$[\mathrm{Q}^2, \mathrm{L}_n^+]_* = [\mathrm{Q}^2, \mathrm{L}_n^+]_{(n,-n)} = 0\,, \quad [\mathrm{Q}^{-2}, \mathrm{L}_n^-]_* = [\mathrm{Q}^{-2}, \mathrm{L}_n^-]_{(-n,n)} = 0\,, \tag{4.9}$$

where these elements are related to the central elements $S_0^{\pm}$ ($S_0^+ = S_0$)

$$S_0^{\pm} = 1 \pm (q - q^{-1}) L_0^{\pm} = \{1 - A_0^{\pm}(q - q^{-1})\}\mathrm{Q}^{\pm 2}\,. \tag{4.10}$$

In the form of unified expression ($\epsilon = \pm$), one may have the relations

$$[\mathrm{Q}^{\epsilon 2k}, \mathrm{L}_n^{\epsilon}]_{(\epsilon nk, -\epsilon nk)} = 0\,, \tag{4.11}$$

and

$$S_0^{\epsilon k} L_n^{\epsilon} = q^{-2\epsilon nk} L_n^{\epsilon} S_0^{\epsilon k}\,, \quad \text{or} \quad [S_0^{\epsilon k}, L_n^{\epsilon}]_* = [S_0^{\epsilon k}, L_n^{\epsilon}]_{(\epsilon nk, -\epsilon nk)} = 0\,. \tag{4.12}$$

From these, we again recognize $S_0^{\pm} \propto Q^{\pm 2}$.

The second remark is about trivial (commutative) representations

$$\mathrm{L}_n'^{\pm} = \frac{\mp g_n}{q - q^{-1}}\,, \quad g_n g_m = g_{n+m}\,. \tag{4.13}$$

Two examples of g_n are given in [43]

$$g_n = c^n \mathrm{H}^n\,, \quad \text{or} \quad g_n = q^{cn^2} \mathrm{Q}^{2cn} \mathrm{H}^n\,. \quad (\text{for const. } c) \tag{4.14}$$

As can be seen in (4.7), $\mathrm{L}_n^{\pm}$ is a linear combination of trivial part H^n and nontrivial part $\mathrm{Q}^{\pm 2}\mathrm{H}^n$. We now recall that $\hat{L}_n^{\pm}$ is also composed of trivial (commutative) $\hat{T}_n^{(0)}$ and nontrivial (non-commutative) $\hat{T}_n^{(\pm 2)}$ parts as previously mentioned below (3.26). In this sense the commutative representation H^n (essentially X^{-n}) plays a key role in $CZ^{\pm}$ algebras. It is also interesting to note that the substitution

$$\hat{T}_n^{(0)} = \frac{q^n}{q - q^{-1}} \mathrm{H}^n\,, \quad \hat{T}_n^{(\pm 2)} = \frac{q^{\pm 2}}{q - q^{-1}} \mathrm{H}^n \mathrm{Q}^{\pm 2}\,, \tag{4.15}$$

satisfy the same algebras as (3.26), (3.27) (3.29) and (3.30). Here we reverted the ordering of $\mathrm{Q}^{\pm 2}\mathrm{H}^n$ due to (4.19).

There is another trivial representation in terms of Y

$$g_n = c^n Y^n \tag{4.16}$$

and we further notice that X^n and Y^n play an important role to understand a relation between quantum plane and the $*$-bracket (2.9) as shall be discussed in Section 4.4. There we show that the commutative representations are nontrivially realized by composite operators of DMT units in two directions on a magnetic lattice. Commuting operators again behave like noncommuting operators in the framework of quantum plane, which generates phase factors when operators are exchanged. It will turn out that the $*$-bracket for X^n and Y^n fit perfectly with this quantum plane picture in the system of TBM.

To close this subsection, we present a matrix representation of CZ^* algebra based on (4.7). In order to complete the set of algebras (3.42), eq. (3.38) the remaining algebra $[L_n^+, L_n^-]_*$ should be satisfied in addition to $CZ^{\pm}$ algebras (3.36).

If we choose $b = 0$ in the matrix representation (4.7) denoting $\tilde{\mathscr{L}}_n^{\pm}$, we have

$$\tilde{\mathscr{L}}_n^{\pm} = \mp\left(\frac{1-\mathrm{Q}^{\pm 2}}{q-q^{-1}} + a_{\pm}\mathrm{Q}^{\pm 2}\right)\mathrm{H}^n\,, \tag{4.17}$$

and this does not satisfy (3.38) but a slightly different one

$$[\,\tilde{\mathscr{L}}_n^{+}\,,\tilde{\mathscr{L}}_m^{-}\,]_{(n+m)} = q^m[n]\tilde{\mathscr{L}}_{n+m}^{+} - q^{-n}[m]\tilde{\mathscr{L}}_{n+m}^{-}\,. \tag{4.18}$$

However, applying the following transformation that keeps $CZ^{\pm}$ unchanged

$$\mathscr{L}_n^{\pm} = \mathrm{H}^n\tilde{\mathscr{L}}_n^{\pm}\mathrm{H}^{-n}\,, \tag{4.19}$$

we verify that (3.38) is certainly satisfied, i.e.,

$$[\,\mathscr{L}_n^{+}\,,\mathscr{L}_m^{-}\,]_{(n+m)} = q^{-m}[n]\mathscr{L}_{n+m}^{+} - q^{n}[m]\mathscr{L}_{n+m}^{-}\,. \tag{4.20}$$

Since $CZ^{\pm}$ is preserved under the transformation (4.19), $\mathscr{L}_n^{\pm}$ also satisfy (3.36) and (3.42). Therefore the CZ^* algebra is confirmed. Note that none of (4.18) and (4.20) holds for $b \neq 0$, and we only consider the $b = 0$ case hereafter.

4.3. Preliminary representations to TBM

This is a preliminary section to discuss connections between CZ^* algebra and TBM (tight binding model) Hamiltonian (A.11). TBM is a two-dimensional lattice model which reproduces electron's Schrödinger equation under static magnetic field in a continuum limit. Our goal (see Section 5) is to show that TBM Hamiltonians can be expressed in the CZ^* generators $\mathscr{L}_{\pm 1}^{\pm}$ following the basic idea presented in Appendix A, however in order to get an overview at the moment we focus our attention to some representations derived from (4.17) and (4.19) prior to detailed investigation.

These representations are relevant to three methods of finding a relationship between CZ^* algebra and TBM: (i) modification of Schrödinger equations according to the structure of quantum planes, (ii) operator factorization, (iii) change of the parameter q in CZ^* algebra. Modifications of the CZ^* are necessary in the latter two cases.

The first representation is given by

$$\mathscr{L}_n^{\pm} = \mp\mathrm{H}^n\,\frac{1 \mp iq^{\pm 2}\mathrm{Q}^{\pm 2}}{q-q^{-1}} = \mp X^{-n}\,\frac{1 \mp iY^{\pm 2}}{q-q^{-1}}\,, \tag{4.21}$$

with the choice

$$a_{\pm} = \frac{1 \mp iq^{\pm 2}}{q-q^{-1}}\,. \tag{4.22}$$

Since TBM Hamiltonian $\hat{H}$ is a linear combination in $X^{\pm 1}$ and $Y^{\pm 1}$ as seen in Appendix A

$$\hat{H}(X,Y;q) = iY^{-1}(X^{-1}-X) + i((X^{-1}-X)Y\,. \tag{4.23}$$

$Y^{\pm 2}$ in this representation is a bit inconvenient. In this case we have to consider a slightly modified Hamiltonian in accordance with a different Schrödinger equation on a quantum plane whose effective length and magnitude of phase fluctuation are doubled compared to the linear Hamiltonian system. (Section 5.4).

