Color-kinematics relation from the Feynman diagram perspective

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Feynman diagrams for gluon tree amplitudes are studied in the Feynman gauge and in any number of spacetime dimensions. The color-kinematics combinations $\Delta = n_s - n_t - n_u$ of numerators are explicitly calculated for $N = 4, 5, 6$ gluons to see whether the color-kinematics relation $\Delta = 0$ is satisfied. This is a tedious task because of the presence of four-gluon vertices, and the large number of Feynman diagrams, numerators, and Δ combinations involved, especially when $N = 6$. For on-shell amplitudes, it is found that $\Delta = 0$ for $N = 4$, but $\Delta \neq 0$ for $N = 5$ and $N = 6$ owing to the presence of the four-gluon vertex. However, a *local* generalized gauge transformation can bring about $\Delta = 0$ for $N = 5$, but not for $N = 6$. This raises the question whether gluon amplitudes satisfying the color-kinematics relation contain nonlocal interactions.

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I. INTRODUCTION

The color-kinematics (CK) duality is a remarkable relation that allows graviton amplitudes to be expressed as a double copy of gluon amplitudes [1], thereby opening up a new way to study gravity. It was discovered for $N = 4$ gluons long ago [2], but it was Bern, Carrasco, and Johansson (BCJ) [3] who generalized it to all N and made the double copy connection. There are many interesting developments and further generalizations since then that can be found in the comprehensive review in Ref. [4], especially when supersymmetry is invoked, and there is also a recent reformulation [5]. This article is confined to the narrow objective of studying the CK relation for pure gluon amplitudes from the Feynman diagram perspective.

Much is known about the CK relation for the gluon amplitude, but its ramification on Feynman diagrams with more than four external legs seems not to have been thoroughly studied. This is partly due to the difficulty of including the four-gluon vertex systematically, and partly because of the large number of Feynman diagrams and partial amplitudes involved. In this article we try to fill in this gap by carrying out these detailed investigations.

The CK relation relies on the fact that the numerator factors n_a in an N-point gluon amplitude $\sum_a \mathfrak{c}_a n_a / Q_a$ are not unique. There are many ways to change from one set of n_a to another set $\tilde{n}_a = n_a - \delta n$ that can keep all the partial amplitudes the same. Such a change is known as a *generalized gauge transformation*, or *gauge transformation* for short. In this expression, $\mathfrak{c}_a$ is the color factor and Q_a is the propagator factor in the Feynman gauge. Certain triplets of $\mathfrak{c}_a$ are related by the Jacobi identity, say in the form $\mathfrak{c}_s - \mathfrak{c}_t - \mathfrak{c}_u = 0$. The CK relation asserts that suitable gauges can be found so that the same relation $n_s - n_t - n_u = 0$ holds also for the numerator factors. Following [6], those n_a satisfying the CK relation are said to be in the *BCJ representation*.

The existence of BCJ representation is guaranteed by a general formula relating such n_a to the partial amplitudes. Unfortunately the n_a so obtained are highly nonlocal. Since locality is a fundamental attribute of quantum field theory, it is important to find out whether locality can be preserved in a BCJ representation. A necessary condition to be local is for every n_a to be a polynomial function of the polarization vectors ϵ_i and the external momenta k_i. The n_a obtained from the general formula are given by rational functions, not polynomials, hence nonlocal. However, the numerators n_a in the Cachazo-He-Yuan (CHY) gluon amplitude [7–9] are polynomial functions that also satisfy the CK relation, so according to this criterion local BCJ representation does exist. However, that may no longer be true when a more stringent criterion of locality is imposed.

There are actually several versions of CHY gluon numerators [10–14], all yielding the same partial amplitudes. They will be collectively denoted as n'_a. A *Mathematica* program is given in Ref. [14] to calculate n'_a in one of these versions, for

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the coupling constant $g = -\frac{1}{2}$. All explicit calculations of n'_a in this article are done using this program.

A more stringent condition for locality is to start from the numerator factors $\bar{n}_a$ computed from Feynman rules and Feynman diagrams. Their direct connection to the Yang-Mills field theory makes $\bar{n}_a$ strictly local. The more stringent criterion for n_a to be local is to ask $\delta n_a = \bar{n}_a - n_a$ to be local, i.e., given by polynomials of ϵ_i and k_i. We shall find that for $N = 4$, $\bar{n}_a$ is already in the BCJ representation, so it is local. For $N = 5$, the CK relation is not satisfied by $\bar{n}_a$, but there are such local gauge transformations bringing it into a BCJ representation. For $N = 6$, $\bar{n}_a$ is not in the BCJ representation, and there is no local gauge transformation that can render it in a BCJ representation. In particular, since n'_a for $N = 6$ satisfy the CK relation, the CHY gluon theory cannot be local according to this more stringent criterion. Possible implication of this nonlocality will be discussed in Sec. VII.

To compute $\bar{n}_a$, four-gluon vertices must be converted into cubic vertices. There is a unique way to do it, but their presence greatly complicates the computations. For example, it is necessary to go to at least $N = 6$ before their effects can be fully seen.

Sec. II contains a review of some known properties of the CK relation for gluons. It also contains a discussion of definitions, conventions, and other general points. Sec. III explains how the four-gluon vertex should be systematically incorporated, and the complication that brings along. It also reviews the Slavnov-Taylor identity which will be used to simplify calculations. Because of the rather lengthy calculations to be carried out in subsequent sections, a detailed summary of results is also presented to serve as a guidance of what is to come.

Four-point Feynman amplitudes are discussed in Sec. IV, five-point in Sec. V, and six-point in Sec. VI. A concluding section can be found in Sec. VII. Lengthy formulas and results are presented in Appendixes A, B, C, and D.

II. REVIEWS, NOTATIONS, AND GENERAL DISCUSSIONS

A. Partial amplitudes

An N-point gluon amplitude $\sum_{\alpha\in S_{N-2}} \mathbb{C}_\alpha A(1\alpha N)$ consists of a sum of product of a Del Duca-Dixon-Maltoni (DDM) color factor $\mathbb{C}_\alpha$ [15] and a color-stripped partial amplitude $A(1\alpha N)$. α is a permutation of $\{2, 3, \ldots, N-1\}$, the sum is taken over all the $(N-2)!$ permutations $\alpha \in S_{N-2}$, and the external lines $1, \alpha_2, \alpha_3, \ldots, \alpha_{N-1}, N$ of $A(1\alpha N)$ are arranged cyclically in that order in a planar diagram.

Each partial amplitude $A(1\alpha N)$ is given by a sum of terms of the form $\pm n_a/Q_a$ that can be computed from Feynman diagrams, or from an S-matrix theory such as the CHY formula. However, the numerator factors $\bar{n}_a$ computed from Feynman diagrams are *not* the same as the numerator factors n'_a computed from the CHY theory, though both necessarily yield the same partial amplitudes $A(1\alpha N)$ on-shell. A change of one set of n_a to another that leaves all $A(1\alpha N)$ unchanged is known as a *generalized gauge transformation*, or simply a gauge transformation.

Partial amplitudes depend on the scalar products of polarization vectors ϵ_i and outgoing momenta k_i of the external lines. We shall call a (generalized) gauge transformation *local* if δn_a, the difference between two n_a's, is a polynomial of these scalar products for every a. Otherwise, it is *nonlocal*.

The following notations are used in this article: $b_{ij} = \epsilon_i \cdot \epsilon_j$, $c_{ij} = \epsilon_i \cdot k_j$, and $s_{ij\cdots\ell} = (k_i + k_j + \cdots + k_\ell)^2 \coloneqq s_I$, where $I = \{i, j, \ldots, \ell\}$ is the unordered set of subscripts of s. Note that b and s are symmetric in their subscripts but c is not. Because of momentum conservation, $s_I = s_{I'}$ if I' is the set of external lines not contained in I. To avoid this ambiguity, most of the time we shall use the set of indices not containing N. Unfortunately, there is no unique way to implement the conservation condition $\sum_{j=1}^N c_{ij} = 0$, so terms involving c may appear a bit unwieldy.

Both *on-shell* and *off-shell* amplitudes will be discussed. An on-shell amplitude obeys $s_i = k_i^2 = 0$ and $c_{ii} = \epsilon_i \cdot k_i = 0$ for all i. Otherwise it is off-shell. For on-shell amplitudes, there are only $D - 2$ independent polarizations ϵ_i in a spacetime dimension D. For off-shell lines, there are D polarization vectors $\epsilon_i^{(\lambda)}, \lambda = 1, \ldots, D$, to be normalized so that

$$(u \cdot \epsilon_i)(\epsilon_i \cdot v) \coloneqq \sum_{\lambda=1}^{D} (u \cdot \epsilon_i^{(\lambda)})(\epsilon_i^{(\lambda)} \cdot v) = u \cdot v \qquad (1)$$

for any i, and for any vectors u and v. The first expression in Eq. (1) is a shorthand in which all polarizations of a repeated off-shell line i are automatically summed. With this convention, if i is an off-shell line, then

$$b_{pi} b_{iq} = b_{pq}, \qquad b_{pi} c_{iq} = c_{pq},$$
$$c_{ip} c_{iq} = k_p \cdot k_q = \frac{1}{2}(s_{pq} - s_p - s_q). \qquad (2)$$

The propagator factor $Q_a = s_{I_1} s_{I_2} \cdots s_{I_{N-3}}$ is made up of a product of $N - 3$ Mandelstam variables such that whenever $p < q$, either $I_p \subset I_q$ or $I_p \cap I_q = \emptyset$. The total number of distinct propagators subject to this constraint is $(2N - 5)!!$, but only C_{N-2} of them appear in any partial amplitude $A(1\alpha N)$. For $N = 4, 5, 6$ which will be studied explicitly, the Catalan number C_{N-2} is respectively $C_{N-2} = 2, 5, 14$, while the total number of distinct propagator factor is $(2N - 5)!! = 3, 15, 105$. Since $C_{N-2}(N-2)! > (2N-5)!!$, the same n_a/Q_a may appear in different partial amplitudes $A(1\alpha N)$, possibly with different signs.

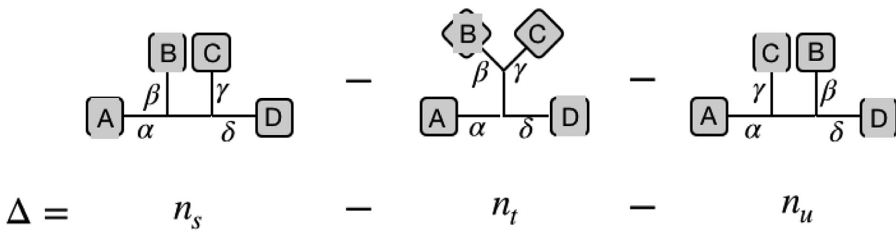

FIG. 1. A CK combination. A, B, C, D represent (possibly empty) tree diagrams.

B. CK relations and CK combinations

Making use of the explicit form $\sum \pm n_a/Q_a$ for $A(1\alpha N)$, the gluon amplitude can also be written as $\sum_a \mathfrak{c}_a n_a/Q_a$, where $\mathfrak{c}_a$ is a color factor obtained from an appropriate combination of the DDM color factors $\mathbb{C}_\alpha$. Jacobi identity demands certain triplets of these $\mathfrak{c}_a$'s to be related, say $\mathfrak{c}_s - \mathfrak{c}_t - \mathfrak{c}_u = 0$. The color-kinematics relation (CK relation) asserts that (the nonunique) numerators n_a can be found for the *on-shell amplitude* so that $n_s - n_t - n_u = 0$ as well.

The Lie algebra structure constants f_{ijk} from which $\mathfrak{c}_a$ is constructed is antisymmetric in its indices, hence the CK relation also demands n_a to have the same antisymmetry as $\mathfrak{c}_a$.

From now on, partial amplitudes will be studied without the accompanying color factors. Even without $\mathfrak{c}_a$, the triplet of numerators n_s, n_t, n_u can still be identified from the diagrams, as illustrated in Figs. 1 and 2. These *numerator diagrams* resemble, but strictly speaking are not, Feynman diagrams. They are diagrams for n_a, with shapes determined by the propagator Q_a. More specifically, the distinct components $s_{\alpha\beta}, s_{\beta\gamma}, s_{\alpha\gamma}$ in the propagator factors

$$Q_s = s_A s_B s_C s_D s_{\alpha\beta}, \qquad Q_t = s_A s_B s_C s_D s_{\beta\gamma},$$
$$Q_u = s_A s_B s_C s_D s_{\alpha\gamma}, \tag{3}$$

give the shapes in Fig. 1. The common factor $P_l \coloneqq s_A s_B s_C s_D$ will be used to label the *CK combination* $\Delta_l = n_s - n_t - n_u$, where s_A, s_B, s_C, s_D represent the products of all inverse propagators in A, B, C, D, respectively. The total number of these common factors is $(2N-5)!!(N-3)/3$, so there are 1, 10, 105 Δ_l's when $N = 4, 5, 6$.

These diagrams contain only cubic vertices which are antisymmetric when two lines are flipped, as shown in Fig. 2. In addition, each n_a must also contain N factors of ϵ and $N-2$ factors of k. Beyond that, n_a could be rather general.

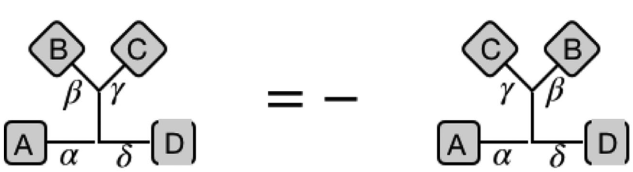

FIG. 2. n_a is antisymmetric when two lines are flipped.

C. Contents of partial amplitudes

A term n_a/Q_a is contained in $A(1\alpha N)$ when $1\alpha_2\alpha_3\cdots\alpha_{N-1}N$ coincides with the ordering of the external lines of its diagram. Owing to antisymmetry at cubic vertices, some external lines may be flipped, at the cost of a minus sign per flip. Flips produce different orderings of the external lines, hence it is possible for a given n_a/Q_a to be contained in several $A(1\alpha N)$, with a relative sign determined by the number of flips. By convention, its sign is chosen to be positive in the "smallest" $A(1\alpha N)$.

The $(N-2)!$ partial amplitudes $A(1\alpha N) = A_m$ will be ordered in a way to be discussed later. The "smallest" A refers to the A_m with the smallest m.

Note that lines 1 and N are special because they always occupy the two ends of the arguments of every A_m. To reflect this, all diagrams will be drawn above a base line with 1 at one end and N at the other end. As a result, lines 1 and N should never be flipped, and none of the lines in α is allowed to flip across the base line connecting 1 and N.

Explicit application of these rules will appear in the sections where the amplitudes for $N = 4, 5, 6$ are discussed.

D. Dimensional consideration and the variety of terms

The numerator n_a contains N factors of ϵ and $N-2$ factors of k. In terms of b, c, s, the allowed monomial combinations are

$$\begin{aligned} N &= 4, \qquad bcc, bbs, \\ N &= 5, \qquad bccc, bbcs, \\ N &= 6, \qquad bcccc, bbccs, bbbss. \end{aligned} \tag{4}$$

More generally, if μ is the largest integer $\le \frac{1}{2}N - 1$, then the allowed terms are $b^i c^{N-2i} s^{i-1}$, for $1 \le i \le \mu$. Terms of the same type can conceivably be combined through momentum conservation, but terms of different types can never be combined.

On dimensional grounds, it is possible to multiply these forms by a dimensionless rational function, such as $s_{12}s_{34}/s_{13}s_{23}$, but then the numerator factor n_a will no longer be a polynomial of b_{ij}, c_{ij}, and s_I, and whatever interaction that gives rise to this n_a may no longer be local.

For $N = 4$, there are six different b_{ij}'s, and three different $b_{ij}b_{kl}$'s. Taking momentum conservation and the on-shell condition into account, there are two c_{ij}'s for each i, and two s_I. So even the four-point numerators n_a contain many different terms each. For $N = 5$, there are 10 different b's, 15 different bb's, 3 different c_{ij}'s for each i, and 5 different s_I. For $N = 6$, there are 15 different b's, 30 different bb's, 15 different bbb's, 4 different c_{ij}'s for each i, and 9 different s_I. In short, each n_a contains a vast number of terms, especially for $N = 6$. This makes the computation of numerators and CK combinations a very tedious task. Fortunately, considerable simplification can be obtained by using symmetries of the Feynman diagrams and the Slavnov-Taylor identity.

E. CK relations in *S*-matrix theories

A short review on the CK relation of gluon tree amplitudes is given in this subsection. See Ref. [4] for a more thorough review. To start with, define an $(N-2)! \times (N-2)!$ *propagator matrix* m whose matrix element $m(\alpha|\beta) \equiv m(1\alpha N|1\beta N)$ is determined by terms common to $A(1\alpha N)$ and $A(1\beta N)$. $m(\alpha|\beta)$ is equal to those common terms, with all n_a set equal to 1, and with a sign which is the product of signs of the corresponding terms in the two amplitudes. In other words, up to a sign, it is the sum of the common propagators of the two amplitudes. This matrix can also be computed directly using the CHY formula for the biadjoint scalar theory.

With the help of the propagator matrix, the on-shell partial amplitudes

$$A(1\alpha N) = \sum_{\beta \in S_{N-2}} m(\alpha|\beta)\nu(\beta) \tag{5}$$

can be related to $(N-2)!$ parameters $\nu(\beta)$, to be referred to as the *fundamental numerator factors*. Since there are the same number of A's and ν's, one might be inclined to regard Eq. (5) as the definition of $\nu(\beta)$ in terms of the $(N-2)!$ known $A(1\alpha N)$'s. Unfortunately such a definition is not unique because the partial amplitudes satisfy a set of BCJ relations [3] to be discussed later, making them not linearly independent, and the propagator matrix m not invertible.

Comparing this expression for $A(1\alpha N)$ with its other expression $\sum \pm n_a/Q_a$, a set of relations between n_a and $\nu(\beta)$ can be derived by equating propagators. To distinguish n_a from the fundamental numerators $\nu(\beta)$, n_a are sometimes referred to as *ordinary numerators*. A set of ordinary numerators may or my not satisfy the CK relation, but if they are determined from Eq. (5), then the CK relation will automatically be fulfilled. In other words, the n_a so determined are in the BCJ representation.

Since there are $(2N-5)!!$ ordinary numerators n_a and $(N-2)!$ parameters $\nu(\beta)$, there must be $N_\Delta = (2N-5)!! - (N-2)!$ on-shell relations between n_a. These are the CK relations. For $N = 4, 5, 6$, this number N_Δ is 1, 9, 81 respectively. They differ from the numbers 1, 10, 105 given in Sec. II B for the number of Δ_l's. The difference of the two sets of numbers, 0, 1, 24, is the number of *trivial CK identities*. These are identities involving CK combinations Δ_l, valid whatever n_a are. Details will be given in later sections.

The relation between n_a and $\nu(\beta)$ can be visualized diagrammatically. To do that, start from a half-ladder diagram shown in Fig. 3. Since no line in this diagram may be flipped, it belongs to a single $A(1\alpha N)$ and appears in a single propagator matrix element $m(\alpha|\alpha)$. As a result, its numerator is equal to a single fundamental numerator: $n_a = \nu(\alpha)$.

Next, consider two half-ladder diagrams whose external lines α and γ coincide except for a single neighboring pair. Identifying the numerators of these two diagrams as n_s and n_u, and using the CK relation shown in Fig 1, one obtains a two-term relation $n_t = n_s - n_u = \nu(\alpha) - \nu(\gamma)$ for n_t. In this way, starting from half-ladder diagrams, one can get progressively to more and more complicated combinations of half-ladder diagrams to express the relations between any n_a and a combination of $\nu(\beta)$'s.

As mentioned above, the on-shell amplitudes satisfy a set of Bern-Carrasco-Johansson relations [3,16–18]. There are $(N-2)! - (N-3)! = (N-3)!(N-3)$ independent relations, given by

$$\begin{aligned} 0 = {} & s_{1\beta}A(1\beta\gamma_1\cdots\gamma_{N-3}N) + \cdots \\ & + (s_{1\beta} + s_{\gamma_1\beta} + \cdots + s_{\gamma_i\beta})A(1\gamma_1\cdots\gamma_i\beta\cdots\gamma_{N-3}N) + \cdots \\ & + (s_{1\beta} + s_{\gamma_1\beta} + \cdots + s_{\gamma_{N-3}\beta})A(1\gamma_1\cdots\gamma_{N-3}\beta N). \end{aligned} \tag{6}$$

Each relation gives rise to a null vector u for the on-shell propagator matrix [19,20]. For $N = 4, 5, 6$, this number of null vectors is 1, 4, 18 respectively. It follows from Eq. (5) that $A(1\alpha N)$ remains unchanged if we add to the column vector $\nu = (\nu(\beta))$ any linear combination of these null vectors u. This flexibility of $\nu(\beta)$ reflects the nonuniqueness of n_a in BCJ representations. Note that the coefficients x_i of these linear combinations do not have to be constants. They can be any function of b_{ij}, c_{ij}, and s_I, including rational and irrational functions. A change of $\nu(\beta)$ causes a gauge transformation of n_a. If x_i are constants or

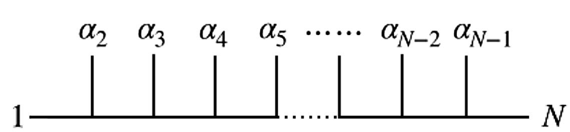

FIG. 3. A half-ladder diagram whose propagator factor is $Q_a = s_{1\alpha_2}s_{1\alpha_2\alpha_3}\cdots s_{1\alpha_2\alpha_3\cdots\alpha_{N-1}}$.

polynomials, the gauge transformation is local. If they are rational or other functions, the transformation is nonlocal.

The validity of the BCJ relation can be seen diagrammatically in the half-ladder diagrams of each $A(1\alpha N)$. In that case, the on-shell kinematical relation $s_{1\beta} + s_{\gamma_1\beta} + \cdots + s_{\gamma_i\beta} = s_{1\gamma_1\cdots\gamma_i\beta} - s_{1\gamma_1\cdots\gamma_i}$ shows that the factor multiplying each $A(1\alpha N)$ is just the difference of two consecutive inverse propagators, thereby causing the terms in Eq. (6) to cancel pairwise.

In the presence of null vectors, $\nu(\beta)$ is not unique, and it remains to be shown that there is at least one set of $\nu(\beta)$ satisfying Eq. (5). Since m has $(N-3)!(N-3)$ null vectors and a rank $(N-3)!$, only a $(N-3)! \times (N-3)!$ submatrix of m can be inverted. If $a = (\alpha_2\alpha_3\cdots\alpha_{N-2}) \in S_{N-3}$ is a permutation of the $(N-3)$ objects $\{2, 3, \ldots, N-1\}\backslash\alpha_{N-1}$, then the $(N-3)! \times (N-3)!$ matrix $\bar{m}$, with matrix elements $\bar{m}(a|b) = m(1a(N-1)N|1bN(N-1))$, has an inverse $\bar{m}^{-1} \coloneqq -S$. If we choose

$$\nu(\beta) = \sum_{c\in S_{N-3}} S(b|c)A(1cN(N-1)), \quad \text{if } \beta = (b, N-1),$$
$$= 0, \quad \text{if } \beta_{N-1} \neq N-1, \tag{7}$$

then Eq. (5) is satisfied [4,9,21], hence it is a solution of Eq. (5).

The matrix element $m(1a(N-1)N|1bN(N-1))$ used above is defined similar to the matrix element $m(1\alpha N|1\beta N)$. It is given by the common terms in $A(1a(N-1)N)$ and $A(1bN(N-1))$, after setting all $n_a = 1$ and taking into account the relative signs. In order for a numerator diagram to be common to these two amplitudes, lines N and $(N-1)$ must merge into a cubic vertex, thereby producing a Q_a factor $s_{(N-1)N}$, as shown in Fig. 4. If o is the internal line at that vertex, then $m(1a(N-1)N|1bN(N-1)) = -m(1ao|1bo)/s_{(N-1)N}$. Since o is off-shell, the $(N-3)! \times (N-3)!$ matrix $m(a|b) \equiv m(1ao|1bo)$ is nonsingular and has an inverse. There is an explicit formula for $S(b|c)$ which will not be displayed here. For our purpose it is sufficient to know that it is a polynomial of s_I of degree $N-3$.

Equivalently, this solution of $\nu(\beta)$ can be obtained by using the CK relations to eliminate all but the half-ladder n_a's, which are equal to some $\nu(\alpha)$. Owing to the BCJ relations, there are only $(N-3)!$ independent partial amplitudes, so only $(N-3)!$ of the $\nu(\alpha)$'s can be determined by Eq. (5). The remaining ones can be anything, including zero. The second line of Eq. (7) is simply a particular way of choosing what is to be set equal to zero.

The numerators obtained in Eq. (7) is highly nonlocal. They are rational functions of ϵ_i and k_i rather than polynomials. If we start from a set of local numerators $\bar{n}_a$, say obtained from Feynman diagrams, then the gauge transformation δn_a taking $\bar{n}_a$ into those obtained from Eq. (7) is also nonlocal.

F. Local and nonlocal gauge transformations

Let δn_a be a (generalized) gauge transformation, i.e., the difference of two sets of n_a that give rise to the same partial amplitudes. If every δn_a is a polynomial function of b_{ij}, c_{ij}, and s_I, then the gauge transformation is *local*, otherwise *nonlocal*.

Suppose $Q_a = s_{I_1}s_{I_2}\cdots s_{I_{N-3}}$, with $I_1 \prec I_2 \prec \cdots \prec I_{N-3}$ (see the next subsection for how I_a is ordered), is the corresponding propagator factor, and μ is the largest integer not exceeding $\frac{1}{2}N-1$. Then δn_a is a *local* gauge transformation only if every δn_a is a multilinear polynomial of $\{s_{I_1}, s_{I_2}, \ldots, s_{I_{N-3}}\}$ of degree μ, with coefficients that are polynomials of b_{ij} and c_{ij}. More precisely,

$$\delta n_a = \sum_{i=1}^{N-3} p_a^{(i-1)} s_{I_i} + \sum_{i=1}^{N-3}\sum_{j=i+1}^{N-3} q_a^{(i-1,j-1)} s_{I_i}s_{I_j} + \cdots + \sum_{i_1=1}^{N-3}\sum_{i_2=i_1+1}^{N-3}\cdots\sum_{i_\mu=i_{\mu-1}+1}^{N-3} r_a^{(i_1-1,i_2-1,\ldots,i_\mu-1)} s_{I_{i_1}}s_{I_{i_2}}\cdots s_{I_{i_\mu}}, \tag{8}$$

where $p, q, \ldots, r$ are polynomial functions of b_{ij} and c_{ij}.

With this form, the product of m s_I in δn_a cancels m factors of s in Q_a, leaving behind a term proportional to the $(m-N+3)$ power of s. To be a gauge transformation, one must find other $\delta n_b/Q_b$ terms in the same $A(1\alpha N)$ with the same s dependence to cancel it. This requirement generates a number of s-independent equations for $p, q, \ldots, r$ that must be satisfied. We shall refer to these equations as the *gauge constraint equations*.

An s-power larger than μ is not allowed in Eq. (8) on dimensional grounds. See Sec. II D. s_J not contained in Q_a is also not allowed because then the equations for $p, q, \ldots, r$ etc. will contain a rational function of s, making the gauge transformation no longer local.

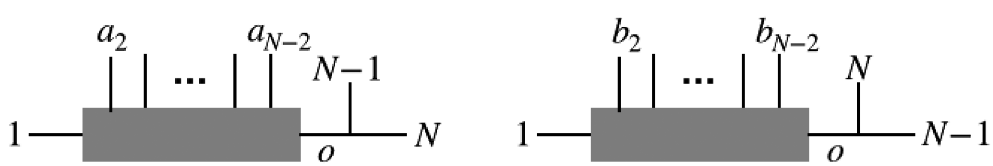

FIG. 4. The pair of tree diagrams used to compute $m(1a(N-1)N|1bN(N-1))$.

G. Canonical ordering

For $N > 4$, there are quite a few partial amplitudes, many propagators Q_a, and a large number of CK combinations Δ_l, so a systematic method is needed to enumerate them. The simplest is to express them as lists, and have the lists ordered canonically.

Each partial amplitude $A(1\alpha N)$ corresponds to an ordered list α, but Q_a and Δ_l are labeled by products of s_I's, with I's being unordered sets of numbers. The first task is therefore to convert an unordered set of numbers into an ordered list, and then use canonical ordering to order the lists.