The second type of representation is for example

$$\mathscr{L}_n^{\pm} = \mp \mathrm{H}^n \frac{1 - q^{\pm 2}\mathrm{Q}^{\pm 2}}{q - q^{-1}} = X^{-n} Y^{\pm 1}[Z], \quad Y = q^Z, \tag{4.24}$$

with the choice

$$a_{\pm} = \frac{1 + q^{\pm 2}}{q - q^{-1}}. \tag{4.25}$$

In this case we can extract the linear TBM Hamiltonian by factoring out the operator $[Z]$ (for details see (5.24)). In order to see the factorization, we rather employ $CZ^{*\prime}$ the following algebra

$$[\, L_n^{\prime -}, L_m^{\prime -}\,]_{(n-m)} = [n-m] L_{n+m}^{\prime -}, \tag{4.26}$$

$$[\, L_n^{+}, L_m^{\prime -}\,]_{(n+m)} = [n] L_{n+m}^{+} - [m] L_{n+m}^{\prime -}, \tag{4.27}$$

by introducing

$$L_n^{\prime -} = q^n L_n^{-} \tag{4.28}$$

where arbitrary representations can be applied.

The third type of candidates are not exactly the $\mathscr{L}_n^{\pm}$ but slightly modified operators $\check{\mathscr{L}}_n^{\pm}$, where we consider the algebra given by $Y^{\frac{1}{2}}$ instead of Y in the first representation (4.21);

$$\check{\mathscr{L}}_n^{\pm} = \mp \mathrm{H}^n \frac{1 \mp i q_1^{\pm 2} \mathrm{Q}_1^{\pm 1}}{q_1 - q_1^{-1}} = \mp X^{-n} \frac{1 \mp i Y_1^{\pm 1}}{q_1 - q_1^{-1}}, \quad Y_1^2 = Y. \tag{4.29}$$

It is obtained by the replacement $q \to q_1 = q^{\frac{1}{2}}$ and by changing matrices $\{Y^2, \mathrm{Q}^2\}$ to $\{Y_1, \mathrm{Q}_1\}$ in (4.21), where the condition $q^N = 1$ should be changed to $q_1^N = 1$ as well as matrix entries q in Y_1 to q_1. Commutation relation $qXY = YX$ is then changed to $q_1 X Y_1 = Y_1 X$. After all, except for the change of power in Y and Q, everything is understood as $q \to q_1 = q^{\frac{1}{2}}$. For the sake of later conveniences, we introduce more general modification

$$\check{\mathscr{L}}_n^{\pm} = \mp X^{-n} \frac{1 \mp i Y_k^{\pm 1}}{q_k - q_k^{-1}}, \tag{4.30}$$

$$Y_k = diag(q_k, q_k^2, \cdots, q_k^N), \quad q_k^N = 1, \quad q_k X Y_k = Y_k X. \tag{4.31}$$

The modified algebra of $\check{\mathscr{L}}_n^{\pm}$ is given by the replacement $q \to q_k$ in $CZ^*(q)$, that leads to a sequence of algebras $CZ^*(q_k)$, and we refer to it as CZ^* family (algebras). We then have

$$[\check{\mathscr{L}}_n^{\pm}, \check{\mathscr{L}}_m^{\pm}]_{*_k} = [n-m]_k \check{\mathscr{L}}_{n+m}^{\pm}, \tag{4.32}$$

$$[\check{\mathscr{L}}_n^{+}, \check{\mathscr{L}}_m^{-}]_{*_k} = q_k^{-m} [n]_k \check{\mathscr{L}}_{n+m}^{+} - q_k^{n} [m]_k \check{\mathscr{L}}_{n+m}^{-}, \tag{4.33}$$

where

$$[n]_k := \frac{q_k^n - q_k^{-n}}{q_k - q_k^{-1}}, \tag{4.34}$$

$$[\check{\mathscr{L}}_n^{+}, \check{\mathscr{L}}_m^{-}]_{*_k} = [\, \check{\mathscr{L}}_n^{+}, \check{\mathscr{L}}_m^{-}\,]_{(n+m)} \Big|_{q \to q^k}. \tag{4.35}$$

We assume $Y_k = Y^k$ and $q_k = q^k$ for $k \geq 2$, and (4.21) multiplied by $q + q^{-1}$ corresponds to the $k = 2$ case. (4.29) is regarded as an exceptional case $k = 1$ with $Y_1^2 = Y$ ($\leftrightarrow q_1^2 = q$). This is the outlined strategy of how to obtain the TBM form $\hat{H}(X, Y_k; q_k)$ from $\check{\mathscr{L}}_n^{\pm}$ by changing the q parameter in CZ^* representations. Details are explained in Section 5.2.

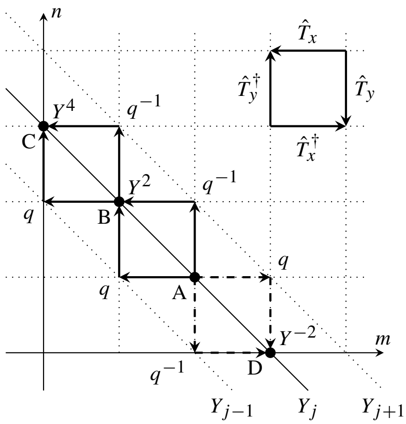

Fig. 1. Translations on Y_j line ($j = m + n$).

4.4. Quantum plane and $*$-bracket

We discuss quantum plane picture of $CZ^{\pm}$ algebra in the framework of discrete magnetic translations $\hat{T}_x$ and $\hat{T}_y$ in TBM. Relations between the discrete magnetic translations (DMT) and the Wyle base matrices X and Y are summarized in Appendix B for the convenience.

Let us consider the matrix expressions of DMT, for example given by (B.6):

$$\hat{T}_x = -iXY\,, \quad \hat{T}_y = -iY^{-1}X \tag{4.36}$$

$$\hat{T}_x^\dagger = iY^{-1}X^{-1}\,, \quad \hat{T}_y^\dagger = iX^{-1}Y \tag{4.37}$$

As shown in Fig. 1, these DMT operators describe the translations on a two-dimensional lattice (m, n) in four directions, respectively.

Consider the points A, B, C and D on the line Y_j for $j = n + m$ fixed to a constant value, and a route of successive movements by DMTs from the point A. There are two shortest ways to get to B from A, namely via Y_{j-1} and via Y_{j+1}. In the case of via Y_{j-1}, we have

$$\hat{T}_y^\dagger \hat{T}_x = (X^{-1}Y)(XY) = qY^2\,, \tag{4.38}$$

and we interpret this relation that Y^2 corresponds to a movement from A to B along Y_j with a phase factor q caused by the fluctuation via Y_{j-1}. If we go from A to C via Y_{j-1} twice, we understand that Y^4 is the moving operator along Y_j, and q^2 the fluctuation phase factor.

Similarly in the case of via Y_{j+1}, we have

$$\hat{T}_x \hat{T}_y^\dagger = (XY)(X^{-1}Y) = q^{-1}Y^2\,, \tag{4.39}$$

and interpret that Y^2 corresponds to a movement from A to B along Y_j with a phase factor q^{-1} caused by the fluctuation via Y_{j+1}. As to the movements from A to D, which is in a opposite direction to B along Y_j, we understand in a parallel way that Y^{-2} corresponds to the movements from A to D with a phase factor $q^{\pm 1}$ caused by the fluctuation via $Y_{j\pm 1}$.

Let us define a positive direction for Y_j as the one with increasing n of the vertical axis, and denote the numbers of fluctuations via Y_{j+1} (resp. Y_{j-1}) by k (resp. l) for positive direction along Y_j (they are denoted by $-k$ and $-l$ for negative direction) when moving to an arbitrary

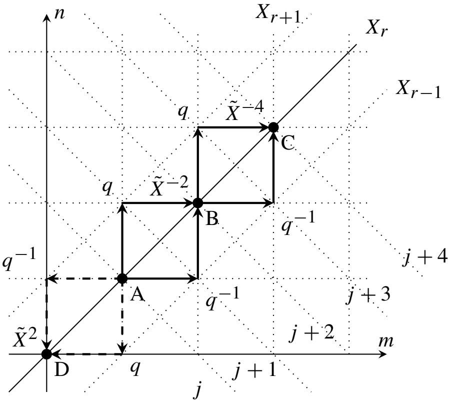

Fig. 2. Translations on X_r line ($r = n - m$).

point which is $k+l$ points away along Y_j. Then we can express the translation operator composed of $k+l$ translations along Y_j in the following way with the total phase factor

$$q^{-k+l} Y^{2(k+l)} . \tag{4.40}$$

For example, in the case of fluctuating once via Y_{j+1} and twice via Y_{j-1} when moving to C from D in Fig. 1, it reads qY^6 since we have $k = 1$ and $l = 2$.

We now define the $*$-product as an ordered product as follows: (i) first put the operators with fluctuations toward a positive direction (i.e., via Y_{j+1}) in a leftward position, and those with negative fluctuations (via Y_{j-1}) in a rightward position. (ii) Second attach phase factors q^{-1} for a positive fluctuation, and q for negative one.

Then we can express (4.40) using the $*$-product definition as

$$(Y^{2k} Y^{2l})_* = q^{l-k} Y^{2(k+l)} . \tag{4.41}$$

By the way, Y^{2n} is nothing but the trivial representation (4.13) with (4.16), and we therefore have the CZ^+ algebra with the definition (4.41)

$$[\mathrm{L}'^+_n, \mathrm{L}'^+_m]_* = [n-m]\mathrm{L}'^+_{n+m} , \qquad \mathrm{L}'^+_n = \frac{-Y^{2n}}{q - q^{-1}} . \tag{4.42}$$

The commuting operator Y^2 on the line Y_j acquires a nontrivial phase factor related to the $*$-product (4.41) as an effect of fluctuations via $Y_{j\pm 1}$. The $*$-product plays the function of projecting a commuting operator product into a noncommuting one. We conclude that this fact is formulated by L'^+, which is a trivial CZ^+ representation. In other words, we have obtained the picture that commuting translation operators on a quantum line Y_j (one-dimensional quantum plane) raise phase factors as an effect of quantum fluctuation of the quantum plane.

To complete the investigation, we have to consider another direction orthogonal to Y_j. The argument is straightforward, but attention should be paid to matrix normalization in order to parallel the discussion above. We thus elaborate on the details with reference to Fig. 2. Let us consider the points A, B, C and D on the line X_r for $r = n - m$, and two-way successive movements by DMTs from A to B via $X_{r\pm 1}$. In the case of via X_{r-1}, we have

$$\hat{T}_y^\dagger \hat{T}_x^\dagger = -(X^{-1}Y)(Y^{-1}X^{-1}) = -X^{-2} = q^{-1}\tilde{X}^{-2} , \qquad \tilde{X} = iXq^{-1/2} , \tag{4.43}$$

which means that $\tilde{X}^{-2}$ corresponds to a movement from A to B along X_r with a phase factor q^{-1} caused by the fluctuation via X_{r-1}. If we go from A to C via X_{r-1} twice, we thus have the moving operator $\tilde{X}^{-4}$ along X_r giving rise to the factor q^{-2}.

Similarly in the case of via X_{r+1}, we have

$$\hat{T}_x^\dagger \hat{T}_y^\dagger = -(Y^{-1}X^{-1})(X^{-1}Y) = -q^2 X^{-2} = q\tilde{X}^{-2} \tag{4.44}$$

which means that $\tilde{X}^{-2}$ corresponds to a movement from A to B along X_r with a phase factor q caused by the fluctuation via X_{r+1}. The movements from A to D, which is in a opposite direction to B along X_r, are understood in a parallel way that $\tilde{X}^2$ corresponds to the movements from A to D with the factor $q^{\pm 1}$ caused by the fluctuation via $X_{r\pm 1}$.

Let us define a positive direction for X_r as the one with increasing n on the vertical axis, and denote the numbers of fluctuations via X_{r+1} (resp. X_{r-1}) by k (resp. l) for positive directions along X_r (they are denoted by $-k$ and $-l$ for negative directions) when moving to an arbitrary point which is $k+l$ points away along X_r. Then we can express the translation operator composed of $k+l$ translations along X_r in the following way with the total phase factor

$$q^{k-l}\tilde{X}^{-2(k+l)}\,. \tag{4.45}$$

For example, in the case of fluctuating once via X_{r+1} and twice via X_{r-1} when moving to C from D in Fig. 2, it reads $q^{-1}\tilde{X}^{-6}$ since we have $k=1$ and $l=2$.

In the same way as Y^2, we here define the ordered product for $\tilde{X}$ as well: (i) first put the operators with fluctuations toward a positive direction (i.e., via X_{r+1}) in a leftward position, and those with negative fluctuations (via X_{r-1}) in a rightward position. (ii) Second attach phase factors q for a positive fluctuation, and q^{-1} for negative one. Note that the phases are inversed compared to the previous Y^2 case.

Then we can express (4.45) using the $*$-product as

$$(\tilde{X}^{-2k}\tilde{X}^{-2l})_* = q^{k-l}\tilde{X}^{-2(k+l)}\,. \tag{4.46}$$

Again, $\tilde{X}^{2n}$ is nothing but a trivial representation (4.13) with (4.14), and we have the CZ^- algebra with the definition (4.46)

$$[\mathrm{L}_n'^-, \mathrm{L}_m'^-]_* = [n-m]\mathrm{L}_{n+m}'^-\,, \qquad \mathrm{L}_n'^- = \frac{\tilde{X}^{-2}}{q-q^{-1}} \tag{4.47}$$

We therefore verify that the same picture as CZ^+ holds. Namely the commuting operator $\tilde{X}^2$ on the quantum line X_r acquires a nontrivial phase factor related to the $*$-product (4.46) as an effect of quantum fluctuations via $X_{r\pm 1}$. Since X_r is orthogonal to Y_j, it can be said that $\mathrm{L}_n'^\pm$ are the algebras belonging to directions orthogonal to each other.

We finally put a remark that $\tilde{X}^{-2}$ increases the position j by 2 along X_r, and effective moving length of $\tilde{X}^{-1}$ may thus amount to $\Delta j = 1$ if one applies a dual lattice. Similarly Y^2 increases r by 2 along Y_j, and thus Y^{-1} may effectively increase by $\Delta r = 1$.

5. CZ^* and TBM Hamiltonians

In Section 5, we show that the matrix representation of TBM corresponds to the Wyle representation of CZ^*, which describes the Hamiltonian sequence covering various magnetic lattices. In Section 5.1, deriving the DMT algebra (exchange, fusion and circulation rules) in TBM, we comment on its relation to the quantum plane picture. In Section 5.2, we show that the TBM

Hamiltonian sequence can be described using the ± 1 modes of the matrix representation of the CZ^* algebra family. Section 5.3 discusses extensions to general modes. The power of X corresponds to the Hamiltonian with the effective spacing of the magnetic lattice expanded from 1 to n. The power of Y corresponds to the Hamiltonian sequence that extends the quantum plane fluctuation (order of q) from 1 to k (Section 5.4). These Hamiltonians can be represented by the CZ^* generators.