An unordered set of three numbers 1, 2, 3 can be written as an ordered list $\{1,2,3\}$. More generally, an unordered set of m numbers can be turned into an ordered list $\{i_1, i_2, \ldots, i_m\}$ with $i_1 < i_2 < \cdots i_m$.

Two different lists are ordered first according to their length, and if they have the same length, according the first unequal numbers appearing in the two lists. For example, $\{5,6\} \prec \{1,2,3\}$, and $\{4,7,8,9\} \prec \{4,7,9,8\}$.

Two lists each with several sublists are first ordered by their number of sublists. If they have the same number of sublists, then they are ordered according to the first unequal sublists. For example, $\{\{5,6\},\{7,8,9\}\} \prec \{\{1,2\},\{3\},\{4\}\}$, and $\{\{3,4,5\},\{4,7,8,9\}\} \prec \{\{3,4,5\},\{4,7,9,8\}\}$.

Mandelstam variables s_I are ordered according to the list I. Thus $s_5 \prec s_{23} \prec s_{24} \prec s_{123}$. Products of k Mandelstam variables $s_{I_1} s_{I_2} \cdots s_{I_k}$ are ordered according to $\{I_1, I_2, \ldots, I_k\}$. In this way, we can order the inverse propagators (a product of $N-3$ Mandelstam variables) Q_a, and the CK combinations (labeled by a product of $N-4$ Mandelstam variables) Δ_l. Similarly, the partial amplitudes can be enumerated according to the list of its arguments. For example, for $N=5$, $A_1 = A(12345)$, $A_2 = A(12435)$, $A_3 = A(13245)$, $A_4 = A(13425)$, $A_5 = A(14235)$, and $A_6 = A(14325)$.

III. FEYNMAN DIAGRAMS

A. Summary of results

The main purpose of this article is to investigate how to obtain the CK relations from the Feynman diagrams of a Yang-Mills field theory, and whether the (generalized) gauge transformation required to accomplish that is local or not.

Feynman gauge is used in the Feynman rules, so propagators are the same as those in scalar theories. Feynman diagrams presented here should be understood as diagrams for the numerator $\bar{n}_a$ obtained from vertex factors alone, without the accompanying propagators $1/Q_a$.

The first obstacle encountered in such a project is the presence of four-gluon (4g) vertices. As they are absent in Fig. 1, they must be converted into antisymmetric cubic vertices. This necessitates the introduction of a "virtual vertex" which does not exist by itself, as two of them must be paired up to be properly defined. Since each diagram in Fig. 1 may contain up to two virtual vertices that must be paired up, the effect of 4g vertices will not be fully revealed until $N=6$ at least, thereby greatly complicating the calculation.

For $N=4$, it has been known for a long time that the CK relation $\bar{\Delta} = \bar{n}_s - \bar{n}_t - \bar{n}_u = 0$ computed from Feynman diagrams holds on-shell [2]. Its off-shell expression which is useful for large N computation will also be computed.

Both the numerator factor $\bar{n}_a$ computed from Feynman diagrams and n'_a computed from the CHY theory satisfy the CK relation, but these two sets of numerators are different. They will be shown to be related by a *local* gauge transformation.

For $N > 4$, the CK combinations $\bar{\Delta}_l$ computed from Feynman numerators $\bar{n}_a$ no longer vanish, even on-shell. Nevertheless, amplitudes $A(1\alpha N)$ so computed can be used in Eq. (7) to obtain a set of $\nu(\beta)$, and from there a set of n_a satisfying the CK relation. However, such n_a's are highly nonlocal, so the gauge transformation $\delta n_a = n_a - \bar{n}_a$ is nonlocal as well. We would like to know whether a local gauge transformation can be found to implement the CK relation. That is, whether a gauge transformation of the form Eq. (8) can cause $\Delta_l \coloneqq \bar{\Delta}_l - \delta\Delta_l = \bar{\Delta}_l - \delta(n_s - n_t - n_u) \simeq 0$. The notation $\simeq$ is used to indicate equality only for on-shell amplitudes. This requires a set of equations for the parameters $p, q, \ldots, r$ in Eq. (8) to satisfy, equations which shall be referred to as the *CK equations*. Since these parameters also need to satisfy the gauge-constraint relations, the locality question is equivalent to the question of whether the set of gauge-constraint and CK equations have a solution.

For $N=5$, which will be discussed in Sec. V, solutions do exist *provided* certain relations between Feynman diagrams with one virtual vertex are satisfied. Direct calculation shows that these relations are indeed satisfied, so there are local gauge transformations for $N=5$ to implement the CK relation.

The case of $N=6$ to be studied in Sec. VI is much more complicated. There are two sets of parameters, p and q. There are $315p$ parameters whose gauge-constraint equations can be explicitly solved, leaving behind 105 free parameters that must also satisfy 210 CK equations. Although there are many more equations than variables, nevertheless the equations are highly degenerate, so solutions could still exist if the $\bar{\Delta}_l$ obey a large set of conditions. It turns out that only some of these conditions are obeyed, but not others, so the p equations have no solution.

There are also $315q$ parameters, which must satisfy 144 gauge-constraint relations, and 450 CK equations. Again it turns out that there are no solutions.

The absence of p solution or q solution tells us that there is no local gauge transformation that can implement the CK relation. As an independent check, the CHY numerator factor n'_a and the difference $\delta n_a = \bar{n}_a - \hat{n}_a$ are computed. Since n'_a obey the CK relation, δn_a must be nonlocal to be consistent with the p, q conclusions. Indeed it is, because it contains s dependences beyond those involved in Q_a.

Thus we conclude that, starting from Feynman diagrams, there is no local gauge transformation capable of implementing the CK relation, but we have to go up to $N = 6$ to show it. Possible implications of this result will be discussed in Sec. VII.

B. Three-gluon, four-gluon, and virtual vertices in partial amplitudes

A three-gluon (3g) vertex T in a color-stripped partial amplitude is given by

$$T(p,q,r) = b_{pq}(c_{rp} - c_{rq}) + b_{qr}(c_{pq} - c_{pr}) + b_{rp}(c_{qr} - c_{qp}). \tag{9}$$

These three terms can each be associated with a diagram in the first row of Fig. 5, where ϵ is depicted by a black dot, and the difference of two momenta is shown as an arrow.

The 3g vertex is cyclic in its arguments, and antisymmetric when two arguments are exchanged: $T(p,q,r) = T(q,r,p) = -T(q,p,r)$.

To convert a four-gluon vertex into cubic vertices, it should be separated into two terms as follows:

$$\begin{aligned} V_4(p,q,r,s) &= 2b_{pr}b_{qs} - b_{pq}b_{rs} - b_{ps}b_{qr} \\ &= \frac{Q(p,q,r,s)s_{pq}}{s_{pq}} + \frac{Q(q,r,s,p)s_{qr}}{s_{qr}}, \\ Q(p,q,r,s) &= b_{pr}b_{qs} - b_{ps}b_{qr} = Q(r,s,p,q). \end{aligned} \tag{10}$$

Each term is depicted by a pair of virtual vertices connected by a dotted line, and each virtual vertex has the antisymmetry of the 3g vertex when two solid lines are flipped. However, a single virtual vertex does not really exist; they must come in pairs to form a Q.

Each cubic vertex in Fig. 1 can be a 3g vertex or a virtual vertex. Feynman diagrams containing virtual vertices can be obtained from purely 3g diagrams by replacing solid propagator lines with dotted lines in all possible ways, subject to the constraint that no two dotted lines may intersect at a vertex.

To save space, commas between arguments of functions like T and Q will often be dropped.

C. CK combinations for Feynman diagrams

With virtual vertices, one or two of the four lines α, β, γ, δ in Fig. 1 may now be dotted. In terms of the original 4g vertices, exposing a dotted line is equivalent to cutting through the middle of a 4g vertex.

Figure 6 shows an example where α is dotted, and an example where both α and γ are dotted. Since two dotted lines are not allowed to intersect, the second row of Fig. 6 has only two instead of three diagrams. Also, for the same reason, diagrams with three or four dotted lines do not exist. The diagrams in Fig. 6 are merely symbolic, because virtual vertices cannot exist by themselves. To make sense of Fig. 6, we must pair up all the virtual vertices, as shown in Fig. 7. This compels us to study Feynman diagrams explicitly also for $N = 5$ and $N = 6$.

These diagrams with dotted lines turn out to be the culprits that cause $\bar{\Delta}_l \neq 0$ for $N = 5$ and $N = 6$.

D. Slavnov-Taylor identity

The Slavnov-Taylor identity can be used to simplify calculation of the CK relation for a larger N. Like the Ward-Takahashi identity in QED, it is a consequence of gauge invariance of the Yang-Mills theory, relating the divergence of a gauge field to the gauge and ghost fields. In Feynman gauge, the Green's function identity is

$$\langle 0|\partial^{\mu_1} A^{a_1}_{\mu_1}(x_1) A^{a_2}_{\mu_2}(x_2)\cdots A^{a_n}_{\mu_n}(x_n)|0\rangle + \sum_{l=2}^{n} \langle 0|\bar{\omega}_{a_1}(x_1) A^{a_2}_{\mu_2}(x_2)\cdots D_{\mu_l}\omega^{a_l}(x_l)\cdots A^{a_n}_{\mu_n}(x_n)|0\rangle = 0, \tag{11}$$

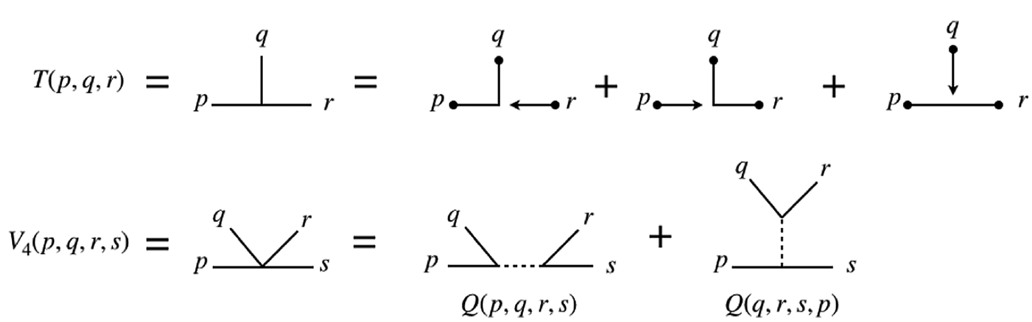

FIG. 5. 3g, 4g, and virtual vertices.

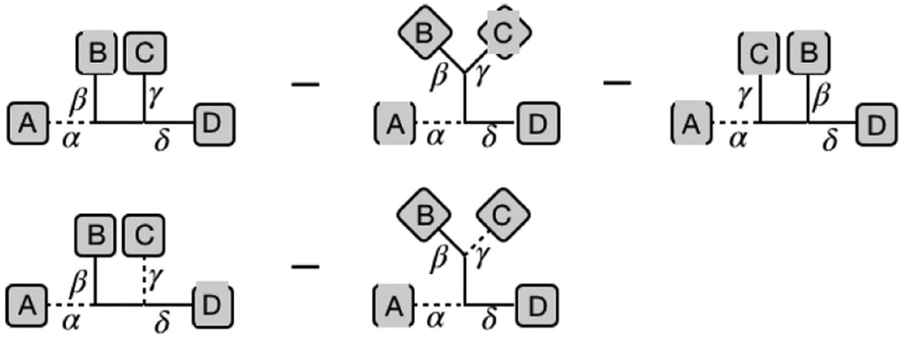

FIG. 6. Virtual vertex in CK combinations.

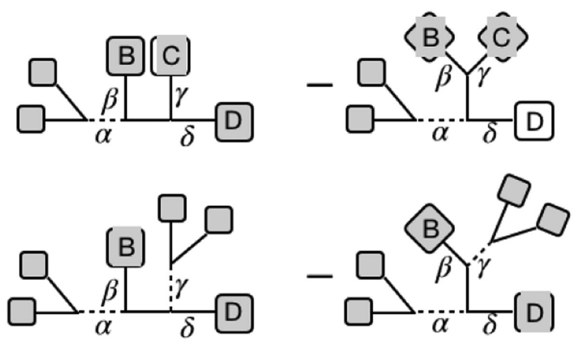

FIG. 7. Completion of the virtual vertices in Fig. 6.

where A is the gluon field, and ω, $\bar{\omega}$ are the ghost and antighost fields, whose covariant derivative is given by $D_\nu \omega^a(x) = \partial_\nu \omega^a - g f^{adc} \omega^d A^c_\nu$.

The diagrammatic expression for Eq. (11) in terms of amplitudes is sketched in Fig. 8. Solid and dashed lines represent gluons and (anti)ghosts, with an arrow running from the antighost end to the ghost end. The left-hand side, with ϵ_i replaced by k_i, is the divergence of the gluon amplitude. The right-hand side is given by covariant derivative terms in momentum space, with a cross ($\times$) representing $\epsilon \cdot k$ and a filled box representing k^2. For on-shell amplitudes, $\epsilon_i \cdot k_i$ and k_i^2 both vanish, hence the gluon amplitude is divergenceless. For off-shell amplitudes, this identity shows how the divergence of a gluon amplitude is related to amplitudes involving gluons and a single ghost line.

Let $\mathcal{M}$ be the gluon amplitude. Singling out the polarization vector $\epsilon_i^{(\lambda)}$ of its ith line, it can be written as $\mathcal{M} = \epsilon^{(\lambda)}_{i\mu} \bar{\mathcal{M}}^\mu = \epsilon_i^{(\lambda)} \cdot \bar{\mathcal{M}}$. If i is an internal line, the replacement of ϵ_i by k_i on the left-hand side of Fig. 8 can be accomplished by $\sum_{\lambda=1}^{D} (\epsilon_i^{(\lambda)} \cdot k_i)(\epsilon_i^{(\lambda)} \cdot \bar{\mathcal{M}})$. In the shorthand notation used in Eq. (1), this is just $c_{ii}\mathcal{M}$. It is in this form that the Slavnov-Taylor identity will be used.

By equating the residues of the amplitudes at various propagator poles, many *numerator Slavnov-Taylor identities* can also be obtained.

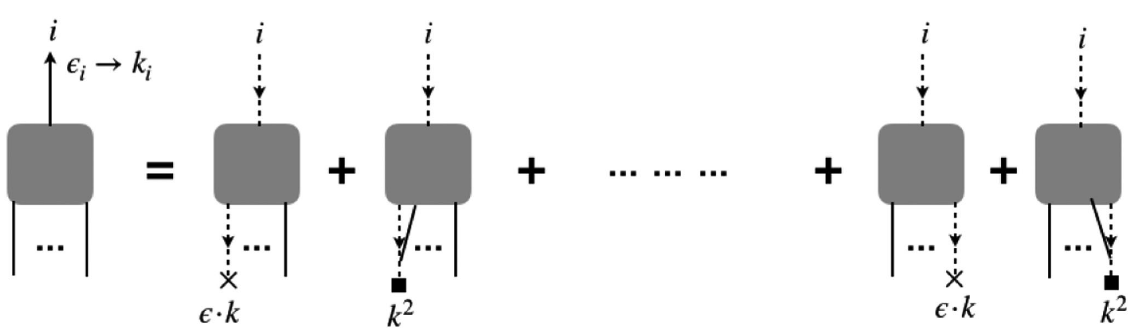

FIG. 8. Slavnov-Taylor identity.

IV. FOUR-POINT AMPLITUDES

The numerators of the partial amplitudes

$$A(1234)=\frac{\bar{n}_s}{s}+\frac{\bar{n}_t}{t},$$
$$A(1324)=-\frac{\bar{n}_t}{t}+\frac{\bar{n}_u}{u},\ \text{with}$$
$$s=s_{12}=s_{34},\quad t=s_{23}=s_{41},\quad u=s_{13}=s_{24},\quad (12)$$

can be computed from the Feynman diagrams in Fig. 9 to be

$$\bar{n}_s=T(129)T(\overline{934})+Q(1234)s\coloneqq P(1234)+Q(1234)s \coloneqq A_4(1234),$$
$$\bar{n}_t=T(419)T(\overline{923})+Q(4123)t=P(4123)+Q(4123)t =A_4(4123),$$
$$\bar{n}_u=T(139)T(\overline{924})+Q(1324)u=P(1324)+Q(1324)u =A_4(1324).\quad (13)$$

In Eq. (13) and in the rest of this article, high numbers such as 9,8 are used to designate internal (off-shell) lines whose polarizations are to be summed over. See Eq. (1) and Eq. (2). The polarizations of 9 and $\bar{9}$ are the same, $\epsilon_9^{(\lambda)}=\epsilon_{\bar{9}}^{(\lambda)}$, but the momenta k_9 and $k_{\bar{9}}$ are opposite to enable all momenta at a vertex to be outgoing. For example, in the $P(1234)=T(129)T(\bar{9}34)$ term, $k_9+k_1+k_2=0$, and $k_{\bar{9}}+k_3+k_4=-k_9+k_3+k_4=0$. The minus sign in front of the $\bar{n}_t/t$ term in $A(1324)$ of Eq. (12) is a consequence of antisymmetry at the $\bar{9}23$ vertex.

The function $P(1234)$ in Eq. (13) can be computed using Feynman rules. The result, valid both on-shell and off-shell, is

$$P(1234)=bcc\,\text{terms}+\frac{1}{2}b_{12}b_{34}(s_{13}-s_{14}-s_{23}+s_{24}).\quad (14)$$

The bcc terms are rather lengthy and will not be displayed here. Suffice to say that they are not zero even for on-shell external lines. However, a direct calculation shows that the CK relation $\bar{\Delta}\coloneqq\bar{n}_s-\bar{n}_t-\bar{n}_u=0$ is valid on-shell, hence a complete cancellation of the bcc terms occurs in $\bar{\Delta}$. Its off-shell expression, containing only bcc terms,

$$\begin{aligned}\bar{\Delta}&\coloneqq\bar{n}_s-\bar{n}_t-\bar{n}_u\\ &=b_{12}[c_{33}(c_{41}-c_{42})-c_{44}(c_{31}-c_{32})]+b_{13}[c_{44}(c_{21}-c_{23})-c_{22}(c_{41}-c_{43})]\\ &\quad+b_{14}[c_{22}(c_{31}-c_{34})-c_{33}(c_{21}-c_{24})]+b_{23}[c_{11}(c_{42}-c_{43})-c_{44}(c_{12}-c_{13})]\\ &\quad+b_{24}[c_{33}(c_{12}-c_{14})-c_{11}(c_{32}-c_{34})]+b_{34}[c_{11}(c_{23}-c_{24})-c_{22}(c_{13}-c_{14})]\\ &\coloneqq d(1234),\end{aligned}\quad (15)$$

will be useful later for larger N computations. An equivalent expression which will also be useful later is

$$d(1234)=c_{11}T(234)-c_{22}T(341)+c_{33}T(412) -c_{44}T(123).\quad (16)$$

Although the bcc terms from each $\bar{n}_a$ are complicated, their combined expression in Eq. (15) is relatively simple. The simplicity stems from combinations of terms of the type shown in Fig. 10, yielding

$$\mathbf{c}(\alpha\beta\gamma\delta)=b_{\alpha\delta}[c_{\beta\beta}(c_{\gamma\alpha}-c_{\gamma\delta})-c_{\gamma\gamma}(c_{\beta\delta}-c_{\beta\alpha})].\quad (17)$$

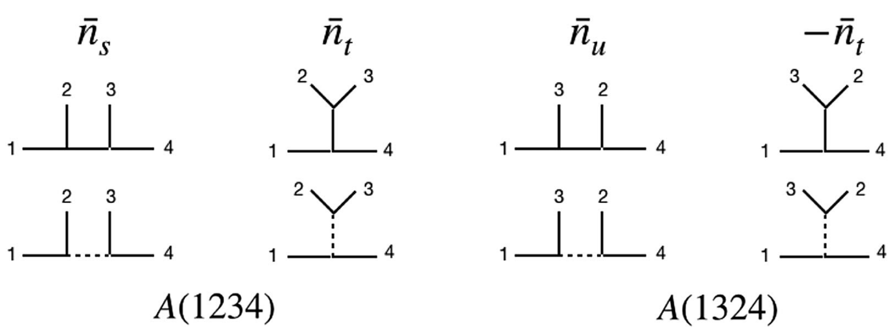

FIG. 9. Feynman diagrams for the four-point color-stripped amplitude.

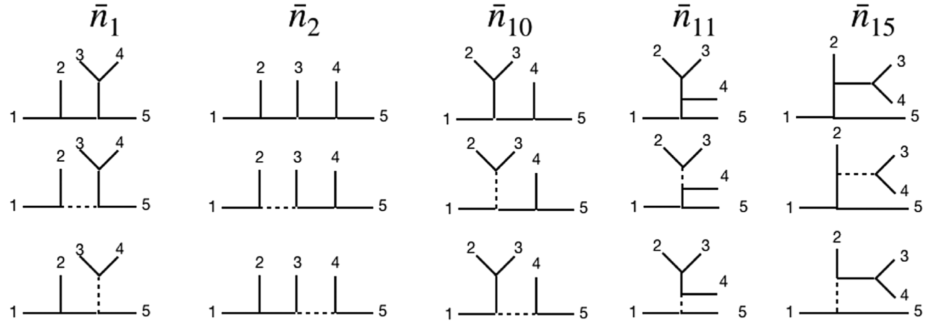

FIG. 10. Diagrams for the c-identity function $\mathbf{c}(\alpha\beta\gamma\delta)$.

Either from the antisymmetry at every vertex or directly from Eq. (15) or Eq. (16), $d(1234)$ can be seen to have a high degree of symmetry,

$$d(1234) = -d(2134) = -d(1243) = -d(1324). \tag{18}$$

Since $\bar{n}_a$ satisfy the on-shell CK relation, they can be related to the fundamental numerators $\nu(\beta)$ by using Eq. (5) and Eq. (12),

$$\begin{pmatrix} A(1234) \\ A(1324) \end{pmatrix} = \begin{pmatrix} 1/s+1/t & -1/t \\ -1/t & 1/u+1/t \end{pmatrix} \begin{pmatrix} \nu(23) \\ \nu(32) \end{pmatrix} = \begin{pmatrix} \bar{n}_s/s+\bar{n}_t/t \\ \bar{n}_u/u-\bar{n}_t/t \end{pmatrix}. \tag{19}$$

The mass matrix $m(\alpha|\beta)$ displayed above is obtained from Fig. 9 and the rules stated at the beginning to Sec. II C. The relation between $\bar{n}_a$ and $\nu(\beta)$ can be read off from Eq. (19) to be

$$\begin{aligned} \bar{n}_s &= \nu(23) + xu(23), \\ \bar{n}_u &= \nu(32) + x(32), \\ \bar{n}_t &= \nu(23) - \nu(32) + x(u(23) - u(32)), \end{aligned} \tag{20}$$

where x is an arbitrary parameter and $u = (u(23), u(32))^T = (s, -u)^T$ is the null vector of the on-shell mass matrix $m(\alpha|\beta)$ because $s+t+u \simeq 0$. See Sec. II E. Since both $\bar{n}_a$ and n'_a are in the BCJ representation, $\delta n_a = \bar{n}_a - n'_a$ must be of the form xu. A direct calculation shows that $x = -(b_{13}b_{24} - b_{12}b_{34})$.

V. FIVE-POINT AMPLITUDE

A. Partial amplitudes and numerators

As discussed in Sec. II A, there are six partial amplitudes for $N = 5$, each containing 5 terms, making a total of 30 terms. Since there are only 15 independent propagators $1/Q_a$, each n_a/Q_a is expected to appear in several partial amplitudes, with signs determined by the antisymmetry of vertices as discussed in Sec. II C.

The explicit enumerations via canonical ordering of $s_I, Q_a = s_{I_1}s_{I_2}$, and n_a are given in Appendix A. With those notations, the six partial amplitudes are

$$\begin{aligned} A_1 &= A(12345) \\ &= +\frac{n_1}{s_{12}s_{34}} + \frac{n_2}{s_{12}s_{123}} + \frac{n_{10}}{s_{23}s_{123}} + \frac{n_{11}}{s_{23}s_{234}} + \frac{n_{15}}{s_{34}s_{234}}, \\ A_2 &= A(12435) \\ &= -\frac{n_1}{s_{12}s_{34}} + \frac{n_3}{s_{12}s_{124}} + \frac{n_{12}}{s_{24}s_{124}} + \frac{n_{13}}{s_{24}s_{234}} - \frac{n_{15}}{s_{34}s_{234}}, \\ A_3 &= A(13245) \\ &= +\frac{n_4}{s_{13}s_{24}} + \frac{n_5}{s_{13}s_{123}} - \frac{n_{10}}{s_{23}s_{123}} - \frac{n_{11}}{s_{23}s_{234}} - \frac{n_{13}}{s_{24}s_{234}}, \\ A_4 &= A(13425) \\ &= -\frac{n_4}{s_{13}s_{24}} + \frac{n_6}{s_{13}s_{134}} + \frac{n_{13}}{s_{24}s_{234}} + \frac{n_{14}}{s_{34}s_{134}} - \frac{n_{15}}{s_{34}s_{234}}, \\ A_5 &= A(14235) \\ &= +\frac{n_7}{s_{14}s_{23}} + \frac{n_8}{s_{14}s_{124}} - \frac{n_{11}}{s_{23}s_{234}} - \frac{n_{12}}{s_{24}s_{124}} - \frac{n_{13}}{s_{24}s_{234}}, \\ A_6 &= A(14325) \\ &= -\frac{n_7}{s_{14}s_{23}} + \frac{n_9}{s_{14}s_{134}} + \frac{n_{11}}{s_{23}s_{234}} - \frac{n_{14}}{s_{34}s_{134}} + \frac{n_{15}}{s_{34}s_{234}}. \end{aligned} \tag{21}$$

The canonically ordered n_a used here are different from those used in Ref. [3]. A dictionary relating the two can be found in Table III of Appendix A.

The Feynman diagrams for $A(12345)$ are shown in Fig. 11. Those for other partial amplitudes can be obtained by a relabeling of the external lines, together with a change of the numerator designations.

The sign of a n_a/Q_a term in an amplitude A_m can be read out from the Feynman diagrams, as discussed in Sec. II C. By convention, the sign of every term in $A_1 = A(12345)$ is positive. The sign of the n_1/Q_1 term in $A_2 = A(12435)$ for example is negative because a flip of 3 and 4 is needed to change 12345 to 12435.