5.1. DMT and quantum plane in TBM

The purpose of this subsection is to verify the quantum plane picture of tight binding model (TBM) by showing that discrete magnetic translations (DMTs) satisfy the same properties as the MTA (exchange, fusion and circulation) of the continuous magnetic translations reviewed in Section 2.2. In contrast to the discussion in Section 4.4, we do not use the matrix representation of DMT. As a result of this picture, TBM can be regarded as a Hamiltonian system constructed on a quantum plane.

TBM Hamiltonian is given by DMT on a two-dimensional lattice as follows [19–21,57]:

$$H = \hat{T}_x + \hat{T}_y + \hat{T}_x^\dagger + \hat{T}_y^\dagger \,, \tag{5.1}$$

$$\hat{T}_x = \sum_{n,m} e^{i\theta^x_{mn}} c^\dagger_{m+1,n} c_{m,n} \,, \qquad \hat{T}_y = \sum_{n,m} e^{i\theta^y_{mn}} c^\dagger_{m,n+1} c_{m,n} \,, \tag{5.2}$$

where $c^\dagger_{m,n}$ and $c_{m,n}$ represent the creation/annihilation operators at each site of (m,n), and θ^x_{mn} and θ^y_{mn} are the AB phase associated with the unit movement length a in each direction of x and y

$$\theta^x_{mn} : (m,n) \to (m+1,n) \,, \qquad \theta^y_{mn} : (m,n) \to (m,n+1) \,. \tag{5.3}$$

Introducing the wave function

$$\Psi = \sum_{m,n} \psi_{m,n} c^\dagger_{m,n} \left|0\right\rangle = \sum_{m,n} \psi_{m,n} \left|\psi_{m,n}\right\rangle \,, \qquad \psi_{m,n} \in \mathbf{C} \,, \tag{5.4}$$

the eigenvalue equation $H\Psi = E\Psi$ with (5.1) is known to reduce to the following Schrödinger equation [19,21]

$$e^{i\theta^x_{m-1,n}} \psi_{m-1,n} + e^{i\theta^y_{m,n-1}} \psi_{m,n-1} + e^{-i\theta^x_{m,n}} \psi_{m+1,n} + e^{-i\theta^y_{m,n}} \psi_{m,n+1} = E\psi_{m,n} \,. \tag{5.5}$$

Note that this reproduces the continuous Schrödinger equation for the Hamiltonian $H' = -tH$ at the order of $\mathcal{O}(a^2)$,

$$\frac{1}{2m}(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A})^2 \psi_{m,n} = \mathcal{E}\psi_{m,n} \,, \qquad \mathcal{E} = \frac{\hbar^2}{2ma^2}\frac{E+4t}{t} \,, \tag{5.6}$$

where $\mathcal{O}(a)$ vanishes in the continuum limit $a \to 0$.

Having the formulae from (5.2) and (5.4),

$$\hat{T}_x \hat{T}_y \left|\psi_{m,n}\right\rangle = e^{i\theta^x_{m,n+1} + i\theta^y_{m,n}} \left|\psi_{m+1,n+1}\right\rangle \,, \tag{5.7}$$

$$\hat{T}_y \hat{T}_x \left|\psi_{m,n}\right\rangle = e^{i\theta^y_{m+1,n} + i\theta^x_{m,n}} \left|\psi_{m+1,n+1}\right\rangle \,, \tag{5.8}$$

we obtain the exchange and circulation algebras

$$\hat{T}_y \hat{T}_x \left|\psi_{m,n}\right\rangle = e^{2\pi i \phi} \hat{T}_x \hat{T}_y \left|\psi_{m,n}\right\rangle , \tag{5.9}$$

$$\hat{T}_y^\dagger \hat{T}_x^\dagger \hat{T}_y \hat{T}_x \left|\psi_{m,n}\right\rangle = e^{2\pi i \phi} \left|\psi_{m,n}\right\rangle , \tag{5.10}$$

where

$$2\pi\phi = (\theta^y_{m+1,n} - \theta^y_{m,n}) - (\theta^x_{m,n+1} - \theta^x_{m,n}) . \tag{5.11}$$

Concerning the fusion algebra, we have to define new composite operator $\hat{T}_{x+y}$ satisfying the following fusion relations with phase factor $\xi_{m,n}$, which will be determined later

$$\hat{T}_x \hat{T}_y \left|\psi_{m,n}\right\rangle := e^{i\xi_{m,n}} \hat{T}_{x+y} \left|\psi_{m,n}\right\rangle , \tag{5.12}$$

$$\hat{T}_y \hat{T}_x \left|\psi_{m,n}\right\rangle := e^{-i\xi_{m,n}} \hat{T}_{x+y} \left|\psi_{m,n}\right\rangle . \tag{5.13}$$

Together (5.7), (5.8) and these, we have

$$\hat{T}_{x+y} \left|\psi_{m,n}\right\rangle = e^{-i\xi_{m,n}} e^{i\theta^x_{m,n+1} + i\theta^y_{m,n}} \left|\psi_{m+1,n+1}\right\rangle \tag{5.14}$$

$$= e^{i\xi_{m,n}} e^{i\theta^y_{m+1,n} + i\theta^x_{m,n}} \left|\psi_{m+1,n+1}\right\rangle . \tag{5.15}$$

Using (5.11), the consistency condition of the r.h.s. of this equation reads

$$2\pi\phi + 2\xi_{m,n} = 0 , \tag{5.16}$$

and therefore we have the fusion algebra

$$\hat{T}_x \hat{T}_y \left|\psi_{m,n}\right\rangle := e^{-i\pi\phi} \hat{T}_{x+y} \left|\psi_{m,n}\right\rangle , \tag{5.17}$$

$$\hat{T}_y \hat{T}_x \left|\psi_{m,n}\right\rangle := e^{i\pi\phi} \hat{T}_{x+y} \left|\psi_{m,n}\right\rangle . \tag{5.18}$$

Introducing the parameter q as

$$q = e^{-i\pi\phi} , \tag{5.19}$$

we summarize the exchange, fusion and circulation as follows

$$\hat{T}_x \hat{T}_y = q^2 \hat{T}_y \hat{T}_x , \quad \hat{T}_x \hat{T}_y = q \hat{T}_{x+y} , \quad \hat{T}_y^\dagger \hat{T}_x^\dagger \hat{T}_y \hat{T}_x = q^{-2} . \tag{5.20}$$

If we combine the fusion and exchange rules into

$$\hat{T}_{x+y} = q^{-1} \hat{T}_x \hat{T}_y = q \hat{T}_y \hat{T}_x , \tag{5.21}$$

this equation can be interpreted as follows: the translation $\hat{T}_{x+y}$ corresponds to the one along the line X_r defined in Section 4.4, and the first and second equalities imply mutually different phase factors $q^{\pm 1}$ in accordance with the fluctuations taking detours to adjacent $X_{r\pm 1}$ and back to X_r, corresponding to different ordering of $\hat{T}_x$ and $\hat{T}_y$ operators. Note that the phase factor q given by (5.19) is not necessarily a constant because ϕ, given by (5.11), depends on its site (m, n). As discussed in Section 4.4, the q can be regarded as the fluctuation of quantum plane, and it is related to the AB phases as seen in (5.11).

This interpretation suggests that the $*$-products are generated by the quantization of space (quantum plane) in view of discretization. It is interesting that the quantum plane picture can be understood as the underlying structure before a periodic condition is taken into account.

5.2. CZ^* and TBM Hamiltonian family

Hereafter we discuss the relations between TBM Hamiltonian $\hat{H}$ and the CZ^* matrix representations $\mathscr{L}^{\pm}_{\pm 1}$ defined in (4.21)-(4.26). Since our three representations (4.21), (4.24) and (4.29) have different Y-dependence from $\hat{H}$ (see (A.11)), it is not straightforward to find their relationships. For convenience of discussion, we explicitly show the dependence on deformation parameters q and matrix sizes N in CZ^* and TBM Hamiltonian $\hat{H}$, such as $CZ^*(q,N)$ and $\hat{H}(q,N')$, where the latter TBM matrix sizes are given by $N'=2Q$. Although the same symbols for q and N are employed in both CZ^* and $\hat{H}$, they are originally introduced independently, and hence generally different. Then denoting the deformation parameter of TBM by q_k when both q are related, we consider the situation that the Hamiltonian $\hat{H}(q_k,Q)$ coincides with $\check{H}_k$ which is a linear combination of $CZ^*(q_k,N)$ generators. Keeping the relation of $CZ(q_k,N)$ to its parent $CZ(q,N)$, and providing a certain relation between N and Q, we are going to determine the values of q and q_k in each case. (In the case of the factorization (4.24), we do not have to consider this issue, because q_k coincides with q, which is nothing but a matrix element of Y of size $N=2Q$.)

Let us first consider the second type representation, that is the factorized form (4.24), where the powers of Y coincide with those in $\hat{H}$ up to the factorization of $[Z]$. In this case the CZ^* algebra is slightly modified to the $CZ^{*'}$ algebra, which is given by $\mathscr{L}^{+}_{n}$ and $\mathscr{L}'^{-}_{1}$

$$\mathscr{L}'^{-}_{n}=q^n\mathscr{L}^{-}_{n}\,, \tag{5.22}$$

satisfying (3.36), (4.26), (4.27). It is convenient to define $\hat{H}_Z$ by using the $CZ^{*'}$ generators as

$$\hat{H}_Z=i(\mathscr{L}^{+}_{1}-\mathscr{L}^{+}_{-1})+i(\mathscr{L}'^{-}_{1}-\mathscr{L}'^{-}_{-1})\,. \tag{5.23}$$

Using the relation $qXY=YX$, we find that $[Z]$ is factorized from $\hat{H}_Z$ as

$$\hat{H}_Z=i(X^{-1}-X)Y[Z]+iY^{-1}(X^{-1}-X)[Z]=\hat{H}[Z] \tag{5.24}$$

and the TBM Hamiltonian (A.11) is therefore related to the CZ^* generators in the following form

$$\hat{H}=\hat{H}_Z[Z]^{-1}\,. \tag{5.25}$$

This means that eigenvalues of $\hat{H}_Z$ are given by the product of the TBM eigenvalues (A.10) and the phase factor matrix $[Z]$,

$$\hat{H}_Z\tilde{\psi}=E[Z]\tilde{\psi}\,,\quad \tilde{\psi}=[Z]^{-1}\psi\,. \tag{5.26}$$

Next, let us consider the case (4.29), which is one of the third types and its corresponding algebra is $CZ^*(q_1)$ defined in (4.32) and (4.33). Defining $\check{H}_1$ in terms of the $CZ^*(q_1)$ generators $\check{\mathscr{L}}^{\pm}_{\pm 1}$

$$\check{H}_1=i(q_1-q_1^{-1})(\check{\mathscr{L}}^{+}_{1}-\check{\mathscr{L}}^{+}_{-1})+i(q_1-q_1^{-1})(\check{\mathscr{L}}^{-}_{1}-\check{\mathscr{L}}^{-}_{-1})\,, \tag{5.27}$$

and substituting (4.29) on the r.h.s. of this, we obtain the same form as the TBM Hamiltonian $\hat{H}$

$$\check{H}_1=i(X^{-1}-X)Y_1+iY_1^{-1}(X^{-1}-X)=\hat{H}(X,Y_1;q_1)\,, \tag{5.28}$$

where Y_1 and q_1 are substituted for Y and q in (4.23). In this representation we have the conditions $X^N=Y_1^N=1$ as well as $Y^N=1$. Both should be satisfied, and it is realized in the following way: Y_1 with the relation $q = q_1^2$ is related to Y set in the CZ^* representation (4.21) by the relation $Y_1^2 = Y$. Notice that it does not mean that (4.21) coincides with (4.29). If we choose $N = 2Q$ remembering that $2Q$ is the matrix size of $\hat{H}$, we have

$$Y_1^{2Q} = 1\,, \quad q_1^{2Q} = 1\,, \quad \therefore q_1 = e^{\pm i\pi\phi}\,, \tag{5.29}$$

as well as for Y and q

$$Y^Q = 1\,, \quad q^Q = 1\,, \quad \therefore q = e^{\pm 2\pi i\phi}\,, \tag{5.30}$$

where the double sign $\pm$ is introduced for a complex conjugation system. Thus we have

$$(Y_1)_{jk} = q_1^j \delta_{jk}\,, \quad (Y)_{jk} = q^j \delta_{jk}\,, \quad Y_1^N = Y^{\frac{N}{2}} = 1\,. \tag{5.31}$$

The matrix representation (4.29) describes the TBM Hamiltonian $\check{H}_1$ given by (5.28) not with $q = e^{-i\pi\phi}$ but with

$$q = e^{\pm 2\pi i\phi}\,, \quad \text{or} \quad q_1 = e^{\pm i\pi\phi}\,. \tag{5.32}$$

This adjustment corresponds to the manipulation to replace q by q_1 in (A.9) adopting $q = e^{-2\pi i\phi}$, instead of using parametrization (A.8) when driving the Schrödinger equation (A.9).

Finally we deal with the rest of all, the first type (4.21) and the third type (4.30) in the same formalism $CZ^*(q_k)$, since (4.21) is a special case of the third type with $k = 2$. Defining $\check{H}_k$ as

$$\check{H}_k = i(q_k - q_k^{-1})(\check{\mathscr{L}}_1^+ - \check{\mathscr{L}}_{-1}^+) + i(q_k - q_k^{-1})(\check{\mathscr{L}}_1^- - \check{\mathscr{L}}_{-1}^-)\,, \tag{5.33}$$

and substituting (4.30) on the r.h.s., we obtain for $k = 2$

$$\check{H}_2 = i(X^{-1} - X)Y_2 + iY_2^{-1}(X^{-1} - X) = \hat{H}(X, Y_2; q_2)\,, \tag{5.34}$$

where $q_2 = q^2$. In this representation we have the condition $X^N = Y_2^N = 1$. Note that $Y^N \neq 1$ this time. Y_2 with the relation $q_2 = q^2$ is related to Y in the CZ^* representation (4.21) by the relation $Y_2 = Y^2$. If we choose $N = Q$, we have

$$Y_2^Q = 1\,, \quad q_2^Q = 1\,, \quad \therefore q_2 = e^{\pm 2\pi i\phi}\,, \tag{5.35}$$

as well as for Y

$$Y^{2Q} = 1\,, \quad q^{2Q} = 1\,, \quad \therefore q = e^{\pm i\pi\phi}\,. \tag{5.36}$$

Thus we have

$$(Y_2)_{jk} = q_2^j \delta_{jk}\,, \quad (Y)_{jk} = q^j \delta_{jk}\,, \quad Y_2^N = Y^{2N} = 1\,, \tag{5.37}$$

and the matrix representation (4.21) hence describes the TBM Hamiltonian $\check{H}_2$ with q given by

$$q = e^{\pm i\pi\phi}\,. \tag{5.38}$$

As to the third representation (4.30), considering CZ^* family for $k \geq 3$ in the same way as above, we verify that the matrix (4.30) describes the TBM Hamiltonian family $\check{H}_k$ given by (5.33)

$$\check{H}_k = i(X^{-1} - X)Y_k + iY_k^{-1}(X^{-1} - X) = \hat{H}(X, Y_k; q_k)\,, \tag{5.39}$$

where Y_k and q_k are given by

$$Y_k = Y^k\,, \quad q_k = q^k = e^{\pm k\pi i\phi}\,. \tag{5.40}$$

5.3. Generalization of $\check{\mathscr{L}}_n^{\pm}$ to other modes $n \neq \pm 1$

As discussed in Section 4, $X^{\pm 2}$ have the effect of increasing or decreasing j by 2 ($\Delta j = 2$) along the quantum line X_r. Reflecting this feature, $\hat{H}(X^2, Y; q)$ is deduced to describe the system of which effective interval Δj is twice that of $\hat{H}(X, Y; q)$. Then denoting another TBM Hamiltonian $\hat{H}(X^2, Y; q)$ by $\hat{H}_2$,

$$\hat{H}_2 = i(X^{-2} - X^2)Y + iY^{-1}(X^{-2} - X^2)\,, \tag{5.41}$$

we can derive the Schrödinger equation

$$i(q^{j+2} + q^{-j})\psi_{j+2} - i(q^{-j} + q^{j-2})\psi_{j-2} = E\psi_j \tag{5.42}$$

by applying $\Delta j = 2$ to the original equation (A.9).