The numerators $\bar{n}_a$ read off from Fig. 11 using the Feynman rules are

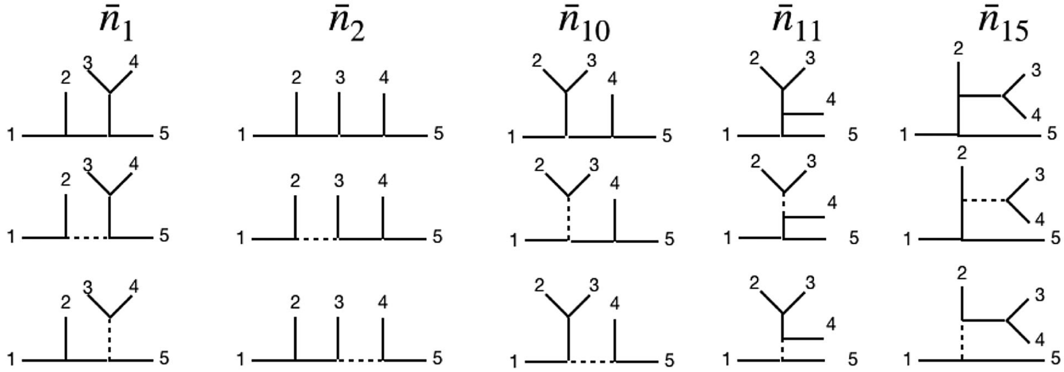

FIG. 11. Feynman diagrams for $A(12345)$.

$$\begin{aligned}
\bar{n}_1 &= T(129)T(\bar{9}\,\bar{8}\,5)T(834) + Q(12\bar{8}5)T(834)s_{12} + T(129)Q(345\bar{9})s_{34},\\
\bar{n}_2 &= T(129)T(\bar{9}3\bar{8})T(845) + Q(123\bar{8})T(845)s_{12} + T(129)Q(45\bar{9}3)s_{123},\\
\bar{n}_{10} &= T(239)T(1\bar{9}\,\bar{8})T(845) + Q(23\bar{8}1)T(845)s_{23} + T(239)Q(451\bar{9})s_{123},\\
\bar{n}_{11} &= T(239)T(\bar{9}4\bar{8})T(851) + Q(234\bar{8})T(851)s_{23} + T(239)Q(51\bar{9}4)s_{234},\\
\bar{n}_{15} &= T(349)T(2\bar{9}\,\bar{8})T(851) + Q(34\bar{8}2)T(851)s_{34} + T(349)Q(2\bar{9}51)s_{234}.
\end{aligned} \tag{22}$$

The other 10 $\bar{n}_a$'s, obtained similarly, are

$$\begin{aligned}
\bar{n}_3 &= T(129)T(\bar{9}4\bar{8})T(835) + Q(124\bar{8})T(835)s_{12} + T(129)Q(35\bar{9}4)s_{124},\\
\bar{n}_{12} &= T(249)T(1\bar{9}\,\bar{8})T(835) + Q(24\bar{8}1)T(835)s_{24} + T(249)Q(351\bar{9})s_{124},\\
\bar{n}_{13} &= T(249)T(\bar{9}3\bar{8})T(851) + Q(243\bar{8})T(851)s_{24} + T(249)Q(51\bar{9}3)s_{234},\\
\bar{n}_4 &= T(139)T(\bar{9}\,\bar{8}\,5)T(824) + Q(13\bar{8}5)T(824)s_{13} + T(139)Q(245\bar{9})s_{24},\\
\bar{n}_5 &= T(139)T(\bar{9}2\bar{8})T(845) + Q(132\bar{8})T(845)s_{13} + T(139)Q(45\bar{9}2)s_{123},\\
\bar{n}_6 &= T(139)T(\bar{9}4\bar{8})T(825) + Q(134\bar{8})T(825)s_{13} + T(139)Q(25\bar{9}4)s_{134},\\
\bar{n}_{14} &= T(349)T(1\bar{9}\,\bar{8})T(825) + Q(34\bar{8}1)T(825)s_{34} + T(349)Q(251\bar{9})s_{134},\\
\bar{n}_7 &= T(149)T(\bar{9}\,\bar{8}\,5)T(823) + Q(14\bar{8}5)T(823)s_{14} + T(149)Q(235\bar{9})s_{23},\\
\bar{n}_8 &= T(149)T(\bar{9}2\bar{8})T(835) + Q(142\bar{8})T(835)s_{14} + T(149)Q(35\bar{9}2)s_{124},\\
\bar{n}_9 &= T(149)T(\bar{9}3\bar{8})T(825) + Q(143\bar{8})T(825)s_{14} + T(149)Q(25\bar{9}3)s_{134}.
\end{aligned} \tag{23}$$

Every $\bar{n}_a$ contains three terms, TTT, QT, and TQ, reflecting the three rows of diagrams in Fig. 11. Owing to antisymmetry at the vertices, the last two can all be expressed in terms of a single function f representing Fig. 12,

$$\begin{aligned}
f(12345) &\coloneqq Q(123\bar{9})T(945)\\
&= b_{23}[b_{45}(-c_{14} + c_{15}) - b_{15}(c_{44} + 2c_{45})\\
&\quad + b_{14}(2c_{54} + c_{55})]\\
&\quad + b_{13}[b_{45}(c_{24} - c_{25}) + b_{25}(c_{44} + 2c_{45})\\
&\quad - b_{24}(2c_{54} + c_{55})],
\end{aligned} \tag{24}$$

which is antisymmetric in the first two arguments and in the last two arguments:

$$f(12345) = -f(21345) = -f(12354). \tag{25}$$

It is also convenient to introduce another function

$$F(12345) \coloneqq f(12345) + f(12453) + f(12534) \tag{26}$$

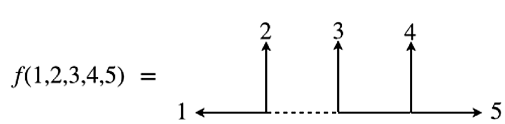

FIG. 12. $f(12345)$.

that appears in CK combinations. Although F is not zero even on-shell, it nevertheless possesses a large amount of symmetry,

$$F(12345) = -F(21345) = -F(12435) = F(12453) = F(12534), \tag{27}$$

needed to simplify calculations.

B. CK combinations

As discussed in Sec. II B, there are 10 CK combinations $\bar{\Delta}_l$. Using the canonical ordering of double Mandelstam variables specified in Appendix A to label them, these CK combinations are

$$\begin{aligned} &\Delta_1 = n_2 - n_1 - n_3, && \Delta_2 = n_5 - n_4 - n_6, \\ &\Delta_3 = n_8 - n_7 - n_9, && \Delta_4 = n_{10} - n_7 - n_{11}, \\ &\Delta_5 = n_{12} - n_4 - n_{13}, && \Delta_6 = n_1 - n_{14} - n_{15}, \\ &\Delta_7 = n_2 - n_5 - n_{10}, && \Delta_8 = n_3 - n_8 - n_{12}, \\ &\Delta_9 = n_6 - n_9 - n_{14}, && \Delta_{10} = n_{11} - n_{13} - n_{15}. \end{aligned} \tag{28}$$

Every n_a occurs twice in these Δ_l, enabling the following combination of Δ_l to be identically zero,

$$\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4 + \Delta_5 + \Delta_6 - \Delta_7 + \Delta_8 - \Delta_9 - \Delta_{10} = 0. \tag{29}$$

Such a relation will henceforth be referred to as a *trivial CK identity*. It is valid on-shell and off-shell because it is identically zero whatever n_a are. There is only one trivial identity in $N = 5$, but there are many in $N = 6$.

Using Eq. (22) to Eq. (27), the CK combinations $\bar{\Delta}_l$ can be calculated to be

$$\begin{aligned} \bar{\Delta}_1 &= T(129)d(45\bar{9}3) + F(12345)s_{12}, \\ \bar{\Delta}_2 &= T(139)d(45\bar{9}2) + F(13245)s_{13}, \\ \bar{\Delta}_3 &= T(149)d(35\bar{9}2) + F(14235)s_{14}, \\ \bar{\Delta}_4 &= T(239)d(451\bar{9}) + F(23154)s_{23}, \\ \bar{\Delta}_5 &= T(249)d(351\bar{9}) + F(24153)s_{24}, \\ \bar{\Delta}_6 &= T(349)d(12\bar{9}5) + F(34512)s_{34}, \\ \bar{\Delta}_7 &= T(459)d(123\bar{9}) + F(45321)s_{123}, \\ \bar{\Delta}_8 &= T(359)d(124\bar{9}) + F(35421)s_{124}, \\ \bar{\Delta}_9 &= T(259)d(134\bar{9}) + F(25431)s_{134}, \\ \bar{\Delta}_{10} &= T(159)d(23\bar{9}4) + F(51432)s_{234}. \end{aligned} \tag{30}$$

All have the form $Td + Fs$. The Td terms are of the form $bccc$ (see Sec. II D); they vanish on-shell on account of the Slavnov-Taylor identity for the following reason. Every term in d given by Eq. (15) contains a c_{ii} for some i, which is zero unless i is off-shell. In the expression Td, internal line 9 is the only one off-shell, hence the only nonvanishing term in d is proportional to c_{99}. By setting $\mathcal{M}$ in Sec. III D to be T and i to be 9, the Slavnov-Taylor identity ensures that the Td terms vanish on-shell.

The Fs terms are of the form $bbcs$ and are not zero even on-shell, thus CK combinations $\bar{\Delta}_l$ computed from Feynman diagrams do not satisfy the CK relation.

Although $\bar{\Delta}_l$ is not quite zero, a large amount of cancellation has already taken place by the vanishing of the $bccc$ terms. The $bccs$ terms do not vanish, but since it is linear in s, there is a chance that $\bar{n}_a$ can be converted into a BCJ representation by a *local* gauge transformation.

C. Local gauge transformation

A generalized gauge transformation $\bar{n}_a \to n_a = \bar{n}_a - \delta n_a$ induces a transformation $\bar{\Delta}_l \to \Delta_l = \bar{\Delta}_l - \delta\Delta_l$, with $\delta\Delta_l = \delta n_s - \delta n_t - \delta n_u$. To bring about the CK relation $\Delta_l = 0$ on-shell, a shift of amount $\delta\Delta_l = \bar{\Delta}_l$ is required. Since $\bar{\Delta}_l$ is of the form Fs, δn_a is also expected to be linear in s. This gives hope that it may be a local gauge transformation.

According to Sec. II F, local gauge transformations for $N = 5$ must have the form

$$\delta n_a = p_a s_a + p'_a s'_a, \tag{31}$$

where $Q_a = s_a s'_a$ is the propagator factor, and by our convention $s_a \prec s'_a$. In other words, δn_a may involve only s_a and s'_a, but no other s. Note that s_a here is *not* a Mandelstam variable with the canonical subscript a. It is defined to be the first Mandelstam factor of Q_a.

An examination of Eq. (21) shows that every $A(1\alpha N)$ remains the same if the three parameters in every one of the following 10 triplets are equal:

$$\begin{aligned} &t_1 = (-p'_1, p'_2, -p'_3), && t_2 = (-p'_4, p'_5, -p'_6), \\ &t_3 = (-p'_7, p'_8, -p'_9), && t_4 = (-p_7, p'_{10}, -p'_{11}), \\ &t_5 = (-p_4, p'_{12}, -p'_{13}), && t_6 = (p_1, -p'_{14}, -p'_{15}), \\ &t_7 = (p_2, -p_5, -p_{10}), && t_8 = (p_3, -p_8, -p_{12}), \\ &t_9 = (p_6, -p_9, -p_{14}), && t_{10} = (p_{11}, -p_{13}, -p_{15}). \end{aligned} \tag{32}$$

For example, take the parameters in the first triplet t_1. The only partial amplitudes affected by these parameters are

$$\delta A_1 = \frac{p'_1}{s_{12}} + \frac{p'_2}{s_{12}}, \qquad \delta A_2 = -\frac{p'_1}{s_{12}} + \frac{p'_3}{s_{12}},$$

and they are zero if the three parameters in t_1 are equal.

The t_1 triplet contains p parameters for the numerators $(\bar{n}_1, \bar{n}_2, \bar{n}_3)$, all members of $\bar{\Delta}_1$. This is why this triplet is labeled t_1. Similarly, the triplet of parameters in t_l are all related to the numerators involved in $\bar{\Delta}_l$.

Using Eq. (31) and Eq. (32), and t_l to represent any of the three p parameters in its triplet, the shift $\delta\Delta_l = \delta n_s - \delta n_t - \delta n_u$ takes on the form

$$\begin{aligned}
\delta\Delta_1 &= \delta(n_2 - n_1 - n_3) \\
&= (t_7 - t_6 - t_8)s_{12} + t_1(s_{34} + s_{123} + s_{124}), \\
\delta\Delta_2 &= \delta(n_5 - n_4 - n_6) \\
&= (-t_7 + t_5 - t_9)s_{13} + t_2(s_{24} + s_{123} + s_{134}), \\
\delta\Delta_3 &= \delta(n_8 - n_7 - n_9) \\
&= (-t_8 + t_4 + t_9)s_{14} + t_3(s_{23} + s_{124} + s_{134}), \\
\delta\Delta_4 &= \delta(n_{10} - n_7 - n_{11}) \\
&= (-t_7 + t_3 - t_{10})s_{23} + t_4(s_{14} + s_{123} + s_{234}), \\
\delta\Delta_5 &= \delta(n_{12} - n_4 - n_{13}) \\
&= (-t_8 + t_2 + t_{10})s_{24} + t_5(s_{13} + s_{124} + s_{234}), \\
\delta\Delta_6 &= \delta(n_1 - n_{14} - n_{15}) \\
&= (-t_1 + t_9 + t_{10})s_{34} + t_6(s_{12} + s_{134} + s_{234}), \\
\delta\Delta_7 &= \delta(n_2 - n_5 - n_{10}) \\
&= (t_1 - t_2 - t_4)s_{123} + t_7(s_{12} + s_{13} + s_{23}), \\
\delta\Delta_8 &= \delta(n_3 - n_8 - n_{12}) \\
&= (-t_1 - t_3 - t_5)s_{124} + t_8(s_{12} + s_{14} + s_{24}), \\
\delta\Delta_9 &= \delta(n_6 - n_9 - n_{14}) \\
&= (-t_2 + t_3 + t_6)s_{134} + t_9(s_{13} + s_{14} + s_{34}), \\
\delta\Delta_{10} &= \delta(n_{11} - n_{13} - n_{15}) \\
&= (-t_4 + t_5 + t_6)s_{234} + t_{10}(s_{23} + s_{24} + s_{34}). \qquad (33)
\end{aligned}$$

These formulas are valid on-shell and off-shell. Using the kinematical formula

$$s_{I_1I_2} + s_{I_1I_3} + s_{I_2I_3} = s_{I_1} + s_{I_2} + s_{I_3} + s_{I_1I_2I_3}, \qquad (34)$$

Eq. (33) when on-shell reduces to

$$\begin{aligned}
\delta\Delta_1 &= (t_7 - t_6 - t_8 + t_1)s_{12}, \\
\delta\Delta_2 &= (-t_7 + t_5 - t_9 + t_2)s_{13}, \\
\delta\Delta_3 &= (-t_8 + t_4 + t_9 + t_3)s_{14}, \\
\delta\Delta_4 &= (-t_7 + t_3 - t_{10} + t_4)s_{23}, \\
\delta\Delta_5 &= (-t_8 + t_2 + t_{10} + t_5)s_{24}, \\
\delta\Delta_6 &= (-t_1 + t_9 + t_{10} + t_6)s_{34}, \\
\delta\Delta_7 &= (t_1 - t_2 - t_4 + t_7)s_{123}, \\
\delta\Delta_8 &= (-t_1 - t_3 - t_5 + t_8)s_{124}, \\
\delta\Delta_9 &= (-t_2 + t_3 + t_6 + t_9)s_{134}, \\
\delta\Delta_{10} &= (-t_4 + t_5 + t_6 + t_{10})s_{234}. \qquad (35)
\end{aligned}$$

In order for $\delta\Delta_l = \bar{\Delta}_l$, the two sides must have the same s dependence. This can be verified to be true by comparing Eq. (30) with Eq. (35). With this matching of the s dependence, the requirement $\delta\Delta_l = \bar{\Delta}_l$ is reduced to a set of linear equations for the 10 unknowns t_l. In matrix form, these CK equations are

$$\sum_{l'=1}^{10} \tau_{ll'} t_{l'} = d_l (1 \le l \le 10) \qquad (36)$$

where d_l is the on-shell expression of $\bar{\Delta}_l$ with the s dependence stripped off, namely,

$$\begin{aligned}
d_1 &= F(12345), & d_2 &= F(13245), \\
d_3 &= F(14235), & d_4 &= F(23154), \\
d_5 &= F(24153), & d_6 &= F(34512), \\
d_7 &= F(45321), & d_8 &= F(35421), \\
d_9 &= F(25431), & d_{10} &= F(51432). \qquad (37)
\end{aligned}$$

The 10×10 matrix

$$\tau = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix} \qquad (38)$$

has rank 5, and has five null vectors

$$\begin{aligned}
v_1 &= (-1, 1, 1, -1, 0, 0, 1, 0, 0, -1), \\
v_2 &= (-1, 0, 1, 0, 0, 0, 1, 0, -1, 0), \\
v_3 &= (0, -1, -1, 0, 0, 0, -1, -1, 0, 0), \\
v_4 &= (-1, 0, 1, -1, 0, -1, 0, 0, 0, 0), \\
v_5 &= (0, 1, 1, -1, -1, 0, 0, 0, 0, 0). \qquad (39)
\end{aligned}$$

As a result, the inhomogeneous terms d_l must satisfy five sum rules $R_x \coloneqq \sum_l (v_x)_l d_l = 0$ in order for Eq. (36) to have a solution. Explicitly, these sum rules are

$$\begin{aligned} R_1 &= -d_1 + d_2 + d_3 - d_4 + d_7 - d_{10} = 0, \\ R_2 &= -d_1 + d_3 + d_7 - d_9 = 0, \\ R_3 &= -d_2 - d_3 - d_7 - d_8 = 0, \\ R_4 &= -d_1 + d_3 - d_4 - d_6 = 0, \\ R_5 &= d_2 + d_3 - d_4 - d_5 = 0. \end{aligned} \tag{40}$$

To check whether the sum rules are satisfied, introduce a D-function defined by

$$\begin{aligned} D(12345) \coloneqq{}& F(12345) + F(23415) + F(34125) \\ &+ F(41235). \end{aligned} \tag{41}$$

In many ways this five-point function D resembles the four-point function d in Eq. (15). It vanishes on-shell, and has the following expression off-shell:

$$\begin{aligned} D(12345) ={}& c_{11}(-b_{23}b_{45} + 2b_{24}b_{35} - b_{25}b_{34}) \\ &+ c_{22}(-b_{34}b_{15} + 2b_{13}b_{45} - b_{14}b_{35}) \\ &+ c_{33}(-b_{12}b_{45} + 2b_{15}b_{24} - b_{14}b_{25}) \\ &+ c_{44}(-b_{15}b_{23} + 2b_{13}b_{25} - b_{12}b_{35}) \\ &- 2c_{55}(-b_{12}b_{34} + 2b_{13}b_{24} - b_{14}b_{23}). \end{aligned} \tag{42}$$

By construction, it is cyclically symmetric in its first four arguments, but it also possesses the following additional symmetries:

$$D(12345) = D(32145) = D(14325) = D(43215). \tag{43}$$

Diagrammatically, D is given by a sum of 12 diagrams of the form Fig. 12.

We are now ready to check the sum rules. Using Eq. (37) and the symmetry of F given in Eq. (27), one gets

$$\begin{aligned} R_1 &= -d_1 + d_2 + d_3 - d_4 + d_7 - d_{10} \\ &= -F(12345) + F(13245) + F(14235) - F(23154) + F(45321) - F(51432) \\ &= F(21534) - F(31452) - F(14532) + F(32154) - F(45312) + F(16324) \\ &= [D(21534)F(53214)] - [D(31452) - F(53142)] = D(21534) - D(31452) \simeq 0, \\ R_2 &= -d_1 + d_3 + d_7 - d_9 = -F(12345) + F(14235) + F(45321) - F(25431) \\ &= F(12543) + F(41253) + F(54123) + F(25413) = D(12543) \simeq 0, \\ R_3 &= -d_2 - d_3 - d_7 - d_8 = -F(13245) - F(14235) - F(45321) - F(35421) \\ &= F(13542) + F(41352) + F(54132) + F(35412) = D(13542) \simeq 0, \\ R_4 &= -d_1 + d_3 - d_4 - d_6 = -F(12345) + F(14235) - F(23154) - F(34512) \\ &= -F(12345) - F(41235) - F(23415) - F(34125) = -D(12345) \simeq 0, \\ R_5 &= d_2 + d_3 - d_4 - d_5 = F(13245) + F(14235) - F(23154) - F(24153) \\ &= F(31425) + F(14235) + F(23145) + F(42315) = D(31425) \simeq 0, \end{aligned} \tag{44}$$

where $\simeq$ means equality on-shell. Thus the sum rules are all true.

With these sum rules satisfied, one can proceed to solve Eq. (36). Since the rank of τ is five, only five t_l can be solved, leaving the other five t_l free. For example, solving for t_4, t_5, t_6, t_7, t_{10}, one gets

$$\begin{aligned} t_4 &= -d_3 - t_3 + t_8 - t_9, \\ t_5 &= d_8 - t_1 - t_3 + t_8, \\ t_6 &= -d_9 + t_2 - t_3 - t_9, \\ t_7 &= -d_3 - d_7 - t_1 + t_2 - t_3 + t_8 - t_9, \\ t_{10} &= d_4 + d_7 + t_1 - t_2 + t_3, \end{aligned} \tag{45}$$

where Eq. (40) has been used to simplify expressions.

Let us briefly summarize the results obtained so far for $N = 5$. The CK combinations $\bar{\Delta}_l$ computed from Feynman diagrams are not zero on-shell, though their $bccc$ terms all vanish. A generalized gauge transformation $\bar{n}_a \to n_a = \bar{n}_a - \delta n_a$ can restore the CK relation, namely, can bring them to a BCJ representation, if the induced change $\delta\Delta_l = \delta(n_s - n_t - n_u)$ is equal to $\bar{\Delta}_l$ for all l. In order for the gauge transformation to be local, the latter must be of the form $\delta n_a = p_a s_a + p'_a s'_a$, with the 30 parameters p_a, p'_a obeying 10 triplet identities, thereby leaving only 10 parameters t_l to accomplish the requirement $\delta\Delta_l = \bar{\Delta}_l$. This requirement has a solution only when the 10 $\bar{\Delta}_l$'s satisfy five sum rules $R_x = 0$ shown in Eq. (40). A detailed calculation using the explicit expressions of $\bar{\Delta}_l$ shows that these sum rules are indeed satisfied. With that, five of the ten t_l's can be solved, leaving the other five t_l's as free parameters. If t_1, t_2, t_3, t_8, t_9 are chosen as free parameters, then the solution for t_4, t_5, t_6, t_7, t_{10} is given in Eq. (45).

These 10 t_l move $\bar{n}_a$ to n_a, satisfying the ten CK equations $\Delta_l = 0$. The solution carries five free t parameters, which is a consequence of the well-known fact that any solution of the homogeneous equation can be added to a particular solution to get a new solution of the inhomogeneous equation. The homogeneous equation $\sum_{l'} \tau_{ll'} t^0_{l'} = 0$ has five independent solutions v_x given by Eq. (39), corresponding to the five free t_l parameters. The addition of v_x is therefore a further gauge transformation in the BCJ representation, keeping all CK relations intact. According to the discussion of Sec. II E, v_x must therefore be related to the null vectors u_i of the mass matrix m, but there are five v_x and only four u_i. Why do they not match?

D. Null vectors of τ and m

As discussed in Sec. II E, the $N = 5$ BCJ relations can be obtained from Eq. (6) to be

$$\begin{aligned}
0 &= A(13245)s_{13} + A(12345)(s_{13} + s_{23}) \\
&\quad + A(12435)(s_{13} + s_{23} + s_{34}), \\
0 &= A(13425)s_{13} + A(14325)(s_{13} + s_{34}) \\
&\quad + A(14235)(s_{13} + s_{23} + s_{34}), \\
0 &= A(14325)s_{14} + A(13425)(s_{14} + s_{34}) \\
&\quad + A(13245)(s_{14} + s_{24} + s_{34}), \\
0 &= A(12345)s_{12} + A(13245)(s_{12} + s_{23}) \\
&\quad + A(13425)(s_{12} + s_{23} + s_{24}).
\end{aligned} \tag{46}$$

These relations give rise to four null vectors of the on-shell propagator matrix m. They are

$$\begin{aligned}
u_1 &= (s_{13} + s_{23}, s_{13} + s_{23} + s_{34}, s_{13}, 0, 0, 0) \\
&= (s_{13} + s_{23}, -s_{124}, s_{13}, 0, 0, 0), \\
u_2 &= (0, 0, 0, s_{13}, s_{13} + s_{23} + s_{34}, s_{13} + s_{34}) \\
&= (0, 0, 0, s_{13}, -s_{124}, s_{13} + s_{34}), \\
u_3 &= (0, 0, s_{14} + s_{24} + s_{34}, s_{14} + s_{34}, 0, s_{14}) \\
&= (0, 0, -s_{123}, s_{14} + s_{34}, 0, s_{14}), \\
u_4 &= (s_{12}, 0, s_{12} + s_{23}, s_{12} + s_{23} + s_{24}, 0, 0) \\
&= (s_{12}, 0, s_{12} + s_{23}, -s_{134}, 0, 0).
\end{aligned} \tag{47}$$

These null vectors enable a generalized gauge transformation $\nu \to \nu - \delta\nu$ on the fundamental numerator factors in the form

$$\delta\nu = \delta \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \\ \nu_5 \\ \nu_6 \end{pmatrix} = \delta \begin{pmatrix} \nu(234) \\ \nu(243) \\ \nu(324) \\ \nu(342) \\ \nu(423) \\ \nu(432) \end{pmatrix} = \sum_{i=1}^{4} x_i u_i$$
$$= \begin{pmatrix} (x_4 - x_1)s_{12} + x_1 s_{123} \\ -x_1 s_{124} \\ (x_1 - x_4)s_{13} + (x_4 - x_3)s_{123} \\ (x_2 - x_3)s_{13} + (x_3 - x_4)s_{134} \\ -x_2 s_{124} \\ (x_3 - x_2)s_{14} + x_2 s_{134} \end{pmatrix}, \tag{48}$$

without altering any of the partial amplitudes given by Eq. (5). In Eq. (48), x_i are arbitrary parameters, and on-shell kinematical relations such as $s_{13} + s_{23} = s_{123} - s_{12}$ have been used. With the relation between n_a and $\nu(\beta)$ discussed in Sec. II E, changes to $\delta\nu(\beta)$ give rise to changes in δn_a. Let us examine the basic changes associated with half-ladder diagrams.

Recall from Sec. II E that the ordinary numerator n_a for a half-ladder diagram in $A(1\alpha N)$ is equal to the fundamental numerator $\nu(\alpha)$. With the enumeration scheme given in Appendix A, these half-ladder n_a are

$$(n_2, n_3, n_5, n_6, n_8, n_9) = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6). \tag{49}$$

Their gauge transformation $\delta n_a = p_a s_a + p'_a s'_a$ expressed in t_l are

$$\begin{aligned}
\delta n_2 &= p_2 s_{12} + p'_2 s_{123} = t_7 s_{12} + t_1 s_{123}, \\
\delta n_3 &= p_3 s_{12} + p'_3 s_{124} = t_8 s_{12} - t_1 s_{124}, \\
\delta n_5 &= p_5 s_{13} + p'_5 s_{123} = -t_7 s_{13} + t_2 s_{123}, \\
\delta n_6 &= p_6 s_{13} + p'_6 s_{134} = t_9 s_{13} - t_2 s_{134}, \\
\delta n_8 &= p_8 s_{14} + p'_8 s_{124} = -t_8 s_{14} + t_3 s_{124}, \\
\delta n_9 &= p_9 s_{12} + p'_9 s_{134} = -t_9 s_{12} - t_3 s_{134}.
\end{aligned} \tag{50}$$

The homogeneous part t^0_l causes change between the n_a's that satisfy the CK relations. For those n_a which are equal to some $\nu(\alpha)$, Eq. (50) with t_l replaced by t^0_l can be identified with Eq. (48). Therefore

$$
\begin{aligned}
&t_7^0 s_{12} + t_1^0 s_{123} = (x_4 - x_1) s_{12} + x_1 s_{123},\\
&t_8^0 s_{12} - t_1^0 s_{124} = -x_1 s_{124},\\
&t_7^0 s_{13} + t_2^0 s_{123} = (x_1 - x_4) s_{13} + (x_4 - x_3) s_{123},\\
&t_9^0 s_{13} - t_2^0 s_{134} = (x_2 - x_3) s_{13} + (x_3 - x_4) s_{134},\\
&t_8^0 s_{14} + t_3^0 s_{124} = -x_2 s_{124},\\
&t_9^0 s_{12} - t_3^0 s_{134} = (x_3 - x_2) s_{14} + x_2 s_{134}.
\end{aligned} \tag{51}
$$

This allows the identification

$$
\begin{aligned}
&t_1^0 = x_1, \qquad t_2^0 = x_4 - x_3, \qquad t_3^0 = -x_2,\\
&t_7^0 = x_4 - x_1, \qquad t_8^0 = 0, \qquad t_9^0 = x_2 - x_3.
\end{aligned} \tag{52}
$$

Let us compare these expressions for t_l^0 with those given by Eq. (45) after setting all $d_l = 0$. There are five free parameters in Eq. (45), $t_1^0, t_2^0, t_3^0, t_8^0, t_9^0$, through which the other five t_l^0 can be obtained. For example, $t_7^0 = -t_1^0 + t_2^0 - t_3^0 + t_8^0 - t_9^0$, which is consistent with Eq. (52). The other five t_l^0 in Eq. (52) should all be free, but that equation shows only four free parameters related to x_i, and $t_8^0 = 0$ is not free. How come? The reason is that t_8^0 is really not free. t_8 can be made to disappear if we make the replacement $t_4' = t_4 - t_8, t_5' = t_5 - t_8, t_7' = t_7 - t_8$ in Eq. (45) and Eq. (35).