The new Hamiltonian $\hat{H}_2$ possesses $U_q(sl_2)$ symmetry:

$$\hat{H}_2 = (q_2 - q_2^{-1})(\mathcal{E}_+ + \mathcal{E}_-)\,, \quad q_2 = q^2\,, \tag{5.43}$$

$$\mathcal{E}_+ = \frac{i}{q_2 - q_2^{-1}}(X^{-2} - X^2)Y\,, \quad \mathcal{E}_- = \frac{i}{q_2 - q_2^{-1}}Y^{-1}(X^{-2} - X^2)\,, \tag{5.44}$$

$$\mathcal{K} = q_2 X^{-4} \tag{5.45}$$

$$[\mathcal{E}_+, \mathcal{E}_-] = \frac{\mathcal{K} - \mathcal{K}^{-1}}{q_2 - q_2^{-1}}\,, \quad \mathcal{K}\mathcal{E}_\pm\mathcal{K}^{-1} = q_2^{\pm 2}\mathcal{E}_\pm\,. \tag{5.46}$$

However it is rather convenient to regard this symmetry as the $n = \pm 2$ parts of CZ^* representation family (4.30) when considering the following Hamiltonian series

$$\hat{H}_n = i(X^{-n} - X^n)Y + iY^{-1}(X^{-n} - X^n)\,, \tag{5.47}$$

which gives the Schrödinger equation with the effective interval $\Delta j = n = 2\nu$

$$i(q^{j+2\nu} + q^{-j})\psi_{j+2\nu} - i(q^{-j} + q^{j-2\nu})\psi_{j-2\nu} = E\psi_j\,. \tag{5.48}$$

Namely, as a generalization of Section 5.2, defining $\check{H}_{(n,k)}$ in terms of the representation (4.30) of the CZ^* family

$$\check{H}_{(n,k)} = i(q_k - q_k^{-1})(\check{\mathscr{L}}_n^+ - \check{\mathscr{L}}_{-n}^+) + i(q_k - q_k^{-1})(\check{\mathscr{L}}_n^- - \check{\mathscr{L}}_{-n}^-)\,, \tag{5.49}$$

$$= \hat{H}_n(X, Y_k; q_k)\,, \tag{5.50}$$

we thus find the connection of the Hamiltonian series $\hat{H}_n$ to the CZ^* family operators $\check{\mathscr{L}}_n^{\pm}$, which are extended from the $n = \pm 1$ modes $\check{\mathscr{L}}_{\pm 1}^{\pm}$.

Note that the previous cases $\check{H}_1$ and $\check{H}_2$ discussed in Section 5.2 belong to the $n = 1$ series of $\check{H}_{(n,k)}$

$$\check{H}_k = \check{H}_{(1,k)} = \hat{H}(X, Y_k; q_k)\,, \quad \hat{H}_1 = \hat{H}\,, \quad k = 1, 2\,. \tag{5.51}$$

5.4. Hamiltonian series with $Y^{\pm k}$ family

Regarding the first type representation (4.21), it may be more convenient to consider the Hamiltonian $\hat{H}(X, Y^2; q)$, instead of the original $\hat{H}(X, Y; q)$. In this way, one may anticipate the avoidance of the complicated discussion in the previous subsection and a more direct correspondence between (4.21) and $\hat{H}(X, Y^2; q)$.

Let us consider the following Hamiltonian family

$$\hat{H}_{(n,k)} = i(X^{-n} - X^n)Y^k + iY^{-k}(X^{-n} - X^n), \tag{5.52}$$

and first we set $n = k = 2$

$$\hat{H}_{(2,2)} = i(X^{-2} - X^2)Y^2 + iY^{-2}(X^{-2} - X^2). \tag{5.53}$$

This leads to the following Schrödinger equation

$$i(q^{2j+4} + q^{-2j})\psi_{j+2} - i(q^{-2j} + q^{2j-4})\psi_{j-2} = E\psi_j, \tag{5.54}$$

and it corresponds to a system whose effective interval Δj and the size of quantum fluctuation q (see §4) are twice those of the original system (A.9), since (5.54) coincides with the equation obtained from (A.9) by the replacements $\Delta j = 1 \to 2$ and $q \to q^2$. It is straightforward to verify that the Hamiltonian (5.52) describes the quantum plane system with the effective interval n and the fluctuation size q^k.

If we introduce the following $\mathscr{H}_n$ operator composed of the generators $\mathscr{L}_n^{\pm}$ given in the CZ^* representation (4.21)

$$\mathscr{H}_n = i(q - q^{-1})(\mathscr{L}_n^{+} - \mathscr{L}_{-n}^{+}) + i(q - q^{-1})(\mathscr{L}_n^{-} - \mathscr{L}_{-n}^{-}), \tag{5.55}$$

we find the relation

$$\mathscr{H}_2 = \hat{H}_{(2,2)} = \hat{H}(X^2, Y_2; q_2). \tag{5.56}$$

The Hamiltonian $\hat{H}_{(2,2)}$ is also given by a combination of the generators of the quantum algebra $U_q(sl_2)$

$$\hat{H}_{(2,2)} = (q_4 - q_4^{-1})(\mathcal{E}'_+ + \mathcal{E}'_-), \quad q_4 = q^4, \tag{5.57}$$

$$\mathcal{E}'_+ = \frac{i}{q_4 - q_4^{-1}}(X^{-2} - X^2)Y^2, \quad \mathcal{E}'_- = \frac{i}{q_4 - q_4^{-1}}Y^{-2}(X^{-2} - X^2), \tag{5.58}$$

$$\mathcal{K}' = q_4 X^{-4} \tag{5.59}$$

$$[\mathcal{E}'_+, \mathcal{E}'_-] = \frac{\mathcal{K}' - \mathcal{K}'^{-1}}{q_4 - q_4^{-1}}, \quad \mathcal{K}'\mathcal{E}'_\pm\mathcal{K}'^{-1} = q_4^{\pm 2}\mathcal{E}'_\pm. \tag{5.60}$$

Needless to say, the CZ^* representation (4.21) is again suitable to describe the relation between $\mathscr{H}_n$ and the Hamiltonian series $\hat{H}_{(n,2)}$

$$\mathscr{H}_n = \hat{H}_{(n,2)} = \hat{H}(X^n, Y_2; q_2). \tag{5.61}$$

We also have its generalization as

$$\hat{H}_{(n,k)} = \hat{H}(X^n, Y_k; q_k). \tag{5.62}$$

As seen above, $\hat{H}_{(n,k)}$ expresses a variety of Hamiltonians designated by combinations of effective interval of magnetic lattice Δj and quantum plane fluctuation q_k. There exists a $U_q(sl_2)$

symmetry in the $\hat{H}_{(n,k)}$ system for each n and k, for example, (5.57) for $\hat{H}_{(2,2)}$, (5.43) for $\hat{H}_{(2,1)}$, and (A.15) for $\hat{H}_{(1,1)}$. On the other hand, it is possible to express the Hamiltonian series in a unified manner such as (5.50), (5.61) and (5.62), if we employ one of the closed algebra systems CZ^* or CZ^* family as observed in (5.49) and (5.55).

6. Conclusions and discussions

In this paper we have focused on the relations between CZ algebras and the quantum plane picture using algebraic properties of DMT as well as MT. The mechanism of generating phase factors is found to be compatible with the $*$-bracket feature of CZ algebras, and it is very convenient to investigate various properties of CZ algebras. As a result, we have clarified some properties that could not be obtained from the q-harmonic oscillators representation (3.10). They are what mechanism determines the phase factor in (2.10), that it is related to fluctuations on the quantum plane, and that there is a certain rule in the method of constructing the CZ generators.

Commutative representation is especially important, and we need a specific pair of commuting and noncommuting operators. In the MT representation, by introducing]the weight of MT and CZ operators, we have presented the definition of $*$-bracket (3.20) that can express the three types of CZ algebras, $CZ^{\pm}$ and CZ^*, in the unified form. The $CZ^{\pm}$ operator (3.31) is a linear combination of commutative $\hat{T}_n^{(0)}$ and noncommutative $\hat{T}_n^{(\pm 2)}$ operators, and the same structure is also found for the DMT matrix representation (4.7). We in fact observed in Section 4.2 that the commutative representations X^n and Y^n play the same role as the MT operator $\hat{T}_n^{(0)}$ as seen in (4.15). The commutative representation shows its significance in connection to the $*$-brackets and the quantum plane picture. The (non)commutativity of the local operator $\hat{T}_n^{(0)}$ and the nonlocal operator $\hat{T}_n^{(\pm 2)}$ is flipped in the $*$-bracket commutator as seen in (3.26). The commutative operator carries the essential role of noncommuting $CZ^{\pm}$ algebraic relations, while the noncommuting operator does the role of deformed $U(1)$ translational group and thereby determines the weight for the $*$-bracket (3.20). The operators X^{-n} and $X^{-n}Y^{\pm 2}$ perform the same functions as $\hat{T}_n^{(0)}$ and $\hat{T}_n^{(\pm 2)}$ as mentioned in (4.15).

The property that commutative operators behave as non-commutative ones (and vice versa) matches the phase fluctuation of the quantum plane, and by considering the commutative DMT representations (4.42) and (4.47), we recognize that the $CZ^{\pm}$ algebras can be described by the $*$-ordered products (4.40) and (4.45) in Section 4.4. In this way, we have provided another definition of $*$-product by the ordered product that counts the number of fluctuations in the positive and negative directions based on the quantum plane picture of TBM. The AB phase (5.11) associated with the movement of particles by DMT is interpreted as the fluctuation of the quantum plane (5.19), and the phase factor generated by the successive operations of DMT is then expressed by a certain ordered product to reproduce the $*$-bracket.

As a glimpse of the quantum plane picture, we have discussed the relations between CZ^* algebra and TBM Hamiltonian series in Section 5. It has been shown that the matrix representation of TBM corresponds to the Wyle representation of CZ^*, which describes a sequence of Hamiltonians covering various magnetic lattices. Each Hamiltonian in the sequence is parameterized by two integers n and k which are the power of the Wyle base X and Y. The parameters n and k correspond to the effective spacing of the magnetic lattice and the fluctuation size of quantum plane (power of q), respectively. All the Hamiltonians can be represented in a unified manner by the CZ^* generators without introducing additional multiple copies of $U_q(sl_2)$ or CZ^* family. (Recall that in Section 4.3 we have introduced the representation sequence of the CZ^* family in order to express the TBM Hamiltonian of quantum fluctuation size of k. The sequence is generated by successively replacing $q \to q^k$. However, considering a single CZ^* algebra with a sequence of Hamiltonians rather than the CZ^* algebra family is physically easier to understand.) In this way, the CZ^* algebra may be regarded as a universal algebra to describe the Hamiltonian series in accordance to various quantum plane settings of n and k.

The similarities between MT and DMT representations found in this paper may suggest a universal property common to various CZ^* representations. The correspondence between the q-differential representation (3.6) and the MT representation (3.11) may reveal a physical meaning of q-differential operators in lattice systems with CZ^* algebraic structure. The matrix representation of TBM Hamiltonian series by Wyle base implies the existence of quantum plane behind the physical systems. All these observations are related to the representations of CZ^* algebra, and we therefore believe that significance of CZ^* algebra has been increased by this paper. We will be able to clarify unsolved issues and universal properties of CZ^* algebra from some properties common to multiple representations including MT and DMT representations as we have done for the question in (2.10).

CRediT authorship contribution statement

Naruhiko Aizawa: Conceptualization, Supervision, Writing – review & editing. **Haru-Tada Sato:** Conceptualization, Investigation, Methodology, Validation, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Quantum group symmetry in TBM

In this appendix, we review the quantum algebra symmetry in TBM, that is, the TBM Hamiltonian is written by $U_q(sl_2)$ raising-lowering operators [19]. We also put a remark on q-inversion symmetry of the Hamiltonian. In order to see the $U_q(sl_2)$ structure, we impose a periodic condition on the Schrödinger equation (5.5), and we then transform (5.5) into a matrix form in use of the Wyle base matrices X and Y.

Let us choose the factors of hopping terms as in [57] so that the gauge invariant condition (5.11) is satisfied

$$\theta^x_{m,n} = -(n+m)\pi\phi\,, \quad \theta^y_{m,n} = (m+n+1)\pi\phi\,, \tag{A.1}$$

and set periodic condition

$$\psi_{m,n} = \psi_{m+Q,n+Q} \tag{A.2}$$

with co-prime integers P, Q and the ratio ϕ

$$\phi = \frac{P}{Q}\,. \tag{A.3}$$

Under these conditions, (5.5) is invariant under the transformation $(m, n) \to (m + Q, n + Q)$. According to the Bloch theorem, we have

$$\psi_{m,n} = e^{imk_x} e^{ink_y} u_{m,n}\,, \quad u_{m,n} = u_{m+Q,n+Q}\,. \tag{A.4}$$

Using magnetic momenta $k_\pm = \frac{1}{2}(k_x \pm k_y)$ and reflecting the exchange symmetry $k_- \leftrightarrow -k_-$ in $n \leftrightarrow m$ with

$$u_{m,n} = u_{n,m} := \psi_{n+m}\,, \tag{A.5}$$

we have

$$\psi_{m,n} = e^{ik_+(m+n)} e^{ik_-(m-n)} \psi_{m+n}\,, \quad \psi_{m+n} = \psi_{m+n+2Q}\,. \tag{A.6}$$

Reducing a degree of freedom with choosing mid band spectrum for $k_\pm$ [57,58]

$$j = m + n\,, \quad (k_+, k_-) = (\frac{\pi}{2}, 0)\,, \tag{A.7}$$

and making q related with ϕ by

$$q = e^{-i\pi\phi}\,, \tag{A.8}$$

(5.5) becomes the following equation [21]

$$-i(q^{j-1} + q^{-j})\psi_{j-1} + i(q^{-j} + q^{j+1})\psi_{j+1} = E\psi_j\,. \tag{A.9}$$