E. Feynman and CHY numerators

Recall the discussion in Sec. II D about the structure of n_a. For $N = 5$, it contains terms of the form *bccc* and *bbcs*, each type possessing many different terms that cannot be combined. That observation of course applies both to $\bar{n}_a$ computed from Feynman diagrams, and n_a' computed from the CHY theory. Since the CK relation is satisfied by n_a', the difference $\delta n_a = \bar{n}_a - n_a'$ must take on the form $p_a s_a + p_a' s_a'$ if it is a local gauge transformation. In other words, their *bccc* terms must completely cancel, and the difference of their *bbcs* terms must involve only s_a and s_a', without any other s_I. Conversely, by computing $\bar{n}_a - n_a'$ explicitly, one can verify whether it is of the form $p_a s_a + p_a' s_a'$ and therefore whether δn_a is indeed a local gauge transformation.

Using the *Mathematica* program in Ref. [14] to compute n_a', and then $\bar{n}_a - n_a'$ explicitly, I have verified that the difference is indeed given by a local gauge transformation. Namely, δn_a is of the form $p_a s_a + p_a' s_a'$. The coefficients p_a and p_a' are rather complicated and will not be reproduced here. This provides an independent confirmation of the general assertion that a local gauge transformation can bring $\bar{n}_a$ into a set of n_a that satisfy the CK relation.

VI. SIX-POINT AMPLITUDE

A. Partial amplitudes and numerators

$N = 6$ can be dealt with in much the same way as $N = 5$, but everything is much lengthier. For that reason, many details are relegated to the appendixes.

There are now $4! = 24$ partial amplitudes, listed in Table IV of Appendix B. There are 25 single Mandelstam variables s_e listed in Table V, 105 double Mandelstam variables used to enumerate Δ_l, listed in Table VI, and 105 triple Mandelstam variables used to enumerate n_a/Q_a, listed in Table VII. Of the 105 possible terms n_a/Q_a, only 14 are contained in each partial amplitude. For example, the amplitude A_1 is given by

$$
\begin{aligned}
A_1 = A(123456) = &\frac{n_2}{s_{12}s_{34}s_{345}} + \frac{n_3}{s_{12}s_{34}s_{1234}} + \frac{n_7}{s_{12}s_{45}s_{123}} + \frac{n_8}{s_{12}s_{45}s_{345}} + \frac{n_{10}}{s_{12}s_{123}s_{1234}}\\
&+ \frac{n_{61}}{s_{23}s_{45}s_{123}} + \frac{n_{63}}{s_{23}s_{45}s_{2345}} + \frac{n_{64}}{s_{23}s_{123}s_{1234}} + \frac{n_{66}}{s_{23}s_{234}s_{1234}} + \frac{n_{67}}{s_{23}s_{234}s_{2345}}\\
&+ \frac{n_{90}}{s_{34}s_{234}s_{1234}} + \frac{n_{91}}{s_{34}s_{234}s_{2345}} + \frac{n_{93}}{s_{34}s_{345}s_{2345}} + \frac{n_{105}}{s_{45}s_{345}s_{2345}}.
\end{aligned} \tag{53}
$$

The others can be obtained by a suitable permutation of lines 2,3,4,5, together with the appropriate renaming of n_a, and a possible minus sign determined by the number of flips as discussed in Sec. II C.

Feynman diagrams for $\bar{n}_a$ are shown in Fig. 17 of Appendix B. On top of those, diagrams containing virtual vertices must be added, like those in the two bottom rows of Fig. 11. However, this time there are four additional rows, three containing a single dotted line, and one containing two dotted lines.

$N = 6$ is the first time when a 3g vertex with three *internal* lines appear. It can be seen in Fig. 17 that the 15 $\bar{n}_a$ with

$$
a \in I_a = \{3,6,9,18,21,24,33,36,39,48,51,54,63,72,81\} \tag{54}
$$

contain such an internal vertex, and the remaining 90 $\bar{n}_a$ with

$$a\in J_a=\{1,2,4,5,7,8,10,11,12,13,14,15,16,17,19,20,22,23,25,26,27,28,29,30,31,32,34,35,37,38,40,\\ 41,42,43,44,45,46,47,49,50,52,53,55,56,57,58,59,60,61,62,64,65,66,67,68,69,70,71,73,74,75,\\ 76,77,78,79,80,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,99,100,101,102,103,104,105\} \quad (55)$$

do not. Since two dotted lines are not allowed to intersect, those $\bar{n}_a$ in I_a do not contain any diagram with two virtual vertices, or two dotted lines, but those in J_a do.

B. CK combinations and trivial CK identities

The 105 CK combinations $\Delta_l = n_s - n_t - n_u$ are listed in Table VIII, with s, t, u specified in row B. The number without a bar on top is s, and the other two numbers with a bar on top are t and u. The Q_a of those three n_a all contain a common double Mandelstam variable, listed in row A. For example, $\Delta_1 = n_3 - n_2 - n_1$, and Q_3, Q_2, Q_1 all contain $s_{12}s_{34} = s_1s_8$, a fact that can be checked directly using Tables IV, V, VI.

The 105 Δ_l's can be divided into groups containing a single common Mandelstam variable s_e in every one of their n_a. These n_a occur twice in each group, enabling the Δ_l in the same group to be combined into trivial CK identities. These trivial CK identities are valid on-shell and off-shell because they are identically zero, no matter what n_a are. There is only one such relation for $N = 5$, but for $N = 6$, there are 25 s_e's, so there are 25 trivial CK identities. They are listed below, with the common s_e of the group shown in parenthesis.

$$\begin{aligned}
&0 = -\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - \Delta_5 + \Delta_6 - \Delta_7 - \Delta_8 + \Delta_9 - \Delta_{10}, (s_{12})\\
&0 = -\Delta_{11} + \Delta_{12} + \Delta_{13} + \Delta_{14} - \Delta_{15} + \Delta_{16} - \Delta_{17} - \Delta_{18} + \Delta_{19} - \Delta_{20}, (s_{13})\\
&0 = \Delta_{22} + \Delta_{23} + \Delta_{24} + \Delta_{26} + \Delta_{29} - (\Delta_{21} + \Delta_{25} + \Delta_{27} + \Delta_{28} + \Delta_{30}), (s_{14})\\
&0 = -\Delta_{31} + \Delta_{32} + \Delta_{33} + \Delta_{34} - \Delta_{35} + \Delta_{36} - \Delta_{37} - \Delta_{38} + \Delta_{39} - \Delta_{40}, (s_{15})\\
&0 = -\Delta_{21} + \Delta_{31} + \Delta_{41} + \Delta_{42} - \Delta_{43} - \Delta_{44} + \Delta_{45} - \Delta_{46} + \Delta_{47} - \Delta_{48}, (s_{23})\\
&0 = -(\Delta_{11} - \Delta_{32} - \Delta_{49} - \Delta_{50} + \Delta_{51} + \Delta_{52} - \Delta_{53} + \Delta_{54} - \Delta_{55} + \Delta_{56}), (s_{24})\\
&0 = -\Delta_{12} + \Delta_{22} + \Delta_{57} + \Delta_{58} - \Delta_{59} - \Delta_{60} + \Delta_{61} - \Delta_{62} + \Delta_{63} - \Delta_{64}, (s_{25})\\
&0 = \Delta_1 + \Delta_{33} + \Delta_{57} + \Delta_{65} - \Delta_{66} - \Delta_{67} + \Delta_{68} - \Delta_{69} - \Delta_{70} - \Delta_{71}, (s_{34})\\
&0 = \Delta_2 + \Delta_{23} + \Delta_{49} + \Delta_{72} - \Delta_{73} - \Delta_{74} + \Delta_{75} - \Delta_{76} - \Delta_{77} - \Delta_{78}, (s_{35})\\
&0 = \Delta_3 - \Delta_{13} - \Delta_{41} - \Delta_{79} + \Delta_{80} + \Delta_{81} + \Delta_{82} + \Delta_{83} - \Delta_{84} - \Delta_{85}, (s_{45})\\
&0 = -\Delta_4 + \Delta_{14} + \Delta_{42} - \Delta_{79} + \Delta_{86} - \Delta_{87}, (s_{123}) \qquad 0 = -\Delta_5 + \Delta_{24} + \Delta_{50} - \Delta_{72} + \Delta_{88} - \Delta_{89}, (s_{124})\\
&0 = -\Delta_6 + \Delta_{34} + \Delta_{58} - \Delta_{65} + \Delta_{90} - \Delta_{91}, (s_{125}) \qquad 0 = -\Delta_{15} + \Delta_{25} - \Delta_{59} + \Delta_{66} + \Delta_{92} - \Delta_{93}, (s_{134})\\
&0 = -\Delta_{16} + \Delta_{35} - \Delta_{51} + \Delta_{73} + \Delta_{94} - \Delta_{95}, (s_{135}) \qquad 0 = -\Delta_{26} + \Delta_{36} - \Delta_{43} + \Delta_{80} + \Delta_{96} - \Delta_{97}, (s_{145})\\
&0 = -\Delta_{37} - \Delta_{44} + \Delta_{52} + \Delta_{67} + \Delta_{98} - \Delta_{99}, (s_{234}) \qquad 0 = -\Delta_{27} - \Delta_{45} + \Delta_{60} + \Delta_{74} + \Delta_{100} - \Delta_{101}, (s_{235})\\
&0 = -\Delta_{17} - \Delta_{53} + \Delta_{61} + \Delta_{81} + \Delta_{102} - \Delta_{103}, (s_{245}) \qquad 0 = \Delta_7 - \Delta_{68} + \Delta_{75} + \Delta_{82} - \Delta_{104} - \Delta_{105}, (s_{345})\\
&0 = \Delta_8 - \Delta_{18} + \Delta_{28} - \Delta_{46} + \Delta_{54} + \Delta_{69} - \Delta_{86} + \Delta_{88} - \Delta_{92} - \Delta_{98}, (s_{1234})\\
&0 = \Delta_9 - \Delta_{19} + \Delta_{38} - \Delta_{47} + \Delta_{62} + \Delta_{76} - \Delta_{87} + \Delta_{90} - \Delta_{94} - \Delta_{100}, (s_{1235})\\
&0 = \Delta_{10} - \Delta_{29} + \Delta_{39} - \Delta_{55} + \Delta_{63} + \Delta_{83} - \Delta_{89} + \Delta_{91} - \Delta_{96} - \Delta_{102}, (s_{1245})\\
&0 = \Delta_{20} - \Delta_{30} + \Delta_{40} - \Delta_{70} + \Delta_{77} + \Delta_{84} - \Delta_{93} + \Delta_{95} - \Delta_{97} - \Delta_{104}, (s_{1345})\\
&0 = \Delta_{48} - \Delta_{56} + \Delta_{64} - \Delta_{71} + \Delta_{78} + \Delta_{85} - \Delta_{99} + \Delta_{101} - \Delta_{103} - \Delta_{105}. (s_{2345})
\end{aligned} \quad (56)$$

C. CK combinations for Feynman diagrams

Recall from Sec. II D that the $\bar{\Delta}_l$ terms are in the form $bcccc$, $bbccs$, $bbbss$. We shall see that the $bcccc$ terms are always zero on-shell, but not the other two kinds.

$\bar{\Delta}_l$ are computed from Feynman diagrams shown in Fig. 17 of Appendix B, plus similar diagrams containing virtual vertices. The computation is similar to those in Eq. (22), Eq. (23), and Eq. (30), but much lengthier. The result, shown in Eq. (B1) in Appendix B4, comes in two types, depending on whether $\bar{\Delta}_l$ contains an $\bar{n}_a$ in I_a or not. Those that do have an l index in the list

$$I_l = \{1,2,3,8,9,10,11,12,13,18,19,20,21,22,23,\\ 28,29,30,31,32,33,38,39,40,41,46,47,48,49,54,\\ 55,56,57,62,63,64,69,70,71,76,77,78,83,84,85\}, \tag{57}$$

and those that do not have an l index in the list

$$J_l = \{4,5,6,7,14,15,16,17,24,25,26,27,34,35,36,\\ 37,42,43,44,45,50,51,52,53,58,59,60,61,65,66,\\ 67,68,72,73,74,75,79,80,81,82,86,87,88,89,90,\\ 91,92,93,94,95,96,97,98,99,100,101,102,103,\\ 104,105\}. \tag{58}$$

I_l contains 45 members and J_l contains and 60 members. They are illustrated below with one example taken from each type.

First, consider $\bar{\Delta}_1 = \bar{n}_3 - \bar{n}_2 - \bar{n}_1$, which is an example of type I_l, with Feynman diagrams shown in Fig. 13. Using Fig. 9 and Eq. (15) for off-shell $N = 4$, the first four rows of Fig. 13 give

$$\begin{aligned}
&\text{Rows 1 and 4. } d(9856)T(\bar{8}34)T(\bar{9}12),\\
&\text{Row 2. } T(\bar{8}34)[f(12856) - f(12658) - f(12586)]s_{34}\\
&\quad = T(\bar{8}34)F(12568)s_{34},\\
&\text{Row 3. } T(\bar{9}12)[f(34965) - f(34569) + f(34659)]s_{12}\\
&\quad = T(\bar{9}12)F(34659)s_{12}.
\end{aligned} \tag{59}$$

To accommodate the fifth row with two dotted lines, a new function defined by Fig. 14 is needed. This function

$$\begin{aligned}
g(123456) &= Q(1239)Q(\bar{9}456)\\
&= b_{23}(b_{16}b_{45} - b_{15}b_{46})\\
&\quad + b_{13}(-b_{26}b_{45} + b_{25}b_{46})
\end{aligned} \tag{60}$$

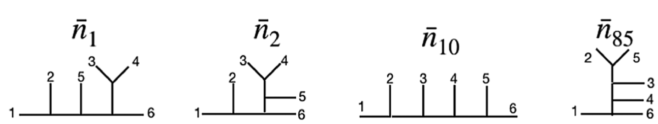

FIG. 13. Feynman diagrams for $\bar{\Delta}_1$.

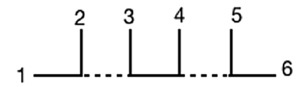

FIG. 14. $g(123456)$.

has the symmetry

$$g(123456) = -g(213456) = -g(123465) = g(654321), \tag{61}$$

and obeys the sum rule

$$g(123456) + g(123564) + g(123645) = 0. \tag{62}$$

What appears in $\bar{\Delta}_l$ is g_A, which is g with the two middle arguments antisymmetrized:

$$g_A(123456) = g(123456) - g(124356). \tag{63}$$

In addition to having the symmetry properties of g shown in Eq. (61), it also obeys

$$g_A(123456) = g_A(341256) = g_A(125634). \tag{64}$$

Putting these together, we get

$$\begin{aligned}
\bar{\Delta}_1 &= s_{12}F(12568)T(\bar{8}34)\\
&\quad + [s_{34}F(34659) + d(9856)T(\bar{8}34)]T(\bar{9}12)\\
&\quad + s_{12}s_{34}g_A(125634)\\
&\simeq s_{12}F(12568)T(\bar{8}34) + s_{34}F(34659)T(\bar{9}12)\\
&\quad + s_{12}s_{34}g_A(125634).
\end{aligned} \tag{65}$$

The second line follows because the $dTT \simeq 0$ on account of the Slavnov-Taylor identity.

Next, consider the type-J_l example $\bar{\Delta}_4 = \bar{n}_{10} - \bar{n}_7 - \bar{n}_{11}$, whose Feynman diagrams are shown in Fig. 15. The difference between this type and the I_l type can be spotted in the fifth row of the diagram. There are now three double-dotted diagrams instead of the previous two in type I_l.

In the present case, the diagrams in Fig. 15 give

$$\begin{aligned}
&\text{Rows 1 and 4. } d(9456)P(123\bar{9}),\\
&\text{Rows 2 and 5. } d(9456)Q(123\bar{9})s_{12},\\
&\text{Row 3. } T(\bar{9}12)[f(93456) - f(93654) - f(93546)]s_{123}\\
&\quad = T(\bar{9}12)F(93456)s_{123},
\end{aligned} \tag{66}$$

where $P(1234)$ is given in Fig. 9 and Fig. 16(a), with its bbs terms shown in Fig. 16(b). Collecting the results, we get

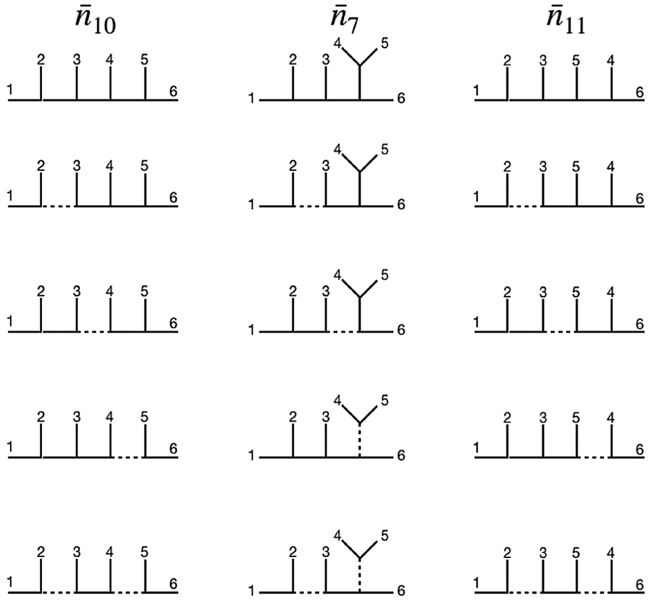

FIG. 15. Feynman diagrams for $\bar{\Delta}_4$.

FIG. 16. (a) the functions $P(1234)$, (b) the bbs part of $P(1234)$.

$$\begin{aligned}\bar{\Delta}_4 &= d(9456)[P(123\bar{9}) + Q(123\bar{9})s_{12}] \\ &\quad + T(\bar{9}12)F(93456)s_{123} \\ &= d(9456)A_4(123\bar{9}) + T(\bar{9}12)F(93456)s_{123}. \quad (67)\end{aligned}$$

Note that the double dotted diagrams in row 5 have conveniently been combined with those in row 2 to make things simple. As a result, there is no ss term in Eq. (67). A different way to explain the absence of the ss term is to appeal to Eq. (62), which shows that the three double dotted diagrams add up to zero.

By a direct calculation, one sees that the $bcccc$ terms of $d(9456)A_4(123\bar{9})$ add up to zero on-shell. This is expected from the numerator Slavnov-Taylor identity because the $bcccc$ terms of $A_4(123\bar{9})$ are the residue of $A(123\bar{9})$ at the pole $1/s_{12}$. The remaining on-shell $bbccs$ terms can be computed with the help of Eq. (13) to Eq. (16) to be

$$\begin{aligned}d(9456)A_4(123\bar{9}) &\simeq c_{99}T(456)A_4(123\bar{9}) \\ &\simeq -T(456)T(123)s_{12}. \quad (68)\end{aligned}$$

The full Slavnov-Taylor identity can be used to verify the correctness of this relation. The identity asserts that $A(1234)=A_4(1234)/s_{12}+A_4(2341)/s_{23}=A_4(1234)/s_{12} - A_4(2314)/s_{23}$ must be zero on-shell when ϵ_4 is replaced by k_4. As a consequence, $\epsilon_4 \to k_4$ on $A_4(1234)$ must produce a result proportional to s_{12}, with a proportionality constant invariant under a cyclic permutation of 1,2,3. This is precisely what Eq. (68) gives.

Caution should be exercised in using Eq. (68). $T(pqr)$ is given by Eq. (9), but there is no momentum conservation between the three lines p, q, r.

Substituting Eq. (68) into Eq. (67), one gets

$$\bar{\Delta}_4 \simeq -s_{12}T(123)T(456) + T(\bar{9}12)F(93456)s_{123}. \quad (69)$$

Let $d_e s_e + d'_e s'_e$ be the on-shell expression of those $\bar{\Delta}_l$ terms linear in s, when $\bar{\Delta}_l$ is labeled by the double Mandelstam variable $P_l = s_e s'_e$ with $s_e \prec s'_e$. Then d_e, d'_e can be read off from Eq. (B1) of Appendix B, together with manipulations similar to Eq. (65) and Eq. (69). The result is shown in Table IX of Appendix B.

D. Local gauge transformations

The vanishing of the $bcccc$ terms in $\bar{\Delta}_l$ leaves it proportional to s and ss. That gives hope that a *local* gauge transformation of the type given in Sec. II F,

$$\delta n_a = p_a s_a + p'_a s'_a + p''_a s''_a + q_a s'_a s''_a + q'_a s_a s''_a + q''_a s_a s'_a \quad (70)$$

may be able to bring $\bar{n}_a$ to a set of n_a satisfying the CK relation $\Delta_l = \bar{\Delta}_l - \delta\Delta_l = 0$. The purpose of this subsection and the next two is to investigate whether this hope is realized.

As discussed in Sec. II F, each of the p and the q parameters must satisfy a set of gauge constraint and a set of CK equations. It turns out that there are many p and many q equations. We shall find that neither the p equations nor the q equations have consistent solutions, hence the gauge transformation used to restore CK relation must be nonlocal. To verify this conclusion independently, and to understand why that happens, we shall also compute the numerators n'_a from the CHY theory. Since the CHY theory is known to satisfy the CK relations. this gauge transformation $\delta n_a = \bar{n}_a - n'_a$ must be nonlocal in order to agree with the conclusion above. This is indeed the case because δn_a contains s dependences beyond those allowed by Eq. (70).

E. p equations

The p parameters of Eq. (70) give rise to a change

$$\frac{\delta n_a}{Q_a} = \frac{p_a}{s'_a s''_a} + \frac{p'_a}{s_a s''_a} + \frac{p''_a}{s_a s'_a}. \quad (71)$$

Similar to the situation in $N = 5$, in order to keep all $\delta A_m = 0$, the $3 \times 105 = 315$ p parameters must form 105 triplets t_l, with the p parameters inside each triplet equal. As in $N = 5$, each triplet is a associated with a Δ_l, and these triplets are

$$
\begin{aligned}
&t_1 = (p''_3, -p''_2, -p''_1), \quad t_2 = (p''_6, -p''_5, -p''_4), \quad t_3 = (p''_7, -p''_8, -p''_9), \quad t_4 = (p''_{10}, -p'_7, -p''_{11}),\\
&t_5 = (p''_{12}, -p'_4, -p''_{13}), \quad t_6 = (p''_{14}, -p'_1, -p''_{15}), \quad t_7 = (p'_2, -p'_5, -p'_8), \quad t_8 = (p'_{10}, -p'_3, -p'_{12}),\\
&t_9 = (p'_{11}, -p'_6, -p'_{14}), \quad t_{10} = (p'_{13}, -p'_9, -p'_{15}), \quad t_{11} = (p''_{18}, -p''_{17}, -p''_{16}), \quad t_{12} = (p''_{21}, -p''_{20}, -p''_{19}),\\
&t_{13} = (p''_{22}, -p''_{23}, -p''_{24}), \quad t_{14} = (p''_{25}, -p'_{22}, -p''_{26}), \quad t_{15} = (p''_{27}, -p'_{19}, -p''_{28}), \quad t_{16} = (p''_{29}, -p'_{16}, -p''_{30}),\\
&t_{17} = (p'_{17}, -p'_{20}, -p'_{23}), \quad t_{18} = (p'_{25}, -p'_{18}, -p'_{27}), \quad t_{19} = (p'_{26}, -p'_{21}, -p'_{29}), \quad t_{20} = (p'_{28}, -p'_{24}, -p'_{30}),\\
&t_{21} = (p''_{33}, -p''_{32}, -p''_{31}), \quad t_{22} = (p''_{36}, -p''_{35}, -p''_{34}), \quad t_{23} = (p''_{37}, -p''_{38}, -p''_{39}), \quad t_{24} = (p''_{40}, -p'_{37}, -p''_{41}),\\
&t_{25} = (p''_{42}, -p'_{34}, -p''_{43}), \quad t_{26} = (p''_{44}, -p'_{31}, -p''_{45}), \quad t_{27} = (p'_{32}, -p'_{35}, -p'_{38}), \quad t_{28} = (p'_{40}, -p'_{33}, -p'_{42}),\\
&t_{29} = (p'_{41}, -p'_{36}, -p'_{44}), \quad t_{30} = (p'_{43}, -p'_{39}, -p'_{45}), \quad t_{31} = (p''_{48}, -p''_{47}, -p''_{46}), \quad t_{32} = (p''_{51}, -p''_{50}, -p''_{49}),\\
&t_{33} = (p''_{52}, -p''_{53}, -p''_{54}), \quad t_{34} = (p''_{55}, -p'_{52}, -p''_{56}), \quad t_{35} = (p''_{57}, -p'_{49}, -p''_{58}), \quad t_{36} = (p''_{59}, -p'_{46}, -p''_{60}),\\
&t_{37} = (p'_{47}, -p'_{50}, -p'_{53}), \quad t_{38} = (p'_{55}, -p'_{48}, -p'_{57}), \quad t_{39} = (p'_{56}, -p'_{51}, -p'_{59}), \quad t_{40} = (p'_{58}, -p'_{54}, -p'_{60}),\\
&t_{41} = (p''_{61}, -p''_{62}, -p''_{63}), \quad t_{42} = (p''_{64}, -p'_{61}, -p''_{65}), \quad t_{43} = (p_{31}, -p_{46}, -p'_{62}), \quad t_{44} = (p''_{66}, -p_{47}, -p''_{67}),\\
&t_{45} = (p''_{68}, -p_{32}, -p''_{69}), \quad t_{46} = (p'_{64}, -p_{33}, -p'_{66}), \quad t_{47} = (p'_{65}, -p_{48}, -p'_{68}), \quad t_{48} = (p'_{67}, -p'_{63}, -p'_{69}),\\
&t_{49} = (p''_{70}, -p''_{71}, -p''_{72}), \quad t_{50} = (p''_{73}, -p'_{70}, -p''_{74}), \quad t_{51} = (p_{16}, -p_{49}, -p'_{71}), \quad t_{52} = (p''_{75}, -p_{50}, -p''_{76}),\\
&t_{53} = (p''_{77}, -p_{17}, -p''_{78}), \quad t_{54} = (p'_{73}, -p_{18}, -p'_{75}), \quad t_{55} = (p'_{74}, -p_{51}, -p'_{77}), \quad t_{56} = (p'_{76}, -p'_{72}, -p'_{78}),\\
&t_{57} = (p''_{79}, -p''_{80}, -p''_{81}), \quad t_{58} = (p''_{82}, -p'_{79}, -p''_{83}), \quad t_{59} = (p_{19}, -p_{34}, -p'_{80}), \quad t_{60} = (p''_{84}, -p_{35}, -p''_{85}),\\
&t_{61} = (p''_{86}, -p_{20}, -p''_{87}), \quad t_{62} = (p'_{82}, -p_{21}, -p'_{84}), \quad t_{63} = (p'_{83}, -p_{36}, -p'_{86}), \quad t_{64} = (p'_{85}, -p'_{81}, -p'_{87}),\\
&t_{65} = (p_1, -p_{52}, -p_{79}), \quad t_{66} = (p''_{88}, -p_{80}, -p''_{89}), \quad t_{67} = (p''_{90}, -p_{53}, -p''_{91}), \quad t_{68} = (p_2, -p''_{92}, -p''_{93}),\\
&t_{69} = (p_3, -p'_{88}, -p'_{90}), \quad t_{70} = (p'_{89}, -p_{54}, -p'_{92}), \quad t_{71} = (p'_{91}, -p_{81}, -p'_{93}), \quad t_{72} = (p_4, -p_{37}, -p_{70}),\\
&t_{73} = (p''_{94}, -p_{71}, -p''_{95}), \quad t_{74} = (p''_{96}, -p_{38}, -p''_{97}), \quad t_{75} = (p_5, -p''_{98}, -p''_{99}), \quad t_{76} = (p_6, -p'_{94}, -p'_{96}),\\
&t_{77} = (p'_{95}, -p_{39}, -p'_{98}), \quad t_{78} = (p'_{97}, -p_{72}, -p'_{99}), \quad t_{79} = (p_7, -p_{22}, -p_{61}), \quad t_{80} = (p''_{100}, -p_{62}, -p''_{101}),\\
&t_{81} = (p''_{102}, -p_{23}, -p''_{103}), \quad t_{82} = (p_8, -p''_{104}, -p''_{105}), \quad t_{83} = (p_9, -p'_{100}, -p'_{102}), \quad t_{84} = (p_{24}, -p'_{101}, -p'_{104}),\\
&t_{85} = (p_{63}, -p'_{103}, -p'_{105}), \quad t_{86} = (p_{10}, -p_{25}, -p_{64}), \quad t_{87} = (p_{11}, -p_{26}, -p_{65}), \quad t_{88} = (p_{12}, -p_{40}, -p_{73}),\\
&t_{89} = (p_{13}, -p_{41}, -p_{74}), \quad t_{90} = (p_{14}, -p_{55}, -p_{82}), \quad t_{91} = (p_{15}, -p_{56}, -p_{83}), \quad t_{92} = (p_{27}, -p_{42}, -p_{88}),\\
&t_{93} = (p_{28}, -p_{43}, -p_{89}), \quad t_{94} = (p_{29}, -p_{57}, -p_{94}), \quad t_{95} = (p_{30}, -p_{58}, -p_{95}), \quad t_{96} = (p_{44}, -p_{59}, -p_{100}),\\
&t_{97} = (p_{45}, -p_{60}, -p_{101}), \quad t_{98} = (p_{66}, -p_{75}, -p_{90}), \quad t_{99} = (p_{67}, -p_{76}, -p_{91}), \quad t_{100} = (p_{68}, -p_{84}, -p_{96}),\\
&t_{101} = (p_{69}, -p_{85}, -p_{97}), \quad t_{102} = (p_{77}, -p_{86}, -p_{102}), \quad t_{103} = (p_{78}, -p_{87}, -p_{103}), \quad t_{104} = (p_{92}, -p_{98}, -p_{104}),\\
&t_{105} = (p_{93}, -p_{99}, -p_{105}). \qquad (72)
\end{aligned}
$$

In particular, the t_1 triplet contains p parameters associated with n_3, n_2, n_1, all components of Δ_1, and the t_4 triplet contains p parameters associated with n_{10}, n_7, n_{11}, all components of Δ_4. The equality of triplet parameters comes from the requirement $\delta A_m = 0$. For example, in order for $\delta A_1 = 0$, all $\delta n_a/Q_a$ terms must cancel. To have the coefficients of $1/s_{12}s_{34}$ cancel, $p''_3 = -p''_2$ is required. To have the coefficients of $1/s_{12}s_{123}$ to cancel, $p''_{10} = -p'_7$ is required.