This is a component expression of $2Q$-dimensional matrix representation of the Schrödinger equation

$$\hat{H}\psi = E\psi\,, \quad \psi = (\psi_1, \psi_2, \cdots, \psi_{2Q})^T \tag{A.10}$$

and the Hamiltonian $\hat{H}$ can be written by the matrices X and Y of size $N = 2Q$ defined in (4.5)

$$\hat{H} = iY^{-1}(X^{-1} - X) + i(X^{-1} - X)Y\,. \tag{A.11}$$

This is known to be written in the $U_q(sl_2)$ generators $E_\pm$ and $K^{\pm 1}$

$$E_+ = \frac{i}{q - q^{-1}}(X^{-1} - X)Y\,, \quad E_- = \frac{i}{q - q^{-1}}Y^{-1}(X^{-1} - X)\,, \tag{A.12}$$

$$K = qX^{-2}\,, \tag{A.13}$$

which satisfy the $U_q(sl_2)$ algebraic relations

$$[E_+, E_-] = \frac{K - K^{-1}}{q - q^{-1}}\,, \quad KE_\pm K^{-1} = q^{\pm 2}E_\pm\,. \tag{A.14}$$

One can check these relations by using $YX = qXY$, and the Hamiltonian reads [19,57]

$$\hat{H} = (q - q^{-1})(E_+ + E_-)\,. \tag{A.15}$$

Now, let us put some remarks on q-inversion symmetry of the Hamiltonian. We use the inverted $q = e^{i\pi\phi}$ till the end of this Appendix. Considering complex conjugation of (5.5), we have the Hamiltonian

$$\hat{H}' = Y\hat{H}Y^{-1} = iY(X^{-1} - X) + i(X^{-1} - X)Y^{-1} \tag{A.16}$$

which looks different from $\hat{H}$. Of course, $\hat{H}$ and $\hat{H}'$ describe the same physics because $\hat{H}'$ is obtained by the replacement of q in (A.11) with q^{-1} (recall that $Y(q^{-1}) = Y^{-1}(q)$). We then notice that for each expression

$$\hat{H}^\dagger = \hat{H}\,, \quad \hat{H}'^\dagger = \hat{H}'\,. \tag{A.17}$$

Transformation of any operator $\mathcal{O}$ to that of q-inverted system $\mathcal{O}'$ is similarly given by

$$\mathcal{O}' = Y\mathcal{O}Y^{-1}\,, \tag{A.18}$$

and operator relations are identical each other. For example, the $U_q(sl_2)$ generators transform as

$$E'_\pm = YE_\pm Y^{-1}\,, \quad K' = YKY^{-1} \tag{A.19}$$

and they satisfy the same relations as (A.14). The inverted Hamiltonian is thus given by the identical form to $\hat{H}$

$$\hat{H}' = (q - q^{-1})(E'_+ + E'_-)\,, \tag{A.20}$$

which is written as the sum of $E'_\pm$ the raising and lowering operators of $U_q(sl_2)$.

Appendix B. Matrix representations of DMT

In Appendix A, we have the matrix representation $\hat{H}$ (see (A.11)) of the TBM Hamiltonian H given in (5.1). This implies that there is a correspondence between the operators $\hat{T}_x$, $\hat{T}_y$ in (5.1) and the matrices X, Y in (A.11). In this appendix, we clarify the correspondence and then we verify the q-inversion symmetry (A.18) and the DMT algebras (5.20) in matrix representation.

In order to see the correspondence, let us consider the j-th component of matrix actions of X and Y on ψ

$$(X^{\pm 1}\psi)_j = \psi_{j\mp 1}\,, \quad (Y^{\pm 1}\psi)_j = q^{\pm j}\psi_j\,. \tag{B.1}$$

These relations mean that X shifts the coordinate j by 1 and Y generates a phase factor q^j.

For all possible products of X and Y appeared in $\hat{H}$ (A.11) and $\hat{H}'$ (A.16), we have

$$(X^{\pm 1}Y\psi)_j = q^{j\mp 1}\psi_{j\mp 1}\,, \quad (Y^{-1}X^{\pm 1}\psi)_j = q^{-j}\psi_{j\mp 1}\,, \tag{B.2}$$

$$(YX^{\pm 1}\psi)_j = q^{j}\psi_{j\mp 1}\,, \quad (X^{\pm 1}Y^{-1}\psi)_j = q^{-j\pm 1}\psi_{j\mp 1}\,. \tag{B.3}$$

On the other hand, acting DMT operators (5.2) on the wave function (5.4), for example

$$\hat{T}_x\Psi = \sum_{m,n} e^{i\theta^x_{m-1,n}}\psi_{m-1,n}\left|\psi_{m,n}\right\rangle \tag{B.4}$$

and extracting the coefficient of $e^{i\frac{\pi}{2}(m+n)}\left|\psi_{m,n}\right\rangle$ (since the overall factor $e^{ij\frac{\pi}{2}}$ is excluded when deriving (A.9)), we obtain

$$\begin{aligned}\hat{T}_x\Psi &\sim e^{-i\frac{\pi}{2}(m+n)}e^{i\theta^x_{m-1,n}}\psi_{m-1,n}\\ &= -iq^{j-1}\psi_{j-1} = -i(XY\psi)_j\,.\end{aligned} \tag{B.5}$$

Repeating the same process for the rest of DMT operators, we have the following correspondence, namely the matrix representation

$$\hat{T}_x \leftrightarrow -iXY\,, \quad \hat{T}_y \leftrightarrow -iY^{-1}X\,, \quad \hat{T}_x^\dagger \leftrightarrow iY^{-1}X^{-1}\,, \quad \hat{T}_y^\dagger \leftrightarrow iX^{-1}Y\,. \tag{B.6}$$

Also for the complex conjugate system ($q = e^{i\pi\phi}$) with changing the mid band condition $k_+ \leftrightarrow -k_+$ and denoting the DMT operators by $\hat{T}'_x$, $\hat{T}'_y$ etc., we have

$$\hat{T}'_x \leftrightarrow -iXY^{-1}\,, \quad \hat{T}'_y \leftrightarrow -iYX\,, \quad \hat{T}'^{\dagger}_x \leftrightarrow iYX^{-1}\,, \quad \hat{T}'^{\dagger}_y \leftrightarrow iX^{-1}Y^{-1}\,. \tag{B.7}$$

This representation is identical to (B.6) under the exchange $Y \leftrightarrow Y^{-1}$ and related to (B.6) by the q-inversion (A.18)

$$\hat{T}'_y = Y\hat{T}_x Y^{-1}\,, \quad \hat{T}'_x = Y\hat{T}_y Y^{-1}\,, \quad \text{etc.} \tag{B.8}$$

This transformation is consistent with the Hamiltonian given by (A.16)

$$\hat{H}' = \hat{T}'_x + \hat{T}'_y + \hat{T}'^{\dagger}_x + \hat{T}'^{\dagger}_y\,. \tag{B.9}$$

Finally we check the DMT algebras (5.20). We have the correspondence of the exchange (5.9) and circulation (5.10) rules using the matrix representation (B.6)

$$\hat{T}_y\hat{T}_x = q^{-2}\hat{T}_x\hat{T}_y\,, \quad \hat{T}^{\dagger}_y\hat{T}^{\dagger}_x\hat{T}_y\hat{T}_x = q^{-2}\,, \tag{B.10}$$

and these coincide with (5.9) and (5.10) if we remember that q is given by (A.8). The fusion rule (5.17)

$$\hat{T}_x\hat{T}_y = q\hat{T}_{x+y} \tag{B.11}$$

can also be reproduced from the Hermitian conjugate of (4.43)

$$\hat{T}_x\hat{T}_y = -X^2 = q\tilde{X}^2 \tag{B.12}$$

and

$$\hat{T}_{x+y} = \tilde{X}^2\,. \tag{B.13}$$

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