In order to have $\Delta_l = \bar{\Delta}_l - \delta\Delta_l = 0$ on-shell for every l, the p parameters must obey a number of CK equations. For example,

$$
\begin{aligned}
\delta\Delta_1 &= \delta(n_3 - n_2 - n_1)\\
&= s_{12}(p_3 - p_2 - p_1) + s_{34}(p'_3 - p'_2 - p'_1) + (p''_3 s_{1234} - p''_2 s_{345} - p''_1 s_{125})\\
&= s_{12}(t_{69} - t_{68} - t_{65}) + s_{34}(-t_8 - t_7 + t_6) + t_1(s_{1234} + s_{345} + s_{125})\\
&= s_{12}(t_{69} - t_{68} - t_{65}) + s_{34}(-t_8 - t_7 + t_6) + t_1(s_{12} + s_{34})\\
&= s_{12}(t_{69} - t_{68} - t_{65} + t_1) + s_{34}(-t_8 - t_7 + t_6 + t_1), \qquad (73)
\end{aligned}
$$

where Eq. (72) and Eq. (34) have been used to get to the final result.

Excluding the ss term which is associated with the q parameter, the on-shell terms in $\bar{\Delta}_1$ shown in Eq. (65) are

$$\bar{\Delta}_1 \simeq s_{12}F(12569)T(\bar{9}34) + s_{34}F(34659)T(\bar{9}12) + O(s^2).$$

A matching of the s-dependent coefficients of Eq. (73) and Eq. (65) yields

$$t_{69} - t_{68} - t_{65} + t_1 = F(12569)T(\bar{9}34) \coloneqq d_1,$$
$$-t_8 - t_7 + t_6 + t_1 = F(34659)T(\bar{9}12) \coloneqq d_1', \quad (74)$$

which are the two equations associated with the requirement $\delta\Delta_1 = \bar{\Delta}_1$. Similarly,

$$\begin{aligned}\delta\Delta_4 &= \delta(n_{10} - n_7 - n_{11}) \\ &= s_{12}(p_{10} - p_7 - p_{11}) + s_{123}(p'_{10} - p''_7 - p''_{11}) + (p''_{10}s_{1234} - p'_7 s_{45} - p'_{11}s_{1235}) \\ &= s_{12}(t_{86} - t_{79} - t_{87}) + s_{123}(t_8 - t_3 - t_9) + t_4(s_{1234} + s_{45} + s_{1235}) \\ &= s_{12}(t_{86} - t_{79} - t_{87}) + s_{123}(t_8 - t_3 - t_9) + t_4 s_{123} \\ &= s_{12}(t_{86} - t_{79} - t_{87} + t_4) + s_{123}(t_8 - t_3 - t_9). \end{aligned} \quad (75)$$

This differs from $\delta\Delta_1$ in that t_4 is added to one of the s terms in the last line, but not the other s term. On the other hand, Eq. (69) gives

$$\bar{\Delta}_4 \simeq -s_{12}T(123)T(456) + T(\bar{9}12)F(93456)s_{123} + O(s^2).$$

A matching of the s-dependent coefficients of Eq. (75) and Eq. (69) yields

$$t_{86} - t_{79} - t_{87} + t_4 = -T(123)T(456) \coloneqq d_4,$$
$$t_8 - t_3 - t_9 = T(\bar{9}12)F(93456) \coloneqq d_4'. \quad (76)$$

More generally, let the $\bar{\Delta}_l$ terms linear in s to be designated as $d_e s_e + d'_e s'_e$, where $P_l = s_e s'_e$ is the double Mandelstam variable specifying $\bar{\Delta}_l$, with $s_e \prec s'_e$. The $2\times 105 = 210$ equations for $\delta\Delta_l = \bar{\Delta}_l$ are listed in Appendix C. For $l \in I_l$, the equations resemble Eq. (74). For $l \in J_l$, they resemble Eq. (76).

These equations can written in a matrix form similar to Eq. (36),

$$\sum_{i'=1}^{105} (\tau)_{ii'} t_{i'} = \bar{d}_i, \quad (77)$$

where $i = (2l-1, 2l)$ with $1 \le l \le 105$, and $\bar{d}_i = (d_e, d'_e)$.

With 105 unknowns t_l and 210 equations in Eq. (77), one might expect to have no solution unless the 210×105 matrix τ is highly degenerate. In fact, it is, with a rank 79. It has 131 left null vectors $u_x (1 \le x \le 131)$ so that $\sum_{i=1}^{210} (u_x)_i (\tau)_{ii'} = 0$. As a result, Eq. (77) has a solution only when all the following 131 sum rules are satisfied:

$$R_x = \sum_{i=1}^{210} (u_x)_i \bar{d}_i = 0. \quad (78)$$

Some of these sum rules have two terms, some four terms, some six terms, but some have many more terms. The shorter sum rules are often satisfied, but the longer ones are generally not. As a result, Eq. (77) has no solution. Details are shown in Appendix C.

F. q equations

The q parameters of Eq. (70) give rise to a change

$$\frac{\delta n_a}{Q_a} = \frac{q_a}{s_a} + \frac{q'_a}{s'_a} + \frac{q''_a}{s''_a}. \quad (79)$$

To be a generalized gauge transformation, it is necessary to keep all $\delta A_m = 0$, thus the $3 \times 105 = 315$ q parameters must satisfy many gauge constraint equations. Unlike the p parameters, the constraint equations for q are much more complicated. For example, in order to cancel the $1/s_{12}$ terms in δA_1, it follows from Eq. (53) that one needs

$$q_2 + q_3 + q_7 + q_8 + q_{10} = 0, \quad (80)$$

and in order to cancel the $1/s_{345}$ terms in δA_1, one needs

$$q''_2 + q''_8 + q'_{93} + q'_{105} = 0. \quad (81)$$

Since each n_a is often contained in several A_m's, these equations are generally coupled. If we go through the cancellation of every $1/s$ term for every δA_m, then there are 144 such homogeneous equations for the $315 q$ parameters to satisfy. These 144 constraint equations are listed in Eq. (D1) in Appendix D.

On top of that, q must also satisfy the many CK equations implementing $\delta\Delta_l = \bar{\Delta}_l$. For example, the equation for $\delta\Delta_1 = \bar{\Delta}_1$ is

$$\begin{aligned}\bar{\Delta}_1 &= s_{12}s_{34}g_A(125634) \coloneqq s_{12}s_{34}g_1 \\ &= \delta\Delta_1 = \delta(n_3 - n_2 - n_1) \\ &= (q''_3 - q''_2 - q''_1)s_{12}s_{34} + s_{12}(q'_3 s_{1234} - q'_2 s_{345} - q'_1 s_{125}) \\ &\quad + s_{34}(q_3 s_{1234} - q_2 s_{345} - q_1 s_{125}). \end{aligned} \quad (82)$$

For the s dependence of $\delta\Delta_1$ and $\bar{\Delta}_1$ to match, one way is put $q_1 = q_1' = q_2 = q_2' = q_3 = q_3' = 0$ so that the last two terms vanish. There is however potentially a more general solution. Using the on-shell version of the kinematical identity Eq. (34),

$$s_{1234} + s_{345} + s_{125} = s_{12} + s_{34},$$

the last line can be reduced to

$$(q_3'' - q_2'' - q_1'')s_{12}s_{34} + q_3' s_{12}(s_{12} + s_{34}) + q_3 s_{34}(s_{12} + s_{34}),$$

provided $q_3' = -q_2' = -q_1'$ and $q_3 = -q_2 = -q_1$, but *a priori* neither set has to vanish. Unfortunately this manipulation does not help because it produces extra terms proportional to s_{12}^2 and s_{34}^2, which could be avoided only when $q_3' = q_3 = 0$. In other words, in order to satisfy Eq. (82), it is necessary to have

$$g_1 = g_A(125634) = q_3'' - q_2'' - q_1'', \quad \text{and} \tag{83}$$

$$0 = q_3' = q_3 = q_2' = q_2 = q_1' = q_1. \tag{84}$$

Although this manipulation does not pan out here, it does work for $\delta\Delta_4 = \bar{\Delta}_4$:

$$\begin{aligned} \bar{\Delta}_4 = 0 \coloneqq s_{12}s_{123}g_4 &= \delta(n_{10} - n_7 - n_{11}) \\ &= s_{12}s_{123}(q_{10}'' - q_7' - q_{11}'') \\ &\quad + s_{12}(q_{10}' s_{1234} - q_7'' s_{45} - q_{11}' s_{1235}) \\ &\quad + s_{123}(q_{10}s_{1234} - q_7 s_{45} - q_{11}s_{1235}). \end{aligned} \tag{85}$$

Again one requires $q_{10}' = -q_7'' = -q_{11}'$ and $q_{10} = -q_7 = -q_{11}$ to be able to combine the extra s terms, but this time

$$s_{1234} + s_{45} + s_{1235} = s_{123} \tag{86}$$

results in one term instead of two, thus avoiding the quadratic s terms appearing above. As a result, as long as $q_{10} = -q_7 = -q_{11} = 0$, the s dependence would match, Eq. (85) would be satisfied if

$$g_4 = 0 = q_{10}'' - q_7' - q_{11}'' + q_{10}', \tag{87}$$

$$0 = q_{10} = -q_7 = -q_{11}, \tag{88}$$

$$q_{10}' = -q_7'' = -q_{11}'. \tag{89}$$

The parameters in the last line must be equal but they do not have to vanish.

In general, the q parameters for $\delta\Delta_l = \bar{\Delta}_l$ need to satisfy a relation similar to those satisfied by $\bar{\Delta}_1$ if $\bar{\Delta}_l$ is of type I_l, or a relation similar to those satisfied by $\bar{\Delta}_4$ if $\bar{\Delta}_l$ is type J_l. These relations come in three groups: X, Y, Z. Group X consists of equations like Eq. (83) and Eq. (87), group Y consists of equations like Eq. (89), with equal q parameters but not necessarily zero, and group Z consists of equations like Eq. (84) and Eq. (87), with zero q parameters. The number of equations in groups X, Y, Z are respectively (105, 120, 225). They are listed in Appendix D.

On top of these, remember that there are also 144 equations derived from the gauge constraints to be satisfied. With so many equations and only $3 \times 105 = 315$ unknowns q_a, q_a', q_a'', one might expect to have no solutions. This is indeed the case. Details are shown in Appendix D.

G. Nonlocal gauge transformation between Feynman and CHY numerators

In order to confirm the conclusion that there is no *local* gauge transformation moving the Feynman numerators $\bar{n}_a$ to a set of n_a satisfying the CK relations, because neither the p equations nor the q equations have solutions, and in order to understand why this is so, we calculate explicitly the CHY fundamental numerator factor $\nu(2345)$. It is equal to the CHY ordinary numerator factor n_{10}', so the gauge transformation $\delta n_{10} = \bar{n}_{10} - n_{10}'$ should be nonlocal in order to agree with the general conclusion. If local, the terms of δn_{10} linear in s may depend only on $s_{12}, s_{123} = s_{12} + s_{13} + s_{23}$, and $s_{1234} = s_{56}$. Its terms quadratic in s may depend on the three nondiagonal quadratic products of these three s, and its s-independent (or $bcccc$) terms must be zero. If any of these is violated, δn_{10} is nonlocal.

Using the *Mathematica* program of Ref. [14] to compute $n_{10}' = \nu(2345)$, and Feynman rules to compute $\bar{n}_{10}$, one indeed gets no $bcccc$ terms in the resulting δn_{10}. However, its linear s dependence and its quadratic s dependence are much more complicated then those terms allowed above, so δn_{10} is indeed nonlocal.

VII. CONCLUSION

Numerator factors $\bar{n}_a$ and CK combinations $\bar{\Delta}_l = \bar{n}_s - \bar{n}_t - \bar{n}_u$ have been computed from the Feynman diagrams of N-particle pure gluon amplitudes. To do so, four-gluon vertices must first be converted into a pair of virtual cubic vertices. The necessity for virtual vertices to come in pairs implies that their effect on $\bar{\Delta}_l$ cannot be fully realized until at least $N = 6$.

For $N = 4$, the CK relation $\bar{\Delta} = 0$ is valid on-shell.

For $N = 5$, there is one trivial CK identity and nine nontrivial CK combinations which are not zero even on-shell. However, a local (generalized) gauge transformation can convert every $\bar{\Delta}_l$ into some Δ_l that satisfies the CK relation, provided five sum rules involving Feynman diagrams with one virtual vertex are obeyed. Explicit calculation shows that the sum rules are fulfilled, thus proving that $\bar{\Delta}_l$ can obey CK relations through a local gauge transformation. This result is verified by computing the difference between Feynman and CHY numerator factors.

For $N = 6$, there are 25 trivial CK identities and 80 nontrivial combinations $\bar{\Delta}_l$ which are not zero on-shell. After a lengthy calculation, one concludes that there is no local (generalized) gauge transformation that can render the

CK relations valid. This conclusion is confirmed also by computing the difference between the Feynman and CHY numerator factors.

Locality is a fundamental attribute of quantum field theory that one would like to preserve. For scalar amplitudes, locality is often deemed to be maintained when a scattering amplitude displays the same propagators as the Feynman tree amplitude. Strictly speaking this is correct only when it is also true off-shell. For a gluon theory, that is not sufficient, for the Yang-Mills coupling with local gauge invariance are also reflected in the numerator factors. Even with a local redefinition of fields or the insertion of total derivative terms, one would still expect the numerator factors to be polynomials of ϵ_i and k_i if the S-matrix theory preserves locality. Since Feynman diagrams derived from the Yang-Mills Lagrangian are by definition local, the gauge transformation between it and the S-matrix theory must also be local. The failure to do so for the CHY gluon theory at $N = 6$ raises the possibility that the inherent interaction in the CHY theory may not be local. The difficulty of extending the CHY gluon amplitude off-shell may be another circumstantial evidence suggesting its nonlocality.

The CHY scalar theories can be extended off-shell to agree with the Feynman amplitudes, and to keep its signature Möbius invariance intact [22,23]. The CHY gluon amplitude can also be extended off-shell maintaining Möbius invariance [13], but that extension does not agree with the off-shell Feynman amplitude, and it does not satisfy the Slavnov-Taylor identity.

It is conceivable that the nonlocality discussed above indeed reflects an inherently nonlocal interaction mechanism in S-matrix theories such as the CHY theory, but it is also possible that this nonlocality can be fixed by an ordinary gauge change, say from the Feynman gauge to an R_ξ gauge. Since both the CK relation and the CHY theory have the ability to obtain the local Einstein theory by doubling the local Yang-Mills theory, at least on-shell, it seems that they ought to remain local, so the apparent nonlocality found for $N = 6$ may just be a gauge artifact. However, further investigation is required to clarify and to determine the true meaning of the nonlocality found in this article.

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APPENDIX A: $N = 5$

The canonical ordering of the 10 Mandelstam variables $s_e = s_I$ is shown in Table I. These are all the s_I that enter into the double Mandelstam-variable expression for Q_a, but they are not all independent. For example, $s_{123} = s_{12} + s_{13} + s_{23}$.

The canonical ordering of the 15 double Mandelstam variables $Q_a = s_{e_1} s_{e_2}$ is shown in Table II.

TABLE I. Ordered Mandelstam variables for $N = 5$.

e	1	2	3	4	5	6	7	8	9	10
s_I	s_{12}	s_{13}	s_{14}	s_{23}	s_{24}	s_{34}	s_{123}	s_{124}	s_{134}	s_{234}

TABLE II. Ordered double Mandelstam variables for $N = 5$.

a	1	2	3	4	5
e_1, e_2	1, 6	1, 7	1, 8	2, 5	2, 7
a	6	7	8	9	10
$e1$, $e2$	2, 9	3, 4	3, 8	3, 9	4, 7
a	11	12	13	14	15
$e1$, $e2$	4, 10	5, 8	5, 10	6, 9	6, 10

TABLE III. n_a correspondence in Eq. (21) and in Ref. [3].

Eq. (21)	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}
Ref. [3]	n_3	n_1	n_{12}	$-n_{10}$	n_{15}	n_9	$-n_7$	n_{14}	n_6	n_4	n_2	n_{13}	n_{11}	$-n_8$	n_5

A dictionary relating the numerator factor n_a used in Eq. (21) and those used in Ref. [3] is given in Table III.

APPENDIX B: $N = 6$

1. Canonical ordering

Table IV gives the canonical ordering of the partial amplitudes $A_m = A(1\alpha N)$. The canonical ordering of the 25 single Mandelstam variables $s_e = s_I$ is shown in Table V. These are all the s_I that enter into the double Mandelstam-variable expression for P_l and the triple Mandelstam-variable expression for Q_a, but they are not all independent. For example, $s_{123} = s_{12} + s_{13} + s_{23}$. The canonical ordering of the 105 double Mandelstam variables $P_l = s_{e_1} s_{e_2}$ is shown in Table VI. The canonical ordering of the 105 triple Mandelstam variables $Q_a = s_{e_1} s_{e_2} s_{e_3}$ is shown in Table VII.

2. Feynman diagrams

Figure 17 shows the 105 diagrams whose propagators are $1/Q_a$.

TABLE IV. Numbering of the partial amplitudes.

m	1	2	3	4	5	6	7	8
$A(1\alpha N)$	$A(123456)$	$A(123546)$	$A(124356)$	$A(124536)$	$A(125346)$	$A(125436)$	$A(132456)$	$A(132546)$

m	9	10	11	12	13	14	15	16
$A(1\alpha N)$	$A(134256)$	$A(134526)$	$A(135246)$	$A(135426)$	$A(142356)$	$A(142536)$	$A(143256)$	$A(143526))$

m	17	18	19	20	21	22	23	24
$A(1\alpha N)$	$A(145236)$	$A(145326)$	$A(152346)$	$A(152436)$	$A(153246)$	$A(153426)$	$A(154236)$	$A(154326)$

TABLE V. Ordered Mandelstam variables $s_e = s_I$ for $N = 6$.

e	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
s_I	s_{12}	s_{13}	s_{14}	s_{15}	s_{23}	s_{24}	s_{25}	s_{34}	s_{35}	s_{45}	s_{123}	s_{124}	s_{125}	s_{134}	s_{135}

e	16	17	18	19	20	21	22	23	24	25
s_I	s_{145}	s_{234}	s_{235}	s_{245}	s_{345}	s_{1234}	s_{1235}	s_{1245}	s_{1345}	s_{2345}

TABLE VI. Ordered double Mandelstam variables $P_l = s_{e_1} s_{e_2}$ for $N = 6$.

l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
e_1, e_2	1, 8	1, 9	1, 10	1, 11	1, 12	1, 13	1, 20	1, 21	1, 22	1, 23	2, 6	2, 7	2, 10	2, 11	2, 14

l	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
e_1, e_2	2, 15	2, 19	2, 21	2, 22	2, 24	3, 5	3, 7	3, 9	3, 12	3, 14	3, 16	3, 18	3, 21	3, 23	3, 24

l	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
e_1, e_2	4, 5	4, 6	4, 8	4, 13	4, 15	4, 16	4, 17	4, 22	4, 23	4, 24	5, 10	5, 11	5, 16	5, 17	5, 18

l	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
e_1, e_2	5, 21	5, 22	5, 25	6, 9	6, 12	6, 15	6, 17	6, 19	6, 21	6, 23	6, 25	7, 8	7, 13	7, 14	7, 18

l	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
e_1, e_2	7, 19	7, 22	7, 23	7, 25	8, 13	8, 14	8, 17	8, 20	8, 21	8, 24	8, 25	9, 12	9, 15	9, 18	9, 20

l	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90
e_1, e_2	9, 22	9, 24	9, 25	10, 11	10, 16	10, 19	10, 20	10, 23	10, 24	10, 25	11, 21	11, 22	12, 21	12, 23	13, 22

l	91	92	93	94	95	96	97	98	99	100	101	102	103	104	105
e_1, e_2	13, 23	14, 21	14, 24	15, 22	15, 24	16, 23	16, 24	17, 21	17, 25	18, 22	18, 25	19, 23	19, 25	20, 24	20, 25

TABLE VII. Ordered triple Mandelstam variables $Q_a = s_{e_1}s_{e_2}s_{e_3}$ for $N = 6$.

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$e_1e_2e_3$	1, 8, 13	1, 8, 20	1, 8, 21	1, 9, 12	1, 9, 20	1, 9, 22	1, 10, 11	1, 10, 20	1, 10, 23	1, 11, 21	1, 11, 22	1, 12, 2	1, 12, 23	1, 13, 22	1, 13, 23

a	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$e_1e_2e_3$	2, 6, 15	2, 6, 19	2, 6, 21	2, 7, 14	2, 7, 19	2, 7, 22	2, 10, 11	2, 10, 19	2, 10, 24	2, 11, 21	2, 11, 22	2, 14, 21	2, 14, 24	2, 15, 22	2, 15, 24

a	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
$e_1e_2e_3$	3, 5, 16	3, 5, 18	3, 5, 21	3, 7, 14	3, 7, 18	3, 7, 23	3, 9, 12	3, 9, 18	3, 9, 24	3, 12, 21	3, 12, 23	3, 14, 21	3, 14, 24	3, 16, 23	3, 16, 24

a	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
$e_1e_2e_3$	4, 5, 16	4, 5, 17	4, 5, 22	4, 6, 15	4, 6, 17	4, 6, 23	4, 8, 13	4, 8, 17	4, 8, 24	4, 13, 22	4, 13, 23	4, 15, 22	4, 15, 24	4, 16, 23	4, 16, 24

a	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
$e_1e_2e_3$	5, 10, 11	5, 10, 16	5, 10, 25	5, 11, 21	5, 11, 22	5, 17, 21	5, 17, 25	5, 18, 22	5, 18, 25	6, 9, 12	6, 9, 15	6, 9, 25	6, 12, 21	6, 12, 23	6, 17, 21

a	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90
$e_1e_2e_3$	6, 17, 25	6, 19, 23	6, 19, 25	7, 8, 13	7, 8, 14	7, 8, 25	7, 13, 22	7, 13, 23	7, 18, 22	7, 18, 25	7, 19, 23	7, 19, 25	8, 14, 21	8, 14, 24	8, 17, 21

a	91	92	93	94	95	96	97	98	99	100	101	102	103	104	105
$e_1e_2e_3$	8, 17, 25	8, 20, 24	8, 20, 25	9, 15, 22	9, 15, 24	9, 18, 22	9, 18, 25	9, 20, 24	9, 20, 25	10, 16, 23	10, 16, 24	10, 19, 23	10, 19, 25	10, 20, 24	10, 20, 25

3. CK combinations

Table VIII gives the 105 CK combinations $\Delta_l = n_s - n_t - n_u$. Row A shows the double Mandelstam variable $s_{e_1}s_{e_2}$ common to Q_s, Q_t, Q_u. Row B shows the a's of the three n_a making up of n_s, n_t, n_u. The one without a bar on top is n_s, the other two are n_t, n_u or n_u, n_t. For example, the table tells us that $\Delta_1 = n_3 - n_1 - n_2$, and that Q_3, Q_1, Q_2 all contain the common factor $s_{12}s_{34}$.

4. $\bar{\Delta}_l$

Using Table VIII and Fig. 17, the CK combinations $\bar{\Delta}_l$ for the Feynman diagrams can be calculated and are shown in Eq. (B1). The coefficients d_e, d'_e of their linear s terms are given in Table IX.

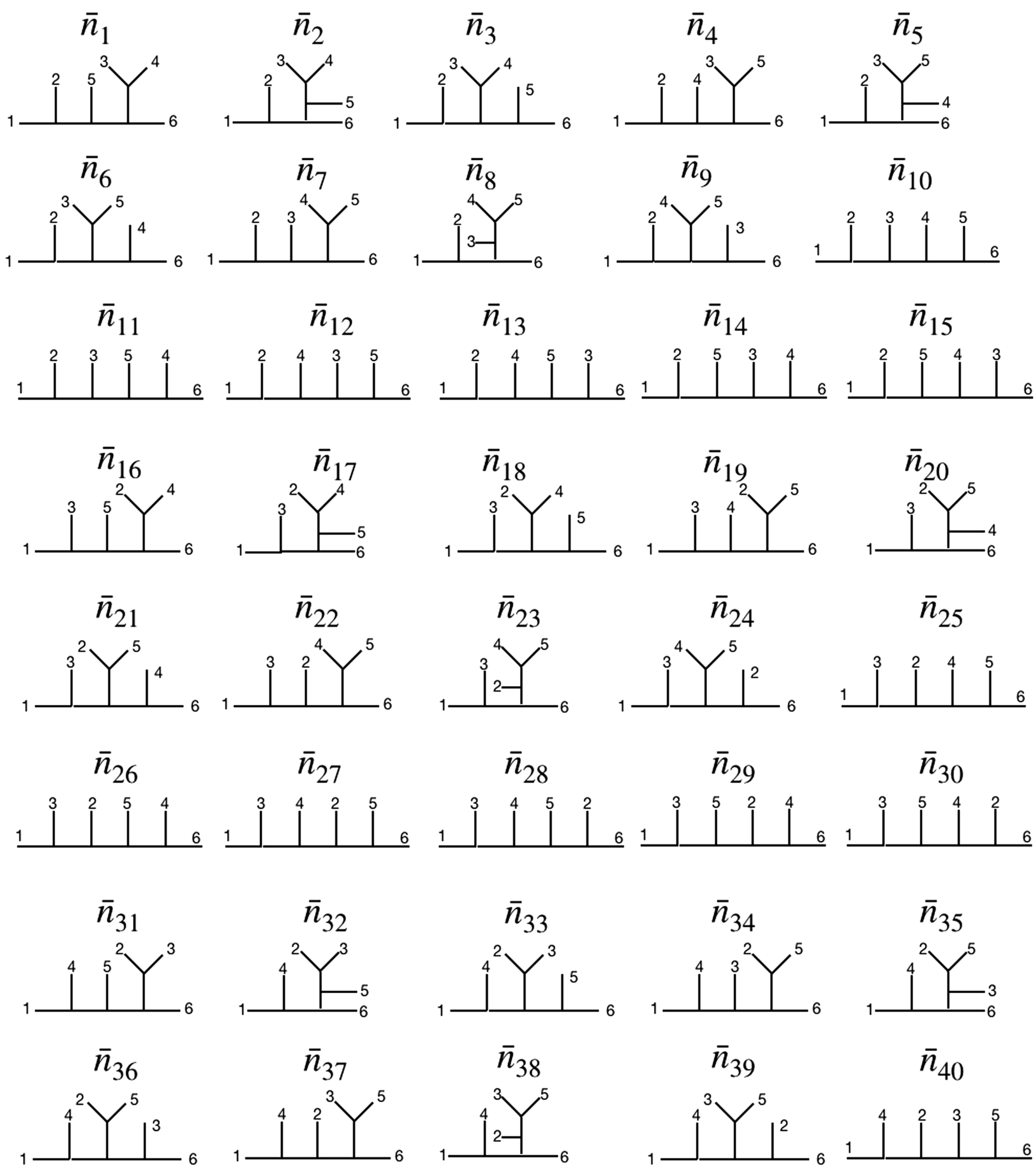

Fig. 17. (Continued).

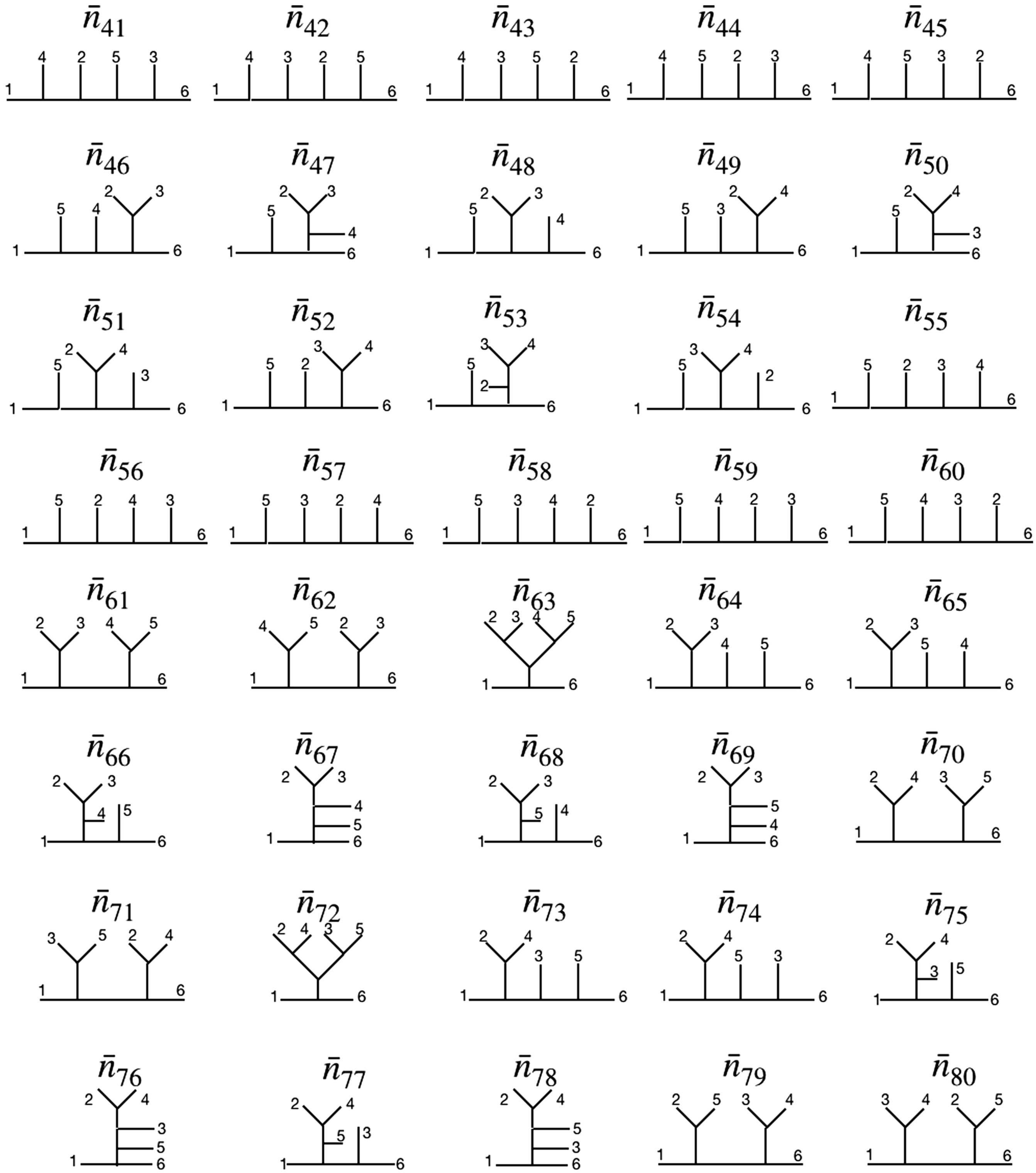

Fig. 17. (Continued).

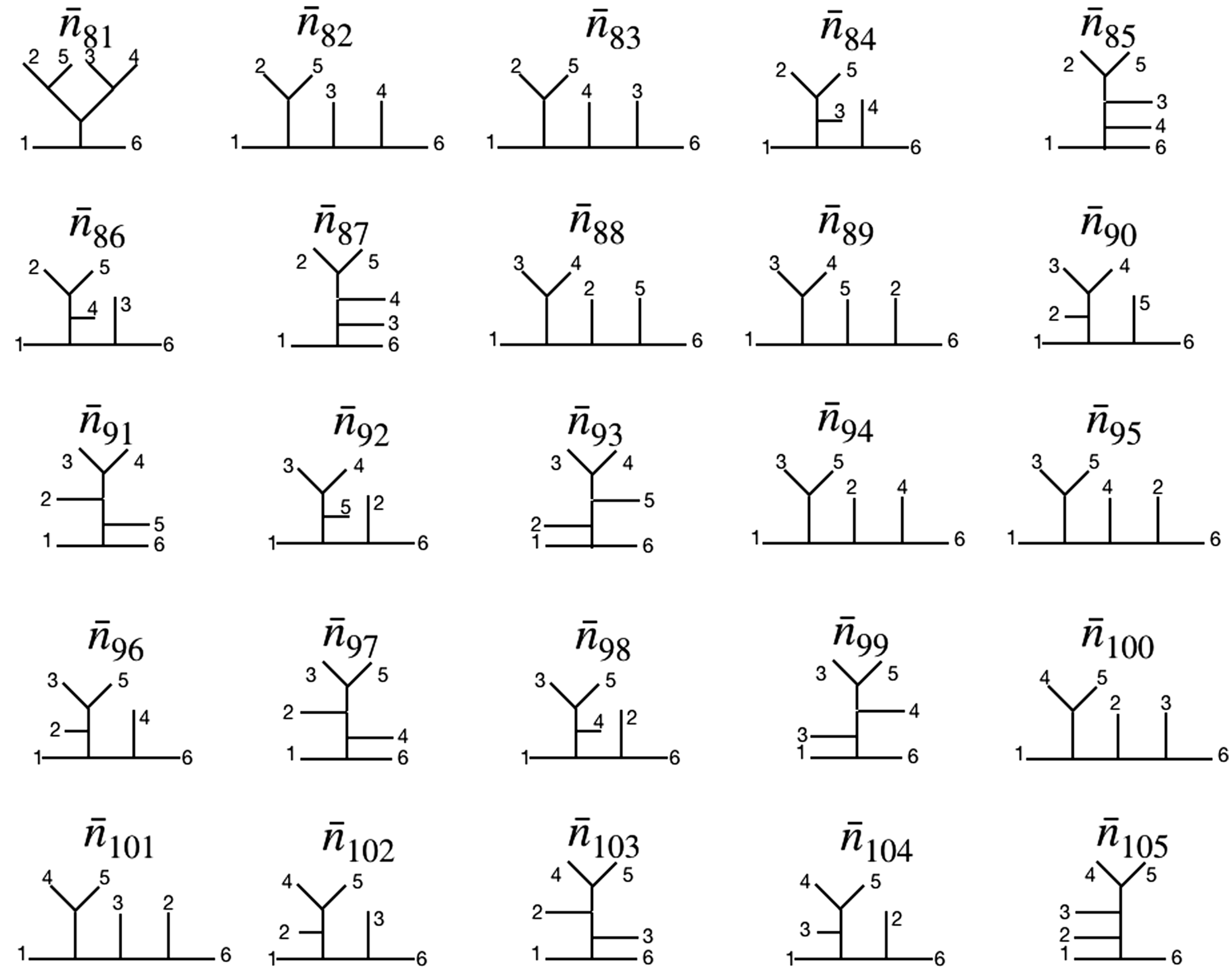

FIG. 17. 3g Feynman diagrams for $N = 6$.

TABLE VIII. Composition of the 105 Δ_l's.

	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}
A	$s_{12}s_{34}$	$s_{12}s_{35}$	$s_{12}s_{45}$	$s_{12}s_{123}$	$s_{12}s_{124}$	$s_{12}s_{125}$	$s_{12}s_{345}$	$s_{12}s_{1234}$	$s_{12}s_{1235}$	$s_{12}s_{1245}$
B	$(\bar{1},\bar{2},3)$	$(\bar{4},\bar{5},6)$	$(7,\bar{8},\bar{9})$	$(\bar{7},10,\overline{11})$	$(\bar{4},12,\overline{13})$	$(\bar{1},14,\overline{15})$	$(2,\bar{5},\bar{8})$	$(\bar{3},10,\overline{12})$	$(\bar{6},11,\overline{14})$	$(\bar{9},13,\overline{15})$

	Δ_{11}	Δ_{12}	Δ_{13}	Δ_{14}	Δ_{15}	Δ_{16}	Δ_{17}	Δ_{18}	Δ_{19}	Δ_{20}
A	$s_{13}s_{24}$	$s_{13}s_{25}$	$s_{13}s_{45}$	$s_{13}s_{123}$	$s_{13}s_{134}$	$s_{13}s_{135}$	$s_{13}s_{245}$	$s_{13}s_{1234}$	$s_{13}s_{1235}$	$s_{13}s_{1345}$
B	$(\overline{16},\overline{17},18)$	$(\overline{19},\overline{20},21)$	$(22,\overline{23},\overline{24})$	$(\overline{22},25,\overline{26})$	$(\overline{19},27,\overline{28})$	$(\overline{16},29,\overline{30})$	$(17,\overline{20},\overline{23})$	$(\overline{18},25,\overline{27})$	$(\overline{21},26,\overline{29})$	$(\overline{24},28,\overline{30})$

	Δ_{21}	Δ_{22}	Δ_{23}	Δ_{24}	Δ_{25}	Δ_{26}	Δ_{27}	Δ_{28}	Δ_{29}	Δ_{30}
A	$s_{14}s_{23}$	$s_{14}s_{25}$	$s_{14}s_{35}$	$s_{14}s_{124}$	$s_{14}s_{134}$	$s_{14}s_{145}$	$s_{14}s_{235}$	$s_{14}s_{1234}$	$s_{14}s_{1245}$	$s_{14}s_{1345}$
B	$(\overline{31},\overline{32},33)$	$(\overline{34},\overline{35},36)$	$(37,\overline{38},\overline{39})$	$(\overline{37},40,\overline{41})$	$(\overline{34},42,\overline{43})$	$(\overline{31},44,\overline{45})$	$(32,\overline{35},\overline{38})$	$(\overline{33},40,\overline{42})$	$(\overline{36},41,\overline{44})$	$(\overline{39},43,\overline{45})$

(Table continued)

TABLE VIII. *(Continued)*

	Δ_{31}	Δ_{32}	Δ_{33}	Δ_{34}	Δ_{35}	Δ_{36}	Δ_{37}	Δ_{38}	Δ_{39}	Δ_{40}
A	$s_{15}s_{23}$	$s_{15}s_{24}$	$s_{15}s_{34}$	$s_{15}s_{125}$	$s_{15}s_{135}$	$s_{15}s_{145}$	$s_{15}s_{234}$	$s_{15}s_{1235}$	$s_{15}s_{1245}$	$s_{15}s_{1345}$
B	$(\overline{46},\overline{47},48)$	$(\overline{49},\overline{50},51)$	$(52,\overline{53},\overline{54})$	$(\overline{52},55,\overline{56})$	$(\overline{49},57,\overline{58})$	$(\overline{46},59,\overline{60})$	$(47,\overline{50},\overline{53})$	$(\overline{48},55,\overline{57})$	$(\overline{51},56,\overline{59})$	$(\overline{54},58,\overline{60})$

	Δ_{41}	Δ_{42}	Δ_{43}	Δ_{44}	Δ_{45}	Δ_{46}	Δ_{47}	Δ_{48}	Δ_{49}	Δ_{50}
A	$s_{23}s_{45}$	$s_{23}s_{123}$	$s_{23}s_{145}$	$s_{23}s_{234}$	$s_{23}s_{235}$	$s_{23}s_{1234}$	$s_{23}s_{1235}$	$s_{23}s_{2345}$	$s_{24}s_{35}$	$s_{24}s_{124}$
B	$(61,\overline{62},\overline{63})$	$(\overline{61},64,\overline{65})$	$(31,\overline{46},\overline{62})$	$(\overline{47},66,\overline{67})$	$(\overline{32},68,\overline{69})$	$(\overline{33},64,\overline{66})$	$(\overline{48},65,\overline{68})$	$(\overline{63},67,\overline{69})$	$(70,\overline{71},\overline{72})$	$(\overline{70},73,\overline{74})$

	Δ_{51}	Δ_{52}	Δ_{53}	Δ_{54}	Δ_{55}	Δ_{56}	Δ_{57}	Δ_{58}	Δ_{59}	Δ_{60}
A	$s_{24}s_{135}$	$s_{24}s_{234}$	$s_{24}s_{245}$	$s_{24}s_{1234}$	$s_{24}s_{1245}$	$s_{24}s_{2345}$	$s_{25}s_{34}$	$s_{25}s_{125}$	$s_{25}s_{134}$	$s_{25}s_{235}$
B	$(16,\overline{49},\overline{71})$	$(\overline{50},75,\overline{76})$	$(\overline{17},77,\overline{78})$	$(\overline{18},73,\overline{75})$	$(\overline{51},74,\overline{77})$	$(\overline{72},76,\overline{78})$	$(79,\overline{80},\overline{81})$	$(\overline{79},82,\overline{83})$	$(19,\overline{34},\overline{80})$	$(\overline{35},84,\overline{85})$

	Δ_{61}	Δ_{62}	Δ_{63}	Δ_{64}	Δ_{65}	Δ_{66}	Δ_{67}	Δ_{68}	Δ_{69}	Δ_{70}
A	$s_{25}s_{245}$	$s_{25}s_{1235}$	$s_{25}s_{1245}$	$s_{25}s_{2345}$	$s_{34}s_{125}$	$s_{34}s_{134}$	$s_{34}s_{234}$	$s_{34}s_{345}$	$s_{34}s_{1234}$	$s_{34}s_{1345}$
B	$(\overline{20},86,\overline{87})$	$(\overline{21},82,\overline{84})$	$(\overline{36},83,\overline{86})$	$(\overline{81},85,\overline{87})$	$(1,\overline{52},\overline{79})$	$(\overline{80},88,\overline{89})$	$(\overline{53},90,\overline{91})$	$(2,\overline{92},\overline{93})$	$(3,\overline{88},\overline{90})$	$(\overline{54},89,\overline{92})$

	Δ_{71}	Δ_{72}	Δ_{73}	Δ_{74}	Δ_{75}	Δ_{76}	Δ_{77}	Δ_{78}	Δ_{79}	Δ_{80}
A	$s_{34}s_{2345}$	$s_{35}s_{124}$	$s_{35}s_{135}$	$s_{35}s_{235}$	$s_{35}s_{345}$	$s_{35}s_{1235}$	$s_{35}s_{1345}$	$s_{35}s_{2345}$	$s_{45}s_{123}$	$s_{45}s_{145}$
B	$(\overline{81},91,\overline{93})$	$(4,\overline{37},\overline{70})$	$(\overline{71},94,\overline{95})$	$(\overline{38},96,\overline{97})$	$(5,\overline{98},\overline{99})$	$(6,\overline{94},\overline{96})$	$(\overline{39},95,\overline{98})$	$(\overline{72},97,\overline{99})$	$(7,\overline{22},\overline{61})$	$(\overline{62},100,\overline{101})$

	Δ_{81}	Δ_{82}	Δ_{83}	Δ_{84}	Δ_{85}	Δ_{86}	Δ_{87}	Δ_{88}	Δ_{89}	Δ_{90}
A	$s_{45}s_{245}$	$s_{45}s_{345}$	$s_{45}s_{1245}$	$s_{45}s_{1345}$	$s_{45}s_{2345}$	$s_{123}s_{1234}$	$s_{123}s_{1235}$	$s_{124}s_{1234}$	$s_{124}s_{1245}$	$s_{125}s_{1235}$
B	$(\overline{23},102,\overline{103})$	$(8,\overline{104},\overline{105})$	$(9,\overline{100},\overline{102})$	$(24,\overline{101},\overline{104})$	$(63,\overline{103},\overline{105})$	$(10,\overline{25},\overline{64})$	$(11,\overline{26},\overline{65})$	$(12,\overline{40},\overline{73})$	$(13,\overline{41},\overline{74})$	$(14,\overline{55},\overline{82})$

	Δ_{91}	Δ_{92}	Δ_{93}	Δ_{94}	Δ_{95}	Δ_{96}	Δ_{97}	Δ_{98}	Δ_{99}	Δ_{100}
A	$s_{125}s_{1245}$	$s_{134}s_{1234}$	$s_{134}s_{1345}$	$s_{135}s_{1235}$	$s_{135}s_{1345}$	$s_{145}s_{1245}$	$s_{145}s_{1345}$	$s_{234}s_{1234}$	$s_{234}s_{2345}$	$s_{235}s_{1235}$
B	$(15,\overline{56},\overline{83})$	$(27,\overline{42},\overline{88})$	$(28,\overline{43},\overline{89})$	$(29,\overline{57},\overline{94})$	$(30,\overline{58},\overline{95})$	$(44,\overline{59},\overline{100})$	$(45,\overline{60},\overline{101})$	$(66,\overline{75},\overline{90})$	$(67,\overline{76},\overline{91})$	$(68,\overline{84},\overline{96})$

	Δ_{101}	Δ_{102}	Δ_{103}	Δ_{104}	Δ_{105}
A	$s_{235}s_{2345}$	$s_{245}s_{1245}$	$s_{245}s_{2345}$	$s_{345}s_{1345}$	$s_{345}s_{2345}$
B	$(69,\overline{85},\overline{97})$	$(77,\overline{86},\overline{102})$	$(78,\overline{87},\overline{103})$	$(92,\overline{98},\overline{104})$	$(93,\overline{99},\overline{105})$

$$\bar{\Delta}_1 = (\bar{1},\bar{2},3) = s_{12}F(12569)T(\bar{9}34) + [s_{34}F(34659) + d(9856)T(\bar{8}34)]T(\bar{9}12) + s_{12}s_{34}g_A(125634),$$

$$\bar{\Delta}_2 = (\bar{4},\bar{5},6) = s_{12}F(12469)T(\bar{9}35) + [s_{35}F(35649) + d(9846)T(\bar{8}35)]T(\bar{9}12) + s_{12}s_{35}g_A(124635),$$

$$\bar{\Delta}_3 = (7,\bar{8},\bar{9}) = s_{12}F(12396)T(\bar{9}45) + [s_{45}F(45693) + d(9386)T(\bar{8}45)]T(\bar{9}12) + s_{12}s_{45}g_A(123654),$$

$$\bar{\Delta}_4 = (\bar{7},10,\overline{11}) = s_{12}A_4(1239))d(\bar{9}456) + s_{123}F(93456)T(\bar{9}12),$$

$$\bar{\Delta}_5 = (\bar{4},12,\overline{13}) = s_{12}A_4(1249)d(\bar{9}356) + s_{124}F(94356)T(\bar{9}12),$$

$$\bar{\Delta}_6 = (\bar{1},14,\overline{15}) = s_{12}A_4(1259)d(\bar{9}346) + s_{125}F(95346)T(\bar{9}12),$$

$$\bar{\Delta}_7 = (2,\bar{5},\bar{8}) = s_{12}A_4(1296)d(\bar{9}345) + s_{345}F(69543)T(\bar{9}12),$$

$$\bar{\Delta}_8 = (\bar{3},10,\overline{12}) = s_{12}F(12349)T(\bar{9}56) + [s_{1234}F(56439) + d(9348)T(\bar{8}56)]T(\bar{9}12) + s_{12}s_{1234}g_A(123456),$$

$$\bar{\Delta}_9 = (\bar{6},11,\overline{14}) = s_{12}F(12359)T(\bar{9}46) + [s_{1235}F(46539) + d(9358)T(\bar{8}46)]T(\bar{9}12) + s_{12}s_{1235}g_A(123546),$$

$$\bar{\Delta}_{10} = (\bar{9},13,\overline{15}) = s_{12}F(12459)T(\bar{9}36) + [s_{1245}F(36549) + d(9458)T(\bar{8}36)]T(\bar{9}12) + s_{12}s_{1245}g_A(124536).$$

$$\bar{\Delta}_{11} = (\overline{16},\overline{17},18) = s_{13}F(13956)T(\bar{9}24) + [s_{24}F(24965) + d(9856)T(\bar{8}24)]T(\bar{9}13) + s_{13}s_{24}g_A(135624),$$

$$\bar{\Delta}_{12} = (\overline{19},\overline{20},21) = s_{13}F(13946)T(\bar{9}25) + [s_{25}F(25964) + d(9846)T(\bar{8}25)]T(\bar{9}13) + s_{13}s_{25}g_A(134625),$$

$\bar{\Delta}_{13} = (22, \overline{23}, \overline{24}) = s_{13}F(13296)T(\bar{9}45) + [s_{45}F(45692) + d(9286)T(\bar{8}45)]T(\bar{9}13) + s_{13}s_{45}g_A(132654),$

$\bar{\Delta}_{14} = (\overline{22}, 25, \overline{26}) = s_{13}A_4(1329)d(\bar{9}456) + s_{123}F(92456)T(\bar{9}13),$

$\bar{\Delta}_{15} = (\overline{19}, 27, \overline{28}) = s_{13}A_4(1349)d(\bar{9}256) + s_{134}F(94256)T(\bar{9}13),$

$\bar{\Delta}_{16} = (\overline{16}, 29, \overline{30}) = s_{13}A_4(1359)d(\bar{9}246) + s_{135}F(95246)T(\bar{9}13),$

$\bar{\Delta}_{17} = (17, \overline{20}, \overline{23}) = s_{13}A_4(1396)d(\bar{9}245) + s_{245}F(69542)T(\bar{9}13),$

$\bar{\Delta}_{18} = (\overline{18}, 25, \overline{27}) = s_{13}F(13249)T(\bar{9}56) + s_{1234}[F(56429) + d(9248)T(\bar{8}56]T(\bar{9}13) + s_{13}s_{1234}g_A(132456),$

$\bar{\Delta}_{19} = (\overline{21}, 26, \overline{29}) = s_{13}F(13259)T(\bar{9}46) + s_{1235}[F(46529) + d(9258)T(\bar{8}46)]T(\bar{9}13) + s_{13}s_{1235}g_A(132546),$

$\bar{\Delta}_{20} = (\overline{24}, 28, \overline{30}) = s_{13}F(13459)T(\bar{9}26) + s_{1345}[F(26549) + d(9458)T(\bar{8}26)]T(\bar{9}13) + s_{13}s_{1345}g_A(134526),$

$\bar{\Delta}_{21} = (\overline{31}, \overline{32}, 33) = s_{14}F(14956)T(\bar{9}23) + [s_{23}F(23965)) + d(9856)T(\bar{8}23)]T(\bar{9}14) + s_{14}s_{23}g_A(145623),$

$\bar{\Delta}_{22} = (\overline{34}, \overline{35}, 36) = s_{14}F(14936)T(\bar{9}25) + [s_{25}F(25963) + d(9836)T(\bar{8}25)]T(\bar{9}14) + s_{14}s_{25}g_A(143625),$

$\bar{\Delta}_{23} = (37, \overline{38}, \overline{39}) = s_{14}F(14296)T(\bar{9}35) + [s_{35}F(35692) + d(9286)T(\bar{8}35)]T(\bar{9}14) + s_{14}s_{35}g_A(142653),$

$\bar{\Delta}_{24} = (\overline{37}, 40, \overline{41}) = s_{14}A_4(1429)d(\bar{9}356) + s_{124}F(92356)T(\bar{9}14),$

$\bar{\Delta}_{25} = (\overline{34}, 42, \overline{43}) = s_{14}A_4(1439)d(\bar{9}256) + s_{134}F(93256)T(\bar{9}14),$

$\bar{\Delta}_{26} = (\overline{31}, 44, \overline{45}) = s_{14}A_4(1459)d(\bar{9}236) + s_{145}F(95236)T(\bar{9}14),$

$\bar{\Delta}_{27} = (32, \overline{35}, \overline{38}) = s_{14}A_4(1496)d(\bar{9}235) + s_{235}F(69532)T(\bar{9}14),$

$\bar{\Delta}_{28} = (\overline{33}, 40, \overline{42}) = s_{14}F(14239)T(\bar{9}56) + s_{1234}[F(56329) + d(9238)T(\bar{8}56)]T(\bar{9}14) + s_{14}s_{1234}g_A(142356),$

$\bar{\Delta}_{29} = (\overline{36}, 41, \overline{44}) = s_{14}F(14259)T(\bar{9}36) + s_{1245}[F(36529) + d(9258)T(\bar{8}36)]T(\bar{9}14) + s_{14}s_{1245}g_A(142536),$

$\bar{\Delta}_{30} = (\overline{39}, 43, \overline{45}) = s_{14}F(14359)T(\bar{9}26) + s_{1345}[F(26539) + d(9358)T(\bar{8}26)]T(\bar{9}14) + s_{14}s_{1345}g_A(143526).$

$\bar{\Delta}_{31} = (\overline{46}, \overline{47}, 48) = s_{15}F(15946)T(\bar{9}23) + [s_{23}F(23964) + d(9846)T(\bar{8}23)]T(\bar{9}15) + s_{15}s_{23}g_A(154623),$

$\bar{\Delta}_{32} = (\overline{49}, \overline{50}, 51) = s_{15}F(15936)T(\bar{9}24) + [s_{24}F(24963) + d(9835)T(\bar{8}24)]T(\bar{9}15) + s_{15}s_{24}g_A(153624),$

$\bar{\Delta}_{33} = (52, \overline{53}, \overline{54}) = s_{15}F(15296)T(\bar{9}34) + [s_{34}F(34692) + d(9286)T(\bar{8}34)]T(\bar{9}15) + s_{15}s_{34}g_A(152643),$

$\bar{\Delta}_{34} = (\overline{52}, 55, \overline{56}) = s_{15}A_4(1529)d(\bar{9}346) + s_{125}F(92346)T(\bar{9}15),$

$\bar{\Delta}_{35} = (\overline{49}, 57, \overline{58}) = s_{15}A_4(1539)d(\bar{9}246) + s_{135}F(93246)T(\bar{9}15),$

$\bar{\Delta}_{36} = (\overline{46}, 59, \overline{60}) = s_{15}A_4(1549)d(\bar{9}236) + s_{145}F(94236)T(\bar{9}15),$

$\bar{\Delta}_{37} = (47, \overline{50}, \overline{53}) = s_{15}A_4(1569)d(\bar{9}234) + s_{234}F(69432)T(\bar{9}15),$

$\bar{\Delta}_{38} = (\overline{48}, 55, \overline{57}) = s_{15}F(15239)T(\bar{9}46) + [s_{1235}F(46329) + d(9238)T(\bar{8}46)]T(\bar{9}15) + s_{15}s_{1235}g_A(152346),$

$\bar{\Delta}_{39} = (\overline{51}, 56, \overline{59}) = s_{15}F(15249)T(\bar{9}36) + [s_{1245}F(36429) + d(9248)T(\bar{8}36)]T(\bar{9}15) + s_{15}s_{1245}g_A(152436),$

$\bar{\Delta}_{40} = (\overline{54}, 58, \overline{60}) = s_{15}F(15349)T(\bar{9}26) + [s_{1345}F(26439) + d(9348)T(\bar{8}26)]T(\bar{9}15) + s_{15}s_{1345}g_A(153426),$

$\bar{\Delta}_{41} = (61, \overline{62}, \overline{63}) = s_{23}F(23169)T(\bar{9}45) + [s_{45}F(45619) + d(1986)T(\bar{8}45)]T(\bar{9}23) + s_{23}s_{45}g_A(231645),$

$\bar{\Delta}_{42} = (\overline{61}, 64, \overline{65}) = s_{23}A_4(2391)d(\bar{9}456) + s_{123}F(19456)T(\bar{9}23),$

$\bar{\Delta}_{43} = (31, \overline{46}, \overline{62}) = s_{23}A_4(2369)d(145\bar{9}) + s_{145}F(96541)T(\bar{9}23),$

$\bar{\Delta}_{44} = (\overline{47}, 66, \overline{67}) = s_{23}A_4(2349)d(\bar{1}\,\bar{9}\,56) + s_{234}F(94165)T(\bar{9}23),$

$\bar{\Delta}_{45} = (\overline{32}, 68, \overline{69}) = s_{23}A_4(2359)d(1\bar{9}46) + s_{235}F(95164)T(\bar{9}23),$

$\bar{\Delta}_{46} = (\overline{33}, 64, \overline{66}) = s_{23}F(23194)T(\bar{9}56) + [s_{1234}F(56491) + d(1948)T(\bar{8}56)]T(\bar{9}23) + s_{23}s_{1234}g_A(231465),$

$\bar{\Delta}_{47} = (\overline{48}, 65, \overline{68}) = s_{23}F(23195)T(\bar{9}46) + [s_{1235}F(46591) + d(1958)T(\bar{8}46)]T(\bar{9}23) + s_{23}s_{1235}g_A(235164),$

$\bar{\Delta}_{48} = (\overline{63}, 67, \overline{69}) = s_{23}F(23459)T(\bar{9}16) + [s_{2345}F(61549) + d(9458)T(\bar{8}61)]T(\bar{9}23) + s_{23}s_{2345}g_A(234561),$

$$\bar{\Delta}_{49}=(70,\overline{71},\overline{72})=s_{24}F(24169)T(\bar{9}35)+[s_{35}F(35619)+d(1986)T(\bar{8}35)]T(\bar{9}24)+s_{23}s_{35}g_A(241653),$$
$$\bar{\Delta}_{50}=(\overline{70},73,\overline{74})=s_{24}A_4(2491)d(\bar{9}356)+s_{124}F(19356)T(\bar{9}24),$$
$$\bar{\Delta}_{51}=(16,\overline{49},\overline{71})=s_{24}A_4(2469)d(135\bar{9})+s_{135}F(96531)T(\bar{9}24),$$
$$\bar{\Delta}_{52}=(\overline{50},75,\overline{76})=s_{24}A_4(2439)d(1\bar{9}56)+s_{234}F(93165)T(\bar{9}24),$$
$$\bar{\Delta}_{53}=(\overline{17},77,\overline{78})=s_{24}A_4(2459)d(1\bar{9}36)+s_{245}F(95163)T(\bar{9}24),$$
$$\bar{\Delta}_{54}=(\overline{18},73,\overline{75})=s_{24}F(24193)T(\bar{9}56)+[s_{1234}F(56391)+d(1938)T(\bar{8}56)]T(\bar{9}24)+s_{24}s_{1234}g_A(241365),$$
$$\bar{\Delta}_{55}=(\overline{51},74,\overline{77})=s_{24}F(24195)T(\bar{9}36)+[s_{1245}F(36591)+d(1958)T(\bar{8}36)]T(\bar{9}24)+s_{24}s_{1245}g_A(241563),$$
$$\bar{\Delta}_{56}=(\overline{72},76,\overline{78})=s_{24}F(24359)T(\bar{9}16)+[s_{2345}F(61539)+d(9358)T(\bar{8}61)]T(\bar{9}24)+s_{24}s_{2345}g_A(615324),$$
$$\bar{\Delta}_{57}=(79,\overline{80},\overline{81})=s_{25}F(25169)T(\bar{9}34)+s_{34}[F(34619)+d(1986)T(\bar{8}34)]T(\bar{9}25)+s_{25}s_{34}g_A(251634),$$
$$\bar{\Delta}_{58}=(\overline{79},82,\overline{83})=s_{25}A_4(2591)d(\bar{9}346)+s_{125}F(19346)T(\bar{9}25),$$
$$\bar{\Delta}_{59}=(19,\overline{34},\overline{80})=s_{25}A_4(2569)d(134\bar{9})+s_{134}F(96431)T(\bar{9}25),$$
$$\bar{\Delta}_{60}=(\overline{35},84,\overline{85})=s_{25}A_4(2539)d(1\bar{9}46)+s_{235}F(93164)T(\bar{9}25),$$
$$\bar{\Delta}_{61}=(\overline{20},86,\overline{87})=s_{25}A_4(2549)d(1\bar{9}36)+s_{245}F(94163)T(\bar{9}25),$$
$$\bar{\Delta}_{62}=(\overline{21},82,\overline{84})=s_{25}F(25193)T(\bar{9}46)+[s_{1235}F(46391)+d(1938)T(\bar{8}46)]T(\bar{9}25)+s_{25}s_{1235}g_A(251364),$$
$$\bar{\Delta}_{63}=(\overline{36},83,\overline{86})=s_{25}F(25194)T(\bar{9}36))+[s_{1245}F(36491)+d(1948)T(\bar{8}36)]T(\bar{9}25)+s_{25}s_{1245}g_A(251463),$$
$$\bar{\Delta}_{64}=(\overline{81},85,\overline{87})=s_{25}F(25349)T(\bar{9}16)+[s_{2345}F(61439)+d(9348)T(\bar{8}61)]T(\bar{9}25)+s_{25}s_{2345}g_A(614325),$$
$$\bar{\Delta}_{65}=(1,\overline{52},\overline{79})=s_{34}A_4(3469)d(125\bar{9})+s_{125}F(96521)T(\bar{9}34),$$
$$\bar{\Delta}_{66}=(\overline{80},88,\overline{89})=s_{34}A_4(1349)d(\bar{9}256)+s_{134}F(19256)T(\bar{9}34),$$
$$\bar{\Delta}_{67}=(\overline{53},90,\overline{91})=s_{34}A_4(3492)d(1\bar{9}56)+s_{234}F(29165)T(\bar{9}34),$$
$$\bar{\Delta}_{68}=(2,\overline{92},\overline{93})=s_{34}A_4(3459)d(12\bar{9}6)+s_{345}F(95612)T(\bar{9}34),$$
$$\bar{\Delta}_{69}=(3,\overline{88},\overline{90})=s_{34}F(34912)T(\bar{9}56)+[s_{1234}F(56921)+d(1298)T(\bar{8}56)]T(\bar{9}34)+s_{34}s_{1234}g_A(341256),$$
$$\bar{\Delta}_{70}=(\overline{54},89,\overline{92})=s_{34}F(34195)T(\bar{9}26)+[s_{1345}F(26591)+d(1958)T(\bar{8}26)]T(\bar{9}34)+s_{34}s_{1345}g_A(341562),$$
$$\bar{\Delta}_{71}=(\overline{81},91,\overline{93})=s_{34}F(34295)T(\bar{9}16)+[s_{2345}F(61592)+d(9258)T(\bar{8}16)]T(\bar{9}34)+s_{34}s_{2345}g_A(342516),$$
$$\bar{\Delta}_{72}=(4,\overline{37},\overline{70})=s_{35}A_4(3569)d(124\bar{9})+s_{124}F(96421)T(\bar{9}35),$$
$$\bar{\Delta}_{73}=(\overline{71},94,\overline{95})=s_{35}A_4(3591)d(\bar{9}246)+s_{135}F(19246)T(\bar{9}35),$$
$$\bar{\Delta}_{74}=(\overline{38},96,\overline{97})=s_{35}A_4(3592)d(1\bar{9}46))+s_{235}F(29164)T(\bar{9}35),$$
$$\bar{\Delta}_{75}=(5,\overline{98},\overline{99})=s_{35}A_4(3549)d(12\bar{9}6)+s_{345}F(94612)T(\bar{9}35),$$
$$\bar{\Delta}_{76}=(6,\overline{94},\overline{96})=s_{35}F(35912)T(\bar{9}46)+[s_{1235}F(46921)+d(1298)T(\bar{8}46)]T(\bar{9}35)+s_{35}s_{1235}g_A(351246),$$
$$\bar{\Delta}_{77}=(\overline{39},95,\overline{98})=s_{35}F(35194)T(\bar{9}26)+[s_{1345}F(26491)+d(1948)T(\bar{8}26)]T(\bar{9}35)+s_{35}s_{1345}g_A(351462),$$
$$\bar{\Delta}_{78}=(\overline{72},97,\overline{99})=s_{35}F(35294)T(\bar{9}16)+[s_{2345}F(61492)+d(9248)T(\bar{8}16)]T(\bar{9}35)+s_{35}s_{2345}g_A(352416),$$
$$\bar{\Delta}_{79}=(7,\overline{22},\overline{61})=s_{45}A_4(4569)d(123\bar{9})+s_{123}F(96321)T(\bar{9}45),$$
$$\bar{\Delta}_{80}=(\overline{62},100,\overline{101})=s_{45}A_4(1459)d(\bar{9}236))+s_{145}F(19236)T(\bar{9}45),$$
$$\bar{\Delta}_{81}=(\overline{23},102,\overline{103})=s_{45}A_4(4592)d(1\bar{9}36)+s_{245}F(29163)T(\bar{9}45),$$
$$\bar{\Delta}_{82}=(8,\overline{104},\overline{105})=s_{45}A_4(4593)d(12\bar{9}6)+s_{345}F(39612)T(\bar{9}45),$$
$$\bar{\Delta}_{83}=(9,\overline{100},\overline{102})=s_{45}F(45912)T(\bar{9}36)+[s_{1245}F(36921)+d(1298)T(\bar{8}36)T]T(\bar{9}45)+s_{45}s_{1245}g_A(451236),$$
$$\bar{\Delta}_{84}=(24,\overline{101},\overline{104})=s_{45}F(45913)T(\bar{9}26)+[s_{1345}F(26931)+d(1398)T(\bar{8}26)T]T(\bar{9}45)+s_{45}s_{1345}g_A(451326),$$
$$\bar{\Delta}_{85}=(63,\overline{103},\overline{105})=s_{45}F(45923)T(\bar{9}61)+[s_{2345}F(61932)+d(2398)T(\bar{8}61)T]T(\bar{9}45)+s_{45}s_{2345}g_A(452361),$$

$$\bar{\Delta}_{86} = (10, \overline{25}, \overline{64}) = s_{123}F(49321)T(\bar{9}56) + s_{1234}A_4(9456)d(123\bar{9}),$$
$$\bar{\Delta}_{87} = (11, \overline{26}, \overline{65}) = s_{123}F(59321)T(\bar{9}46) + s_{1235}A_4(9546)d(123\bar{9}),$$
$$\bar{\Delta}_{88} = (12, \overline{40}, \overline{73}) = s_{124}F(93421)T(\bar{9}56) + s_{1234}A_4(9356)d(124\bar{9}),$$
$$\bar{\Delta}_{89} = (13, \overline{41}, \overline{74}) = s_{124}F(95421)T(\bar{9}36) + s_{1245}A_4(9536)d(124\bar{9}),$$
$$\bar{\Delta}_{90} = (14, \overline{55}, \overline{82}) = s_{125}F(39521)T(\bar{9}46) + s_{1235}A_4(9346)d(125\bar{9}),$$
$$\bar{\Delta}_{91} = (15, \overline{56}, \overline{83}) = s_{125}F(49521)T(\bar{9}36) + s_{1245}A_4(3694)d(125\bar{9}),$$
$$\bar{\Delta}_{92} = (27, \overline{42}, \overline{88}) = s_{134}F(29431)T(\bar{9}56) + s_{1234}A_4(5692)d(134\bar{9}),$$
$$\bar{\Delta}_{93} = (28, \overline{43}, \overline{89}) = s_{134}F(59431)T(\bar{9}26) + s_{1345}A_4(2695)d(134\bar{9}),$$
$$\bar{\Delta}_{94} = (29, \overline{57}, \overline{94}) = s_{135}F(29531)T(\bar{9}46) + s_{1235}A_4(4692)d(135\bar{9}),$$
$$\bar{\Delta}_{95} = (30, \overline{58}, \overline{95}) = s_{135}F(49531)T(\bar{9}26) + s_{1345}A_4(2694)d(135\bar{9}),$$
$$\bar{\Delta}_{96} = (44, \overline{59}, \overline{100}) = s_{145}F(29541)T(\bar{9}36) + s_{1245}A_4(3692)d(145\bar{9}),$$
$$\bar{\Delta}_{97} = (45, \overline{60}, \overline{101}) = s_{145}F(39541)T(\bar{9}26) + s_{1345}A_4(2693)d(145\bar{9}),$$
$$\bar{\Delta}_{98} = (66, \overline{75}, \overline{90}) = s_{234}F(91432)T(\bar{9}56) + s_{1234}A_4(5619)d(\bar{9}234),$$
$$\bar{\Delta}_{99} = (67, \overline{76}, \overline{91}) = s_{234}F(59432)T(\bar{9}61) + s_{2345}A_4(6195)d(\bar{9}243),$$
$$\bar{\Delta}_{100} = (68, \overline{84}, \overline{96}) = s_{235}F(91532)T(\bar{9}46) + s_{1235}A_4(4619)d(\bar{9}235),$$
$$\bar{\Delta}_{101} = (69, \overline{85}, \overline{97}) = s_{235}F(49532)T(\bar{9}61) + s_{2345}A_4(6194)d(\bar{9}235),$$
$$\bar{\Delta}_{102} = (77, \overline{86}, \overline{102}) = s_{245}F(91542)T(\bar{9}36) + s_{1245}A_4(3619)d(\bar{9}245),$$
$$\bar{\Delta}_{103} = (78, \overline{87}, \overline{103}) = s_{245}F(39542)T(\bar{9}61) + s_{2345}A_4(6193)d(\bar{9}245),$$
$$\bar{\Delta}_{104} = (92, \overline{98}, \overline{104}) = s_{345}F(91543)T(\bar{9}26) + s_{1345}A_4(2619)d(\bar{9}345),$$
$$\bar{\Delta}_{105} = (93, \overline{99}, \overline{105}) = s_{345}F(92543)T(\bar{9}61) + s_{2345}A_4(6129)d(\bar{9}345). \quad \text{(B1)}$$

TABLE IX. A list of d_e and d'_e in $\bar{\Delta}_l = d_e s_e + d'_e s'_e + O(s^2)$, with $s_e \prec s'_e$.

l	d_e	d'_e	l	d_e	d'_e
1	$F(12569)T(\bar{9}34)$	$F(34659)T(\bar{9}12)$	2	$F(12469)T(\bar{9}35)$	$F(35649)T(\bar{9}12)$
3	$F(12396)T(\bar{9}45)$	$F(45693)T(\bar{9}12)$	4	$-T(123)T(456)$	$F(93456)T(\bar{9}12)$
5	$-T(124)T(356)$	$F(94356)T(\bar{9}12)$	6	$-T(125)T(346)$	$F(95346)T(\bar{9}12)$
7	$-T(126)T(354)$	$F(69543)T(\bar{9}12)$	8	$F(12349)T(\bar{9}56)$	$F(56439)T(\bar{9}12)$
9	$F(12359)T(\bar{9}46)$	$F(46539)T(\bar{9}12)$	10	$F(12459)T(\bar{9}36)$	$F(36549)T(\bar{9}12)$
11	$F(13956)T(\bar{9}24)$	$F(24965)T(\bar{9}13)$	12	$F(13946)T(\bar{9}25)$	$F(25964)T(\bar{9}13)$
13	$F(13296)T(\bar{9}45)$	$F(45692)T(\bar{9}13)$	14	$-T(132)T(456)$	$F(92456)T(\bar{9}13)$
15	$-T(134)T(256)$	$F(94256)T(\bar{9}13)$	16	$-T(135)T(246)$	$F(95246)T(\bar{9}13)$
17	$-T(136)T(254)$	$F(69542)T(\bar{9}13)$	18	$F(13249)T(\bar{9}56)$	$F(56429)T(\bar{9}13)$
19	$F(13259)T(\bar{9}46)$	$F(46529)T(\bar{9}13)$	20	$F(13459)T(\bar{9}26)$	$F(26549)T(\bar{9}13)$
21	$F(14936)T(\bar{9}25)$	$F(23965)T(\bar{9}14)$	22	$F(14936)T(\bar{9}25)$	$F(25963)T(\bar{9}14)$
23	$F(14296)T(\bar{9}35)$	$F(35692)T(\bar{9}14)$	24	$-T(142)T(356)$	$F(92356)T(\bar{9}14)$
25	$-T(143)T(256)$	$F(93256)T(\bar{9}14)$	26	$-T(145)T(236)$	$F(95236)T(\bar{9}14)$
27	$-T(146)T(253)$	$F(69532)T(\bar{9}14)$	28	$F(14239)T(\bar{9}56)$	$F(56329)T(\bar{9}14)$
29	$F(14259)T(\bar{9}36)$	$F(36529)T(\bar{9}14)$	30	$F(14359)T(\bar{9}26)$	$F(26539)T(\bar{9}14)$
31	$F(15946)T(\bar{9}23)$	$F(23964)T(\bar{9}15)$	32	$F(15936)T(\bar{9}24)$	$F(24963)T(\bar{9}15)$

(Table continued)

TABLE IX. *(Continued)*

l	d_e	d'_e	l	d_e	d'_e
33	$F(15296)T(\bar{9}34)$	$F(34692)T(\bar{9}15)$	34	$-T(152)T(346)$	$F(92346)T(\bar{9}15)$
35	$-T(153)T(246)$	$F(93246)T(\bar{9}15)$	36	$-T(154)T(236)$	$F(94236)T(\bar{9}15)$
37	$-T(156)T(234)$	$F(69432)T(\bar{9}15)$	38	$F(15239)T(\bar{9}46)$	$F(46329)T(\bar{9}15)$
39	$F(15249)T(\bar{9}36)$	$F(36429)T(\bar{9}15)$	40	$F(15349)T(\bar{9}26)$	$F(26439)T(\bar{9}15)$
41	$F(23169)T(\bar{9}45)$	$F(45619)T(\bar{9}23)$	42	$-T(231)T(564)$	$F(19456)T(\bar{9}23)$
43	$-T(236)T(541)$	$F(96541)T(\bar{9}23)$	44	$-T(234)T(165)$	$F(94165)T(\bar{9}23)$
45	$-T(235)T(164)$	$F(95164)T(\bar{9}23)$	46	$F(23194)T(\bar{9}56)$	$F(56491)T(\bar{9}23)$
47	$F(23195)T(\bar{9}46)$	$F(46591)T(\bar{9}23)$	48	$F(23459)T(\bar{9}16)$	$F(61549)T(\bar{9}23)$
49	$F(24169)T(\bar{9}35)$	$F(35619)T(\bar{9}24)$	50	$-T(241)T(365)$	$F(19356)T(\bar{9}24)$
51	$-T(246)T(531)$	$F(96531)T(\bar{9}24)$	52	$-T(243)T(165)$	$F(93165)T(\bar{9}24)$
53	$-T(245)T(163)$	$F(96153)T(\bar{9}24)$	54	$F(24193)T(\bar{9}56)$	$F(56391)T(\bar{9}24)$
55	$F(24195)T(\bar{9}36)$	$F(36591)T(\bar{9}24)$	56	$F(24359)T(\bar{9}16)$	$F(61539)T(\bar{9}24)$
57	$F(25169)T(\bar{9}34)$	$F(34619)T(\bar{9}25)$	58	$-T(251)T(364)$	$F(19346)T(\bar{9}25)$
59	$-T(256)T(134)$	$F(96431)T(\bar{9}25)$	60	$-T(253)T(164)$	$F(93164)T(\bar{9}25)$
61	$-T(254)T(163)$	$F(94163)T(\bar{9}25)$	62	$F(25193)T(\bar{9}46)$	$F(46391)T(\bar{9}25)$
63	$F(25194)T(\bar{9}36)$	$F(36491)T(\bar{9}25)$	64	$F(25349)T(\bar{9}16)$	$F(61439)T(\bar{9}25)$
65	$-T(346)T(521)$	$F(96521)T(\bar{9}34)$	66	$-T(341)T(265)$	$F(19256)T(\bar{9}34)$
67	$-T(342)T(165)$	$F(29165)T(\bar{9}34)$	68	$-T(345)T(612)$	$F(95612)T(\bar{9}34)$
69	$F(34912)T(\bar{9}56)$	$F(56921)T(\bar{9}34)$	70	$F(34195)T(\bar{9}26)$	$F(26591)T(\bar{9}34)$
71	$F(34295)T(\bar{9}16)$	$F(61592)T(\bar{9}34)$	72	$-T(356)T(421)$	$F(96421)T(\bar{9}35)$
73	$-T(351)T(264)$	$F(19246)T(\bar{9}35)$	74	$-T(352)T(146)$	$F(29164)T(\bar{9}35)$
75	$-T(354)T(612)$	$F(94612)T(\bar{9}35)$	76	$F(35912)T(\bar{9}46)$	$F(46921)T(\bar{9}35)$
77	$F(35194)T(\bar{9}26)$	$F(26491)T(\bar{9}35)$	78	$F(35294)T(\bar{9}16)$	$F(61492)T(\bar{9}35)$
79	$-T(456)T(321)$	$F(96321)T(\bar{9}45)$	80	$-T(451)T(263)$	$F(19236)T(\bar{9}45)$
81	$-T(452)T(136)$	$F(29163)T(\bar{9}45)$	82	$-T(453)T(621)$	$F(39612)T(\bar{9}45)$
83	$F(45912)T(\bar{9}36)$	$F(36921)T(\bar{9}45)$	84	$F(45913)T(\bar{9}26)$	$F(26931)T(\bar{9}45)$
85	$F(45923)T(\bar{9}61)$	$F(61932)T(\bar{9}45)$	86	$F(49321)T(\bar{9}56)$	$-T(654)T(321)$
87	$F(59321)T(\bar{9}46)$	$-T(645)T(321)$	88	$F(93421)T(\bar{9}56)$	$-T(653)T(421)$
89	$F(95421)T(\bar{9}36)$	$-T(635)T(421)$	90	$F(39521)T(\bar{9}46)$	$-T(643)T(521)$
91	$F(49521)T(\bar{9}36)$	$-T(364)T(512)$	92	$F(29431)T(\bar{9}56)$	$-T(562)T(413)$
93	$F(59431)T(\bar{9}26)$	$-T(625)T(431)$	94	$F(29531)T(\bar{9}46)$	$-T(642)T(531)$
95	$F(49531)T(\bar{9}26)$	$-T(624)T(531)$	96	$F(29541)T(\bar{9}36)$	$-T(632)T(541)$
97	$F(39541)T(\bar{9}26)$	$-T(623)T(541)$	98	$F(91432)T(\bar{9}56)$	$-T(561)T(234)$
99	$F(59432)T(\bar{9}61)$	$-T(165)T(243)$	100	$F(91532)T(\bar{9}46)$	$-T(461)T(235)$
101	$F(49532)T(\bar{9}61)$	$-T(164)T(235)$	102	$F(91542)T(\bar{9}36)$	$-T(361)T(245)$
103	$F(39542)T(\bar{9}61)$	$-T(163)T(245)$	104	$F(91543)T(\bar{9}26)$	$-T(261)T(345)$
105	$F(92543)T(\bar{9}61)$	$-T(612)T(345)$			

APPENDIX C: p EQUATIONS FOR $N = 6$

1. $\delta\Delta_l = \bar{\Delta}_l$ equations

The 210 equations for parameters p, or t_l, can be obtained from the list E below by setting $E_{2l-1} = d_l$ and $E_{2l} = d'_l$ $(1 \le l \le 105)$. Each t combination in E_i is expressed as a signed list of m. For example, $E_1 = (1, -65, -68, 69)$ and $E_2 = (1, 6, -7, -8)$ corresponds to Eq. (74)

$$t_1 - t_{65} - t_{68} + t_{69} = d_1, \qquad t_1 + t_6 - t_7 - t_8 = d'_1.$$

The list E with 210 members is

$$
\begin{aligned}
E=\{&(1,-65,-68,69),(1,6,-7,-8),(2,-72,-75,76),(2,5,7,-9),(3,79,-82,-83),(3,-4,7,10),\\
&(-79,86,-87),(-3,4,8,-9),(-72,88,-89),(2,5,-8,-10),(-65,90,-91),(1,6,-9,10),\\
&(68,-75,-82),(-1,2,3,7),(8,-69,86,-88),(-1,4,-5,8),(9,-76,87,-90),(-2,-4,-6,9),\\
&(10,-83,89,-91),(3,-5,6,10),(11,-51,53,-54),(11,16,-17,-18),(12,-59,61,-62),(12,15,17,-19),\\
&(13,-79,81,-84),(13,-14,17,20),(79,-86,87),(-13,14,18,-19),(-59,92,-93),(12,15,-18,-20),\\
&(-51,94,-95),(11,16,-19,20),(-53,61,81),(-11,12,13,17),(18,54,-86,-92),(-11,14,-15,18),\\
&(19,62,-87,-94),(-12,-14,-16,19),(20,-84,93,-95),(13,-15,16,20),(21,-43,45,-46),(21,26,-27,-28),\\
&(22,59,60,-63),(22,25,27,-29),(23,-72,74,77),(23,-24,27,30),(72,-88,89),(-23,24,28,-29),\\
&(59,-92,93),(22,25,-28,-30),(-43,96,-97),(21,26,-29,30),(-45,60,74),(-21,22,23,27),\\
&(28,46,-88,92),(-21,24,-25,28),(29,63,-89,-96),(-22,-24,-26,29),(30,77,-93,-97),(23,-25,26,30),\\
&(31,43,44,-47),(31,36,-37,-38),(32,51,52,-55),(32,35,37,-39),(33,-65,67,70),(33,-34,37,40),\\
&(65,-90,91),(-33,34,38,-39),(51,-94,95),(32,35,-38,-40),(43,-96,97),(31,36,-39,40),\\
&(-44,52,67),(-31,32,33,37),(38,47,-90,94),(-31,34,-35,38),(39,55,-91,96),(-32,-34,-36,39),\\
&(40,70,-95,97),(33,-35,36,40),(41,-79,80,-85),(41,-42,43,48),(79,-86,87),(-41,42,46,-47),\\
&(-26,36,80),(-21,31,41,43),(-37,98,-99),(31,44,-46,-48),(-27,100,-101),(21,45,-47,48),\\
&(28,46,-86,-98),(-21,42,-44,46),(38,47,-87,-100),(-31,-42,-45,47),(48,-85,99,-101),(41,-44,45,48),\\
&(49,-72,73,78),(49,-50,51,56),(72,-88,89),(-49,50,54,-55),(-16,35,73),(-11,32,49,51),\\
&(37,-98,99),(32,52,-54,-56),(-17,102,-103),(11,53,-55,56),(18,54,-88,98),(-11,50,-52,54),\\
&(39,55,-89,-102),(-32,-50,-53,55),(56,78,-99,-103),(49,-52,53,56),(57,-65,66,71),(57,-58,59,64),\\
&(65,-90,91),(-57,58,62,-63),(-15,25,66),(-12,22,57,59),(27,-100,101),(22,60,-62,-64),\\
&(17,-102,103),(12,61,-63,64),(19,62,-90,100),(-12,58,-60,62),(29,63,-91,102),(-22,-58,-61,63),\\
&(64,71,-101,103),(57,-60,61,64),(-6,34,58),(-1,-33,-57,65),(59,-92,93),(57,66,-69,-70),\\
&(37,-98,99),(33,67,-69,-71),(7,-104,-105),(-1,68,70,71),(-8,69,92,98),(1,-66,-67,69),\\
&(40,70,-93,-104),(33,-66,68,70),(64,71,-99,-105),(57,-67,68,71),(-5,24,50),(-2,-23,-49,72),\\
&(51,-94,95),(49,73,-76,-77),(27,-100,101),(23,74,-76,-78),(-7,104,105),(-2,75,77,78),\\
&(-9,76,94,100),(2,-73,74,76),(30,77,-95,104),(23,-73,75,77),(56,78,-101,105),(49,-74,75,78),\\
&(-4,14,42),(3,-13,-41,79),(43,-96,97),(41,80,-83,84),(17,-102,103),(13,81,-83,85),\\
&(-7,104,105),(-3,82,84,85),(-10,83,96,102),(-3,-80,-81,83),(-20,84,97,104),(-13,80,82,84),\\
&(-48,85,103,105),(-41,81,82,85),(8,-18,-46,86),(4,-14,-42),(9,-19,-47,87),(-4,14,42),\\
&(-8,-28,-54,88),(5,-24,-50),(10,-29,-55,89),(-5,24,50),(-9,-38,-62,90),(6,-34,-58),\\
&(-10,-39,-63,91),(-6,34,58),(-18,28,69,92),(15,-25,-66),(20,-30,-70,93),(-15,25,66),\\
&(-19,38,76,94),(16,-35,-73),(-20,-40,-77,95),(-16,35,73),(-29,39,83,96),(26,-36,-80),\\
&(-30,40,84,97),(-26,36,80),(-46,54,69,98),(44,-52,-67),(48,-56,-71,99),(-44,52,67),\\
&(-47,62,76,100),(45,-60,-74),(-48,-64,-78,101),(-45,60,74),(-55,63,83,102),(53,-61,-81),\\
&(-56,64,85,103),(-53,61,81),(-70,77,84,104),(-68,75,82),(-71,78,85,105),(-68,75,82)\}.
\end{aligned}
\tag{C1}
$$

2. Solutions

The $210t$ equations in Eq. (C1) can be written in the matrix form $\tau \cdot t = \bar{d}$ as in Eq. (77). The 210×105 matrix τ turns out to have a rank of 79, thus possessing 131 left null vectors u_x so that $u_x \cdot \tau = 0$. As a result, for Eq. (C1) to have a solution, the inhomogeneous terms $\bar{d}_i$ must satisfy 131 sum rules of the form $R_x \coloneqq u_x \cdot \bar{d} = 0$. Of the 131 sum rules, 40 of them have two terms, 37 of them have four terms, eight of them have six terms, and the rest, 46, of them have at least 13 terms. It turns out that sum rules are often satisfied when the number of terms are less or equal to six, but they are generally not satisfied when many terms are present. Hence the t equations Eq. (C1) have no solutions.

To illustrate this statement, let us pick four null vectors, with 2, 4, 6, 13 nonzero entries, for a detailed illustration. First, let us use the t equations to verify directly that these are indeed null vectors. Once verified, their respective sum rules are then checked.

Every null vector u_x has 210 components. For those with very few nonzero entries, it is more economical to express u_x by displaying the position of its nonzero entries, together with its coefficient (usually -1) if it is not 1.

(1) $u_1 = (13, 210)$. This null vector has only two nonzero entries, at position 13 and 210, both with value 1. Since $E_{13} = (68, -75, -82)$, meaning $t_{68} - t_{75} - t_{82}$, and $E_{210} = (-68, 75, 82)$, meaning $-t_{68} + t_{75} + t_{82}$, (the left-hand side of) these two equations obviously add up to zero, proving that $u_1 = (13, 210)$ is indeed a null vector.

(2) $u_{131} = (-4, 6, 8, 10)$. The four nonzero entries are

$$-E_4 = -(2,5,7,-9), \qquad E_6 = (3,-4,7,10), \qquad E_8 = (-3,4,8,-9), \qquad E_{10} = (2,5,-8,-10).$$

These four equations do add up to be zero, so u_{131} is indeed a null vector.

(3) $u_{126} = (-2, 4, -6, -8, -14, 20)$. The six nonzero entries are

$$\begin{aligned} &-E_2 = -(1,6,-7,-8), \qquad E_4 = (2,5,7,-9), \qquad -E_6 = -(3,-4,7,10),\\ &-E_8 = -(-3,4,8,-9), \qquad -E_{14} = -(-1,2,3,7), \qquad E_{20} = (3,-5,6,10). \end{aligned}$$

They can be seen to add up to zero, so u_{126} is a null vector.

(4) $u_{127} = (1, -3, -5, -6, -7, -8, 9, -11, 13, 14, 15, -17, 19)$. The 13 nonzero entries are

$$\begin{aligned} &E_1 = (1,-65,-68,69), \qquad -E_3 = -(2,-72,-75,76), \qquad -E_5 = -(3,79,-82,-83),\\ &-E_6 = -(3,-4,7,10), \qquad -E_7 = -(-79,86,-87), \qquad -E_8 = -(-3,4,8,-9),\\ &E_9 = (-72,88,-89), \qquad -E_{11} = -(-65,90,-91), \qquad E_{13} = (68,-75,-82),\\ &E_{14} = (-1,2,3,7), \qquad E_{15} = (8,-69,86,-88), \qquad -E_{17} = -(9,-76,87,-90),\\ &E_{19} = (10,-83,89,-91). \end{aligned} \tag{C2}$$

Again, these 13 equations can be seen to add up to zero, proving that u_{127} is a null vector.

Having thus verified the correctness of the null vectors, let us now use the explicit expressions of (d_e, d'_e) in Table IX to check whether the corresponding sum rules are satisfied.

(1) $R_1 = \bar{d}_{13} + \bar{d}_{210} = d_7 + d'_{105} = -T(126)T(354) - T(612)T(345) = 0$ because of the symmetries of T.

(2)

$$\begin{aligned} R_{131} &= -\bar{d}_4 + \bar{d}_6 + \bar{d}_8 + \bar{d}_{10} = -d'_2 + d'_3 + d'_4 + d'_5\\ &= (-F(35649) + F(45693) + F(93456) + F(94356))T(\bar{9}12)\\ &= (-F(35496) - F(54936) - F(93546) - F(49356))T(\bar{9}12)\\ &= -D(35496)T(\bar{9}12) \simeq 0, \end{aligned}$$

where Eq. (27), Eq. (41), and the numerator Slavnov-Taylor identity have been used.

(3)

$$
\begin{aligned}
R_{126} &= -\bar{d}_2 + \bar{d}_4 - \bar{d}_6 - \bar{d}_8 - \bar{d}_{14} + \bar{d}_{20} = -d'_1 + d'_2 - d'_3 - d'_4 - d'_7 + d'_{10} \\
&= (-F(34659) + F(35649) - F(45693) - F(93456) - F(69543) + F(36549))T(\bar{9}12) \\
&= (F(34695) - F(53649) - F(45368) + F(93465) + F(69345) - F(36459))T(\bar{9}12) \\
&= (D(34695) - D(53649))T(\bar{9}12) \simeq 0.
\end{aligned}
$$

(4)

$$
\begin{aligned}
R_{127} &= \bar{d}_1 - \bar{d}_3 - \bar{d}_5 - \bar{d}_6 - \bar{d}_7 - \bar{d}_8 + \bar{d}_9 - \bar{d}_{11} + \bar{d}_{13} + \bar{d}_{14} + \bar{d}_{15} - \bar{d}_{17} + \bar{d}_{19} \\
&= d_1 - d_2 - d_3 - d'_3 - d_4 - d'_4 + d_5 - d_6 + d_7 + d'_7 + d_8 - d_9 + d_{10} \neq 0
\end{aligned}
$$

as can be seen from Table IX.

APPENDIX D: q EQUATIONS FOR $N=6$

This appendix provides supplementary material for Sec. VI F. Please see the main text for some of the definitions used below.

1. Gauge constraint equations

The following notations will be used for the q parameters: $q_a \to a_1, q'_a \to a_2, q''_a \to a_3$. In terms of these notations, the 144 constraint equations are given below by setting each entry to be zero. For example, Eq. (80) is given by the sixth entry below, and Eq. (81) is given by the ninth entry.

$$
\begin{aligned}
&1_1-2_1-5_1-6_1+14_1, -1_1+2_1+8_1-9_1+15_1, 1_2-2_2+79_2+81_2-93_1, 1_3+14_2+79_3+82_2, \\
&\quad -1_3+15_2-79_3+83_2, 2_1+3_1+7_1+8_1+10_1, -2_1-3_1+4_1-5_1+12_1, 2_2+3_2+90_1+91_1+93_1, \\
&2_3+8_3+93_2+105_2, -2_3-5_3-93_2-99_2, 3_3+10_3+64_3+66_3+90_3, -3_3+12_3+73_3+75_3-90_3, \\
&\quad -4_1+5_1-8_1+9_1+13_1, 4_2-5_2+70_2+72_2-99_1, 4_3+12_2+70_3+73_2, -4_3+13_2-70_3+74_2, \\
&5_1+6_1-7_1-8_1+11_1, 5_2+6_2+96_1+97_1+99_1, 5_3-8_3+99_2-105_2, 6_3+11_3+65_3+68_3+96_3, \\
&\quad -6_3+14_3+82_3+84_3-96_3, 7_2+8_2+61_2+63_2+105_1, 7_3+10_2+61_3+64_2, -7_3+11_2-61_3+65_2, \\
&\quad -8_2+9_2+102_1+103_1-105_1, 9_3+13_3+74_3+77_3+102_3, -9_3+15_3+83_3+86_3-102_3, 16_1-17_1-20_1-21_1+29_1, \\
&\quad -16_1+17_1+23_1-24_1+30_1, 16_2-17_2+71_1-72_1+78_1, 16_3+29_2+71_3+94_2, -16_3+30_2-71_3+95_2+17_1, \\
&18_1+22_1+23_1+25_1, -17_1-18_1+19_1-20_1+27_1, 17_2+18_2-75_1-76_1-78_1, 17_3+23_3-78_2-103_2, \\
&\quad -17_3-20_3+78_2+87_2, 18_3+25_3-64_3-66_3-75_3, -18_3+27_3+75_3+88_3-90_3, -19_1+20_1-23_1+24_1+28_1, \\
&19_2-20_2+80_1-81_1+87_1, 19_3+27_2+80_3+88_2, -19_3+28_2-80_3+89_2, 20_1+21_1-22_1-23_1+26_1, \\
&20_2+21_2-84_1-85_1-87_1, 20_3-23_3-87_2+103_2, 21_3+26_3-65_3-68_3-84_3, -21_3+29_3+84_3+94_3-96_3, \\
&22_2+23_2-61_2-63_2-103_1, 22_3+25_2-61_3-64_2, -22_3+26_2+61_3-65_2, -23_2+24_2+103_1+104_1-105_1, \\
&24_3+28_3+89_3+92_3+104_3, -24_3+30_3+95_3+98_3-104_3, 31_1-32_1-35_1-36_1+44_1, -31_1+32_1+38_1-39_1+45_1, \\
&31_2-32_2+62_1-63_1+69_1, 31_3+44_2+62_3+100_2, -31_3+45_2-62_3+101_2, 32_1+33_1+37_1+38_1+40_1, \\
&\quad -32_1-33_1+34_1-35_1+42_1, 32_2+33_2-66_1-67_1-69_1, 32_3+38_3-69_2-97_2, -32_3-35_3+69_2+85_2+33_3, \\
&40_3-66_3-73_3-75_3, -33_3+42_3+66_3-88_3+90_3, -34_1+35_1-38_1+39_1+43_1, 34_2-35_2-80_1+81_1+85_1, \\
&34_3+42_2-80_3-88_2, -34_3+43_2+80_3-89_2, 35_1+36_1-37_1-38_1+41_1, 35_2+36_2-85_1-86_1-87_1, \\
&35_3-38_3-85_2+97_2, 36_3+41_3-74_3-77_3-86_3, -36_3+44_3+86_3+100_3-102_3, 37_2+38_2-70_2+72_2-97_1, \\
&37_3+40_2-70_3-73_2, -37_3+41_2+70_3-74_2, -38_2+39_2+97_1-98_1+99_1, 39_3+43_3-89_3-92_3-98_3,
\end{aligned}
$$

$$-39_3+45_3+98_3+101_3-104_3,46_1-47_1-50_1-51_1+59_1,-46_1+47_1+53_1-54_1+60_1,46_2-47_2-62_1+63_1+67_1,$$
$$46_3+59_2-62_3-100_2,-46_3+60_2+62_3-101_2,47_1+48_1+52_1+53_1+55_1,-47_1-48_1+49_1-50_1+57_1,$$
$$47_2+48_2-67_1-68_1-69_1,47_3+53_3-67_2-91_2,-47_3-50_3+67_2+76_2,48_3+55_3-68_3-82_3-84_3,$$
$$-48_3+57_3+68_3-94_3+96_3,-49_1+50_1-53_1+54_1+58_1,49_2-50_2-71_1+72_1+76_1,49_3+57_2-71_3-94_2,$$
$$-49_3+58_2+71_3-95_2,50_1+51_1-52_1-53_1+56_1,50_2+51_2-76_1-77_1-78_1,50_3-53_3-76_2+91_2,$$
$$51_3+56_3-77_3-83_3-86_3,-51_3+59_3+77_3-100_3+102_3,52_2+53_2-79_2-81_2-91_1,52_3+55_2-79_3-82_2,$$
$$-52_3+56_2+79_3-83_2,-53_2+54_2+91_1-92_1+93_1,54_3+58_3-92_3-95_3-98_3,-54_3+60_3+92_3-101_3+104_3,$$
$$61_1+63_1+64_1+66_1+67_1,-61_1-63_1+65_1+68_1+69_1,62_2-63_2+100_1-102_1-103_1,-62_2+63_2+101_1-104_1+105_1,$$
$$63_3+67_3+91_3+93_3+105_3,-63_3+69_3+97_3+99_3-105_3,-63_3-67_3-76_3-78_3-103_3,63_3-69_3-85_3-87_3+103_3,$$
$$66_2+67_2+90_2+91_2,-66_2-67_2-75_2-76_2,-67_3-69_3-72_3-76_3-97_3,67_3+69_3+81_3+85_3+91_3,$$
$$68_2+69_2+96_2+97_2,-68_2-69_2-84_2-85_2,70_1+72_1+73_1+75_1+76_1,-70_1-72_1+74_1+77_1+78_1,$$
$$71_2-72_2+94_1-96_1-97_1,-71_2+72_2+95_1+98_1-99_1,72_3+76_3-91_3-93_3-99_3,-72_3+78_3+99_3+103_3-105_3,$$
$$-72_3+78_3+85_3+87_3-97_3,75_2+76_2-90_2-91_2,76_3+78_3-81_3+87_3-91_3,77_2+78_2+102_2+103_2,$$
$$-77_2-78_2-86_2-87_2,79_1+81_1+82_1+84_1+85_1,-79_1-81_1+83_1+86_1+87_1,80_2-81_2+88_1-90_1-91_1,$$
$$-80_2+81_2+89_1+92_1-93_1,81_3+85_3-93_3-97_3-99_3,-81_3+87_3+93_3-103_3+105_3,84_2+85_2-96_2-97_2,$$
$$86_2+87_2-102_2-103_2,92_2-93_2+104_2-105_2,-92_2+93_2-98_2+99_2,98_2-99_2-104_2+105_2. \tag{D1}$$

2. $\delta\Delta_l = \bar{\Delta}_l$ equations

There are three groups of equations, X, Y, Z. Group X equations consist of setting the lth member of the following list to g_l, where g_l is the ss coefficient of $\bar{\Delta}_l$ in Eq. (B1). For example, Eq. (83) is given by the first member of X, and Eq. (87) is given by the fourth member of X. A member of X has three terms if $\bar{\Delta}_l$ belongs to I_l, and has four terms of $\bar{\Delta}_l$ belongs to J_l. Altogether X has 105 members.

$$X=\{-1_3,-2_3+3_3,-4_3-5_3+6_3,7_3-8_3-9_3,-7_2+10_3-11_3-7_3,-4_2+12_3-13_3-4_3,$$
$$-1_2+14_3-15_3-1_3,2_2-5_2-8_2+2_3,-3_2+10_2-12_2,-6_2+11_2-14_2,-9_2+13_2-15_2,$$
$$-16_3-17_3+18_3,-19_3-20_3+21_3,22_3-23_3-24_3-22_2,25_3-26_3-22_3,-19_2+27_3-28_3-19_3,$$
$$-16_2+29_3-30_3-16_3,17_2-20_2-23_2+17_3,-18_2+25_2-27_2,-21_2+26_2-29_2,-24_2+28_2-30_2,$$
$$-31_3-32_3+33_3,-34_3-35_3+36_3,37_3-38_3-39_3,-37_2+40_3-41_3-37_3,-34_2+42_3-43_3-34_3,$$
$$-31_2+44_3-45_3-31_3,32_2-35_2-38_2+32_3,-33_2+40_2-42_2,-36_2+41_2-44_2,-39_2+43_2-45_2,$$
$$-46_3-47_3+48_3,-49_3-50_3+51_3,52_3-53_3-54_3,-52_2+55_3-56_3-52_3,-49_2+57_3-58_3-49_3,$$
$$-46_2+59_3-60_3-46_3,47_2-50_2-53_2+47_3,-48_2+55_2-57_2,-51_2+56_2-59_2,-54_2+58_2-60_2,$$
$$61_3-62_3-63_3,-61_2+64_3-65_3-61_3,31_1-46_1-62_2+31_3,-47_1+66_3-67_3-47_3,-32_1+68_3-69_3-32_3,$$
$$-33_1+64_2-66_2,-48_1+65_2-68_2,-63_2+67_2-69_2,70_3-71_3-72_3,-70_2+73_3-74_3-70_3,$$
$$16_1-49_1-71_2+16_3,-50_1+75_3-76_3-50_3,-17_1+77_3-78_3-17_3,-18_1+73_2-75_2,-51_1+74_2-77_2,$$
$$-72_2+76_2-78_2,79_3-80_3-81_3,-79_2+82_3-83_3-79_3,19_1-34_1-80_2-19_3,-35_1+84_3-85_3-35_3,$$
$$-20_1+86_3-87_3-20_3,-21_1+82_2-84_2,-36_1+83_2-86_2,-81_2+85_2-87_2,1_1-52_1-79_1+1_3,$$
$$-80_1+88_3-89_3-80_3,-53_1+90_3-91_3-53_3,2_1-92_3-93_3+2_3,3_1-88_2-90_2,-54_1+89_2-92_2,$$
$$-81_1+91_2-93_2,4_1-37_1-70_1+4_3,-71_1+94_3-95_3-71_3,-38_1+96_3-97_3-38_3,5_1-98_3-99_3+5_3,$$

$$\begin{aligned}
&6_1-94_2-96_2,-39_1+95_2-98_2,-72_1+97_2-99_2,7_1-22_1-61_1+7_3,-62_1+100_3-101_3-62_3,\\
&-23_1+102_3-103_3-23_3,8_1-104_3-105_3+8_3,9_1-100_2-102_2,24_1-101_2-104_2,63_1-103_2-105_2,\\
&10_1-25_1-64_1+10_2,11_1-26_1-65_1+11_2,12_1-40_1-73_1+12_2,13_1-41_1-74_1+13_2,14_1-55_1-82_1+14_2,\\
&15_1-56_1-83_1+15_2,27_1-42_1-88_1+27_2,28_1-43_1-89_1+28_2,29_1-57_1-94_1+29_2,30_1-58_1-95_1+30_2,\\
&44_1-59_1-100_1+44_2,45_1-60_1-101_1+45_2,66_1-75_1-90_1+66_2,67_1-76_1-91_1+67_2,68_1-84_1-96_1+68_2,\\
&69_1-85_1-97_1+69_2,77_1-86_1-102_1+77_2,78_1-87_1-103_1+78_2,92_1-98_1-104_1+92_2,93_1-99_1-105_1+93_2.\}
\end{aligned} \tag{D2}$$

Group Y consists of 60 pairs of equations, namely 120 equations. They are listed below. The three quantities within each pair of parentheses are equal. These equations occur only for $\bar{\Delta}_l$ in class J_l. For example, Eq. (89) is given by the first member of Y.

$$\begin{aligned}
Y=\{&(-7_3,10_2,-11_2),(-4_3,12_2,-13_2),(-1_3,14_2,-15_2),(2_3,-5_3,\bar{8}_3),(-2_3,25_2,-26_2),\\
&(-19_3,27_2,-28_2),(-16_3,29_2,-30_2),(17_3,-20_3,-23_3),(-37_3,40_2,-41_2),(-34_3,42_2,-43_2),\\
&(-31_3,44_2,-45_2),(32_3,-35_3,-38_3),(-52_3,55_2,-56_2),(-49_3,57_2,-58_2),(-46_3,59_2,-60_2),\\
&(47_3,-50_3,-53_31),(-61_3,64_2,-65_2),(31_3,-46_3,-62,3),(-47_3,66_2,-67_2),(-32_3,68_2,-69_2),\\
&(-70_3,73_2,-74_2),(16_3,-49_3,-71_3),(-50_3,75_2,-76_2),(-17_3,77_2,-78_2),(-79_3,82_2,-83_2),\\
&(19_3,-34_3,-80_3),(-35_3,84_2,-85_2),(-20_3,86_2,-87_2),(1_3,-52_3,-79_3),(-80_3,88_2,-89_2),\\
&(-53_3,90_2,-91_2),(2_3,-92_2,-93_2),(4_3,-37_3,-70_31),(-71_3,94_2,-95_2),(-38_3,96_2,-97_2),\\
&(5_3,-98_2,-99_2),(7_3,-22_3,-61_3),(-62_3,100_2,-101_2),(-23_3,102_2,-103_2),(8_3,-104_2,-105_2),\\
&(10_2,-25_2,-64_2),(11_2,-26_2,-65_2),(12_2,-40_2,-73_2),(13_2,-41_2,-74_2),(14_2,-55_2,-82_2),\\
&(15_2,-56_2,-83_2),(27_2,-42_2,-88_2),(28_2,-43_2,-89_2),(29_2,-57_2,-94_2),(30_2,-58_2,-95_2),\\
&(44_2,-59_2,-100_2),(45_2,-60_2,-101_2),(66_2,-75_2,-90_2),(67_2,-76_2,-91_2),(68_2,-84_2,-96_2),\\
&(69_2,-85_2,-97_2),(77_2,-86_2,-102_2),(78_2,-87_2,-103_2),(92_2,-98_2,-104_2),(93_2,-99_2,-105_2)\}.
\end{aligned} \tag{D3}$$

Group Z consists of the following $225q$ parameters which are zero.

$$\begin{aligned}
Z=\{&1_2,1_1,2_2,2_1,3_3,3_2,3_1,4_2,4_1,5_2,5_1,6_3,6_2,6_1,7_2,7_1,8_2,8_1,9_3,9_2,\\
&9_1,10_3,10_1,11_3,11_1,12_3,12_1,13_3,13_1,14_3,14_1,15_3,15_1,16_2,16_1,17_2,17_1,18_3,18_2,18_1,\\
&19_2,19_1,20_2,20_1,21_3,21_2,21_1,22_2,22_1,23_2,23_1,24_3,24_2,24_1,25_3,25_1,26_3,26_1,27_3,27_1,\\
&28_3,28_1,29_3,29_1,30_3,30_1,31_2,31_1,32_2,32_1,33_3,33_2,33_1,34_2,34_1,35_2,35_1,36_3,36_2,36_1,\\
&37_2,37_1,38_2,38_1,39_3,39_2,39_1,40_3,40_1,41_3,41_1,42_3,42_1,43_3,43_1,44_3,44_1,45_3,45_1,46_2,\\
&46_1,47_2,47_1,48_3,48_2,48_1,49_2,49_1,50_2,50_1,51_3,51_2,51_1,52_2,52_1,53_2,53_1,54_3,54_2,54_1,\\
&55_3,55_1,56_3,56_1,57_3,57_1,58_3,58_1,59_3,59_1,60_3,60_1,61_2,61_1,62_2,62_1,63_3,63_2,63_1,64_3,\\
&64_1,65_3,65_1,66_3,66_1,67_3,67_1,68_3,68_1,69_3,69_1,70_2,70_1,71_2,71_1,72_3,72_2,72_1,73_3,73_1,\\
&74_3,74_1,75_3,75_1,76_3,76_1,77_3,77_1,78_3,78_1,79_2,79_1,80_2,80_1,81_3,81_2,81_1,82_3,82_1,83_3,\\
&83_1,84_3,84_1,85_3,85_1,86_3,86_1,87_3,87_1,88_3,88_1,89_3,89_1,90_3,90_1,91_3,91_1,92_3,92_1,93_3,\\
&93_1,94_3,94_1,95_3,95_1,96_3,96_1,97_3,97_1,98_3,98_1,99_3,99_1,100_3,100_1,101_3,101_1,102_3,102_1,103_3,\\
&103_1,104_3,104_1,105_3,105_1\}.
\end{aligned} \tag{D4}$$

For example, Eq. (84) corresponds to members $1_1, 1_2, 2_1, 2_2, 3_1, 3_2$ of Z, and Eq. (88) corresponds to members $7_1, 10_1, 11_1$ of Z.

3. Solutions

The $3 \times 105 = 315$ q parameters must satisfy the 144 gauge constraint equations Eq. (D1), the 105 equations of group X, the 120 equations of group Y, and the 225 equations of group Z. With the equations numbers so much larger than the number of q parameters, there would be no solutions unless these equations are highly degenerate.

To show that there is really no solution, one can proceed as follows. By substituting Z into Y, more q parameters must vanish. Substituting these vanishing parameters into X, one gets 105 equations relating q's on the left-hand side to g_l on the right-hand side. Since $g_l = 0$ for the $60l$'s with $l \in J_l$, the combination of q on the left must be zero as well. It turns out that for those l, there is always a single q term left on the left, so in this way one gets even more zero q parameters. With all these vanishing q parameters substituted into the left-hand side of the remaining $45X$ equations for $l \in I_l$, where $g_l \neq 0$, there can be no solutions if any of the left-hand side of the 45 equations is identically zero, namely, has no nonvanishing q parameters left. This turns out to be indeed the case for $l = 7, 8, 13, 14, 19, 20, 37, 40, 43, 44, 45$. Thus even without ever using the 144 gauge constraint equations Eq. (D1), one sees that the X, Y, Z solutions have no solutions as long as $g_l \neq 0$ for $l \in J_l$.

[1] H. Kawai, D. C. Lewellen, and S. H. H. Tye, A relation between tree amplitudes of closed and open strings, Nucl. Phys. **B269**, 1 (1986).

[2] D.-p. Zhu, Zeros in scattering amplitudes and the structure of non-Abelian gauge theories, Phys. Rev. D **22**, 2266 (1980).

[3] Z. Bern, J. J. M. Carrasco, and H. Johansson, New relations for gauge-theory amplitudes, Phys. Rev. D **78**, 085011 (2008).

[4] Z. Bern, J. J. Carrasco, M. Chiodaroli, H. Johansson, and R. Roiban, The duality between color and kinematics and its applications, arXiv:1909.01358.

[5] C. Cheung and J. Mangan, Covariant color-kinematics duality, J. High Energy Phys. 11 (2021) 069.

[6] J. Broedel and J. J. M. Carrasco, Virtuous trees at five and six points for Yang-Mills and gravity, Phys. Rev. D **84**, 085009 (2011).

[7] F. Cachazo, S. He, and E. Y. Yuan, Scattering equations and Kawai-Lewellen-Tye orthogonality, Phys. Rev. D **90**, 065001 (2014).

[8] F. Cachazo, S. He, and E. Y. Yuan, Scattering of Massless Particles in Arbitrary Dimensions, Phys. Rev. Lett. **113**, 171601 (2014).

[9] F. Cachazo, S. He, and E. Y. Yuan, Scattering of massless particles: Scalars, gluons and gravitons, J. High Energy Phys. 07 (2014) 033.

[10] C. S. Lam and Y.-P. Yao, Evaluation of the CHY gauge amplitude, Phys. Rev. D **93**, 105008 (2016).

[11] Y.-J. Du and F. Teng, BCJ numerators from reduced Pfaffian, J. High Energy Phys. 04 (2017) 033.

[12] C. S. Lam, Pfaffian diagrams for gluon tree amplitudes, Phys. Rev. D **98**, 076002 (2018).

[13] C. S. Lam, Off-shell Yang-Mills amplitude in the CHY formalism, Phys. Rev. D **100**, 045009 (2019).

[14] A. Edison and F. Teng, Efficient calculation of crossing symmetric BCJ tree numerators, J. High Energy Phys. 12 (2020) 138.

[15] V. Del Duca, L. J. Dixon, and F. Maltoni, New color decompositions for gauge amplitudes at tree and loop level, Nucl. Phys. **B571**, 51 (2000).

[16] N. E. J. Bjerrum-Bohr, P. H. Damgaard, and P. Vanhove, Minimal Basis for Gauge Theory Amplitudes, Phys. Rev. Lett. **103**, 161602 (2009).

[17] S. Stieberger, Open and closed vs pure open string disk amplitudes, arXiv:0907.2211.

[18] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard, and P. Vanhove, Monodromy and Jacobi-like relations for color-ordered amplitudes, J. High Energy Phys. 06 (2010) 003.

[19] D. Vaman and Y.-P. Yao, Constraints and generalized gauge transformations on tree-level gluon and graviton amplitudes, J. High Energy Phys. 11 (2010) 028.

[20] D. Vaman and Y.-P. Yao, Color kinematic symmetric (BCJ) numerators in a light-like gauge, J. High Energy Phys. 12 (2014) 036.

[21] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard, and P. Vanhove, The momentum kernel of gauge and gravity theories, J. High Energy Phys. 01 (2011) 001.

[22] C. S. Lam and Y.-P. Yao, Off-shell CHY amplitudes, Nucl. Phys. **B907**, 678 (2016).

[23] C. S. Lam, CHY theory for several fields, Phys. Rev. D **102**, 025018 (2020).